RAIRO-Theor. Inf. Appl. 50 (2016) 263–272
DOI: 10.1051/ita/2016027

Available online at:
www.rairo-ita.org

THE AVERAGE SCATTERING NUMBER OF GRAPHS

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Abstract. The scattering number of a graph is a measure of the vulnerability of a graph. In this paper we investigate a refinement that involves the average of a local version of the parameter. If v is a vertex in a connected graph G, then $sc_v(G) = \max\{\omega(G - S_v) - |S_v|\}$, where the maximum is taken over all disconnecting sets S_v of G that contain v. The average scattering number of G denoted by $sc_{av}(G)$, is defined as $sc_{av}(G) = \frac{\Sigma_{v \in V(G)} sc_v(G)}{n}$, where n will denote the number of vertices in graph G. Like the scattering number itself, this is a measure of the vulnerability of a graph, but it is more sensitive. Next, the relations between average scattering number and other parameters are determined. The average scattering number of some graph classes are obtained. Moreover, some results about the average scattering number of graphs obtained by graph operations are given.

Mathematics Subject Classification. 05C40, 05C69, 68M10, 68R10.

1. Introduction

In this paper we consider only finite and undirected graphs, and have no loops or multiple edges. Let $G = (V;E)$ be a connected graph and v a vertex in G. Furthermore, S will denote a proper subset of V, and S_v will denote one that contains v. In graph theory, isomorphism of graphs G and H is a bijection between the vertex sets of G and H $f : V(G) \to V(H)$ such that any two vertices u and v of G are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H. This kind of bijection is generally called edge-preserving bijection, in accordance with the general notion of isomorphism being a structure-preserving bijection. If an isomorphism exists between two graphs, then the graphs are called isomorphic and we write $G \cong H$. Let $deg(u)$ denote the degree of the vertex u in G.

It is known that communication systems are often exposed to failures and attacks. So robustness of the network topology is a key aspect in the design of computer networks. The stability of a communication network, composed of processing nodes and communication links, is of prime importance to network designers. Parameters used to measure vulnerability include connectivity [3], average lower connectivity [1], and scattering number [7].

The connectivity of a graph G is the minimum number of vertices whose removal from G results in a disconnected or trivial graph and is denoted by $\kappa(G)$. A. [1] introduced the concept of average lower connectivity.

Keywords and phrases. Connectivity, rupture degree, scattering number, average lower domination number, average lower independence number.

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Article published by EDP Sciences

For a vertex v of a graph G, the lower connectivity, denoted by $s_v(G)$, is the smallest number of vertices that contains v and those vertices whose deletion from G produces a disconnected or a trivial graph. The average lower connectivity denoted by $\kappa_{av}(G)$, is the value $\frac{\Sigma_{v\in V(G)} s_v(G)}{n}$, $\Sigma_{v\in V(G)} s_v(G)$ will denote the sum over all vertices of G. The toughness [2] $t(G)$ of G is defined by $t(G) = \min\{\frac{|S|}{\omega(G-S)} : S \subseteq V(G) \textit{ and } \omega(G-S) > 1\}$ where $\omega(G-S)$ denotes the number of components in $G-S$.

The concept of scattering number was introduced by Jung in [7]. In 1989, Ouyang *et al.* [8] for the first time proposed to use the scattering number of graphs to measure the vulnerability of networks. They obtained some basic results on scattering number of trees and an analysis of the scattering number of Harary graphs. In [9], the authors gave some results on the relationship between the scattering number and some other parameters of graph. Unlike the other measures, the scattering number shows not only the difficulty to break down the network but also the damage that has been caused.

The scattering number $sc(G)$ of G is defined by

$$sc(G) = \max\{\omega(G-S) - |S| : S \subset V(G), \omega(G-S) \geq 2\}$$

where $\omega(G-S)$ denotes the number of components in $G-S$. A set S such that $\omega(G-S) \neq 1$ and $\omega(G-S)-|S| = sc(G)$ is called a scattering set of G.

The average parameters have been found to be more useful in some circumstance than the corresponding measures based on worst-case situation [6]. Therefore, incorporating the concept of the scattering number and the idea of the average lower connectivity, we introduce a new graph parameter called the average scattering number in this paper.

1.1. The average scattering number

We investigate a refinement that involves the local scattering number. If v is a vertex in a connected graph G, then $sc_v(G) = \max\{\omega(G-S_v) - |S_v|, \omega(G-S_v) \geq 2\}$, where $\omega(G-S_v)$ denotes the number of components of the graph $G-S_v$ and the maximum is taken over all disconnecting sets S_v of G that contain v.

Note that for a graph G of order n, $sc(G) = -n$ if and only if G is isomorphic to the complete graph K_n [4]. In particular, the local scattering number of a complete graph K_n is defined to be $-n$.

The average scattering number of G denoted by $sc_{av}(G)$, is defined as

$$sc_{av}(G) = \frac{\Sigma_{v\in V(G)} sc_v(G)}{n},$$

where $\Sigma_{v\in V(G)} sc_v(G)$ will denote the sum over all vertices of G.

Clearly, for any graph G, $sc(G) = \max\{sc_v(G) : v \in V(G)\}$. A local scattering set of G is any (strict) subset S_v of $V(G)$ for which $\omega(G-S_v) - |S_v| = sc_v(G)$. In particular, the average scattering number of a complete graph K_n is defined to be $-n$.

For example, consider the graph G in Figure 1, where $|V(G)| = 5$ and $|E(G)| = 5$. It can be easily seen that $sc_{u_1} = sc_{u_4} = sc_{u_5} = 0$ and $sc_{u_2} = sc_{u_3} = 1$. It follows that $sc_{av}(G) = (0+1+1+0+0)/5 = 2/5$.

1.2. Motivation

Given two graphs, one can ask the following question: is the average scattering number a suitable parameter, regarding vulnerability? In other words, does the average scattering number distinguish between them?

Example 1.1. It can be easily seen that the connectivity of a path $P_n\,(n \geq 4)$ and a star $K_{1,n-1}\,(n \geq 4)$ are equal: $\kappa(P_n) = \kappa(K_{1,n-1}) = 1$. On the other hand, the average scattering numbers of a path $P_n\,(n \geq 4)$ and a star $K_{1,n-1}\,(n \geq 4)$ are different: $sc_{av}(K_{1,n-1}) = \frac{n^2-4n+2}{n}$ and $sc_{av}(P_n) = \frac{n-2}{n}$.

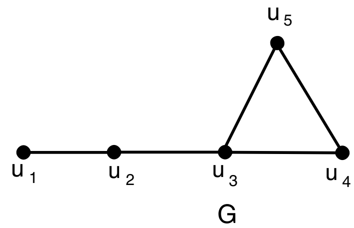

FIGURE 1. The graph G.

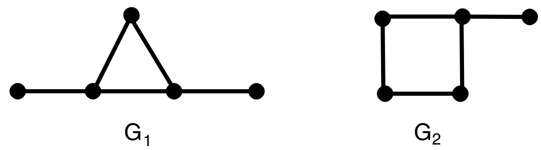

FIGURE 2. The graphs G_1 and G_2.

Example 1.2. It can be easily seen that the scattering number of a path P_7 and a complete bipartite $K_{3,4}$ are equal: $sc(P_7) = sc(K_{3,4}) = 1$. On the other hand, the average scattering number of a path P_7 and a complete bipartite $K_{3,4}$ are different: $sc_{av}(K_{3,4}) = \frac{-1}{7}$ and $sc_{av}(P_7) = \frac{5}{7}$.

Example 1.3. Let G_1 and G_2 be the graphs presented in Figure 2.

It can be easily seen that the connectivity and scattering number of these graphs are equal

$$\kappa(G_1) = \kappa(G_2) = sc(G_1) = sc(G_2) = 1.$$

On the other hand, the average scattering number of G_1 and G_2 are different

$$sc_{av}(G_1) = 2/5;$$

$$sc_{av}(G_2) = 1/5.$$

The results could be checked by readers.

Another example, $sc(P_n) = 1$, but $sc_{av}(P_n) = \frac{n-2}{n}$. It is easy to see that the scattering number of P_n is always constant value. On the other hand, the average scattering number of P_n is not constant, that is always variable value.

These examples means that the average scattering number can be more efficient compared with the other vulnerability parameters. If we want to choose the stabler graph among the graphs which have the same order and the same size, one way is to choose the graph with minimum average scattering number. Graphs with large average scattering number are more vulnerable. In order to reconstruct a disrupted network easily, the number of connected components, formed after the vertices deleted, should be possibly small.

In Section 2, we give theorems related to average scattering number and graph parameters. In Section 3, the average scattering number of some graph classes are obtained. In Section 4, some results about the average scattering number of graphs obtained by graph operations are given.

2. Bounds for average scattering number

In this section, we give theorems related to average scattering number and graph parameters.

Lemma 2.1. *Let $G=(V,E)$ be a connected graph. If $sc_v(G)\leq k$ for all $v\in V$, then $sc_{av}(G)\leq k$. Moreover, if $sc_v(G)\geq k$ for all $v\in V$, then $sc_{av}(G)\geq k$.*

Theorem 2.2. *If G is a noncomplete graph of order n, then*

$$sc_{av}(G)\leq sc(G).$$

Proof. For any graph G, $sc(G)=\max\{sc_v(G):v\in V(G)\}$ and so we have

$$sc_{av}(G)\leq sc(G).$$ □

Theorem 2.3. *If G is a noncomplete graph of order n with the scattering set is unique, then*

$$sc_{av}(G)<sc(G).$$

Proof. Let $v\in V(G)$ and S_1 be a scattering set of G. If $v\in S_1$, then we get $sc_v(G)=sc(G)$. If $v\notin S_1$, then we get $sc_v(G)<sc(G)$. So we have

$$sc_{av}(G)<sc(G).$$ □

Theorem 2.4. *If G is a noncomplete graph of order n, covering number β and independence number α, then*

$$sc_{av}(G)\geq \alpha-\beta-2.$$

Proof. Let $v\in V(G)$ and S_v be a cutset of G. It can be easily seen that there is a local scattering set S_v^* of G such that $|S_v^*|=\beta+1$ and then we have $\omega(G-S_v^*)=\alpha-1$. So

$$sc_v(G)=\max\{\omega(G-S_v)-|S_v|\}\geq\omega(G-S_v^*)-|S_v^*|=(\alpha-1)-(\beta+1)$$

$$sc_v(G)\geq\alpha-\beta-2.$$

By Lemma 2.1, we get

$$sc_{av}(G)\geq\alpha-\beta-2.$$ □

Theorem 2.5. *If G is a noncomplete graph of order n, covering number β, independence number α and $\beta=\alpha$, then*

$$sc_{av}(G)\geq -1.$$

Proof. Let $v\in V(G)$, S_v be a cutset of G and M be a minimum covering set of G.

If $v\in V(M)$, then it can be easily seen that there is a local scattering set S_v^* such that $|S_v^*|=\beta$. Then $\omega(G-S_v^*)=\alpha$ and we have

$$sc_v(G)=\max\{\omega(G-S_v)-|S_v|\}\geq\omega(G-S_v^*)-|S_v^*|=\alpha-\beta=0, \tag{2.1}$$

for $v\in V(M)$.

If $v\notin V(M)$, then it can be easily seen that there is a local scattering set S_v^* such that $|S_v^*|=\beta+1$. Then $\omega(G-S_v^*)=\alpha-1$ and we have

$$sc_v(G)=\max\{\omega(G-S_v)-|S_v|\}\geq\omega(G-S_v^*)-|S_v^*|=(\alpha-1)-(\beta+1)=-2, \tag{2.2}$$

for $v\notin V(M)$.

Thus, by (2.1), (2.2) and Lemma 2.1,

$$sc_{av}(G) \geq \frac{0.\beta + (-2).\alpha}{n} = \frac{-2.\alpha}{2.\alpha} = -1. \qquad \square$$

Proposition 2.6. *If G is a noncomplete graph of order n and covering number β, then*

$$sc_{av}(G) \geq 1 - \beta.$$

Theorem 2.7. *If G is a noncomplete graph of order n, independence number α, and toughness t, then*

$$sc_{av}(G) \leq \begin{cases} 2(1-t), & \text{if} \quad t > 1\,; \\ \alpha(1-t), & \text{if} \quad t \leq 1. \end{cases}$$

Proof. Let S be a scattering set of G. By Theorem 2.2, we have

$$sc_{av}(G) \leq sc(G) = \omega(G-S) - |S|$$

$$\frac{sc_{av}(G)}{\omega(G-S)} \leq 1 - \frac{|S|}{\omega(G-S)}$$

By the definition of toughness $\frac{|S|}{\omega(G-S)} \geq t$ for any scattering set S. Then

$$\frac{sc_{av}(G)}{\omega(G-S)} \leq 1 - t.$$

So we have,

$$sc_{av}(G) \leq \omega(G-S)(1-t)$$

We know that $2 \leq \omega(G-S) \leq \alpha$ for any scattering set S. If $t > 1$ then $(1-t)$ is negative and so $\omega(G-S)$ should be minimum, $2 \leq \omega(G-S)$ and we have

$$sc_{av}(G) \leq 2(1-t).$$

If $t \leq 1$ then $(1-t)$ is positive and so $\omega(G-S)$ should be maximum, $\omega(G-S) \leq \alpha$ and we have

$$sc_{av}(G) \leq \alpha(1-t). \qquad \square$$

Corollary 2.8. *If G is a noncomplete graph of order n and independence number α, then*

$$sc_{av}(G) \leq \alpha.$$

Proposition 2.9. *If G is a noncomplete r-regular graph, then*

$$sc_{av}(G) \geq -r.$$

Lemma 2.10. *Let H be a spanning subgraph of a noncomplete connected graph G, then*

$$sc_{av}(H) \geq sc_{av}(G).$$

Theorem 2.11 [10]**.** *Let G be a noncomplete connected graph of order $n(\geq 3)$;*

(a) *the length of a longest path is p, then $sc(G) \leq n - p$.*
(b) $sc(G) \leq n - 2\kappa$.
(c) $sc(G) \leq \alpha - \kappa$.

Theorem 2.12. *If G is a noncomplete graph of order $n \geq 4$ and longest path length p, then*

$$sc_{av}(G) \leq n - p.$$

Proof. By Theorems 2.2 and 2.11, we have

$$sc_{av}(G) \leq n - p.$$

□

Theorem 2.13. *If G is a noncomplete graph of order n, then*

$$sc_{av}(G) \leq n - 2\kappa.$$

Proof. The proof is similar to Theorem 2.12. □

Theorem 2.14. *If G is a noncomplete graph of order n, connectivity κ and independence number α, then*

$$sc_{av}(G) \leq \alpha - \kappa.$$

Proof. The proof is similar to Theorem 2.12. □

3. The average scattering number of some graph classes

In this section, the average scattering number of some graph classes are obtained.

Theorem 3.1. *If T is a tree of order $n \geq 4$ having k leaves, then*

$$sc_{av}(T) \geq \frac{n-k}{n}\cdot$$

Proof. Let $v \in V(T)$ and S_v^* be a local scattering set of T. Let W be the set of the k leaves of T.

If $v \in W$ and $|S_v^*| = r$ then we have $\omega(T - S_v^*) \geq r$ and we have

$$sc_v(T) = \omega(T - S_v^*) - |S_v^*| \geq r - r = 0$$

for k leaves.

If $v \notin W$ and $|S_v^*| = r$ then we have $\omega(T - S_v^*) \geq r + 1$ and we get

$$sc_v(T) = \omega(T - S_v^*) - |S_v^*| \geq r + 1 - r = 1$$

for $n - k$ vertices with degree at least 2. Thus,

$$sc_{av}(T) \geq \frac{k.0 + (n-k).1}{n} = \frac{n-k}{n}\cdot$$

□

Theorem 3.2. *If P_n is a path graph of order $n \geq 4$, then*

$$sc_{av}(P_n) = \frac{n-2}{n}\cdot$$

Proof. Let $v \in V(P_n)$ and S_v^* be a local scattering set of P_n. A path graph has 2 vertices with degree 1 and $n-2$ vertices with degree 2. Let $V(P_n)$ be partitioned into V_1 and V_2 such that V_1 contains the 2 vertices of degree one in P_n and V_2 contains all the remaining vertices.

If $v \in V_1$ then $|S_v^*| = 2$ and $\omega(P_n - S_v^*) = 2$. So $sc_v(P_n) = 0$.

If $v \in V_2$ then $|S_v^*| = 1$ and $\omega(P_n - S_v^*) = 2$. So $sc_v(P_n) = 1$ for $n - 2$ vertices with degree 2. Thus,

$$sc_{av}(P_n) = \frac{2.0 + (n-2).1}{n} = \frac{n-2}{n}\cdot$$

□

Using Theorem 3.1, we show that the path P_n has the minimum average scattering number among all trees of order $n \geq 4$.

Theorem 3.3. *If C_n is a cycle graph of order $n \geq 4$, then*

$$sc_{av}(C_n) = 0.$$

Proof. The vertices of C_n be $c_1, c_2, ..., c_n$ in order along the cycle. For every vertex of C_n, if $|S_v| = r$ then $\omega(C_n - S_v) \leq r$. So we have $sc_v(C_n) \leq 0$.

It can be easily seen that there is a local scattering set S_v^* of C_n such that $|S_v^*| = 2$ where if $v = c_i$ or $v = c_{i+2}$ then $S_v^* = \{c_i, c_{i+2}\}$. Hence, we have $\omega(C_n - S_v^*) = 2$. So we get $sc_v(C_n) = 0$ for n vertices of C_n. From the definition of average scattering number we have,

$$sc_{av}(C_n) = \frac{n.0}{n} = 0. \qquad \square$$

Theorem 3.4. *If $K_{a,b}$ is a complete bipartite graph of order $a + b$ $(2 \leq a \leq b)$, then*

$$sc_{av}(K_{a,b}) = \begin{cases} \dfrac{b^2 - a^2 - 2b}{a+b}, & \text{if} \quad a < b; \\ 0\,, \quad \text{if} \quad a = b. \end{cases}$$

Proof. Let $v \in V(K_{a,b})$ and S_v be a cutset of $K_{a,b}$. Let the partite sets of $K_{a,b}$ be A and B with $|A| = a$ and $|B| = b$. We distinguish two cases.

Case 1. For every vertex of $K_{a,b}$, if $a = b$, then $|S_v| = a = b$ and $\omega(K_{a,b} - S_v) = a$. Therefore, $sc_v(K_{a,b}) = 0$, so $sc_{av}(K_{a,b}) = 0$.

Case 2. For $v \in A$, a minimum cutset of $K_{a,b}$ that contains v must be A, so $|S_v| = a$ and $\omega(K_{a,b} - S_v) = b$. Therefore, we get $sc_v(K_{a,b}) = b - a$. On the other hand, for $v \in B$, a minimum cutset of $K_{a,b}$ that contains v must be $A \cup \{v\}$, so $|S_v| = a+1$ and $\omega(K_{a,b} - S_v) = b-1$. Hence, we have $sc_v(K_{a,b}) = b-a-2$. Elementary computation yields the result. $\square$

Theorem 3.5. *The average scattering of the star $K_{1,n-1}$ $(n \geq 4)$ is $\frac{n^2-4n+2}{n}$.*

Proof. Let $v \in V(K_{1,n-1})$ and S_v^* be a local scattering set of $K_{1,n-1}$. A star graph has one vertex with degree $n-1$ and $n-2$ vertices with degree 1.

If $deg(v) = n-1$, then $|S_v^*| = 1$ and $\omega(K_{1,n-1} - S_v^*) = n-1$. So $sc_v(K_{1,n-1}) = n-2$.

If $deg(v) = 1$, then $|S_v^*| = 2$ and $\omega(K_{1,n-1} - S_v^*) = n-2$. So $sc_v(K_{1,n-1}) = n-4$ for $n-1$ vertices with degree one.

Thus,

$$sc_{av}(K_{1,n-1}) = \frac{1.(n-2) + (n-1).(n-4)}{n} = \frac{n^2 - 4n + 2}{n}. \qquad \square$$

4. Graph operations

In this section we consider some of the graph operations such as power, join, cartesian product of graphs. The union operation is not taken into consideration since it is disconnected.

4.1. Power

We begin with the definition of the power of a graph.

Definition 4.1. [5] The $k-th$ power G^k of a connected graph G is that graph with $V(G^k) = V(G)$ for which $uv \in E(G^k)$ if and only if $1 \leq d_G(u,v) \leq k$.

Theorem 4.2. *If G is a graph of order n and diameter d, then*

$$sc_{av}(G) \geq sc_{av}(G^2) \geq sc_{av}(G^3) \geq \ldots \geq sc_{av}(G^d) = -n.$$

Proof. Since for any graph G and positive integer i, G^i is a subgraph of G^{i+1}, it follows from Lemma 2.10 that $sc_{av}(G) \geq sc_{av}(G^2) \geq sc_{av}(G^3) \geq \ldots \geq sc_{av}(G^d)$. If a graph has diameter d, then its $d-th$ power is the complete graph and $sc_{av}(G^d) = -n$. □

4.2. Join

In this section, we consider some results on the average scattering number of the join of two graphs.

Definition 4.3. [5] The join $G = G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union G_1 union G_2 together with all the edges joining V_1 and V_2.

Theorem 4.4. *Let G and H be two noncomplete connected graphs of order $a(\geq 4)$ and $b(\geq 4)$, respectively. Then*

$$sc_{av}(G+H) \geq \frac{1}{a+b}(a(sc(H)-a) + b(sc(G)-b)).$$

Proof. If $v \in V(G)$, take as Y some scattering set of H and let $S = V(G) \cup Y$. Clearly, S is a cutset of $G+H$ and

$$\omega((G+H)-S) - |S| = \omega(H-Y) - |Y \cup V(G)| = sc(H) - a.$$

Therefore, for any local scattering set S_v,

$$\omega((G+H)-S_v) - |S_v| \geq \omega((G+H)-S) - |S| = sc(H) - a.$$

If $v \in V(H)$, take as X some scattering set of G and let $S = V(H) \cup X$. Clearly, S is a cutset of $G+H$ and

$$\omega((G+H)-S) - |S| = \omega(G-X) - |X \cup V(H)| = sc(G) - b.$$

Therefore, for any local scattering set S_v,

$$\omega((G+H)-S_v) - |S_v| \geq \omega((G+H)-S) - |S| = sc(G) - b.$$

Thus,

$$sc_{av}(G+H) = \frac{1}{a+b}(\Sigma_{v \in V(G)} sc_v(G+H) + \Sigma_{v \in V(H)} sc_v(G+H))$$

$$sc_{av}(G+H) \geq \frac{1}{a+b}(a(sc(H)-a) + b(sc(G)-b)).$$ □

Theorem 4.5. *If W_n is a wheel graph of order n, then*

$$sc_{av}(W_n) = -1.$$

Proof. Since $W_n \cong K_1 + C_{n-1}$, the wheel graph W_n has n vertices. Let $v \in V(W_n)$ and S_v^* be a local scattering set of W_n. The cardinality of S_v^* local scattering set is always the same for every vertex of any W_n and equals 3 and $\omega(W_n - S_v^*) = 2$. Hence, we have $sc_v(W_n) = -1$. It follows from the definition of average scattering number that

$$sc_{av}(W_n) = \frac{n.(-1)}{n} = -1.$$

□

Proposition 4.6. *Let $a(\geq 4)$ be a positive integer, then*

$$sc_{av}(K_1 + P_a) = \frac{-2}{a+1}\cdot$$

Proof. The proof is similar to Theorem 4.5. □

4.3. Cartesian product

Now we give the definition of Cartesian product.

Definition 4.7. [5] The Cartesian product $G_1 \times G_2$ of graphs G_1 and G_2 has $V(G_1) \times V(G_2)$ as its vertex set and (u_1, u_2) is adjacent to (v_1, v_2) if either $u_1 = v_1$ and u_2 is adjacent to v_2 or $u_2 = v_2$ and u_1 is adjacent to v_1.

Theorem 4.8. *Let a and b be positive integers, then*

$$sc_{av}(P_a \times P_b) = \begin{cases} \dfrac{-1}{ab}, & \text{if } a \text{ and } b \text{ are odd;} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let $v \in V(P_a \times P_b)$ and S_v be a scattering set of $P_a \times P_b$. We distinguish two cases.

Case 1. We can assume a and b are odd. Let M be a minimum covering set of $P_a \times P_b$. If $v \in V(M)$, then it can be easily seen that there is a scattering set S_v of $P_a \times P_b$ such that $|S_v| = \frac{ab-1}{2}$, where S_v contains all the vertices of a minimum covering set of $P_a \times P_b$. Then $\omega((P_a \times P_b) - S_v) = \frac{ab+1}{2}$ and we get

$$sc_v(P_a \times P_b) = \frac{ab+1}{2} - \frac{ab-1}{2} = 1, \tag{4.1}$$

for $v \in V(M)$.
If $v \notin V(M)$, then $|S_v| = \frac{ab+1}{2}$ and $\omega((P_a \times P_b) - S_v) = \frac{ab-1}{2}$. Thus,

$$sc_v(P_a \times P_b) = \frac{ab-1}{2} - \frac{ab+1}{2} = -1, \tag{4.2}$$

for $v \notin V(M)$.
Therefore, by (4.1) and (4.2),

$$sc_{av}(P_a \times P_b) = \frac{1}{ab}\left(\frac{ab-1}{2}.1 + \frac{ab+1}{2}.(-1)\right) = \frac{-1}{ab}\cdot$$

Case 2. We can assume a or b are even. It is easy to find a Hamilton cycle in $P_a \times P_b$. It follows from Lemma 2.10 and Theorem 3.3 that $sc_{av}(P_a \times P_b) \leq 0$. On the other hand, if a or b are even, then we know that $\alpha = \beta$. It can be easily seen that there is a scattering set S_v of $P_a \times P_b$ such that $|S_v| = \frac{ab}{2}$, where S_v contains all the vertices of a minimum covering set of $P_a \times P_b$. Then $\omega((P_a \times P_b) - S_v) = \frac{ab}{2}$ and we get

$$sc_v(P_a \times P_b) = 0,$$

for $v \in V(P_a \times P_b)$. So we have

$$sc_{av}(P_a \times P_b) = 0.$$

□

5. Conclusion

In this study, a new graph theoretical parameter namely the average scattering number has been presented for the network vulnerability. The present parameter has been constructed by summing of the local scattering number of every vertex of a graph divided by the number of vertices of the graph. Additionally, the stability of popular interconnection networks has been studied and their average scattering numbers have been computed. If we want to choose the stabler graph among the graphs which have the same order and the same size, one way is to choose the graph with minimum average scattering number.

Acknowledgements. The author would like to express their deepest gratitude to the anonymous referees for the constructive suggestions and comments that improve the quality of this paper.

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Communicated by C. de Figueiredo.
Received March 20, 2016. Accepted November 22, 2016.