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REDUCTION IN NON-$(k+1)$-POWER-FREE MORPHISMS

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Abstract. Under some hypotheses, if the image by a morphism of a $(k+1)$-power-free word contains a $(k+1)$-power, we can reduce this word to obtain a new word with the same scheme. These hypotheses are satisfied in the case of uniform morphisms. This allows us to state that, when $k \geq 4$, a k-power-free uniform morphism is a $(k+1)$-power-free morphism.

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1. Introduction

A word without two consecutive occurrences of the same factor is called a square-free word. From the seminal papers of Thue [21,22] (see also [2]), we know how to build infinite square-free words on a three-letter alphabet but also infinite overlap-free words on a two-letter alphabet, that is, words that do not have any factor of the form $auaua$ with a a letter and u a word.

Most of the explicitly built infinite square-free words or infinite overlap-free words (for instance in [8,17,22]) are obtained by iterating a morphism. They are generated as fixed points of free monoid morphisms. Indeed, a non-erasing ($\forall x \in A$, $f(x) \neq \varepsilon$) endomorphism f on an alphabet A such that $f(a) = au$ with $u \neq \varepsilon$ satisfies $f^{n+1}(a) = f^n(a)f^n(u)$ for every positive integer n. Consequently, $f^n(a)$ is a prefix of $f^{n+1}(a)$ and we can define the infinite word $\overline{a} = \lim_{n\to+\infty} f^n(a)$. We say that $\overline{a}$ is the (infinite) word generated by f.

The study of infinite square-free words [1,6] and overlap-free words [9] generated by morphisms was extended to words avoiding other repetitions: cube u^3 [10] and more generally k-power u^k [19].

Although there exist other ways, the most common method to produce square-free words remains to start with a letter and to iterate a endomorphism. Some type of morphisms appears: the morphisms that preserve the absence of repetitions. Note that if a morphism preserves the absence of squares, that is to say, if the image of a square-free word by this morphism is also square-free, then the sequence generated will be square-free. Two different kinds of morphisms may be considered: those that generate square-free words and those that preserve the absence of square, called square-free morphisms. The study of square-free morphisms is thus a specific part of the previous problem. This definition can be extended to k-powers with $k \geq 3$, and also to sesqui-powers or to fractional powers.

Several methods exist to verify whether a morphism is square-free [5], overlap-free [3, 18], cube-free [20] or k-power-free [12, 13, 23]. In this search to verify whether a morphism preserves the absence of repetitions,

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the uniform morphisms, that is, those where the images of the letters have the same length, give specific results [7, 11].

In line with this approach, a natural question arises: is a k-power-free morphism also a $(k+1)$-power-free morphism? In other words, if the image of every k-power-free word by a morphism f is k-power-free, is the image of a $(k+1)$-power-free word also $(k+1)$-power-free? The answer already exists for the Thue-Morse morphism [4]. It is k-power-free for every integer $k > 2$.

In the search for an answer, equations of words (Lem. 3.1) appear in the initial case of a non-$(k+1)$-power-free morphism. We give some conditions (Lem. 3.9) under which we can simplify the initial equations. We call this simplification a reduction of the initial word. We construct a new word whose image contains a $(k+1)$-power but with a strictly lower length. The fact that the powers are synchronised (Lem. 2.12) appears as a particular case and will allow us to conclude for uniform morphisms (Prop. 4.1).

2. Preliminaries

Let us recall some basic notions of Combinatorics of words.

2.1. Words

An *alphabet* A is a finite set of symbols called *letters*. Since an alphabet with one element is of limited interest to us, we always assume that the cardinality of alphabets is at least two. A *word* over A is a finite sequence of letters from A. The *empty word* ε is the empty sequence of letters. Equipped with the concatenation operation, the set A^* of words over A is a free monoid with ε as neutral element and A as set of generators.

Given a non-empty word $u = a_1 \dots a_n$, with $a_i \in A$ for every integer i from 1 to n, the *length* of u denoted by $|u|$ is the integer n, that is, the number of letters of u. By convention, we have $|\varepsilon| = 0$. The *mirror image* of u, denoted by $\tilde{u}$, is the word $a_n \dots a_2 a_1$.

A word u is a *factor* of a word v if there exist two (possibly empty) words p and s such that $v = pus$. We denote by $\mathrm{Fcts}(v)$ the set of all factors of v. If $u \in \mathrm{Fcts}(v)$, we also say that v *contains* the word u (as a factor). If $p = \varepsilon$, u is a *prefix* of v. If $s = \varepsilon$, u is a *suffix* of v. If $u \neq v$, u is a *proper* factor of v. If u, p, and s are non-empty words, u is an *internal* factor of v.

Two words u and v are *conjugated* if $u = t_1 t_2$ and $v = t_2 t_1$ for two (possibly empty) words t_1 and t_2.

Let w be a non-empty word and let i, j be two integers such that $0 \leq i - 1 \leq j \leq |w|$. We denote by $w[i \dots j]$ the factor of w such that $|w[i \dots j]| = j - i + 1$ and $w = pw[i \dots j]s$ for two words s and p satisfying $|p| = i - 1$. Note that, when $j = i - 1$, we have $w[i \dots j] = \varepsilon$. When $i = j$, we also denote by $w[i]$ the factor $w[i \dots i]$, which is the ith letter of w. In particular, $w[1]$ and $w[|w|]$ are respectively the first and the last letter of w.

Powers of a word are defined inductively by $u^0 = \varepsilon$, and for every integer $n \geq 1$, $u^n = uu^{n-1}$. Given an integer $k \geq 2$, since the case ε^k is of little interest, we call a k-power any word u^k with $u \neq \varepsilon$. Given an integer $k \geq 2$, a word is *k-power-free* if it does not contain any k-power as factor. A *primitive* word is a word that is not a k-power of another word whatever the integer $k \geq 2$. A (non-empty) k-power v^k is called *pure* if any proper factor of v^k is k-power-free. In particular, we say that v^k is a pure k-power of a word w if $v^k \in \mathrm{Fcts}(w)$ and v^k is pure. Repeating the fact that a non-pure k-power contains a k-power, which is itself pure or not, we obtain that any k-power contains a pure k-power. Moreover, if v^k is a pure k-power then v is primitive but the converse does not hold. Let us also remark that a word cannot start with two different pure k-powers.

The following proposition gives the well-known solutions (see [15]) to an elementary equation on words and will be widely used in the following sections:

Proposition 2.1. *Let A be an alphabet and u, v, w three words over A.*

(1) *If $vu = uw$ and $v \neq \varepsilon$ then there exist two words r and s over A and an integer n such that $u = r(sr)^n$, $v = rs$ and $w = sr$.*
(2) *If $vu = uv$, then there exist a word w over A and two integers n and p such that $u = w^n$ and $v = w^p$.*

We also need three other properties on words. The first one is an immediate consequence of Proposition 2.1(2).

Lemma 2.2 ([12, 14])**.** *If a non-empty word v is an internal factor of vv,* i.e.*, if there exist two non-empty words x and y such that $vv = xvy$, then there exist a non-empty word t and two integers $i, j \geq 1$ such that $x = t^i$, $y = t^j$, and $v = t^{i+j}$.*

We also use a well-known result on combinatorics on words:

Proposition 2.3 (Fine and Wilf [15, 16])**.** *Let x and y be two words. If a power of x and a power of y have a common prefix of length at least equal to $|x| + |y| - gcd(|x|, |y|)$ then x and y are powers of the same word.*

As a consequence of Proposition 2.3, we obtain:

Corollary 2.4 (Keränen [12])**.** *Let x and y be two words. If a power of x and a power of y have a common factor of length at least equal to $|x| + |y| - gcd(|x|, |y|)$ then there exist two words t_1 and t_2 such that x is a power of t_1t_2 and y is a power of t_2t_1 with t_1t_2 and t_2t_1 primitive words. Furthermore, if $|x| > |y|$ then x is not primitive.*

2.2. Morphisms

Let A and B be two alphabets. A *morphism* f from A^* to B^* is a mapping from A^* to B^* such that $f(uv) = f(u)f(v)$ for all words u, v over A. When B has no importance, we say that f is a morphism on A or that f is defined on A.

Given an integer L, f is *L-uniform* if $|f(a)| = L$ for every letter a in A. A morphism f is *uniform* if it is L-uniform for some integer $L \geq 0$. Given a set X of words over A, and given a morphism f on A, we denote by $f(X)$ the set $\{f(w) \mid w \in X\}$.

A morphism f on A is *k-power-free* if and only if $f(w)$ is k-power-free for every k-power-free word w over A. For instance, the *empty morphism* ϵ ($\forall a \in A$, $\epsilon(a) = \varepsilon$) or the *identity endomorphism* Id ($\forall a \in A$, $Id(a) = a$) are k-power-free.

We say that a morphism is *non-erasing* if, for all letters $a \in A$, $f(a) \neq \varepsilon$. The empty morphism ϵ is the only morphism that is both erasing and k-power-free. Indeed, for any non-empty erasing morphism f, there exist two different letters a and b in A (remember Card$(A) \geq 2$) such that $f(a) \neq \varepsilon$, $f(b) = \varepsilon$, and so $f(aba^{k-1})$ contains a k-power.

A morphism on A is called *prefix* (resp. *suffix*) if, for all different letters a and b in A, the word $f(a)$ is not a prefix (resp. not a suffix) of $f(b)$. A prefix (resp. suffix) morphism is non-erasing. A morphism is *bifix* if it is prefix and suffix.

Given a morphism f on A, the *mirror morphism* $\tilde{f}$ of f is defined for all words w over A, by $\tilde{f}(w) = \widetilde{f(\tilde{w})}$. In particular, $\tilde{f}(a) = \widetilde{f(a)}$ for every letter a in A. Note that f is k-power-free if and only if $\tilde{f}$ is k-power-free.

Proofs of the three following lemmas are left to the reader.

Lemma 2.5. *Let f be a bifix morphism on an alphabet A and let u, v, w, and t be words over A.*

The equality $f(u) = f(v)p$ with p be a prefix of $f(w)$ implies $u = vw'$ for a prefix w' of w such that $f(w') = p$.

And the equality $f(u) = sf(v)$ with s a suffix of $f(t)$ implies $u = t'v$ for a suffix t' of t such that $f(t') = s$.

Lemma 2.6. *Let f be a prefix morphism on an alphabet A, let u and v be words over A, and let a and b be letters in A. Furthermore, let p_1 (resp. p_2) be a prefix of $f(a)$ (resp. of $f(b)$). If $(p_1; p_2) \neq (\varepsilon; f(b))$ and if $(p_1; p_2) \neq (f(a); \varepsilon)$ then the equality $f(u)p_1 = f(v)p_2$ implies $u = v$ and $p_1 = p_2$.*

Lemma 2.7. *Let f be a suffix morphism on an alphabet A, let u and v be words over A, and let a and b be letters in A. Furthermore, let s_1 (resp. s_2) be a suffix of $f(a)$ (resp. of $f(b)$). If $(s_1; s_2) \neq (\varepsilon; f(b))$ and if $(s_1; s_2) \neq (f(a); \varepsilon)$ then the equality $s_1f(u) = s_2f(v)$ implies $u = v$ and $s_1 = s_2$.*

Definition 2.8. A morphism f from A^* to B^* is a *ps-morphism* (Keränen [12] called f a ps-code) if and only if the equalities

$$f(a) = ps,\ f(b) = ps' \text{ and } f(c) = p's$$

with $a, b, c \in A$ (possibly $c = b$) and $p,\ s,\ p',\ s' \in B^*$ imply $b = a$ or $c = a$.

Obviously, taking $c = b$, and $s = \varepsilon$ in a first time and $p = \varepsilon$ in a second time, we obtain that a ps-morphism is a bifix morphism.

Lemma 2.9 ([12,14])**.** *If f is not a ps-morphism then f is not a k-power-free morphism for every integer $k \geq 2$.*

Lemma 2.10. *Let f be a ps-morphism from A^* to B^* and let u, v and w be words over A such that $f(u) = \delta\beta$, $f(v) = \alpha\beta$, and $f(w) = \alpha\gamma$ for some non-empty words α, β, γ, and δ over B. Then it implies $v = v_1 a v_2$, $u = u_1 b v_2$, and $w = v_1 c w_2$ for some words v_1, v_2, u_1, and w_2, and some letters a, b, and c. Moreover, we have either $b = a$ or $c = a$.*

Furthermore, if $|\delta| < |f(u[1])|$ then $u_1 = \varepsilon$ and if $|\gamma| < |f(w[|w|])|$ then $w_2 = \varepsilon$.

Proof. Let us recall that, as any ps-morphism, f is bifix.

Let $v[1 \dots i]$ be the shortest prefix of v such that α is a prefix of $f(v[1 \dots i])$. Since $\alpha \neq \varepsilon$, we have $v[1 \dots i] \neq \varepsilon$, *i.e.*, $i \geq 1$. We set $v_1 = v[1 \dots i-1]$, $v_2 = v[i+1 \dots |v|]$, and $a = v[i]$. There exist two words $p \neq \varepsilon$ and s $(\neq f(a))$ such that $f(a) = ps$, $\alpha = f(v_1)p$ and $\beta = sf(v_2)$.

Let $u[j \dots |u|] (\neq \varepsilon)$ be the shortest suffix of u such that β is a suffix of $f(u[j \dots |u|])$. There exist two words $s_1 \neq \varepsilon$ and p_1 $(\neq f(u[j]))$ such that $f(u[j]) = p_1 s_1$ and $\beta = s_1 f(u[j+1 \dots |u|])$. In particular, if $|\delta| < |f(u[1])|$ then $p_1 = \delta (\neq \varepsilon)$ and $j = 1$.

Let $w[1 \dots \ell]$ be the shortest prefix of w such that α is a prefix of $f(w[1 \dots \ell])$. We set $w_2 = w[\ell+1 \dots |w|]$ and $c = w[\ell]$. There exist two words $p_2 \neq \varepsilon$ and s_2 $(\neq f(c))$ such that $f(c) = p_2 s_2$, $\alpha = f(w[1 \dots \ell-1])p_2$, and $\gamma = s_2 f(w_2)$. In particular, if $|\gamma| < |f(w[|w|])|$ then $\ell = |w|$, $s_2 = \gamma (\neq \varepsilon)$, and $w_2 = \varepsilon$.

If $s \neq \varepsilon$, we set $u_1 = u[1 \dots j-1]$ and $b = u[j]$. Let us note that, if $|\delta| < |f(u[1])|$, we obtain $u_1 = \varepsilon$. By Lemma 2.7, since f is bifix, the equality $(\beta =) sf(v_2) = s_1 f(u[j+1 \dots |u|])$, with $(s; s_1) \neq (\varepsilon; f(b))$, implies $u[j+1 \dots |u|] = v_2$ and $s = s_1$, *i.e.*, $u = u_1 b v_2$. Furthermore, since $p, p_2 \neq \varepsilon$, we obtain $(p; p_2) \neq (\varepsilon; f(c))$ and $(p; p_2) \neq (f(a); \varepsilon)$. By Lemma 2.6, the equality $(\alpha =) f(v_1)p = f(w[1 \dots \ell-1])p_2$ implies $p = p_2$ and $v_1 = w[1 \dots \ell-1]$, that is, $w = v_1 c w_2$. So we have $f(a) = ps$, $f(b) = p_1 s$, and $f(c) = ps_2$. Since f is a ps-morphism, then $b = a$ or $c = a$.

If $s = \varepsilon$ then $\beta = f(v_2) = s_1 f(u[j+1 \dots |u|])$ with $s_1 \neq \varepsilon$. By Lemma 2.5, we obtain $s_1 = f(u[j])$, $p_1 = \varepsilon$, and $v_2 = u[j \dots |u|]$. Since $\delta \neq \varepsilon$, it follows that $j \geq 2$ and so $|\delta| \geq |f(u[1])|$. We set $u_1 = u[1 \dots j-2]$ and $b = u[j-1]$. We have $u = u_1 b v_2$ but also $p = f(a)$ and $f(v_1 a) = \alpha = f(w[1 \dots \ell-1])p_2$ with $p_2 \neq \varepsilon$. Since f is bifix, by Lemma 2.5, we obtain $s_2 = \varepsilon$ and $w[1 \dots \ell-1]c = v_1 a$, *i.e.*, $c = a$ and $w = v_1 a w_2$. □

Assuming $f(\overline{w}) = pu^k s$ for a factor $\overline{w}$ of a word w, and assuming that $\overline{w}$ contains a factor w_0 such that $|f(w_0)| = |u|$, we show in Lemma 2.12 that $\overline{w}$ necessarily contains a k-power w'^k such that $f(w')$ is a conjugate of u. We will say that $f(w)$ contains a synchronised k-power u^k. More precisely:

Definition 2.11. Let $k \geq 2$ be an integer. Let f be a morphism from A^* to B^*, w be a word over A and u be a non-empty word over B such that $f(w)$ contains the k-power u^k. Let $\overline{w}$ be a shortest factor of w whose image by f contains u^k, *i.e.*, $f(\overline{w}) = pu^k s$ with $|p| < |f(\overline{w}[1])|$ and $|s| < |f(\overline{w}[|\overline{w}|])|$.

We say that $f(w)$ and u^k are synchronised if there exist three words w_0, w_1, and w_2 such that $|f(w_0)| = |u|$ and $\overline{w} = w_1 w_0 w_2$ with $p = \varepsilon$ if $w_1 = \varepsilon$, and $s = \varepsilon$ if $w_2 = \varepsilon$.

The following lemma and its proof are based on Reduction 2 of the proof of Theorem 5.1 in [23].

Lemma 2.12. *Let $k \geq 2$ be an integer. If f is a ps-morphism and if $f(w)$ contains a synchronised k-power then w contains a k-power.*

Remark 2.13. More precisely, we prove that $\overline{w}$ starts or ends with a k-power whose image by f is a conjugate of the synchronised k-power.

Proof. Let u be the word such that $f(w)$ and u^k are synchronised, let $\overline{w}$ be the shortest factor of w whose image by f contains u^k, and let w_0 be a factor of $\overline{w}$ such that $|f(w_0)| = |u|$.

There exist a proper prefix p of $f(\overline{w}[1])$ and a proper suffix s of $f(\overline{w}[|\overline{w}|])$ such that $f(\overline{w}) = pu^k s$. Moreover, there exist two integers $0 \leq \ell < m \leq |\overline{w}|$ such that $\overline{w}[\ell+1 \dots m] = w_0$.

If $\ell = 0$, *i.e.*, $\overline{w}$ starts with w_0, then $p = \varepsilon$ and $f(\overline{w})$ starts with u. By Lemma 2.5, we obtain $u = f(w_0)$ and that $\overline{w}$ starts with w_0^k, *i.e.*, w contains a k-power. If $m = |\overline{w}|$, *i.e.*, $\overline{w}$ ends with w_0, then, in a similar way, we obtain that $\overline{w}$ ends with w_0^k.

From now, let us assume that $0 < \ell < m < |\overline{w}|$, *i.e.*, w_0 is an internal factor of $\overline{w}$. It implies that $f(w_0)$ is an internal factor of u^k. In particular, it means that $f(w_0)$ and u are conjugated.

For every integer j in $[0,k]$, let i_j be the smallest integer such that pu^j is a prefix of $f(\overline{w}[1 \dots i_j])$, that is, $|f(\overline{w}[1 \dots i_j - 1])| < pu^j \leq |f(\overline{w}[1 \dots i_j])|$ (except the special case $j = 0$ and $p = \varepsilon$ where the first inequality is not strict). We have $i_0 = 1$ and $i_k = |\overline{w}|$. There exist words p_j ($\neq \varepsilon$ when $j \neq 0$) and s_j such that $f(\overline{w}[i_j]) = p_j s_j$ for every $j \in [0,k]$, $p = p_1$, $s = s_k$, and $u = s_j f(\overline{w}[i_j + 1 \dots i_{j+1} - 1])p_{j+1}$ for every $j \in [0, k-1]$.

Let us first remark that $|s_q| = |s_n|$ for two integers $0 \leq q, n \leq k-1$ implies $|p_{q+1}| = |p_{n+1}|$ (the converse also holds using Lemma 2.7 and the fact that s_q and s_n are not images of a letter). Indeed, since $u = s_q f(\overline{w}[i_q + 1 \dots i_{q+1} - 1])p_{q+1} = s_n f(\overline{w}[i_n + 1 \dots i_{n+1} - 1])p_{n+1}$, we obtain $s_q = s_n$ and $f(\overline{w}[i_q + 1 \dots i_{q+1} - 1])p_{q+1} = f(\overline{w}[i_n + 1 \dots i_{n+1} - 1])p_{n+1}$ with $p_{q+1} \neq \varepsilon$ and $p_{n+1} \neq \varepsilon$. By Lemma 2.6, since f is bifix, we have $\overline{w}[i_q + 1 \dots i_{i+1} - 1] = \overline{w}[i_n + 1 \dots i_{n+1} - 1]$ and $p_{q+1} = p_{n+1}$.

Let δ be the integer such that $\ell \in [i_\delta, i_{\delta+1}[$.

The equalities $|s_\delta f(\overline{w}[i_\delta + 1 \dots \ell])| = |u| - |f(\overline{w}[\ell + 1 \dots i_{\delta+1} - 1])p_{\delta+1}| = |f(\overline{w}[\ell + 1 \dots m])| - |f(\overline{w}[\ell + 1 \dots i_{\delta+1} - 1])p_{\delta+1}| = |s_{\delta+1} f(\overline{w}[i_{\delta+1} + 1 \dots m])| (\leq |u|)$ hold. But the words $s_\delta f(\overline{w}[i_\delta + 1 \dots \ell])$ and $s_{\delta+1} f(\overline{w}[i_{\delta+1} + 1 \dots m])$ are both prefixes of u. Consequently, $s_\delta f(\overline{w}[i_\delta + 1 \dots \ell]) = s_{\delta+1} f(\overline{w}[i_{\delta+1} + 1 \dots m])$.

If $\delta = 0$ and $p_0 = p = \varepsilon$ then $s_\delta (= s_0) = f(\overline{w}[i_\delta])$ and $f(\overline{w}[i_\delta \dots \ell]) = s_{\delta+1} f(\overline{w}[i_{\delta+1} + 1 \dots m])$ with $s_{\delta+1} \neq f(\overline{w}[i_{\delta+1}])$. By Lemma 2.5, we obtain $s_1 (= s_{\delta+1}) = \varepsilon$, $p_1 (= p_{\delta+1}) = f(\overline{w}[i_{\delta+1}])$, and $u = f(\overline{w}[1 \dots i_1])$. Again by Lemma 2.5 and by induction, it implies that $\overline{w}$ starts with $(\overline{w}[1 \dots i_1])^k$ with $|f(\overline{w}[1 \dots i_1])| = |u|$, *i.e.*, $f(\overline{w}[1 \dots i_1])$ is a conjugate of u.

From now let us assume $\delta \neq 0$ or $p \neq \varepsilon$. Since f is bifix, $s_\delta \neq f(\overline{w}[i_\delta])$ and $s_{\delta+1} \neq f(\overline{w}[i_{\delta+1}])$, by Lemma 2.7, we obtain $s_\delta = s_{\delta+1}$. Thus, we have $p_{\delta+1} = p_{\delta+2}$ for an integer δ such that $0 \leq \delta \leq k-2$.

We will now show that, for every integer r such that $1 \leq r \leq \delta + 1$, we necessarily have $p_r = p_{\delta+1}$.

By contradiction, let us assume that there exists an integer r satisfying $1 \leq r \leq \delta + 1$ and $p_r \neq p_{\delta+1}$, and let us choose the greatest one. By this way, $p_{r+1} = p_{r+2} (= p_{\delta+1})$.

It follows that $s_r f(\overline{w}[i_r + 1 \dots i_{r+1} - 1]) = s_{r+1} f(\overline{w}[i_{r+1} + 1 \dots i_{r+2} - 1])$. Since $s_r \neq f(\overline{w}[i_r])$ and $s_{r+1} \neq f(\overline{w}[i_{r+1}])$, by Lemma 2.7, we obtain $s_r = s_{r+1}$. But p_r and p_{r+1} are both suffixes of u. Thus, one of the two different words p_r or p_{r+1} is a (proper) suffix of the other. It means that one the two different words $f(\overline{w}[i_r])$ or $f(\overline{w}[i_{r+1}])$ is a (proper) suffix of the other, a contradiction with the fact that f is bifix.

In a similar way, we prove that, for every integer r in $[\delta + 1, k-1]$, we have $s_r = s_\delta$ with $s_r \neq f(\overline{w}[i_r])$. And it follows that $p_r = p_\delta$ for every integer r in $[\delta + 2, k]$.

Consequently, we have $p_q = p_\delta = p_1$ and $s_0 f(\overline{w}[2 \dots i_1 - 1])p_1 = u = s_{q-1} f(\overline{w}[i_{q-1} + 1 \dots i_q - 1])p_q$ for all integers q in $[1, k]$.

If $s_0 = f(\overline{w}[1])$, since f is bifix and by Lemma 2.5, it follows that $\overline{w}[i_{q-1} + 1 \dots i_q] = \overline{w}[1 \dots i_1]$ and $s_{q-1} = \varepsilon$ for all $2 \leq q \leq k$, that is, w starts with $(\overline{w}[1 \dots i_1])^k$ where $f(\overline{w}[1 \dots i_1])$ is a conjugate of u.

If $s_0 \neq f(\overline{w}[1])$, since f is bifix and by Lemma 2.7, then we obtain $\overline{w}[i_{q-1} + 1 \dots i_q - 1] = \overline{w}[2 \dots i_1 - 1]$ and $s_{q-1} = s_0$ for all $2 \leq q \leq k$. In particular, it means that $s_0 = s_1$.

Therefore, $\overline{w} = \overline{w}[1](\overline{w}[2 \dots i_1 - 1]\overline{w}[i_1])^{r-1}\overline{w}[2 \dots i_1 - 1]\overline{w}[|\overline{w}|]$ with $f(\overline{w}[1]) = ps_1$, $f(\overline{w}[i_1]) = p_1 s_1$, and $f(\overline{w}[|\overline{w}|]) = p_1 s$. Since f is a ps-morphism, it means that $\overline{w}[1] = \overline{w}[i_1]$ or $\overline{w}[|\overline{w}|] = \overline{w}[i_1]$, *i.e.*,

$\overline{w} = (\overline{w}[1\dots i_1-1])^r\overline{w}[|\overline{w}|]$ or $\overline{w} = \overline{w}[1](\overline{w}[2\dots i_1])^k$. Hence, the word $\overline{w}$ starts or ends with a k-power whose image is a conjugate of u. □

Lemma 2.14. *Let $k \geq 4$ be an integer. The image of a pure k-power by a k-power-free morphism is also a pure k-power.*

Proof. Let f be a k-power-free morphism on A and let v^k be a pure k-power over A.

If $f(v)^k$ was not a pure k-power then there would exist a pure k-power $u^k \in \mathrm{Fcts}\left(f(v)^k\right)$ such that $|u| < |f(v)|$.

Since f is k-power-free and since the three words (proper factors of v^k) $v[2\dots|v|]v^{k-2}v[1\dots|v|-1]$, $v^{k-1}v[1\dots|v|-1]$, and $v[2\dots|v|]v^{k-1}$ are k-power-free, we obtain $|u^k| > |f(v[2\dots|v|]v^{k-2}v[1\dots|v|-1])| \geq 2|f(v)| > |u| + |f(v)|$. By Corollary 2.4, $f(v)$ and u are powers of conjugated words and $f(v)$ is not primitive, a contradiction with the hypotheses. □

3. Reduction of a power

3.1. About k-power-free morphisms

Even if it seems not obvious, hypotheses of Lemma 3.1 appear almost immediately when the image of a word by a morphism contains a $(k+1)$-power.

Lemma 3.1. *Let $k \geq 4$ be an integer. Let f be a ps-morphism from A^* to B^*. Let v and T be non-empty words over A such that v^k is a pure k-power. Let us assume that $f(T) = \pi_1 f(v)^k\sigma_2$ with $|\pi_1| < |f(T[1])|$ and $|\sigma_2| < |f(T[|T|])|$. Then one of the following holds:*

- $(P.1)$: *There exist a pure k-power x^k, a word y over A, and a word Z over B such that*
  - $(P.1.1)$: *$T = x^k y$, $|y| \leq 1$, $f(y) = \pi_1\sigma_2$, $f(x) = \pi_1 Z$, and $f(v) = Z\pi_1$*
  - $(P.1.2)$: *or $T = yx^k$, $|y| = 1$, $f(y) = \pi_1\sigma_2$, $f(x) = Z\sigma_2$, and $f(v) = \sigma_2 Z$.*
- $(P.2)$: *There exist a pure k-power x^k and a non-empty word y over A such that*
  - $(P.2.1)$: *$T = x^k y$ with $|f(x^{k-1})| < |\pi_1 f(v)|$*
  - $(P.2.2)$: *or $T = yx^k$ with $|f(x^{k-1})| < |f(v)\sigma_2|$.*
- $(P.3)$: *f is not k-power-free.*

Proof. If T is k-power-free then f is not k-power-free, it ends the proof.

So T contains at least one k-power. Among the k-powers of T, we choose one whose image by f is a shortest. We can write $T = y_1x^ky_2$ where $|f(x)| = \min\{|f(x')| \text{ where } {x'}^k \in \mathrm{Fcts}(T)\}$. By this definition, since f is bifix (as any ps-morphism) and so non-erasing, x^k is a pure k-power. Otherwise, x^k (and T) would contain a proper factor $\breve{x}^k$ with $f(\breve{x})^k$ a proper factor of $f(x)^k$, that is, $|f(\breve{x})| < |f(x)|$, a contradiction with the definition of x.

Case 1: A power of $f(x)$ and a power of $f(v)$ have a common factor of length at least $|f(x)| + |f(v)|$.

In a first time, we are going to list two cases where this situation necessarily holds.

If $y_1 \neq \varepsilon$ and $y_2 \neq \varepsilon$, since $|\pi_1| < |f(T[1])| \leq |f(y_1)|$ and $|\sigma_2| < |f(T[|T|])| \leq |f(y_2)|$, we obtain that $f(x)^k$ is an internal factor of $f(v)^k$. It follows that $|f(x)| < |f(v)|$. If $|f(x)^k| < |f(v)^{k-2}|$, by a length criterion, we obtain that $f(x)^k$ is an internal factor of $f(v)^{k-1}$ with v^{k-1} k-power-free, that is, f is not k-power-free. Thus, it is bound to $|f(x)^k| \geq |f(v)^{k-2}| \geq |f(v^2)| \geq |f(v)| + |f(x)|$.

If $y_1 = \varepsilon$ and $y_2 = \varepsilon$ then $T = x^k$, $f(x)^k = \pi_1 f(v)^k\sigma_2$, π_1 is a prefix of $f(x[1])$, σ_2 a suffix of $f(x[|x|])$, and $|f(x)| \geq |f(v)|$. The word $f(x)^{k-2} \in \mathrm{Fcts}\left(f(v)^k\right)$ is a common factor of powers of the two words $f(x)$ and $f(v)$. Furthermore, we have $|f(x)|^{k-2} \geq |f(x)^2| \geq |f(x)| + |f(v)|$.

Let us now really deal with this Case 1. By Corollary 2.4, there exist two words t_1 and t_2 and two integers p and r such that $f(v) = (t_1t_2)^p$ and $f(x) = (t_2t_1)^r$ with t_1t_2 and t_2t_1 primitive words.

If $p \geq 2$ then $f(v^{\lceil k/2\rceil})$ contains a k-power. Indeed, we have $f(v^{\lceil k/2\rceil}) = (t_1t_2)^{p\times\lceil k/2\rceil}$ with $p \times \lceil k/2\rceil \geq k$. In the same way, if $r \geq 2$ then $f(x^{\lceil k/2\rceil})$ contains a k-power. But $v^{\lceil k/2\rceil}$ (a proper factor of v^k) and $x^{\lceil k/2\rceil}$ (a proper factor of x^k) are both k-power-free, *i.e.*, f is not k-power-free.

So we can assume that $p = r = 1$. We have $f(T) = f(y_1)(t_2t_1)^k f(y_2) = \pi_1(t_1t_2)^k\sigma_2$ with $|f(y_1y_2)| = |\pi_1\sigma_2|$, $|\pi_1| < |f(T[1])| = |f((y_1x)[1])|$, and $|\sigma_2| < |f(T[|T|])| = |f((xy_2)[|xy_2|])|$.

If $y_2 \neq \varepsilon$, we obtain $|f(y_2)| > |\sigma_2|$, hence, $|f(y_1)| < |\pi_1|$. It means that $y_1 = \varepsilon$. Furthermore, $|\pi_1| < |f(x)| = |t_2t_1|$ and $f(T) = f(x^ky_2)$ starts with $(t_2t_1)^k$ and $\pi_1(t_1t_2)^k$. Since t_1t_2 is not an internal factor of $(t_1t_2)^2$ (we know that $f(v) = t_1t_2$ is primitive) and by a length criterion, we necessarily obtain $t_2 = \pi_1$ and $f(y_2) = \pi_1\sigma_2$. It follows that $f(v) = t_1\pi_1$ and $f(x) = \pi_1t_1$. Since $|t_2| = |\pi_1| < |f(T[1])| = |f(x[1])|$ and $|\sigma_2| < |f(y_2[|y_2|])|$, if $|y_2| \geq 2$, we obtain $y_2[1 \dots |y_2|-1] \neq \varepsilon$ and that $f(y_2[1 \dots |y_2|-1])$ is a prefix of $t_2 = \pi_1$ itself a prefix $f(x[1])$. This is in contradiction with the fact that f is bifix. So $|y_2| = 1$ and $|f(y_2)| - |\sigma_2| = |t_2| \leq |f(v)|$.

In the same way, when $y_1 \neq \varepsilon$, we successively obtain $y_2 = \varepsilon$, $\sigma_2 = t_1$, $f(y_1) = \pi_1\sigma_2$, $f(v) = \sigma_2t_2$, $f(x) = t_2\sigma_2$, $|y_1| = 1$, and $|f(y_1)| - |\pi_1| = |t_1| \leq |f(v)|$.

If $y_1 = y_2 = \varepsilon$ then $\pi_1 = \sigma_2 = \varepsilon$, $t_1t_2 = t_2t_1$ (*i.e.*, $x = v$), and $T = x^k$.

Case 2: Any power of $f(x)$ and any power of $f(v)$ do not have common factor of length at least $|f(x)| + |f(v)|$.

If $y_1 = \varepsilon$ and $y_2 \neq \varepsilon$, we have $T = x^ky_2$ and $f(x)f(x)^{k-1}f(y_2) = \pi_1 f(v)^k\sigma_2$. But π_1 is a prefix of $f(T[1]) = f(x[1])$. Consequently, there exists a word σ_1 such that $f(x[1]) = \pi_1\sigma_1$. Hence, $\sigma_1 f(x[2 \dots |x|])f(x^{k-1}) \in \mathrm{Fcts}\left(f(x^k)\right) \cap \mathrm{Fcts}\left(f(v^k)\right)$. Furthermore, $|\sigma_1 f(x[2 \dots |x|])f(x^{k-2})| < |f(v)|$ and $|f(x^{k-1})| < |\pi_1 f(v)|$.

In a same way, if $y_1 \neq \varepsilon$ and $y_2 = \varepsilon$, we obtain $T = y_1x^k$ and $|f(x^{k-1})| < |f(v)\sigma_2|$. □

By Lemma 2.12 and Remark 2.13, we immediately obtain:

Corollary 3.2. *With hypotheses and notations of Lemma 3.1, if $f(T)$ and $f(v)^k$ are synchronised (this is obviously the case when f is a uniform ps-morphism) then either f is not k-power-free or T satisfies $(P.1)$.*

Corollary 3.3. *Let $k \geq 4$ be an integer. Let f be a ps-morphism from A^* to B^*. Let v^k and t^k be two pure k-powers over A. Let us assume that $f(t^k) = \pi_1 f(v)^k\sigma_2$ with $|\pi_1| < |f(t[1])|$ and $|\sigma_2| < |f(t[|t|])|$. If $\pi_1 \neq \varepsilon$ or if $\sigma_2 \neq \varepsilon$ then f is not k-power-free.*

Proof. By Lemma 3.1, if f is k-power-free then t^k satisfies $(P.1)$ or $(P.2)$. Since t^k is a pure k-power, it follows that t^k can only satisfy $(P.1.1)$ with $|y| = 0$. But this contradicts the fact that $|f(y)| = |\pi_1\sigma_2| > 0$. □

Corollary 3.4. *Let $k \geq 4$ be an integer. Let f be a ps-morphism from A^* to B^*. Let v and T be non-empty words over A such that v^k is a pure k-power. Let us assume that $f(T) = \pi_1 f(v)^{k+1}\sigma_2$ with $|\pi_1| < |f(T[1])|$ and $|\sigma_2| < |f(T[|T|])|$. Then either f is not k-power-free or there exist a pure k-power x^k, a word Y over A and a word Z over B such that*

$(P.1.1)'$: *$T = x^{k+1}Y$, $|Y| \leq 1$, $f(Y) = \pi_1\sigma_2$, $f(x) = \pi_1 Z$, and $f(v) = Z\pi_1$*
$(P.1.2)'$: *or $T = Yx^{k+1}$, $|Y| = 1$, $f(Y) = \pi_1\sigma_2$, $f(x) = Z\sigma_2$, and $f(v) = \sigma_2 Z$.*

Proof. Let T_1 be the shortest prefix of T such that $f(T_1)$ starts with $\pi_1 f(v)^k$, *i.e.*, $f(T_1) = \pi_1 f(v)^k\sigma_2'$ with $|\sigma_2'| < |f(T_1[|T_1|])|$ and let $\overline{T}_1$ be the word such that $T = T_1\overline{T}_1$.

Let T_2 be the shortest suffix of T such that $f(T_2)$ ends with $f(v)^k\sigma_2$, *i.e.*, $f(T_2) = \pi_1' f(v)^k\sigma_2$ with $|\pi_1'| < |f(T_2[1])|$ and let $\overline{T}_2$ be the word such that $T = \overline{T}_2T_2$.

By Lemma 3.1, either f is not k-power-free or each of the words T_1 and T_2 satisfies one of the condition $(P.1.1)$, $(P.1.2)$, $(P.2.1)$, or $(P.2.2)$.

If T_1 satisfies $(P.1.1)$, that is, if there exist a pure k-power x^k, a word y over A, and a word Z over B such that $T_1 = x^ky$, $|y| \leq 1$, $f(y) = \pi_1\sigma_2'$, $f(x) = \pi_1 Z$, and $f(v) = Z\pi_1$, then we obtain $f(T) = f(x^ky\overline{T}_1) = (\pi_1 Z)^k\pi_1\sigma_2' f(\overline{T}_1) = \pi_1(Z\pi_1)^{k+1}\sigma_2$. It means that $\sigma_2' f(\overline{T}_1)$ starts with Z and $f(y\overline{T}_1)$ starts with $\pi_1 Z = f(x)$. Since f is injective, we obtain that $y\overline{T}_1$ starts with x. Hence, there exist a word Y such that $y\overline{T}_1 = xY$ with $f(Y) = \pi_1\sigma_2$. If $|Y| \geq 2$, since $|\sigma_2| < |f(T[|T|])| = |f(Y[|Y|])|$, then $f(Y[1 \dots |Y|-1])$ is a prefix of π_1 itself a prefix of $f(x[|1|])$. This contradicts the fact that f is bifix. It follows that T satisfies $(P.1.1)'$.

If T_1 satisfies $(P.1.2)$ then we obtain $f(v) = \sigma_2' Z$ and $f(T) = f(yx^k\overline{T}_1) = \pi_1\sigma_2'(Z\sigma_2')^k f(\overline{T}_1) = \pi_1(\sigma_2' Z)^{k+1}\sigma_2$. That is, $f(\overline{T}_1) = Z\sigma_2$ with $|\sigma_2| < |f(T[|T|])| = |f(\overline{T}_1[|\overline{T}_1|])|$. Moreover, $f(y) = \pi_1\sigma_2'$ with

$|\pi_1| < |f(T[1])| = |f(y[1])| = |f(y)|$ and $f(x) = Z\sigma_2'$. By Lemma 2.10 and since $|\sigma_2'| < |f(T_1[|T_1|])| = |f(x[|x|])|$, we obtain $x = x_1 a$ and $\overline{T}_1 = x_1 c$ for some word x_1 and some letters a and c with either $y = a$ or $c = a$. If $c = a$ then $\overline{T}_1 = x$, $\sigma_2' = \sigma_2$, and T satisfies $(P.1.2)'$. If $y = a$ then it means that $f(x)$ ends with $f(y) = f(a) = \pi_1\sigma_2'$. It implies that Z ends with π_1. Thus, there exist a word Z_1 such that $Z = Z_1\pi_1$ and $f(x_1) = Z_1$. Since $f(\overline{T}_1) = Z\sigma_2 = Z_1\pi_1\sigma_2 = f(x_1)f(c)$, we obtain $f(c) = \pi_1\sigma_2$. Taking ax_1 for x, c for Y, and $\sigma_2' Z_1$ for Z, we obtain that T satisfies $(P.1.1)'$.

In the same way, if T_2 satisfies $(P.1.2)$ then T satisfies $(P.1.2)'$ and if T_2 satisfies $(P.1.1)$ then T satisfies $(P.1.1)'$ or $(P.1.2)'$.

If T_1 satisfies $(P.2.2)$, that is, if $T_1 = yx^k$ with $|f(x^{k-1})| < |f(v)\sigma_2'|$, then, by definition of T_1, we obtain $|\sigma_2'| < |f(x)| \leq |f(x)^{k-2}| < |f(v)|$ and $|f(yx^{k-1}x[1\ldots|x|-1])| < |\pi_1 f(v)^k|$. It follows $|f(x\overline{T}_1)| \geq |f(x[|x|]\overline{T}_1)| > |f(v)\sigma_2|$ and it implies that $\overline{T}_1 \neq \varepsilon$. Thus, $f(x)^k$ is an internal factor of $f(v^{k-1})$: f is not k-power-free.

In the same way, if T_2 satisfies $(P.2.1)$ then f is not k-power-free.

Let us now assume that T_1 satisfies $(P.2.1)$ and T_2 satisfies $(P.2.2)$, *i.e.*, there exist two pure k-powers x^k and x'^k, and two non-empty words y and y' over A such that $T_1 = x^k y$ with $|f(x^{k-1})| < |\pi_1 f(v)|$ and $T_2 = y'x'^k$ with $|f(x'^{k-1})| < |f(v)\sigma_2|$. In particular, $|f(x)| < \frac{1}{2}|\pi_1 f(v)| < |\pi_1| + \frac{1}{2}|f(v)|$ and $|f(x')| < |\sigma_2| + \frac{1}{2}|f(v)|$. It follows that $|f(T[2\ldots|T|-1])| \geq |f(T)| - |f(x)| - |f(x')| > |f(v)^k|$: there exists a word V', which is a conjugate of $f(v)$, and such that $f(T[2\ldots|T|-1])$ contains V'^k. If $T[2\ldots|T|-1]$ is k-power-free then f is not. Thus, $T[2\ldots|T|-1]$ contains a pure k-power t^k. But $f(t)^k$ is an internal factor of $f(v)^{k+1}$. So if $|f(t)^k| \leq |f(t)| + |f(v)|$ then $f(t)^k$ is factor of $f(v^3)$ with v^3 k-power-free. Hence, f is not k-power-free. If $|f(t)^k| > |f(t)| + |f(v)|$, by Corollary 2.4, then there exist two words t_1, t_2, and two integers p, q such that $f(t) = (t_1t_2)^p$ and $f(v) = (t_2t_1)^q$ with t_1t_2 and t_2t_1 primitive words. Moreover, since v^k and t^k are pure k-powers, we obtain $p = q = 1$. Let T' and T'' be the non-empty words such that $T = T't^kT''$. We have $f(T) = f(T')(t_1t_2)^k f(T'') = \pi_1(t_2t_1)^{k+1}\sigma_2$ with $|\pi_1| < |f(T'[1])|$ and $|\sigma_2| < |f(T''[|T''|])|$. Since t_1t_2 is a primitive word, t_1t_2 is not an internal factor of $(t_1t_2)^2$. So $f(T') = \pi_1 t_2$ and $f(T'') = t_1\sigma_2$. By Lemma 2.10, it implies $t = v_1 a v_2$, $T' = bv_2$, and $T'' = v_1 c$ for some words v_1, v_2, u_1 and w_2, and some letters a, b and c with either $b = a$ or $c = a$. That is, $T = (av_2v_1)^{k+1}c$ or $T = b(v_2v_1a)^{k+1}$. Hence, T satisfies $(P.1.1)'$ or $(P.1.2)'$. □

3.2. Equations of reduction

When $f(w) = pu^\kappa s$, the different occurrences of u give us equations on the images of the factors of w. Some equations can be reduced:

Lemma 3.5. *Let $\alpha_1, \alpha_2, \beta_1, \beta_1', \beta_2, \gamma_1, \gamma_2$ be words over an alphabet B such that $|\beta_1| = |\beta_2| \neq 0$, β_1' is a proper suffix of β_1, and $0 \leq |\alpha_2| - |\alpha_1| \leq |\beta_1'|$.*

Under these hypotheses, the equality $\alpha_2\beta_2\gamma_2 = \alpha_1\beta_1'\beta_1\gamma_1$ implies $\alpha_2\gamma_2 = \alpha_1\beta_1'\gamma_1$.

Proof. Let us set $w = \alpha_1\beta_1'\beta_1\gamma_1 = \alpha_2\beta_2\gamma_2$.

The words α_1, α_2 are both prefixes of w. Since $|\alpha_2| \geq |\alpha_1|$, the word α_1 is a prefix of α_2. Hence, there exists a word α_2' such that $\alpha_2 = \alpha_1\alpha_2'$ with $|\alpha_2'| = |\alpha_2| - |\alpha_1| \leq |\beta_1'|$.

We have $|\gamma_2| - |\gamma_1| = |w| - |\alpha_2\beta_2| - |\gamma_1| = |\beta_1'| - (|\alpha_2| - |\alpha_1|)$ so $0 \leq |\gamma_2| - |\gamma_1| \leq |\beta_1'|$. The words γ_1, γ_2 are both suffixes of w. Consequently, there exists a word γ_2' such that $\gamma_2 = \gamma_2'\gamma_1$ with $0 \leq |\gamma_2'| \leq |\beta_1'|$.

The equality $\alpha_2\beta_2\gamma_2 = \alpha_1\beta_1'\beta_1\gamma_1$ becomes $\alpha_1\alpha_2'\beta_2\gamma_2'\gamma_1 = \alpha_1\beta_1'\beta_1\gamma_1$, that is, $\alpha_2'\beta_2\gamma_2' = \beta_1'\beta_1$. But $|\alpha_2'| + |\gamma_2'| = |\alpha_2'\beta_2\gamma_2'| - |\beta_2| = |\beta_1'\beta_1| - |\beta_1| = |\beta_1'|$. Thus, α_2' is a prefix of β_1' and γ_2' is a suffix of β_1 so of β_1' with $|\alpha_2'| + |\gamma_2'| = |\beta_1'|$, that is, $\alpha_2'\gamma_2' = \beta_1'$.

It follows that $\alpha_2\gamma_2 = \alpha_1\alpha_2'\gamma_2'\gamma_1 = \alpha_1\beta_1'\gamma_1$. □

The situation described in Figure 1 is an example of one case where the hypotheses of the following lemma are satisfied. Figure 2 deals with point (4) of Remark 3.7.

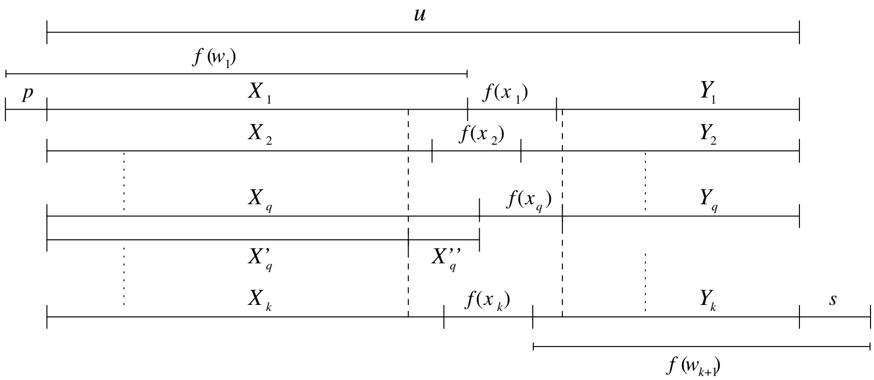

FIGURE 1. Reduction of a power.

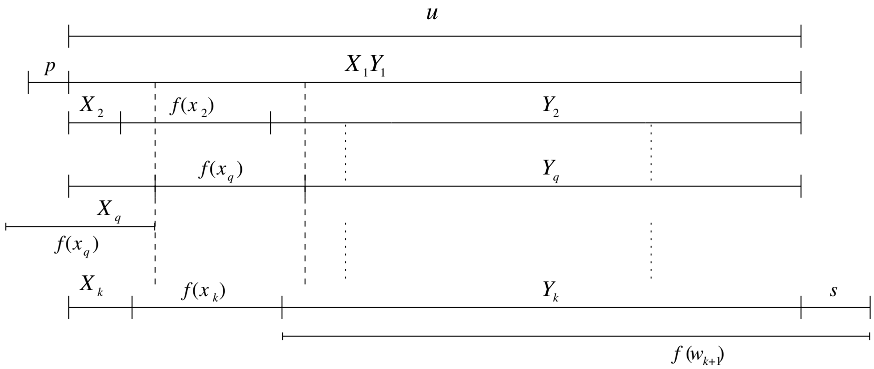

FIGURE 2. Case (4) of Remark 3.7.

Lemma 3.6. *Let $\kappa \geq 3$ be an integer. Let f be a morphism from A^* to B^*. Let $(w_i)_{i=1\dots\kappa+1}$ and $(x_i)_{i=1\dots\kappa}$ be words over A such that $|f(x_i)| = |f(x_j)| \neq 0$ for all integers i, j in $[1, \kappa]$.*

We denote by w the word $w_1x_1 \dots w_\kappa x_\kappa w_{\kappa+1}$.

We assume that there exist words u, p, s, $(X_i)_{i=1\dots\kappa}$, and $(Y_i)_{i=1\dots\kappa}$ over B such that $f(w_1) = pX_1$, $f(w_{\kappa+1}) = Y_\kappa s$, and $f(w_i) = Y_{i-1}X_i$ for all $2 \leq i \leq \kappa$. Moreover, we assume that, for all integers i in $[1, \kappa]$, we have $u = X_i f(x_i) Y_i$. It means that $f(w) = pu^\kappa s$.

Let us also assume that there exists an integer q such that, for every integer i in $[1, \kappa]$, $0 \leq |X_q| - |X_i| \leq |X''_q|$ where X''_q is a common suffix of X_q and $f(x_q)$. Then the word $\breve{w} = w_1w_2 \dots w_\kappa w_{\kappa+1}$ satisfies $f(\breve{w}) = p\breve{u}^\kappa s$ with $\breve{u} = X_iY_i$ for every integer i in $[1, \kappa]$.

In particular, $f(\breve{w})$ and $\breve{u}^\kappa$ are synchronised only if $f(w)$ and u^κ are synchronised.

We say that we have reduced w. And, before proving Lemma 3.6, let us first consider, in the following remark, some special cases of reduction. Point (4) will be treated in the proof of Lemma 3.6. Point (5) is the mirror image of point (4). And point (6) is a combination of points (4) and (5).

Remark 3.7.

(1) Using the mirror image and exchanging $|X_q|$ the maximum of $|X_i|$ by the maximum $|Y_q|$ of $|Y_i|$ (*i.e.*, $|X_q|$ is the minimum of $|X_i|$), the condition "$0 \leq |X_q| - |X_i| \leq |X''_q|$ where X''_q is a common suffix of X_q and $f(x_q)$" of Lemma 3.6 can be replaced by "$0 \leq |Y_q| - |Y_i| \leq |Y'_q|$ where Y'_q is a common prefix of Y_q and $f(x_q)$".

(2) A prefix u_1 of u is also a prefix of $\breve{u}$ if $|u_1| < |X_q|$ and a suffix u_2 of u is also a suffix of $\breve{u}$ if $|u_2| < \max |Y_j|$.

(3) If, instead of $u = X_\kappa f(x_\kappa) Y_\kappa$, we only have that $X_\kappa f(x_\kappa)Y_\kappa$ is a prefix of u then $f(\breve{w}) = p\breve{u}^{\kappa-1}X_\kappa Y_\kappa s$ with $X_\kappa Y_\kappa$ prefix of $\breve{u}$.
(4) If $q \neq 1$ and X_q is a suffix of $f(x_q)$, *i.e.*, $X'_q = \varepsilon$ (see Fig. 2), then we do not need x_1 and optionally not w_1 in the hypotheses of Lemma 3.6. Conclusion remains true with $u = X_1Y_1$, $w'_2 = w_1w_2$ or w_2, $f(w'_2) = pX_1Y_1X_2$, $w = w'_2x_2w_3 \dots w_\kappa x_\kappa w_{\kappa+1}$, and $\breve{w}$ a (not necessarily proper) suffix of $w'_2w_3 \dots w_\kappa w_{\kappa+1}$
(5) If $q \neq \kappa$ and Y_q is a prefix of $f(x_q)$ then we do not need x_κ and optionally not $w_{\kappa+1}$ in the hypotheses of Lemma 3.6. Conclusion remains true with $u = X_\kappa Y_\kappa$, $w'_\kappa = w_\kappa w_{\kappa+1}$ or w_κ, $f(w'_\kappa) = Y_{\kappa-1}X_\kappa Y_\kappa s$, $w = w_1x_1w_2 \dots w_{\kappa-1}x_{\kappa-1}w'_\kappa$, and $\breve{w}$ a (not necessarily proper) prefix of $w_1w_2 \dots w_{\kappa-1}w'_\kappa$.
(6) If $q \neq 1$, $q \neq \kappa$, X_q is a suffix of $f(x_q)$, and Y_q is a prefix of $f(x_q)$ then we do not need neither x_1 nor x_κ in the hypotheses of Lemma 3.6. Conclusion remains true with $u = X_1Y_1 = X_\kappa Y_\kappa$, $w'_2 = w_1w_2$ or w_2, $w'_\kappa = w_\kappa w_{\kappa+1}$ or w_κ, $f(w'_2) = pX_1Y_1X_2$, $f(w'_\kappa) = Y_{\kappa-1}X_\kappa Y_\kappa s$, $w = w'_2x_2w_3 \dots w_{\kappa-1}x_{\kappa-1}w'_\kappa$, and $\breve{w}$ a (not necessarily proper) factor of $w'_2w_3 \dots w_{\kappa-1}w'_\kappa$.

Proof. Without loss of generality, we can assume that $|p| < |f(w_1[1])|$ and $|s| < |f(w_{\kappa+1}[|w_{\kappa+1}|])|$. Let X'_q be the word such that $X_q = X'_qX''_q$. For every integer $i \in [1, \kappa]$, we have $X_if(x_i)Y_i = X'_qX''_qf(x_q)Y_q(= u)$. Since $0 \leq |X_i| - |X'_q| \leq |X''_q|$, by Lemma 3.5, we obtain $X_iY_i = X_qY_q$ (it is $\breve{u}$).

That is, $f(\breve{w}) = pX_1Y_1X_2 \dots Y_{\kappa-1}X_\kappa Y_\kappa s = p\breve{u}^\kappa s$ and $|\breve{u}| = |u| - |f(x_q)|$.

Let us treat point (4) of Remark 3.7, that is, $q \neq 1$, X_q is a suffix of $f(x_q)$ (*i.e.*, $X'_q = \varepsilon$), $u = X_1Y_1$, $w'_2 = w_1w_2$ or w_2, $f(w'_2) = pX_1Y_1X_2$, and $w = w'_2x_2 \dots w_\kappa x_\kappa w_{\kappa+1}$. Let $\overline{X}_q$ be the word such that $f(x_q) = \overline{X}_qX_q$. We have $|X_i| \leq |X_q| \leq |f(x_q)|$ for every integer i in $[2, \kappa]$. Since $X_if(x_i)$ and $X_qf(x_q)$ are both prefixes of u, let z_i be the prefix of X_q such that $X_q\overline{X}_qz_i = X_if(x_i)$. We have $|z_i| = |X_if(x_i)| - |X_q\overline{X}_q| = |X_i|$. Thus, $X_i = z_i$ is a suffix of $f(x_i)$. It follows that, for all integers i in $[2, \kappa]$, $X_iY_i = \breve{u}$ (as $X_if(x_i)Y_i$) is a suffix of $pu = pX_1Y_1$. Hence, $f(\breve{w})$ ends with $\breve{u}^\kappa s$.

Even if it seems elementary, the delicate point of this proof is the property of synchronization. In the next part, we are interested in it. This will also give basic ideas of the proof for the specific assumptions of Remark 3.7. A re-reading of the general case adjusting conditions (most frequently considering a suffix of w_1 or w_1w_2 in Case (4), and a prefix of $w_{\kappa+1}$ or of $w_\kappa w_{\kappa+1}$ in Case (5)) gives solutions to theses specific cases.

If $f(\breve{w})$ and $\breve{u}^\kappa$ are synchronised, there exist two integers $0 \leq \ell < m \leq |\breve{w}|$ such that $|\breve{u}| = |f(\breve{w}[\ell+1 \dots m])| = |f(\breve{w}[1 \dots m])| - |f(\breve{w}[1 \dots \ell])|$ and, specifically, $p = \varepsilon$ when $l = 0$, and $s = \varepsilon$ when $m = |\breve{w}|$.

If $l = 0$ then $\breve{u} = f(\breve{w}[1 \dots m]) = X_1Y_1$. Since f is injective, there exists a prefix w'_2 of w_2 such that $f(w_1) = X_1$ and $f(w'_2) = Y_1$. It follows that w starts with $w_1x_1w'_2$ and $f(w)$ starts with $f(w_1x_1w'_2) = X_1f(x_1)Y_1 = u$. Hence, $f(w)$ and u^κ are synchronised.

In a similar way, if $m = |\breve{w}|$ then $f(w)$ and u^κ are synchronised.

From now, we assume that $0 < \ell < m < |\breve{w}|$ and let $r \geq 1$ be the integer such that $|p\breve{u}^{r-1}| \leq |f(\breve{w}[1 \dots \ell])| < |p\breve{u}^r|$. Let us recall that the words $\breve{w}[1 \dots \ell]$ and $w_1 \dots w_r$ are both prefixes of $\breve{w}$ and that $f(w_1 \dots w_r) = p\breve{u}^{r-1}X_r$.

Case 1: $|f(\breve{w}[1 \dots \ell])| \leq |p\breve{u}^{r-1}| + \min\{|X_r|; |X_{r+1}|\}$.

We have $|f(\breve{w}[1 \dots \ell])| \leq |p\breve{u}^{r-1}X_r| = |f(w_1 \dots w_r)|$ and so $\breve{w}[1 \dots \ell]$ is a prefix of $w_1 \dots w_r$. More precisely, since $|f(w_1 \dots w_{r-1})| \leq |p\breve{u}^{r-1}| \leq |f(\breve{w}[1 \dots \ell])|$, there exists a suffix y_r of w_r such that $\breve{w}[1 \dots \ell]y_r = w_1 \dots w_r$. Furthermore, since $|f(\breve{w}[1 \dots m])| = |\breve{u}| + |f(\breve{w}[1 \dots \ell])|$, we obtain $|p\breve{u}^r| \leq |f(\breve{w}[1 \dots m])| \leq |p\breve{u}^r| + |X_{r+1}|$. There exists a suffix y_{r+1} of w_{r+1} such that $\breve{w}[1 \dots m]y_{r+1} = w_1 \dots w_rw_{r+1}$. In particular, we have $|f(w_{r+1})| = |f(\breve{w}[1 \dots m])| + |f(y_{r+1})| - |f(\breve{w}[1 \dots \ell])| - |f(y_r)| = |\breve{u}| + |f(y_{r+1})| - |f(y_r)|$.

Since y_r is a suffix of w_r and y_{r+1} is a suffix of w_{r+1}, let i be the integer such that $w[1 \dots i]y_r = w_1x_1w_2 \dots x_{r-1}w_r$ and let j be the integer such that $w[1 \dots j]y_{r+1} = w_1x_1w_2 \dots w_rx_rw_{r+1}$. Since $i = 0$ implies $\ell = 0$ and since $j = |w|$ implies $m = |w|$, we have $0 < i < j < |w|$. Furthermore, $|f(w[i+1 \dots j])| = |f(w[1 \dots j])| - |f(w[1 \dots i])| = |f(x_rw_{r+1})| - |f(y_{r+1})| + |f(y_r)| = |\breve{u}| + |f(x_r)| = |\breve{u}| + |f(x_q)| = |u|$. That is, $f(w)$ and u^κ are synchronised.

Case 2: $|f(\breve{w}[1 \dots \ell])| \geq |p\breve{u}^{r-1}| + \max\{|X_r|; |X_{r+1}|\}$.

The inequalities $|p\check{u}^{r-1}X_r| \leq |f(\check{w}[1\ldots\ell])| < |p\check{u}^r|$ mean that $|f(w_1\ldots w_r)| \leq |f(\check{w}[1\ldots\ell])| < |f(w_1\ldots w_{r+1})|$. Consequently, there exists a prefix z_{r+1} of w_{r+1} such that $\check{w}[1\ldots\ell] = w_1\ldots w_r z_{r+1}$. Since $|p\check{u}^r X_{r+1}| \leq |f(\check{w}[1\ldots m])| < |p\check{u}^{r+1}|$, there exists a prefix z_{r+2} of w_{r+2} such that $\check{w}[1\ldots m] = w_1\ldots w_{r+1}z_{r+2}$. We have $|\check{u}| = |f(\check{w}[1\ldots m])| - |f(\check{w}[1\ldots\ell])| = |f(w_{r+1}z_{r+2})| - |f(z_{r+1})|$.

Let i be the integer such that $w[1\ldots i] = w_1x_1\ldots w_rx_rz_{r+1}$ and let j be the integer such that $w[1\ldots j] = w_1x_1\ldots w_rx_rw_{r+1}x_{r+1}z_{r+2}$. Since $0<\ell<m<|\check{w}|$, we have $0<i<j<|w|$. Furthermore, $|f(w[i+1\ldots j])| = |f(w[1\ldots j])| - |f(w[1\ldots i])| = |f(w_{r+1}x_{r+1})| + |f(z_{r+2})| - |f(z_{r+1})| = |\check{u}| + |f(x_{r+1})| = |\check{u}| + |f(x_q)| = |u|$. That is, $f(w)$ and u^κ are synchronised.

Case 3: $\min\{|X_r|;|X_{r+1}|\} < |f(\check{w}[1\ldots\ell])| - |p\check{u}^{r-1}| < \max\{|X_r|;|X_{r+1}|\}$.

In particular, it means that $|X_r| \neq |X_{r+1}|$.

If $|X_r| < |X_{r+1}|$ then we obtain $|f(w_1\ldots w_r)| = |p\check{u}^{r-1}X_r| < |f(\check{w}[1\ldots\ell])| < |p\check{u}^r| \leq |f(w_1\ldots w_{r+1})|$ and $|f(w_1\ldots w_r)| \leq |p\check{u}^r| < |f(\check{w}[1\ldots m])| < |p\check{u}^rX_{r+1}| = |f(w_1\ldots w_{r+1})|$. Thus, there exists a prefix z_{r+1} of w_{r+1} such that $\check{w}[1\ldots\ell] = w_1\ldots w_rz_{r+1}$ and there exists a suffix y_{r+1} of w_{r+1} such that $\check{w}[1\ldots m]y_{r+1} = w_1\ldots w_rw_{r+1}$. So we have $|\check{u}| = |f(\check{w}[1\ldots m])| - |f(\check{w}[1\ldots\ell])| = |f(w_{r+1})| - |f(z_{r+1})| - |f(y_{r+1})|$.

Since $0 < |f(\check{w}[1\ldots\ell])| - |f(w_1\ldots w_r)| = |f(\check{w}[1\ldots\ell])| - |p\check{u}^{r-1}X_r| < |X_{r+1}| - |X_r|$, we obtain $|X_rf(z_{r+1})| < |X_{r+1}|$ and so the word $X_rf(z_{r+1})$ is a prefix of X_{r+1}. Since $||X_r| - |X_{r+1}|| \leq |X''_q| \leq |f(x_q)|$, it follows that $|X_rf(x_r)| \geq |X_{r+1}|$. But X_{r+1} is a prefix of $X_rf(x_r)$ (they are both prefixes of u). So $f(z_{r+1})$ is a prefix of $f(x_r)$. By Lemma 2.6, it implies that z_{r+1} is a prefix of x_r.

Let i be the integer such that $w[1\ldots i] = w_1x_1\ldots w_rz_{r+1}$ and let j be the integer such that $w[1\ldots j]y_{r+1} = w_1x_1w_2\ldots w_rx_rw_{r+1}$.

As above, we obtain $0<i<j<|w|$ and $|f(w[i+1\ldots j])| = |f(w[1\ldots j])| - |f(w[1\ldots i])| = |f(x_rw_{r+1})| - |f(y_{r+1})| - |f(z_{r+1})| = |\check{u}| + |f(x_r)| = |\check{u}| + |f(x_q)| = |u|$. That is, $f(w)$ and u^κ are synchronised.

Using the fact that z_{r+2} is a prefix of $f(x_{r+1})$, the case $|X_r| > |X_{r+1}|$ is solved in the same way. □

For every positive integer ℓ, since $|f(x_i)| = |f(x_j)|$ is equivalent to $|f(x_i^\ell)| = |f(x_j^\ell)|$ and since a prefix (resp. a suffix) of $f(x_i)$ is a prefix (resp. a suffix of $f(x_i^\ell)$), we immediately obtain the following Corollary which will be the central point of the proof of Proposition 4.1.

Corollary 3.8 (Method of reduction)**.** *Let $\kappa \geq 3$ and $\ell \geq 1$ be two integers, let α be an integer in $\{1,2\}$ and let β be an integer in $\{\kappa-1,\kappa\}$*

Let f be a morphism from A^* to B^* and let $(w_i)_{i=\alpha\ldots\beta+1}$, $(x_i)_{i=\alpha\ldots\beta}$ be words over A such that $|f(x_i)| = |f(x_j)| \neq 0$ for all integers i,j in $[\alpha,\beta]$.

We denote by w the word $w_\alpha x_\alpha^\ell\ldots w_\beta x_\beta^\ell w_{\beta+1}$.

We assume that there exist u, p, s, $(X_i)_{i=\alpha\ldots\beta}$, and $(Y_i)_{i=\alpha\ldots\beta}$ words over B such that $f(w_i) = Y_{i-1}X_i$ for all integers i in $[1+\alpha;\beta]$. Furthermore, we also assume that $f(w_\alpha) = pu^{\alpha-1}X_1$ and $f(w_{\beta+1}) = Y_\kappa u^{\kappa-\beta}s$ where $u = X_if(x_i^\ell)Y_i(\neq\varepsilon)$ for all integers i in $[\alpha,\beta]$. It means that $f(w) = pu^\kappa s$.

Finally, we assume that there exists an integer q such that, for every integer i in $[\alpha,\beta]$, $0 \leq |X_q| - |X_i| \leq |X''_q|$ where X''_q is a common suffix of X_q and $f(x_q)$, $0 \leq |X_q| - |X_i| \leq |f(x_q)|$ when $\alpha = 2$, or $0 \leq |Y_i| - |Y_q| \leq |f(x_q)|$ when $\beta = \kappa - 1$.

Then, for every integer $0 \leq \phi < \ell$, the word $\check{w} = w_\alpha x_\alpha^\phi\ldots w_\beta x_\beta^\phi w_{\beta+1}$ satisfies $f(\check{w}) = p\check{u}^\kappa s$ with $\check{u} = X_if(x_i^\phi)Y_i$ for every integer i in $[1;\kappa]$.

In particular, $f(\check{w})$ and $\check{u}^\kappa$ are synchronised only if $f(w)$ and u^κ are synchronised.

3.3. Situations of reduction

Let $k \geq 3$ be an integer and let $\kappa \in \{k;k+1\}$. Let f be a morphism from A^* to B^* and let ω be a word over A such that $f(\omega) = pU^\kappa S$ for some words p, S, and $U \neq \varepsilon$ over B such that $|p| < |f(\omega[1])|$. Moreover, we assume $|S| < |f(\omega[|\omega|])|$ when $\kappa = k+1$. It is important to note that, when $\kappa = k$, the word S is not necessarily a proper suffix of $f(\omega[|\omega|])$.

For every integer j in $[1, \kappa+1]$, let i_j the smallest integer such that pU^{j-1} is a prefix of $f(\omega[1 \dots i_j])$. We have $i_1 = 1$ and there exist words p_j and s_j such that $f(\omega[i_j]) = p_j s_j$, $p_1 = p$, $s_{\kappa+1}$ is a prefix of S ($s_{\kappa+1} = S$ when $\kappa = k+1$), $p_j \neq \varepsilon$ if $j \neq 1$, and $s_1 \neq \varepsilon$. Furthermore, we have $f(\omega[1 \dots i_j]) = pU^{j-1}s_j$ for every integer j in $[1, \kappa+1]$ and $U = s_j f(\omega[i_j+1 \dots i_{j+1}-1])p_{j+1}$ for every integer j in $[1, \kappa]$.

Since a factor of ω can appear many times in ω, it is necessary to indicate which exact factor we are going to work with. If $\omega[n \dots m] = z$, we set $n_{\rm z} = n$ and $m_{\rm z} = m$. This fixes the considered occurrence of z in ω. For every positive integer α, if $\omega[n \dots m] = z^\alpha$, we also set $n_{\rm z} = n$ and $m_{\rm z} = m$ without specifying α. It is the same notation as the case $\alpha = 1$; we will precise only if necessary.

To simplify notations, let us recall that, given two integers $1 \leq n_{\rm z} \leq m_{\rm z} \leq |\omega|$, the word $\omega[n_{\rm z} \dots m_{\rm z}] = z^\alpha$ define two words z_p and z_s such that $\omega = z_p z^\alpha z_s$, with $n_{\rm z} = |z_p| + 1$ and $m_{\rm z} = |z_p z^\alpha|$. This means that $z_p = \omega[1 \dots n_{\rm z}-1]$ and $z_s = \omega[m_{\rm z}+1 \dots |\omega|]$.

Given two integers $1 \leq n_{\rm z} \leq m_{\rm z} \leq i_{\kappa+1}$, we also define a word $D_{\rm z}$ and three integers $\lambda_{\rm z}$, $d_{\rm z}$, and $c_{\rm z}$ (even if $c_{\rm z}$ is not used in this section). Eventually, we will precise $D_{{\rm z},\omega}$, $\lambda_{{\rm z},\omega}$, $d_{{\rm z},\omega}$, and $c_{{\rm z},\omega}$ if a doubt may occur. Briefly, $\lambda_{\rm z}$ is the integer such that $f(\omega[n_{\rm z} \dots m_{\rm z}]) = f(z^\alpha)$ starts in the ${\lambda_{\rm z}}^{\rm th}$ occurrence of U; $d_{\rm z}$ indicates if the first occurrence of $f(z)$ in $f(\omega[n_{\rm z} \dots m_{\rm z}])$ covers or not two consecutive occurrences of U; $c_{\rm z}$ is the number of occurrences of U covers by $f(\omega[n_{\rm z} \dots m_{\rm z}])$ and $D_{\rm z}$ is a prefix of U such that $f(z_p z)$ ends with $D_{\rm z}$ or $D_{\rm z} f(z)$.

More precisely, if $n_{\rm z} = 1$, *i.e.*, z is a prefix of ω, then we set $\lambda_{\rm z} = 0$, $d_{\rm z} = 1$, and $D_{\rm z}$ is the word such that $f(z) = pD_{\rm z}$. When $n_{\rm z} \geq 2$, let $\lambda_{\rm z}$ be the integer such that $n_{\rm z} \in]i_{\lambda_{\rm z}}; i_{\lambda_{\rm z}+1}]$, *i.e.*, $|pU^{\lambda_{\rm z}-1}| \leq |f(\omega[1 \dots n_{\rm z}-1])| = |f(z_p)| < |pU^{\lambda_{\rm z}}|$. If $|f(z_p z)| \leq |pU^{\lambda_{\rm z}}|$ then let $d_{\rm z} = 0$ otherwise let $d_{\rm z} = 1$. Let $D_{\rm z}$ be the word such that $f(z_p z^{d_{\rm z}}) = pU^{\lambda_{\rm z}-1+d_{\rm z}} D_{\rm z}$. It means that $D_{\rm z} = s_{\lambda_{\rm z}} f(\omega[i_{\lambda_{\rm z}}+1 \dots n_{\rm z}-1])$ when $d_{\rm z} = 0$ and $s_{\lambda_{\rm z}} f(\omega[i_{\lambda_{\rm z}}+1 \dots n_{\rm z}-1])f(z) = UD_{\rm z}$ when $d_{\rm z} = 1$. In particular, $D_{\rm z}$ is a proper suffix of $f(z)$ when $d_{\rm z} = 1$. Finally, $c_{\rm z}$ is the lowest integer such that $|f(\omega[1 \dots m_{\rm z}])| \leq |pU^{\lambda_{\rm z}+c_{\rm z}-1}|$.

It is important to remark that, if $\omega[n_{\rm z} \dots m_{\rm z}] = z^\alpha$, the integers $n_{\rm z}$ and $m_{\rm z}$ define z^α and z. But, since we may have several occurrences of z^α in ω, we do not have the contrary. In other words, the equality $z = z'$ not necessarily implies $n_{\rm z} = n_{\rm z'}$ or $m_{\rm z} = m_{\rm z'}$. In the same vein, $\lambda_{\rm z}$, $d_{\rm z}$, $c_{\rm z}$, and $D_{\rm z}$ depend on $n_{\rm z}$ and $m_{\rm z}$ but not directly of z. But if no question exists over the considered factor of ω or if the choice of the considered factor does not matter, we will write z^α instead of $\omega[n_{\rm z} \dots m_{\rm z}]$.

For every integer $\alpha \geq 2$ and for every word $\omega[n_{\rm z} \dots m_{\rm z}] = z^\alpha$ with $n_{\rm z}, m_{\rm z} \in [1, i_{\kappa+1}]$, the word $f(\omega[n_{\rm y} \dots m_{\rm y}]) = f(y^\alpha) = f(y)^\alpha$ with $n_{\rm y}, m_{\rm y} \in [1, i_{\kappa+1}]$ is a *conjugated shift to the left* of $f(\omega[n_{\rm z} \dots m_{\rm z}]) = f(z^\alpha) = f(z)^\alpha$ (in $f(\omega)$) if there exist two words $t_1 \neq \varepsilon$ and t_2 such that $f(y) = t_2 t_1$, $f(z) = t_1 t_2$, and if we have one of the following conditions:

(i) $D_{\rm z} = D_{\rm y} t_2$ when $d_{\rm y} = d_{\rm z}$
(ii) $D_{\rm y} = D_{\rm z} t_1$ when $d_{\rm y} = 1$ and $d_{\rm z} = 0$
(iii) $D_{\rm y} f(y) t_2 = UD_{\rm z}$ when $d_{\rm y} = 0$ and $d_{\rm z} = 1$

Let us remark that conditions (2) and (3) imply $|D_{\rm z}| < |t_2|$. Taking $t_2 = \varepsilon$, let us also note that $f(z^\alpha)$ is a conjugated shift to the left of itself.

We say that $f(y)^\alpha$ is a *conjugated shift to the right* of $f(z)^\alpha$ if $f(z)^\alpha$ is a conjugated shift to the left of $f(y)^\alpha$. We simply say that $f(y)^\alpha$ is a *conjugated shift* of $f(z)^\alpha$ if it is a conjugated shift to the left or to the right of $f(z)^\alpha$.

For a general use of conjugated shifts of $f(z)^\alpha$, we will switch the roles of t_1 and t_2 in the definition and the conditions (1) to (3) for a conjugated shift to the right.

For any pure k-power $\omega[n_{\rm v} \dots m_{\rm v}] = v^k$ of ω, there are $k-2$ choices for the factor v^3 in v^k. We denote by $v^3_{(\beta)}$ the β^{th} factor of v^3 in v^k, that is, $\omega[n_{\rm v} \dots m_{\rm v}] = v^{\beta-1} v^3_{(\beta)} v^{k-\beta-2}$ with $1 \leq \beta \leq k-2$.

We will focus on theses different cubes v^3 but without specifying β in this section.

For every factor $\omega[n_{\rm v} \dots m_{\rm v}] = v^3$ of $\omega[1 \dots i_{\kappa+1}]$ and, for every integer $j \in [1; \kappa]$, let $L_{j,v}$ be the set of the words $\omega[n_{\rm x} \dots m_{\rm x}] = x^3$ such that $f(\omega[n_{\rm x} \dots m_{\rm x}]) = f(x)^3$ is a conjugated shift to the left of $f(\omega[n_{\rm v} \dots m_{\rm v}]) = f(v)^3$ with $\lambda_{\rm x} = j$ if $d_{\rm x} = d_{\rm v} = 0$ and $\lambda_{\rm x} = j-1$ otherwise.

We also denote by $R_{j,v}$ the set of the words $\omega[n_\mathrm{x}\dots m_\mathrm{x}] = x^3$ such that $f(\omega[n_\mathrm{x}\dots m_\mathrm{x}]) = f(x)^3$ is a conjugated shift to the right of $f(v)^3$ with $\lambda_\mathrm{x} = j - d_\mathrm{v} \times d_\mathrm{x}$.

If $\omega[n_{\mathrm{x_j}}\dots m_{\mathrm{x_j}}] = x_j^3$ is a word in $L_{j,v} \cup R_{j,v}$, we denote by $t_{1,j}$ and $t_{2,j}$ the words such that $f(v) = t_{1,j}t_{2,j}$ and $f(x_j) = t_{2,j}t_{1,j}$.

If j_0 is an integer such that $\omega[n_\mathrm{v}\dots m_\mathrm{v}] = v^3 \in L_{j_0,v}(\cup R_{j_0,v})$, we will always assume that $n_{\mathrm{x_{j_0}}} = n_\mathrm{v}$ and $m_{\mathrm{x_{j_0}}} = m_\mathrm{v}$, that is, $x_{j_0} = v$.

Lemma 3.9. *We use all previous definitions and notations of this section. In particular, v^3 is a chosen factor of a pure k-power v^k. When one of the four following situations holds, there exist a word $\breve{\omega}$ such that $f(\breve{\omega}) = p'(U')^\kappa S'$ for some words p', S', and $U' \neq \varepsilon$ over B satisfying $|p'| < |f(\breve{\omega}[1])|$, $0 < |U'| < |U|$, and $f(\breve{\omega})$ and $(U')^\kappa$ are synchronised if $f(\omega)$ and U^κ are synchronised.*

(1) $d_\mathrm{v} = 1$, $|D_\mathrm{v} f(v)^2| < |U|$, *and* $L_{j,v} \cup R_{j,v} \neq \emptyset$ *for every integer* $j \in [2,\kappa]$.
(2) $d_\mathrm{v} = 1$, $L_{j,v} \cup R_{j,v} \neq \emptyset$ *for every integer* $j \in [2,\kappa-1]$, *and there exists a positive integer ϕ such that* $\omega[n_\mathrm{v}\dots|\omega|]$ *starts with* $v^{\phi+2}$ *and* $\sup\{2|f(v)|; |D_\mathrm{v} f(v)^\phi|\} \leq |U| < |D_\mathrm{v} f(v)^{\phi+1}|$.
(3) $d_\mathrm{v} = 0$, $|D_\mathrm{v} f(v)^2| \leq |U|$, *and* $L_{j,v} \cup R_{j,v} \neq \emptyset$ *for every integer* $j \in [1,\kappa]$.
(4) $d_\mathrm{v} = 0$, $|U| < |D_\mathrm{v} f(v)^2| < |D_\mathrm{v} U|$, *and* $L_{j,v} \cup R_{j,v} \neq \emptyset$ *for every integer* $j \in [1,\kappa-1]$.

Proof. For every integer j, let $\omega[n_{\mathrm{x_j}}\dots m_{\mathrm{x_j}}] = x_j^3$ be a word in $L_{j,v} \cup R_{j,v}$.

Case (1): $d_\mathrm{v} = 1$, $|D_\mathrm{v} f(v)^2| < |U|$, and $L_{j,v} \cup R_{j,v} \neq \emptyset$ for every integer $j \in [2,\kappa]$.

If $x_j^3 \in L_{j,v}$ and $d_{\mathrm{x_j}} = 1$ (including $x_{j_0} = v$) or if $x_j^3 \in R_{j,v}$, let X_j be the word $D_{\mathrm{x_j}}$ and let e_j be the integer $d_{\mathrm{x_j}}$. If $x_j^3 \in L_{j,v}$ and $d_{\mathrm{x_j}} = 0$, let X_j be the suffix of $f(x_j)$ such that $D_{\mathrm{x_j}} f(x_j)^2 = UX_j$ and let $e_j = 2$. Let q be an integer such that $|X_q| = \max\{|X_j|; j \in [2;\kappa]\}$. For all integers $j \in [2,\kappa]$, if $d_{\mathrm{x_j}} = 0$ with $x_j^3 \in L_{j,v}$, or if $d_{\mathrm{x_j}} = 1$, then, by definitions, we have that X_j is a suffix of $f(x_j)$. If $d_{\mathrm{x_j}} = 0$ with $x_j^3 \in R_{j,v}$ then it means that $D_\mathrm{v} = D_{\mathrm{x_j}} t_{2,j}$. But D_v is a suffix of $f(v) = t_{1,j}t_{2,j}$. So it implies that $X_j = D_{\mathrm{x_j}}$ is a suffix of $t_{1,j}$ and of $f(x_j) = t_{2,j}t_{1,j}$.

In particular, X_q is a suffix of $f(x_q)$. It follows that $0 \leq |X_q| - |X_j| \leq |X_q| \leq |f(x_q)|$ for all integers $j \in [2,\kappa]$. Furthermore, if $d_{\mathrm{x_j}} = 0$ with $x_j^3 \in R_{j,v}$ then $\lambda_{\mathrm{x_j}} = j$, and $\lambda_{\mathrm{x_j}} = j-1$ otherwise. It follows that $f(\omega[1\dots n_{\mathrm{x_j}}-1]) f(x_j^{e_j}) = pU^{j-1}X_j$.

Since $|X_j f(x_j)| \leq 2|f(x_j)| = |f(v)^2| \leq |U|$, it follows that $X_j f(x_j)$ is a prefix of U. Hence, there exists a word Y_j such that $U = X_j f(x_j) Y_j$ for all integers $j \in [2,\kappa]$. Let w_2 be the prefix of ω such that $f(w_2) = pUX_2$, *i.e.*, $w_2 = \omega[1\dots n_{\mathrm{x_2}}-1]x_2^{e_2}$ and let $w_{\kappa+1}$ be the suffix of ω such that $f(w_{\kappa+1}) = Y_\kappa S$, *i.e.*, $\omega = \omega[1\dots n_{\mathrm{x_\kappa}}-1]x_\kappa^{1+e_\kappa} w_{\kappa+1}$. In particular, we have $f(\omega[n_{\mathrm{x_j}}\dots n_{\mathrm{x_{j+1}}}-1]) f(x_{j+1}^{e_{j+1}}) = f(x_j^{1+e_j}) Y_j X_{j+1}$ for all integers $j \in [2,\kappa-1]$. Since f is bifix, it implies that there exists a word w_j such that $f(w_j) = Y_{j-1}X_j$ for all integers $j \in [3,\kappa]$.

In summary, we obtain $\omega = w_2x_2w_3x_3\dots w_\kappa x_\kappa w_{\kappa+1}$, $f(\omega) = pU^\kappa S$ with $U = X_j f(x_j) Y_j$ for all integers $j \in [2,\kappa]$. Moreover, there exists an integer $q \in [2,\kappa]$ such that $0 \leq |X_q| - |X_j| \leq |X_q| \leq |f(x_q)|$ and X_q is a suffix of $f(x_q)$.

By Corollary 3.8 (or Lem. 3.6 and using Rem. 3.7(4)), in particular the property of synchronised words, we can reduce $f(\omega)$. More precisely, let U' be the non-empty word X_qY_q and let $\overline{w}_2$ be the shortest suffix of w_2 such that $f(\overline{w}_2)$ ends with $U'X_2$ and let $\breve{\omega}$ be the word $\overline{w}_2w_3\dots w_\kappa w_{\kappa+1}$. We obtain $f(\breve{\omega}) = p'(U')^\kappa S$ with $|p'| < |f(\breve{\omega}[1])|$ and $|U'| = |U| - |f(x_q)| < |U|$.

Fact 1: Let us note that U' is a suffix of U and, if U starts with a word z prefix of $D_\mathrm{v} f(v)$ (for instance X_q) then z is also a prefix of U'.

Fact 2: For all integers $j \in [1,\kappa]$, if $x_j^3 \in L_{j,v}$ then w_j ends with x_j and, in addition, if $d_{\mathrm{x_j}} = 1$ then w_{j+1} starts with x_j. If $x_j^3 \in R_{j,v}$ then w_{j+1} starts with x_j and, in addition, if $d_{\mathrm{x_j}} = 1$ then w_j ends with x_j.

Fact 3: If there exists an integer j_1 such that $x_{j_1}^3 \in L_{j_1,v}$ with $d_{\mathrm{x_{j_1}}} = 0$ and if there exists an integer j_2 such that $x_{j_2}^3 \in R_{j_2,v}$ with $d_{\mathrm{x_{j_2}}} = 0$ then w_{j_1+1} starts with x_{j_1} and w_{j_2} ends with x_{j_2}.

Case (2): $d_v = 1$, $L_{j,v} \cup R_{j,v} \neq \emptyset$ for every integer $j \in [2, \kappa - 1]$, and there exists a positive integer ϕ such that $\omega[n_v \dots |\omega|]$ starts with $v^{\phi+2}$ and $\sup\{2|f(v)|; |D_v f(v)^{\phi}|\} \leq |U| < |D_v f(v)^{\phi+1}|$.

In this case, U is a prefix of $D_v f(v)^{\phi+1}$.

For every integer $j \in [2, \kappa - 1]$, we define X_j and e_j as Case (1) and we obtain that X_j is also a suffix of $f(x_j)$ (thus of $f(x_j)^{\phi+1}$).

If $x_j^3 \in L_{j,v}$ with $d_{x_j} = 1$, then U is a prefix of the word $D_v f(v)^{\phi+1} = X_j t_{2,j}(t_{1,j}t_{2,j})^{\phi+1}$ and so of $X_j f(x_j)^{\phi+2}$. If $x_j^3 \in L_{j,v}$ with $d_{x_j} = 0$, since U^2 is a prefix of $UD_v f(v)^{\phi+1} = UD_v(t_{1,j}t_{2,j})^{\phi+1} = D_{x_j}(t_{2,j}t_{1,j})^{\phi+2}t_{2,j} = UX_j f(x_j)^{\phi} t_{2,j}$, it follows that U is a prefix of $X_j f(x_j)^{\phi+1}$.

In the same way, we show that U is a prefix of $X_j f(x_j)^{\phi+1}$ when $x_j^3 \in R_{j,v}$.

Let q be an integer such that $|X_q| = \max\{|X_j|; j \in [2; \kappa - 1]\}$. If $x_j^3 \in L_{j,v}$ with $d_{x_j} = 1$, or if $x_j^3 \in R_{j,v}$ with $d_{x_j} = 0$ then $|X_j| \leq |X_{j_0}|$. Thus, if $q \neq j_0$, either $x_j^3 \in L_{j,v}$ with $d_{x_j} = 0$, or $x_j^3 \in R_{j,v}$ with $d_{x_j} = 1$. Let δ be the greatest integer such that $|X_q f(x_q)^{\delta}| \leq |U| < |X_q f(x_q)^{\delta+1}|$.

For every integer $j \in [2, \kappa - 1]$, since $|X_j f(x_j)^{\delta}| \leq |X_q f(x_q)^{\delta}| \leq |U|$, there exists a word Y_j such that $U = X_j f(x_j)^{\delta} Y_j$. Since U is a prefix of $X_q f(x_q)^{\phi+2}$, we obtain $U = X_q f(x_q)^{\delta} Y_q$ with Y_q a prefix of $f(x_q)$.

Let w_2 be the prefix of ω such that $f(w_2) = pUX_2$, let w_κ be the suffix of ω such that $f(w_\kappa) = Y_{\kappa-1}US$ and, for all integers $j \in [3, \kappa - 1]$, let w_j be the word such that $f(w_j) = Y_{j-1}X_j$.

By Corollary 3.8 (or Lem. 3.6 and using Rem. 3.7(6)), we can reduce $f(\omega)$. More precisely, let U' be non-empty the word $X_q Y_q$. Accordingly, U' is both a prefix and a suffix of U. Let $\overline{w}_2$ be the shortest suffix of w_2 such that $f(\overline{w}_2)$ ends with $U'X_2$ and let $\breve{\omega}$ be the word $\overline{w}_2 w_3 \dots w_{\kappa-1} w_\kappa$. We obtain $f(\breve{\omega}) = p(U')^{\kappa-1}US$ and so it starts with $p(U')^{\kappa}$ where $|U'| = |U| - |f(x_q)| < |U|$.

Case (3): $d_v = 0$, $|D_v f(v)^2| \leq |U|$, and $L_{j,v} \cup R_{j,v} \neq \emptyset$ for every integer $j \in [1, \kappa]$.

For every integer $j \in [1, \kappa]$, let X_j be the word $D_{x_j} f(x_j)$ if $x_j^3 \in L_{j,v}$ with $d_{x_j} = 0$ (including $x_{j_0} = v$), or the word D_{x_j} if $x_j^3 \in L_{j,v}$ with $d_{x_j} = 1$, or if $x_j^3 \in R_{j,v}$.

If $x_j^3 \in L_{j,v}$, let $e_j = 1$, and if $x_j^3 \in R_{j,v}$, let $e_j = 0$.

For any word $x_j^3 \in R_{j,v}$, since $|D_v f(v)^2| \leq |U|$, we necessarily have $d_{x_j} = 0$. Furthermore, $0 \leq |X_{j_0}| - |X_j| = |t_{2,j}| < |f(x_{j_0})| = |f(v)|$.

If $x_j^3 \in L_{j,v}$ and $d_{x_j} = 0$, we have $X_{j_0} = D_v f(v) = D_{x_j} t_{2,j} t_{1,j} t_{2,j} = X_j t_{2,j}$ and so $0 \leq |X_{j_0}| - |X_j| = |t_{2,j}| < |f(x_{j_0})|$. If $x_j^3 \in L_{j,v}$ and $d_{x_j} = 1$, we have $X_j t_{2,j} = D_{x_j} t_{2,j} = D_v t_{1,j} t_{2,j} = X_{j_0}$ and so $0 \leq |X_{j_0}| - |X_j| = |t_{2,j}| < |f(x_{j_0})| = |f(v)|$.

We have $|X_{j_0}| = \max\{|X_j|; j \in [1; \kappa]\}$ and $f(\omega[1 \dots n_{x_j} - 1]) f(x_j^{e_j}) = pU^{j-1}X_j$ for all integers $j \in [1, \kappa]$.

Since $|X_j f(x_j)| \leq |D_v f(v)^2| \leq |U|$, the word $X_j f(x_j)$ is a prefix of U. Thus, there exist words Y_j such that $U = X_j f(x_j) Y_j$ for all j in $[1, \kappa]$. Let w_1 be the word $\omega[1 \dots n_{x_1} - 1]x_1^{e_1}$ and let $w_{\kappa+1}$ be the word such that $\omega[n_{x_\kappa} \dots |\omega|] = x_\kappa^{1+e_\kappa} w_{\kappa+1}$. In particular, we have $f(w_1) = pX_1$, $f(w_{\kappa+1}) = Y_\kappa S$ and, for every integer $j \in [1, \kappa - 1]$, $f(\omega[n_{x_j} \dots n_{x_{j+1}} - 1]) f(x_{j+1}^{e_{j+1}}) = f(x_j^{1+e_j}) Y_j X_{j+1}$. Since f is bifix, it implies that there exists a word w_j such that $f(w_j) = Y_{j-1}X_j$ for all integers $j \in [2, \kappa]$.

By Corollary 3.8 (or Lem. 3.6), we can reduce $f(\omega)$. More precisely, $\breve{\omega} = w_1 w_2 \dots w_\kappa w_{\kappa+1}$ and $U' = X_i Y_i (\neq \varepsilon)$ for all integers $i \in [1, \kappa]$. We obtain $f(\breve{\omega}) = p'(U')^{\kappa} S$ with $|p'| = |p| < |f(W[1])| = |f(w_1[1])| = |f(\breve{\omega}[1])|$ and $|U'| = |U| - |f(x_q)| < |U|$.

Fact 1: A prefix of U of length at most $\max\{|X_i|\} = |X_{j_0}| = |D_v f(v)|$ is also a prefix of U' and a suffix of U of length at most $\max\{|Y_i|\}$ is also a suffix of U'.

Fact 2: If $|D_v f(v^3)| \leq |U|$ (*i.e.*, Y_q starts with $f(v)$), we can work with $e_j + 1$ instead of e_j and we obtain that a prefix of U of length at most $\max\{|X_j|\} = |X_{j_0}| = |D_v f(v^2)|$ is also a prefix of U'.

Fact 3: If $L_{j,v} \neq \emptyset$ for every integer $j \in [1, \kappa]$ then w_j ends with x_j and w_{j+1} starts with x_j.

Case (4): $d_v = 0$, $|U| < |D_v f(v)^2| < |D_v U|$ and $L_{j,v} \cup R_{j,v} \neq \emptyset$ for every integer $j \in [1, \kappa - 1]$.

Let us recall that, by definition, $|D_v f(v)| \leq |U|$.

Let S_2 be the set of integers j such that there exists a word x_j^3 in $R_{j,v}$ with $d_{x_j} = 1$ but no word in $R_{j,v}$ with $d_{x_j} = 0$ and no word in $L_{j,v}$.

Case 4.1: $d_{\mathrm{v}}=0$, $|U|<|D_{\mathrm{v}}f(v)^2|<|D_{\mathrm{v}}U|$, $S_2=\emptyset$, and $L_{j,v}\cup R_{j,v}\neq\emptyset$ for every integer $j\in[1,\kappa-1]$.

If $x_j^3\in R_{j,v}$, if $x_j^3\in L_{j,v}$ with $d_{\mathrm{x_j}}=1$, or if $x_j^3\in L_{j,v}$ with $d_{\mathrm{x_j}}=0$ and $|D_{\mathrm{x_j}}f(x_j)^2|\geq|U|$ then let X_j be the word $D_{\mathrm{x_j}}$ and let $e_j=d_{\mathrm{x_j}}$. If $x_j^3\in L_{j,v}$ with $d_{\mathrm{x_j}}=0$ and $|D_{\mathrm{x_j}}f(x_j)^2|<|U|$ then let X_j be the word $D_{\mathrm{x_j}}f(x_j)$ and let $e_j=1$. For all integers $j\in[1,\kappa-1]$, we have $f(\omega[1\ldots n_{\mathrm{x_j}}-1])f(x_j^{e_j})=pU^{j-1}X_j$.

For all integers $j\in[1,\kappa-1]$, $X_jf(x_j)$ is a prefix of U. Consequently, there exists a word Y_j such that $U=X_jf(x_j)Y_j$. Since $|Uf(x_j)|>|X_jf(x_j)^2|\geq|U|$, we obtain that $X_jf(x_j)^2$ is a prefix of U^2. It follows that Y_j is a prefix of $f(x_j)$.

Let q be an integer such that $|X_q|=\max\{|X_j|;j\in[1;\kappa-1]\}$. In particular, we have $|Y_q|\leq|f(x_q)|$ and $0\leq|X_q|-|X_j|=|Y_j|-|Y_q|\leq|f(x_j)|=|f(x_q)|$ for every integer j in $[1;\kappa-1]$.

Let w_1 be the word $\omega[1\ldots n_{\mathrm{x_1}}-1]x_1^{e_1}$ and let w_κ be the word such that $\omega[n_{\mathrm{x_{\kappa-1}}}\ldots|\omega|]=x_{\kappa-1}^{1+e_{\kappa-1}}w_\kappa$. We have $f(w_1)=pX_1$, $f(w_\kappa)=Y_{\kappa-1}US$. We obtain $f(\omega[n_{\mathrm{x_j}}\ldots n_{\mathrm{x_{j+1}}}-1])f(x_{j+1}^{e_{j+1}})=f(x_j^{1+e_j})Y_jX_{j+1}$ for all integers $j\in[1,\kappa-2]$. Since f is bifix, it implies that there exists a word w_j such that $f(w_j)=Y_{j-1}X_j$ for all integers $j\in[2,\kappa-1]$.

By Lemma 3.6 and using Remark 3.7(5), we can reduce $f(\omega)$.

The non-empty word $U'=X_\kappa Y_\kappa$ is a prefix of U and any suffix of U of length at most $\max\{|Y_i|\}$ is also a prefix of U'. We take $\breve{\omega}=w_1w_2\ldots w_\kappa$. Hence, $f(\breve{\omega})=p(U')^{\kappa-1}US$ starts with $p'U'^{\kappa}$ with $|p'|=|p|<|f(W[1])|=|f(w_1[1])|=|f(\breve{\omega}[1])|$ and $|U'|<|U|$.

Case 4.2: $d_{\mathrm{v}}=0$, $|U|<|D_{\mathrm{v}}f(v)^2|<|D_{\mathrm{v}}U|$, $S_2\neq\emptyset$, and $L_{j,v}\cup R_{j,v}\neq\emptyset$ for every integer $j\in[1,\kappa-1]$.

If $j\in S_2$, let X_j be the word $D_{\mathrm{x_j}}$ and let $e_j=1$.

If $j\notin S_2$, we assume that if $x_j^3\in R_{j,v}$ then $d_{\mathrm{x_j}}=0$ else we take $x_j^3\in L_{j,v}$. If $x_j^3\in R_{j,v}$ (with $d_{\mathrm{x_j}}=0$), or if $x_j^3\in L_{j,v}$ with $d_{\mathrm{x_j}}=0$ and $|D_{\mathrm{x_j}}f(x_j)^2|>|U|$ (for instance x_{j_0}), let X_j be the word such that $D_{\mathrm{x_j}}f(x_j)^2=UX_j$ and let $e_j=2$. If $x_j^3\in L_{j,v}$ with $d_{\mathrm{x_j}}=1$, or $d_{\mathrm{x_j}}=0$ and $|D_{\mathrm{x_j}}f(x_j)^2|\leq|U|$, let X_j be the word such that $D_{\mathrm{x_j}}f(x_j)^3=UX_j$ and let $e_j=3$. For all integers $j\in[1,\kappa-1]$, we have $f(\omega[1\ldots n_{\mathrm{x_j}}-1])f(x_j^{e_j})=pU^jX_j$. Especially, the word X_j is a suffix of $f(x_j)$ for every integer $j\in[1,\kappa-1]$.

Let j_1 be an integer in S_2, *i.e.*, $x_{j_1}^3\in R_{j,v}$ and $d_{\mathrm{x_{j_1}}}=1$. Hence, U^2 starts with $X_{j_1}f(x_{j_1}^2)$. By definition, we have $UX_{j_0}=D_{\mathrm{v}}f(v)^2=UX_{j_1}t_{2,j}$. For any word $x_j^3\in L_{j,v}$ with $d_{\mathrm{x_j}}=1$, or with $d_{\mathrm{x_j}}=0$ and $|D_{\mathrm{x_j}}f(x_j)^2|\leq|U|$, always by definitions, we obtain $UX_j=UX_{j_0}t_{1,j}$. It follows that $|f(x_{j_1})|=|f(x_j)|\geq|X_j|>|X_{j_0}|\geq|X_{j_1}|$. Furthermore, the words $f(x_j)$ and $f(x_{j_1})$ are conjugated.

Let $\tau_{2,j}$ be the non-empty suffix of X_j (and of $f(x_j)$) such that $X_j=X_{j_1}\tau_{2,j}$ and let $\tau_{1,j}$ be the word such that $f(x_j)=\tau_{1,j}\tau_{2,j}$. Since UX_{j_1} ends with $\tau_{1,j}$, we obtain $f(x_{j_1})=\tau_{2,j}\tau_{1,j}$. Thus, U^2 starts with $X_{j_1}(\tau_{2,j}\tau_{1,j})^2=X_jf(x_j)\tau_{1,j}$. Since f is bifix, it implies that $\omega[m_{\mathrm{x_j}}+1\ldots|\omega|]$ also starts with x_j. In other words, x_j^3 is followed by x_j in ω.

Let q be an integer such that $|X_q|=\max\{|X_j|;j\in[1;\kappa-1]\}$. In particular, $0\leq|X_q|-|X_j|\leq|f(x_q)|$.

For all integers $j\in[1,\kappa-1]$, $X_jf(x_j)$ is a prefix of U. Consequently, there exists a word Y_j such that $U=X_jf(x_j)Y_j$.

Let w_2 be the prefix of ω such that $f(w_2)=pUX_1$, that is, $w_2=\omega[1\ldots n_{\mathrm{x_1}}-1]x_1^{e_1}$ and let $w_{\kappa+1}$ be the suffix of ω such that $f(w_{\kappa+1})=Y_\kappa S$, that is, $\omega=\omega[1\ldots n_{\mathrm{x_\kappa}}-1]x_\kappa^{1+e_\kappa}w_{\kappa+1}$. Accordingly, for all integers $j\in[1,\kappa-2]$, we have $f(\omega[n_{\mathrm{x_j}}\ldots n_{\mathrm{x_{j+1}}}-1])f(x_{j+1}^{e_{j+1}})=f(x_j^{1+e_j})Y_jX_{j+1}$. Since f is bifix, it implies that there exists a word w_j such that $f(w_j)=Y_{j-1}X_j$ for all integers $j\in[3,\kappa]$.

By Lemma 3.6 and using Remark 3.7(4), we can reduce $f(\omega)$. Reduction is almost the same that case where $d_{\mathrm{v}}=1$, $|D_{\mathrm{v}}f(v)^2|<|U|$, and $L_{j,v}\cup R_{j,v}\neq\emptyset$ for every integer $j\in[2,\kappa]$. Let us note that U' is a suffix of U and that any prefix of U of length at most $\max\{|X_j\}$ is also a prefix of U'. □

4. Special case of uniform morphisms

As a consequence of Corollary 3.2 and using Lemma 3.9, we will be able to reduce a word whose image by a k-power-free uniform morphism contains a $(k+1)$-power. We obtain the following result.

Proposition 4.1. *Let A and B be two alphabets and let $k \geq 4$ be an integer. A k-power-free uniform morphism is a $(k+1)$-power-free morphism.*

Proof. Let f be a uniform morphism from A^* to B^*. We assume that f is not $(k+1)$-power-free and we want to show that f is not k-power-free.

The morphism f must be a ps-morphism. Otherwise, f is not k-power-free, it ends the proof.

Let w be a shortest $(k+1)$-power-free word whose image by f contains a $(k+1)$-power. Hence, $f(w) = pu^{k+1}s$ for two words p and s and a non-empty word u over B.

If $f(w)$ and u^{k+1} are synchronised, by Lemma 2.12, then w contains a $(k+1)$-power, a contradiction.

Now, let us assume that f is a ps-morphism, and that $f(w)$ and u^{k+1} are not synchronised. In particular, it implies that f is bifix and injective.

The central point of this proof is that, starting with w and u, we use iteratively reduction of Lemma 3.8 (that is, of Lemma 3.6 and including the special cases of Rem. 3.7) on the word whose image contains a $(k+1)$-power in such a way that there is no reduction left. That is, no situation of the hypotheses of Lemma 3.8 can be founded after this procedure. We obtain new words W and U such that $f(W) = pU^{k+1}s$ with p a proper prefix of $W[1]$, s a proper suffix of $W[|W|]$ and $f(W)$ and U^{k+1} are not synchronised.

We will show that either f is not k-power-free, or $f(W)$ and U can again be reduced using Lemma 3.8, a contradiction.

We focus on the fact that W necessarily contains a k-power. Indeed, since whatever the conjugate U_c of U, $f(W)$ contains U_c^k, the contrary ends the proof, f is not k-power-free. Moreover, if $W \neq w$, *i.e.*, $|W| < |w|$, then, by definition of w, it means that W contains a $(k+1)$-power.

Step 1: For any pure k-power v^k of W, the words U^{k+1} and $f(v)^k$ do not have any common factor of length at least $|U| + |f(v)|$.

By contradiction, let us assume that U^{k+1} and $f(v)^k$ have a common factor of length at least $|U|+|f(v)|$. By Corollary 2.4, there exist two words t_1 and t_2, and two integers r and q such that $f(v) = (t_1t_2)^r$ and $U = (t_2t_1)^q$ with t_1t_2 and t_2t_1 primitive words.

If $r \geq 2$ then $f(v^{k-1}) = (t_1t_2)^{(k-1)\times r}$ with $(k-1) \times r \geq 2k-2$. And, since $k \geq 3$, we have $2k-2 \geq k$. Therefore, $f(v^{k-1})$ contains a k-power with v^{k-1} k-power-free by definition of v, f is not k-power-free.

If $r = 1$ then it implies $q \geq k-1$. Otherwise, v^q would be an internal factor of v^k and thus of W with $|f(v)^q| = |U|$. Hence, $f(W)$ and U^k would be synchronised. Thus, if $W = v_1v^kv_2$ for some words v_1 and v_2 then $f(W) = f(v_1)(t_1t_2)^kf(v_2) = pU^{k+1}s = p(t_2t_1)^{q\times(k+1)}s$ with $q \geq k-1$.

Let x be the greatest integer such that $p(t_2t_1)^x$ is a prefix of $f(v_1v)$ and let y be the greatest integer such that $(t_2t_1)^ys$ is a suffix of $f(vv_2)$. There exist four words t'_p, t''_p, t'_s, and t''_s such that $t_2t_1 = t'_pt''_p = t'_st''_s$, $f(v_1v) = p(t_2t_1)^xt'_p$, $f(vv_2) = t''_s(t_2t_1)^ys$, and $f(v^{k-2}) = t''_p(t_2t_1)^{q(k+1)-x-y-2}t'_s$.

If $x = 0$ then $|f(v_1)| < |p|$. It implies $v_1 = \varepsilon$. Consequently, $f(v_1v^2) = pt'_pf(v) = (t_1t_2)^2$ starts with a prefix of $p(t_2t_1)^2$. Since t_2t_1 is a primitive word, by Lemma 2.2, we obtain that (t_2t_1) is not an internal factor of $(t_2t_1)^2$. It implies $p = t_1$ and $t'_p = t_2$. In the same way, if $y = 0$, we obtain $s = t_2$ and $t''_s = t_1$.

Since $f(v_1v)$ ends with t_1t_2 and since $f(vv_2)$ starts with t_1t_2, if $x \geq 1$ and $t'_p \neq t_2$, or if $y \geq 1$ and $t''_s \neq t_1$ then (t_2t_1) is an internal factor of $(t_2t_1)^2$. By Lemma 2.2, t_2t_1 is not a primitive word, a contradiction with the definition of t_2t_1.

Consequently, $t'_p = t_2 = t'_s$, $t''_p = t_1 = t''_s$, $f(v_1v) = pt_2f(v)^x$, $f(vv_2) = f(v)^yt_1s$, and $x+y+k-2 = q\times(k+1)-1$. Since f is bifix, it follows that $f(v_1v)$ ends with $f(v)^x$ and $f(vv_2)$ starts with $f(v)^y$. It implies that $v^{q\times(k+1)-1}$ is an internal factor of W with $q\times(k+1)-1 \geq q$. Thus, v^q is an internal factor W with $|f(v)^q| = |U|$, *i.e.*, $f(W)$ and u^k are synchronised, a contradiction with the hypotheses.

Step 2: $W[2\ldots|W|-1]$ contains a k-power and so a pure-k-power.

By contradiction, let us assume that $W[2\ldots|W|-1]$ is k-power-free. It implies that W starts or ends with a pure k-power. Let s_1 and p_{k+2} be the words such that $f(W[1]) = ps_1$ and $f(W[|W|]) = p_{k+2}s$, that is, $U^{k+1} = s_1f(W[2\ldots|W|-1])p_{k+2}$.

If $|s_1| \leq |U^k|$ then there exists a word U_c such that s_1U_c is the prefix of $s_1f(W[2\dots|W|-1])p_{k+2} = U^{k+1}$ of length $|s_1U|$. Trivially, the word U_c is a conjugate of U (and $|U_c| = |U|$).

If $|s_1| + |p_{k+2}| \leq |U|$, we naturally have $|s_1| \leq |U^k|$. Moreover $|s_1| + |U_c^k| + |p_{k+2}| \leq |U^{k+1}|$. It means that $f(W[2\dots|W|-1])$ starts with U_c^k. Since $W[2\dots|W|-1]$ is a k-power-free word, it ends the proof, f is not k-power-free.

Let us now study the case where $|s_1| + |p_{k+2}| > |U|$.

Let us recall that, since we assume that $W[2\dots|W|-1]$ is k-power-free, any pure k-power of $W = W[1\dots|W|]$ is necessarily a prefix or a suffix of it.

If W starts with a pure k-power v^k, let W_{com} be the greatest prefix of $s_1f(v[2\dots|v|])f(v^{k-1})$ that is a factor of U^{k+1} so a common factor of a power of $f(v)$ and a power of U. Let us note that if $W = v^k$ then $W_{com} = U^{k+1}$ otherwise $W_{com} = s_1f(v[2\dots|v|])f(v^{k-1})$.

If $|W_{com}| \geq |U| + |f(v)|$, by Corollary 2.4, there exist two words t_1 and t_2, and two integers r and q such that $f(v) = (t_1t_2)^r$ and $U = (t_2t_1)^q$ with t_1t_2 and t_2t_1 primitive words. Since v^k is a pure k-power, it follows that $r = 1$. Otherwise, f is not k-power-free. Since $f(W[2\dots|W|-1])$ contains $U^{\lceil k/2\rceil} = (t_2t_1)^{q\times\lceil k/2\rceil}$ if $q \geq 2$ then f is not k-power-free. It follows that $r = q = 1$ and $|f(v)| = |U|$, a contradiction with the assumption that $f(W)$ and U are not synchronised.

So we have $|W_{com}| < |U| + |f(v)|$. By definition of W_{com}, if $W = v^k$ then $W_{com} = U^{k+1} = s_1f(v[2\dots|v|])f(v^{k-2})f(v[1\dots|v|-1])p_{k+2}$ would be a common factor of $f(v)^k$ and U^{k+1} with $|W_{com}| \geq |s_1| + |f(v)| + |p_{k+2}| > |f(v)| + |U|$, a contradiction. It follows that $W \neq v^k$ and $|W_{com}| = |f(v)| + |s_1f(v[2\dots|v|])f(v^{k-2})| > |f(v)| + 2|s_1|$. So it implies $|s_1| < |U|/2$.

In the case where W ends with a k-power v'^k, we similarly obtain $|p_{k+2}| < |U|/2$.

If W starts with a k-power then $|s_1| < |U|/2$ and, since $|s_1| + |p_{k+2}| > |U|$, it implies $|p_{k+2}| > |U|/2$, hence, $W[2\dots|W|]$ is k-power-free. But $f(W[1\dots|W|])$ starts with $ps_1U_c^k$, *i.e.*, $f(W[2\dots|W|])$ contains the k-power U_c^k, *i.e.*, f is not k-power-free.

In the same way, if W ends with a k-power, we obtain either a contradiction with the assumptions or that f is not k-power-free.

Step 3: For any pure k-power $v^k \in \text{Fcts}(W[2\dots|W|-1])$, the word $f(v)^k$ is an internal factor of U^3 and $|f(v^{k-1})| < |U|$.

For any pure k-power $v^k \in \text{Fcts}(W[2\dots|W|-1])$, the word $f(v)^k$ is an internal factor of U^{k+1}. So $|f(v)^k| < |U| + |f(v)|$, *i.e.*, $|f(v)^{k-1}| < |U|$. In particular, we obtain $|f(v)| < \frac{1}{2}|U|$ and $|f(v)^k| < \frac{3}{2}|U|$. That is, $f(v)^k$ is an internal factor of U^3. It implies $c_{\text{v}} = 1, 2$ or 3.

Let us recall that, for every integer $j \in [1; k+2-c_{\text{v}}]$, $f(v)^k$ is an internal factor of $p_jU^{c_{\text{v}}}s_{j+c_{\text{v}}}$. Thus, if $\widehat{v}_j$ is the shortest factor of $W[i_j\dots i_{j+c_{\text{v}}}]$ such that $f(\widehat{v}_j)$ contains $f(v)^k$ then, by Corollary 3.2, $\widehat{v}_j$ satisfies property $(P.1)$ for all integers $j \in [1; k+2-c_{\text{v}}]$. More precisely, there exist a letter y and a word x_j such that $|f(v)| = |f(x_j)|$, and $\widehat{v}_j = x_j^ky$ or $\widehat{v}_j = yx_j^k$.

We are going to see that it implies that W can be reduced, a final contradiction.

Let us recall that we denote by $z^3_{(\beta)}$ or $(z^3)_{(\beta)}$ the βth factor of z^3 in a k-power z^k, that is, $z^k = z^{\beta-1}z^3_{(\beta)}z^{k-\beta-2}$ with $1 \leq \beta \leq k-2$.

Case 3.1: $c_{\text{v}} = 3$

We necessarily have $d_{\text{v}} = 1$ and $|D_{\text{v}}f(v^{k-2})|(\leq |f(v^{k-1})|) < |U| \leq |D_{\text{v}}f(v^{k-1})|$. For every integer $j \in [1; k-1]$, if $\widehat{v}_j$ satisfies $(P.1.1)$ then $(x_j^3)_{(1)} \in L_{j+1,v_{(1)}}$ and if $\widehat{v}_j$ satisfies $(P.1.2)$ then $(x_j^3)_{(1)} \in R_{j+1,v_{(1)}}$ In other words, we have $L_{j+1,v_{(1)}} \cup R_{j+1,v_{(1)}} \neq \emptyset$ with $j+1 \in [2; k]$. By Lemma 3.9(2), we can reduce W.

Case 3.2: $c_{\text{v}} \neq 3$ and there exists a positive integer β $(\leq k-2)$ such that $d_{\text{v}_{(\beta)}} = 1$

We necessarily have $c_{\text{v}} = 2$ thus $k+2-c_{\text{v}} = k$. For every integer $j \in [1; k]$, if $\widehat{v}_j$ satisfies $(P.1.1)$ then $(x_j^3)_{(\beta)} \in L_{j+1,v_{(\beta)}}$ and if $\widehat{v}_j$ satisfies $(P.1.2)$ then $(x_j^3)_{(\beta)} \in R_{j+1,v_{(\beta)}}$. That is, $L_{j,v_{(\beta)}} \cup R_{j,v_{(\beta)}} \neq \emptyset$ for every integer $j \in [2; k+1]$. By Lemma 3.9(1), a reduction can be done.

Case 3.3: $c_{\text{v}} \neq 3$ and, for every positive integer β $(\leq k-2)$, we have $d_{\text{v}_{(\beta)}} = 0$

If $c_{\mathrm{v}_{(1)}} = 1$ then $|D_{\mathrm{v}_{(1)}} f(v_{(1)})^2| \leq |U|$ and $L_{j,v_{(1)}} \cup R_{j,v_{(1)}} \neq \emptyset$ for every integer $j \in [1; k+1]$. By Lemma 3.9(3), a reduction can be done.

If $c_{\mathrm{v}_{(1)}} = 2$, there exists an integer ϕ such that $|U| < |D_{\mathrm{v}_{(\phi)}} f(v_\phi)^2|$ and $L_{j,v_{(\phi)}} \cup R_{j,v_{(\phi)}} \neq \emptyset$ for every integer $j \in [1; k]$. By Lemma 3.9(4), a reduction can be done. □

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