

REDUCING THE GRADEDNESS PROBLEM OF STRING REWRITING SYSTEMS TO A TERMINATION PROBLEM*

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Abstract. A finite string rewriting system (SRS) is called *graded* if every word over its alphabet is equivalent to only a finite number of other words. We consider the problem of deciding whether a given finite SRS is graded. We show that, in general, this problem is not decidable. Moreover we show that for many SRSs (including all one-rule SRSs), one can convert the SRS to another SRS such that the original one is graded if and only if the converted one is terminating. Since there are computer programs that can decide for many cases whether a given SRS is terminating or not, this can give us a method to prove automatically if a given SRS is graded or not.

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1. INTRODUCTION

A monoid M is called *graded with respect to S* where S is a finite set of generators, if every element of M can be written as a word over S in only finitely many ways. The notion of a graded monoid was introduced by Margolis *et al.* in [7], where they discussed the membership problem in the case of a graded monoid. A finite string rewriting system (SRS) $\langle A \mid R \rangle$ is called *graded* if every word is equivalent to only a finite number of other words. In other words, $\langle A \mid R \rangle$ is graded if and only if the monoid it presents is graded with respect to A . It was proved

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in [7] that if M is a graded monoid then any finite SRS $\langle A \mid R \rangle$ that presents M is graded as long as any letter $a \in A$ is not equivalent to 1. From a combinatorial point of view, graded monoids have some useful properties. For instance, they have a decidable word problem. Note that any free monoid A^* is a graded monoid with the special property that any word can be written in only one way using the generators A . Given a finite SRS $\langle A \mid R \rangle$, we want to decide whether it is graded or not. We will show that, in general, this question is not decidable. Our main result is, that certain types of SRSs can be converted into another SRS, such that the original SRS is graded if and only if the other SRS is terminating. There is a lot of research about proving termination for SRSs, and there are computer programs that can solve automatically many cases [5, 11]. So this can give us a method to prove automatically if a given SRS is graded or not.

2. PRELIMINARIES

Let A be a finite alphabet. The free monoid generated by A is denoted by A^* and the free semigroup is denoted by A^+ .

Let M be a monoid. A *congruence* on M is an equivalence relation R with the property that aRb and cRd implies $acRbd$. If R is a congruence, then M/R is a monoid with respect to the multiplication $[a]_R \cdot [b]_R = [ab]_R$ where $[x]_R$ denotes the equivalence class of x .

Let A be a set and let $R = \{(u_i, v_i) \mid i \in I\}$ be a relation on A^* . The tuple $T = \langle A \mid R \rangle$ is called a *string rewriting system* (SRS) and it is usually written $\langle A \mid u_i \rightarrow v_i \rangle$, where every $u_i \rightarrow v_i$ is called a *rule*. We denote the left hand sides of the rules and the right hand sides by $\text{lhs}(T)$ and $\text{rhs}(T)$ respectively, that is, $\text{lhs}(T) = \{u_i \mid i \in I\}$ and $\text{rhs}(T) = \{v_i \mid i \in I\}$. We will only discuss *finite* SRSs, *i.e.*, both A and R are finite. The *single-step reduction relation* induced by R is denoted by \rightarrow_R and defined by

$$w \rightarrow_R w' \text{ if } w = xuy \text{ and } w' = xvy \text{ for some } x, y \in A^*, (u, v) \in R.$$

We denote by \leftrightarrow_R and $\xrightarrow{*}_R$ the symmetric and the transitive reflexive closures of \rightarrow_R . We also use $\overset{*}{\leftrightarrow}_R$ to denote the reflexive symmetric transitive closure of \rightarrow_R . Note that $\overset{*}{\leftrightarrow}_R$ is the congruence generated by R , that is, the least congruence that contains R . We say that two words w, w' over A are *equivalent* if $w \overset{*}{\leftrightarrow}_R w'$. Usually we will omit the R and write $\rightarrow, \leftrightarrow, \xrightarrow{*}$ and $\overset{*}{\leftrightarrow}$. We will say that $\langle A \mid R \rangle$ *presents* the monoid $A^*/\overset{*}{\leftrightarrow}$, or that $\langle A \mid R \rangle$ is a *presentation* of $A^*/\overset{*}{\leftrightarrow}$ with generators A and rules R . If only the generated monoid is of interest, the convention is to write rules in the form $u = v$ instead of $u \rightarrow v$.

The *reduction graph* of an SRS $T = \langle A \mid R \rangle$, denoted G_T , is the directed graph (V, E) where $V = A^*$ and $(a, b) \in E$ if $a \rightarrow_R b$. The *conversion graph* of T , denoted C_T , is the undirected graph (V, E) where $V = A^*$ and $(a, b) \in E$ if $a \leftrightarrow_R b$. A path in the reduction (conversion) graph is called a *reduction (conversion)* of T . We sometimes call a reduction (conversion) of length 1, a *step*. Note that, since

there are only a finite number of rules, the degree of every vertex in the reduction (conversion) graph is finite. In other words, every vertex has only a finite number of adjacent vertices.

An SRS $\langle A \mid R \rangle$ is called *confluent* if $u \xrightarrow{*} x$ and $u \xrightarrow{*} y$ implies that there is a $v \in A^*$ such that both $x \xrightarrow{*} v$ and $y \xrightarrow{*} v$. It is called *terminating* if there is no infinite sequence $x_1, x_2, x_3 \dots$ of words such that $x_i \rightarrow x_{i+1}$ for all $i > 0$. In other words, T is terminating if it has no infinite reduction. It is well-known ([3], Lems. 2.2.4 and 2.2.5) that an SRS is terminating if and only if \rightarrow is acyclic and globally finite. Acyclic means that there are no cycles $u \rightarrow \dots \rightarrow u$, i.e., the transitive closure of \rightarrow is irreflexive. Globally finite means that for every $u \in A^*$ the number of v such that $u \xrightarrow{*} v$ is finite.

We denote by $\text{OVL}^*(u, v)$ and $\text{OVL}(u, v)$ the sets of overlaps and proper overlaps of u with v , respectively.

$$\text{OVL}^*(u, v) = \{w \in A^+ \mid \exists x, y \in A^* : u = xw \wedge v = wy\}$$

and

$$\text{OVL}(u, v) = \{w \in A^+ \mid \exists x, y \in A^* : u = xw \wedge v = wy \wedge x \neq 1 \wedge y \neq 1\}.$$

In addition, $\text{OVL}^*(u) = \text{OVL}^*(u, u)$ and $\text{OVL}(u) = \text{OVL}(u, u)$ are the sets of overlaps and proper overlaps of u with itself. Note that $\text{OVL}^*(u) = \text{OVL}(u) \cup \{u\}$.

It is undecidable whether or not a given finite string-rewriting system is confluent or terminating ([4], Thms. 2.5.13 and 2.5.14). It is known [10] that a one-rule SRS $T = \langle A \mid u \rightarrow v \rangle$ such that v is not a factor of u , is confluent if and only if $\text{OVL}(u) \subseteq \text{OVL}(v)$.

Assume that w is a factor of x . Note that w may appear in x more than once. When we want to distinguish a specific factor w of x , we will speak of the factor w of x at position p where p is an integer between 0 and $|x|$. Position p in x means the location between y and z in yz if $x = yz$ and $|y| = p$. For instance, in the word $abaab$, the factor $\underline{a}baab$ is at position 0 where the factor $aba\underline{a}b$ is at position 3.

Let $T = \langle A \mid R \rangle$ be an SRS. When regarding a reduction or conversion in G_T or C_T we will often be interested not only in the words involved but in the specific rewritings as well. We will do so by writing the respected rule below the arrow and the position above the arrow. Hence, when we rewrite the factor u of w_1 at position p to v and we get the word w_2 , we will write this as $w_1 \xrightarrow[p]{u \rightarrow v} w_2$. For instance, there is a difference between $aa \xrightarrow[a \rightarrow ab]{0} aba$ and $aa \xrightarrow[a \rightarrow ba]{1} aba$, or even between $aa \xrightarrow[a \rightarrow aa]{0} aaa$ and $aa \xrightarrow[a \rightarrow aa]{1} aaa$ (both steps use the rule $a \rightarrow aa$ but in different positions).

3. GRADED SRSs

Definition 3.1. An SRS $T = \langle A \mid R \rangle$ is called *graded* if every $w \in A^*$ is equivalent to only a finite number of other words.

In other words, T is graded if and only if every connected component in its conversion graph is finite. If $T = \langle A \mid u_i \rightarrow v_i \rangle$ is a length preserving finite SRS (i.e. $|u_i| = |v_i|$ for every i), then it is clearly graded since there are only a finite number of words of a given length. On the other hand, any finite SRS of the form $\langle A \mid u \rightarrow v \rangle$ where u is a proper factor of v is clearly not graded. More generally, it is clear that any acyclic non-terminating SRS is not graded.

Definition 3.2. Let M be a monoid generated by a finite set $S \subseteq M$. M is *graded with respect to S* if every member of M can be written as a word over S in only finitely many ways. M is called *graded* if it is graded with respect to some set of generators S .

The term graded comes from a property proved in [7]: if M is graded with respect to S , then the function $\lambda_S : M \rightarrow \mathbb{N}$ defined by

$$\lambda_S(g) = \max\{k \mid g = s_1 s_2 \dots s_k, \text{ for some } s_i \in S, \quad i = 1, \dots, k\}$$

is a grading function for M , that is, $\lambda_S(m) = k$ if and only if $m \in T^k \setminus T^{k+1}$ where $T = M \setminus \{1\}$.

It is clear that if $\langle A \mid R \rangle$ is a graded SRS then the monoid it presents is also graded (with respect to A).

Note that if M is graded with respect to S and $e \in M$ is an idempotent then $e = 1$. For assume not and e can be written as a non-trivial word over S , say, $e = s_1 s_2 \dots s_k$ where $s_i \in S$. Then $(s_1 s_2 \dots s_k)^l$ are all equivalent for every $l \geq 1$ which contradicts the gradedness of M .

The following are immediate corollaries of ([7], Props. 1.6 and 1.15).

Proposition 3.3. *Let $\langle A \mid R \rangle$ be a finite SRS that presents the monoid M . If every $a \in A$ is not equivalent to 1 in M then $\langle A \mid R \rangle$ is a graded SRS if and only if M is a graded monoid.*

Note that the condition that no $a \in A$ is equivalent to 1 is essential. For instance, the non-graded SRS $T = \langle a, b \mid 1 \rightarrow b \rangle$ presents the free (hence graded) monoid $\mathbb{N} = \langle a \rangle$.

Proposition 3.4. *Let $T = \langle A \mid u_i \rightarrow v_i \rangle$ be an acyclic confluent SRS. Then T is graded if and only if both $\langle A \mid u_i \rightarrow v_i \rangle$ and its converse $\langle A \mid v_i \rightarrow u_i \rangle$ are terminating.*

Remark 3.5. If a one-rule SRS $\langle A \mid u \rightarrow v \rangle$ is length increasing, i.e., $|u| < |v|$, then its converse is clearly terminating. Such SRS is confluent if and only if $\text{OVL}(u) \subseteq \text{OVL}(v)$. In particular, T is confluent if it is non-overlapping, i.e., $\text{OVL}(u) = \emptyset$. Hence, Proposition 3.4 implies that gradedness and termination are equivalent for non-overlapping length-increasing one-rule SRSs. The decidability status of the termination problem for such SRSs is still open.

Example 3.6. Consider the SRS $T = \langle a, b \mid a^m b^l \rightarrow b^k a^n \rangle$ where $m, l, k, n \geq 1$. Since

$$\text{OVL}(a^m b^l) = \text{OVL}(b^k a^n) = \emptyset$$

both $\langle a, b \mid a^m b^l \rightarrow b^k a^n \rangle$ and $\langle a, b \mid b^k a^n \rightarrow a^m b^l \rangle$ are confluent and they are clearly acyclic. According to [12] the SRS $\langle a, b \mid a^m b^l \rightarrow b^k a^n \rangle$ is terminating if and only if one of the following holds:

1. $n \leq m$ or $k \leq l$.
2. $m \leq n \leq 2m$ and $k \not\equiv 0 \pmod{l}$.
3. $l \leq k \leq 2l$ and $n \not\equiv 0 \pmod{m}$.

Using Proposition 3.4 we can deduce a criterion for the gradedness of T .

4. UNDECIDABILITY RESULT

Let the gradedness problem for the class \mathcal{C} of SRSs be the following problem. Instance: some $T \in \mathcal{C}$. Question: is T graded? We will show that the gradedness problem for the class of all finite SRSs is undecidable.

We will use a generalization of Markov’s theorem due to [6], regarding monoids with word problem decidable in linear time.

Definition 4.1. Let \mathcal{C} be some class of monoids. A property P is called a *Markov property with respect to \mathcal{C}* if the following conditions hold

1. If M_1 satisfies P and $M_1 \cong M_2$ then M_2 satisfies P .
2. There is a monoid $M \in \mathcal{C}$ that satisfies P .
3. There is a monoid $M_1 \in \mathcal{C}$ that does not satisfy P and cannot be embedded in a monoid $M_2 \in \mathcal{C}$ that satisfies P .

It is well-known that if \mathcal{C} is the class of all finitely presented monoids, then any Markov property is undecidable. It is proved in ([6], Thm. 3.2) that even if \mathcal{C} is the class of all finitely presented monoids with word problem decidable in linear time, any Markov property is undecidable. Note that being a graded monoid is a Markov property, since conditions 1 and 2 clearly hold and condition 3 is fulfilled by any monoid with an idempotent other than the identity. For instance, take the monoid presented by $\langle a \mid a \rightarrow aa \rangle$.

Corollary 4.2. *The gradedness problem for monoids is undecidable for the class of finitely presented monoids that have a word problem decidable in linear time.*

Definition 4.3. For an SRS $T = \langle A \mid R \rangle$ let $\text{trim}(T)$ be defined by $\text{trim}(T) = \langle A' \mid R' \rangle$ where B denotes the set of letters from A that are equivalent to 1, $A' = A \setminus B$ and R' is derived from R by deleting all occurrences of letters from B .

Note that T and $\text{trim}(T)$ present the same monoid.

Lemma 4.4. *The monoid M presented by the finite SRS T is graded if and only if the SRS $\text{trim}(T)$ is graded.*

Proof. Since $\text{trim}(T) = \langle A' \mid R' \rangle$ also presents M and no letter from A' is equivalent to 1 the statement follows immediately from Proposition 3.3. \square

Proposition 4.5. *The gradedness problem for SRSs is undecidable for the class of finite SRSs that have a word problem decidable in linear time.*

Proof. Assume it is decidable. Let M be a monoid presented by $T = \langle A \mid R \rangle$ such that the word problem for this SRS is decidable in linear time. We can compute $\text{trim}(T)$ since the word problem for T is decidable. It is clear that the word problem for $\text{trim}(T)$ is also decidable in linear time so we can decide if $\text{trim}(T)$ is graded. By Lemma 4.4 we can decide whether M is graded. This contradicts Corollary 4.2. \square

Remark 4.6. We remark that similar problems for Term Rewriting Systems (TRS) are discussed in [8]. It is proved ([8], Thm. 1) that the problem of determining, given a finite TRS and a term t , whether the equivalence class of t is finite is undecidable, even if the reduction relation is terminating and the left hand sides of the rules are non-overlapping. It is also undecidable whether all equivalence classes are finite. On the other hand ([8], Thm. 2), both problems are decidable for TRSs of ground terms.

5. CONVERTING INTO TERMINATION PROBLEM

In this section we will prove that certain types of SRSs T can be converted into another SRS S_T , such that T is graded if and only if S_T is terminating. This will be done in several steps. In Section 5.1 we show that under some assumptions on T , T is graded if and only if it has no infinite conversion without reversals (notion that will be defined in Def. 5.2). In Section 5.2 we define compatible SRSs and prove that if S is an SRS compatible with T , $\text{lhs}(S) \cap \text{rhs}(S) = \emptyset$ and there is no infinite conversion of T without reversals, then S is terminating. In Section 5.3 we show that (under some assumptions on T) we can construct from T a specific SRS S_T which satisfies the above condition. In addition, we prove that if S_T is terminating then there is no infinite conversion of T without reversals. So we get the desired result.

5.1. REVERSALS IN A CONVERSION

From now on, we will always assume that the empty word does not appear in the rules of the SRSs we discuss. Since SRSs with the empty word in the rules are clearly not graded, this does not restrict the generality of our results.

Definition 5.1. Let T be an SRS and let

$$w_1 \xrightarrow[u \rightarrow v]{p} w_2$$

be some step in G_T or C_T . Assume that z is a factor of w_1 at position q with factorization $w_1 = x_1zy_1$. We say that z is *not affected* in this step if the following

conditions hold:

- w_2 can be factored as $w_2 = x_2zy_2$ for some $x_2, y_2 \in A^*$.
- One of the following two conditions hold
 - $y_1 = y_2$ and $x_1 \xrightarrow[u \rightarrow v]{p} x_2$.
 - $x_1 = x_2$ and $y_1 \xrightarrow[u \rightarrow v]{p-|x_1z|} y_2$.

The position of z in w_2 (that is, $|x_2|$) will be called the corresponding position in w_2 . Similarly, if P is a reduction (conversion) of T

$$w_1 \xrightarrow[u_1 \rightarrow v_1]{p_1} w_2 \xrightarrow[u_2 \rightarrow v_2]{p_2} w_3 \rightarrow \dots \rightarrow w_n$$

we say that the factor z at position q_1 in w_1 is *not affected* in this reduction (conversion) if z is not affected in the step $w_1 \xrightarrow[u_1 \rightarrow v_1]{p_1} w_2$ with corresponding position q_2 in w_2 and z at position q_2 in w_2 is not affected in the step $w_2 \xrightarrow[u_2 \rightarrow v_2]{p_2} w_3$ and so on. Eventually z has a corresponding position q_n in w_n .

Definition 5.2. Let $T = \langle A \mid R \rangle$ be an SRS. Let P be a finite conversion of T

$$w_0 \rightarrow w_1 \rightarrow \dots \rightarrow w_n \quad (2 \leq n).$$

We will say that P is a *reversal* if the following conditions hold.

- The first step is $w_0 \xrightarrow[u \rightarrow v]{p} w_1$.
- The factor v at position p of w_1 is not affected in the conversion

$$w_1 \rightarrow w_2 \rightarrow \dots \rightarrow w_{n-1}$$

where p' is the corresponding position of v in w_{n-1} .

- The last step is $w_{n-1} \xrightarrow[v \rightarrow u]{p'} w_n$.

In other words, we rewrite u to v , then we do not “affect” the factor v for $n - 2$ steps and then rewrite it back to u . Similarly, if P is a conversion (finite or infinite) of T

$$w_0 \rightarrow w_1 \rightarrow \dots \rightarrow w_k \rightarrow \dots$$

We will say that P has a *reversal* if there are some $i, j \in \mathbb{N}$ ($i + 1 < j$) such that

$$w_i \rightarrow w_{i+1} \rightarrow \dots \rightarrow w_j$$

is a reversal.

We will present some results from [1] about reversals.

Lemma 5.3 ([1], Chap. II Lem. 1). *Let $T = \langle A \mid R \rangle$ be an SRS. Let P be a conversion of T , between the words $x, y \in A^*$. If P has reversals, then there is a shorter conversion between x and y that has no reversals.*

Recall that a multigraph is a graph $G = (V, E)$, where E is a multiset. In other words, multiple edges are permitted.

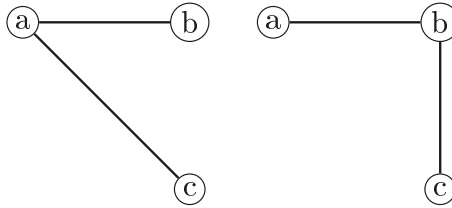
Definition 5.4. The *left pair* of the rule $u_i \rightarrow v_i$ is the unordered pair of letters $\{a, b\}$, where a is the first letter of the word u_i and b is the first letter of the word v_i . If $\{a, b\}$ are the last letters of u_i, v_i they are called the *right pair* of the rule $u_i \rightarrow v_i$.

Definition 5.5. Let T be a finite SRS $\langle A \mid R \rangle$. The *left (right) graph* of T is the multigraph whose vertices are the letters A and whose edges are the left (right) pairs of the rules in R . If there are n rules with the same left (right) pair the respective edge occurs n times in the left (right) graph. We will say that T has no *left (right) cycles* if its left (right) graph has no cycles.

Example 5.6. The left and right graphs of the SRS

$$T = \langle a, b, c \mid ab \rightarrow ba^2, ac \rightarrow c^2b \rangle$$

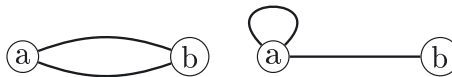
are:



So both graphs have no cycles. The left and right graphs of the SRS

$$T = \langle a, b \mid aa \rightarrow ba, aa \rightarrow bb \rangle$$

are:



In this case, both graphs have cycles.

Proposition 5.7 ([1], Chap. II, Lem. 2). *Let $T = \langle A \mid R \rangle$ be an SRS with no left cycles. If there exists $a \in A$ and a conversion*

$$w_0 = aw'_0 \rightarrow w_1 \rightarrow \dots \rightarrow aw'_k = w_k$$

with no reversals, then for all $0 < r < k$, there exists $w'_r \in A^$ such that*

$$w_r = aw'_r$$

and

$$w'_0 \rightarrow w'_1 \rightarrow \dots \rightarrow w'_{k-1} \rightarrow w'_k$$

is also a conversion without reversals. A dual result holds for SRSs with no right cycles.

Definition 5.8. We will call an SRS *border-acyclic* if it has no left cycles or no right cycles.

Corollary 5.9. Let $T = \langle A \mid R \rangle$ be a border-acyclic SRS. Let P be a conversion

$$w_0 \rightarrow w_1 \rightarrow \dots \rightarrow w_{k-1} \rightarrow w_k$$

with no reversals and assume that $w_0 = w_k$. Then, $k = 0$ and the conversion consists of only one element. In other words, a conversion without reversals does not contain repeated vertices.

Proof. Using Proposition 5.7 $|w_0|$ times we get a conversion

$$1 \rightarrow x_1 \rightarrow \dots \rightarrow x_{k-1} \rightarrow 1$$

but $|u_i|, |v_i| > 0$ so there can be no steps $1 \rightarrow x_1$. So $k = 0$. □

Now we can prove one direction of our argument.

Corollary 5.10. Let $T = \langle A \mid R \rangle$ be a border-acyclic SRS. If T has an infinite conversion without reversals then it is not a graded SRS.

Proof. It is clear that T is graded if and only if every connected component of C_T is finite. Assume that there is an infinite conversion without reversals in C_T . By Corollary 5.9 it does not contain repeated vertices, so it contains an infinite number of different vertices. Hence, T cannot be graded. □

The converse of Corollary 5.10 is also true, even without the requirement of border-acyclicity. In order to prove it we need another notion.

Definition 5.11. Let G be an undirected graph. The *distance* between two vertices x and y is the length (that is, the number of steps) of a shortest path from x to y .

Definition 5.12. Let G be an undirected graph and let P be a path (finite or infinite) in G .

$$P = x_0 \rightarrow x_1 \rightarrow \dots \rightarrow x_k \rightarrow \dots$$

The path P is called *geodesic* if for all k , the distance between x_0 and x_k is k .

Note that if P is geodesic then the distance between x_i and x_j for $i < j$ is $j - i$.

Let G be an undirected graph and let x_0 be a vertex. Denote by $\text{Adj}(x_0)$ the set of adjacent vertices of x_0 , and more generally, $\text{Adj}(X) = \bigcup_{x \in X} \text{Adj}(x)$ where X

is any set of vertices. Also denote by $B_k(x_0)$ the set of vertices whose distance from x_0 is less than or equal to k . It is clear that $B_0(x_0) = \{x_0\}$ and $B_{k+1}(x_0) = B_k(x_0) \cup \text{Adj}(B_k(x_0))$.

Lemma 5.13. *Let $T = \langle A \mid u_i \rightarrow v_i \rangle$ be a non-graded SRS such that $|u_i|, |v_i| > 0$. Let H be an infinite connected component of C_T . Then, for every $k \in \mathbb{N}$ and every vertex $x_0 \in H$, C_T contains a geodesic conversion of length $k \in \mathbb{N}$ that starts with x_0 .*

Proof. Note that every vertex has a finite degree since the number of rules in T is finite. The equality $B_{k+1}(x_0) = B_k(x_0) \cup \text{Adj}(B_k(x_0))$ implies that $B_k(x_0)$ is finite for any $k \in \mathbb{N}$. We claim that for every $k \in \mathbb{N}$ there is a vertex x_k whose distance from x_0 is k . For assume there are no such vertices, then $B_k(x_0) = B_{k-1}(x_0)$. Hence $B_m(x_0) = B_{k-1}(x_0)$ for every $m > k - 1$. So

$$H = \bigcup_{m \in \mathbb{N}} B_m(x_0) = B_{k-1}(x_0)$$

which contradicts the fact that H is infinite. So we have proved that there is an x_k whose distance from x_0 is k . A conversion of length k from x_0 to x_k has to be geodesic, so we are done. □

Proposition 5.14. *Let $T = \langle A \mid u_i \rightarrow v_i \rangle$ be an SRS such that $|u_i|, |v_i| > 0$. T is non-graded if and only if it has an infinite geodesic conversion.*

Remark 5.15. Note that the “only if” part of Proposition 5.14 is slightly stronger than the well-known König’s lemma, which says that there is an infinite simple conversion.

Proof. If T has an infinite geodesic conversion $x_0 \rightarrow x_1 \rightarrow \dots$ then the set $\{x_i \mid i \geq 0\}$ is infinite whence T is not graded. For the other direction, we will build the required conversion step by step. The first step is to define $P_0 = x_0$, where x_0 belongs to an infinite connected component of C_T . P_0 is a conversion of length 0. Note that by Lemma 5.13, the length of geodesic conversions that starts with x_0 is not bounded above. In the i th step, P_i will be a geodesic conversion of length i , starting from x_0 with the property that the length of geodesic conversions starting with P_i is not bounded. Denote this conversions by

$$P_i = x_0 \rightarrow \dots \rightarrow x_i.$$

In the $(i + 1)$ st step, we observe that there is a finite number of vertices y such that

$$P_y = x_0 \rightarrow \dots \rightarrow x_i \rightarrow y$$

is a geodesic conversion (for one thing, they all have to be in $\text{Adj}(x_i)$ which is a finite set). For at least one of them, say y_0 , the length of geodesic conversions starting with P_y is not bounded above (otherwise, the length of geodesic conversions starting with P_i will be bounded above). Then define $x_{i+1} = y_0$ and $P_{i+1} = P_{y_0}$. Continuing this process we can build an infinite geodesic conversion that starts with x_0 . □

To conclude this section, we have the following theorem.

Theorem 5.16. *Let $T = \langle A \mid u_i \rightarrow v_i \rangle$ be an SRS such that $|u_i|, |v_i| > 0$. If T is not graded, it has an infinite conversion without reversals. If T is border-acyclic, then the converse also holds.*

Proof. By Lemma 5.3, geodesic conversions cannot have reversals so the first statement follows from Proposition 5.14. The second statement is Corollary 5.10. \square

Remark 5.17. A graded SRS T cannot have an infinite geodesic conversion by Proposition 5.14. However, if T is not border-acyclic, it may have an infinite conversion without reversals. For instance, consider the SRS

$$T = \langle a, b \mid a^2b^2a \rightarrow a^5b, \quad b^2a^3 \rightarrow a^3ba^2 \rangle.$$

Define $f : \{a, b\}^* \rightarrow \mathbb{N}$ by

$$f(u) = |u|_a + 2|u|_b$$

where $|u|_a$ and $|u|_b$ are the number of times that a and b appear in u . Since

$$f(a^2b^2a) = f(a^5b) = 7, \quad f(b^2a^3) = f(a^3ba^2) = 7$$

it is clear that f is constant on equivalence classes, hence the length of words in any equivalence class is bounded and T is graded. On the other hand, the infinite conversion

$$a^5ba^2 \xrightarrow[a^5b \rightarrow a^2b^2a]{0} a^2b^2a^3 \xrightarrow[b^2a^3 \rightarrow a^3ba^2]{2} a^5ba^2 \longrightarrow a^2b^2a^3 \longrightarrow a^5ba^2 \longrightarrow \dots$$

has no reversals.

5.2. COMPATIBLE SRSs

Let T be an SRS. Now we start the construction of another SRS that is not terminating (*i.e.*, has an infinite reduction) if and only if T has an infinite conversion without reversals. In this section, we present a type of SRS that satisfies one direction of the required result. Namely, that if it is not terminating, T has an infinite conversion without reversals.

Definition 5.18. Let $T = \langle A \mid R \rangle$ be an SRS. Let Λ be a set and let S be an SRS over the alphabet $A \times \Lambda$. We will call the elements of Λ *labels*. S is called *compatible with T* if $\pi_A(u) \leftrightarrow_R \pi_A(v)$ for every rule $u \rightarrow v$ in S . Here $\pi_A : (A \times \Lambda)^* \rightarrow A^*$ denotes the standard projection to the first component extended to a homomorphism.

In a compatible SRS S , it is clear that any step $w_1 \xrightarrow[u \rightarrow v]{p} w_2$ in the reduction graph of S , G_S , corresponds to a step

$$\pi_A(w_1) \xrightarrow[\pi_A(u) \rightarrow \pi_A(v)]{p} \pi_A(w_2)$$

in the conversion graph of T, C_T . In order to simplify notation, when the set A is clear we will usually write \bar{u} instead of $\pi_A(u)$.

It is clear that if z at position q_1 of w_1 is not affected in the step $w_1 \xrightarrow[u \rightarrow v]{p} w_2$ and the corresponding position in w_2 is q_2 then \bar{z} at position q_1 of \bar{w}_1 is not affected in the step $\bar{w}_1 \xrightarrow[u \rightarrow \bar{v}]{p} \bar{w}_2$ and the corresponding position in \bar{w}_2 is also q_2 .

Moreover, assume that $w_1 \xrightarrow[u \rightarrow v]{p} w_2$ and z' at position q_1 of \bar{w}_1 is not affected in the step $\bar{w}_1 \xrightarrow[u \rightarrow \bar{v}]{p} \bar{w}_2$ and the corresponding position in \bar{w}_2 is q_2 . Then it is clear that the factor z of w_1 at position q_1 and length $|z'|$ satisfies that $\bar{z} = z'$, it is not affected in the step $w_1 \xrightarrow[u \rightarrow v]{p} w_2$ and the corresponding position in w_2 is q_2 .

Proposition 5.19. *Let $T = \langle A \mid R \rangle$ be an SRS and let S be a compatible SRS over $A \times \Lambda$ such that $\text{lhs}(S) \cap \text{rhs}(S) = \emptyset$. If P is a finite reduction of S*

$$w_0 \rightarrow w_1 \rightarrow \dots \rightarrow w_n$$

then the corresponding conversion \bar{P} of T

$$\bar{w}_0 \rightarrow \bar{w}_1 \rightarrow \dots \rightarrow \bar{w}_n$$

is not a reversal.

Proof. Assume that \bar{P} is a reversal. This means that for some rule $u \rightarrow v$ in S the first step of \bar{P} is $\bar{w}_0 \xrightarrow[u \rightarrow \bar{v}]{p} \bar{w}_1$. The factor \bar{v} at position p is not affected in the conversion

$$\bar{w}_1 \rightarrow \bar{w}_2 \rightarrow \dots \rightarrow \bar{w}_{n-1}$$

where p' is the corresponding position in \bar{w}_{n-1} and the last step is $\bar{w}_{n-1} \xrightarrow[v \rightarrow \bar{u}]{p'} \bar{w}_n$.

Hence, we can also say that in the reduction P the factor v at position p of w_1 is not affected in the reduction

$$w_1 \rightarrow w_2 \rightarrow \dots \rightarrow w_{n-1}$$

and the corresponding position of v in w_{n-1} is p' . Also the last step is $w_{n-1} \xrightarrow[v \rightarrow z]{p'} w_n$, where $\bar{z} = \bar{u}$ (note that in general $z \neq u$ because they might have different labels). But this means that both $u \rightarrow v$ and $v \rightarrow z$ are rules in S in contradiction to the assumption. \square

Corollary 5.20. *Let $T = \langle A \mid R \rangle$ be an SRS and let S be a compatible SRS over $A \times \Lambda$ such that $\text{lhs}(S) \cap \text{rhs}(S) = \emptyset$. If S is not terminating then T has an infinite conversion without reversals.*

5.3. THE SRS S_T

For a given SRS T , we will construct another SRS S_T and we will prove that (under some assumptions on T), T is graded if and only if S_T is terminating.

We will use the following set of labels $\Lambda = \{\mathbf{s}, \mathbf{m}, \mathbf{e}\}$ that stand for *start*, *middle* and *end*. We denote by π_A and π_Λ the projections to the first and second components extended to homomorphisms. In order to simplify notation, when dealing with words over $A \times \Lambda$ we will write the labels in subscript. For instance, if we want to write the word $w \in (A \times \Lambda)^*$ such that $\pi_A(w) = abb$ and $\pi_\Lambda(w) = \mathbf{mes}$, we will write $a_{\mathbf{m}}b_{\mathbf{e}}b_{\mathbf{s}}$ or $(abb)_{\mathbf{mes}}$ instead of $(a, \mathbf{m})(b, \mathbf{e})(b, \mathbf{s})$. Hence, when writing u_α as a word over $A \times \Lambda$ we always mean that $u \in A^*$, $\alpha \in \Lambda^*$ and $|u| = |\alpha|$. If $u \in A^*$ and $|u| > 1$ we will write $u_{\mathbf{SME}}$ for the word $u_{\mathbf{s}\mathbf{m}\dots\mathbf{m}\mathbf{e}}$, that is, the first label is \mathbf{s} , the last is \mathbf{e} and the $|u| - 2$ middle labels (no labels if $|u| = 2$) are \mathbf{m} .

Definition 5.21. Let $T = \langle A \mid u_i \rightarrow v_i \rangle$ be an SRS with a finite set of rules ($1 \leq i \leq m$), and assume that $|u_i|, |v_i| > 1$ for all i . An SRS over the alphabet $A \times \Lambda$, denoted S_T , will be defined in the following way: For every rule $u \rightarrow v$ of T , the SRS S_T has the following rules:

$$u_\alpha \rightarrow v_{\mathbf{SME}}$$

$$v_\beta \rightarrow u_{\mathbf{SME}}$$

where α, β can be any words over Λ as long as $u_\alpha \neq u_{\mathbf{SME}}$ and $v_\beta \neq v_{\mathbf{SME}}$. The total number of rules in S_T is $\sum_{i=1}^m (3^{|u_i|} + 3^{|v_i|} - 2)$.

Example 5.22. If $T = \langle a, b \mid ab \rightarrow bbaa \rangle$ an example for a reduction of S_T will be:

$$a_{\mathbf{m}}a_{\mathbf{m}}b_{\mathbf{m}} \xrightarrow{a_{\mathbf{m}}b_{\mathbf{m}} \rightarrow b_{\mathbf{s}}b_{\mathbf{m}}a_{\mathbf{m}}a_{\mathbf{e}}}^1 a_{\mathbf{m}}b_{\mathbf{s}}b_{\mathbf{m}}a_{\mathbf{m}}a_{\mathbf{e}} \rightarrow$$

$$\rightarrow b_{\mathbf{s}}b_{\mathbf{m}}a_{\mathbf{m}}a_{\mathbf{e}}b_{\mathbf{m}}a_{\mathbf{m}}a_{\mathbf{e}} \rightarrow b_{\mathbf{s}}b_{\mathbf{m}}a_{\mathbf{m}}b_{\mathbf{s}}b_{\mathbf{m}}a_{\mathbf{m}}a_{\mathbf{e}}a_{\mathbf{m}}a_{\mathbf{e}} \rightarrow \dots$$

Note that S_T is compatible with T and that $\text{lhs}(S_T) \cap \text{rhs}(S_T) = \emptyset$ so Corollary 5.20 holds for S_T . The goal of this section is to prove the other direction, namely, that if T has an infinite conversion without reversals, then S_T is not terminating (although we will need another condition on T for this to be true).

The first step is to prove the following property of S_T : given a reduction of S_T

$$x_0 \rightarrow x_1 \rightarrow \dots \rightarrow x_k$$

such that $\pi_A(x_0) = \mathbf{m} \dots \mathbf{m}$, and assume that $w_{\mathbf{SME}}$ is a factor of x_k ($w \in A^*$). Then, all the letters of $w_{\mathbf{SME}}$ “appear” in the reduction at the same step. More precisely:

Lemma 5.23. *Given a reduction P of S_T*

$$x_0 \rightarrow x_1 \rightarrow \dots \rightarrow x_k$$

such that $\pi_A(x_0) = \mathbf{m} \dots \mathbf{m}$, and assume that $w_{\mathbf{SME}}$ is a factor of x_k at position p ($w \in A^$). Then, there is a $j \leq k$ such that the j th step in the reduction is*

$$x_{j-1} \xrightarrow[u_\alpha \rightarrow w_{\mathbf{SME}}]{q} x_j$$

(where $u_\alpha \rightarrow w_{\mathbf{SME}}$ is one of the rules of S_T), the factor $w_{\mathbf{SME}}$ of x_j at position q is not affected in the reduction $x_j \rightarrow \dots \rightarrow x_k$ and the corresponding position in x_k is p .

Before we can give a precise proof, we will introduce another tool. If P is a reduction of S_T , then we will define a sequence of words over $A \times \Lambda \times \mathbb{N}$, denoted \tilde{P} . The part over $A \times \Lambda$ will be precisely P and we will number the letters in the path according to the step in which they “appear” in P . For instance, let $T = \langle a, b \mid ab \rightarrow bbaa \rangle$ and let P be the reduction of S_T :

$$a_{\mathbf{m}}a_{\mathbf{m}}b_{\mathbf{m}} \rightarrow a_{\mathbf{m}}b_{\mathbf{s}}b_{\mathbf{m}}a_{\mathbf{m}}a_{\mathbf{e}} \rightarrow b_{\mathbf{s}}b_{\mathbf{m}}a_{\mathbf{m}}a_{\mathbf{e}}b_{\mathbf{m}}a_{\mathbf{m}}a_{\mathbf{e}} \rightarrow b_{\mathbf{s}}b_{\mathbf{m}}a_{\mathbf{m}}b_{\mathbf{s}}b_{\mathbf{m}}a_{\mathbf{m}}a_{\mathbf{e}}a_{\mathbf{m}}a_{\mathbf{e}} \rightarrow \dots$$

then \tilde{P} is:

$$\begin{aligned} a_{\mathbf{m},0}a_{\mathbf{m},0}b_{\mathbf{m},0} &\rightarrow a_{\mathbf{m},0}b_{\mathbf{s},1}b_{\mathbf{m},1}a_{\mathbf{m},1}a_{\mathbf{e},1} \rightarrow b_{\mathbf{s},2}b_{\mathbf{m},2}a_{\mathbf{m},2}a_{\mathbf{e},2}b_{\mathbf{m},1}a_{\mathbf{m},1}a_{\mathbf{e},1} \rightarrow \\ &\rightarrow b_{\mathbf{s},2}b_{\mathbf{m},2}a_{\mathbf{m},2}b_{\mathbf{s},3}b_{\mathbf{m},3}a_{\mathbf{m},3}a_{\mathbf{e},3}a_{\mathbf{m},1}a_{\mathbf{e},1} \rightarrow \dots \end{aligned}$$

We will call these numbers *origin labels* because they show in which step the letter originated.

Now we can prove Lemma 5.23.

Proof of Lemma 5.23. Let \tilde{P} be the reduction over $A \times \Lambda \times \mathbb{N}$ corresponding to P (as above):

$$\tilde{x}_0 \rightarrow \tilde{x}_1 \rightarrow \dots \rightarrow \tilde{x}_k.$$

Write $w_{\mathbf{SME},\mu}$ for the factor of \tilde{x}_k corresponding to $w_{\mathbf{SME}}$ (where $\mu \in \{0, 1, \dots, k\}^*$). We will prove the result by showing that there is an $0 < j \leq k$ such that $\mu = j \dots j$. In other words, every origin label of μ is the same (positive) number.

First note the following property which follows from the structure of S_T : if

$$(a_1, \lambda_1, \mu_1)(a_2, \lambda_2, \mu_2)$$

(where $a_i \in A$, $\lambda_i \in \Lambda$, and $\mu_i \in \{0, 1, \dots, k\}$) are two adjacent letters in \tilde{x}_k such that $\mu_1 < \mu_2$ then $\lambda_2 = \mathbf{s}$. For if $\lambda_2 \neq \mathbf{s}$ then (a_2, λ_2, μ_2) is not the leftmost letter

that has been written in the μ_2 th step, hence, a letter of a “later” stage (hence with grater or equal origin label) should be to the left of (a_2, λ_2, μ_2) . Similarly, if $\mu_1 > \mu_2$ then $\lambda_1 = \mathbf{e}$. Now we can complete the proof: assuming that $w_{\mathbf{SME}, \mu}$ is of length l , let us write explicitly its letters:

$$w_{\mathbf{SME}, \mu} = (w_1, \lambda_1, \mu_1)(w_2, \lambda_2, \mu_2) \dots (w_l, \lambda_l, \mu_l)$$

where $\lambda_1 = \mathbf{s}$, $\lambda_l = \mathbf{e}$ and $\lambda_2 = \dots = \lambda_{l-1} = \mathbf{m}$. Now, define $j = \max\{\mu_1, \dots, \mu_l\}$ and let $\mu_{r_1}, \dots, \mu_{r_2}$ be a maximal sequence of j -s in μ , that is,

$$\mu_{r_1} = \mu_{r_1+1} = \dots = \mu_{r_2} = j$$

and if $r_1 > 1$ ($r_2 < l$) then $\mu_{r_1-1} < j$ ($\mu_{r_2+1} < j$). Now, assume that $r_1 > 1$ so $\mu_{r_1-1} < \mu_{r_1}$ and hence $\lambda_{r_1} = \mathbf{s}$ by the above observation, which contradicts the fact that $\lambda_{r_1} = \mathbf{m}$. So $r_1 = 1$ and similarly we prove that $r_2 = l$. So we have proved that $\mu = j \dots j$ and it remains to show that $j > 0$. Indeed, if $j = 0$ then $w_{\mathbf{SME}, \mu}$ is a factor of \tilde{x}_0 and then $\pi_A(w_{\mathbf{SME}, \mu}) = \mathbf{m} \dots \mathbf{m} \neq \mathbf{sm} \dots \mathbf{me}$, a contradiction. \square

Now we can prove our desired result:

Proposition 5.24. *Let T be an SRS $\langle A \mid u_i \rightarrow v_i \rangle$ such that $|u_i|, |v_i| > 1$ for all i and any word u_i, v_i from the rules of T , appear in the rules only once. If T has an infinite conversion without reversals, then S_T is non-terminating.*

Proof. Assume that P is an infinite conversion without reversals:

$$x_0 \rightarrow x_1 \rightarrow \dots \rightarrow x_k \rightarrow \dots$$

We will build an infinite reduction of S_T in the following way:

First define $y_0 = (x_0)_{\mathbf{m} \dots \mathbf{m}}$ it is clear that $\overline{y_0} = x_0$. Now, assume that we have already defined

$$y_0 \rightarrow y_1 \rightarrow \dots \rightarrow y_{k-1}$$

such that $\overline{y_i} = x_i$ for every i , and assume that the k th step of P is

$$x_{k-1} \xrightarrow[u \rightarrow v]{p} x_k$$

where $u \rightarrow v$ or $v \rightarrow u$ is a rule of T . Denote the factor of y_{k-1} at position p with length $|u|$ by u_α ($\overline{u_\alpha} = u$). If $u_\alpha \neq u_{\mathbf{SME}}$ the k th step in the reduction of S_P will be

$$y_{k-1} \xrightarrow[u_\alpha \rightarrow v_{\mathbf{SME}}]{p} y_k.$$

This leaves to prove that $u_\alpha \neq u_{\text{SME}}$. Indeed, if $u_\alpha = u_{\text{SME}}$, then according to Lemma 5.23, there is some j such that the j th step is:

$$y_{j-1} \xrightarrow[z_\beta \rightarrow u_{\text{SME}}]{q} y_j,$$

the factor u_{SME} of y_j at position q is not affected in the reduction

$$y_j \rightarrow \dots \rightarrow y_{k-1}$$

and the corresponding position in y_{k-1} is p . This says, that the j th step in P is:

$$x_{j-1} \xrightarrow[z \rightarrow u]{q} x_j$$

($z = \overline{z_\beta}$), the factor u of x_j at position q is not affected in

$$x_j \rightarrow \dots \rightarrow x_{k-1},$$

and the corresponding position in x_{k-1} is p . Note also that either $u \rightarrow v$ or $v \rightarrow u$ is a rule in T , and either $u \rightarrow z$ or $z \rightarrow u$ is a rule in T . The condition that no word appears in the rules twice says that this is the same rule. Hence $z = v$, and then

$$x_{j-1} \rightarrow \dots \rightarrow x_{k-1} \rightarrow x_k$$

is a reversal in P , a contradiction. □

To conclude, we will state our main result:

Theorem 5.25. *Let T be an SRS $\langle A \mid u_i \rightarrow v_i \rangle$ such that $|u_i|, |v_i| > 1$ for all i and any word u_i, v_i from the rules of T , appear in the rules only once. If S_T is terminating then T is graded. If T is border-acyclic, then the converse also holds.*

Proof. Combine Theorem 5.16, Corollary 5.20 and Proposition 5.24. □

Example 5.26. Consider the SRS $T = \langle a, b, c \mid cc \rightarrow cbacb \rangle$. $\text{OVL}(cc) = \{c\}$ and $\text{OVL}(cbacb) = \{cb\}$ so it does not have a confluent orientation and we cannot use Proposition 3.4 to determine if it is graded. Note that T does not have right cycles so by Theorem 5.25 it is graded if and only if S_T is terminating. The SRS S_T has $3^2 + 3^5 - 2 = 250$ rules over the letters $\{a_s, a_m, a_e, b_s, b_m, b_e, c_s, c_m, c_e\}$. We have used the application called TORPA [11] to prove that S_T is terminating, hence, T is graded. Regarding the termination status of S_T , there is another point worth mentioning. It is a common situation that simple argument can reduce the number of rules we need to check. In our case, note that if $w \in \text{rhs}(S_T)$ then w does not contain the letters a_s, a_e and b_s because the words of the rules start with c and end with b, c . Hence, if $w_1 \rightarrow w_2$ is a rule in S_T such that w_1 contains a_s, a_e or b_s it cannot be used an infinite number of times. So we can omit those rules and

see that S_T is terminating if and only if the SRS with the following 43 rules is terminating:

$$\begin{array}{lll}
 c_s c_s \rightarrow c_s b_m a_m c_m b_e & c_s c_m \rightarrow c_s b_m a_m c_m b_e & c_m c_s \rightarrow c_s b_m a_m c_m b_e \\
 c_m c_m \rightarrow c_s b_m a_m c_m b_e & c_m c_e \rightarrow c_s b_m a_m c_m b_e & c_e c_s \rightarrow c_s b_m a_m c_m b_e \\
 c_e c_m \rightarrow c_s b_m a_m c_m b_e & c_e c_e \rightarrow c_s b_m a_m c_m b_e & \\
 c_s b_m a_m c_s b_m \rightarrow c_s c_e & c_s b_m a_m c_s b_e \rightarrow c_s c_e & c_s b_m a_m c_m b_m \rightarrow c_s c_e \\
 c_s b_m a_m c_e b_m \rightarrow c_s c_e & c_s b_m a_m c_e b_e \rightarrow c_s c_e & c_s b_e a_m c_s b_m \rightarrow c_s c_e \\
 c_s b_e a_m c_s b_e \rightarrow c_s c_e & c_s b_e a_m c_m b_m \rightarrow c_s c_e & c_s b_e a_m c_m b_e \rightarrow c_s c_e \\
 c_s b_e a_m c_e b_m \rightarrow c_s c_e & c_s b_e a_m c_e b_e \rightarrow c_s c_e & c_m b_m a_m c_s b_m \rightarrow c_s c_e \\
 c_m b_m a_m c_s b_e \rightarrow c_s c_e & c_m b_m a_m c_m b_m \rightarrow c_s c_e & c_m b_m a_m c_m b_e \rightarrow c_s c_e \\
 c_m b_m a_m c_e b_m \rightarrow c_s c_e & c_m b_m a_m c_e b_e \rightarrow c_s c_e & c_m b_e a_m c_s b_m \rightarrow c_s c_e \\
 c_m b_e a_m c_s b_e \rightarrow c_s c_e & c_m b_e a_m c_m b_m \rightarrow c_s c_e & c_m b_e a_m c_m b_e \rightarrow c_s c_e \\
 c_m b_e a_m c_e b_m \rightarrow c_s c_e & c_m b_e a_m c_e b_e \rightarrow c_s c_e & c_e b_m a_m c_s b_m \rightarrow c_s c_e \\
 c_e b_m a_m c_s b_e \rightarrow c_s c_e & c_e b_m a_m c_m b_m \rightarrow c_s c_e & c_e b_m a_m c_m b_e \rightarrow c_s c_e \\
 c_e b_m a_m c_e b_m \rightarrow c_s c_e & c_e b_m a_m c_e b_e \rightarrow c_s c_e & c_e b_e a_m c_s b_m \rightarrow c_s c_e \\
 c_e b_e a_m c_s b_e \rightarrow c_s c_e & c_e b_e a_m c_m b_m \rightarrow c_s c_e & c_e b_e a_m c_m b_e \rightarrow c_s c_e \\
 c_e b_e a_m c_e b_m \rightarrow c_s c_e & c_e b_e a_m c_e b_e \rightarrow c_s c_e &
 \end{array}$$

As mentioned above, this SRS is terminating.

As another example, we can consider the SRS $T = \langle a, b \mid aba \rightarrow abbaab \rangle$. Again, we can construct the SRS S_T and using the same argument as before we can omit rules that contain b_s . We obtain an SRS with 232 rules, the termination question of which TORPA currently cannot answer.

6. ONE-RULE SRSs

For Theorem 5.25 to be true, we had some conditions on the SRS. We will show in this section that for a one-rule SRS $T = \langle A \mid u \rightarrow v \rangle$, these conditions do not restrict the generality of our results. Since the gradedness problem for $\langle A \mid u \rightarrow v \rangle$ and $\langle A \mid v \rightarrow u \rangle$ is the same, we may assume without loss of generality that $|u| \leq |v|$. Moreover, if $|u| = |v|$ then T is trivially graded so we may assume $|u| < |v|$. T is not graded if u is a factor of v , in particular, it is not graded if $u = 1$. If $|u| = 1$ then T is confluent so Proposition 3.4 implies that T is graded if and only if T is terminating, *i.e.*, if u is not a factor of v . Hence, considering the conditions of Theorem 5.25, we see that if $|u| \leq 1$ or if two words in the rules are equal, the gradedness problem is trivial. Now, regarding the last condition that was left, T may have both left and right cycles, but we will show in this section that we can use Adyan reductions [2] in order to reduce it to a one-rule SRS \hat{T} such that \hat{T} is border-acyclic and \hat{T} is graded if and only if T is. We will mention that Shikishima–Tsuji *et al.* [9] have already used this approach for the termination

problem, and some of the arguments below are similar to theirs. We will describe briefly the Adyan reductions and show that in each reduction, the reduced SRS is graded if and only if the original SRS is graded.

6.1. FIRST REDUCTION

Let $T = \langle A \mid u \rightarrow v \rangle$ be a one-rule SRS. We will say that T is *unbordered* if

$$\text{OVL}^*(u) \cap \text{OVL}^*(v) = \emptyset$$

otherwise, it is called *bordered*.

The first Adyan reduction constructs from a bordered SRS an unbordered SRS.

Let $T = \langle A \mid u \rightarrow v \rangle$ be a bordered SRS. Define K to be the shortest element of $\text{OVL}^*(u) \cap \text{OVL}^*(v)$. It is easy to check that K exists and that $\text{OVL}(K) = \emptyset$. A word $w \in A^*$ is called *bordered with K* if $w = K$ or it can be written as

$$w = Kw'K$$

for some $w' \in A^*$, we will denote by Bord_K the set of all words bordered with K . Note that both u and v are bordered with K .

Every word $w \in A^*$ that has K as a factor, can be written in the form

$$w = w_L \bar{w} w_R$$

where $\bar{w} \in \text{Bord}_K$ and K is not a factor of w_L, w_R . It is also clear that this decomposition is unique.

The following is a restatement of a fact that was observed in ([2], Thm. 3)

Lemma 6.1. *Let $w \in A^*$ be a word that has K as a factor. Then, the connected components of w and \bar{w} (in C_T) have the same number of vertices.*

Proof. Since $u, v \in \text{Bord}_K$ and $\text{OVL}(K) = \emptyset$, it is easy to check that if two words $x, y \in A^*$ are adjacent in C_T then $x_L = y_L, x_R = y_R$ and \bar{x} is adjacent to \bar{y} . Hence, $z \mapsto \bar{z}$ is the required bijection and its inverse is $z \mapsto w_L z w_R$. □

Corollary 6.2. *T is not graded if and only if there is some $w \in \text{Bord}_K$ such that the connected component of w in C_T is infinite.*

Proof. Trivial from the previous lemma when we note that if K is not a factor of x , then its connected component has one element only (x itself). □

Let B be an infinite set of new letters:

$$B = \{b_1, b_2, \dots, b_i, \dots\} \quad (B \cap A = \emptyset).$$

We will enumerate all words $w \in A^*$ without K as a factor:

$$R_1, R_2, \dots, R_i, \dots$$

Adyan and Oganessian define a bijection $\varphi_K : \text{Bord}_K \rightarrow B^*$, (which is not a monoid homomorphism) inductively by $\varphi_K(K) = 1$ and if $x = x_1 R_i K$ where $x_1 \in \text{Bord}_K$ then $\varphi_K(x) = \varphi_K(x_1) b_i$. It can be seen that $|\varphi_K(w)| < |w|$ for all $w \in \text{Bord}_K$.

Now, we define an SRS $T_1 = \langle B \mid \varphi_K(u) \rightarrow \varphi_K(v) \rangle$.

Lemma 6.3 [2]. *If $w_1, w_2 \in \text{Bord}_K$ then w_1 and w_2 are in the same connected component of C_T if and only if $\varphi_K(w_1)$ and $\varphi_K(w_2)$ are in the same connected component of C_{T_1} .*

Corollary 6.4. *T is graded if and only if T_1 is graded.*

Proof. Clear from Corollary 6.2 and the fact that φ_K is a bijection. □

The SRS T_1 has a significant flaw, that is, it is not finite. But this can be solved easily. Let $C \subseteq B$ be the finite set of letters from B that occur in $\varphi_K(u), \varphi_K(v)$ (which are finite words) and define $T_2 = \langle C \mid \varphi_K(u) \rightarrow \varphi_K(v) \rangle$.

Lemma 6.5. *T_1 is graded if and only if T_2 is graded.*

Proof. If T_1 is graded then it is clear that T_2 is graded as well. In the other direction, assume that T_2 is graded and denote by $\overset{*}{\leftarrow}_1$ and $\overset{*}{\leftarrow}_2$ the congruences that $\varphi_K(u) \rightarrow \varphi_K(v)$ generate in B^* and C^* respectively. Let $w \in B^*$ be a word. w can be written as

$$w = x_0 y_0 x_1 y_1 \dots x_k y_k x_{k+1}$$

where

$$x_0, x_{k+1} \in (B \setminus C)^* \quad x_j \in (B \setminus C)^+ \quad y_i \in C^+ \quad (1 \leq j \leq k, \quad 0 \leq i \leq k).$$

Since the rule $\varphi_K(u) \rightarrow \varphi_K(v)$ contains only letters from C , it is clear that $w \overset{*}{\leftarrow}_1 w'$ if and only if

$$w' = x_0 y'_0 x_1 y'_1 \dots x_k y'_k x_{k+1}$$

and

$$y_i \overset{*}{\leftarrow}_2 y'_i \quad (0 \leq i \leq k).$$

Since by assumption the connected component of y_i is finite for all $0 \leq i \leq k$, the connected component of w is finite as well. □

Corollary 6.6. *If $T = \langle A \mid u \rightarrow v \rangle$ is a bordered SRS, we can construct from T another one-rule SRS $T_2 = \langle A' \mid u' \rightarrow v' \rangle$ with $|u'v'| < |uv|$ such that T is graded if and only if T_2 is graded.*

Of course, T_2 might be bordered (actually, it will be bordered, unless $|\text{OVL}^*(u) \cap \text{OVL}^*(v)| = 1$). If T_2 is bordered we can repeat this process using Corollary 6.6 until we get an SRS $\tilde{T} = \langle \tilde{A} \mid \tilde{u} \rightarrow \tilde{v} \rangle$ with $\text{OVL}^*(\tilde{u}) \cap \text{OVL}^*(\tilde{v}) = \emptyset$.

So we can conclude this section:

Corollary 6.7. *Let $T = \langle A \mid u \rightarrow v \rangle$ be a one-rule SRS. We can construct from T an unbordered one-rule SRS $\tilde{T} = \langle \tilde{A} \mid \tilde{u} \rightarrow \tilde{v} \rangle$ such that T is graded if and only if \tilde{T} is graded.*

6.2. SECOND REDUCTION

Definition 6.8. A one-rule SRS $T = \langle A \mid u \rightarrow v \rangle$ is called *left (right) noncancellative* if the first (last) letters of u and v are different.

The second Adyan reduction constructs from an unbordered SRS a new SRS which is left or right noncancellative.

Definition 6.9. Let $T = \langle A \mid u \rightarrow v \rangle$ be a one-rule SRS and let $K \in A^*$ be a word such that $\text{OVL}(K) = \emptyset$. We will say that T satisfies condition $\alpha(K)$ if

$$\text{OVL}(u, K) = \text{OVL}(K, u) = \text{OVL}(v, K) = \text{OVL}(K, v) = \emptyset$$

and u, v are not proper factors of K .

Lemma 6.10. *Let $T = \langle A \mid u \rightarrow v \rangle$ be an SRS that satisfies $\alpha(K)$ and assume that K is not a factor of u or v . Then T is graded if and only if the connected component (in C_T) of any word w that does not contain K as a factor is finite.*

Proof. If T is graded then clearly any connected component is finite. In the other direction, choose some $x \in A^*$. It is clear that x can be written as

$$x = x_1 K x_2 K \dots K x_k$$

where K is not a factor of any x_i . If $x \xleftrightarrow{*} x'$ it is clear from the assumptions that there are x'_i such that:

$$x' = x'_1 K x'_2 K \dots K x'_k$$

$x_i \xleftrightarrow{*} x'_i$ and K is not a factor of x'_i .

Since the connected component of every x_i is finite, the connected component of x is finite as well. □

Consider again an SRS $T = \langle A \mid u \rightarrow v \rangle$ that satisfies $\alpha(K)$, define a new alphabet

$$D = A \cup \{b\} \text{ where } b \notin A.$$

Adyan and Oganesyan define [2] a bijection $\psi_K : A^* \rightarrow D^* \setminus D^* K D^*$ in the following way:

If $x \in A^* \setminus A^* K A^*$ then $\psi_K(x) = x$. Otherwise we can write x as

$$x = x_1 K x_2 K \dots K x_k$$

where $x_i \in A^* \setminus A^* K A^*$. In this case, $\psi_K(x) = x_1 b x_2 b \dots b x_k$.

Let $T = \langle A \mid u \rightarrow v \rangle$ be a one-rule SRS that satisfies $\alpha(K)$ for some $K \in A^*$. We will define a new SRS

$$T_3 = \langle D \mid \psi_K(u) \rightarrow \psi_K(v) \rangle.$$

Note that T_3 also satisfies $\alpha(K)$.

Lemma 6.11 ([2], Lem. 1). *Let \leftrightarrow_1^* and \leftrightarrow_2^* be the congruences on A^* and D^* defined by $u \rightarrow v$ and $\psi_K(u) \rightarrow \psi_K(v)$ respectively. Then,*

$$x \leftrightarrow_1^* y \Leftrightarrow \psi_K(x) \leftrightarrow_2^* \psi_K(y) \quad \forall x, y \in A^*.$$

Corollary 6.12. *T is graded if and only if T_3 is graded.*

Proof. Assume that T is not graded, that is, there is $x \in A^*$ with infinite connected component in C_T . Since $\psi_K : A^* \rightarrow D^* \setminus D^* K D^*$ is a bijection, Proposition 6.11 implies that the connected component of $\psi_K(x)$ in G_{T_3} is infinite as well. Hence T_3 is not graded. In the other direction, assume that T_3 is not graded. Since T_3 satisfies $\alpha(K)$ and $\psi_K(u), \psi_K(v)$ does not contain K as a factor, Lemma 6.10 implies that there is a $w \in D^* \setminus D^* K D^*$ with an infinite connected component. Again, since ψ_K is a bijection, this implies that there is a $x \in A^*$ with an infinite connected component. Hence, T is not graded. \square

Theorem 6.13 ([2], Thm. 4).

Let $T = \langle A \mid u \rightarrow v \rangle$ be an unbordered one-rule SRS. Then, we can construct a word $K \in A^$ such that $\text{OVL}(K) = \emptyset$, T satisfies $\alpha(K)$ and the words $\psi_K(u), \psi_K(v)$ have different first letter or different last letter.*

Remark 6.14. For the sake of completeness, we will describe how K is chosen. Let R and S denote the shortest elements of $\text{OVL}^*(u), \text{OVL}^*(v)$ respectively (recall that $\text{OVL}^*(x)$ cannot be empty since $x \in \text{OVL}^*(x)$, moreover $R \neq S$ since T is unbordered). If $\text{OVL}^*(R, S) \neq \emptyset$ or $\text{OVL}^*(S, R) \neq \emptyset$ then choose K to be the shortest element of $\text{OVL}^*(R, S)$ or $\text{OVL}^*(S, R)$. If $\text{OVL}^*(R, S) = \text{OVL}^*(S, R) = \emptyset$ and without loss of generality $|S| \leq |R|$ then choose $K = S$.

As a corollary we have our main theorem for this section.

Theorem 6.15. *Let $T = \langle A \mid u \rightarrow v \rangle$ be a one-rule SRS. Then, we can construct from T another one-rule SRS, $\hat{T} = \langle \hat{A} \mid \hat{u} \rightarrow \hat{v} \rangle$, such that T is graded if and only if \hat{T} is graded and \hat{T} is border-acyclic.*

Proof. By Corollary 6.7 we can construct an unbordered SRS \tilde{T} such that T is graded if and only if \tilde{T} is graded. By Theorem 6.13 we can construct from \tilde{T} another SRS $\hat{T} = \langle \hat{A} \mid \hat{u} \rightarrow \hat{v} \rangle$. Again \hat{T} is graded if and only if \tilde{T} is graded, and the fact that u, v have different first letter or different last letter implies that \hat{T} is border-acyclic. \square

Example 6.16. Consider the SRS $T = \langle a, b \mid abab \rightarrow abbaabb \rangle$. It has both left and right cycles. Choose $K = ab$ and note that $\text{OVL}(K) = \emptyset$ and T satisfies $\alpha(K)$. So we can construct \hat{T} which in this case it is: $\hat{T} = \langle a, b, c \mid cc \rightarrow cbacb \rangle$. \hat{T} has no right cycles and by Example 5.26 it is graded so T is graded as well.

7. CONCLUSION

We showed that under certain conditions on an SRS, its gradedness problem can be converted into a termination problem of a related SRS. We also showed, using Adyan reductions, that the gradedness problem of any one-rule SRS can be reduced into a termination problem. A natural question is whether we can reduce the gradedness problem of any SRS to a termination problem, without restrictive conditions. Another question is whether one can do the converse, that is, reduce the termination problem into the gradedness problem. We proved that gradedness is an undecidable property for finite SRSs. It is natural to ask whether gradedness is decidable for one-rule SRSs. Note that gradedness and termination are equivalent for confluent length-increasing one-rule SRSs, and the decidability status of the termination problem of such SRSs is still open.

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