

NEW BOUNDS ON THE EDGE-BANDWIDTH OF TRIANGULAR GRIDS*

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Abstract. The edge-bandwidth of a graph G is the bandwidth of the line graph of G . Determining the edge-bandwidth $B'(T_n)$ of triangular grids T_n is an open problem posed in 2006. Previously, an upper bound and an asymptotic lower bound were found to be $3n - 1$ and $3n - o(n)$ respectively. In this paper we provide a lower bound $3n - \lceil n/2 \rceil$ and show that it gives the exact values of $B'(T_n)$ for $1 \leq n \leq 8$ and $n = 10$. Also, we show the upper bound $3n - 5$ for $n \geq 10$.

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1. INTRODUCTION

The bandwidth problem for graphs has been extensively studied due to its connections with theoretical and applied topics in sparse matrix computation, VLSI designs, network communications, and other areas (see surveys [6, 8]).

Given a simple graph G with vertex set $V(G)$ and edge set $E(G)$, a bijection $f : V(G) \rightarrow \{1, 2, \dots, |V(G)|\}$ is called a *labeling* of G . For a given labeling f of G , the *bandwidth* of f for G is defined by

$$B(G, f) := \max \{|f(u) - f(v)| : uv \in E(G)\}.$$

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The bandwidth of G , denoted $B(G)$, is $\min \{B(G, f) : f \text{ is a labeling of } G\}$. A labeling f attaining this minimum value is called an *optimal labeling*.

The bandwidth problem is a well-known NP-complete problem in graph theory. Much work has been done for determining the bandwidth of special graphs. The bandwidth of triangular grids T_n was a difficult problem in this direction, posed by D. West in 1993. Here T_n is a graph with vertex set $\{(x, y, z) \in \mathbf{Z}^3 : x + y + z = n, x, y, z \geq 0\}$ and two vertices (x, y, z) and (x', y', z') are joined by an edge if $|x - x'| + |y - y'| + |z - z'| = 2$ (they agree in one coordinate and differ by 1 in the other two coordinates). Hochberg *et al.* [11] have proven that $B(T_n) = n + 1$. By a similar method of this paper, [13] determined the cutwidth of T_n (the cutwidth of G is the minimum, over all labelings, of the maximum number of pairwise overlapping edges).

As a special case of bandwidth, the *edge-bandwidth* $B'(G)$ of G is the bandwidth of the line graph $L(G)$ of G . In other words, we consider an *edge-labeling* (bijection) $\eta : E(G) \rightarrow \{1, 2, \dots, |E(G)|\}$ and the edge-bandwidth of η for G is defined by

$$B'(G, \eta) := \max \{|\eta(e) - \eta(e')| : e, e' \in E(G), e \text{ and } e' \text{ are adjacent in } G\}.$$

The edge-bandwidth of G , denoted $B'(G)$, is $\min \{B'(G, \eta) : \eta \text{ is an edge-labeling of } G\}$.

The edge-bandwidth problem is also NP-complete, which is implied by the in-approximation result for bandwidth (see [2]). Much interest on edge-bandwidth has been paid to several classes of graph-products and grids (see [2, 3, 14] for details). Besides, other special graphs were considered, for example, [12] for K_n , $K_{n,n}$, caterpillars and others; [9] for theta graphs; [5] for n -cubes, butterfly graphs, and complete k -ary trees. In particular, determining the edge-bandwidth of triangular grids T_n is an open problem [3]. Later, Akhtar, Jiang, and Pritikin [1] have shown that $3n - o(n) \leq B'(T_n) \leq 3n - 1$. Moreover, they have claimed that the upper bound can be improved to $3n - 5$ when $n \geq 18$.

The goal of this paper is to improve the lower and upper bounds on $B'(T_n)$. The main results are as follows. First, we present a lower bound $3n - \lceil n/2 \rceil$. Second, we establish a labeling algorithm that implies the upper bound $3n - \lceil n/2 \rceil$ for $1 \leq n \leq 8$ and $n = 10$. Consequently, we obtain the exact value $B'(T_n) = 3n - \lceil n/2 \rceil$ for $1 \leq n \leq 8$ and $n = 10$. This should be a basic formula for the edge-bandwidth of T_n of small n . As n increases, we can only obtain some estimations $3n - \lceil n/2 \rceil \leq B'(T_n) \leq 3n - \lfloor n/2 \rfloor$ for $9 \leq n \leq 11$ and $B'(T_n) \leq 3n - 5$ for $n \geq 10$. The latter improves the claim of [1]. More exact values and better bounds are worthy of further study.

2. TRIANGULAR GRIDS

We consider the triangular grid T_n for a given integer $n > 0$. We may draw T_n on the plane by taking the vertex set as $V(T_n) = \{(x, y) \in \mathbf{Z}^2 : x + y \leq n, x, y \geq 0\}$ and two vertices (x, y) and (x', y') are adjacent if $|x - x'| + |y - y'| = 1$ or if $|x - x'| + |y - y'| = 2$ and $x + y = x' + y'$ (see Fig. 1). Moreover, the three

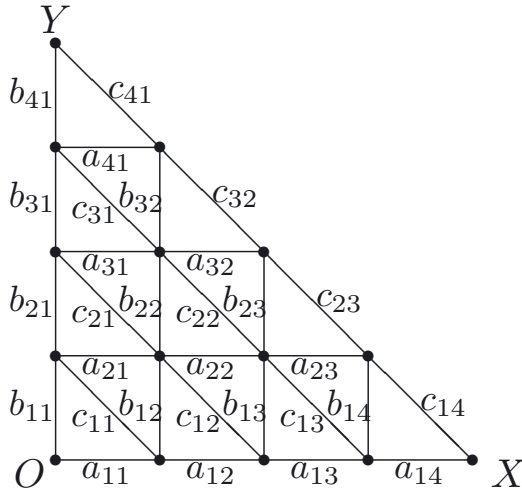


FIGURE 1. Representation of T_4 .

apices of the triangular region are denoted by O, X, Y , which are $(0, 0), (n, 0), (0, n)$ respectively. Accordingly, the set of vertices on the horizontal side OX is $V_{OX} = \{(x, y) \in V(T_n) : y = 0\}$; the set of vertices on the vertical side OY is $V_{OY} = \{(x, y) \in V(T_n) : x = 0\}$; and that on the slant side XY is $V_{XY} = \{(x, y) \in V(T_n) : x + y = n\}$.

For the edge set $E(T_n)$, we denote by a_{ij} the edge between $(j - 1, i - 1)$ and $(j, i - 1)$, by b_{ij} the edge between $(j - 1, i - 1)$ and $(j - 1, i)$, and by c_{ij} the edge between $(j - 1, i)$ and $(j, i - 1)$. They are partitioned into three parts according to three directions:

$$R_i := \{a_{ij} : 1 \leq j \leq n - i + 1\}, 1 \leq i \leq n$$

$$Q_j := \{b_{ij} : 1 \leq i \leq n - j + 1\}, 1 \leq j \leq n$$

$$L_k := \{c_{ij} : i + j = k + 1\}, 1 \leq k \leq n,$$

where R_i 's are called *rows* along the horizontal lines, Q_j 's are called *columns* along the vertical lines, and L_k 's are called *diagonals* along the slant lines. In particular, R_1 is the horizontal side OX , Q_1 is the vertical side OY , and L_n is the slant side XY . Therefore, $|E(T_n)| = 3(1 + 2 + \dots + n) = \frac{3}{2}n(n + 1)$. Figure 1 is an example.

Moreover, we define

$$R'_i := \{b_{ij} : 1 \leq j \leq n - i + 1\} \cup \{c_{ij} : 1 \leq j \leq n - i + 1\}, 1 \leq i \leq n$$

$$Q'_j := \{a_{ij} : 1 \leq i \leq n - j + 1\} \cup \{c_{ij} : 1 \leq i \leq n - j + 1\}, 1 \leq j \leq n$$

$$L'_k := \{a_{ij} : i + j = k + 1\} \cup \{b_{ij} : i + j = k + 1\}, 1 \leq k \leq n,$$

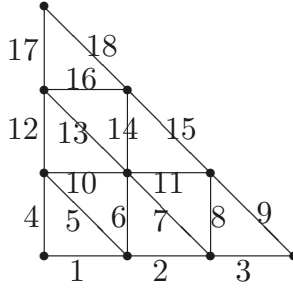


FIGURE 2. ‘Row by Row’ labeling for T_3 .

where R'_i, Q'_j, L'_k are also called rows, columns, and diagonals, respectively. Note that each row R_i or R'_i (each column Q_j or Q'_j , each diagonal L_k or L'_k) induces a path in T_n .

For $1 \leq i \leq n$, let $\tilde{R}_i = R_i \cup R'_i$, called the i th *extended row*. It forms a chain of $n - i + 1$ edge-disjoint triangles. For example, \tilde{R}_1 in Figure 1 shows the 4 edge-disjoint triangles. All extended rows \tilde{R}_i ($1 \leq i \leq n$) constitute a partition of $E(T_n)$. Similarly, $\tilde{Q}_j = Q_j \cup Q'_j$ for $1 \leq j \leq n$ is the j th *extended column* and they also constitute a partition of $E(T_n)$. In the same way, we define the *extended diagonals* $\tilde{L}_k = L_k \cup L'_k$ for $1 \leq k \leq n$.

A simple labeling for T_n is the following ‘Row by Row’ labeling η : we label all edges in the order of $R_1, R'_1, R_2, R'_2, \dots, R_n, R'_n$ by integers $1, 2, \dots, e(n)$ where $e(n) = |E(T_n)| = \frac{3}{2}n(n+1)$. An example for $n = 3$ is depicted in Figure 2. It is easy to see that $B'(T_n, \eta) = |R'_1 \cup R_2| = 3n - 1$. Thus we obtain an obvious upper bound $B'(T_n) \leq 3n - 1$ [1]. We shall see later that this upper bound is tight if and only if $n \leq 2$.

3. LOWER BOUNDS

We first state some traditional notations. For a graph G , let S be a subset of $E(G)$ and $\bar{S} = E(G) \setminus S$. The *outer boundary* or *neighbor set* of S is defined by $\partial^+(S) := \{e \in \bar{S} : e \text{ is adjacent to some edge } e' \in S\}$. Likewise, the *inner boundary* of S is defined by $\partial^-(S) := \{e \in S : e \text{ is adjacent to some edge } e' \in \bar{S}\}$. Clearly, $\partial^+(S) \subseteq \bar{S}$, $\partial^-(S) \subseteq S$, and $\partial^-(S) = \partial^+(\bar{S})$.

Moreover, for a subset $S \subseteq E(G)$, let $V(S)$ denote the set of vertices incident with the edges in S , which is the vertex set of the edge-induced subgraph $G[S]$. We define the *vertex boundary* of S by $\delta(S) := V(\partial^-(S)) \cap V(\partial^+(S))$, that is, the set of vertices incident with the edges in the inner boundary $\partial^-(S)$ and with the edges in the outer boundary $\partial^+(S)$. Further, we define the *total boundary* of S by $\hat{\partial}(S) := \partial^-(S) \cup \partial^+(S)$, which is the set of edges covered by the vertices in $\delta(S)$.

Now we consider any edge-labeling $\eta : E(G) \rightarrow \{1, 2, \dots, m\}$ where $m = |E(G)|$. For $1 \leq i \leq m$, let $S_i := \{e \in E(G) : \eta(e) \leq i\}$, which denotes the set of edges

with the first i labels. The following is the well-known ‘isoperimetric inequality’ due to Harper [10] (see also [1, 3, 7, 14]).

Lemma 3.1 ([10]). *For an edge-labeling $\eta : E(G) \rightarrow \{1, 2, \dots, m\}$ of G , it holds that*

$$B'(G, \eta) \geq \max_{1 \leq i \leq m} \max \{|\partial^-(S_i)|, |\partial^+(S_i)|\} \geq \max_{1 \leq i \leq m} \left\lceil \frac{1}{2} |\hat{\partial}(S_i)| \right\rceil.$$

We shall apply this approach for determining lower bounds to the triangular grid T_n . For a subset $S \subseteq E(T_n)$, S is called a *condensed subset* if the following property is satisfied: For the edge sequence $\sigma = (e_1, e_2, \dots, e_t)$ of any row R_i, R'_i or column Q_j, Q'_j (which is the edge sequence of a path), if $e_l \in S$ for some l with $1 \leq l \leq t$, then $e_h \in S$ for all h with $1 \leq h \leq l$. That is to say, the edges in S are ‘pressed’ as close to the vertical side OY and to the horizontal side OX as possible.

Lemma 3.2. *For any subset $S \subseteq E(T_n)$, there exists a condensed subset \hat{S} such that $|\hat{S}| = |S|$, $|\partial^-(\hat{S})| \leq |\partial^-(S)|$ and $|\partial^+(\hat{S})| \leq |\partial^+(S)|$.*

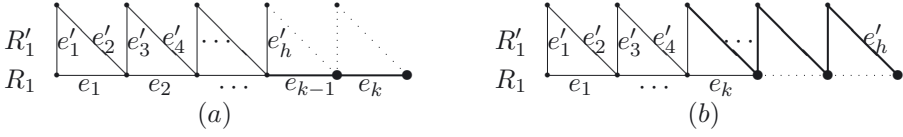
Proof. We show the assertion on the inner boundary $\partial^-(S)$ (the assertion on the outer boundary $\partial^+(S)$ is symmetric). For any subset $S \subseteq E(T_n)$, we first define a subset S' by the following ‘left-shift’ operation (following the technique of [7]). Let $\sigma = (e_1, e_2, \dots, e_t)$ be the edge sequence of any row R_i or R'_i . If $|S \cap \sigma| = k$, then we define $S' \cap \sigma := \{e_1, e_2, \dots, e_k\}$ (the edges of S in σ are transformed into the first k edges in σ). Carrying out this transformation in each row, the resulting subset S' is called the ‘left-shift’ of S . Next, we define a subset \hat{S} from S' by the following ‘down-shift’ operation. Let $\sigma = (e_1, e_2, \dots, e_t)$ be the edge sequence of any column Q_j or Q'_j . If $|S' \cap \sigma| = k$, then we define $\hat{S} \cap \sigma := \{e_1, e_2, \dots, e_k\}$. In this way, \hat{S} is called the ‘down-shift’ of S' . It can be seen that $|\hat{S}| = |S'| = |S|$ and \hat{S} is condensed.

We proceed to prove $|\partial^-(\hat{S})| \leq |\partial^-(S')| \leq |\partial^-(S)|$. We first show $|\partial^-(S')| \leq |\partial^-(S)|$ by induction on n . When $n = 1$, the assertion is trivial. Assume now that $n > 1$ and the assertion holds for smaller n . We start considering R_1 and R'_1 .

Claim 1. $|\partial^-(S') \cap R_1| \leq |\partial^-(S) \cap R_1|$.

Suppose that $S' \cap R_1 = \{e_1, e_2, \dots, e_k\}$ and $S' \cap R'_1 = \{e'_1, e'_2, \dots, e'_h\}$. If $h \geq 2k$, then $\partial^-(S') \cap R_1 = \{e_k\}$, thus $|\partial^-(S') \cap R_1| = 1 \leq |\partial^-(S) \cap R_1|$, as required. So we may assume that $h < 2k$. Then $e_j \in \partial^-(S') \cap R_1$ if and only if $j \geq \lceil \frac{h}{2} \rceil$. Thus $|\partial^-(S') \cap R_1| = k - \lceil \frac{h}{2} \rceil + 1$. An example is shown in Figure 3a, where the dotted lines in R'_1 represent the edges in $\partial^+(S') \cap R'_1$, the heavy lines in R_1 represent the edges in $\partial^-(S') \cap R_1$, and the black points stand for the vertices in the vertex boundary $\delta(S')$.

On the other hand, the edges in $S \cap R_1$ or $S \cap R'_1$ are not necessarily consecutive. We may denote $R_1 = \{a_1, a_2, \dots, a_n\}$ and $R'_1 = \{b_1, c_1, \dots, b_n, c_n\}$. If $a_i \in S \setminus \partial^-(S)$, then $b_i, c_i \in S$. Moreover, for the last edge $a_i \in S \setminus \partial^-(S)$ in this row, we

FIGURE 3. Two adjacent rows R_1 and R'_1 .

have one more $b_{i+1} \in S$. Suppose $x = |(S \setminus \partial^-(S)) \cap R_1|$. Then $h = |S \cap R'_1| \geq 2x + 1$, thus $x \leq \lfloor \frac{h-1}{2} \rfloor = \lceil \frac{h}{2} \rceil - 1$. Therefore $|\partial^-(S) \cap R_1| = k - x \geq k - \lceil \frac{h}{2} \rceil + 1 = |\partial^-(S') \cap R_1|$, and Claim 1 is proved.

Claim 2. $|\partial^-(S') \cap R'_1| \leq |\partial^-(S) \cap R'_1|$.

In fact, if $|\partial^-(S') \cap R'_1| \leq 1$, then the claim is trivial. Otherwise, there are three cases:

- (a) Each edge of $\partial^-(S') \cap R'_1$ is adjacent to an edge in $\partial^+(S') \cap R_1$;
- (b) Each edge of $\partial^-(S') \cap R'_1$ is adjacent to an edge in $\partial^+(S') \cap R_2$;
- (c) Each edge of $\partial^-(S') \cap R'_1$ is adjacent to an edge in $\partial^+(S') \cap R'_2$.

Here, Case (c) can be reduced to Case (b), since they are equivalent to that each edge is incident with a vertex in $\delta(S') \cup R_2$. Moreover, cases (a) and (b) are symmetrical. So we may only consider case (a). An example is shown in Figure 3b, where the dotted lines represent the edges in $\partial^+(S') \cap R_1$, the heavy lines represent the edges in $\partial^-(S') \cap R'_1$, and the black points stand for the vertices in $\delta(S')$.

Suppose that $S' \cap R_1 = \{e_1, e_2, \dots, e_k\}$ and $S' \cap R'_1 = \{e'_1, e'_2, \dots, e'_h\}$. Then $2k < h$ and $e'_j \in \partial^-(S') \cap R'_1$ if and only if $j \geq 2k$. Thus $|\partial^-(S') \cap R'_1| = h - 2k + 1$. On the other hand, denote $R_1 = \{a_1, a_2, \dots, a_n\}$ and $R'_1 = \{b_1, c_1, \dots, b_n, c_n\}$. If b_i or $c_i \in S \setminus \partial^-(S)$, then $a_i \in S$. Moreover, for the last edge $a_i \in S$ in this row, $c_i \in \partial^-(S)$. Suppose $y = |(S \setminus \partial^-(S)) \cap R'_1|$. Then $y \leq 2k - 1$. Hence $|\partial^-(S) \cap R'_1| = h - y \geq h - 2k + 1 = |\partial^-(S') \cap R'_1|$, and Claim 2 follows.

Now we construct a graph T_{n-1} by deleting \hat{R}_1 from T_n . It follows from the induction hypothesis that $|\partial^-(S')| \leq |\partial^-(S)|$ for T_{n-1} . By combining Claim 1 and Claim 2, we show that $|\partial^-(S')| \leq |\partial^-(S)|$ for T_n .

Furthermore, $|\partial^-(\hat{S})| \leq |\partial^-(S')|$ can be shown symmetrically by considering the columns. This completes the proof. \square

Theorem 3.3. For the triangular grid T_n , it holds that $B'(T_n) \geq 3n - \lceil n/2 \rceil$.

Proof. By Lemma 3.1, it suffices to show that for any edge-labeling η , there exists an index i with $1 \leq i \leq e(n)$ such that $\max\{|\partial^-(S_i)|, |\partial^+(S_i)|\} \geq 3n - \lceil n/2 \rceil$. As the index i increases from 1 to $e(n)$, we consider the first moment that S_i touches the slant side XY (except X, Y). More precisely, we take $k = \min\{i : \delta(S_i) \cap (V_{XY} \setminus \{X, Y\}) \neq \emptyset\}$. By Lemma 3.2, we assume that such a set S_k is condensed. We proceed to show that $\max\{|\partial^-(S_k)|, |\partial^+(S_k)|\} \geq \lceil \frac{1}{2} |\hat{\partial}(S_k)| \rceil \geq 3n - \lceil n/2 \rceil$. There are three cases to consider.

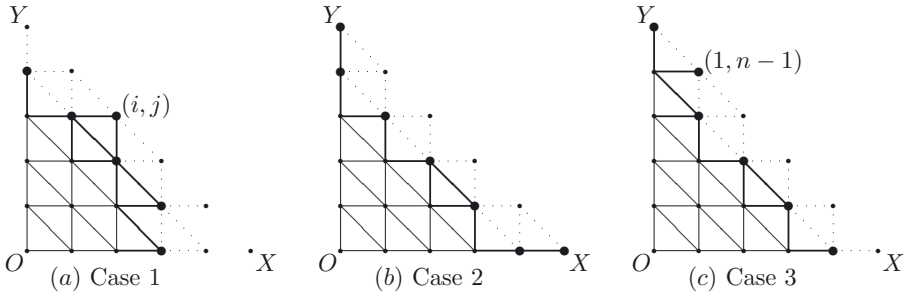


FIGURE 4. Three cases in the proof of Theorem 3.3.

Case 1: $X, Y \notin \delta(S_k)$. An example of $n = 5$ is depicted in Figure 4a. In three cases of Figure 4, the edges in the subset S_k are shown by solid lines, the edges in $\partial^-(S_k)$ by heavy lines, those in $\partial^+(S_k)$ by dotted lines, and the vertices in $\delta(S_k)$ are represented by black points.

We evaluate the number of edges in $\hat{\partial}(S_k) = \partial^-(S_k) \cup \partial^+(S_k)$, namely the edges covered by $\delta(S_k)$. Suppose that $(i, j) \in \delta(S_k) \cap (V_{XY} \setminus \{X, Y\})$. Then $i + j = n$. It is evident that in each column from column Q_1 to column Q_i and in each row from row R_1 to row R_j , there is at least one vertex in $\delta(S_k)$. Hence $|\delta(S_k)| \geq n + 1$. Without loss of generality, we assume that $|\delta(S_k)| = n + 1$ (for otherwise we can get a greater lower bound). Now we consider the subgraph H of T_n induced by $\hat{\partial}(S_k)$. In this subgraph H , each vertex of $\delta(S_k)$ on the side OX, OY , or XY has degree 4, and the $n - 2$ other vertices have degree 6. Moreover, there are at most $n + 1$ edges between the vertices of $\delta(S_k)$. Therefore, we have

$$\begin{aligned} \left\lceil \frac{1}{2} |\hat{\partial}(S_k)| \right\rceil &\geq \left\lceil \frac{1}{2} (4 \times 3 + 6(n - 2) - n - 1) \right\rceil \\ &= \left\lceil \frac{1}{2} (6n - n - 1) \right\rceil = 3n - \left\lfloor \frac{n + 1}{2} \right\rfloor = 3n - \left\lfloor \frac{n}{2} \right\rfloor. \end{aligned}$$

Case 2: $X, Y \in \delta(S_k)$. Then there is an index $i < k$ such that $X, Y \in \delta(S_i)$ and $\delta(S_i) \cap (V_{XY} \setminus \{X, Y\}) = \emptyset$. We replace k by i . An example of $n = 5$ is depicted in Figure 4b. In addition to X and Y , each row has at least one vertex in $\delta(S_i)$. Hence $|\delta(S_i)| \geq n + 2$. We may assume $|\delta(S_i)| = n + 2$. Then in the subgraph H induced by $\hat{\partial}(S_i)$, we have that X and Y have degree 2, each vertex on the side OX or OY has degree 4, and the $n - 2$ other vertices have degree 6. Besides, there are at most $n + 1$ edges between the vertices of $\delta(S_i)$. Hence

$$\left\lceil \frac{1}{2} |\hat{\partial}(S_i)| \right\rceil \geq \left\lceil \frac{1}{2} (2 \times 2 + 4 \times 2 + 6(n - 2) - n - 1) \right\rceil = 3n - \left\lfloor \frac{n}{2} \right\rfloor.$$

Case 3: $Y \in \delta(S_k)$ and $X \notin \delta(S_k)$. Suppose that $(i, j) \in \delta(S_k) \cap (V_{XY} \setminus \{X, Y\})$ as before. If $i, j > 1$, then the proof is similar to that of Case 1 (we can even

get greater lower bound). Now we need only consider the case where $i = 1$ and $j = n - 1$. An example is depicted in Figure 4c. In the subgraph H induced by $\hat{\partial}(S_k)$, we have that $|\delta(S_k)| = n + 1$, Y has degree 2, each vertex on the side OX or XY has degree 4, and the $n - 2$ other vertices have degree 6. Besides, there are at most n edges between the vertices of $\delta(S_k)$. Therefore, we have

$$\begin{aligned} \left\lceil \frac{1}{2} |\hat{\partial}(S_k)| \right\rceil &\geq \left\lceil \frac{1}{2} (2 + 4 \times 2 + 6(n - 2) - n) \right\rceil \\ &= \left\lceil \frac{1}{2} (6n - n - 2) \right\rceil = 3n - \left\lfloor \frac{n}{2} \right\rfloor - 1. \end{aligned}$$

If $|\partial^-(S_k)| \neq |\partial^+(S_k)|$, then $\max\{|\partial^-(S_k)|, |\partial^+(S_k)|\} > \frac{1}{2} |\hat{\partial}(S_k)| \geq 3n - n/2 - 1$, thus we have $\max\{|\partial^-(S_k)|, |\partial^+(S_k)|\} \geq 3n - \lceil n/2 \rceil$. Hence we are left to consider the case where $|\partial^-(S_k)| = |\partial^+(S_k)| = 3n - n/2 - 1$ (where n is even), and try to increase the lower bound by 1. In this situation, each row R_i ($1 \leq i \leq n$) has exact one vertex in $\delta(S_k)$.

Let e_i be such that $\eta(e_i) = i$ ($1 \leq i \leq e(n)$). Then $S_k = \{e_1, e_2, \dots, e_k\}$. By the choice of k , we have $e_k = a_{n,1}$ or $e_k = b_{n-1,2}$. By symmetry, we consider that $e_k = a_{n,1} = uv$ with $v = (1, n - 1) \in \delta(S_k)$ and $b_{n-1,2} \notin S_k$. Now take the edge e_{k+1} into account. By assumption, S_k and S_{k+1} are condensed. Hence e_{k+1} must be contained in $\partial^+(S_k)$ and it is being taken from $\partial^+(S_k)$ to $\partial^-(S_{k+1})$. If the vertex boundary $\delta(S_k)$ is unchanged, then we have $|\partial^-(S_{k+1})| = |\partial^-(S_k)| + 1 = 3n - n/2$, thus the desired lower bound is proved. For example, if $e_{k+1} = c_{ij}$ is a slant edge, then $a_{ij}, b_{ij} \in S_k$ (since S_{k+1} is condensed). Thus two ends of e_{k+1} are contained in $\delta(S_k)$. When e_{k+1} is added to S_{k+1} , then $\delta(S_k)$ is unchanged and the assertion follows. Consequently, we may assume that $e_{k+1} = uv$ is a horizontal edge (the case of vertical edge being symmetric). Herein, $u \in \delta(S_k)$ and when e_{k+1} is added to S_{k+1} , we have $u \notin \delta(S_{k+1})$ and $v \in \delta(S_{k+1})$. Then, while e_{k+1} is taken from $\partial^+(S_k)$ to $\partial^-(S_{k+1})$, at least two edges incident with v are added to $\partial^+(S_{k+1})$. Thus $|\partial^+(S_{k+1})| \geq |\partial^+(S_k)| + 1 = 3n - n/2$, the assertion follows. \square

4. UPPER BOUNDS AND EXACT VALUES

In order to show that the lower bound proved in Section 3 is tight, we present a labeling algorithm, called *Slide Labeling*, described as follows.

Step 1: Partitioning. We partition $E(T_n)$ into n subsets E_1, E_2, \dots, E_n as follows:

- For $1 \leq k < \lfloor n/2 \rfloor$, set $E_k = \tilde{L}_k$.
- For $k = \lfloor n/2 \rfloor$, set $E_k = (Q_1 \cup \tilde{L}_k) \setminus (E_1 \cup \dots \cup E_{k-1})$.
- For $k = \lfloor n/2 \rfloor + 1$, set $E_k = (Q'_1 \cup L'_k) \setminus (E_1 \cup \dots \cup E_{k-1})$.
- For $\lfloor n/2 \rfloor + 1 < k < n$, set $E_k = (\tilde{Q}_r \cup L_{k-1} \cup L'_k) \setminus (E_1 \cup \dots \cup E_{k-1})$ where $r = k - \lfloor n/2 \rfloor$.
- Finally, set $E_n = (L_{n-1} \cup \tilde{L}_n) \setminus (E_1 \cup \dots \cup E_{n-1})$.

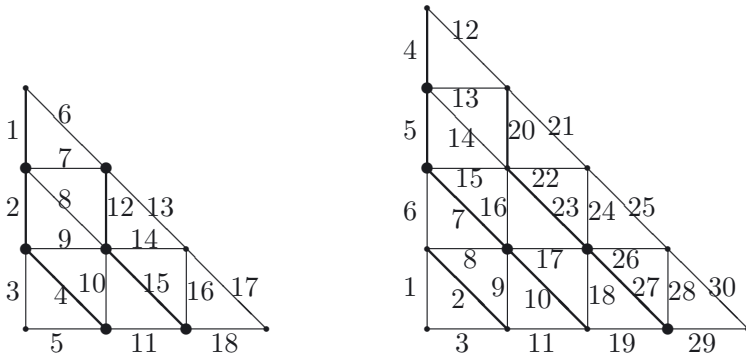


FIGURE 5. Slide labelings for T_3 and T_4 .

We call each subset of E_1, E_2, \dots, E_n a *level set*. Roughly speaking, n levels E_1, E_2, \dots, E_n are divided into two parts: In the first half, each level E_k ($k \leq \lfloor n/2 \rfloor$) consists of a slant row L'_k with a straight-line L_k (boundary) on the right (only $E_{\lfloor n/2 \rfloor}$ has a part of Q_1). In the second half, each level E_k ($k > \lfloor n/2 \rfloor + 1$) consists of a column Q'_r and a slant row L'_k with straight-lines Q_r and L_{k-1} (boundary) on the left. The level $E_{\lfloor n/2 \rfloor + 1}$ is a transition band which has no straight-lines as boundary. Take $n = 4$ as an example (see Fig. 1), then $E_1 = \{b_{11}, c_{11}, a_{11}\}$, $E_2 = \{b_{41}, b_{31}, b_{21}, c_{21}, a_{21}, b_{12}, c_{12}, a_{12}\}$, $E_3 = \{c_{41}, a_{41}, c_{31}, a_{31}, b_{22}, a_{22}, b_{13}, a_{13}\}$, and $E_4 = \{b_{32}, c_{32}, a_{32}, c_{22}, b_{23}, c_{23}, a_{23}, c_{13}, b_{14}, c_{14}, a_{14}\}$.

Step 2: Labeling. We label all edges level by level in the order of (E_1, E_2, \dots, E_n) and in each level E_k , we label the edges from left to right and from top to bottom, in a way of sliding down a slope. Examples can be seen in Figures 5–10.

Theorem 4.1. $B'(T_n) \leq 3n - \lceil n/2 \rceil$ for $1 \leq n \leq 8$ and $n = 10$.

Proof. When $n = 1$, T_n is a cycle and $B'(T_n) = 2 = 3n - 1$. When $n = 2$, $B'(T_n) \leq 5 = 3n - 1$ can be obtained by the Row by Row labeling of Figure 2. Consider $n \geq 3$ as follows.

We first show that $B'(T_n) \leq 3n - \lceil n/2 \rceil = 3n - 2$ for $3 \leq n \leq 4$ by the slide labelings η shown in Figure 5, in which $B'(T_3, \eta) = 7$ and $B'(T_4, \eta) = 10$, respectively. For T_3 , we have $|E_1| = 5, |E_2| = 6, |E_3| = 7$ and so $B'(T_3, \eta) \leq 7$. On the other hand, this maximum label-difference 7 is attainable. Here, we use the black points to represent the vertices for which two incident edges have the maximum label-difference. For T_4 , we have $|E_1| = 3, |E_2| = |E_3| = 8, |E_4| = 11$ and $B'(T_4, \eta) = 10$ by adjusting the last two labels.

We next show that $B'(T_n) \leq 3n - \lceil n/2 \rceil = 3n - 3$ for $5 \leq n \leq 6$. The slide labelings η are shown in Figure 6, in which $B'(T_5, \eta) = 12$ and $B'(T_6, \eta) = 15$, respectively. For the former, $|E_1| = 3, |E_2| = 9, |E_3| = 10, |E_4| = 12, |E_5| = 11$ and the maximum label-difference 12 can be checked directly (see the black points).

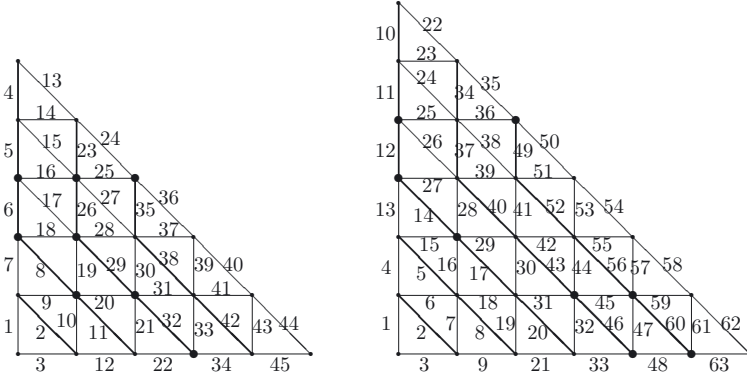


FIGURE 6. Slide labelings for T_5 and T_6 .

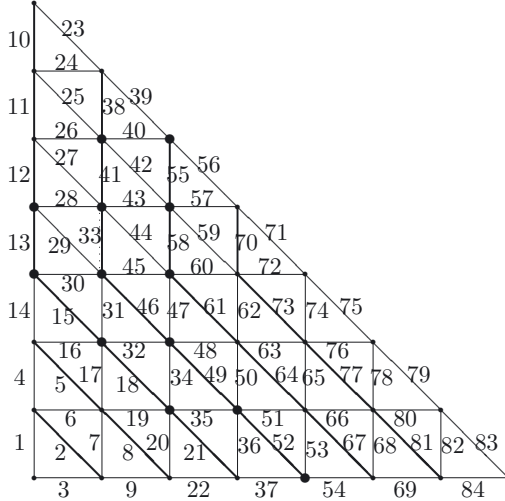


FIGURE 7. Slide labeling for T_7 .

For the latter, $|E_1| = 3, |E_2| = 6, |E_3| = |E_4| = 12, |E_5| = |E_6| = 15$ and the maximum label-difference is 15.

We further show that $B'(T_n) \leq 3n - \lceil n/2 \rceil = 3n - 4$ for $7 \leq n \leq 8$, namely $B'(T_7, \eta) = 17$ and $B'(T_8, \eta) = 20$. However, the slide labelings should be modified slightly. For T_7 , we have $|E_1| = 3, |E_2| = 6, |E_3| = 13, |E_4| = 14, |E_5| = 18$, and $|E_6| = |E_7| = 15$ originally (notice that $|E_5| = 3(n - 1) = 18 > 17$). In order to make $\max_{1 \leq i \leq 7} |E_i| \leq 17$, we take out one edge (say $b_{\lfloor n/2 \rfloor + 1, 2}$) from E_5 and add it to E_4 . The slide labeling is almost the same as before, only the label of the new edge in E_4 is adjusted (see Fig. 7). For T_8 , we also have $|E_6| = 3(n - 1) = 21 > 20$.

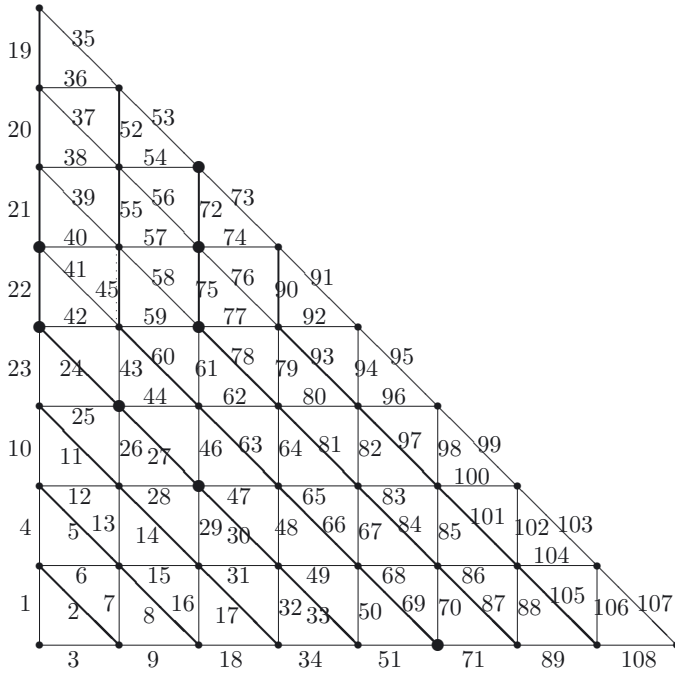


FIGURE 8. Slide labeling for T_8 .

So we take out one edge (say $b_{\lfloor n/2 \rfloor + 1, 2}$) from E_6 and add it to E_5 , and the label of the new edge in E_5 is adjusted (see Fig. 8).

We skip the case of $n = 9$ and finally show that $B'(T_{10}) \leq 3n - \lfloor n/2 \rfloor = 3n - 5 = 25$ for $n = 10$. The slide labeling is shown in Figure 9, where we take out two edges ($b_{\lfloor n/2 \rfloor + 1, 2}$ and $c_{\lfloor n/2 \rfloor, 2}$) from E_7 and add them to E_6 . Meanwhile, the labels of the new edges in E_6 are adjusted. The edge-bandwidth $B'(T_{10}, \eta) = 3n - 5 = 25$ can be checked directly. This completes the proof. \square

By combining the lower bound in Theorem 3.3 and the upper bound in Theorem 4.1, we obtain the exact values as follows.

Theorem 4.2. $B'(T_n) = 3n - \lfloor n/2 \rfloor$ for $1 \leq n \leq 8$ and $n = 10$.

Furthermore, by the slide labeling of T_{10} (namely the one of Fig. 9), we can obtain that $B'(T_n) \leq 3n - 5$ for $n \geq 10$. In fact, we partition $E(T_n)$ into n subsets E_1, E_2, \dots, E_n , where $|E_1| = 3, |E_2| = 9, \dots, |E_{\lfloor n/2 \rfloor}| = 3\lfloor n/2 \rfloor + \lfloor n/2 \rfloor, |E_{\lfloor n/2 \rfloor + 1}| = 2n, |E_{\lfloor n/2 \rfloor + 2}| = 3(n - 1), |E_{\lfloor n/2 \rfloor + 3}| = 3(n - 2), \dots, |E_n| = 3(\lfloor n/2 \rfloor + 1) + \lfloor n/2 \rfloor$. Note that $\max_{1 \leq i \leq n} |E_i| = |E_{\lfloor n/2 \rfloor + 2}| = 3(n - 1)$. As in the case of T_{10} , we take out two edges ($b_{\lfloor n/2 \rfloor + 1, 2}$ and $c_{\lfloor n/2 \rfloor, 2}$) from $E_{\lfloor n/2 \rfloor + 2}$ and add them to $E_{\lfloor n/2 \rfloor + 1}$. Then the maximum size of the level sets E_i is $3(n - 1) - 2 = 3n - 5$.

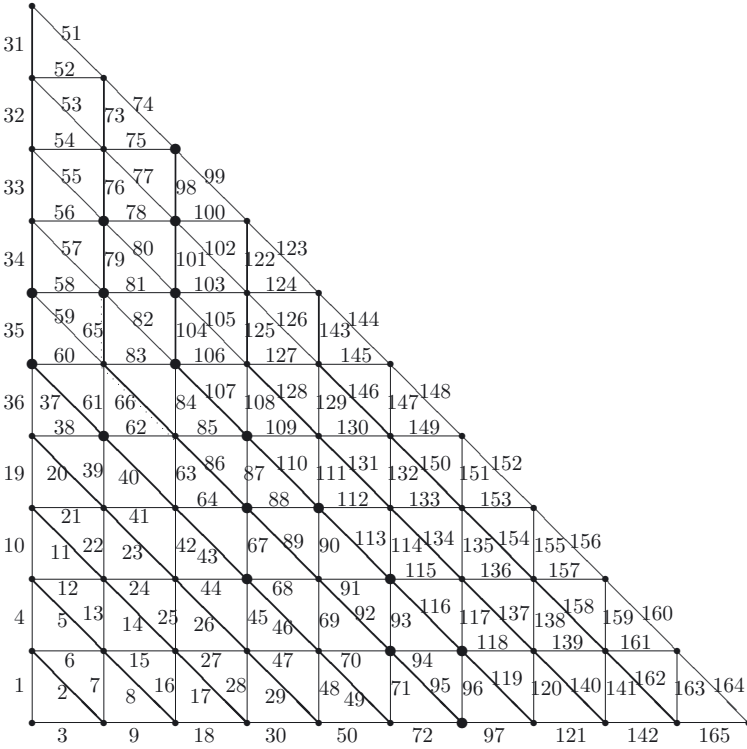


FIGURE 9. Slide labeling for T_{10} .

When we label the edges level by level, the labeling process is similar to that of Figure 9. Thus we obtain the following.

Proposition 4.3. $B'(T_n) \leq 3n - 5$ for $n \geq 10$.

This improves the claim of [1] that $B'(T_n) \leq 3n - 5$ when $n \geq 18$.

Until now, we are left to consider the case of $n = 9$. In fact, we can show the following.

Proposition 4.4. $B'(T_n) \leq 3n - \lfloor n/2 \rfloor$ for $9 \leq n \leq 11$.

Proof. This is true for $n = 10$ since $\lfloor n/2 \rfloor = \lceil n/2 \rceil$ for even n . For $n = 9$ and $B'(T_9) \leq 3n - 4 = 23$, the slide labeling is shown in Figure 10, which is similar to that of T_8 in Figure 8, where we take out one edge $(b_{\lfloor n/2 \rfloor + 1, 2})$ from E_6 and add it to E_5 . As to T_{11} with $B'(T_{11}) \leq 3n - 5 = 28$, the labeling has been described before Proposition 4.3, which is similar to that of T_{10} in Figure 9. \square

To summarize, we have the exact value $B'(T_n) = 3n - \lfloor n/2 \rfloor$ for $1 \leq n \leq 8$ (and $n = 10$) and a variant $3n - \lceil n/2 \rceil \leq B'(T_n) \leq 3n - \lfloor n/2 \rfloor$ for $9 \leq n \leq 11$.

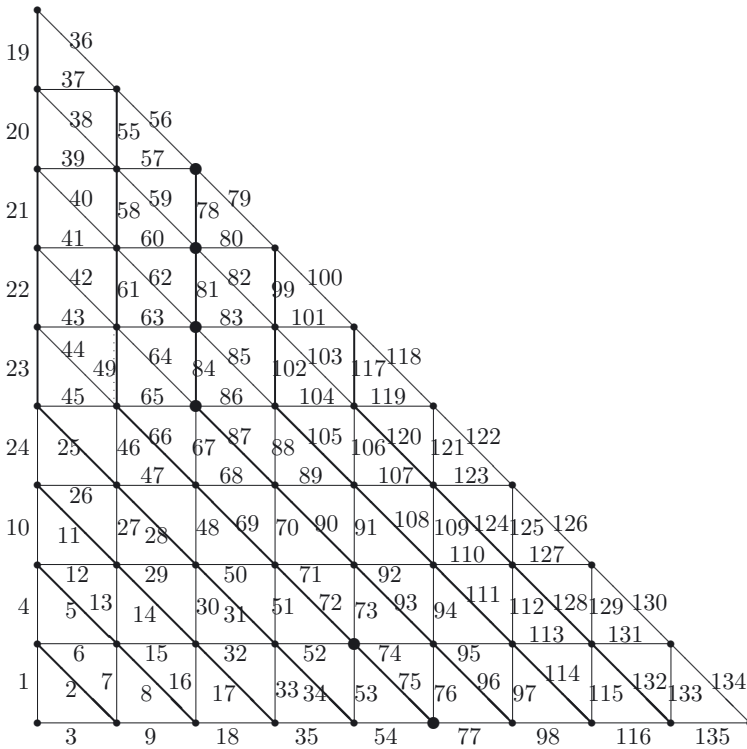


FIGURE 10. Slide labeling for T_9 .

It seems that the formula of edge-bandwidth $B'(T_n)$ would change gradually as n increases, by the action of other unknown tight bounds.

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