RAIRO-Theor. Inf. Appl. 43 (2009) 615–624
DOI: 10.1051/ita/2009005 Available online at:
www.rairo-ita.org

SQUARES AND CUBES IN STURMIAN SEQUENCES

Artūras Dubickas[1]

Abstract. We prove that every Sturmian word ω has infinitely many prefixes of the form $U_n V_n^3$, where $|U_n| < 2.855|V_n|$ and $\lim_{n\to\infty} |V_n| = \infty$. In passing, we give a very simple proof of the known fact that every Sturmian word begins in arbitrarily long squares.

Mathematics Subject Classification. 68R15.

1. Introduction

Let $\mathcal{A}$ be a finite alphabet of letters and let ω be an infinite sequence of elements from $\mathcal{A}$. Using the terminology of combinatorics on words, ω is called an infinite *word* over $\mathcal{A}$, any string of its consecutive letters is called its *factor*, and any factor of ω starting from the first letter of ω is called its *prefix*.

For every positive integer n, let $p(\omega, n)$ be the number of distinct factors of ω of length n. Obviously, $1 \leqslant p(\omega, n) \leqslant |\mathcal{A}|^n$ for each $n \geqslant 1$. By an old result of Morse and Hedlund [23], for any word ω over $\mathcal{A}$, the complexity function $p(\omega, n)$ is either bounded by an absolute constant independent of n (iff the word ω is ultimately periodic) or $p(\omega, n) \geqslant n+1$ for each $n \geqslant 1$. The words ω for which $p(\omega, n) = n+1$ for every $n \in \mathbb{N}$ exist and are called *Sturmian* words. Clearly, $p(\omega, 1) = 2$ implies that a Sturmian word ω must be an infinite word over an alphabet of two letters. It is well-known that the *Fibonacci word*

$$f = 0100101001001010010100100101001001\ldots,$$

which is the limit $f = \lim_{n\to\infty} f_n$ of the sequence of words $f_{-1} = 1, f_0 = 0$ and $f_{n+1} = f_n f_{n-1}$ for $n \geqslant 0$, is Sturmian. See a survey [9] for some extremal properties of the Fibonacci word. Sturmian sequences (also known as Beatty sequences) appear in symbolic dynamics, ergodic theory, number theory, computer graphics,

Keywords and phrases. Sturmian word, block-complexity, stammering word.

[1] Department of Mathematics and Informatics, Vilnius University, Naugarduko 24, Vilnius 03225, Lithuania; arturas.dubickas@mif.vu.lt

Article published by EDP Sciences

pattern recognition, crystallography, etc. See, for instance, [7,10,12,16,17,25,26]. For a more systematic exposition one can consult Chapter 2 in [20], Chapter 10 in [5] and also a collective book under pseudonym of Pytheas Fogg [24].

Given an infinite word ω and a finite factor w of ω, it is often important to know the highest power of w which appears as a factor of ω. Let $|w|$ be the length of the word w. Then, for any fixed real number $\tau > 0$, the τth power of a finite word w is the word of length $\lceil \tau|w|\rceil$ given by $w^\tau = w^{\lfloor \tau \rfloor} u$, where u is the prefix of w of length $\lceil (\tau - \lfloor \tau \rfloor)|w|\rceil$. For example, $01001^{2.1} = 01001010010$. Let τ_n be the supremum taken over $\tau \geqslant 1$ such that w^τ is a factor of ω for at least one factor w of ω satisfying $|w| = n$. (It is possible that $\tau_n = \infty$ for some fixed $n \in \mathbb{N}$.) Then the quantity $\limsup_{n\to\infty} \tau_n$ is called the *index* of ω. It is known that the index of every Sturmian word is at least 3 (see [6,22,26] or Chap. 2 in [20]). On the other hand, by Theorem 1.2 of [8], there exist Sturmian words with index equal to 3. The index of ω is often called a *critical exponent* of α and sometimes is defined as $\sup_{n\geqslant 1} \tau_n$. In the sense of this definition, it was shown recently that each number $\alpha > 1$ is a critical exponent of some infinite word [19] and that each number $\alpha > 2$ is a critical exponent of some infinite word over an alphabet of two letters [11].

For some applications, it is important not only to know whether a word ω has a finite or infinite index and how large this index (or critical exponent) is, but one also needs to determine how far from the beginning of the word ω a non-trivial power w^τ with $\tau > 1$ occurs. For example, the fact that a non-trivial power of a longer and longer word occurs not far from the beginning of an infinite word is crucial in [1]. It is proved there that if α is a Pisot number or a Salem number and $\omega = (d_k)_{k\geqslant 1}$ is a bounded sequence of integers, which is stammering (see the definition below), then the number $\sum_{k=1}^{\infty} d_k \alpha^{-k}$ either belongs to the field $\mathbb{Q}(\alpha)$ or is transcendental. (See also [15] for earlier work and [3] for subsequent work related to this old problem of digit distribution of an irrational algebraic number in base $b \geqslant 2$.) It is remarked in [1] that if α is an arbitrary algebraic number then for the same conclusion a somewhat stronger condition on the word ω is required. The paper [14] related to an unsolved Mahler's problem [21] about the powers of $3/2$ modulo 1 is another example where this kind of information is necessary for Sturmian words ω. More precisely, in [14] one needs to estimate the smallest value of the supremum $\sup_{\sigma\geqslant 0,\ \tau\geqslant 2} \frac{\tau+\sigma}{1+\sigma}$ taken over all Sturmian words ω, where ω has infinitely many prefixes of the form uv^τ, with $|u| \leqslant \sigma|v|$.

Let σ and τ be two real numbers satisfying $0 \leqslant \sigma < \infty$ and $\tau > 1$. Motivated by [1] (see also [3]), we say that an infinite word (sequence) ω over an alphabet $\mathcal{A}$ is a (σ,τ)-*stammering word* (or a (σ,τ)-*stammering sequence*) if there exist two sequences of finite words $(U_n)_{n\geqslant 1}$ and $(V_n)_{n\geqslant 1}$ over $\mathcal{A}$ such that

(i) for any $n \geqslant 1$ the word $U_n V_n^\tau$ is a prefix of ω;
(ii) $|U_n| \leqslant \sigma|V_n|$ for every $n \geqslant 1$;
(iii) $|V_n| \to \infty$ as $n \to \infty$.

By the definition given in [1], a word ω is called a *stammering word* if it is a (σ,τ)-stammering word for some fixed pair (σ,τ), where $0 \leqslant \sigma < \infty$ and $\tau > 1$. We remark that in terms of our definition it is proved in [1] that if for a word ω

there is an integer $t \geqslant 2$ such that $p(\omega, n) \leqslant tn$ for infinitely many $n \in \mathbb{N}$ then ω is a $(4t, 1 + 1/t)$-stammering word.

Theorem 1. *Every Sturmian word is a* $(0, 2)$*-stammering word.*

Theorem 1 is known. See, *e.g.*, [4] or [13] for two different proofs. In general, the constant 2 cannot be replaced by $2 + \varepsilon$ with $\varepsilon > 0$ (see Thm. 1.1 in [8]). We give the proof of Theorem 1 in just few lines (after some preliminaries in Sect. 2).

The main result of this paper is the following:

Theorem 2. *Every Sturmian word is a* $(2.855, 3)$*-stammering word.*

In the proof of Theorem 2 we do not use the concepts of the *slope* α, where α in an irrational number satisfying $0 < \alpha < 1$, and the *intercept* ϱ of the Sturmian word ω, whose nth symbol over the alphabet $\{0, 1\}$ is given as the difference $\lfloor \alpha(n+1)+\varrho \rfloor - \lfloor \alpha n+\varrho \rfloor$ (see [23] or Chap. 2 in [20]). Since we need some information on the prefix of a Sturmian word ω before a factor that is a cube occurs, the problem cannot be reduced to the study of characteristic Sturmian word (*i.e.*, $\varrho = 0$) with the same slope and then observing that the word ω has the same factors as the corresponding characteristic word (as is usually done).

The proof of Theorem 2 is completely self-contained. The only simple fact we use in the preliminary Section 2 is that the word ω over an alphabet $\{a, b\}$ is Sturmian if and only if ω is aperiodic and for every finite (possibly empty) factor w of ω at most one of the words awa and bwb is the factor of ω (see, *e.g.*, Prop. 2.1.3 and Thm. 2.1.5 in [20]).

2. Sturmian words

Lemma 3. *Let* ω *be a Sturmian word over* $\{a, b\}$ *that starts with the letter* a*. Then there is a unique integer* $k \geqslant 0$ *such that* ω *is composed of the blocks* $A = ab^{k+1}$ *and* $B = ab^k$ *only. The word* ω' *obtained from* ω *by replacing* ab^{k+1} *with* A *and* ab^k *with* B *is a Sturmian word over* $\{A, B\}$*.*

Proof. The word ω can be expressed in the form $ab^{k_1}ab^{k_2}ab^{k_3}\ldots$ with some integer $k_1, k_2, k_3, \ldots \geqslant 0$. Let $k = \min\{k_1, k_2, k_3, \ldots\}$. Note that b^{k+2} cannot be a factor of ω, because then both $ab^k a$ and b^{k+2} would be factors of ω, a contradiction. So ω is composed of the blocks $B = ab^k$ and $A = ab^{k+1}$ only.

Consider the word ω' over $\{A, B\}$ obtained from ω. Clearly, ω' is aperiodic. If it is not Sturmian then there exists a word X over $\{A, B\}$ such that AXA and BXB are factors of ω'. Thus either $BXBB$ or $BXBA$ is a factor of ω'. In both cases, for some word Y over $\{a, b\}$ obtained from X by replacing A by ab^{k+1} and B by ab^k, the words $b^{k+1}Yab^{k+1} = bb^kYab^kb$ and ab^kYab^ka are factors of ω, a contradiction. □

We say that ω' is the *block-word* of the Sturmian word ω. Lemma 3 also follows from a more general result of Justin and Vuillon [18] (see also [27]).

Theorem 4. *Let $\omega = \omega_0$ be a Sturmian word over $\{A_0, B_0\}$ and let $(\omega_k)_{k\geqslant 1}$ be a sequence of words such that each ω_k is the block-word of ω_{k-1}. Then there is a unique sequence of integers $s_1, s_2, s_3, \ldots \geqslant 0$ such that ω_k is a Sturmian word over the alphabet $\{A_k, B_k\}$, where*

$$A_k = U_{k-1}V_{k-1}^{s_k+1}, \;\; B_k = U_{k-1}V_{k-1}^{s_k} \quad \textit{with} \quad \{U_{k-1}, V_{k-1}\} = \{A_{k-1}, B_{k-1}\}$$

for each $k \geqslant 1$. In particular, B_k is a prefix of A_k for every $k \geqslant 1$, so $|A_k| > |B_k|$, where $|A_k|$ and $|B_k|$ denote the lengths of the words A_k, B_k in the alphabet $\{A_0, B_0\}$. Moreover, for infinitely many $k \in \mathbb{N}$, we have $|A_k| < 2|B_k|$. Finally, $|A_k|, |B_k| \to \infty$ as $k \to \infty$.

Proof. The sequence of Sturmian block-words $(\omega_k)_{k\geqslant 1}$ exists, by Lemma 3. If the first letter of ω_{k-1} is A_{k-1} then, by Lemma 3, $A_k = A_{k-1}B_{k-1}^{s_k+1}, B_k = A_{k-1}B_{k-1}^{s_k}$, where $s_k \geqslant 0$. Therefore,

$$\frac{|A_k|}{|B_k|} = \frac{|A_{k-1}| + (s_k+1)|B_{k-1}|}{|A_{k-1}| + s_k|B_{k-1}|} < 2.$$

Suppose the first letter of ω_{k-1} is B_{k-1} for all sufficiently large k. Then $A_k = B_{k-1}A_{k-1}^{s_k+1}$, $B_k = B_{k-1}A_{k-1}^{s_k}$. If $s_k \geqslant 1$ for infinitely many $k \in \mathbb{N}$ then, for those k, we have

$$\frac{|A_k|}{|B_k|} = \frac{|B_{k-1}| + (s_k+1)|A_{k-1}|}{|B_{k-1}| + s_k|A_{k-1}|} < 2.$$

Hence, in both cases, $|A_k| < 2|B_k|$ for infinitely many $k \in \mathbb{N}$.

Alternatively, there exists a positive integer t such that, firstly, the first letter of ω_{k-1} is B_{k-1} and, secondly, $A_k = B_{k-1}A_{k-1}, B_k = B_{k-1}$ for every $k \geqslant t$. We will show that this is impossible. Indeed, let $l \geqslant 1$ be an integer such that ω_{t-1} has a prefix $B_{t-1}^l A_{t-1}$. Then the words ω_k, where $k = t-1, \ldots, t+l-2$, begin with B_k (all equal to B_{t-1}). The word ω_{t+l-2} begins with $B_{t+l-2}A_{t+l-2}$. By our assumption, ω_{t+l-1} begins with B_{t+l-1}, hence $B_{t+l-1} = B_{t+l-2}A_{t+l-2}^{s_{t+l-1}}$ and $A_{t+l-1} = B_{t+l-2}A_{t+l-2}^{s_{t+l-1}+1}$ with some $s_{t+l-1} \geqslant 1$, a contradiction.

Finally, it is clear that $|A_k| \to \infty$ as $k \to \infty$. Furthermore, $|B_k| \to \infty$ as $k \to \infty$, because the sequence $(|B_k|)_{k\geqslant 0}$ is non-decreasing and, as we just proved, $|B_k| > |A_k|/2$ for infinitely many $k \in \mathbb{N}$. □

3. Proofs of Theorems 1 and 2

Proof of Theorem 1: Let k be a sufficiently large integer. If the word ω_k begins with the letter B_k then B_k^2 is a prefix of ω_k, because B_k is a prefix of A_k. Suppose that ω_k begins with A_k. Then, by Lemma 3, the word ω_k consists of the blocks $A_kB_k^{s+1}$ and $A_kB_k^s$ only, where $s = s_{k+1} \geqslant 0$. Clearly, $(A_kB_k^s)^2$ is a prefix of ω_k, unless ω_k begins with the block $A_kB_k^{s+1}$. However, if it begins with $A_kB_k^{s+1}$

then, independent on whether the second block is $A_kB_k^{s+1}$ or $A_kB_k^s$, the word ω_k begins with $(A_kB_k^{s+1})^2$, because B_k is a prefix of A_k. Since, by Theorem 4, $|A_k|, |B_k| \to \infty$ as $k \to \infty$, this proves that every Sturmian word ω begins in arbitrarily long squares. □

Proof of Theorem 2: Let k be any of those (infinitely many) k's for which $|B_k| < |A_k| < 2|B_k|$. For brevity, let us write A and B for A_k and B_k, respectively, so that $|B| < |A| < 2|B|$. Below, without further notice, we shall use the fact that B is a prefix of A.

By Lemma 3, the word $\omega = \omega_k$ consists either of the blocks AB^{s+1} and AB^s only or of the blocks BA^{s+1} and BA^s only, where $s = s_{k+1} \geqslant 0$. Suppose first that we have the blocks AB^{s+1} and AB^s, where $s \geqslant 2$. Then AB^3 is a prefix of this word, because B is a prefix of A. Also, $|A| < 2|B|$. So if there are infinitely many such cases then, by Theorem 4 claiming that $|B_k| \to \infty$ as $k \to \infty$, ω is a $(2, 3)$-stammering word, which is more than required. Another simple case is when we have the blocks BA^{s+1} and BA^s, where $s \geqslant 3$, only. Then BA^3 is a prefix of this word and $|B| < |A|$. So if there are infinitely many such cases then ω is a $(1, 3)$-stammering word, which is more than required.

We claim that in the remaining cases, listed in the table below, we have either a cube occurring as a prefix of ω (in which case ω is a $(0, 3)$-stammering word) or ω has one of the prefixes listed in the third column of the table. Note that each prefix there has the form UV^3, where U and V are some words over the alphabet $\{A, B\}$. The maximal value of the quotient $|U|/|V|$ is given in the last column of the table. For each UV^3, the upper bound for the constant $|U|/|V|$ is calculated using the inequality $|B| < |A| < 2|B|$.

1	AB^2, AB	$AB^3, A(BAB)^3, (AB)^2(BA)^3, (AB)^2BA(BAB)^3$	9/4
$2a$	AB, A	$A(AB)^3, A^2BA(AB)^3, ABA^3, ABA(AB)^3$, or case 1	7/3
$2b$	AB, A	A^2BA^3	3
3	BA^3, BA^2	$BA^3, B(A^2BA)^3, BA^2BA(A^2B)^3$	5/3
4	BA^2, BA	$B(ABA)^3, (BA)^2(AB)^3, BA(AB)^3, BA^2BA(AB)^3$	8/3
$5a$	BA, B	$BA(BAB)^3, (BA)^2B(BA)^3, (BA)^2BBA(BAB)^3$	5/2
$5b$	BA, B	BAB^3	3

We begin with case 1, when ω consists of the blocks AB^2 and AB. If the first block is AB^2 then AB^3 is a prefix of ω. It is one of the values listed in the third column of the first row. Suppose that AB is the first block. If the next block is AB again then ω begins with $(AB)^3$, which is a cube. Alternatively, the next block is AB^2, so ω has one of the two prefixes $ABAB^2AB$ or $ABAB^2AB^2$. In the latter case, independent of the third block, $A(BAB)^3$ a prefix of ω (which is in the table). Suppose that the prefix is $ABAB^2AB$. If the next block is AB then $(AB)^2(BA)^3$ is a prefix of ω. Let AB^2 be the next block. Then two possibilities are $ABAB^2ABAB^2AB$ and $ABAB^2ABAB^2AB^2$. The first possibility gives the prefix $(ABAB^2)^3$ which is a cube, whereas the second possibility gives $(AB)^2BA(BAB)^3$.

From $|B| < |A| < 2|B|$, we find that the quotients

$$\frac{|A|}{|B|}, \frac{|A|}{|A|+2|B|}, \frac{2|A|+2|B|}{|A|+|B|}, \frac{3|A|+3|B|}{|A|+2|B|},$$

are all smaller than 9/4.

Consider case 2 when ω consists of the blocks AB and A. If the word ω begins with AA then it begins with a cube A^3. Similarly, if ω begins with $ABAB$ then a cube $(AB)^3$ is a prefix of ω. So there are two possibilities AAB or ABA. As above it is easy to see that $AABAB$ gives $A(AB)^3$, which is one of the prefixes in the corresponding row. Otherwise, $AABA$ splits into $AABAA$ (which gives the prefix A^2BA^3) and $AABAAB$. Here, the next block A leads to $(A^2B)^3$. Assume that the next block is AB. Then the prefix $AABAABAB$ leads to the prefix $A^2BA(AB)^3$. The second possibility ABA gives ABA^3 if the next block is A. Otherwise, we have the following two cases $ABAABAB$ (which leads to $ABA(AB)^3$) and $ABAABA$. In the latter case, the next AB leads to the cube $(ABA)^3$, whereas the next A gives $ABAABAA$. Although this leads to $ABAABA^3$, we do not stop here, because the prefix $ABAAB$ before the cube A^3 occurs is too large. Instead, since ω_k begins with $(AB)A(AB)A^2 = (A_kB_k)A_k(A_kB_k)A_k^2$, we observe that, by Theorem 4, the word ω_{k+1} consists of the blocks $A_{k+1} = AB$ and $B_{k+1} = A$ only. Its prefix in the alphabet $\{A_{k+1}, B_{k+1}\}$ is $A_{k+1}B_{k+1}A_{k+1}B_{k+1}^2$. Here,

$$|A_{k+1}|/|B_{k+1}| = (|A|+|B|)/|A| \in (3/2, 2) \subset (1, 2),$$

so $|B_{k+1}| < |A_{k+1}| < 2|B_{k+1}|$ and we are back to the case 1 for the word ω_{k+1} instead of ω_k. Now, since $|B| < |A| < 2|B|$, the quotients

$$\frac{|A|}{|A|+|B|}, \frac{3|A|+|B|}{|A|+|B|}, \frac{|A|+|B|}{|A|}, \frac{2|A|+|B|}{|A|+|B|}$$

(calculated for $A(AB)^3, A^2BA(AB)^3, ABA^3, ABA(AB)^3$, respectively) are all smaller than 7/3. For the prefix A^2BA^3 the quotient $(2|A|+|B|)/|A|$ is at most 3. This is greater than 2.855, so we split case 2 into two subcases $2a$ and $2b$. The subcase $2b$ will be analyzed later.

Consider case 3 when ω consists of the blocks BA^3 and BA^2. The first block BA^3 is the first prefix of the third row. If BA^2 is followed by BA^2 then ω starts with $(BA^2)^3$. So the first two blocks are BA^2 and BA^3, giving BA^2BA^3. If the next block is BA^3 then ω begins with $B(A^2BA)^3$. The alternative case leads to $BA^2BA^3BA^2$. Independent of the fourth block, this leads to the prefix $BA^2BA(A^2B)^3$. This time,

$$\max\left(\frac{|B|}{|A|}, \frac{|B|}{3|A|+|B|}, \frac{3|A|+2|B|}{2|A|+|B|}\right) < \frac{5}{3}.$$

In case 4 we have the blocks BA^2 and BA. If the first two blocks are BA and BA then ω begins with a cube $(BA)^3$. Suppose ω begins with $BABA^2$. The next block

BA^2 leads to the prefix $B(ABA)^3$, whereas the next block BA gives $BABA^2BA$, which leads to $(BA)^2(AB)^3$. Next, let us consider the beginning BA^2BA. Independent of the next block, this leads to the prefix $(BA)(AB)^3$. The remaining case is BA^2BA^2. If ω does not begin with a cube, the next block must be BA. The beginning BA^2BA^2BA leads to the prefix $BA^2BA(AB)^3$. We have

$$\max\left(\frac{|B|}{2|A|+|B|}, \frac{2|A|+2|B|}{|A|+|B|}, \frac{|A|+|B|}{|A|+|B|}, \frac{3|A|+2|B|}{|A|+|B|}\right) < \frac{8}{3}.$$

Finally, in case 5, we have the blocks BA and B. Both BB and BBA lead to the prefix B^3. The beginning BAB leads to the prefix BAB^3. Let the first two blocks be BA and BA. If ω_k does not start in a cube the next block must be B. Since, by Theorem 4, the sequence ω_{k+1} is Sturmian, the fourth block must be BA, *i.e.*, we have $BABABBA$. By the same argument, if the next block is B, it must be followed by BA. The prefix $BABABBABBA$ leads to $BA(BAB)^3$. Now suppose that the fifth block is BA, *i.e.*, we have $(BA)^2B(BA)^2$. In case the sixth block is BA, we obtain $(BA)^2B(BA)^3$. Otherwise, if the sixth block is B, we get $(BA)^2B(BA)^2B$. Seventh block must be BA again. If the eight block is BA then ω begins in a cube, so suppose that the eighth block is B. Then by the above argument it must be followed by BA, giving $(BA)^2B(BA)^2BBABBA$. This leads to the prefix $(BA)^2BBA(BAB)^3$. Now, from $|B| < |A| < 2|B|$, we obtain

$$\max\left(\frac{|A|+|B|}{|A|+2|B|}, \frac{2|A|+3|B|}{|A|+|B|}, \frac{3|A|+4|B|}{|A|+2|B|}\right) < \frac{5}{2}.$$

For the prefix BAB^3 the quotient $(|A|+|B|)/|A|$ is at most 3. Since this is greater than 2.855, we split case 5 into two subcases $5a$ and $5b$.

This would finish the proof of the theorem with even better constant 8/3, unless for each sufficiently large k in the word ω_k with $|B_k| < |A_k| < 2|B_k|$ we have either case $2b$ or case $5b$. Indeed, then the cases $1, 2a, 3, 4, 5a$ show that the word ω has infinitely many prefixes of the form $U_nV_n^3$ with $|U_n| < 8|V_n|/3$ and $\lim_{n\to\infty}|V_n| = \infty$.

To complete the proof assume that there is a k_0 such that for each $k \geqslant k_0$ satisfying $1 < q_k := |A_k|/|B_k| < 2$ the word $A_k^2B_kA_k^3$ is a prefix of the word ω_k consisting of the blocks A_kB_k and A_k (case $2b$) or $B_kA_kB_k^3$ is a prefix of ω_k consisting of the blocks B_kA_k and B_k (case $5b$).

Let $\delta = (3\sqrt{5}-5)/10 = 0.17082\ldots$ be the root of

$$\delta^2 + \delta = 1/5.$$

If there are infinitely many k's for which we have case $2b$ and $q_k \geqslant 1+\delta$, then the proof is completed, because $A_k^2B_kA_k^3$ is a prefix of ω_k and

$$(2|A_k| + |B_k|)/|A_k| = 2 + 1/q_k \leqslant 2 + 1/(1+\delta) = 2 + 5\delta < 2.855$$

for each such k. Similarly, if there are infinitely many k's for which we have case $5b$ and $q_k \leqslant 1 + 5\delta < 1.855$, then the proof is also completed, because $B_kA_kB_k^3$ is a prefix of ω_k and

$$(|A_k| + |B_k|)/|B_k| = 1 + q_k \leqslant 2 + 5\delta < 2.855$$

for each such k. So we can assume that $q_k < 1 + \delta$ in case $2b$ and $q_k > 1 + 5\delta$ in case $5b$. In particular, no $k \geqslant k_0$ exists for which

$$1 + \delta \leqslant q_k \leqslant 1 + 5\delta.$$

Clearly, in case $2b$ the word ω_k is composed of the blocks $A_{k+1} = A_k B_k$ and $B_{k+1} = A_k$, so for the next word ω_{k+1} using $1 < q_k < 1 + \delta$ we obtain

$$q_{k+1} = |A_{k+1}|/|B_{k+1}| = 1 + |B_k|/|A_k| = 1 + 1/q_k \in (1 + 5\delta, 2).$$

Consequently, the word ω_{k+1} satisfies the condition $5b$, namely, ω_{k+1} consists of the blocks $A_{k+2} = B_{k+1}A_{k+1}$ and $B_{k+2} = B_{k+1}$ and one of its prefixes must be $B_{k+1}A_{k+1}B_{k+1}^3$. By Lemma 3, the next block-word consists of the blocks

$$A_{k+3} = B_{k+1}A_{k+1}B_{k+1}^{s+1} \quad \text{and} \quad B_{k+3} = B_{k+1}A_{k+1}B_{k+1}^{s}$$

for some integer $s \geqslant 2$. If $s \geqslant 4$, then $B_{k+1}A_{k+1}(B_{k+1}^2)^3$ is a prefix of ω. So the bound

$$\frac{|A_{k+1}| + |B_{k+1}|}{2|B_{k+1}|} = \frac{1}{2} + \frac{q_{k+1}}{2} < \frac{3}{2} = 1.5 < 2.855$$

gives the required estimate. Otherwise, let $2 \leqslant s \leqslant 3$. Then using $q_{k+1} = 1 + 1/q_k > 1 + 1/(1+\delta) = 1 + 5\delta$ we obtain

$$q_{k+3} = \frac{|A_{k+1}| + (s+2)|B_{k+1}|}{|A_{k+1}| + (s+1)|B_{k+1}|} = \frac{q_{k+1} + s + 2}{q_{k+1} + s + 1} \geqslant \frac{q_{k+1} + 5}{q_{k+1} + 4} > \frac{6 + 5\delta}{5 + 5\delta} = 1 + \delta$$

and

$$q_{k+3} = \frac{q_{k+1} + s + 2}{q_{k+1} + s + 1} = 1 + \frac{1}{q_{k+1} + s + 1} < 1.25.$$

It follows that for some $k \geqslant k_0$ we have $q_k \in [1 + \delta, 1.25] \subset [1 + \delta, 1 + 5\delta]$, a contradiction. This completes the proof of the theorem. □

In fact, we proved Theorem 2 with the constant

$$2 + 5\delta = \frac{3\sqrt{5} - 1}{2} = 2.8541\ldots$$

which is slightly smaller than 2.855.

4. Concluding remarks

We already observed in Section 1 that the constant 3 of Theorem 2 is optimal. More precisely, for every $\varepsilon > 0$, there exists a Sturmian word which is not a $(\sigma, 3+\varepsilon)$-stammering word for every $\sigma \geqslant 0$. The constant 2.855 in Theorem 2 is not optimal! By some further analysis of different prefixes that can occur as prefixes of a Sturmian word ω before a cube this constant can be reduced. We do not know the best possible constant. However, one can show that the Fibonacci word is a $((\sqrt{5}+1)/2, 3)$-stammering word but is not a $((\sqrt{5}+1)/2-\varepsilon, 3)$-stammering word for every positive number ε.

Given any $\tau \leqslant 3$, let $\sigma(\tau)$ be the infimum over all $\sigma \geqslant 0$ such that every Sturmian word is a (σ, τ)-stammering word. By Theorem 1, $\sigma(\tau) = 0$ for $\tau \leqslant 2$. Theorem 2 combined with the above observation implies that $1.618 < \sigma(3) < 2.855$.

Problem 1. *Evaluate $\sigma(\tau)$ for each $\tau \in (2, 3]$.*

One can also consider a similar problem if τ is not fixed. Following [2], we say that an infinite word (sequence) ω over an alphabet $\mathcal{A}$ satisfies Condition $(*)_\varrho$ if there exist two sequences of finite words $(U_n)_{n\geqslant 1}$ and $(V_n)_{n\geqslant 1}$ over $\mathcal{A}$ and a sequence of positive real numbers $(\tau_n)_{n\geqslant 1}$ such that

(i) for any $n \geqslant 1$ the word $U_n V_n^{\tau_n}$ is a prefix of ω;
(ii) $|U_n V_n^{\tau_n}| \geqslant \varrho |U_n V_n|$ for every $n \geqslant 1$;
(iii) $|V_n^{\tau_n}| \to \infty$ as $n \to \infty$.

Then the *Diophantine exponent* of ω, $\mathrm{Dio}(\omega)$, is defined as the supremum of the real numbers ϱ for which ω satisfies Condition $(*)_\varrho$.

Problem 2. *Evaluate $D(S) := \inf_{\omega - Sturmian} Dio(\omega)$.*

Obviously, if some word is a (σ, τ)-stammering word for a fixed pair (σ, τ) then it satisfies Condition $(*)_\varrho$ for $\varrho = (\sigma + \tau)/(\sigma + 1)$. Hence

$$D(S) \geqslant \sup_{\tau \in [2,3]} \frac{\sigma(\tau) + \tau}{\sigma(\tau) + 1}.$$

Selecting $\tau = 2$ we obtain $D(S) \geqslant 2$. We do not know whether $D(S) = 2$ or $D(S) > 2$. The inequality $D(S) > 2$ (if proved) has some applications to Mahler's problem: one can use the same method as in [14].

Acknowledgements. I thank both referees for very careful reading of the manuscript and for some useful references. This research was partially supported by the Lithuanian State Science and Studies Foundation.

References

[1] B. Adamczewski and Y. Bugeaud, On the complexity of algebraic numbers I. Expansion in integer bases. *Ann. Math.* **165** (2007) 547–565.
[2] B. Adamczewski and Y. Bugeaud, Dynamics for β-shifts and Diophantine approximation. *Ergod. Theory Dyn. Syst.* **27** (2007) 1695–1711.
[3] B. Adamczewski and N. Rampersad, On patterns occuring in binary algebraic numbers. *Proc. Amer. Math. Soc.* **136** (2008) 3105–3109.

[4] J.-P. Allouche, J.P. Davison, M. Queffélec and L.Q. Zamboni, Transcendence of Sturmian or morphic continued fractions. *J. Number Theory* **91** (2001) 39–66.
[5] J.-P. Allouche and J. Shallit, *Automatic sequences, Theory, applications, generalizations*. CUP, Cambridge (2003).
[6] J. Berstel, *On the index of Sturmian words*. In *Jewels are Forever, Contributions on theoretical computer science in honor of Arto Salomaa*, J. Karhumäki et al., eds. Springer, Berlin (1999) 287–294.
[7] J. Berstel and J. Karhumäki, *Combinatorics on words – a tutorial*, in *Current trends in theoretical computer science, The challenge of the new century,* Vol. 2*, Formal models and semantics*, G. Paun, G. Rozenberg, A. Salomaa, eds. World Scientific, River Edge, NJ (2004) 415–475.
[8] V. Berthé, C. Holton and L.Q. Zamboni, Initial powers of Sturmian sequences. *Acta Arith.* **122** (2006) 315–347.
[9] J. Cassaigne, On extremal properties of the Fibonacci word. *RAIRO-Theor. Inf. Appl.* **42** (2008) 701–715.
[10] E. Coven and G. Hedlund, Sequences with minimal block growth. *Math. Syst. Theor.* **7** (1973) 138–153.
[11] J.D. Currie and N. Rampersad, For each $\alpha > 2$ there is an infinite binary word with critical exponent α, *Electron. J. Combin.* **15** (2008) 5 p.
[12] A. De Luca, Sturmian words: structure, combinatorics and their arithmetics. *Theoret. Comput. Sci.* **183** (1997) 45–82.
[13] D. Damanik, R. Killip and D. Lenz, Uniform spectral properties of one-dimensional quasicrystals, III. α-continuity. *Commun. Math. Phys.* **212** (2000) 191–204.
[14] A. Dubickas, Powers of a rational number modulo 1 cannot lie in a small interval (to appear).
[15] S. Ferenczi and C. Mauduit, Transcendence of numbers with low complexity expansion. *J. Number Theory* **67** (1997) 146–161.
[16] A.S. Fraenkel, M. Mushkin and U. Tassa, Determination of $\lfloor n\theta \rfloor$ by its sequence of differences. *Canad. Math. Bull.* **21** (1978) 441–446.
[17] S. Ito and S. Yasutomi, On continued fractions, substitutions and characteristic sequences. *Jpn J. Math.* **16** (1990) 287–306.
[18] J. Justin and L. Vuillon, Return words in Sturmian and episturmian words. *RAIRO-Theor. Inf. Appl.* **34** (2000) 343–356.
[19] D. Krieger and J. Shallit, Every real number greater than 1 is a critical exponent. *Theoret. Comput. Sci.* **381** (2007) 177–182.
[20] M. Lothaire, *Algebraic combinatorics on words,* Encyclopedia of Mathematics and Its Applications, Vol. 90. CUP, Cambridge (2002).
[21] K. Mahler, An unsolved problem on the powers of 3/2. *J. Austral. Math. Soc.* **8** (1968) 313–321.
[22] F. Mignosi, On the number of factors of Sturmian words. *Theoret. Comput. Sci.* **82** (1991) 71–84.
[23] M. Morse and G.A. Hedlund, Symbolic dynamics II: Sturmian sequences. *Amer. J. Math.* **62** (1940) 1–42.
[24] N. Pytheas Fogg, Substitutions in dynamics, arithmetics and combinatorics. *Lect. Notes Math.* **1794** (2002).
[25] K.B. Stolarsky, Beatty sequences, continued fractions, and certain shift operators. *Canad. Math. Bull.* **19** (1976) 473–482.
[26] D. Vandeth, Sturmian words and words with a critical exponent. *Theoret. Comput. Sci.* **242** (2000) 283–300.
[27] L. Vuillon, A characterization of Sturmian words by return words. *Eur. J. Combin.* **22** (2001) 263–275.

Communicated by J. Berstel.
Received December 1st, 2008. Accepted February 5, 2009.