ESAIM: COCV 14 (2008) 211–232
DOI: 10.1051/cocv:2007049

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

ON LIPSCHITZ TRUNCATIONS OF SOBOLEV FUNCTIONS (WITH VARIABLE EXPONENT) AND THEIR SELECTED APPLICATIONS

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Abstract. We study properties of Lipschitz truncations of Sobolev functions with constant and variable exponent. As non-trivial applications we use the Lipschitz truncations to provide a simplified proof of an existence result for incompressible power-law like fluids presented in [Frehse *et al., SIAM J. Math. Anal.* **34** (2003) 1064–1083]. We also establish new existence results to a class of incompressible electro-rheological fluids.

Mathematics Subject Classification. 35J55, 35J65, 35J70, 35Q35, 76D99

Received April 14, 2006. Revised August 25, 2006.
Published online September 21, 2007.

1. Introduction

Let λ be a large positive number, $p \geq 1$. Sobolev-functions from $W_0^{1,p}$ can be approximated by λ-Lipschitz functions that coincide with the originals up to sets of small Lebesgue measure. The Lebesgue measure of these non-coincidence sets is bounded by the Lebesgue measure of the sets where the Hardy-Littlewood maximal function of the gradients are above λ. See for example [3,19,23,26,29,38].

Lipschitz truncations of Sobolev functions are used in various areas of analysis in different aspects. To name a few, we refer to the articles with applications in the calculus of variations [1,19,20,27,35,36], in the existence theory of partial differential equations [13,18,23,34,37] and in the regularity theory [2,14].

The purpose of this article is four-fold. First of all, in Section 2 we recall, survey, and strengthen properties of $W_0^{1,\infty}$-truncations of $W_0^{1,p}$-functions that are useful from the point of view of the existence theory concerning nonlinear PDE's. We illustrate the potential of this tool by establishing the weak stability for the system of p-Laplace equations with very general right-hand sides.

Then, in Section 3 we exploit Lipschitz truncations in the analysis of steady flows of generalized power-law fluids. In this case we reprove in a simplified way the existence results established in [18].

Keywords and phrases. Lipschitz truncation of $W_0^{1,p}/W_0^{1,p(\cdot)}$-functions, existence, weak solution, incompressible fluid, power-law fluid, electro-rheological fluid.

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Supported by the Czech Science Foundation, the project GAČR 201/03/0934, and by MSMT, the project MSM 0021620839.
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Supported by SFB 611 at the University of Bonn.

Article published by EDP Sciences

Next, in order to apply this method to a class of electro-rheological fluids which are characterized by power-law index varying with the spatial variables we extend the Lipschitz truncation method to Sobolev functions of variable exponents $W^{1,p(\cdot)}$. The properties of Lipschitz truncations are presented in Section 4.

Finally, we establish new existence results to an electro-rheological fluid model in Section 5.

We wish to mention that our main interest in investigating properties of Lipschitz truncations of Sobolev functions comes from studies of equations describing flows of certain incompressible fluids. In order to explain how the properties of Lipschitz truncations can be used in the analysis of nonlinear partial differential equations to those readers who are not familiar with (or not interested in) analysis of generalized incompressible Navier-Stokes equations we decided to consider first the following problem: for a given vector field $\mathbf{F} = (F_1, \ldots, F_d)$, to find $\mathbf{v} = (v_1, \ldots, v_d)$ solving[1]

$$\begin{aligned} -\operatorname{div}\big(|\mathbf{D}\mathbf{v}|^{p-2}\mathbf{D}\mathbf{v}\big) &= \mathbf{F} \qquad \text{in } \Omega \subset \mathbb{R}^d, \\ \mathbf{v} &= \mathbf{0} \qquad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

Here Ω is a bounded domain with Lipschitz boundary, $p > 1$ and $\mathbf{D}\mathbf{v}$ denotes either the gradient of $\mathbf{v}$ or its symmetric part.

If $p \neq 2$, (1.1) represents a non-linear problem. A key issue in the proof of the existence of a weak solution to (1.1) is the stability of weak solutions with respect to weak convergence. This property, called *weak stability of* (1.1), can be made more precise in the following way: assume that we have $\mathbf{v}^n$ enjoying the properties

$$\int_\Omega |\mathbf{D}\mathbf{v}^n|^{p-2}\mathbf{D}\mathbf{v}^n \cdot \mathbf{D}\varphi \,\mathrm{d}x = \langle \mathbf{F}^n, \varphi\rangle \quad \text{for all suitable } \varphi, \tag{1.2}$$

and

$$\begin{aligned} \int_\Omega |\mathbf{D}\mathbf{v}^n|^p \,\mathrm{d}x \le K < \infty \quad &\text{for all } n \in \mathbb{N}, \\ \langle \mathbf{F}^n, \varphi\rangle \to \langle \mathbf{F}, \varphi\rangle \quad &\text{for all suitable } \varphi. \end{aligned} \tag{1.3}$$

The uniform estimate $(1.3)_1$ implies (modulo a suitably taken subsequence) that

$$\mathbf{v}^n \rightharpoonup \mathbf{v} \qquad \text{weakly in } W_0^{1,p}(\Omega)^d. \tag{1.4}$$

If $\mathbf{v}$ is also a weak solution to (1.1) then we say that system (1.1) posseses the weak stability property.

Setting $\mathbf{T}(\mathbf{B}) := |\mathbf{B}|^{p-2}\mathbf{B}$ ($p' = \frac{p}{p-1}$), we can reformulate our task differently. Noticing that for $p' = \frac{p}{p-1}$ the uniform bound $(1.3)_1$ implies that

$$\int_\Omega |\mathbf{T}(\mathbf{D}\mathbf{v}^n)|^{p'} \,\mathrm{d}x \le c(K), \tag{1.5}$$

we conclude that $\mathbf{T}(\mathbf{D}\mathbf{v}^n) \rightharpoonup \chi$ weakly in $L^{p'}(\Omega)^{d\times d}$ (at least for a subsequence). The weak stability of (1.1) is thus tantamount to show that $\mathbf{T}(\mathbf{D}\mathbf{v}) = \chi$.

To provide an affirmative answer to the issue of stability of weak solutions, it is enough to show that for a not relabeled subsequence

$$\limsup_{k\to\infty} \int_\Omega \big(\mathbf{T}(\mathbf{D}\mathbf{v}^n) - \mathbf{T}(\mathbf{D}\mathbf{v})\big) \cdot \mathbf{D}(\mathbf{v}^n - \mathbf{v}) \,\mathrm{d}x = 0. \tag{1.6}$$

[1]In (1.1) we could replace the p-Laplace operator by any p-coercive, strictly monotone operator of $(p-1)$-growth.

Indeed, knowing that $\mathbf{T}$ is strictly monotone, *i.e.*,

$$(\mathbf{T}(\zeta) - \mathbf{T}(\mathbf{z})) \cdot (\zeta - \mathbf{z}) > \mathbf{0} \qquad \text{for all } \zeta, \mathbf{z} \in \mathbb{R}^{\mathbf{d}\times\mathbf{d}} \quad (\zeta \neq \mathbf{z}),$$

one concludes from (1.6) that

$$\mathbf{D}\mathbf{v}^n \to \mathbf{D}\mathbf{v} \qquad \text{almost everywhere in } \Omega, \tag{1.7}$$

at least for a not relabeled subsequence. Vitali's theorem then completes the proof allowing to pass to the limit in the nonlinear term.

Note that (1.6) can be weakened, still giving (1.7), see also [4]. More precisely, to obtain (1.7) it is enough to show for some $0 < \theta \le 1$ that there is a not relabelled subsequence of $\{\mathbf{v}^n\}$ such that

$$\limsup_{n\to\infty} \int_\Omega \Big(\big(\mathbf{T}(\mathbf{D}\mathbf{v}^n) - \mathbf{T}(\mathbf{D}\mathbf{v})\big) \cdot \mathbf{D}(\mathbf{v}^n - \mathbf{v}) \Big)^\theta \,\mathrm{d}x = 0. \tag{1.8}$$

We distinguish two cases how to achieve (1.6), or (1.8) respectively.

Simple case. The problem is simply solvable if we assume that $\mathbf{F}^n, \mathbf{F} \in (W_0^{1,p}(\Omega)^d)^*$ and $\mathbf{F}^n \to \mathbf{F}$ strongly in $(W_0^{1,p}(\Omega)^d)^*$. In fact, to obtain (1.6), it is natural to take $\varphi = \mathbf{v}^n - \mathbf{v}$ in (1.2), which is a suitable test function (all terms are meaningful). Then we obtain, after subtracting the term $\int_\Omega \mathbf{T}(\mathbf{D}\mathbf{v}) \cdot \mathbf{D}(\mathbf{v}^n - \mathbf{v})\,\mathrm{d}x$ from both sides of the equation

$$\int_\Omega \big(\mathbf{T}(\mathbf{D}\mathbf{v}^n) - \mathbf{T}(\mathbf{D}\mathbf{v})\big) \cdot \mathbf{D}(\mathbf{v}^n - \mathbf{v})\,\mathrm{d}x = \langle \mathbf{F}^n, \mathbf{v}^n - \mathbf{v}\rangle - \int_\Omega \mathbf{T}(\mathbf{D}\mathbf{v}) \cdot \mathbf{D}(\mathbf{v}^n - \mathbf{v})\,\mathrm{d}x.$$

For $n \to \infty$, the right-hand side vanishes due to weak convergence of $\{\mathbf{v}^n\}$ and strong convergence of $\{\mathbf{F}^n\}$, and (1.6) follows.

Difficult case. More difficult and also more interesting is the case when

$$\mathbf{F}^n = \mathrm{div}\mathbf{G}^n \qquad \text{with } \mathbf{G}^n \to \mathbf{G} \text{ strongly } \in L^1(\Omega)^{d\times d}. \tag{1.9}$$

Then $\mathbf{u}^n := \mathbf{v}^n - \mathbf{v}$ is not anymore a suitable test function since $\langle \mathrm{div}\mathbf{G}, \mathbf{u}^n\rangle$ or $-\langle \mathbf{G}, \nabla\mathbf{u}^n\rangle$ do not have a clear meaning. However, we can replace $\mathbf{u}^n$ by its Lipschitz truncation and conjecture that uniform smallness of the integrand on the sets where the Lipschitz truncation differs from $\mathbf{u}^n$ can lead to (1.8). Note that $\mathbf{F}^n = \mathrm{div}\mathbf{G}^n$ with $\{\mathbf{G}^n\}$ bounded in $L^1(\Omega)^{d\times d}$ is not sufficient for the estimate $(1.3)_1$. However, in our applications in Theorems 3.1 and 5.1 the right hand side will have additional structure (due to the incompressibility constraint involved in the problem) to ensure the validity of $(1.3)_1$.

To proceed further, we need to study carefully the properties of Lipschitz truncations of Sobolev functions. This is the subject of the next section, where we also complete the proof of the weak stability of (1.1) in the difficult case.

2. Lipschitz truncations of standard Sobolev functions

Let $Z \subset \mathbb{R}^d$. Then $Z^\complement$ denotes $\mathbb{R}^d \setminus Z$ and $|Z|$ denotes the d-dimensional Lebesgue measure of Z.

Assumption 2.1. We assume that $\Omega \subset \mathbb{R}^d$ is an open bounded set with the property: there exists a constant $A_1 \ge 1$ such that for all $x \in \Omega$

$$|B_{2\,\mathrm{dist}(x,\Omega^\complement)}(x)| \le A_1\, |B_{2\,\mathrm{dist}(x,\Omega^\complement)}(x) \cap \Omega^\complement|. \tag{2.1}$$

Remark 2.2. If $\Omega \subset \mathbb{R}^d$ is an open bounded set with Lipschitz boundary then Ω satisfies Assumption 2.1.

For any $p \in [1, \infty)$, we use standard notation for the Lebesgue spaces $(L_p(\Omega), \| \cdot \|_p)$ and the Sobolev spaces $(W_0^{1,p}(\Omega), \| \cdot \|_{1,p})$, being the completions of smooth, compactly supported functions w.r.t. the relevant norms. If X is a Banach space of scalar functions then X^d and $X^{d \times d}$ stand for the spaces of vector-valued or tensor-valued functions whose components belong to X.

For $f \in L^1(\mathbb{R}^d)$, we define the Hardy-Littlewood maximal function as usual through

$$(Mf)(x) := \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| \, \mathrm{d}y =: \sup_{r>0} \langle f \rangle_{B_r(x)}.$$

Similarly, for $u \in W^{1,1}(\mathbb{R}^d)$ we define $M(\nabla u) := M(|\nabla u|)$ and for $\mathbf{u} \in W^{1,1}(\mathbb{R}^d)^d$ we set $M(\mathbf{Du}) := M(|\mathbf{Du}|)$.

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^d$ satisfy Assumption 2.1. Let $\mathbf{v} \in W_0^{1,1}(\Omega)^d$. Then for every $\theta, \lambda > 0$ there exist truncations $\mathbf{v}_{\theta,\lambda} \in W_0^{1,\infty}(\Omega)^d$ such that*

$$||\mathbf{v}_{\theta,\lambda}||_\infty \leq \theta, \tag{2.2}$$

$$||\nabla \mathbf{v}_{\theta,\lambda}||_\infty \leq c_1 \, A_1 \, \lambda, \tag{2.3}$$

where $c_1 > 0$ does only depend on the dimension d. Moreover, up to a nullset (a set of Lebesgue measure zero)

$$\{\mathbf{v}_{\theta,\lambda} \neq \mathbf{v}\} \subset \Omega \cap \big(\{M\mathbf{v} > \theta\} \cup \{M(\nabla \mathbf{v}) > \lambda\} \big). \tag{2.4}$$

Theorem 2.3 summarizes the facts established earlier in original papers [3] or [23], and presented in the monograph [26], among others. Since Theorem 2.3 serves as a basic stone in proving Theorem 2.5 (for standard Sobolev functions) and Theorem 4.4 (for functions from the Sobolev space with variable exponent), we give a proof of Theorem 2.3 here for the sake of completness. Before doing so we recall the following extension theorem ([17], p. 201 or also [15], p. 80 and [26], p. 40 for the scalar case).

Lemma 2.4. *Let $\mathbf{v} : E \to \mathbb{R}^m$, defined on a nonempty set $E \subset \mathbb{R}^d$, be such that for certain $\lambda > 0$ and $\theta > 0$ and for all $x, y \in E$*

$$|\mathbf{v}(y) - \mathbf{v}(x)|_{\mathbb{R}^m} \leq \lambda |y - x|_{\mathbb{R}^d} \quad \text{and} \quad |\mathbf{v}(x)|_{\mathbb{R}^m} \leq \theta \, . \tag{2.5}$$

Then there is an extension $\mathbf{v}_{\theta,\lambda} : \mathbb{R}^d \to \mathbb{R}^m$ fulfilling (2.5) for all $x, y \in \mathbb{R}^d$, and $\mathbf{v}_{\theta,\lambda} = \mathbf{v}$ on E.

Let us return to the proof of Theorem 2.3.

Proof of Theorem 2.3. We first extend $\mathbf{v}$ by zero outside of Ω and obtain $\mathbf{v} \in W_0^{1,1}(\mathbb{R}^d)^d$.

The following facts are proved *e.g.* in [26]: for a function $\mathbf{h} \in W_0^{1,1}(\mathbb{R}^d)$ let $\mathcal{L}(\mathbf{h})$ be the set of its Lebesgue points. Then $|\mathcal{L}(\mathbf{h})^{\complement}| = 0$, and for all balls $B_r(x_0) \subset \mathbb{R}^d$ and for all $\xi, \zeta \in \mathcal{L}(\mathbf{h}) \cap B_r(x_0)$ it holds

$$\begin{aligned} |\mathbf{h}(\xi) - \langle \mathbf{h} \rangle_{B_r(x_0)}| &\leq c\, r \, M(\nabla \mathbf{h})(\xi), \\ |\mathbf{h}(\zeta) - \langle \mathbf{h} \rangle_{B_r(x_0)}| &\leq c\, r \, M(\nabla \mathbf{h})(\zeta), \end{aligned} \tag{2.6}$$

which implies that

$$|\mathbf{h}(\xi) - \mathbf{h}(\zeta)| \leq c\, r \, \big(M(\nabla \mathbf{h})(\xi) + M(\nabla \mathbf{h})(\zeta) \big).$$

Then for any $x, y \in \mathcal{L}(\mathbf{h})$ we take $x_0 = x$, $r = 2|y - x|$, $\xi = x$ and $\zeta = y$ in the above inequality and obtain

$$|\mathbf{h}(x) - \mathbf{h}(y)| \leq c \, |x - y| \, \big(M(\nabla \mathbf{h})(x) + M(\nabla \mathbf{h})(y) \big). \tag{2.7}$$

For $\lambda > 0$ we define

$$H_{\theta,\lambda} := \mathcal{L}(\mathbf{v}) \cap \{M\mathbf{v} \leq \theta\} \cap \{M(\nabla \mathbf{v}) \leq \lambda\}.$$

Then it follows from (2.7) that for all $x, y \in H_{\theta,\lambda}$

$$|\mathbf{v}(x) - \mathbf{v}(y)| \leq c\,\lambda\,|x-y| \ \text{ and } |\mathbf{v}(x)| \leq \theta. \tag{2.8}$$

If $\Omega = \mathbb{R}^d$, the statements of Theorem 2.3 follow from Lemma 2.4 applied to $E = H_{\theta,\lambda}$.

If $\Omega \neq \mathbb{R}^d$, we need to proceed more carefully in order to arrange that the Lipschitz truncations vanish on the boundary. Let $x \in H_{\theta,\lambda} \cap \Omega$ and $r := 2\,\mathrm{dist}(x, \Omega^{\complement})$. Then by Assumption 2.1 and since $\mathbf{v}$ is zero on $\Omega^{\complement}$ we have

$$\begin{aligned} \fint_{B_r(x)} |\mathbf{v}(z) - \langle \mathbf{v}\rangle_{B_r(x)}|\,\mathrm{d}z &\geq \frac{1}{|B_r(x)|}\int_{B_r(x)\cap\Omega^{\complement}} |\mathbf{v}(z) - \langle \mathbf{v}\rangle_{B_r(x)}|\,\mathrm{d}z \\ &= \frac{|B_r(x)\cap\Omega^{\complement}|}{|B_r(x)|}|\langle \mathbf{v}\rangle_{B_r(x)}| \\ &\geq \frac{1}{A_1}\,|\langle \mathbf{v}\rangle_{B_r(x)}|. \end{aligned} \tag{2.9}$$

By a variant of the Poincaré inequality, *e.g.* in [26],

$$\fint_{B_r(x)} |\mathbf{h}(z) - \langle \mathbf{h}\rangle_{B_r(x)}|\,\mathrm{d}z \leq c\,r \fint_{B_r(x)} |\nabla \mathbf{h}(z)|\,\mathrm{d}z$$

we observe from (2.9) that for $x \in H_{\theta,\lambda} \cap \Omega$

$$|\langle \mathbf{v}\rangle_{B_r(x)}| \leq c\,A_1\,r \fint_{B_r(x)} |\nabla \mathbf{v}(z)|\,\mathrm{d}z \leq c\,A_1\,r\,M(\nabla\mathbf{v})(x) \leq c\,A_1\,r\,\lambda.$$

Consequently, using also (2.6), we obtain

$$|\mathbf{v}(x)| \leq c\,r\,M(\nabla\mathbf{v})(x) + |\langle \mathbf{v}\rangle_{B_r(x)}| \leq c\,A_1\,r\,\lambda. \tag{2.10}$$

It follows from (2.10) that for all $x \in H_{\theta,\lambda} \cap \Omega$ and all $y \in \Omega^{\complement}$ holds

$$|\mathbf{v}(x) - \mathbf{v}(y)| = |\mathbf{v}(x)| \leq c\,A_1\,\mathrm{dist}(x, \Omega^{\complement})\,\lambda \leq c\,A_1\,|x-y|\,\lambda. \tag{2.11}$$

Since $\mathbf{v}$ is zero on $\Omega^{\complement}$ it follows from (2.8) and (2.11) that

$$|\mathbf{v}(x) - \mathbf{v}(y)| \leq c\,A_1\,|x-y|\,\lambda \qquad \text{for all } x, y \in H_{\theta,\lambda} \cup \Omega^{\complement}. \tag{2.12}$$

In other words, we have shown that $\mathbf{v}$ is Lipschitz continuous on $G_{\theta,\lambda} := H_{\theta,\lambda} \cup \Omega^{\complement}$ with Lipschitz constant bounded by $c\,A_1\,\lambda$. Since, $M\mathbf{v} \leq \theta$ on $H_{\theta,\lambda}$ and $\mathbf{v} = \mathbf{0}$ on $\Omega^{\complement}$, we also have $|\mathbf{v}| \leq \theta$ on $G_{\theta,\lambda}$. Therefore, applying Lemma 2.4 to $E = G_{\theta,\lambda}$ there exists an extension $\mathbf{v}_{\theta,\lambda} \in W^{1,\infty}(\mathbb{R}^d)$ of $\mathbf{v}|_{G_{\theta,\lambda}}$ with $\mathbf{v}(x) = \mathbf{v}_{\theta,\lambda}(x)$ for all $x \in G_{\theta,\lambda}$, $||\nabla \mathbf{v}_{\theta,\lambda}||_\infty \leq c\,A_1\,\lambda$, and $||\mathbf{v}_{\theta,\lambda}||_\infty \leq \theta$. This proves (2.2) and (2.3). From $\mathbf{v}_{\theta,\lambda} = \mathbf{0}$ on $\Omega^{\complement}$ (since it is contained in $G_{\theta,\lambda}$) we conclude that $\mathbf{v}_{\theta,\lambda} \in W^{1,\infty}_0(\Omega)$. Finally, (2.4) follows observing that $\mathbf{v} = \mathbf{v}_{\theta,\lambda}$ on $G_{\theta,\lambda}$,

$|\mathcal{L}(\mathbf{v})^{\complement}| = 0$, and

$$G^{\complement}_{\theta,\lambda} = \Omega \cap H^{\complement}_{\theta,\lambda} = \Omega \cap \big(\mathcal{L}(\mathbf{v})^{\complement} \cup \{M\mathbf{v} > \theta\} \cup \{M(\nabla\mathbf{v}) > \lambda\}\big).$$

The proof of Theorem 2.3 is complete. □

Theorem 2.5. *Let $1 < p < \infty$. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain which satisfies Assumption 2.1. Let $\mathbf{u}^n \in W^{1,p}_0(\Omega)^d$ be such that $\mathbf{u}^n \rightharpoonup \mathbf{0}$ weakly in $W^{1,p}_0(\Omega)^d$ as $n \to \infty$. Set*

$$K := \sup_n ||\mathbf{u}^n||_{1,p} < \infty, \tag{2.13}$$

$$\gamma_n := ||\mathbf{u}^n||_p \to 0 \qquad (n \to \infty). \tag{2.14}$$

Let $\theta_n > 0$ be such that (e.g. $\theta_n := \sqrt{\gamma_n}$)

$$\theta_n \to 0 \qquad \text{and} \qquad \frac{\gamma_n}{\theta_n} \to 0 \qquad (n \to \infty).$$

Let $\mu_j := 2^{2^j}$. Then there exist a sequence $\lambda_{n,j} > 0$ with

$$\mu_j \le \lambda_{n,j} \le \mu_{j+1}, \tag{2.15}$$

and a sequence $\mathbf{u}^{n,j} \in W^{1,\infty}_0(\Omega)^d$ such that for all $j, n \in \mathbb{N}$

$$||\mathbf{u}^{n,j}||_\infty \le \theta_n \to 0 \qquad (n \to \infty), \tag{2.16}$$

$$||\nabla\mathbf{u}^{n,j}||_\infty \le c\,\lambda_{n,j} \le c\,\mu_{j+1}. \tag{2.17}$$

Moreover, up to a nullset

$$\{\mathbf{u}^{n,j} \neq \mathbf{u}^n\} \subset \Omega \cap \big(\{M\mathbf{u}^n > \theta_n\} \cup \{M(\nabla\mathbf{u}^n) > 2\,\lambda_{n,j}\}\big). \tag{2.18}$$

For all $j \in \mathbb{N}$ and $n \to \infty$

$$\mathbf{u}^{n,j} \to \mathbf{0} \quad \text{strongly in } L^s(\Omega)^d \text{ for all } s \in [1,\infty], \tag{2.19}$$

$$\mathbf{u}^{n,j} \rightharpoonup \mathbf{0} \quad \text{weakly in } W^{1,s}_0(\Omega)^d \text{ for all } s \in [1,\infty), \tag{2.20}$$

$$\nabla\mathbf{u}^{n,j} \overset{*}{\rightharpoonup} \mathbf{0} \quad *\text{-weakly in } L^\infty(\Omega)^d. \tag{2.21}$$

Furthermore, for all $n, j \in \mathbb{N}$

$$||\nabla\mathbf{u}^{n,j}\,\chi_{\{\mathbf{u}^{n,j}\neq\mathbf{u}^n\}}||_p \le c\,||\lambda_{n,j}\chi_{\{\mathbf{u}^{n,j}\neq\mathbf{u}^n\}}||_p \le c\,\frac{\gamma_n}{\theta_n}\,\mu_{j+1} + c\,\epsilon_j, \tag{2.22}$$

where $\epsilon_j := K\,2^{-j/p}$ vanishes as $j \to \infty$. The constant c depends on Ω via Assumption 2.1.

The assertions (2.16)–(2.21) summarize the properties of Lipschitz truncations established earlier in [3] and [23]. To our best knowledge, the estimate (2.22) seems to be new. More specifically, the Acerbi-Fusco approximation lemma says, see [3], that $|\{u^{n,\lambda_{n,j}} \neq u^n\}| \le \frac{C\|u^n\|^p_{1,p}}{\lambda^p_{n,j}}$. Applying this estimate we obtain

$$\|\nabla u^{n,\lambda_{n,j}}\,\chi_{\{u^{n,\lambda_{n,j}}\neq u^n\}}\|_p \le \lambda_{n,j}|\{u^{n,\lambda_{n,j}} \neq u^n\}|^{1/p} \le C\|u^n\|_{1,p} \le K\,.$$

Thus one concludes just boundedness of the above term from the Acerbi-Fusco approximation lemma while (2.22) says that for suitable Lipschitz truncations this term can be so small as needed.

Proof of Theorem 2.5. First, observe that (2.13) and (2.14) are direct consequences of $\mathbf{u}^n \rightharpoonup \mathbf{0}$ in $W^{1,p}(\Omega)^d$ and the compact embedding of $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$.

Since $1 < p < \infty$ the Hardy-Littlewood maximal operator M is continuous from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$. This and (2.13) imply

$$\sup_n \int_\Omega |M\mathbf{u}^n|^p \,\mathrm{d}x + \sup_n \int_\Omega |M(\nabla \mathbf{u}^n)|^p \,\mathrm{d}x \le c\,K^p. \tag{2.23}$$

Next, we observe that for $g \in L^p(\mathbb{R}^d)$ with $||g||_p \le K$ we have

$$\begin{aligned} K^p \ge ||g||_p^p &= \int_{\mathbb{R}^d} |g(x)|^p \,\mathrm{d}x = p\int_{\mathbb{R}^d}\int_0^\infty t^{p-1}\,\chi_{\{|g|>t\}}\,\mathrm{d}t\,\mathrm{d}x \\ &= p\int_{\mathbb{R}^d}\sum_{m\in\mathbb{Z}}\int_{2^m}^{2^{m+1}} t^{p-1}\,\chi_{\{|g|>t\}}\,\mathrm{d}t\,\mathrm{d}x \\ &\ge \int_{\mathbb{R}^d}\sum_{m\in\mathbb{Z}}\big(2^m\big)^p\,\chi_{\{|g|>2^{m+1}\}}\,\mathrm{d}x \\ &\ge \int_{\mathbb{R}^d}\sum_{m\in\mathbb{N}}\big(2^m\big)^p\,\chi_{\{|g|>2^{m+1}\}}\,\mathrm{d}x \\ &= \sum_{j\in\mathbb{N}}\sum_{k=2^j}^{2^{j+1}-1}\int_{\mathbb{R}^d}\big(2^k\big)^p\,\chi_{\{|g|>2^{k+1}\}}\,\mathrm{d}x. \end{aligned} \tag{2.24}$$

The choice $g = M(\nabla \mathbf{u}^n)$ implies

$$\sum_{j\in\mathbb{N}}\sum_{k=2^j}^{2^{j+1}-1}\int_{\mathbb{R}^d}\big(2^k\big)^p\,\chi_{\{|M(\nabla\mathbf{u}^n)|>2\cdot 2^k\}}\,\mathrm{d}x \le K^p.$$

Especially, for all $j, n \in \mathbb{N}$

$$\sum_{k=2^j}^{2^{j+1}-1}\int_{\mathbb{R}^d}\big(2^k\big)^p\,\chi_{\{|M(\nabla\mathbf{u}^n)|>2\cdot 2^k\}}\,\mathrm{d}x \le K^p.$$

Since the sum contains 2^j summands, there is at least one index $k_{n,j}$ such that

$$\int_{\mathbb{R}^d}\big(2^{k_{n,j}}\big)^p\,\chi_{\{|M(\nabla\mathbf{u}^n)|>2\cdot 2^{k_{n,j}}\}}\,\mathrm{d}x \le K^p\,2^{-j}. \tag{2.25}$$

Define $\lambda_{n,j} := 2^{k_{n,j}}$ and $\mu_j := 2^{2^j}$. Then

$$\mu_j = 2^{2^j} \le \lambda_{n,j} < 2^{2^{j+1}} = \mu_{j+1} \tag{2.26}$$

and we conclude from (2.25) that

$$\int_{\mathbb{R}^d}\big(\lambda_{n,j}\big)^p\,\chi_{\{|M(\nabla\mathbf{u}^n)|>2\,\lambda_{n,j}\}}\,\mathrm{d}x \le K^p\,2^{-j}. \tag{2.27}$$

Next, we notice that

$$\begin{aligned}
&\int (\lambda_{n,j})^p \, \chi_{\{M\mathbf{u}^n>\theta_n\}\cup\{M(\nabla\mathbf{u}^n)>2\,\lambda_{n,j}\}} \,\mathrm{d}x \\
&\le \left(\frac{\lambda_{n,j}}{\theta_n}\right)^p \int \theta_n^p \, \chi_{\{M\mathbf{u}^n>\theta_n\}} \,\mathrm{d}x + \int (\lambda_{n,j})^p \, \chi_{\{M(\nabla\mathbf{u}^n)>2\,\lambda_{n,j}\}} \,\mathrm{d}x \\
&\le \left(\frac{\lambda_{n,j}}{\theta_n}\right)^p ||M\mathbf{u}^n||_p^p + K^p \, 2^{-j}. \\
&\le c \left(\frac{\lambda_{n,j}}{\theta_n}\right)^p ||\mathbf{u}^n||_p^p + K^p \, 2^{-j}. \\
&= c \left(\frac{\lambda_{n,j}\gamma_n}{\theta_n}\right)^p + K^p \, 2^{-j}.
\end{aligned} \tag{2.28}$$

For each $n, j \in \mathbb{N}$ we apply Theorem 2.3 and set

$$\mathbf{u}^{n,j} := (\mathbf{u}^n)_{\theta_n,\lambda_{n,j}}.$$

Due to Theorem 2.3 (with θ_n and $2\,\lambda_{n,j}$) we have for all $n, j \in \mathbb{N}$

$$||\mathbf{u}^{n,j}||_\infty \le \theta_n, \tag{2.29}$$

$$||\nabla\mathbf{u}^{n,j}||_\infty \le 2\,c_1\,A_1\,\lambda_{n,j} =: c\,\lambda_{n,j} \le c\,\mu_{j+1} \tag{2.30}$$

and up to a nullset

$$\{\mathbf{u}^{n,j} \ne \mathbf{u}^n\} \subset \Omega \cap \big(\{M\mathbf{u}^n > \theta_n\} \cup \{M(\nabla\mathbf{u}^n) > 2\,\lambda_{n,j}\}\big). \tag{2.31}$$

Using (2.28), (2.30), and (2.31) we observe

$$||\nabla\mathbf{u}^{n,j}\,\chi_{\{\mathbf{u}^{n,j}\ne\mathbf{u}^n\}}||_p^p \le c||\lambda_{n,j}\,\chi_{\{\mathbf{u}^{n,j}\ne\mathbf{u}^n\}}||_p^p \le c\left(\frac{\lambda_{n,j}\gamma_n}{\theta_n}\right)^p + cK^p\,2^{-j}. \tag{2.32}$$

Taking the p-th root of (2.32) with the help of (2.26) we conclude (2.22).

Since $\mathcal{D}(\Omega)$ is dense in $L^{s'}(\Omega)$ for all $s' \in [1,\infty)$ and (2.29) implies that

$$\int_\Omega \nabla\mathbf{u}^{n,j}\,\varphi\,\mathrm{d}x = -\int_\Omega \mathbf{u}^{n,j}\,\nabla\varphi\,\mathrm{d}x \to 0 \text{ as } n\to\infty, \text{ for all } \varphi \in \mathcal{D}(\Omega),$$

(2.20) and (2.21) follow for $s \in (1,\infty]$ using also (2.30). The case $s = 1$ then also follows. □

We complete this section by proving the weak stability of (1.1) in the case when $\mathbf{F} = \operatorname{div}\mathbf{G}$ with $\mathbf{G} \in L^1(\Omega)^{d\times d}$. It means that we have $\{\mathbf{v}^n\}$ such that (1.2), (1.3), (1.4), (1.5) and (1.9) hold and we want to prove (1.8). Recall that the choice $\varphi = \mathbf{u}^n$, where $\mathbf{u}^n := \mathbf{v}^n - \mathbf{v}$, is not admissible test function in (1.2). Observing, however, that $\{\mathbf{u}^n\}$ fulfills the assumptions of Theorem 2.5, its application leads to the sequence $\{\mathbf{u}^{n,j}\}$ possessing the properties (2.16)–(2.22); in particular, $\mathbf{u}^{n,j} \in W_0^{1,\infty}(\Omega)^d$ is an admissible (suitable) test function. Inserting $\varphi = \mathbf{u}^{n,j}$ into (1.2) we obtain

$$\int_\Omega \big(\mathbf{T}(\mathbf{D}\mathbf{v}^n) - \mathbf{T}(\mathbf{D}\mathbf{v})\big)\cdot\big(\mathbf{D}\mathbf{u}^{n,j}\big)\,\mathrm{d}x = -\int_\Omega \big((\mathbf{G}^n - \mathbf{G}) + \mathbf{G} + \mathbf{T}(\mathbf{D}\mathbf{v})\big)\cdot\big(\mathbf{D}\mathbf{u}^{n,j}\big)\,\mathrm{d}x \tag{2.33}$$

and the term at the right hand side vanishes as $n \to \infty$ thanks to (2.21) and (1.9). Especially, we have

$$\lim_{n\to\infty} \int_\Omega \big(\mathbf{T}(\mathbf{D}\mathbf{v}^n) - \mathbf{T}(\mathbf{D}\mathbf{v}), \mathbf{D}\mathbf{u}^{n,j}\big)\,\mathrm{d}x = 0. \tag{2.34}$$

We will show below in Lemma 2.6 that (2.34) or even the weaker condition (2.35) implies exactly condition (1.8) that remained to complete the weak stability of (1.1) in the case (1.9) (compare the discussion around (1.5)–(1.8) for details).

Lemma 2.6. *Let Ω and p be as in Theorem 2.5. Let $\mathbf{v}^n, \mathbf{v} \in W_0^{1,p}(\Omega)$ with $\mathbf{v}^n \rightharpoonup \mathbf{v}$ in $W_0^{1,p}(\Omega)$. Let $\mathbf{u}^n := \mathbf{v}^n - \mathbf{v}$ and let $\mathbf{u}^{n,j}$ be the approximations of $\mathbf{u}^n$ as in Theorem 2.5. Assume that for all $j \in \mathbb{N}$ we have*

$$\lim_{n\to\infty} \int_\Omega \big(\mathbf{T}(\mathbf{D}\mathbf{v}^n) - \mathbf{T}(\mathbf{D}\mathbf{v}), \mathbf{D}\mathbf{u}^{n,j}\big)\,\mathrm{d}x \le \delta_j, \tag{2.35}$$

where $\lim_{j\to\infty} \delta_j = 0$. Then for any $0 < \theta < 1$

$$\limsup_{n\to\infty} \int_\Omega \Big[\big(\mathbf{T}(\mathbf{D}\mathbf{v}^n) - \mathbf{T}(\mathbf{D}\mathbf{v})\big) \cdot (\mathbf{D}\mathbf{v}^n - \mathbf{D}\mathbf{v})\Big]^\theta\,\mathrm{d}x = 0.$$

Proof. For all $j \in \mathbb{N}$, (2.35) implies that

$$\begin{aligned}
\limsup_{n\to\infty} I_n &:= \limsup_{n\to\infty} \int_{\{\mathbf{u}^{n,j}=\mathbf{u}^n\}} \big(\mathbf{T}(\mathbf{D}\mathbf{v}^n) - \mathbf{T}(\mathbf{D}\mathbf{v})\big) \cdot \big(\mathbf{D}\mathbf{u}^n\big)\,\mathrm{d}x \\
&\le \limsup_{n\to\infty} \left| \int_{\{\mathbf{u}^{n,j}\neq\mathbf{u}^n\}} \big(\mathbf{T}(\mathbf{D}\mathbf{v}^n) - \mathbf{T}(\mathbf{D}\mathbf{v})\big) \cdot \big(\mathbf{D}\mathbf{u}^{n,j}\big)\,\mathrm{d}x \right| + \delta_j \\
&= \limsup_{n\to\infty} \left| \int_\Omega \big(\mathbf{T}(\mathbf{D}\mathbf{v}^n) - \mathbf{T}(\mathbf{D}\mathbf{v})\big) \cdot \big(\mathbf{D}\mathbf{u}^{n,j}\big)\chi_{\{\mathbf{u}^{n,j}\neq\mathbf{u}^n\}}\,\mathrm{d}x \right| + \delta_j.
\end{aligned}$$

Note that since $\mathbf{v}^n \rightharpoonup \mathbf{v}$ in $W_0^{1,p}(\Omega)$, also $\mathbf{v}$ satisfies $(1.3)_1$ and (1.5). Applying Hölder's inequality to the last integral, and using (1.5) and (2.22) valid for all $j \in \mathbb{N}$ with $\frac{\gamma_n}{\theta_n} \to 0$ as $n \to \infty$, we obtain

$$\begin{aligned}
\limsup_{n\to\infty} I_n &\le c(K) \limsup_{n\to\infty} \|\nabla \mathbf{u}^{n,j} \chi_{\{\mathbf{u}^{n,j}\neq\mathbf{u}^n\}}\|_p + \delta_j \\
&\le c(K) \limsup_{n\to\infty} \left(c\,\frac{\gamma_n}{\theta_n}\,\mu_{j+1} + c\,\epsilon_j + \delta_j \right) \\
&\le c(K)\,\epsilon_j + \delta_j,
\end{aligned} \tag{2.36}$$

with μ_j, ϵ_j as in Theorem 2.5. Since the last estimate holds for all $j \in \mathbb{N}$ and $\lim_{j\to\infty} \epsilon_j = \lim_{j\to\infty} \delta_j = 0$, we finally conclude from (2.36) that

$$\limsup_{n\to\infty} I_n = 0. \tag{2.37}$$

Then with Hölder's inequality

$$\begin{aligned}
&\int_\Omega \Big[\big(\mathbf{T}(\mathbf{D}\mathbf{v}^n)-\mathbf{T}(\mathbf{D}\mathbf{v})\big)\cdot(\mathbf{D}\mathbf{u}^n)\Big]^\theta \,\mathrm{d}x\\
&\quad=\bigg(\int_{\{\mathbf{u}^n=\mathbf{u}^{n,j}\}}\big|\big(\mathbf{T}(\mathbf{D}\mathbf{v}^n)-\mathbf{T}(\mathbf{D}\mathbf{v})\big)\cdot(\mathbf{D}\mathbf{u}^n)\big|\,\mathrm{d}x\bigg)^\theta\,|\Omega|^{1-\theta}\\
&\quad=+\bigg(\int_{\{\mathbf{u}^n\neq\mathbf{u}^{n,j}\}}\big|\big(\mathbf{T}(\mathbf{D}\mathbf{v}^n)-\mathbf{T}(\mathbf{D}\mathbf{v})\big)\cdot(\mathbf{D}\mathbf{u}^n)\big|\,\mathrm{d}x\bigg)^\theta\,\big|\{\mathbf{u}^n\neq\mathbf{u}^{n,j}\}\big|^{1-\theta}\\
&\quad=: Y_{n,j,1}+Y_{n,j,2},
\end{aligned}$$

where $j\in\mathbb{N}$ is arbitrary. Since $(\mathbf{T}(\mathbf{D}\mathbf{v}^n)-\mathbf{T}(\mathbf{D}\mathbf{v}))\cdot(\mathbf{D}\mathbf{u}^n)\geq 0$, we have

$$Y_{n,j,1}\leq (I_n)^\theta\,|\Omega|^{1-\theta}.$$

And therefore with (2.37)

$$\limsup_{n\to\infty} Y_{n,j,1}=0. \tag{2.38}$$

On the other hand from (2.22), $L^p(\Omega)\hookrightarrow L^1(\Omega)$, and $\lambda_{n,j}\geq 1$ we deduce

$$\begin{aligned}
\limsup_{n\to\infty}|\{\mathbf{u}^n\neq\mathbf{u}^{n,j}\}| &= \limsup_{n\to\infty}||\chi_{\{\mathbf{u}^n\neq\mathbf{u}^{n,j}\}}||_1\\
&\leq \limsup_{n\to\infty} c\,\lambda_{n,j}^{-1}\,||\lambda_{n,j}\,\chi_{\{\mathbf{u}^n\neq\mathbf{u}^{n,j}\}}||_p\\
&\leq \limsup_{n\to\infty} c\,||\lambda_{n,j}\,\chi_{\{\mathbf{u}^n\neq\mathbf{u}^{n,j}\}}||_p\\
&\leq c\,\epsilon_j.
\end{aligned} \tag{2.39}$$

Now, Hölder's inequality, $(1.3)_1$, (1.5), and (2.39) prove

$$\begin{aligned}
Y_{n,j,2} &\leq c(K)\,\big(\limsup_{n\to\infty}|\{\mathbf{u}^n\neq\mathbf{u}^{n,j}\}|\big)^{1-\theta}\\
&\leq c(K)\,(\epsilon_j)^{1-\theta}.
\end{aligned} \tag{2.40}$$

Since $j\in\mathbb{N}$ is arbitrary and $\lim_{j\to\infty}\epsilon_j=0$, we get from (2.40) and (2.38)

$$\limsup_{n\to\infty}\int_\Omega\Big[\big(\mathbf{T}(\mathbf{D}\mathbf{v}^n)-\mathbf{T}(\mathbf{D}\mathbf{v})\big)\cdot(\mathbf{D}\mathbf{v}^n-\mathbf{D}\mathbf{v})\Big]^\theta\,\mathrm{d}x=0.$$

This proves Lemma 2.6. □

3. An application: existence result for power-law fluids

We consider the following problem of nonlinear fluid mechanics. For $\Omega\subset\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$ we look for $(\mathbf{v},\mathrm{p}):\Omega\to\mathbb{R}^d\times\mathbb{R}$, representing the velocity and the pressure, satisfying

$$\operatorname{div}(\mathbf{v}\otimes\mathbf{v})-\operatorname{div}\big(\mathbf{T}(\mathbf{D}\mathbf{v})\big)=-\nabla\mathrm{p}+\mathbf{f},\quad \operatorname{div}\,\mathbf{v}=0\text{ in }\Omega \tag{3.1}$$

and

$$\mathbf{v}=\mathbf{0}\text{ on }\partial\Omega, \tag{3.2}$$

where $\mathbf{f}:\Omega\to\mathbb{R}^d$ is given, $\mathbf{Dv}$ denotes the symmetric part of the gradient of $\mathbf{v}$, and $\mathbf{T}:\mathbb{R}^{d\times d}_{\mathrm{sym}}\to\mathbb{R}^{d\times d}_{\mathrm{sym}}$ is a known continuous function having the following properties: for fixed $p\in(1,\infty)$ there are certain positive constants C_1 and C_2 such that for all $\eta\in\mathbb{R}^{d\times d}_{\mathrm{sym}}$

$$\mathbf{T}(\eta)\cdot\eta\geq C_1(|\eta|^p-1)\,, \tag{3.3}$$

$$|\mathbf{T}(\eta)|\leq C_2(|\eta|+1)^{p-1} \tag{3.4}$$

and for all $\eta_1,\eta_2\in\mathbb{R}^{d\times d}_{\mathrm{sym}}$

$$(\mathbf{T}(\eta_1)-\mathbf{T}(\eta_2))\cdot(\eta_1-\eta_2)>0 \text{ if } \eta_1\neq\eta_2. \tag{3.5}$$

System (3.1)–(3.2) describes steady flows of incompressible fluids exhibiting no-slip on the boundary. The fluid is non-Newtonian as its viscosity is not constant and depends on $|\mathbf{Dv}|$, the quantity that reduces in a simple shear flow to the shear rate. A special class of such fluids with shear rate dependent viscosity are the power-law fluids for which $\mathbf{T}$, the Cauchy stress, takes the form $\mathbf{T}(\eta)=\nu_0|\eta|^{p-2}\eta$.

Our aim here is to reprove, in a simpler way, the result established in [18]. In [18] and in [25], the reader can find details related to mechanical and mathematical aspects of the considered system and related results dealing with an analysis of (3.1)–(3.2) as well.

Theorem 3.1. *Let $p>\frac{2d}{d+2}$, $d\geq 2$. Let $\Omega\subset\mathbb{R}^d$ be an open, bounded, connected set with Lipschitz boundary. Assume that $\mathbf{f}\in(W^{1,p}_0(\Omega)^d)^*$ and (3.3)–(3.5) hold. Set $s:=\min\{p',dp/(2(d-p)\}$ if $p<d$ and $s:=p'$ otherwise. Then there exists a weak solution $(\mathbf{v},\mathrm{p})$ to (3.1)–(3.2) such that*

$$\mathbf{v}\in W^{1,p}_0(\Omega)^d \quad \textit{and} \quad \mathrm{p}\in L^s(\Omega), \tag{3.6}$$

$$\operatorname{div}\mathbf{v}=0 \ \textit{a.e. in } \Omega \textit{ and } \int_\Omega \mathrm{p}\,\mathrm{d}x=0, \tag{3.7}$$

$$(\mathbf{T}(\mathbf{Dv}),\mathbf{D}\varphi)=(\mathbf{v}\otimes\mathbf{v},\mathbf{D}\varphi)+(\mathrm{p},\operatorname{div}\varphi)+\langle\mathbf{f},\varphi\rangle \quad \textit{for all } \varphi\in W^{1,\infty}_0(\Omega)^d. \tag{3.8}$$

Proof. Let us for a fixed $p\in(\frac{2d}{d+2},d)$ and $q=\frac{2p}{p-1}=2p'$ consider $\mathbf{v}^n\in W^{1,p}_0(\Omega)\cap L^q(\Omega)$ satisfying $\operatorname{div}\mathbf{v}^n=0$ *a.e.* in Ω and

$$(\mathbf{T}(\mathbf{Dv}^n),\mathbf{D}\varphi)+\frac{1}{n}(|\mathbf{v}^n|^{q-2}\mathbf{v}^n,\varphi)=\langle\mathbf{f},\varphi\rangle+(\mathbf{v}^n\otimes\mathbf{v}^n,\mathbf{D}\varphi)$$
$$\text{for all } \varphi\in W^{1,p}_0(\Omega)^d\cap L^q(\Omega)^d,\ \operatorname{div}\varphi=0. \tag{3.9}$$

Moreover, all $\mathbf{v}^n$ satisfy the uniform estimate[2]

$$\|\mathbf{Dv}^n\|_p^p+\|\nabla\mathbf{v}^n\|_p^p+\frac{1}{n}\|\mathbf{v}^n\|_q^q\leq K \tag{3.10}$$

and consequently, due to the growth condition (3.4) and Sobolev's embedding theorem

$$\|\mathbf{T}(\mathbf{Dv}^n)\|_{p'}\leq c(K), \tag{3.11}$$

$$\|\mathbf{v}^n\|_{\frac{dp}{d-p}}\leq c(K), \tag{3.12}$$

$$\|\mathbf{v}^n\otimes\mathbf{v}^n\|_{\frac{dp}{2(d-p)}}\leq c(K). \tag{3.13}$$

[2]To verify it, take $\varphi=\mathbf{v}^n$ in (3.9) and apply basic inequalities including the Korn one.

The existence of $\mathbf{v}^n$ solving (3.9) for $n \in \mathbb{N}$ is standard and can be proved, for example, *via* Galerkin approximations combined with the monotone operator theory and the compactness for the velocity. An important feature and the advantage of this approximation consists in the fact that the space of test functions coincides with the space where the solution is constructed. The choice of the value for q is due to the quadratic term since for $n \in \mathbb{N}$

$$(\mathbf{v}^n \otimes \mathbf{v}^n, \mathbf{D}\varphi) \leq \|\mathbf{v}^n\|_{2p'}^2 \|\mathbf{D}\varphi\|_p = \|\mathbf{v}^n\|_q^2 \|\mathbf{D}\varphi\|_p \leq C(n).$$

Obviously, the estimate (3.10) implies the existence of $\mathbf{v} \in W_0^{1,p}(\Omega)$, and a (not relabeled) subsequence $\{\mathbf{v}^n\}$ such that

$$\mathbf{v}^n \rightharpoonup \mathbf{v} \qquad \text{weakly in } W_0^{1,p}(\Omega)^d, \tag{3.14}$$

$$\frac{1}{n}(|\mathbf{v}^n|^{q-2}\mathbf{v}^n, \varphi) \to 0 \qquad \text{for all } \varphi \in L^\infty(\Omega)^d, \tag{3.15}$$

and due to the compact embedding theorem

$$\mathbf{v}^n \to \mathbf{v} \quad \text{strongly in } L^\sigma(\Omega)^d \text{ for all } \sigma \in [1, \frac{dp}{d-p}). \tag{3.16}$$

In particular,

$$\mathbf{v}^n \to \mathbf{v} \quad \text{strongly in } L^2(\Omega)^d \text{ provided that } p > \frac{2d}{d+2}, \tag{3.17}$$

which implies that

$$(\mathbf{v}^n \otimes \mathbf{v}^n, \mathbf{D}\varphi) \to (\mathbf{v} \otimes \mathbf{v}, \mathbf{D}\varphi) \quad \text{for all } \varphi \in W_0^{1,\infty}(\Omega)^d. \tag{3.18}$$

Next goal is to prove that also

$$(\mathbf{T}(\mathbf{D}\mathbf{v}^n), \mathbf{D}\varphi) \to (\mathbf{T}(\mathbf{D}\mathbf{v}), \mathbf{D}\varphi) \quad \text{for all } \varphi \in W_0^{1,\infty}(\Omega)^d. \tag{3.19}$$

It suffices, by virtue of (3.10), (3.11) and Vitali's theorem, to show at least for a subsequence that

$$\mathbf{D}\mathbf{v}^n \to \mathbf{D}\mathbf{v} \quad a.e. \text{ in } \Omega. \tag{3.20}$$

This follows, see for example [8] for details, from (3.5) provided that for a certain $\theta \in (0, 1]$

$$\limsup_{n\to\infty} \int_\Omega \Big((\mathbf{T}(\mathbf{D}\mathbf{v}^n) - \mathbf{T}(\mathbf{D}\mathbf{v})) \cdot (\mathbf{D}\mathbf{v}^n - \mathbf{D}\mathbf{v}) \Big)^\theta \, \mathrm{d}x = 0. \tag{3.21}$$

To verify (3.21) (even with $\theta = 1$) we take $\varphi = \mathbf{v}^n - \mathbf{v}$ in (3.9) and let $n \to \infty$. It is then easy to observe that (3.21) is a consequence of

$$\limsup_{n\to\infty} |(\mathbf{v}^n \otimes \mathbf{v}^n, \mathbf{D}(\mathbf{v}^n - \mathbf{v}))| = 0\,. \tag{3.22}$$

Since $(\mathbf{v}^n \otimes \mathbf{v}^n, \mathbf{D}(\mathbf{v}^n - \mathbf{v})) = (\mathbf{v}^n \otimes \mathbf{v}^n, \nabla(\mathbf{v}^n - \mathbf{v})) = -(\mathbf{v}^n \otimes (\mathbf{v}^n - \mathbf{v}), \nabla\mathbf{v}) = -(\mathbf{v}^n \otimes (\mathbf{v}^n - \mathbf{v}), \mathbf{D}\mathbf{v}))$, (3.22) follows from (3.10), (3.12), (3.13) and Hölder's inequality, provided that

$$p > \frac{3d}{d+2}\cdot \tag{3.23}$$

In order to establish the existence result also for

$$p \in \Big(\frac{2d}{d+2}, \frac{3d}{d+2}\Big], \tag{3.24}$$

we first notice that owing to (3.10) and (3.14) the functions

$$\mathbf{u}^n := \mathbf{v}^n - \mathbf{v}$$

fulfill the assumptions of Theorem 2.5 and we conclude the existence of a sequence $\{\mathbf{u}^{n,j}\}$ possessing the properties (2.16)–(2.22).

Note that the functions $\mathbf{u}^{n,j}$ are in general not divergence free on the set $\{\mathbf{u}^n \neq \mathbf{u}^{n,j}\}$ and we have to correct them in order to use them as a test function in (3.9). For $1 < \sigma < \infty$ define

$$L^\sigma_0(\Omega) := \Big\{ h \in L^\sigma(\Omega) \, : \, \int_\Omega h \, \mathrm{d}x = 0 \Big\}.$$

Since $\partial\Omega$ is Lipschitz, according to [5], there exists an linear operator $\mathcal{B}$ such that for all $\sigma \in (1,\infty)$ we have $\mathcal{B} : L^\sigma_0(\Omega) \to W^{1,\sigma}_0(\Omega)^d$ continuously and $\mathrm{div}(\mathcal{B}h) = h$. In particular for all $\sigma \in (1,\infty)$ and all $h \in L^\sigma_0(\Omega)$ we have

$$\begin{aligned} &\mathrm{div}(\mathcal{B}h) = h, \\ &||\mathcal{B}h||_{1,\sigma} \le c\,||h||_\sigma, \end{aligned} \tag{3.25}$$

where the constant depends only on Ω and σ. We define

$$\psi^{n,j} := \mathcal{B}(\mathrm{div}\,\mathbf{u}^{n,j}) = \mathcal{B}(\chi_{\{\mathbf{u}^n \neq \mathbf{u}^{n,j}\}} \mathrm{div}\,\mathbf{u}^{n,j})$$

Then

$$||\psi^{n,j}||_{1,p} \le c\,||\mathrm{div}\,\mathbf{u}^{n,j}\,\chi_{\{\mathbf{u}^n \neq \mathbf{u}^{n,j}\}}||_p.$$

Consequently, (3.14) and (2.16)–(2.22) yield for $j \in \mathbb{N}$, $n \to \infty$,

$$\psi^{n,j} \rightharpoonup 0 \quad \text{weakly in } W^{1,\sigma}(\Omega)^d \quad \text{for all } \sigma \in (1,\infty), \tag{3.26}$$

$$\psi^{n,j} \to 0 \quad \text{strongly in } L^\sigma(\Omega)^d \quad \text{for all } \sigma \in (1,\infty), \tag{3.27}$$

and

$$\begin{aligned} \limsup_{n\to\infty} \|\psi^{n,j}\|_{1,p} &\le c \limsup_{n\to\infty} \Big(\|\mathrm{div}\,\mathbf{u}^{n,j}\,\chi_{\{\mathbf{u}^n \neq \mathbf{u}^{n,j}\}}\|_p \Big) \\ &\le c \limsup_{n\to\infty} \Big(\|\nabla \mathbf{u}^{n,j} \chi_{\{\mathbf{u}^n \neq \mathbf{u}^{n,j}\}}\|_p \Big) \\ &\le c\,\epsilon_j \end{aligned} \tag{3.28}$$

with $\epsilon_j := K\,2^{-\frac{j}{p}}$. Note that we have used in (3.26) that a continuous linear operator preserves weak convergence.

Next, we take in (3.9) φ of the form

$$\varphi^{n,j} = \mathbf{u}^{n,j} - \psi^{n,j}. \tag{3.29}$$

Note that $\varphi^{n,j} \in W_0^{1,s'}(\Omega)^d \cap L^q(\Omega)^d$ and by (3.25)

$$\mathrm{div}\varphi^{n,j} = 0. \tag{3.30}$$

Note that due to (3.26) and (3.27) we have for $j \in \mathbb{N}$, $n \to \infty$

$$\varphi^{n,j} \rightharpoonup 0 \quad \text{weakly in } W^{1,\sigma}(\Omega)^d \quad \text{for all } \sigma \in (1,\infty), \tag{3.31}$$
$$\varphi^{n,j} \to 0 \quad \text{strongly in } L^{\sigma}(\Omega)^d \quad \text{for all } \sigma \in (1,\infty). \tag{3.32}$$

The weak formulation of the approximative problem (3.9) with $\varphi^{n,j}$ as a test function can be rewritten as

$$\begin{aligned}(\mathbf{T}(\mathbf{D}\mathbf{v}^n) - \mathbf{T}(\mathbf{D}\mathbf{v}), \mathbf{D}\mathbf{u}^{n,j}) &= (\mathbf{T}(\mathbf{D}\mathbf{v}^n), \mathbf{D}\psi^{n,j}) \\ &\quad - (\mathbf{T}(\mathbf{D}\mathbf{v}), \mathbf{D}\mathbf{u}^{n,j}) \\ &\quad - \frac{1}{n}(|\mathbf{v}^n|^{q-2}\mathbf{v}^n, \varphi^{n,j}) + \langle \mathbf{f}, \varphi^{n,j}\rangle \\ &\quad + (\mathbf{v}^n \otimes \mathbf{v}^n, \mathbf{D}\varphi^{n,j}) \\ &:= J^1_{n,j} + J^2_{n,j} + J^3_{n,j} + J^4_{n,j}.\end{aligned} \tag{3.33}$$

From $W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$ (since $p > \frac{2d}{d+2}$) and (3.14) we deduce

$$\mathbf{v}^n \otimes \mathbf{v}^n \to \mathbf{v} \otimes \mathbf{v} \qquad \text{in } L^2(\Omega).$$

Letting $n \to \infty$, we observe from (3.10) and (3.18) that

$$\lim_{n\to\infty} (J^2_{n,j} + J^3_{n,j} + J^4_{n,j}) = 0\,. \tag{3.34}$$

On the other hand with Hölder's inequality, (3.11), and (3.28)

$$\limsup_{n\to\infty} J^1_{n,j} \le c(K)\,\epsilon_j. \tag{3.35}$$

Overall, (3.33), (3.34), and (3.35) imply for all $j \in \mathbb{N}$

$$\limsup_{n\to\infty} (\mathbf{T}(\mathbf{D}\mathbf{v}^n) - \mathbf{T}(\mathbf{D}\mathbf{v}), \mathbf{D}\mathbf{u}^{n,j}) \le c(K)\,\epsilon_j. \tag{3.36}$$

Now, (3.21) follows immediately from (3.36) and Lemma 2.6. This proves the validity of (3.9). This and (3.16), (3.18), as well as (3.19) prove that

$$(\mathbf{T}(\mathbf{D}\mathbf{v}), \mathbf{D}\varphi) = \langle \mathbf{f}, \varphi\rangle + (\mathbf{v}\otimes\mathbf{v}, \mathbf{D}\varphi) \quad \text{for all } \varphi \in W_0^{1,\infty}(\Omega)^d, \quad \mathrm{div}\,\varphi = 0. \tag{3.37}$$

Next, we apply deRham's theorem and the Nečas theorem on Sobolev spaces with negative exponents to reconstruct the pressure. Especially, there is $\mathrm{p} \in L_0^s(\Omega)$ fulfilling

$$\begin{aligned}(\mathbf{T}(\mathbf{D}\mathbf{v}),\mathbf{D}\varphi) = \langle \mathbf{f}, \varphi\rangle + (\mathbf{v}\otimes\mathbf{v}, \mathbf{D}\varphi) + (\mathrm{p}, \mathrm{div}\varphi) \\ \text{for all } \varphi \in W_0^{1,\infty}(\Omega)^d.\end{aligned} \tag{3.38}$$

The proof of Theorem 3.1 is complete. □

4. Lipschitz truncations of variable exponent Sobolev functions

In this section we will give a brief introduction to the Lebesgue and Sobolev space with variable exponents. We refer the interested reader to [16,22] and the literature cited below.

Let $\Omega \subset \mathbb{R}^d$ be an open set. By $B_r(x)$ we denote a ball in $\mathbb{R}^d$ with radius r and center x. We write B_r if the center is not important. Let $p\colon \Omega \to [1,\infty)$ be a measurable bounded function, called a variable exponent on Ω, and denote $p^+ = \text{ess sup} p(x)$ and $p^- = \text{ess inf} p(x)$. For the sake of simplicity we will always assume that $1 < p^- \leq p^+ < \infty$. We define the *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions $f : \Omega \to \mathbb{R}$ for which the modular

$$\varrho_{L^{p(\cdot)}(\Omega)}(f) = \int_\Omega |f(x)|^{p(x)} \,\mathrm{d}x$$

is finite. We define the Luxemburg norm on this space by

$$||f||_{L^{p(\cdot)}(\Omega)} = \inf \left\{\lambda > 0\colon \varrho_{L^{p(\cdot)}(\Omega)}(f/\lambda) \leq 1\right\},$$

which is just the Minkowski functional of the absolutely convex set $\{f : \varrho_{L^{p(\cdot)}(\Omega)}(f) \leq 1\}$. Equipped with this norm the set $L^{p(\cdot)}$ is a Banach space. Since $L^{p(\cdot)}(\Omega) \to L^{p-}(\Omega)$ we can define the *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ as the subspace of $L^{p(\cdot)}(\Omega)$ of functions f whose distributional gradient exists and satisfies $\nabla f \in L^{p(\cdot)}(\Omega)$. The norm $||f||_{W^{1,p(\cdot)}(\Omega)} = ||f||_{L^{p(\cdot)}(\Omega)} + ||\nabla f||_{L^{p(\cdot)}(\Omega)}$ makes $W^{1,p(\cdot)}(\Omega)$ a Banach space. If there is no misunderstandig will write $||\cdot||_{p(\cdot)}$ and $||\cdot||_{1,p(\cdot)}$ for the norms of the spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$. Due to $1 < p^- \leq p^+ < \infty$ the spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ are reflexive. The dual of $L^{p(\cdot)}$ is $L^{p'(\cdot)}$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for all $x \in \Omega$.

For fixed exponent spaces we have a very simple relationship between norm and modular. In the variable exponent case this is not so. However, we have the following useful property:

$$\varrho_{p(\cdot)}(f) \leq 1 \text{ if and only if } ||f||_{p(\cdot)} \leq 1. \tag{4.1}$$

We say that a variable exponent $p : \Omega \to [1,\infty)$ is (*locally*) log-*Hölder continuous* if there exists a constant $c > 0$ such that

$$|p(x) - p(y)| \leq \frac{c}{\log(1/|x-y|)},$$

for all points $x, y \in \Omega$ with $|x - y| < \frac{1}{2}$. (Note that this local continuity condition is uniform in Ω.) We say that p is *globally* log-*Hölder continuous* if it is locally log-Hölder continuous and there exist constants $c > 0$ and $p_\infty \in [1,\infty)$ such that for all points $x \in \Omega$ we have

$$|p(x) - p_\infty| \leq \frac{c}{\log(e + |x|)}.$$

The following simple fact is proven *e.g.* in [7,11]:

Proposition 4.1. *Let $\Omega \subset \mathbb{R}^d$. If p is globally* log-*Hölder continuous on Ω, then there exists an extension $\tilde{p}$ such that $\tilde{p}$ is globally* log-*Hölder continuous on $\mathbb{R}^d$ and $\tilde{p}^- = p^-$, $\tilde{p}^+ = p^+$.*

For $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, we define the non-centered maximal function of f by

$$Mf(x) := \sup_{B \ni x} \fint_B |f(y)| \,\mathrm{d}y,$$

where the maximum is taken over all balls $B \subset \mathbb{R}^d$ which contain x. The following proposition is proved in [6,9].

Proposition 4.2. *Let* $p\colon \mathbb{R}^d \to [1,\infty)$ *be a variable exponent with* $1 < p^- \le p^+ < \infty$ *which is globally* log-*Hölder continuous. Then the Hardy-Littlewood maximal operator* M *is continuous from* $L^{p(\cdot)}(\mathbb{R}^d)$ *to* $L^{p(\cdot)}(\mathbb{R}^d)$.

Global log-Hölder continuity is the best possible modulus of continuity to imply the boundedness of the maximal operator, see [6,30]. But for other, weaker results see [9,24,28]. If the maximal operator is bounded, then it follows easily that $C_0^\infty(\mathbb{R}^d)$ is dense in $W^{1,p(\cdot)}(\mathbb{R}^d)$.

The following Corollary is a consequence of Propositions 4.1 and 4.2. It can be used to verify the assumptions on p for the Lipschitz truncation Theorem 4.4 below.

Corollary 4.3. *Let* Ω *be bounded with Lipschitz boundary and let* $p : \Omega \to [1,\infty)$ *be* log-*Hölder continuous with* $1 < p^- \le p^+ < \infty$. *Then there exist an extension* $\tilde{p} : \mathbb{R}^d \to [1,\infty)$ *with* $1 < \tilde{p}^- \le \tilde{p}^+ < \infty$ *such that the Hardy-Littlewood maximal operator* M *is continuous from* $L^{\tilde{p}(\cdot)}(\mathbb{R}^d)$ *to* $L^{\tilde{p}(\cdot)}(\mathbb{R}^d)$.

We are now prepared to generalize the results on Lipschitz truncations of standard Sobolev functions established in Section 2 to Sobolev spaces with variable exponents.

Theorem 4.4. *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded domain which satisfies Assumption 2.1 and let* $p : \mathbb{R}^d \to [1,\infty)$ *with* $1 < p^- \le p^+ < \infty$ *be such that* M *is continuous from* $L^{p(\cdot)}(\mathbb{R}^d)$ *to* $L^{p(\cdot)}(\mathbb{R}^d)$. *Let* $\mathbf{v}^n \in W_0^{1,p(\cdot)}(\Omega)$ *be such that* $\mathbf{v}^n \rightharpoonup \mathbf{0}$ *weakly in* $W_0^{1,p(\cdot)}(\Omega)$ *as* $n \to \infty$. *Set*

$$K := \sup_n ||\mathbf{v}^n||_{1,p(\cdot)} < \infty, \tag{4.2}$$

$$\gamma_n := ||\mathbf{v}^n||_{p(\cdot)} \to 0 \qquad (n \to \infty). \tag{4.3}$$

Let $\theta_n > 0$ *be such that (e.g.* $\theta_n := \sqrt{\gamma_n}$*)*

$$\theta_n \to 0 \qquad \text{and} \qquad \frac{\gamma_n}{\theta_n} \to 0 \qquad (n \to \infty).$$

Then there exist sequences μ_j *and* $\lambda_{n,j} > 1$ *such that for all* $n,\ j \in \mathbb{N}$

$$\mu_j \le \lambda_{n,j} \le \mu_{j+1} \tag{4.4}$$

and a sequence $\mathbf{v}^{n,j} \in W_0^{1,\infty}(\Omega)$ *such that for all* $j,\ n \in \mathbb{N}$

$$||\mathbf{v}^{n,j}||_\infty \le \theta_n \to 0 \qquad (n \to \infty), \tag{4.5}$$

$$||\nabla \mathbf{v}^{n,j}||_\infty \le c\,\lambda_{n,j} \le c\,\mu_{j+1}. \tag{4.6}$$

Moreover, up to a nullset

$$\{\mathbf{v}^{n,j} \ne \mathbf{v}^n\} \subset \Omega \cap \big(\{M\mathbf{v}^n > \theta_n\} \cup \{M(\nabla \mathbf{v}^n) > 2\,\lambda_{n,j}\}\big). \tag{4.7}$$

For all $j \in \mathbb{N}$ *and* $n \to \infty$

$$\mathbf{v}^{n,j} \to \mathbf{0} \quad \text{strongly in } L^s(\Omega)^d \text{ for all } s \in [1,\infty], \tag{4.8}$$

$$\mathbf{v}^{n,j} \rightharpoonup \mathbf{0} \quad \text{weakly in } W_0^{1,s}(\Omega)^d \text{ for all } s \in [1,\infty), \tag{4.9}$$

$$\nabla \mathbf{v}^{n,j} \stackrel{*}{\rightharpoonup} 0 \quad *\text{-weakly in } L^\infty(\Omega)^{d\times d}. \tag{4.10}$$

Furthermore, there exists a sequence $\epsilon_j > 0$ *with* $\epsilon_j \to 0$ *for* $j \to \infty$ *such that for all* $n, j \in \mathbb{N}$

$$||\nabla \mathbf{v}^{n,j}\,\chi_{\{\mathbf{v}^{n,j}\ne\mathbf{v}^n\}}||_{p(\cdot)} \le c\,||\lambda_{n,j}\chi_{\{\mathbf{v}^{n,j}\ne\mathbf{v}^n\}}||_{p(\cdot)} \le c\,\frac{\gamma_n}{\theta_n}\,\mu_{j+1} + \epsilon_j. \tag{4.11}$$

It is possible to choose $\epsilon_j := 2^{-j/p^+}$. *The constant* c *depends on* Ω *via Assumption 2.1.*

Proof. From Lemma 5.5 of [10] it follows that $W_0^{1,p(\cdot)}(\Omega)$ embeds compactly into $L^{p(\cdot)}(\Omega)$. Therefore, from $\mathbf{v}^n \rightharpoonup \mathbf{0}$ in $W_0^{1,p(\cdot)}(\Omega)^d$ we deduce $\mathbf{v}^n \to \mathbf{0}$ in $L^{p(\cdot)}(\Omega)^d$. So (4.2) and (4.3) are just direct consequences of $\mathbf{v}^n \rightharpoonup \mathbf{0}$ in $W_0^{1,p(\cdot)}(\Omega)^d$.

Now, (4.2) and the continuity of the Hardy-Littlewood maximal function imply

$$\sup_n |M\mathbf{v}^n|_{p(\cdot)} + \sup_n |M(\nabla \mathbf{v}^n)|_{p(\cdot)} \le c\,K, \tag{4.12}$$

so (4.1) implies

$$\sup_n \int |M\mathbf{v}^n/(c\,K)|^{p(x)}\,\mathrm{d}x + \sup_n \int |M(\nabla \mathbf{v})^n/(c\,K)|^{p(x)}\,\mathrm{d}x \le 1.$$

Next, we observe that for $g \in L^{p(\cdot)}(\mathbb{R}^d)$ with $||g||_{p(\cdot)} \le 1$ we have

$$\begin{aligned}
1 \ge \int_{\mathbb{R}^d} |g(x)|^{p(x)}\,\mathrm{d}x &= \int_{\mathbb{R}^d}\int_0^\infty p(x)\,t^{p(x)-1}\,\chi_{\{|g|>t\}}\,\mathrm{d}t\,\mathrm{d}x \\
&\ge \int_{\mathbb{R}^d}\sum_{m\in\mathbb{Z}}\int_{2^m}^{2^{m+1}} t^{p(x)-1}\,\chi_{\{|g|>t\}}\,\mathrm{d}t\,\mathrm{d}x \\
&\ge \int_{\mathbb{R}^d}\sum_{m\in\mathbb{Z}} \big(2^m\big)^{p(x)}\,\chi_{\{|g|>2^{m+1}\}}\,\mathrm{d}x \\
&\ge \int_{\mathbb{R}^d}\sum_{m\in\mathbb{N}} \big(2^m\big)^{p(x)}\,\chi_{\{|g|>2^{m+1}\}}\,\mathrm{d}x \\
&= \sum_{j\in\mathbb{N}}\sum_{k=2^j}^{2^{j+1}-1}\int_{\mathbb{R}^d} \big(2^k\big)^{p(x)}\,\chi_{\{|g|>2^{k+1}\}}\,\mathrm{d}x.
\end{aligned} \tag{4.13}$$

The choice $g = M(\nabla \mathbf{v}^n)/(c\,K)$ implies

$$\sum_{j\in\mathbb{N}}\sum_{k=2^j}^{2^{j+1}-1}\int_{\mathbb{R}^d} \big(2^k\big)^{p}\,\chi_{\{|M(\nabla \mathbf{v}^n/(c\,K))|>2\cdot 2^k\}}\,\mathrm{d}x \le 1.$$

Especially, for all $j, n \in \mathbb{N}$

$$\sum_{k=2^j}^{2^{j+1}-1}\int_{\mathbb{R}^d} \big(2^k\big)^{p(x)}\,\chi_{\{|M(\nabla \mathbf{v}^n/(c\,K))|>2\cdot 2^k\}}\,\mathrm{d}x \le 1.$$

Since the sum contains 2^j summands, there is at least one index $k_{n,j}$ such that

$$\int_{\mathbb{R}^d} \big(2^{k_{n,j}}\big)^{p(x)}\,\chi_{\{|M(\nabla \mathbf{v}^n/(c\,K))|>2\cdot 2^{k_{n,j}}\}}\,\mathrm{d}x \le 2^{-j}. \tag{4.14}$$

Let $\epsilon_j := 2^{-j/p^+}$ then $\lim_{j\to\infty}\epsilon_j = 0$. By definition of the norm $||\cdot||_{p(\cdot)}$ and $p^+ < \infty$ it follows from (4.14) that

$$||2^{k_{n,j}}\,\chi_{\{|M(\nabla \mathbf{v}^n/(c\,K))|>2\cdot 2^{k_{n,j}}\}}||_{p(\cdot)}\,\mathrm{d}x \le \epsilon_j. \tag{4.15}$$

Define $\lambda_{n,j} := 2^{k_{n,j}}$ and $\mu_j := 2^{2^j}$. Then

$$\mu_j = 2^{2^j} \leq \lambda_{n,j} < 2^{2^{j+1}} = \mu_{j+1} \tag{4.16}$$

and we conclude from (4.15) that

$$||\lambda_{n,j}\, \chi_{\{|M(\nabla \mathbf{v}^n)|>2\cdot c\, K\, \lambda_{n,j}\}}||_{p(\cdot)}\, \mathrm{d}x \leq \epsilon_j. \tag{4.17}$$

Next, we notice that

$$\begin{aligned}
&||\lambda_{n,j}\, \chi_{\{M\mathbf{v}^n>\theta_n\}\cup\{M(\nabla\mathbf{v}^n)>2\,c\,K\,\lambda_{n,j}\}}\, \mathrm{d}x||_{p(\cdot)} \\
&\leq \frac{\lambda_{n,j}}{\theta_n}\, ||\theta_n\, \chi_{\{M\mathbf{v}^n>\theta_n\}}||_{p(\cdot)} + ||\lambda_{n,j}\, \chi_{\{M(\nabla\mathbf{v}^n)>2\,c\,K\,\lambda_{n,j}\}}||_{p(\cdot)} \\
&\leq \frac{\lambda_{n,j}}{\theta_n}\, ||M\mathbf{v}^n||_{p(\cdot)} + \epsilon_j \\
&\leq c\,\frac{\lambda_{n,j}}{\theta_n}\, c\, ||\mathbf{v}^n||_{p(\cdot)} + \epsilon_j \\
&= c\,\frac{\gamma_n}{\theta_n}\, \lambda_{n,j} + \epsilon_j \\
&\leq c\,\frac{\gamma_n}{\theta_n}\, \mu_{j+1} + \epsilon_j.
\end{aligned} \tag{4.18}$$

For each $n, j \in \mathbb{N}$ we apply Theorem 2.5 and set

$$\mathbf{v}^{n,j} := (\mathbf{v}^n)_{\theta_n,\lambda_{n,j}}.$$

Due to Theorem 2.5 (with θ_n and $2\,c\,K\,\lambda_{n,j}$) we have for all $n, j \in \mathbb{N}$

$$||\mathbf{v}^{n,j}||_\infty \leq \theta_n, \tag{4.19}$$

$$||\nabla\mathbf{v}^{n,j}||_\infty \leq 2\,c\,K\,c_1\,A_1\,\lambda_{n,j} =: c\,K\,\lambda_{n,j} \leq c\,K\,\mu_{j+1} \tag{4.20}$$

and up to a nullset

$$\{\mathbf{v}^{n,j} \neq \mathbf{v}^n\} \subset \Omega \cap \big(\{M\mathbf{v}^n > \theta_n\} \cup \{M(\nabla\mathbf{v}^n) > 2\,c\,K\,\lambda_{n,j}\}\big). \tag{4.21}$$

Using (4.18), (4.20), and (4.21) we observe

$$||\nabla\mathbf{v}^{n,j}\, \chi_{\{\mathbf{v}^{n,j}\neq\mathbf{v}^n\}}||_{p(\cdot)} \leq ||\lambda_{n,j}\, \chi_{\{\mathbf{v}^{n,j}\neq\mathbf{v}^n\}}||_{p(\cdot)} \leq c\,\frac{\gamma_n}{\theta_n}\, \mu_{j+1} + \epsilon_j. \tag{4.22}$$

This proves (4.11).

Since $\mathcal{D}(\Omega)$ is dense in $L^{s'}(\Omega)$ for all $s' \in [1,\infty)$ and (4.20) implies that

$$\int_\Omega \nabla\mathbf{v}^n_{\theta_n,\lambda}\, \varphi\, \mathrm{d}x = -\int_\Omega \mathbf{v}^n_{\theta_n,\lambda}\, \nabla\varphi\, \mathrm{d}x \to 0 \text{ as } n\to\infty, \text{ for all } \varphi \in \mathcal{D}(\Omega),$$

(4.9) and (4.10) follow for $s \in (1,\infty]$. The case $s = 1$ then also follows. □

Remark 4.5. We would like to remark that Theorem 4.4 can easily be extended to other spaces such as weighted L^p spaces. Let for example $1 < p < \infty$ and $\omega \in A_p$, where A_p denotes the Muckenhoupt class. Then M is a continuous operator on $L^p(\mathbb{R}^d; \omega\, \mathrm{d}x)$. As consequence Theorem 4.4 remains true if we replace $L^{p(\cdot)}$ by $L^p(\mathbb{R}^d; \omega\, \mathrm{d}x)$ and $W^{1,p(\cdot)}$ by $W^{1,p}(\mathbb{R}^d; \omega\, \mathrm{d}x)$.

5. An application: existence result for electro-rheological fluids

In Section 3 we have studied the system

$$\operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div}\big(\mathbf{T}(\mathbf{D}\mathbf{v})\big) = -\nabla \mathrm{p} + \mathbf{f}, \quad \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega \tag{5.1}$$

and

$$\mathbf{v} = \mathbf{0} \text{ on } \partial\Omega \tag{5.2}$$

under the assumption that p appearing in (3.3) and (3.10) is constant, with $1 < p < \infty$. Motivated by a model introduced in [31,32] to describe motions of electrorheological fluids and that has been further studied in [33], we are also interested in the case, where p is a function of spatial variables. Electrorheological fluids are a special type of smart fluids which change their material properties due to the application of an electric field. In the model in [32] p is not a constant but a function of the electric field $\mathbf{E}$, *i.e.* $p = p(|\mathbf{E}|^2)$. The interested reader can find the full model for electrorheolgical fluids in [33]. The electric field itself is a solution to the quasi–static Maxwell equations and is not influenced by the motion of the fluid. Thus, we can separate the Maxwell equation from (5.1) and to study, for a given function $p : \Omega \to (1, \infty)$, system (5.1) with $\mathbf{T} : \Omega \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ satisfying for all $x \in \Omega$, $\eta \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$

$$\mathbf{T}(x, \eta) \cdot \eta \geq C_1\big(|\eta|^{p(x)} - 1\big), \tag{5.3}$$

$$|\mathbf{T}(x, \eta)| \leq C_2\big(|\eta| + 1\big)^{p(x)-1} \tag{5.4}$$

and for all $\eta_1, \eta_2 \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$

$$(\mathbf{T}(x, \eta_1) - \mathbf{T}(x, \eta_2)) \cdot (\eta_1 - \eta_2) > 0 \text{ if } \eta_1 \neq \eta_2. \tag{5.5}$$

This model comprises all the mathematical difficulties of the full system for electrorheological fluids (considered in [33]) and the results below can be directly extended to the general case.

Due to the nature of the Maxwell equations it is reasonable to consider that p is Lipschitz continuous. Nevertheless, we are able to handle the case where p is just log-Hölder continuous on $\overline{\Omega}$.

Theorem 5.1. *For $d \geq 2$ let $\Omega \subset \mathbb{R}^d$ be an open, bounded, connected set with Lipschitz boundary $\partial\Omega$ and let $p : \Omega \to (1, \infty)$ be globally* log*-Hölder continuous with $\frac{2d}{d+2} < p^- \leq p^+ < \infty$. Assume that $\mathbf{f} \in (W^{1,p(\cdot)}_0(\Omega))^*$ and (5.3)–(5.5) hold.*

Set $s := \min\{(p^+)', dp^-/(2(d - p^-)\}$ if $p^- < d$ and $s := (p^+)'$ otherwise. Then there exists a weak solution $(\mathbf{v}, \mathrm{p})$ to (5.1)–(5.2) such that

$$\mathbf{v} \in W^{1,p(\cdot)}_0(\Omega)^d \quad \textit{and} \quad \mathrm{p} \in L^s(\Omega), \tag{5.6}$$

$$\operatorname{div} \mathbf{v} = 0 \textit{ a.e. in } \Omega \textit{ and } \int_\Omega \mathrm{p}\, \mathrm{d}x = 0, \tag{5.7}$$

$$(\mathbf{T}(\mathbf{D}\mathbf{v}), \mathbf{D}\varphi) = (\mathbf{v} \otimes \mathbf{v}, \nabla\varphi) + (\mathrm{p}, \operatorname{div} \varphi) + \langle \mathbf{f}, \varphi\rangle \quad \textit{for all } \varphi \in W^{1,\infty}_0(\Omega)^d. \tag{5.8}$$

The existence of weak solutions to system (5.1)–(5.2) under the assumptions (5.3)–(5.5) was first proved in [33] for the case $p^- > \frac{3d}{d+2}$. This was extended in [21] to the case $p^- > \frac{2d}{d+1}$.

Due to Corollary 4.3 we can assume that p is defined on $\mathbb{R}^d$ such that

$$1 < p^- \leq p^+ < \infty \tag{5.9}$$

and that

$$M : L^{p(\cdot)}(\mathbb{R}^d) \to L^{p(\cdot)}(\mathbb{R}^d) \qquad \text{is continuous.} \tag{5.10}$$

In order to prove Theorem 5.1 we will need a few auxiliary results. All these results are solely based on (5.9) and (5.10).

Proposition 5.2 ([10], compact embeddings)**.** *Let $\Omega \subset \mathbb{R}^d$ be as in Theorem 5.1 and let $p : \Omega \to (1,\infty)$ satisfy (5.9) and (5.10). Then the embedding $W_0^{1,p(\cdot)}(\Omega) \to L^{p(\cdot)}(\Omega)$ is compact. Moreover, for $1 \le q < \infty$ with $\frac{1}{p^-} - \frac{1}{d} < \frac{1}{q}$ the embedding $W_0^{1,p(\cdot)}(\Omega) \to L^q(\Omega)$ is compact.*

Proposition 5.3 ([12], Korn inequality)**.** *Let Ω, p be as in Proposition 5.2. Then for all $\mathbf{u} \in W^{1,p(\cdot)}(\Omega)$ holds*

$$||\nabla \mathbf{u}||_{p(\cdot)} \le c\, ||\mathbf{D}\mathbf{u}||_{p(\cdot)}.$$

Define

$$L_0^{p(\cdot)}(\Omega) := \left\{ f \in L^{p(\cdot)}(\Omega) : \int_\Omega f(x)\,\mathrm{d}x = 0 \right\}.$$

Proposition 5.4 ([12,21], divergence equation)**.** *Let Ω, p be as in Proposition 5.2 and let $\mathcal{B}$ denote the operator of (3.25). Then $\mathcal{B}$ is continuous from $L_0^{p(\cdot)}(\Omega)$ to $W^{1,p(\cdot)}(\Omega)^d$ and for each $f \in L_0^{p(\cdot)}(\Omega)$,*

$$\begin{aligned} \operatorname{div}(\mathcal{B}h) &= h, \\ ||\mathcal{B}h||_{1,p(\cdot)} &\le c\, ||h||_{p(\cdot)}. \end{aligned} \tag{5.11}$$

We will further need the following facts: Let $A \subset L^{p(\cdot)}(\Omega)$.
Then $\sup_{f\in A} ||f||_{p(\cdot)} < \infty$ if and only if $\sup_{f\in A} \int |f(x)|^{p(x)}\,\mathrm{d}x < \infty$. For $g \in L^{p(\cdot)}$ and $h \in L^{p'(\cdot)}$ the following assertions analogous to the standard Hölder and Young inequality hold:

$$|(g,h)| \le 2\, ||g||_{p(\cdot)}\, ||h||_{p'(\cdot)}, \tag{5.12}$$

$$(g,h) \le \int_\Omega |g(x)|^{p(x)}\,\mathrm{d}x + \int_\Omega |h(x)|^{p'(x)}\,\mathrm{d}x. \tag{5.13}$$

Lemma 5.5. *Let Ω and p be as in Theorem 5.1. Let $\mathbf{v}^n, \mathbf{v} \in W_0^{1,p(\cdot)}(\Omega)$ with $\mathbf{v}^n \rightharpoonup \mathbf{v}$ in $W_0^{1,p(\cdot)}(\Omega)$. Let $\mathbf{u}^n := \mathbf{v}^n - \mathbf{v}$ and let $\mathbf{u}^{n,j}$ be the approximations of $\mathbf{u}^n$ as in Theorem 2.5. Assume that for all $j \in \mathbb{N}$ we have*

$$\lim_{n\to\infty} \big(\mathbf{T}(\mathbf{D}\mathbf{v}^n) - \mathbf{T}(\mathbf{D}\mathbf{v}), \mathbf{D}\mathbf{u}^{n,j}\big) \le \delta_j, \tag{5.14}$$

where $\lim_{j\to\infty} \delta_j = 0$. Then for any $0 < \theta < 1$

$$\limsup_{n\to\infty} \int_\Omega \Big[\big(\mathbf{T}(\mathbf{D}\mathbf{v}^n) - \mathbf{T}(\mathbf{D}\mathbf{v})\big) \cdot (\mathbf{D}\mathbf{v}^n - \mathbf{D}\mathbf{v})\Big]^\theta \,\mathrm{d}x = 0.$$

Proof. The proof is exactly as the one of Lemma 2.6 with replace $||\cdot||_p$ by $||\cdot||_{p(\cdot)}$ and use Theorem 4.4 instead of Theorem 2.5. □

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. With the help of Propositions 5.2, 5.3, and 5.4 and Theorem 4.4 the proof of Theorem 5.1 is almost exactly as the one for Theorem 3.1. Let us indicate the changes only: Instead of inequality (3.10) we will rather write

$$\int_\Omega |\mathbf{D}\mathbf{v}^n(x)|^{p(x)}\,\mathrm{d}x + \frac{1}{n}||\mathbf{v}^n||_q^q \leq c.$$

Then Proposition 5.3 implies that also

$$\int_\Omega |\nabla\mathbf{v}^n(x)|^{p(x)}\,\mathrm{d}x \leq c.$$

The next change in the proof will be in (3.16) and (3.17), which is now a consequence of Propositon 5.2. Here we have used that $\frac{2d}{d+2} < p^-$.

Note that as in the case p constant the proof gets slightly easier if $p^- > \frac{3d}{d+2}$, see (3.22) and (3.23). We will omit this simplification here, since the other method covers the general case $\frac{2d}{d+2} < p^- \leq p^+ < \infty$.

To define the truncations $\mathbf{u}^{n,j}$ we will just use Theorem 4.4 instead of Theorem 2.5. Especially, we have

$$\limsup_{n\to\infty}\Big(\|\nabla\mathbf{u}^{n,j}\chi_{\{\mathbf{u}^n\neq\mathbf{u}^{n,j}\}}\|_{p(\cdot)}\Big) \leq \epsilon_j$$

with $\epsilon_j \to 0$ for $j \to \infty$.

We will then use Proposition 5.4 to get the corresponding result of (3.25), *i.e.*

$$||\psi^{n,j}||_{1,p(\cdot)} \leq c\,||\mathrm{div}\mathbf{u}^{n,j}\,\chi_{\{\mathbf{u}^n\neq\mathbf{u}^{n,j}\}}||_{p(\cdot)} \leq c\,\epsilon_j.$$

Now, the proof of (3.36) will be the same as for p constant if we use (5.13) and (5.12) as a substitute for the standard Hölder's inequality. Then (3.37) follows as before, if we use Lemma 5.5 instead of Lemma 2.6.

From $W_0^{1,p^+}(\Omega) \to W_0^{1,p(\cdot)}(\Omega)$ we deduce $W^{-1,p'(\cdot)}(\Omega) \to W^{-1,(p^+)'}(\Omega)$. With this embedding we can reconstruct the pressure just as in the case p constant. The proof of Theorem 5.1 is complete. □

References

[1] E. Acerbi and N. Fusco, Semicontinuity problems in the calculus of variations. *Arch. Rational Mech. Anal.* **86** (1984) 125–145.

[2] E. Acerbi and N. Fusco, A regularity theorem for minimizers of quasiconvex integrals. *Arch. Rational Mech. Anal.* **99** (1987) 261–281.

[3] E. Acerbi and N. Fusco, An approximation lemma for $W^{1,p}$ functions, in *Material instabilities in continuum mechanics (Edinburgh, 1985–1986)*, Oxford Sci. Publ., Oxford Univ. Press, New York (1988) 1–5.

[4] L. Boccardo and F. Murat, Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations. *Nonlinear Anal.* **19** (1992) 581–597.

[5] M.E. Bogovskiĭ, Solutions of some problems of vector analysis, associated with the operators div and grad, in *Theory of cubature formulas and the application of functional analysis to problems of mathematical physics (Russian)* **149**, Akad. Nauk SSSR Sibirsk. Otdel. Inst. Mat., Novosibirsk (1980) 5–40.

[6] D. Cruz-Uribe, A. Fiorenza and C.J. Neugebauer, The maximal function on variable L^p spaces. *Ann. Acad. Sci. Fenn. Math.* **28** (2003) 223–238.

[7] D. Cruz-Uribe, A. Fiorenza, J.M. Martell and C. Peréz, The boundedness of classical operators on variable L^p spaces. *Ann. Acad. Sci. Fenn. Math.* **31** (2006) 239–264.

[8] G. Dal Maso and F. Murat, Almost everywhere convergence of gradients of solutions to nonlinear elliptic systems. *Nonlinear Anal.* **31** (1998) 405–412.

[9] L. Diening, Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$. *Math. Inequal. Appl.* **7** (2004) 245–253.

[10] L. Diening, Riesz potential and Sobolev embeddings of generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$. *Math. Nachrichten* **268** (2004) 31–43.

[11] L. Diening and P. Hästö, Variable exponent trace spaces. *Studia Math.* (2007) to appear.

[12] L. Diening and M. Růžička, Calderón-Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics. *J. Reine Angew. Math.* **563** (2003) 197–220.

[13] G. Dolzmann, N. Hungerbühler and S. Müller, Uniqueness and maximal regularity for nonlinear elliptic systems of n-Laplace type with measure valued right hand side. *J. Reine Angew. Math.* **520** (2000) 1–35.
[14] F. Duzaar and G. Mingione, The p-harmonic approximation and the regularity of p-harmonic maps. *Calc. Var. Partial Diff. Eq.* **20** (2004) 235–256.
[15] L.C. Evans and R.F. Gariepy, *Measure theory and fine properties of functions.* CRC Press, Boca Raton, FL, (1992).
[16] X. Fan and D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. *J. Math. Anal. Appl.* **263** (2001) 424–446.
[17] H. Federer, *Geometric Measure Theory.* Band 153 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin-Heidelberg-New York (1969).
[18] J. Frehse, J. Málek, and M. Steinhauer, On analysis of steady flows of fluids with shear-dependent viscosity based on the Lipschitz truncation method. *SIAM J. Math. Anal.* **34** (2003) 1064–1083 (electronic).
[19] M. Giaquinta, G. Modica and J. Souček, *Cartesian currents in the calculus of variations. I*, vol. 37 of *Ergebnisse der Mathematik. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin (1998).
[20] L. Greco, T. Iwaniec and C. Sbordone, Variational integrals of nearly linear growth. *Diff. Int. Eq.* **10** (1997) 687–716.
[21] A. Huber, *Die Divergenzgleichung in gewichteten Räumen und Flüssigkeiten mit $p(\cdot)$-Struktur.* Ph.D. thesis, University of Freiburg, Germany (2005).
[22] O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$. *Czechoslovak Math. J.* **41** (1991) 592–618.
[23] R. Landes, Quasimonotone versus pseudomonotone. *Proc. Roy. Soc. Edinburgh Sect. A* **126** (1996) 705–717.
[24] A. Lerner, Some remarks on the Hardy-Littlewood maximal function on variable L^p spaces. *Math. Z.* **251** (2005) 509–521.
[25] J. Málek and K.R. Rajagopal, Mathematical issues concerning the Navier-Stokes equations and some of its generalizations, in *Evolutionary Equations*, volume 2 of *Handbook of differential equations*, C. Dafermos and E. Feireisl Eds., Elsevier B. V. (2005) 371–459.
[26] J. Malý and W.P. Ziemer, *Fine regularity of solutions of elliptic partial differential equations.* American Mathematical Society, Providence, RI (1997).
[27] S. Müller, A sharp version of Zhang's theorem on truncating sequences of gradients. *Trans. Amer. Math. Soc.* **351** (1999) 4585–4597.
[28] A. Nekvinda, Hardy-Littlewood maximal operator on $L^{p(x)}(\mathbb{R})$. *Math. Inequal. Appl.* **7** (2004) 255–265.
[29] P. Pedregal, *Parametrized measures and variational principles.* Progress in Nonlinear Diff. Eq. Applications, Birkhäuser Verlag, Basel (1997).
[30] L. Pick and M. Růžička, An example of a space $L^{p(x)}$ on which the Hardy-Littlewood maximal operator is not bounded. *Expo. Math.* **19** (2001) 369–371.
[31] K.R. Rajagopal and M. Růžička, On the modeling of electrorheological materials. *Mech. Res. Commun.* **23** (1996) 401–407.
[32] K.R. Rajagopal and M. Růžička, Mathematical modeling of electrorheological materials. *Cont. Mech. Thermodyn.* **13** (2001) 59–78.
[33] M. Růžička, *Electrorheological fluids: modeling and mathematical theory*, *Lect. Notes Math.* **1748**. Springer-Verlag, Berlin (2000).
[34] K. Zhang, On the Dirichlet problem for a class of quasilinear elliptic systems of partial differential equations in divergence form, in *Partial differential equations (Tianjin, 1986)*, *Lect. Notes Math.* **1306** (1988) 262–277.
[35] K. Zhang, Biting theorems for Jacobians and their applications. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **7** (1990) 345–365.
[36] K. Zhang, A construction of quasiconvex functions with linear growth at infinity. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **19** (1992) 313–326.
[37] K. Zhang, Remarks on perturbated systems with critical growth. *Nonlinear Anal.* **18** (1992) 1167–1179.
[38] W.P. Ziemer. *Weakly differentiable functions. Sobolev spaces and functions of bounded variation, Graduate Texts in Mathematics* **120**. Springer-Verlag, Berlin (1989) 308.