

## SHAPE AND TOPOLOGY OPTIMIZATION OF THE ROBUST COMPLIANCE VIA THE LEVEL SET METHOD

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**Abstract.** The goal of this paper is to study the so-called worst-case or robust optimal design problem for minimal compliance. In the context of linear elasticity we seek an optimal shape which minimizes the largest, or worst, compliance when the loads are subject to some unknown perturbations. We first prove that, for a fixed shape, there exists indeed a worst perturbation (possibly non unique) that we characterize as the maximizer of a nonlinear energy. We also propose a stable algorithm to compute it. Then, in the framework of Hadamard method, we compute the directional shape derivative of this criterion which is used in a numerical algorithm, based on the level set method, to find optimal shapes that minimize the worst-case compliance. Since this criterion is usually merely directionally differentiable, we introduce a semidefinite programming approach to select the best descent direction at each step of a gradient method. Numerical examples are given in 2-d and 3-d.

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### 1. INTRODUCTION

Shape optimization is a widely addressed problem. Methods can be roughly divided into two classes: Topology and Geometric optimization methods. Examples of topology optimization methods are the homogenization method [1, 8, 25], the SIMP method [7], the vector variational method [19]. These latter methods amount to optimize a material density for generalized composite designs. Other topology optimization methods are the bubble method or topological gradient approach [2, 12, 13, 24], which amounts to create holes in a given shape, and the thin ligament method [17], not yet implemented, which creates thin ligaments between two parts of the boundary. Geometric optimization methods amount to move the boundaries of an initial domain [16, 20, 23]. Recently introduced in this context, the level set method [4, 5, 18, 22, 27] allows also to change the topology. We shall use again the level set method in the numerical applications of the present work.

The most studied criterion in shape optimization is the compliance, which is defined as the work done by the load or equivalently as the elastic stored energy. Indeed, compliance is a convenient measure of the global rigidity of a structure. However, if a shape is optimized with respect to one load, it has no reason to be stable with respect to perturbations of that load. Knowing how a shape is sensitive to perturbations and performing

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a “worst-case” shape optimization is of the highest engineering interest. This is our motivation for studying the optimal design problem of minimal compliance for a structure submitted to a given load which is subject to unknown perturbations. Such a problem is classical in control theory (where it is called robust control) and has been first studied in topology structural optimization by [9] (see also [1] p. 294, [8, 10]).

To make things clear, take a domain  $\Omega$  whose boundary  $\partial\Omega$  is decomposed into two complementary subsets  $\Gamma_D$  and  $\Gamma_N$  where Dirichlet homogeneous boundary conditions are imposed on  $\Gamma_D$  and Neumann boundary conditions on  $\Gamma_N$ . Let  $d = 2, 3$  be the dimension and denote by  $H_D^1(\Omega)^d$  the set of vector fields belonging to  $H^1(\Omega)^d$  which vanish on  $\Gamma_D$ . Assume that  $meas(\Gamma_D)$  is positive so there is no rigid displacement in  $H_D^1(\Omega)^d$ .

For a given load  $\mathbf{f} = (f_1, f_2)$  with vector fields  $f_1 \in L^2(\mathbb{R}^d)^d$  and  $f_2 \in H^1(\mathbb{R}^d)^d$ , and for a given displacement  $u \in H_D^1(\Omega)^d$ , the elastic energy is defined by:

$$E_\Omega(u, \mathbf{f}) = - \int_\Omega Ae(u) : e(u) + 2 \int_\Omega f_1 \cdot u + 2 \int_{\Gamma_N} f_2 \cdot u,$$

where  $e(u) = 1/2(\nabla u + \nabla u^T)$  is the strain tensor and  $A$  is the fourth-order elasticity tensor such that  $Ae(u)$  is the stress tensor. The compliance is defined by:

$$C_\Omega(\mathbf{f}) = \max_{u \in H_D^1(\Omega)^d} E_\Omega(u, \mathbf{f}),$$

and we want to optimize with respect to  $\Omega$  the robust or worst-case compliance defined by:

$$\mathcal{J}(\Omega) = \max_{\tau \text{ perturbation}} C_\Omega(\mathbf{f} + (\chi\tau, 0)), \quad (1)$$

where  $\tau \in L^2(\mathbb{R}^d)^d$  is an unknown perturbation load, and  $\chi$  is the characteristic function of a given smooth subset of  $\mathbb{R}^d$ , which allows to localize the perturbations. The set of perturbations  $\tau$  has to be further constrained. As an example, for a given real value  $m \in \mathbb{R}^+$ , we ask that  $\|\tau\|_{L^2(\mathbb{R}^d)^d} \leq m$ . Other choices are of course possible.

The first issue, that we address in Section 2, is to prove that the right hand side of (1) is well-posed, namely that there exists a perturbation  $\tau$  which is a maximizer in the right hand side of (1) and to characterize it. The main result is presented in Section 2.1.1. Our approach is to change the problem into the maximization of a nonlinear functional  $F(u)$  (Sect. 2.1), to show that the maximum is attained at a point where  $F$  is differentiable (Sect. 2.2), to use the Euler-Lagrange equation to characterize the set of critical points (Sect. 2.3), to characterize the critical points that can be maximizers (Sect. 2.4) and finally to give an algorithm that computes the maximizers (Sect. 2.5).

Then, Sections 3 and 4 are devoted to the following shape optimization problem:

$$\inf_{\Omega \subset D} (\mathcal{J}(\Omega) + \eta|\Omega|),$$

where  $D$  is a given bounded domain and  $\eta$  is a Lagrange multiplier for a volume constraint. We shall not attempt to prove that there exist optimal shapes. Rather we content ourselves in the rigorous derivation of the shape derivative of this objective function and then use it into a level set method for numerically computing optimal shapes in  $d = 2, 3$  space dimensions. In practice, the resulting optimal designs are much more stable (from a mechanical point of view) than the optimal shapes obtained by standard single load compliance minimization.

## 2. THE DIRECT PROBLEM

In this section, we study the problem of robust compliance when the domain is fixed. We prove that the problem is well-posed and we characterize the set of worst perturbations so that we can differentiate this criterion with respect to the domain in a latter section. For the sake of simplicity, the dependence with respect to the domain is dropped.

- Let  $V$  and  $H$  be two Hilbert spaces such that  $V$  is compactly and densely embedded in  $H$ . We denote by  $(\cdot, \cdot)$  the dual product on  $V$  and  $V'$ . Because  $H$  is identified with its dual, the scalar product of  $H$  is also denoted  $(\cdot, \cdot)$ . The norm in  $V$  is denoted by  $\|\cdot\|_V$  and the subscript is dropped for the norm in  $H$ .
- Let  $\mathbf{A} : V \rightarrow V'$  be a linear coercive, bounded symmetric operator,  $\mathbf{f} \in V'$  and define the energy as:  $E(v, \mathbf{f}) = -(\mathbf{A}v, v) + 2(\mathbf{f}, v)$ .
- Let  $\mathbf{B} : H \rightarrow H$  be a continuous linear operator that characterizes the location of the perturbation and  $\mathbf{B}^*$  its adjoint. In the introduction  $\mathbf{B}$  was simply the self-adjoint operator of multiplication by the characteristic function  $\chi$ , but other choices are possible. The perturbation  $\mathbf{B}^*\tau$  is further constrained to be not too large in the sense that  $\tau$  must satisfy the bound  $\|\tau\| \leq m$  for a fixed positive number  $m$ .

**Definition 2.1.** The robust compliance  $\mathcal{J}$  is defined as:

$$\mathcal{J} = \sup_{\|\tau\| \leq m} \max_{u \in V} E(u, \mathbf{f} + \mathbf{B}^*\tau).$$

The link with the introduction is:

- $V = H_D^1(\Omega)^d$ ,  $H = L^2(\Omega)^d$ ,  $V' = H^{-1}(\Omega)^d \times H^{-1/2}(\Gamma_N)^d$ ;
- $\mathbf{A}$  is the operator such that  $(\mathbf{A}v, w) = \int_{\Omega} A e(v) : e(w) \, dx$ ;
- $\mathbf{f}$  is the element of  $V'$  such that  $(\mathbf{f}, v) = \int_{\Omega} f_1 \cdot v \, dx + \int_{\Gamma_N} f_2 \cdot v \, ds$ ;
- $\mathbf{B} = \mathbf{B}^*$  is the multiplication by the characteristic function  $\chi$ .

**Remark 2.2.**  $\mathbf{B}$  could have been chosen in a different way: let  $S$  be an arbitrary Hilbert space and  $\mathbf{B}^* : S \rightarrow V'$  so that  $\mathbf{B} : V \rightarrow S'$ . As long as  $\mathbf{B}^*\mathbf{B} : V \rightarrow V'$  is continuous and compact, then the following development stands with obvious changes. In this case, the perturbations are defined as  $\mathbf{B}^*\tau$  with  $\|\tau\|_S \leq m$ . In the elasticity setting, it allows to deal with perturbations localized on a smooth subset  $\Gamma$  of the boundary of  $\Omega$  and such that  $\|\tau\|_{L^2(\Gamma)} \leq m$ .

## 2.1. Equivalent formulation

The functional  $\mathcal{J}$  is defined as a double maximum, one on the  $u$  variable and one on the  $\tau$  variable. They can be permuted and solved in one of the variables to give rise to a nonlinear problem in the other variable.

**Proposition 2.3.** Let  $F(u) = -(\mathbf{A}u, u) + 2(\mathbf{f}, u) + 2m\|\mathbf{B}u\|$  and  $G(\tau) = (\mathbf{f} + \mathbf{B}^*\tau, \mathbf{A}^{-1}(\mathbf{f} + \mathbf{B}^*\tau))$ . They satisfy

$$\sup_{u \in V} F(u) = \mathcal{J} = \sup_{\|\tau\| \leq m} G(\tau). \quad (2)$$

*Proof.* We write

$$\begin{aligned} \mathcal{J} &= \sup_{\|\tau\| \leq m} \max_{u \in V} E(u, \mathbf{f} + \mathbf{B}^*\tau) = \sup_{u \in V} \sup_{\|\tau\| \leq m} E(u, \mathbf{f} + \mathbf{B}^*\tau) \\ &= \sup_{u \in V} \left( -(\mathbf{A}u, u) + 2(\mathbf{f}, u) + 2 \sup_{\|\tau\| \leq m} (\mathbf{B}^*\tau, u) \right). \end{aligned}$$

Thanks to the Cauchy-Schwarz inequality, the latest supremum is attained by  $\tau = \frac{m}{\|\mathbf{B}u\|} \mathbf{B}u$ . The problem to study is then:

$$\sup_{u \in V} \left\{ F(u) = -(\mathbf{A}u, u) + 2(\mathbf{f}, u) + 2m\|\mathbf{B}u\| \right\}. \quad (3)$$

The formulation in  $\tau$  is obtained in the same way, remarking that, for a given  $\tau$ , the optimal  $u$  is  $u = \mathbf{A}^{-1}(\mathbf{f} + \mathbf{B}^*\tau)$ .  $\square$

The maximization problem in the right hand side of (2) is a quadratic problem in  $\tau$  with a norm-constraint, *i.e.* a trust-region problem. Despite the impressive literature on trust-region problems, we did not find a proof

of existence of maximizers in the infinite dimensional case. Indeed the existence proof and computation of maximizers of a trust-region problem rely on a duality approach for which there is no duality gap. The proof of absence of gap relies on the so-called  $S$  procedure which is a pure finite-dimensional method (see [26] for this approach and all the references herein). We do not see how to extend straightforwardly this technical argument to the infinite-dimensional setting.

We prefer to use the formulation in  $u$  (*i.e.* Eq. (3)) because we can use tight bounds in our algorithm (see Sect. 2.5.2) which have no equivalent in the trust-region setting (*i.e.* the formulation in  $\tau$ ). Nevertheless, there is a correspondence between trust-region problems and the robust compliance problem. The results advanced in papers dealing with trust-region problems will match the result proven here.

**Remark 2.4.** Setting  $m = 0$  in (3) leads to the standard compliance problem. Setting  $\mathbf{f} = 0$  leads to an eigenvalue problem (the so-called Auchmuty variational principle [6]). There is uniqueness of the maximizer in the first case but not in the second one in full generality. This difference will be seen in the characterization of the maximizers. The problem of robust optimization can be interpreted as an intermediate problem between the standard compliance problem and the eigenvalue problem.

### 2.1.1. Statement of the result

Each implicit assertion of this section is to be proven later.

**Definition 2.5.** Consider the generalized eigenproblem: find  $u \in V$  and  $\rho \in \mathbb{R}$  such that  $\mathbf{A}u = \rho\mathbf{B}^*\mathbf{B}u$ . We define

- the eigenspaces:  $E_\rho = \text{Ker}(\mathbf{A} - \rho\mathbf{B}^*\mathbf{B})$ ,
- the set of eigenvalues:  $Sp(\mathbf{A}, \mathbf{B}) = \{\rho \text{ s.t. } E_\rho \neq \{0\}\}$ ,
- the smallest eigenvalue:  $\lambda_1 = \min\{\lambda \in Sp(\mathbf{A}, \mathbf{B})\}$ .

**Definition 2.6** (Fredholm alternative). For a given  $\rho$  such that  $\mathbf{f} \perp E_\rho$ , we denote by  $u_\rho$  the unique solution of

$$\begin{aligned} \mathbf{A}u_\rho &= \mathbf{f} + \rho\mathbf{B}^*\mathbf{B}u_\rho, \\ \mathbf{A}u_\rho &\in (E_\rho)^\perp. \end{aligned}$$

Recall that  $E_\rho$  being a subspace of  $V$ , its orthogonal  $(E_\rho)^\perp$  is a subspace of the dual  $V'$ . With these notations we can now state our main result in this section.

**Theorem 2.7** (Robust compliance). *Let  $F(u)$  be defined by (3). The set of maximizers of  $F$  is:*

$$\{u_{\rho^*} + r\omega \text{ s.t. } \omega \in E_{\lambda_1} \text{ and } \|\mathbf{B}\omega\| = 1\}$$

where  $\rho^*$  and  $r$  are defined as:

- $(\rho^*, r) = (\lambda_1, m/\lambda_1 - \|\mathbf{B}u_{\lambda_1}\|)$  when  $\mathbf{f} \perp E_{\lambda_1}$  and  $\lambda_1\|\mathbf{B}u_{\lambda_1}\| \leq m$
- $(\rho^*, r) = (s, 0)$  when the above condition are not satisfied. In this case,  $s$  can be defined in two equivalent ways:
  - $s$  is the unique real in  $]0, \lambda_1[$  such that  $s\|\mathbf{B}u_s\| = m$ ;
  - $s$  is the only critical point on  $]0, \lambda_1[$  of the convex function

$$\rho \mapsto (\mathbf{f}, u_\rho) + \frac{m^2}{\rho}.$$

## 2.2. The maximum is a critical value

The first thing to prove is that  $F(u)$  admits a maximizer and that the maximum is a critical value. Then, we will be able to work with the critical point equation. The case  $\mathbf{B} \equiv 0$  being obvious, we shall consider non zero perturbation operators.

**Proposition 2.8.** *Assume  $\mathbf{B}$  is not equal to 0 on  $V$ . There exists at least one maximizer of (3). Furthermore, no maximizer belongs to  $\text{Ker}(\mathbf{B}^*\mathbf{B})$ , so that every maximizer  $u$  of  $F$  satisfies the Euler-Lagrange equation:*

$$\mathbf{A}u = \mathbf{f} + \frac{m}{\|\mathbf{B}u\|} \mathbf{B}^*\mathbf{B}u. \quad (4)$$

*Proof.* Denoting by  $\nu > 0$  the coercivity constant of  $\mathbf{A}$ , it is easily seen that  $F$  is negative outside the ball of radius  $\gamma = 2 \frac{\|\mathbf{f}\| + m\|\mathbf{B}\|}{\nu} > 0$ . Since  $F(0) = 0$ , the maximum is attained inside this ball. Using the coercivity of  $\mathbf{A}$ , the continuity of  $\mathbf{B}^*\mathbf{B}$ , and the compact embedding of  $V$  in  $H$ , the functional  $F$  can be shown to be upper semi-continuous. Then, the maximum of  $F$  is attained.

In order to show that the maximum is a critical value, we have to prove that no maximizer of  $F$  can belong to the set of points where  $F$  is not differentiable, *i.e.* on  $\text{Ker}(\mathbf{B}^*\mathbf{B})$ . Suppose  $u \in \text{Ker}(\mathbf{B}^*\mathbf{B})$  is a maximizer of  $F$ , then

$$F(u) = -(\mathbf{A}u, u) + 2(\mathbf{f}, u) \leq (\mathbf{f}, \mathbf{A}^{-1}\mathbf{f}) \leq (\mathbf{f}, \mathbf{A}^{-1}\mathbf{f}) + 2m\|\mathbf{B}\mathbf{A}^{-1}\mathbf{f}\| = F(\mathbf{A}^{-1}\mathbf{f}).$$

The inequalities are then equalities. Then  $u = \mathbf{A}^{-1}\mathbf{f} \in \text{Ker}(\mathbf{B}^*\mathbf{B})$ . For any  $v^\perp \in \text{Ker}(\mathbf{B}^*\mathbf{B})^\perp$  it is easy to show that  $F(\mathbf{A}^{-1}\mathbf{f} + tv^\perp) > F(\mathbf{A}^{-1}\mathbf{f}) = F(u)$  for sufficiently small  $t$ , which contradicts the maximal character of  $u$ .  $\square$

### 2.3. Characterization of the set of critical points

In this section, we solve the critical point equation (4) by splitting it into two sub-equations:

$$\mathbf{A}u = \mathbf{f} + \rho \mathbf{B}^*\mathbf{B}u, \quad (5)$$

$$\rho = \frac{m}{\|\mathbf{B}u\|}. \quad (6)$$

For any given  $\rho$ , we solve (5) in terms of  $u$ . Then we use the characterization of  $u$  when  $\rho$  is fixed to find the admissible  $\rho$ .

#### 2.3.1. The Fredholm alternative

For a given  $\rho$ , the problem of finding  $u$  solution of (5) is a direct application of the Fredholm alternative:

**Proposition 2.9.** *Define the eigenspaces as  $E_\lambda = \text{Ker}(\mathbf{A} - \lambda \mathbf{B}^*\mathbf{B})$  and the spectrum as  $Sp(\mathbf{A}, \mathbf{B}) = \{\lambda \in \mathbb{R} \text{ s.t. } E_\lambda \neq \{0\}\}$ . For given  $\mathbf{f} \in V'$  and  $\rho \in \mathbb{R}$ , consider the following equation in  $u$ :*

$$(\mathbf{A} - \rho \mathbf{B}^*\mathbf{B})u = \mathbf{f}. \quad (5)$$

*The set of solutions is given by the Fredholm alternative:*

- if  $\rho \notin Sp(\mathbf{A}, \mathbf{B})$  then there exists a unique solution  $\{u_\rho\}$ ;
- if  $\rho \in Sp(\mathbf{A}, \mathbf{B})$  then
  - if  $\mathbf{f} \notin E_\rho$  then there is no solution;
  - if  $\mathbf{f} \in E_\rho$  then the set of solutions is  $\{u_\rho\} + E_\rho$ .

*In the last case,  $u_\rho$  is chosen as the unique normalized solution, *i.e.*  $\mathbf{A}u_\rho \in E_\rho^\perp$ .*

*Proof.* By multiplying (5) by  $\mathbf{A}^{1/2}u$  we obtain:

$$(1 - \rho \mathbf{A}^{-1/2} \mathbf{B}^* \mathbf{B} \mathbf{A}^{-1/2}) \mathbf{A}^{1/2}u = \mathbf{A}^{-1/2}\mathbf{f}$$

which is a shifted problem posed on the Hilbert space  $H$ .

Because  $\Lambda = \mathbf{A}^{-1/2} \mathbf{B}^* \mathbf{B} \mathbf{A}^{-1/2}$  is a compact self-adjoint operator from  $H$  onto  $H$ ,  $\Lambda$  admits an orthonormalized eigenbasis of  $H$ . This eigenbasis is orthogonal to  $\Lambda$  and the corresponding eigenvalues form a countable sequence that converges to 0. Each eigenspace (except maybe for the kernel of  $\Lambda$ ) is of finite dimension. Once shifted, the solution to equation (5) obeys to the Fredholm alternative which is Proposition 2.9.  $\square$

**Remark 2.10.** The definition of  $\mathbf{A}^{1/2}$  may be quite subtle if  $\mathbf{A}$  is considered as an operator acting from  $V$  into  $V'$ . It is easier to understand  $\mathbf{A}$  as an unbounded operator from  $H \rightarrow H$  and to take  $\mathbf{f} \in H$ .

When  $\mathbf{f} \perp E_\rho$  and  $\rho$  is an eigenvalue, there are multiple choices for the definition of  $u_\rho$ . We chose here  $\mathbf{A}u_\rho \in E_\rho^\perp$  so that  $u_\rho$  has the following property:

**Lemma 2.11.** *If  $\rho \in Sp(\mathbf{A}, \mathbf{B})$  and  $\mathbf{f} \perp E_\rho$  then  $\forall \omega \in E_\rho$  we have:*

$$\|\mathbf{B}u_\rho + \mathbf{B}\omega\| = \|\mathbf{B}u_\rho\| + \|\mathbf{B}\omega\|. \quad (7)$$

*Proof.* If  $\omega \in E_\rho$ , then  $\mathbf{B}\omega$  is proportional to  $\mathbf{A}\omega$  and the symmetry of the operator  $\mathbf{A}$  together with the orthogonality condition of  $u_\rho$  yields the orthogonality of  $\mathbf{B}^*\mathbf{B}\omega$  and  $u_\rho$ .  $\square$

We now shift back the spectral decomposition of  $\Lambda$  to obtain a result in terms of  $\mathbf{A}$  and  $\mathbf{B}$ .

**Proposition 2.12.** *There exist  $I_1$  and  $I_2$ , two complementary subsets of  $\mathbb{N}$ , a sequence of real positive numbers  $(\lambda_i)_{i \in I_1}$  and  $(e_i)_{i \in \mathbb{N}}$  a basis of  $V$  such that:*

$$\begin{aligned} \mathbf{A}e_i &= \lambda_i \mathbf{B}^* \mathbf{B}e_i & \forall i \in I_1 \\ \mathbf{B}^* \mathbf{B}e_i &= 0 & \forall i \in I_2 \\ (\mathbf{A}e_i, e_j) &= \delta_{i,j} & \forall i, j \\ (\mathbf{B}e_i, \mathbf{B}e_j) &= 0 & \forall i \neq j \\ \mathbf{f} &= \sum_{i \in \mathbb{N}} \mathbf{f}_i \mathbf{A}e_i \quad \text{where } \mathbf{f}_i = (\mathbf{f}, e_i) & \forall \mathbf{f} \in V'. \end{aligned}$$

The normalized solution  $u_\rho$  then admits the following expression, to which we will refer as “the analytical expression” of  $u_\rho$ :

$$u_\rho = \sum_{i \in I_1} \frac{\lambda_i \mathbf{f}_i}{\lambda_i - \rho} e_i + \sum_{i \in I_2} \mathbf{f}_i e_i, \quad (8)$$

with the convention that, if there exists an  $i$  such that  $\rho = \lambda_i$ , then  $\mathbf{f}_i = 0$  (or else  $u_\rho$  do not exist) and the corresponding term  $\frac{\lambda_i \mathbf{f}_i}{\lambda_i - \rho}$  is equal to 0.

### 2.3.2. The set of critical points

After (5) has been solved for a given  $\rho$ , we can now solve (6) and find the coupled solution  $(\rho, u)$ .

**Definition 2.13.** Let  $Y_1$  and  $Y_2$  be the following subsets of  $\mathbb{R}$ :

$$\begin{aligned} Y_1 &= \{\rho \in \mathbb{R} \text{ s.t. } \rho \|\mathbf{B}u_\rho\| = m\}, \\ Y_2 &= \{\rho \in Sp(\mathbf{A}, \mathbf{B}) \text{ s.t. } \mathbf{f} \perp E_\rho \text{ and } \rho \|\mathbf{B}u_\rho\| \leq m\}. \end{aligned}$$

**Proposition 2.14.** *The couple  $(u, \rho)$  is a critical point of  $F$  in the sense that it is a solution of equations (5) and (6), if and only if*

$$\begin{aligned} u &= u_\rho & \text{and } \rho \in Y_1 \\ \text{or} \\ u &\in \{u_\rho + r\omega; \omega \in E_\rho, \|\mathbf{B}\omega\| = 1 \text{ and } r = \frac{m}{\rho} - \|\mathbf{B}u_\rho\|\} & \text{and } \rho \in Y_2 \end{aligned}$$

where  $u_\rho$  is defined in Definition 2.6.

*Proof.* The proof is a simple albeit lengthy computation, based on Proposition 2.9 which characterizes the solutions of (5). We omit this proof that can be found in [14].  $\square$

## 2.4. Finding the adequate critical points

### 2.4.1. Restriction on the possible maximizers

We prove that the value of  $F(u)$  at a critical point  $(u, \rho)$  (which solves (5) and (6)) only depends on  $\rho$ . We also prove that if we want to maximize  $F$ , then  $\rho$  should be taken the smallest possible.

**Proposition 2.15.** *For any  $(u_1, \rho_1)$  and  $(u_2, \rho_2)$  critical points of  $F$  we have:*

$$F(u_1) \geq F(u_2) \iff \rho_1 \leq \rho_2 \quad \text{and} \quad F(u_1) = F(u_2) \iff \rho_1 = \rho_2. \quad (9)$$

Therefore the only possible maximizers are the  $(u, \rho^*)$  critical points such that  $\rho^* = \min(Y_1 \cup Y_2)$ . Once  $\rho^*$  is found, Proposition 2.14 gives the adequate  $u$ .

*Proof* Let  $u_1$  and  $u_2$  be two critical points of  $F$ . They satisfy

$$\mathbf{A}u_i = \mathbf{f} + \frac{m}{\|\mathbf{B}u_i\|} \mathbf{B}^* \mathbf{B}u_i \quad i = 1, 2 \quad (10)$$

$$F(u_i) = (\mathbf{f}, u_i) + m \|\mathbf{B}u_i\| \quad i = 1, 2. \quad (11)$$

Using the symmetry of  $\mathbf{A}$  and  $\mathbf{B}^* \mathbf{B}$  and equation (10) we obtain:

$$(\mathbf{f}, u_2) + \frac{m}{\|\mathbf{B}u_1\|} (\mathbf{B}^* \mathbf{B}u_1, u_2) = (\mathbf{f}, u_1) + \frac{m}{\|\mathbf{B}u_2\|} (\mathbf{B}^* \mathbf{B}u_2, u_1) \quad (12)$$

*i.e.*

$$(\mathbf{f}, u_1 - u_2) = m \frac{\|\mathbf{B}u_2\| - \|\mathbf{B}u_1\|}{\|\mathbf{B}u_1\| \|\mathbf{B}u_2\|} (\mathbf{B}^* \mathbf{B}u_1, u_2). \quad (13)$$

Combining equations (13) and (11), we obtain

$$\begin{aligned} F(u_1) - F(u_2) &= (\mathbf{f}, u_1 - u_2) + m \|\mathbf{B}u_1\| - m \|\mathbf{B}u_2\| \\ &= m \frac{\|\mathbf{B}u_1\| - \|\mathbf{B}u_2\|}{\|\mathbf{B}u_1\| \|\mathbf{B}u_2\|} [\|\mathbf{B}u_1\| \|\mathbf{B}u_2\| - (\mathbf{B}u_1, \mathbf{B}u_2)]. \end{aligned}$$

Thanks to Cauchy-Schwarz inequality,  $F(u_1) - F(u_2)$  is of the same sign than  $\|\mathbf{B}u_1\| - \|\mathbf{B}u_2\|$ . Moreover, if  $F(u_1) - F(u_2) = 0$  while  $\|\mathbf{B}u_1\| \neq \|\mathbf{B}u_2\|$ , then  $\mathbf{B}u_1$  and  $\mathbf{B}u_2$  are collinear. In this last case, equation (10) shows that  $u_1 = u_2$  which is impossible.  $\square$

### 2.4.2. Finding $\min(Y_1 \cup Y_2)$

Recalling Proposition 2.15, we now want to characterize  $\rho^* = \min(Y_1 \cup Y_2)$ , the only value of  $\rho$  which can give rise to a maximizer. For that purpose, we use an auxiliary function  $g$ .

**Definition 2.16.** Define  $Z$ , the set of  $\rho$  for which the Fredholm alternative (Prop. 2.9) does not admit solutions, *i.e.*

$$Z = \{\rho \in Sp(\mathbf{A}, \mathbf{B}) \text{ such that } \mathbf{f} \notin E_\rho\} \quad (14)$$

and call  $\bar{\lambda}$  the smallest element of  $Z$ .

We define the auxiliary function  $g$  on  $\mathbb{R}_+^* \setminus Z$  by

$$g(\rho) = (\mathbf{f}, u_\rho) + m^2/\rho \quad (15)$$

where  $u_\rho$  is defined by Definition 2.6.

**Proposition 2.17.** *The function  $g$  belongs to  $C^\infty(\mathbb{R}_+^* \setminus Z)$  and is convex on  $(0, \bar{\lambda})$ . Moreover  $g$  tends to  $+\infty$  in  $0^+$  and  $\bar{\lambda}^-$ . Its derivative is equal to*

$$g'(\rho) = \|\mathbf{B}u_\rho\|^2 - m^2/\rho^2. \quad (16)$$

*Proof.* The eigenbasis decomposition of Proposition 2.12 yields

$$g(\rho) = \sum_{i \in I_1} \frac{\lambda_i(\mathbf{f}_i)^2}{\lambda_i - \rho} + \sum_{i \in I_2} (\mathbf{f}_i)^2 + m^2/\rho. \quad (17)$$

Computing the second-order derivative shows that  $g$  is convex. Equation (16) is a simple consequence of using the ‘‘analytical expression of  $u_\rho$ ’’ (8) in the first-order derivative of (17).  $\square$

**Proposition 2.18.** *Let  $\rho^* = \min(Y_1 \cup Y_2)$  and  $s$  be the only critical point of the convex function  $g$  on  $]0, \bar{\lambda}[$ . The following alternative holds:*

$$\begin{aligned} \text{if } (\mathbf{f} \perp E_{\lambda_1} \text{ and } g'(\lambda_1) \leq 0) \quad \text{then } \rho^* = \lambda_1, \\ \text{otherwise } \rho^* = s, \end{aligned}$$

where  $\lambda_1 = \min Sp(\mathbf{A}, \mathbf{B})$  is the smallest eigenvalue of the spectrum as introduced in Definition 2.5.

*Proof.* Using (16) and Definition 2.13 of  $Y_1$  and  $Y_2$ , the following characterization of the set  $Y_1 \cup Y_2$  holds:

$$\begin{aligned} \rho \in Y_1 &\iff g'(\rho) = 0 \\ \rho \in Y_2 &\iff g'(\rho) \leq 0 \text{ and } \rho \in Sp(\mathbf{A}, \mathbf{B}) \setminus Z. \end{aligned}$$

Since  $g$  is convex over  $]0, \bar{\lambda}[$ , we have  $\min Y_1 = s$ . Due to the convexity of  $g$ , any  $0 < \rho < s$  verifies  $g'(\rho) < 0$ . So that  $\rho^* = \min(Y_1 \cup Y_2)$  is not equal to  $s$  if and only if  $\rho^* = \min Y_2 = \min(Sp(\mathbf{A}, \mathbf{B}) \setminus Z)$  and  $\rho^* < s$ . In this case  $\min(Sp(\mathbf{A}, \mathbf{B}) \setminus Z) < s < \bar{\lambda}$ . Recall that  $\bar{\lambda} = \min Z$  so that  $\rho^* \neq s$  if and only if  $\rho^* = \lambda_1 = \min Sp(\mathbf{A}, \mathbf{B}) \notin Z$  and  $\lambda_1 < s$ . The condition  $\lambda_1 = \min Sp(\mathbf{A}, \mathbf{B}) \notin Z$  is by definition of  $Z$ , equivalent to  $\mathbf{f} \perp E_{\lambda_1}$ .

Then  $\rho^* \neq s$  if and only if  $\rho^* = \lambda_1$  and  $\lambda_1 < s$  (i.e.  $g'(\lambda_1) < 0$  by convexity) and  $\mathbf{f} \perp E_{\lambda_1}$ .  $\square$

We can now state the theorem announced in Section 2.1.1.

**Theorem 2.19** (Robust compliance). *Let  $F(u)$  be defined by (3), the set of maximizers of  $F$  is:*

$$\{u_{\rho^*} + r\omega \text{ s.t. } \omega \in E_{\lambda_1} \text{ and } \|\mathbf{B}\omega\| = 1\}$$

where  $\rho^*$  and  $r$  are defined as:

- $(\rho^*, r) = (\lambda_1, m/\lambda_1 - \|\mathbf{B}u_{\lambda_1}\|)$  when  $\mathbf{f} \perp E_{\lambda_1}$  and  $\lambda_1 \|\mathbf{B}u_{\lambda_1}\| \leq m$
- $(\rho^*, r) = (s, 0)$  when the above condition are not satisfied.

*Proof.* The result is obtained by using Proposition 2.15 to state that the maximizers of  $F$  are the critical points corresponding to  $\rho^* = \min(Y_1 \cup Y_2)$ , Proposition 2.18 that characterizes  $\rho^*$ , equation (16) to change the condition  $g'(\lambda_1) \leq 0$  of Proposition 2.18 into the condition  $\|\mathbf{B}u_{\lambda_1}\| \leq m/\lambda_1$  and finally by using Proposition 2.14 that characterizes the set of critical points of  $F$  when  $\rho^*$  is given.  $\square$

## 2.5. An algorithm for the direct problem

It must be kept in mind that  $\rho^*$  may only take two values:  $s$  (as defined in Prop. 2.18 as the only critical point on  $]0, \bar{\lambda}[$  of the convex function  $g$ ) and  $\lambda_1$  (as defined in Def. 2.5 as the smallest eigenvalue). Computing  $\lambda_1$  is a standard problem in numerical analysis, so it is difficult to compute  $\rho^*$  merely when  $\rho^* \neq \lambda_1$ . Even in this case, one needs to compute the first eigenvalue  $\lambda_1$  in order to check in which case we are.



The standard idea would be to calculate the maximizer of (3) by using a standard gradient or fixed-point algorithm. Unfortunately we can exhibit simple cases where there exist local maxima that are not global (see the Appendices). That is why we propose here a stable algorithm that minimizes  $g$  (a convex function) over the bounded set  $(0, \lambda_1)$ . We will also obtain more accurate bounds on  $\rho^*$  than 0 and  $\lambda_1$ .

### 2.5.1. The Newton method

The first thing to do is to compute numerically  $\lambda_1$ , the smallest eigenvalue together with its eigenspace  $E_{\lambda_1}$ . If  $\mathbf{f}$  belongs to  $(E_{\lambda_1})^\perp$ , one has to compute  $u_{\lambda_1}$ , the unique solution to  $(\mathbf{A} - \lambda_1 \mathbf{B}^* \mathbf{B})u_{\lambda_1} = \mathbf{f}$  and  $\mathbf{A}u_{\lambda_1} \in (E_{\lambda_1})^\perp$ . If  $\|\mathbf{B}u_{\lambda_1}\| \leq m/\lambda_1$  then we are done.

If these conditions are not met, we have to find  $s$  which is the minimizer of  $g$  on  $]0, \lambda_1[$ . Recall that  $g$  is convex on this interval and is given by:

$$\begin{aligned} u_\rho &= (\mathbf{A} - \rho \mathbf{B}^* \mathbf{B})^{-1} \mathbf{f} \\ v_\rho &= \mathbf{A}^{-1} \mathbf{B}^* \mathbf{B} (\mathbf{A} - \rho \mathbf{B}^* \mathbf{B})^{-1} \mathbf{A} u_\rho \\ g(\rho) &= (\mathbf{f}, u_\rho) + m^2/\rho \\ g'(\rho) &= \|\mathbf{B}u_\rho\| - m^2/\rho^2 \\ g''(\rho) &= 2(\mathbf{B}^* \mathbf{B} u_\rho, \mathbf{B}^* \mathbf{B} v_\rho) + 2m^2/\rho^3, \end{aligned} \tag{18}$$

where the last equation is obtained by two derivations of the analytical expression of  $g$  (17) and identification using Proposition 2.12.

**Remark 2.20.** The auxiliary function  $v_\rho = \mathbf{A}^{-1} \mathbf{B}^* \mathbf{B} (\mathbf{A} - \rho \mathbf{B}^* \mathbf{B})^{-1} \mathbf{A} u_\rho$  seems *a priori* difficult to compute. But if the LU or Cholesky factorization of  $\mathbf{A} - \rho \mathbf{B}^* \mathbf{B}$  has already been obtained when computing  $u_\rho$ , then it is costless to reuse it. The inverse power algorithm that computes  $\lambda_1$  already used this factorization of  $\mathbf{A}$  and we just have to store it along the iterations.

Since the second derivative of  $g$  can be easily computed, we can use a Newton method to compute the minimum  $s$  of  $g$ . Then, Theorem 2.19 gives the single maximizer  $\{u_s\}$ .

### 2.5.2. Bounds on $s$

Because each iteration of the Newton method can be costly, it is a good idea to find some bounds on  $\rho^* = s$  in order to have a good starting point for the algorithm. These bounds will be called  $s_m$  and  $s_M$  such that  $s_m \leq s \leq s_M$ .

Assume that we already computed  $n$  eigenvectors  $(e_i)_{i \in I}$  and eigenvalues  $(\lambda_i)_{i \in I}$  with  $I = \{i_k\}_{k=1 \dots n}$ . Recall that  $I \subset I_1$  where  $I_1$  is defined as in Proposition 2.12. We also have computed the coordinates  $\mathbf{f}_i = (\mathbf{f}, e_i)$  where  $e_i$  are normalized by  $(\mathbf{A}e_i, e_j) = \delta_{i,j}$ .

**Proposition 2.21.** Let  $\lambda_M = \max_{i \in I} \lambda_i$  and  $(s_M, s_m)$  be the solutions of

$$\begin{aligned} 0 \leq s_M \leq \lambda_1 \text{ and } \sum_{i \in I} \frac{\lambda_i \mathbf{f}_i^2}{(\lambda_i - s_M)^2} &= m^2/s_M \\ 0 \leq s_m \leq \lambda_1 \text{ and } \frac{\lambda_M \alpha}{(\lambda_M - s_m)^2} + \sum_{i \in I} \frac{\lambda_i \mathbf{f}_i^2}{(\lambda_i - s_m)^2} &= m^2/s_m \end{aligned}$$

where  $\alpha$  is defined by:

$$\alpha = \min \left\{ (\mathbf{A}^{-1} \mathbf{f}, \mathbf{f}) - \sum_{i \in I} (\mathbf{f}_i)^2, \lambda_M (\mathbf{B}^* \mathbf{B} \mathbf{A}^{-1} \mathbf{f}, \mathbf{A}^{-1} \mathbf{f}) - \sum_{i \in I} (\mathbf{f}_i)^2 \frac{\lambda_M}{\lambda_i} \right\},$$

then  $s_m \leq s \leq s_M$  with an equality if and only if  $Sp(\mathbf{A}, \mathbf{B}) \setminus Z \subset \{\lambda_i, i \in I\}$ .

**Remark 2.22.** Computing  $s_M$  and  $s_m$  is not a difficult task. Indeed, arrange  $I = \{i_k\}_{k=1\dots n}$  in increasing order, define  $\tilde{\mathbf{A}} = \text{diag}(\lambda_{i_k})_{k=1\dots n}$  and  $\tilde{\mathbf{B}}$  the identity matrix of rank  $n$ , then  $s_M$  and  $s_m$  are respectively the solutions of the robust optimization problem in finite dimension  $n$  with a source term equal to

$$\begin{aligned}\mathbf{f}_m &= [\sqrt{\lambda_{i_1}} \mathbf{f}_{i_1} \quad \sqrt{\lambda_{i_2}} \mathbf{f}_{i_2} \quad \dots \quad \sqrt{\lambda_{i_{n-1}}} \mathbf{f}_{i_{n-1}} \quad \sqrt{\lambda_{i_n}} \mathbf{f}_{i_n}], \\ \mathbf{f}_M &= [\sqrt{\lambda_{i_1}} \mathbf{f}_{i_1} \quad \sqrt{\lambda_{i_2}} \mathbf{f}_{i_2} \quad \dots \quad \sqrt{\lambda_{i_{n-1}}} \mathbf{f}_{i_{n-1}} \quad \sqrt{\lambda_{i_n} (\alpha + \mathbf{f}_{i_n})}].\end{aligned}$$

Solving these robust optimization problems is straightforward. The matrices  $\tilde{\mathbf{A}}$  and  $\tilde{\mathbf{B}}$  are diagonal thus we can use the explicit formula (8) without any extra computations for getting the spectrum  $Sp(\tilde{\mathbf{A}}, \tilde{\mathbf{B}})$ .

*Proof of Proposition 2.21.* We first prove that  $s_M \geq s$ . We have

$$\sum_{i \in I_1} \frac{\lambda_i(\mathbf{f}_i)^2}{(\lambda_i - s_M)^2} \geq \sum_{i \in I} \frac{\lambda_i(\mathbf{f}_i)^2}{(\lambda_i - s_M)^2} = m^2/s_M$$

so that  $g'(s_M) \geq 0$ . Since  $\lambda_1 \geq s_M \geq 0$ ,  $g$  is convex on  $(0, \lambda_1)$  and  $g'(s) = 0$ , we deduce  $s_M \geq s$ .

To prove the other inequality  $s_m \leq s$ , we must first find an upper estimate of

$$K = \sum_{i \in I_1 \setminus I} \frac{(\mathbf{f}_i)^2}{(\lambda_i^{1/2} - \lambda_i^{-1/2} s_m)^2} = \sum_{i \in I_1 \setminus I} \frac{(\mathbf{f}_i)^2/\lambda_i}{(1 - \lambda_i^{-1} s_m)^2}.$$

We rely on the following identities

$$\begin{aligned}\sum_{i \in I_1 \setminus I} (\mathbf{f}_i)^2 &= (\mathbf{A}^{-1} \mathbf{f}, \mathbf{f}) - \sum_{i \in I} (\mathbf{f}_i)^2 - \sum_{i \in I_2} (\mathbf{f}_i)^2, \\ \sum_{i \in I_1 \setminus I} (\mathbf{f}_i)^2/\lambda_i &= (\mathbf{B}^* \mathbf{B} \mathbf{A}^{-1} \mathbf{f}, \mathbf{A}^{-1} \mathbf{f}) - \sum_{i \in I} (\mathbf{f}_i)^2/\lambda_i,\end{aligned}$$

which yield

$$\begin{aligned}K &= \sum_{i \in I_1 \setminus I} \frac{\mathbf{f}_i^2}{(\lambda_i^{1/2} - \lambda_i^{-1/2} s_m)^2} \\ &\leq \frac{\sum_{i \in I_1 \setminus I} \mathbf{f}_i^2}{(\lambda_M^{1/2} - \lambda_M^{-1/2} s_m)^2} \leq \lambda_M \frac{(\mathbf{A}^{-1} \mathbf{f}, \mathbf{f}) - \sum_{i \in I} \mathbf{f}_i^2}{(\lambda_M - s_m)^2} \\ K &= \sum_{i \in I_1 \setminus I} \frac{\mathbf{f}_i^2/\lambda_i}{(1 - s_m/\lambda_i)^2} \\ &\leq \frac{\sum_{i \in I_1 \setminus I} \mathbf{f}_i^2/\lambda_i}{(1 - s_m/\lambda_M)^2} = \lambda_M^2 \frac{(\mathbf{B}^* \mathbf{B} \mathbf{A}^{-1} \mathbf{f}, \mathbf{A}^{-1} \mathbf{f}) - \sum_{i \in I} \mathbf{f}_i^2/\lambda_i}{(\lambda_M - s_m)^2}.\end{aligned}$$

We can now conclude that

$$\begin{aligned} \sum_{i \in I_1} \frac{\lambda_i \mathbf{f}_i^2}{(\lambda_i - s_m)^2} &= \sum_{i \in I} \frac{\lambda_i \mathbf{f}_i^2}{(\lambda_i - s_m)^2} + \sum_{i \in I_1 \setminus I} \frac{\lambda_i \mathbf{f}_i^2}{(\lambda_i - s_m)^2} \\ &= \sum_{i \in I} \frac{\lambda_i \mathbf{f}_i^2}{(\lambda_i - s_m)^2} + K \\ &\leq \sum_{i \in I} \frac{\lambda_i \mathbf{f}_i^2}{(\lambda_i - s_m)^2} + \frac{\lambda_M \alpha}{(\lambda_M - s_m)^2} = m^2/s_m, \end{aligned}$$

so that  $g'(s_m) = \sum_{i \in I_1} \frac{\lambda_i \mathbf{f}_i^2}{(\lambda_i - s_m)^2} - m^2/s_m \leq 0$  and therefore  $s_m \leq s$ .  $\square$

### 3. SHAPE DIFFERENTIATION

#### 3.1. Setting of the problem

The goal of this section is to compute a shape derivative of the robust compliance. This shape derivative will be later used in a numerical algorithm, the level set method, to optimize the domain. Recall from Section 2 that we consider the following shape optimization problem

$$\min_{\Omega \subset D} (\mathcal{J}(\Omega) + \eta|\Omega|)$$

where  $\eta$  is a given Lagrange multiplier for a volume constraint,  $D$  is a fixed bounded domain, and  $\mathcal{J}$  is the robust compliance defined by the following maximization problem

$$\mathcal{J}(\Omega) = \max_{u \in H_D^1(\Omega)^d} \left[ \int_{\Omega} (2f_1 \cdot u - Ae(u) : e(u)) \, dx + 2 \int_{\Gamma_N} f_2 \cdot u \, ds + 2m \left( \int_{\Omega} \chi u \cdot u \, dx \right)^{1/2} \right]. \quad (19)$$

We denote by  $\mathcal{M}$  the set of  $u$  that realize the maximum in the above expression. We reinterpret the abstract Theorem 2.7 as:

**Proposition 3.1.** *For a given smooth bounded open set  $\Omega$ , the set  $\mathcal{M}$  of maximizers of  $\mathcal{J}$  is given by:*

$$\mathcal{M} = \{u_{\rho^*} + r\omega \text{ such that } \omega \in E_{\lambda_1} \text{ and } \|\chi\omega\|_{L^2} = 1\}$$

with  $(\rho^*, r) = (s, 0)$  when the maximizer is unique,  
 $(\rho^*, r) = (\lambda_1, \frac{m}{\lambda_1} - \|\chi u_{\rho^*}\|_{L^2})$  when there are multiple maximizers.

We briefly recall the notion of shape differentiation, a standard tool that can be found in [16, 23].

**Definition 3.2.** For any vector field  $\theta \in W^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d)$  define

$$\Omega_{\theta} = (\text{Id} + \theta) \circ \Omega = \{x + \theta(x) \text{ such that } x \in \Omega\}.$$

For a given function  $\mathcal{J}(\Omega)$ , its shape derivative  $\mathcal{J}'$  is defined (when it exists) as the differential of the following functional  $J$  at 0

$$\begin{aligned} J: W^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d) &\rightarrow \mathbb{R} \\ \theta &\mapsto J(\theta) = \mathcal{J}(\Omega_{\theta}). \end{aligned}$$

Whether  $J$  admits a directional derivative or a Fréchet derivative, we say that  $\mathcal{J}'$  is a directional or Fréchet derivative.

### 3.2. A general derivation theorem

#### Definition 3.3.

– Let  $\alpha > 0$ ,  $\beta_1 > 0$  be constants and  $\mathbb{L}$  the space of linear unbounded self-adjoint operators from  $H \rightarrow H$  such that:

$$\forall L \in \mathbb{L}, \quad \forall u \in V \quad \beta_1 \|u\|_V^2 \geq (Lu, u)_H \geq \alpha \|u\|_V^2.$$

(In particular, the operators in  $\mathbb{L}$  have a “uniformly” compact resolvent in  $H$ , are uniformly coercive and continuous in  $V$ .)

– Let  $\beta_2 > 0$  be a constant and  $\mathbb{M}$  the space of continuous positive linear self-adjoint operators from  $H \rightarrow H$  uniformly continuous with constant  $\beta_2$ , *i.e.*:

$$\forall M \in \mathbb{M} \quad , \forall u \in H \quad \beta_2 \|u\|_H^2 \geq (Mu, u)_H \geq 0.$$

– Define the norm in those two spaces as follows: if  $N \in \mathbb{L}$  or  $N \in \mathbb{M}$  then

$$\|N\| = \max_{u \in V} \frac{(Nu, u)_H}{\|u\|_V^2} \quad (20)$$

*i.e.*, we endow  $\mathbb{M}$  with the natural norm of  $\mathbb{L}$ .

– Let  $X$  be a given Banach space  $X \rightarrow \mathbb{L} \times \mathbb{M} \times V'$ ,  $\theta \mapsto (\mathbf{L}(\theta), \mathbf{M}(\theta), \mathbf{l}(\theta))$  be a Fréchet differentiable mapping (for  $\mathbf{L}$  and  $\mathbf{M}$ , the differentiability is understood with respect to the norm just defined). Define  $D\mathbf{L}(\theta_0) \cdot \theta$  the derivative of the operator  $\mathbf{L}$  with respect to  $\theta$  at the point  $\theta_0$  applied to  $\theta$ . And define  $\mathbf{L}'(\theta) = D\mathbf{L}(0) \cdot \theta$ . Use consistent notations for the operator  $\mathbf{l}$  and  $\mathbf{M}$ .

We shall give precise examples of  $\mathbf{L}$ ,  $\mathbf{M}$  and  $\mathbf{l}$  in Proposition 3.6 below. For the moment, let us simply indicate that, typically,  $\mathbf{L}(0)$  is the elasticity operator  $\mathbf{A}$ ,  $\mathbf{M}(0)$  is the perturbation localization operator  $\mathbf{B}^* \mathbf{B}$  and  $\mathbf{l}(0)$  is the applied force for the actual shape  $\Omega_0$ .

**Theorem 3.4.** *Suppose that  $\mathbf{L}$  and  $\mathbf{M}$  and  $\mathbf{l}$  satisfy the assumptions of Definition 3.3. Suppose also that there exists a neighborhood  $U_0$  of 0 in  $X$  and a constant  $C$  such that for all  $\theta \in W^{1,\infty}$ ,  $\forall \theta_n, \theta_0 \in U_0$  the following boundedness and continuity properties hold:*

$$\|D\mathbf{L}(\theta_0) \cdot \theta\| \leq C \|\theta\|_{W^{1,\infty}} \quad (21)$$

$$\|[D\mathbf{L}(\theta_n) - D\mathbf{L}(\theta_0)] \cdot \theta\| / \|\theta\|_{W^{1,\infty}} \rightarrow 0 \quad \text{when } \theta_n \rightarrow \theta_0, \quad (22)$$

and assume that equations (21) and (22) hold true for  $\mathbf{M}$  and  $\mathbf{l}$  too. Define the following robust compliance functional:

$$J(\theta) = \max_{u \in V} \left\{ F(\theta, u) = -(\mathbf{L}(\theta)u, u) + 2(\mathbf{l}(\theta), u) + 2m(\mathbf{M}(\theta)u, u)^{1/2} \right\},$$

and let  $\mathcal{M}_0$  be the set of maximizers of  $J(0)$ . Then, for all  $\theta_0 \in X$ ,  $J(\theta)$  is directionally differentiable at the point  $\theta = 0$  in the direction  $\theta_0$  and the value of the directional derivative is

$$J'(\theta_0) = \max_{u \in \mathcal{M}_0} \left[ -(\mathbf{L}'(\theta_0)u, u) + 2(\mathbf{l}'(\theta_0), u) + \rho^*(\mathbf{M}'(\theta_0)u, u) \right]$$

where either  $\rho^* = \lambda_1$ , the smallest eigenvalue of  $Sp(\mathbf{L}(0), \mathbf{M}(0))$ , when there are multiple maximizers, or  $\rho^* = s = m/(\mathbf{M}(0)u_s, u_s)^{1/2}$  when there is a single maximizer.

*Proof (sketch).* In order to rigorously prove Theorem 3.4, Clarke’s subgradient theory [11] has to be used. The existence of subgradients is guaranteed by the uniform bounds imposed on the operators in Definition 3.3. The directional derivative of  $J$  is just obtained by taking the supremum on the subgradient. Details can be found in Appendix 7.  $\square$

### 3.3. Application to shape optimization

The goal of this Section is to compute the directional derivative of the robust compliance. In order to simplify the notations, only volume forces are considered (*i.e.*  $f_2 = 0$  in Sect. 2).

**Theorem 3.5.** *The robust compliance (19) is directionally differentiable with respect to the domain (Def. 3.2) and the value of the directional derivative in the direction  $\theta$  is*

$$\mathcal{J}'(\theta) = \max_{u \in \mathcal{M}_0} \int_{\partial\Omega} (\theta \cdot n) [v(u, u) + 2f_1 \cdot u] ds \quad (23)$$

where  $\mathcal{M}_0$  is the set of maximizers in the definition of  $\mathcal{J}(\Omega)$  and  $v$  is a bilinear term defined by

$$\begin{aligned} v(u, w) &= -Ae(u) : e(w) + \rho^* \chi u \cdot w && \text{on } \Gamma_N \\ v(u, w) &= Ae(u) : e(w) && \text{on } \Gamma_D \end{aligned}$$

where  $\rho^*$  is the common value of  $\frac{m}{\|\chi u\|}$  for every  $u$  in  $\mathcal{M}_0$ .

Theorem 3.4 cannot be directly applied here, because the spaces  $\mathbf{L}$  and  $\mathbf{M}$ , introduced in Definition 3.3, are assumed to be invariant with respect to  $\theta$ . So far, our formulation of the shape optimization problem has been Eulerian in the sense that the elasticity operator  $\mathbf{A}$  is defined from  $H_D^1(\Omega)^d \rightarrow H^{-1}(\Omega)^d \times H^{-1/2}(\Gamma_N)^d$  where  $\Omega$  is the current shape. By varying the displacement field  $\theta$ , we move the shape  $\Omega_\theta$ , and thus the function spaces on which  $\mathbf{A}$  is defined are changing with  $\theta$ . Therefore, we must first introduce a Lagrangian formulation of the problem by mapping back to the reference shape  $\Omega$  all objects defined on  $\Omega_\theta$  (this is a standard practice in shape optimization, see *e.g.* [16, 23]). We thus introduced transported or Lagrangian operators  $\mathbf{L}$  and  $\mathbf{M}$  as follows.

**Proposition 3.6.** *Let  $(\cdot, \cdot)$  denotes the scalar product of  $L^2(\Omega)$ . For every  $\theta$  in  $W^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d)$ , small enough, the map  $T = \text{Id} + \theta$  is a Lipschitz diffeomorphism and we can define:*

- $\mathbf{L}(\theta) \in \mathbf{L}$  such that  $(\mathbf{L}(\theta)v, w) = \int_{\Omega_\theta} Ae(v \circ T^{-1}) : e(w \circ T^{-1}) dx$ ;
- $\mathbf{M}(\theta) \in \mathbf{M}$  such that  $(\mathbf{M}(\theta)v, w) = \int_{\Omega_\theta} \chi(v \circ T^{-1}) \cdot (w \circ T^{-1})$ ;
- $\mathbf{l}(\theta) \in V'$  such that  $(\mathbf{l}(\theta), v) = \int_{\Omega_\theta} f_1 \cdot (v \circ T^{-1}) dx$ .

Consequently the robust compliance is given by

$$\mathcal{J}(\Omega_\theta) = J(\theta) = \max_{u \in H^1(\Omega)^d} \left[ -(\mathbf{L}(\theta)u, u) + 2(\mathbf{l}(\theta), u) + 2m(\mathbf{M}(\theta)u, u)^{1/2} \right].$$

*Proof.* The fact that  $T = \text{Id} + \theta$  is a Lipschitz diffeomorphism for small  $\theta$  is well known [16, 23]. As a consequence, the following equivalence holds true:  $u \circ T^{-1} \in H^1(\Omega_\theta)^d \iff u \in H^1(\Omega)^d$ .

Then a simple computation shows that

$$\begin{aligned} J(\theta) &= \max_{u \in H^1(\Omega)^d} \left[ -(\mathbf{L}(\theta)u, u) + 2(\mathbf{l}(\theta), u) + 2m(\mathbf{M}(\theta)u, u)^{1/2} \right] \\ &= \max_{u \in H^1(\Omega_\theta)^d} \left[ -(\mathbf{L}(\theta)u \circ T, u \circ T) + 2(\mathbf{l}(\theta), u \circ T) + 2m(\mathbf{M}(\theta)u \circ T, u \circ T)^{1/2} \right] \\ &= \max_{u \in H^1(\Omega_\theta)^d} \left[ - \int_{\Omega_\theta} Ae(u) : e(u) dx + 2 \int_{\Omega_\theta} f_1 \cdot u dx + 2m \left( \int_{\Omega_\theta} \chi u \cdot u dx \right)^{1/2} \right] \\ &= \mathcal{J}(\Omega_\theta). \end{aligned}$$

We now prove that  $\mathbf{L}(\theta)$  belongs to  $\mathbb{L}$  for small  $\theta$  (the other claims of Prop. 3.6 are proved in the same way). Recall the symmetries of the elasticity tensor  $A$ ,  $A^{ijkl} = A^{jikl} = A^{ijlk} = A^{klij}$ , so that, using Einstein summation convention,  $A^{ijkl}(\partial_j u^i)(\partial_l v^k) = Ae(u) : e(v)$ . We have

$$\begin{aligned} (\mathbf{L}(\theta)v, w) &= \int_{\Omega_\theta} A^{ijkl} \partial_j (v \circ T^{-1})^i \partial_l (w \circ T^{-1})^k \, dx \\ &= \int_{\Omega} |\det \nabla T| A^{ijkl} \partial_j (v \circ T^{-1})^i \circ T \partial_l (w \circ T^{-1})^k \circ T \, dx \\ &= \int_{\Omega} |\det \nabla T| A^{ijkl} (\partial_s v^i \partial_j (T^{-1})^s) (\partial_m w^l \partial_k (T^{-1})^m) \, dx \\ &= \int_{\Omega} C^{iskm}(\theta) \partial_s v^j \partial_m w^l \, dx \end{aligned}$$

with

$$\begin{aligned} C^{iskm}(\theta) &= A^{iskm} + C_\theta^{iskm} + o(\|\theta\|_{W^{1,\infty}}) \\ C_\theta^{iskm} &= (\partial_l \theta^l) A^{iskm} - A^{ijkm} (\partial_j \theta^s) - A^{iskl} (\partial_l \theta^m). \end{aligned}$$

These coefficients are differentiable with respect to  $\theta$ . This proves that the operator  $\mathbf{L}(\theta)$  is, at least for small  $\theta$ , uniformly coercive and bounded with respect to  $\theta$ .  $\square$

The above computation gives also the value of the derivative  $\mathbf{L}'(\theta)$ :

$$(\mathbf{L}'(\theta)v, w) = \int_{\Omega} C_\theta^{iskm} \partial_s v^j \partial_m w^l \, dx. \quad (24)$$

Similarly we have

$$\begin{aligned} (\mathbf{I}'(\theta), v) &= \int_{\Omega} [\operatorname{div}(\theta) f_1 \cdot v + f_1 \cdot (\nabla v \theta)] \, dx \\ (\mathbf{M}'(\theta)v, w) &= \int_{\Omega} \chi [\operatorname{div}(\theta)v \cdot w + (\nabla v \theta) \cdot w + v \cdot (\nabla w \theta)] \, dx. \end{aligned}$$

*Proof of Theorem 3.5.* In order to apply Theorem 3.4 to  $J$ , it remains to prove the boundedness and continuity condition of the derivative. It can be proved using the formula for the  $D\mathbf{L}(\theta_0) \cdot \theta$  derivative of  $\mathbf{L}$  at the point  $\theta_0$  in the direction  $\theta$ , which is obtained in the same way than (24) (valid for  $\theta_0 = 0$ ) and which is given by:

$$([D\mathbf{L}(\theta_0) \cdot \theta]v, w) = \int_{\Omega_{\theta_0}} C_\theta^{iskm} \partial_s (v^j \circ (\operatorname{Id} + \theta_0)^{-1}) \partial_m (w^l \circ (\operatorname{Id} + \theta_0)^{-1}) \, dx.$$

The term  $C_\theta^{iskm} / \|\theta\|_{W^{1,\infty}}$  is uniformly bounded in  $L^\infty$  norm by some  $D$ . Hence, for a small ball  $U_0$  in  $W^{1,\infty}$  centered in 0, the derivative is continuous and uniformly bounded in the norm (20):

$$\forall \theta_0 \in U_0 \quad \|D\mathbf{L}(\theta_0) \cdot \theta\| \leq C \|\theta\|_{W^{1,\infty}}$$

where  $C$  is a constant that depends only on the radius of the ball  $U_0$ . Hence the Lipschitz condition.

Theorem 3.4 can then be applied to  $J$  which is locally equal to  $\mathcal{J}(\Omega_\theta)$ . In order to prove Theorem 3.5, it is sufficient to show that

$$-(\mathbf{L}'(\theta)u, u) + 2(\mathbf{I}'(\theta), u) + \rho^*(\mathbf{M}'(\theta)u, u) = \int_{\partial\Omega} (\theta \cdot n)[v(u, u) + 2f_1 \cdot u] \quad (25)$$

where  $v(\cdot, \cdot)$  is defined in Theorem 3.5. It is a lengthy algebraic computation. First, we perform an integration by part on  $\theta$  in the left hand side of (25) which yields two terms, a volume integral in  $\Omega$  and a surface integral on  $\partial\Omega$ . The volume integral cancels out because  $u$  satisfies the Eulerian equations (5) and (6). The surface integral is equal to

$$\int_{\partial\Omega} ((\theta \cdot n) [-Ae(u): e(u) + \rho^* \chi u \cdot u + 2f_1 \cdot u] + 2A^{ijkl} \partial_s u^i \theta^s n^j \partial_l u^k) ds \quad (26)$$

where the last term takes different values on the Neumann part and on the Dirichlet part of the boundary. On the Neumann part, the homogeneous condition  $Ae(u) \cdot n = 0$  means that  $A^{ijkl} \partial_l u^k n^j = 0$ . Therefore this term is equal to 0. On the Dirichlet part of the boundary, since  $\nabla u = \frac{\partial u}{\partial n} n$ , we have

$$\partial_s u^i \theta^s n^j = \frac{\partial u^i}{\partial n} n^s \theta^s n^j = \partial_j u^i (\theta \cdot n),$$

so that the last term in the integrand of (26) is equal to  $2(\theta \cdot n) Ae(u): e(u)$ .  $\square$

#### 4. THE ALGORITHM OF SHAPE OPTIMIZATION

The function  $\theta \mapsto \mathcal{J}(\Omega_\theta)$  and the Lagrangian  $\mathcal{L}(\Omega_\theta) = \mathcal{J}(\Omega_\theta) + \eta|\Omega_\theta|$ , admit a directional derivative with respect to  $\theta \in W^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d)$  given by:

$$\mathcal{L}'(\theta) = \mathcal{J}'(\theta) + \int_{\partial\Omega} (\theta \cdot n) \eta ds. \quad (27)$$

The goal of this section is to develop a gradient-based algorithm for the minimization of  $\mathcal{L}(\Omega_\theta)$ . There are two main steps in this algorithm: the choice of a special descent direction and a level set method that moves the domain. We do not describe the level set theory here, and refer to [18, 21] or [4] for an extensive explanation of this method applied to shape optimization. The choice of a descent direction follows [15] where it has been introduced for optimization of multiple eigenvalues.

As usual, the directional derivative  $\mathcal{L}'(\theta)$  only depends on the value of  $(\theta \cdot n)$  on the boundary of  $\Omega$  that we shall now denote by  $\Theta$ . More precisely, according to (27) and Theorem 3.5, it has the following structure:  $\mathcal{L}'(\theta) = g(\theta \cdot n)$  where  $g$  is a function defined by

$$g(\Theta) = \max_{u \in \mathcal{M}_0} \int_{\partial\Omega} \Theta [v(u, u) + 2f_1 \cdot u + \eta] ds. \quad (28)$$

Introduce an Hilbert space  $\mathcal{H}$  (see [15] for the choices of  $\mathcal{H}$ ). Here we choose  $\mathcal{H} = H^1(D)$ . Perform a steepest-descent algorithm on the space  $\mathcal{H}$ , *i.e.* find a descent direction  $\Theta^* \in \mathcal{H}$  solution of the following problem

$$g(\Theta^*) = \min_{\|\Theta\|_{\mathcal{H}}=1} g(\Theta). \quad (29)$$

In this section, we will show that this problem is a Semi Definite Programming (SDP) problem in low dimension that is easily solved.

##### 4.1. Reduction to low dimension

The goal of this Section is to reduce the problem (29) to a problem in low dimension where the number of variables are of the same order than the dimension of the first eigenspace. This so-called ‘‘low-dimension’’ problem is then solved in Section 4.2 by an SDP method.

Recall that the set of maximizers of  $F(u)$  is given by Theorem 2.19 (to which we refer for the notations) as

$$\mathcal{M}_0 = \{u_{\rho^*} + r\omega \text{ such that } \omega \in E_{\lambda_1} \text{ and } \|\mathbf{B}\omega\| = 1\} \quad (30)$$

with  $(\rho^*, r) = (s, 0)$  when the maximizer is unique, and  $(\rho^*, r) = (\lambda_1, m/\lambda_1 - \|\mathbf{B}u_{\rho^*}\|)$  when there are multiple maximizers.

**Definition 4.1.** The scalar product below is implicitly assumed to be that of  $\mathcal{H}$  (that we chose to be  $\mathcal{H} = H^1(D)$ ).

- Define  $(\omega_i)_{i=1,\dots,d}$  a basis of  $E_{\lambda_1}$  orthonormalized by  $(B\omega_i, B\omega_j) = \delta_{ij}$ .
- Define  $(a_{ij})_{i,j=1,\dots,d}, (b_i)_{i=1,\dots,d}$  and  $c$  in  $\mathcal{H}$  by

$$\begin{aligned} (a_{ij}, X) &= \int_{\partial\Omega} X r^2 v(\omega_i, \omega_j) ds \\ (b_i, X) &= \int_{\partial\Omega} X r [v(u_{\rho^*}, \omega_i) + f_1 \cdot \omega_i] ds \\ (c, X) &= \int_{\partial\Omega} X [v(u_{\rho^*}, u_{\rho^*}) + 2f_1 \cdot u_{\rho^*} + \eta] ds \end{aligned}$$

where  $v(\cdot, \cdot)$  is defined in Theorem 3.5 as the bilinear part of the directional derivative.

- Define  $(h_k)_{k=1,\dots,m}$  an orthonormal basis of  $\text{Span}(a_{ij}, b_i, c)_{i,j}$  (a subspace of  $\mathcal{H}$  of finite dimension  $m$ ), and for any  $i, j$  let  $(a_{ij}^k)_{k=1,\dots,m}$  (resp.  $(b_i^k)_{k=1,\dots,m}$  and  $(c^k)_{k=1,\dots,m}$ ) be the coordinates of  $a_{ij}$  (resp.  $b_i$  and  $c$ ) in the basis  $(h_k)$ .

**Proposition 4.2.** *An optimal descent direction  $\Theta^*$ , solution of (29), is given by*

$$\Theta^* = \sum_{k=1}^m X_k^* h_k + \Theta^\perp$$

where  $(h_k)$  is the basis introduced in Definition 4.1,  $\Theta^\perp$  is any vector of adequate norm in  $[\text{Span}(h_k)_k]^\perp$  and  $X_k^*$  are minimizers of

$$\min_{\sum_{k=1}^m (X_k)^2 \leq 1} \max_{\sum_{i=1}^d (\gamma_i)^2 = 1} [\gamma_i \gamma_j a_{ij}^k X_k + 2\gamma_i b_i^k X_k + c^k X_k]. \quad (31)$$

*Proof.* Recall that  $g$  is defined in (28) as:

$$g(\Theta) = \max_{u \in \mathcal{M}_0} \int_{\partial\Omega} \Theta [v(u, u) + 2f_1 \cdot u + \eta] ds$$

where  $v$  is bilinear. Using the characterization (30) of  $\mathcal{M}_0$  we have:

$$g(\Theta) = \max_{\substack{\omega \in E_{\lambda_1} \\ \|\mathbf{B}\omega\|=1}} \left[ \int_{\partial\Omega} \Theta [v(u_{\rho^*} + r\omega, u_{\rho^*} + r\omega) + 2f_1 \cdot u_{\rho^*} + 2f_1 \cdot r\omega + \eta] ds \right].$$

Decomposing  $\omega = \sum_i \gamma_i \omega_i$  and using Definition 4.1, we have:

$$g(\Theta) = \max_{\sum_i \gamma_i^2 = 1} [\gamma_i \gamma_j (a_{ij}, \Theta) + 2\gamma_i (b_i, \Theta) + (c, \Theta)].$$

Decomposing the descent direction in  $\Theta = \sum_{k=1}^m X_k h_k + \Theta^\perp$ , then the  $X_k$  are minimizers of (31).  $\square$



## 4.2. The semi definite programming problem

We solve the low dimensional problem (31) by a SDP (Semi Definite Programming) method. We refer to [26] for an introduction to SDP problems. We shall only recall here that a SDP problem is of the following type: given a matrix  $E(Y)$  whose coefficients depend linearly on the vector  $Y$  and a given vector  $Y_0$ , find the vector  $Y^*$  which realizes the minimum

$$\min_{E(Y) \geq 0} Y^T Y_0$$

where  $\geq$  stands for “symmetric nonnegative”.

### Definition 4.3.

- For any  $X = (X_1, \dots, X_m)$  and  $(z, w) \in \mathbb{R}^2$ , let  $Y = [X, w, z]$ .
- Let  $A(X)$  be the  $d \times d$  matrix  $A(X)_{ij} = a_{ij}^k X_k$ , let  $B(X)$  be the  $d \times 1$  matrix  $B(X)_i = b_i^k X_k$ , let  $C(X)$  be the scalar  $C(X) = c^k X_k$ , and let

$$D(Y) = \begin{bmatrix} -A(X) + z\text{Id} & B(X) \\ B^*(X) & -C(X) - z + w \end{bmatrix}, \quad E(Y) = \begin{bmatrix} D(Y) & 0 & 0 \\ 0 & \text{Id} & X \\ 0 & X^T & 1 \end{bmatrix},$$

where the coefficients of  $E$  depend linearly on  $Y = [X, w, z]$ .

- Let  $Y^* = [X^*, w^*, z^*]$  be the solution of the following SDP problem:

$$\min_{E(Y) \geq 0} w. \tag{32}$$

We now prove that problems (29) and (32) are equivalent. Let us emphasize that (32) is easy to solve in practice and that its computational time is negligible compared to the computational time of the evaluation of the robust compliance.

**Proposition 4.4.** *Let  $Y^* = [X^*, w^*, z^*]$  be the solution of the SDP problem (32). Then  $(X_k^*)_k$ , the coordinates of  $X^*$ , are minimizers of problem (31). Hence, in view of Proposition 4.2, a descent direction  $\Theta^*$ , solution of (29), is given by*

$$\Theta^* = \sum_{k=1}^m X_k^* h_k.$$

*Proof.* First notice that the matrices  $A$ ,  $B$  and  $C$  are defined so that

$$\begin{aligned} & \min_{\sum_k (X_k)^2 \leq 1} \max_{\sum_i (\gamma_i)^2 = 1} [\gamma_i \gamma_j a_{ij}^k X_k + 2\gamma_i b_i^k X_k + c^k X_k] \\ &= \min_{X^T X \leq 1} \max_{\gamma^T \gamma = 1} [\gamma^T A(X) \gamma + 2\gamma^T B(X) + C(X)]. \end{aligned}$$

Let us drop the  $X$  dependence for a moment. For any matrix  $A$  and vector  $B$ , it is known (see e.g. [26], p. 229) that the following trust-region problems are equivalent

$$\max_{\gamma^T \gamma = 1} [\gamma^T A \gamma + 2\gamma^T B] = \min_{-A+z\text{Id} \geq 0} [B^T (-A + z\text{Id})^{-1} B + z],$$

with the convention that  $B$  must belong to the range of  $(-A + z\text{Id})$ . Thus the following problems are equivalent

$$\begin{aligned} & \max_{\gamma^T \gamma = 1} [\gamma^T A \gamma + 2\gamma^T B + C] = \min_{-A+z\text{Id} \geq 0} [B^T (-A + z\text{Id})^{-1} B + z + C] \\ &= \min_{-A+z\text{Id} \geq 0} w = \min_{D \geq 0} w, \tag{33} \\ & -C - z + w \geq B^T (-A + z\text{Id})^{-1} B \end{aligned}$$

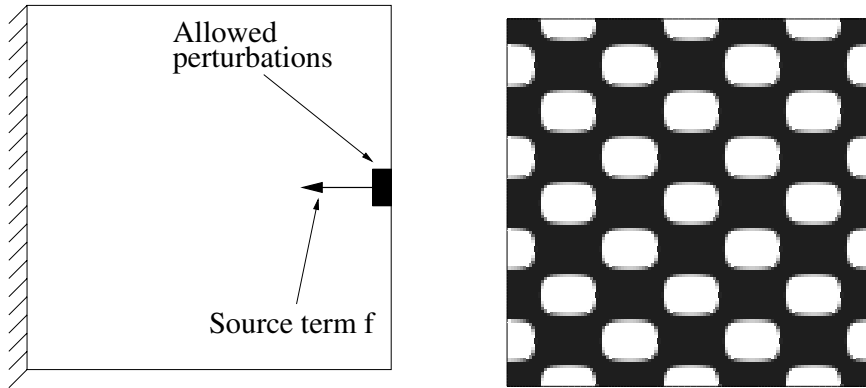


FIGURE 1. The beam to cantilever problem and its initialization.

where  $D = \begin{bmatrix} -A + z\text{Id} & B \\ B^T & -C - z + w \end{bmatrix}$ . Thanks to (33), problem (31) (and hence (29)) are equivalent to

$$\min_{X^T X \leq 1} \min_{D(Y) \geq 0} w.$$

The constraint  $E(Y) \geq 0$  is equivalent to  $X^T X \leq 1$  and  $D(Y) \geq 0$ . So that (31) is equivalent to (32).  $\square$

## 5. NUMERICAL RESULTS

### 5.1. Beam to cantilever transition

The working domain is a  $1 \times 1$  square discretized by a  $120 \times 120$  quadrangular mesh. The shape is clamped on the left wall and a unit horizontal load is applied in the middle of the right wall. Vertical perturbations are allowed in a non-optimizable box (the black box of Fig. 1 left) in the middle of the right wall. We perform several tests with an increasing parameter  $m$ , *i.e.* increasing the norm of the allowed perturbations. For each of the nine following tests, the same initialization is used (Fig. 1 right). Lagrange multipliers are adjusted so that the optimal shapes have always a volume equal to 0.2. The other parameters are the same for each test and are set to 1 for the Young modulus and to 0.3 for the Poisson's ratio. The weak-material approximation is used to mimic the void. Its Young modulus is set to  $10^{-4}$  with the same Poisson's ratio than the plain material.

Figure 2 shows the solutions for increasing  $m$  ( $m$  increases from left to right and then from top to bottom). When  $m$  is equal to 0 the robust-compliance problem is a standard compliance problem whose solution is a beam. It can be seen that the upper-left shape is close to a beam ( $m$  is the smallest) and the lower-right shape is close to a cantilever ( $m$  is the biggest). When  $m$  is large, the force term is negligible and the problem becomes an eigenvalue problem (see Rem. 2.4). The optimal shape is then close to a classical cantilever.

### 5.2. The wheel-bridge

On a  $16 \times 12$  rectangle, discretized by a  $160 \times 120$  quadrangular mesh, we study the case of the wheel-bridge. Namely, a unit vertical load is applied on the middle of the bottom of the domain and the shape is only constrained by a zero vertical displacement on the two sides of the bottom of the bridge (the gray boxes of Fig. 3 left). To eliminate rigid displacement, we enforce a null horizontal displacement at the node where the load is applied. Vertical perturbations are located on the black boxes of Figure 3 left.

As for the previous example, the test was run for several different values of the parameter  $m$ . The same initialization was used for the different parameters. Optimal shapes are displayed in Figure 4.

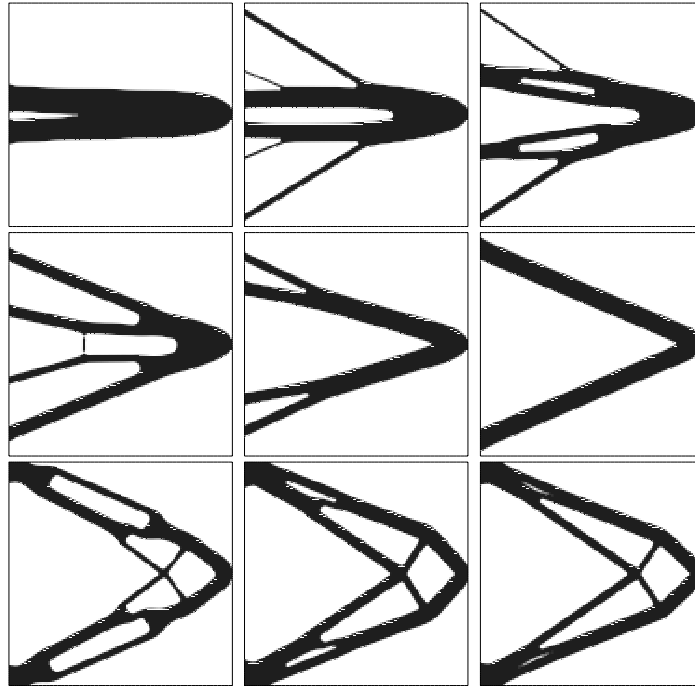


FIGURE 2. Different values of  $m = 10^{-3}, 5 \times 10^{-3}, 7 \times 10^{-3}, 8.5 \times 10^{-3}, 10^{-2}, 2 \times 10^{-2}, 10^{-1}, 1, 4$  in the beam-to-cantilever problem.

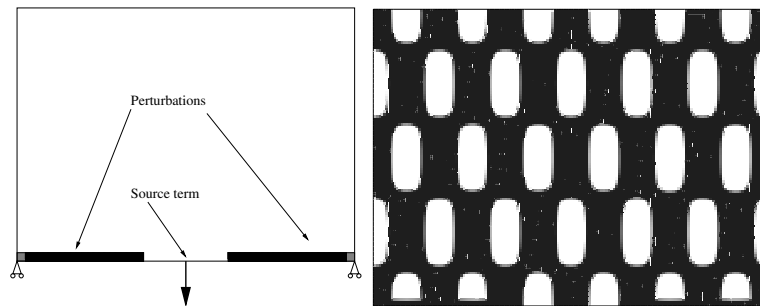


FIGURE 3. The wheel-bridge problem and its initialization.

We also performed a numerical test for the multiple loads wheel-bridge (see Fig. 5). Namely three sets of vertical forces are applied, one on the middle of the bridge and two uniform volume loads, each being located on one black box of Figure 3. The optimization criterion is then the sum of the three compliances, the  $L^2$ -norm of the three loads being the same. The optimal shapes of the robust compliance and of the multiple loads problem are similar: this is in accordance with the fact that the worst perturbation for the optimal shape is made of two vertical symmetric set of forces (and not asymmetric as pulling on one side and pushing on the other side).

### 5.3. The 2-d mast

The optimal mast problem is a T-shaped working domain meshed with 3600 square cells. The T shape has an height of 6, a width of 2 at the bottom and 4 at the top. The bottom of the shape is fixed while two vertical loads are applied at the lower corner of the horizontal branch of the T (see Fig. 6). This test was run in [2] for

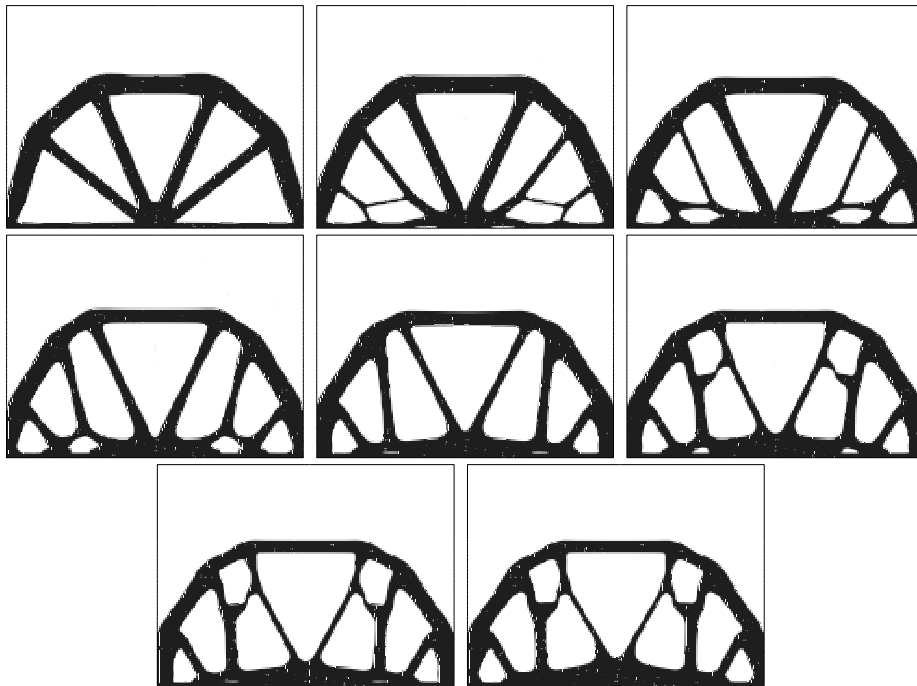


FIGURE 4. Different values of  $m = 10^{-3}, 0.1, 0.3, 0.5, 1.3, 1.5, 2.3, 6$  in the wheel-bridge problem.

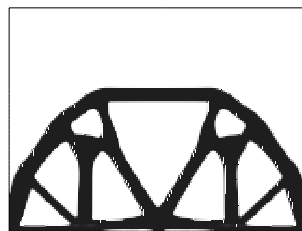


FIGURE 5. The multiloaded wheel-bridge optimal shape.

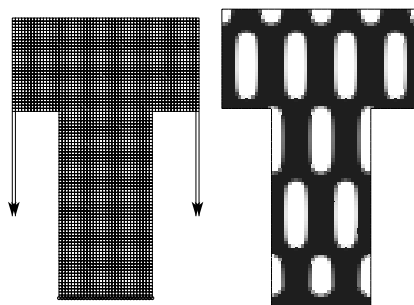


FIGURE 6. Mesh and initialization of the 2-d mast problem.

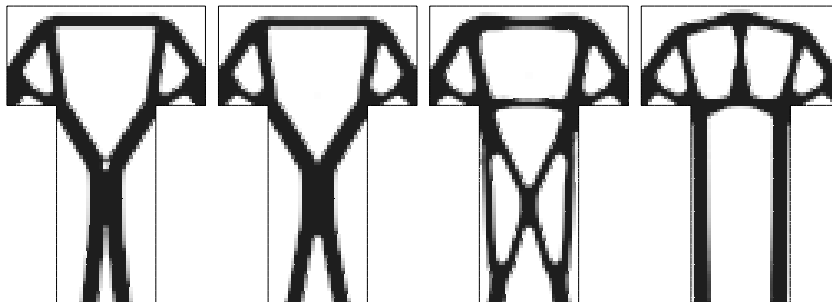


FIGURE 7. The perturbation parameter  $m$  is successively equal to 0, 0.01, 0.05, 0.1 for the 2d-mast problem (vertical perturbations).

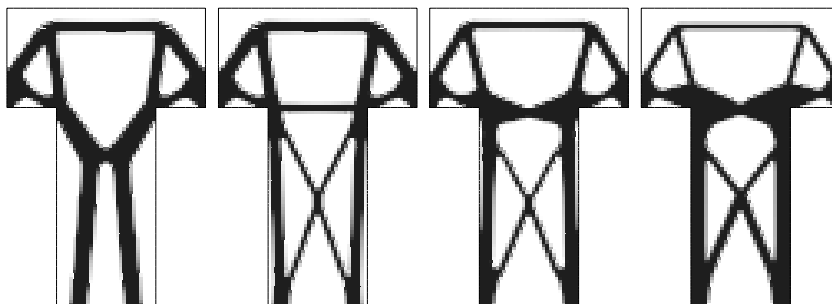


FIGURE 8. Increasing values of  $m$  from 0.02, 0.03, 0.1 to 0.2 for the 2d-mast problem (vertical and horizontal perturbations).

single load compliance minimization. Vertical perturbations are allowed where the loads are sets. Figure 7 shows the different optimal shapes for different values of the parameter  $m$ .

The 2-d mast test has also been performed with perturbations both horizontal and vertical. Figure 8 shows the solutions for increasing values of  $m$ .

We have here a good example of how different sets of allowed perturbations (different operators  $\mathbf{B}$ ) give different results. Optimal shapes of Figure 7 are less sensitive to vertical perturbations than the one of Figure 8 but more to horizontal ones. The Lagrange multiplier for the volume constraint  $\eta$  is higher when the set of allowed perturbations is broader, *i.e.* for the second set of optimal shapes.

#### 5.4. The 3-d beam

A  $40 \times 40 \times 21$  domain is discretized by a  $20 \times 20 \times 21$  quadrangular mesh. Denoting by  $(x_1, x_2, x_3)$  the spatial coordinates, the horizontal plane  $x_3 = 0$  is submitted to an homogeneous Dirichlet boundary condition and a normal vertical unit load is applied on the middle of the plane  $x_3 = 21$ . Perturbations are allowed everywhere and in all directions. The Young modulus is 1 in the shape and  $10^{-3}$  in the void. The Poisson ratio is 0.3 everywhere. For all the numerical tests, the same initialization is used and the Lagrange multiplier is set so that the different shapes have the same volume: 14% of the total volume of the working domain. Optimal shape for increasing  $m$  are displayed in Figure 9.

It is interesting to notice that, if the value of  $m$  is increased too much, then the algorithm will have the tendency to remove the upper part of the shape. This behavior of the algorithm is well-known in optimization of the first eigenvalue (see [15] for the same test with eigenvalue optimization). A common way to avoid this problem is to put an heavier mass-tip at the top. The present test was designed so that we could avoid this trick

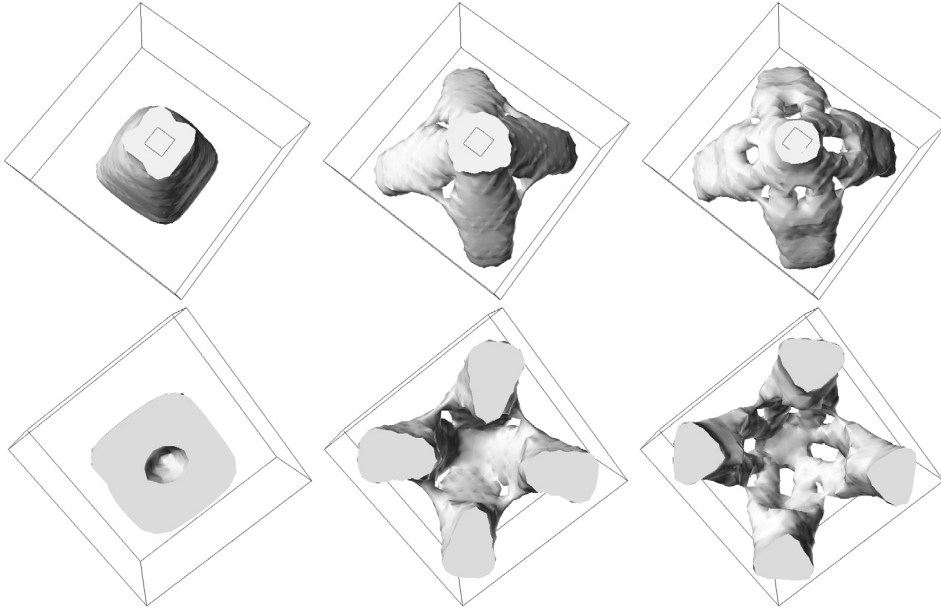


FIGURE 9. Upper and bottom view of the optimal shapes for  $m = 10^{-2}, 3 \times 10^{-2}$  and  $5 \times 10^{-2}$ .

of the mass-tip by imposing a balance between the optimization of the first eigenvalue and the optimization with respect to the load.

### 5.5. The 3-d chair

This problem has also been studied in [3]. The four bottom corners are fixed while the back and the seat of the chair are not subject to optimization and support pressure loads. The pressure applied on the back of the chair is 5 times smaller than the pressure applied on the seat. Perturbations are allowed both on the back and on the seat and are of the following type:

$$\mathbf{B}^* \tau = \chi_S \tau_3 e_3 + 0.2 \chi_B \tau_1 e_1 \quad \text{with } \|\tau\|_{L^2(D)^3} \leq m,$$

where  $(e_1, e_2, e_3)$  is the usual Cartesian basis of  $\mathbb{R}^3$  and  $\tau_i$  are the corresponding coordinates of  $\tau$ .  $\chi_S$  denotes the characteristic function of the seat while  $\chi_B$  is the one of the back. Two optimal shapes, for  $m = 0$  and  $m = 1$ , are displayed on Figure 10. The robust-compliance optimal chair has a more complex topology and is more stable. The topology of the robust-compliance optimal chair is not the same than the topology of the multiple loads optimal chair (see [3]). The robust chair seems more stable than the multiple loads chair.

### 5.6. Computational performances

In terms of computational complexity the robust compliance optimization is definitely more expensive than single load compliance minimization. However, it should be not too expensive compared to multiple loads compliance minimization with many loads or the maximization of the first eigenvalues. Indeed, the most costly step in the optimization process is the solution of the direct problem, as detailed in Section 2.5, which can be done quite efficiently as we shall see. The other step of computing a descent direction is not so expensive since there is no need of an adjoint state and the choice of the best descent direction is an SDP problem in small dimension, as explained in Section 4.

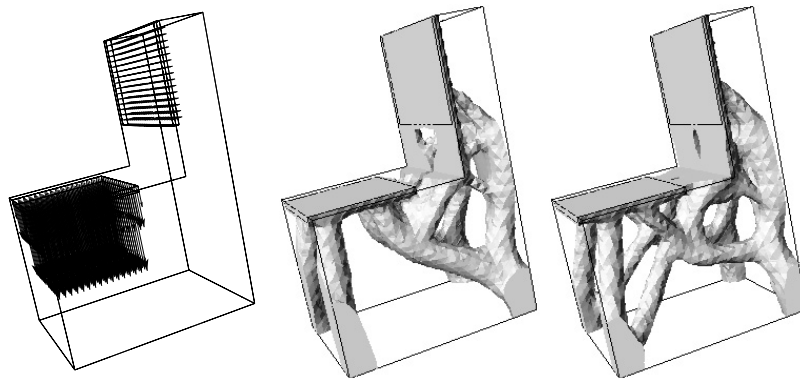


FIGURE 10. Loading and optimal shape of the chair for  $m = 0$  (left) and  $m = 1$  (right).

We now explain an efficient way of solving algorithm (18) for the direct problem of robust compliance. A prerequisite of this algorithm is to compute the first eigenvalue and first eigenfunction (possibly multiple ones) of

$$(\mathbf{A} - \lambda_1 \mathbf{B}^* \mathbf{B})v_1 = 0.$$

This usually requires factorizing the matrix  $\mathbf{A}$  which is the bottleneck in terms of CPU and memory requirement. Let us show that it can be the only matrix factorization of the algorithm. The other steps of this Newton method are the computation of

$$u_\rho = (\mathbf{A} - \rho \mathbf{B}^* \mathbf{B})^{-1} \mathbf{f} \quad \text{and} \quad v_\rho = \mathbf{A}^{-1} \mathbf{B}^* \mathbf{B} (\mathbf{A} - \rho \mathbf{B}^* \mathbf{B})^{-1} \mathbf{A} u_\rho,$$

for successive values of  $\rho < \lambda_1$  (usually less than 10 to 15 iterations are needed). Knowing the factorization of  $\mathbf{A}$ , the computations of  $u_\rho$  and  $v_\rho$ , which require a linear solve with the matrix  $(\mathbf{A} - \rho \mathbf{B}^* \mathbf{B})$  can easily be done by an iterative solver using  $\mathbf{A}$  as a preconditioner. Since the matrix  $\mathbf{B}$  corresponds to the discretization of a characteristic function, the condition number of the preconditioned matrix should be not too large and the resulting convergence of iterative methods very fast.

In truth we have not implemented such an iterative solver and we content ourselves in factorizing the matrix  $(\mathbf{A} - \rho \mathbf{B}^* \mathbf{B})$  at each iteration of the Newton method (which is, of course, time consuming but much simpler to implement). Recall that in practice the number of Newton iterations is around 10 to 15, so each time the direct problem is solved by Newton method (*i.e.*  $\rho^* = s$ ) we perform 10 to 15 matrix factorization where 1 would be enough. Taking into account the fact that the Newton method is not always used (when  $\rho^* = \lambda_1$ ), we guess our computational times are probably larger by a factor of, at least, 5 to 10 than the best possible performances.

For 2-d test cases the robust compliance optimization usually requires of the order of 100 iterations (*i.e.* direct problems to solve). For example, the wheel-bridge problem took around 5000 seconds for the robust compliance, while only 500 s were necessary for the multiple loads optimization (with approximately the same number of iterations). Convergence was detected when the relative error of the  $L^2$  norm of two successive iterates of the material density is smaller than  $10^{-3}$ .

As an example of a 3-d test case, let us discuss the 3-d chair problem. The standard compliance problem is solved in 80 iterations for a total of 1 hour and 35 minutes of computation time while the robust compliance problem is solved with 112 iterations in 7 hours on a standard PC (Pentium 4 at 2.6GHz).

We expect that, with an efficient implementation, our robust compliance optimization algorithm should be comparable in complexity to the standard multiple loads compliance minimization or the eigenvalues maximization problem.

## 6. CONCLUSION

We proved an existence result for the worst perturbation of a given set of loads  $\mathbf{f}$ . We also characterized the set of worst perturbations. We gave a stable algorithm that computes the set of worst perturbations and the robust compliance. This algorithm relies on a Newton type method and we gave efficient bounds on the critical point of the Newton method. Thanks to those bounds, in all the tests made, we did not need more than a small number of iterations (of the order of 10 to 15) inside the Newton method. Each iteration requires solving a linear system with a matrix that does not change too much. This could be done efficiently by an iterative method. We thus expect that an efficient implementation makes the problem of finding the worst perturbation of a given load of the same order of computational difficulty than computing the compliance for that load.

We then proved the existence of a shape directional derivative for the robust compliance problem and used a local SDP problem in order to deduce a regularized and extended velocity for the level set method. Our numerical algorithm successfully finds optimal shapes that are stable with respect to unknown perturbations in the loading. Optimal designs are new and different from the previous designs obtained by multiple loads optimization. They are, of course, more stable than those obtained by single load compliance optimization.

However we do not improve the level set algorithm in its standard drawbacks. Amongst them are the dependency with respect to initialization and the dependency with respect to some numerical parameters. Those drawbacks are linked to the possibility for the level set method (which is a simple steepest descent algorithm here) to fall into local minima, often corresponding to different topologies. One possible remedy is to incorporate a topological gradient step as in [2]. We did not try this idea so far in the context of robust optimization.

Eventually, since the robust compliance problem is self-adjoint and is somehow intermediate between the standard compliance and a generalized eigenvalue problem, it can be fully analysed in the context of homogenization theory (see Th. 4.1.30 in [1]). This relaxation theory bears the advantage of proving the existence of a global minimizer. However, a numerical implementation of the robust compliance objective function in the setting of the homogenization theory has yet to be performed.

## 7. APPENDICES

### Appendix A: Existence of a local maximum for the direct problem of the robust compliance

We exhibit a simple problem of robust compliance where a local maximum is not a global maximum. Let  $x \in \mathbb{R}$  be a parameter that will be chosen large enough. Let  $m = 3\sqrt{x^2 + 1}$  and

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} x \\ 1 \end{bmatrix}, \quad u = \begin{bmatrix} -2x \\ 2 \end{bmatrix}.$$

It is easy to check that  $u$  is indeed a critical point of  $F(u) = -(\mathbf{A}u, u) + 2(\mathbf{f}, u) + 2m\|u\|$ , in the sense that  $\partial_u F = -\mathbf{A}u + \mathbf{f} + \frac{m}{\|u\|}u = 0$ .

In order to prove that  $u$  is a local maximum, we have to prove that the Hessian of  $F$  is definite negative. The Hessian is given by:

$$\partial_u^2 F(v, w) = -(\mathbf{A}v, w) + \frac{m}{\|u\|}(v, w) - \frac{m^3}{\|u\|^3}(v, u)(w, u)$$

which as a matrix is equal to:

$$\partial_u^2 F = \frac{1}{2(x^2 + 1)} \begin{bmatrix} (x^2 + 1) - 3x^2 & 3x \\ 3x & -(x^2 + 1) - 3 \end{bmatrix}.$$



The trace of  $\partial_u^2 F$  is equal to  $-3/2$  and setting  $x$  big enough gives  $\det(\partial_u^2 F) > 0$  which means that the two eigenvalues of the matrix are strictly negative and that  $u$  is a local maximum. But  $u$  can not be a global maximum because

$$F(2x, 2) = 12x^2 + 8 > 4x^2 + 8 = F(-2x, 2) = F(u).$$

## Appendix B: Proof of Theorem 3.4

This section is devoted to the proof of Theorem 3.4 which is done in three steps. First we state Clarke's subgradient result, then we apply it to the robust compliance, and thirdly we compute the collection of directional derivatives when the subgradient is given.

### Clarke's subgradient theorem

We simply state Theorem (2.8.2) of [11].

**Definition 7.1.** Let  $U \subset X$  be a subset of a Banach space  $X$ ,  $S$  be a metrisable space and  $F : X \times S \rightarrow \mathbb{R}$ . For any  $\theta \in U \subset X$  define

$$J(\theta) = \max_{u \in S} F(u, \theta).$$

We introduce the following notations:

- $\partial_\theta F(u, \theta_0) \in X^*$  is the differential of  $\theta \mapsto F(u, \theta)$  at the point  $\theta_0$ .
- $\mathcal{M}(\theta) = \{u \in S \text{ s.t. } F(u, \theta) = J(\theta)\}$ , *i.e.* the set of maximizers.
- $P[\mathcal{M}(\theta)]$  is the set of Radon probability measures supported by  $\mathcal{M}(\theta)$ .
- $\left\langle \int_S \partial_\theta F(u, \theta_0) \mu(du), v \right\rangle_{\langle X^*, X \rangle} = \int_S \langle \partial_\theta F(u, \theta_0), v \rangle_{\langle X^*, X \rangle} \mu(du)$ .

**Theorem 7.2.** Under assumptions (i) to (vi) below

- (i)  $S$  is sequentially compact;
- (ii)  $\forall \theta \in U$  the mapping  $u \mapsto F(u, \theta)$  is upper semi-continuous;
- (iii)  $\exists K > 0$  independent of  $u$  such that,  $\forall u \in S$ , the mapping  $\theta \mapsto F(u, \theta)$  is Lipschitz of rank  $K$ ;
- (iv)  $\forall \theta \in U$ , the set  $\{F(u, \theta)\}_{u \in S}$  is bounded;
- (v)  $\forall (\theta_0, u) \in U \times S$ ,  $\partial_\theta F(u, \theta_0)$  exists;
- (vi) the mapping  $(\theta_0, u) \mapsto \partial_\theta F(u, \theta_0)$ , from  $U \times S$  into  $X^*$ , is continuous, there exists a subgradient of  $J$  at the point  $\theta_0 = 0$  given by

$$\partial J(0) = \left\{ \int_S \partial_\theta F(u, 0) \mu(du) \text{ s.t. } \mu \in P[\mathcal{M}(0)] \right\}. \quad (34)$$

### The subgradient of $J$

We now apply Theorem 7.2 to the functional

$$F(u, \theta) = -(\mathbf{L}(\theta)u, u) + 2(\mathbf{l}(\theta), u) + 2m(\mathbf{M}(\theta)u, u)^{1/2}$$

such that the robust compliance is just

$$J(\theta) = \max_{u \in V} F(u, \theta) = \max_{u \in H} F(u, \theta) \quad \forall \theta \in X.$$

Unfortunately, the hypotheses of Theorem 7.2 are not satisfied for  $(S, U) = (H, X)$ . Therefore, we have to find some subsets  $S \subset H$  and  $U \subset X$  for which the assumptions of Theorem 7.2 hold true. We give in Lemma 7.1 some sufficient conditions that imply Theorem 7.2. In Proposition 7.3, we build some subsets to which we can apply Lemma 7.1.

**Lemma 7.1.** *Recall that  $\mathcal{M}(\theta)$  is the set of maximizers  $u$  of  $F(u, \theta)$  over  $H$ . Let  $U$  be a neighborhood of 0 in  $X$ , and  $S \subset H$  be such that*

- (1)  $\forall \theta \in U, \forall u \in \mathcal{M}(\theta), u$  belongs to  $S$ ;
- (2)  $\exists \gamma > 0$  such that,  $\forall u \in S, \forall \theta \in U$ , we have  $\|u\|_V \leq \gamma$ ;
- (3)  $\exists \beta > 0$  such that,  $\forall u \in S, \forall \theta \in U$ , we have  $(\mathbf{M}(\theta)u, u) \geq \beta$ .

Then,  $\forall \theta \in U, J(\theta) = \max_{u \in S} F(u, \theta)$  and  $J(\theta)$  has a subgradient at  $\theta = 0$ , given by Theorem 7.2.

*Proof.* Property (1) of  $S$  implies that  $J(\theta) = \max_{u \in H} F(u, \theta) = \max_{u \in S} F(u, \theta)$ . We check that  $S$  satisfies the assumptions of Theorem 7.2. Using the norm of  $H$ , property (ii) is immediate. Property (iv) holds true because  $S$  is a bounded set in  $V$ . Property (2) of  $S$  gives the compactness with respect to the norm of  $H$ . Of course  $S$  is metrisable in the  $H$  topology. So that (i) is satisfied for the topology of  $H$ .

The differential  $\partial_\theta F$  is defined as

$$\partial_\theta F(u, \theta_0) \cdot \theta = -(D\mathbf{L}(\theta_0) \cdot \theta u, u) + 2(D\mathbf{I}(\theta_0) \cdot \theta u, u) + m \frac{(D\mathbf{M}(\theta_0) \cdot \theta u, u)}{(\mathbf{M}(\theta_0)u, u)^{1/2}}$$

where the differentials  $D\mathbf{L}(\theta_0), D\mathbf{M}(\theta_0)$  and  $D\mathbf{I}(\theta_0)$  are defined in Definition 3.3. Assumption (3) on  $S$  implies that  $\partial_\theta F(u, \theta_0)$  is defined everywhere, this proves property (v).

We prove (iv) for the term  $\frac{(D\mathbf{M}(\theta_0) \cdot \theta u, u)}{(\mathbf{M}(\theta_0)u, u)^{1/2}}$ , which, thanks to property (3) is continuous as long as  $(D\mathbf{M}(\theta_0) \cdot \theta u, u)$  and  $(\mathbf{M}(\theta_0)u, u)$  are continuous. For  $D\mathbf{M}(\theta_0)$ , it is an hypothesis of Theorem 3.4. For  $(\mathbf{M}(\theta_0)u, u)$ , Theorem 3.4 stipulates that  $\exists U_0$  so that  $\theta_0 \mapsto \mathbf{M}(\theta_0)$  is Lipschitz continuous with respect to the norm defined in Definition 3.3:

$$\|\mathbf{M}(\theta_0)\| = \max_{v \in V} \frac{(\mathbf{M}(\theta_0)v, v)}{\|v\|_V^2}.$$

This implies, using assumption (2), and the fact that  $S \subset V, (u, \theta) \mapsto (\mathbf{M}(\theta_0)u, u)$  is Lipschitz continuous on  $S \times U$ . This proves property (iv).

Property (v) is proved in the same way. If  $u$  is fixed, assumption (3) states that  $\theta \mapsto F(u, \theta)$  is Lipschitz as long the three functionals  $\theta \mapsto (\mathbf{A}(\theta)u, u)$  (resp.  $(\mathbf{M}(\theta)u, u); (\mathbf{I}(\theta)u, u)$ ) are Lipschitz. This is a direct consequence of the Lipschitz property of Theorem 3.4.  $\square$

**Proposition 7.3.** *There exists a subset  $U \subset X$  such that if  $S = \cup_{\theta \in U} \mathcal{M}(\theta)$ , then  $(S, U)$  verifies the assumption of Lemma 7.1.*

*Proof.* The subset  $U$  is defined as a ball of center 0 and radius  $\tau$ . The value of  $\tau$  will be adjusted later. For every  $\tau$ , Property (1) is immediate. We remark that  $F(u, \theta) \geq F(0, \theta) = 0, \forall \theta, \forall u \in \mathcal{M}(\theta)$ , so that

$$-(\mathbf{L}(\theta)u, u) + 2(\mathbf{I}(\theta), u) + 2m(\mathbf{M}(\theta)u, u)^{1/2} \geq 0.$$

Using the uniform properties of  $\mathbf{L}$  and  $\mathbf{M}$ , we deduce

$$\alpha \|u\|_V^2 \leq 2\|\mathbf{I}(\theta)\|_{V'} \|u\|_V + 2m\sqrt{\beta_2} \|u\|_H.$$

The continuous injection of  $V$  into  $H$ , and a uniform bound on  $\|\mathbf{l}(\theta)\|_V, \forall \theta \in U$  yields the uniform upper bound (2). Then,  $\forall \theta \in U$ , a maximizer  $u$  of  $F(u, \theta)$  over  $H$ , satisfies the Euler optimality condition

$$(\mathbf{M}(\theta)u, u)^{1/2} = \frac{m}{\rho_\theta}$$

where  $\rho_\theta$  is always smaller than the first eigenvalue of the generalized eigenproblem  $\mathbf{L}(\theta)u = \lambda\mathbf{M}(\theta)u$ . Expressing the first eigenvalue as a Rayleigh quotient leads to

$$\lambda(\theta) \leq \min_{u \in H} \frac{(\mathbf{L}(\theta)u, u)}{(\mathbf{M}(\theta)u, u)}.$$

For any  $v \in V$  such that  $(\mathbf{M}(0)v, v)$  is not zero, upon a restriction on  $\tau$ , there exist constants  $c_1 > 0$  and  $c_2 > 0$  independent of  $\theta$  such that  $(\mathbf{M}(\theta)v, v) \geq c_1$  and  $(\mathbf{L}(\theta)v, v) \leq c_2$  for all  $\theta$  in  $U$ . Then  $\lambda(\theta) \leq c_2/c_1$  and

$$\forall \theta \in U, \forall u \in \mathcal{M}(\theta), \quad (\mathbf{M}(\theta)u, u)^{1/2} = \frac{m}{\rho_\theta} \geq \frac{m}{\lambda(\theta)} \geq \frac{mc_1}{c_2}$$

which is nothing else than the lower bound (3). □

### Directional derivative

Let  $\theta$  be given and denote by  $J(\theta)'$  the directional derivative of  $J(\theta)$  in the direction  $\theta$  at the point  $\theta = 0$ . The proof of Theorem 3.4 is reduced to proving the following result.

**Proposition 7.4.** *For any direction  $\theta$ , the directional derivative of the robust compliance at 0 exists and is equal to*

$$J(\theta)' = \max_{u \in \mathcal{M}(0)} [-(\mathbf{L}(\theta)'u, u) + 2(\mathbf{l}(\theta)', u) + \rho^*(\mathbf{M}(\theta)'u, u)]. \quad (35)$$

*Proof.* Thanks to Lemma 7.1, the subgradient of the robust compliance  $J$  exists at  $\theta_0 = 0$ . A standard corollary is that the directional derivative of  $F$  at 0 in the direction  $\theta \in X$  exists and is equal to

$$J(\theta)' = \max_{\zeta \in \partial J(0)} \langle \zeta, \theta \rangle_{\langle X^*, X \rangle}.$$

Recall that, if  $\zeta \in \partial J(0)$ , there exists  $\mu \in P[\mathcal{M}(0)]$  such that

$$\langle \zeta, \theta \rangle_{\langle X^*, X \rangle} = \int_S \langle \partial_\theta F(u, 0), \theta \rangle_{\langle X^*, X \rangle} \mu(du). \quad (36)$$

First, there exists  $u^* \in \mathcal{M}(0)$  such that

$$\langle \partial_\theta F(u^*, 0), \theta \rangle_{\langle X^*, X \rangle} = \max_{u \in \mathcal{M}(0)} \langle \partial_\theta F(u, 0), \theta \rangle_{\langle X^*, X \rangle}$$

since

$$\langle \partial_\theta F(u, 0), \theta \rangle_{\langle X^*, X \rangle} = -(D\mathbf{L}(0) \cdot \theta_0 u, u) + 2(D\mathbf{l}(0) \cdot \theta_0, u) + \rho^*(D\mathbf{M}(0) \cdot \theta_0 u, u)$$

and the set of maximizers  $\mathcal{M}(0)$  is finite dimensional and bounded (see Th. 2.19), thus compact.

Second, whatever the measure  $d\mu$ , the convex sum (36) is always lower or equal to  $\langle \partial_\theta F(u^*, 0), \theta \rangle$  and the maximum of such convex sums over all probability measures is therefore attained and equal to  $\langle \partial_\theta F(u^*, 0), \theta \rangle$  when  $d\mu$  is a Dirac mass whose support is  $u^*$ . This yields formula (35) with the notation  $D\mathbf{L}(0) \cdot \theta_0 = \mathbf{L}(\theta_0)'$ . □

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