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PARTIAL REGULARITY FOR ANISOTROPIC FUNCTIONALS OF HIGHER ORDER

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Abstract. We prove a $C^{k,\alpha}$ partial regularity result for local minimizers of variational integrals of the type $I(u) = \int_\Omega f(D^k u(x))\mathrm{d}x$, assuming that the integrand f satisfies (p,q) growth conditions.

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1. Introduction

Higher order variational functionals, emerging in the study of problems from materials science and engineering, have attracted a great deal of attention in last few years [4,5,7]. In particular, the regularity of minimizers of such functionals has been studied very recently. In [15] and [16] the partial $C^{k,\alpha}$ regularity has been established for quasiconvex integrals with a p-power growth with respect to the gradient and in [3] for convex integrals having subquadratic nonstandard growth condition, only in dimension 2.

The aim of this paper is to establish the partial regularity of minimizers of integral functionals of the type

$$I(u) = \int_\Omega f(D^k u(x))\mathrm{d}x \tag{1}$$

where Ω is a bounded subset of $\mathbb{R}^n$, $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$, $N \geq 1$, $k > 1$ and f is a C^2 convex integrand satisfying the non standard growth condition:

$$C|\xi|^p \leq f(\xi) \leq L(1 + |\xi|^q) \tag{2}$$

with $p < q$, without restriction on the dimension and on the order of derivatives involved, in the superquadratic case.

Nonstandard growth conditions have been introduced by Marcellini, in the scalar case for $k = 1$. He observed that, even in the scalar case, minimizers of (1) may fail to be regular (see [13,17,18]), when q is too large with respect to p. On the other hand, one can prove regularity of scalar minimizers of (1) if q is not too far away from p (see *e.g.* [19] and its references). More precisely, in [19] it is shown that if one writes down the Euler equation for the functional I, under suitable assumptions on p and q, the Moser iteration argument still works, thus leading to a *sup* estimate for the gradient Du of the minimizer.

Keywords and phrases. Partial regularity, non standard growth, higher order derivatives.

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Clearly this approach can not be carried on in the vector valued case, *i.e.* when $N > 1$. First regularity results for systems are proved in [1] and [20] under special structure assumptions and in [22] in a more general setting. Moreover, higher integrability results for the gradient of the minimizers of (1) are avalaible in the vectorial case (see the references in [2, 8, 9]).

In this paper we prove that, for $k > 1$, differently from all previous quoted results, if f satisfies (2) and the strong ellipticity assumption

$$\langle D^2 f(\xi)\eta, \eta\rangle \geq \gamma(1 + |\xi|^2)^{\frac{p-2}{2}}|\eta|^2 \tag{3}$$

where

$$2 \leq p < q < \min\left\{p + 1, \frac{pn}{n-1}\right\}, \tag{4}$$

a minimizer $u \in W^{k,p}(\Omega; \mathbb{R}^N)$ of functional (1) is $C^{k,\alpha}$ for all $\alpha < 1$ in an open set $\Omega_0 \subset \Omega$ such that $\operatorname{meas}(\Omega \setminus \Omega_0) = 0$.

We point out that apart from condition (4), no special structure assumption is needed on f and the condition on the exponents does not depend on k, *i.e.* the order of derivatives involved.

The proof of our result goes through a more or less standard blow-up argument aimed to establish a decay estimate on the excess function for the k- order derivatives

$$U(x_0, r) = \fint_{B_r(x_0)} |D^k u - (D^k u)_{x_0,r}|^2 + |D^k u - (D^k u)_{x_0,r}|^p \mathrm{d}x.$$

Here, first order techniques have to be combined with new theoretical arguments needed to face the analytical and geometrical constraints of higher order derivatives. In particular, the essential tool is a lemma due to Fonseca and Malý (see [11] and also Lem. 2.4 below) which makes possible to connect in an annulus $B_r \setminus B_s$ two $W^{k,p}$ functions v and w with a more regular function function $z \in W^{k,q}(B_r \setminus B_s)$ with $p < q < \frac{pn}{n-1}$.

2. Statements and preliminary lemmas

Let us consider the functional

$$I(u) = \int_\Omega f(D^k u(x))\mathrm{d}x$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $n \geq 2$. Let $f : \mathbb{R}^{MN} \to \mathbb{R}$, where $M = \frac{(n+k-1)!}{k!(n-1)!}$ and $N \geq 2$, satisfy the following assumptions:

$$f \in C^2 \tag{H1}$$

$$C|\xi|^p \leq f(\xi) \leq L(1 + |\xi|^q) \tag{H2}$$

$$\langle D^2 f(\xi)\eta, \eta\rangle \geq \gamma(1 + |\xi|^2)^{\frac{p-2}{2}}|\eta|^2 \tag{H3}$$

where

$$2 \leq p < q < \min\left\{p + 1, \frac{pn}{n-1}\right\}.$$

It is well known that

$$|Df(\xi)| \leq c(1 + |\xi|^{q-1}). \tag{H4}$$

We say that $u \in W^{k,p}(\Omega, \mathbb{R}^N)$ is a minimizer of I if

$$I(u) \leq I(u + v)$$

for any $v \in u + W_0^{k,p}(\Omega; \mathbb{R}^N)$.

Remark 2.1. If u is a local minimizer of I and $\phi \in C_0^k(\Omega; I\!R^N)$ from the minimality condition one has for any $\varepsilon > 0$

$$0 \leq \int_\Omega [f(D^k u + \varepsilon D^k \phi) - f(D^k u)]\mathrm{d}x = \varepsilon \int_\Omega \mathrm{d}x \int_0^1 \frac{\partial f}{\partial \xi_\alpha^i}(D^k u + \varepsilon t D^k \phi) D_\alpha \phi^i \mathrm{d}t$$

where $|\alpha| = k$. Dividing this inequality by ε, and letting ε go to zero, from (H4) and the assumption $q \leq p+1$ we get

$$\int_\Omega \frac{\partial f}{\partial \xi_\alpha^i}(D^k u) D_\alpha \phi^i \mathrm{d}x \geq 0$$

and therefore, by the arbitrariness of ϕ, the usual Euler-Lagrange system holds:

$$\int_\Omega \frac{\partial f}{\partial \xi_\alpha^i}(D^k u) D_\alpha \phi^i \mathrm{d}x = 0 \qquad \forall \phi \in C_0^k(\Omega; I\!R^N).$$

The aim of this paper is proving the following

Theorem 2.1. *Let f satisfy the assumptions* (H1)–(H3) *and let $u \in W^{k,p}(\Omega; I\!R^N)$ be a minimizer of I. Then there exists an open subset Ω_0 of Ω such that*

$$\mathrm{meas}(\Omega \setminus \Omega_0) = 0$$

and

$$u \in C^{k,\alpha}(\Omega_0, I\!R^N) \quad \textit{for all} \quad \alpha < 1.$$

In what follows, we will denote by u a $W^{k,p}(\Omega; I\!R^N)$ minimizer of the integral functional (1) and assume that its integrand f satisfies (H1)–(H3). We set for every $B_r(x_0) \subset \Omega$

$$⨍_{B_r(x_0)} g = (g)_{x_0,r} = \frac{1}{\mathrm{meas}(B_r(x_0))} \int_{B_r(x_0)} g.$$

Moreover, given $p > 1$ and $u \in W^{k,p}(\Omega; I\!R^N)$, $k \geq 1$, we will denote by $P(y) = P_u(x, R, y)$ the unique polynomial of degree $k-1$ such that

$$\int_{B_r(x)} D^l(u(y) - P(y))\mathrm{d}y = 0 \qquad l = 1, \ldots, k-1.$$

Its coefficients depend on x, R and also on the derivatives of u (see [12]). When no confusion will arise, we will omit the dependence of P on x, R and u.

Next lemma can be found in [11], (Th. 3.3), in a slightly different form.

Lemma 2.2. *Let $v \in W^{k,p}(B_1(0))$ and $0 < s < r < 1$. There exists a linear operator $T:\ W^{k,p}(B_1(0)) \to W^{k,p}(B_1(0))$ such that*

$$Tv = v \qquad \text{on} \quad (B_1 \setminus B_r) \cup B_s$$

and for all $\mu > 0$, for all $q < p\frac{n}{n-1}$

$$
\begin{aligned}
&\left(\int_{B_r\setminus B_s}|D^kTv|^2\right)^{\frac12}+\mu\left(\int_{B_r\setminus B_s}|D^kTv|^q\right)^{\frac1q}\\
&\qquad\le C(r-s)^\rho\Bigg\{\left[\sup_{t\in(s,r)}(t-s)^{-\frac12}\left(\int_{B_t\setminus B_s}|D^kv|^2\right)^{\frac12}+\sup_{t\in(s,r)}(r-t)^{-\frac12}\left(\int_{B_t\setminus B_s}|D^kv|^2\right)^{\frac12}\right]\\
&\qquad\qquad+\mu\left[\sup_{t\in(s,r)}(t-s)^{-\frac1p}\left(\int_{B_r\setminus B_t}|D^kv|^p\right)^{\frac1p}+\sup_{t\in(s,r)}(r-t)^{-\frac1p}\left(\int_{B_r\setminus B_t}|D^kv|^p\right)^{\frac1p}\right]\Bigg\} \qquad (5)
\end{aligned}
$$

where $C=C(n,p,q)>0$, and $\rho=\rho(n,p,q)>0$.

Let us recall an elementary Lemma proved in [10].

Lemma 2.3. *Let ψ be a continuous nondecreasing function on an interval $[a,b]$, $a<b$. There exist $a'\in[a,a+\frac13(b-a)]$, $b'\in[b-\frac13(b-a),b]$ such that $a\le a'<b'\le b$ and*

$$
\begin{aligned}
\frac{\psi(t)-\psi(a')}{t-a'} &\le 3\,\frac{(\psi(b)-\psi(a))}{b-a}\\
\frac{\psi(b')-\psi(t)}{b'-t} &\le 3\,\frac{(\psi(b)-\psi(a))}{b-a}
\end{aligned}
$$

for all $t\in(a',b')$.

Finally, combining the previous two lemmas we obtain a generalization to the case of higher order derivatives of Lemma 2.4 in [10]. We give the proof here for completeness.

Lemma 2.4. *Let $v,\ w\in W^{k,p}(B_1(0))$ and $\frac14<s<r<1$. Fix $p<q<\frac{np}{n-1}$, for all $\mu>0$ and $m\in I\!N$ there exist a function $z\in W^{k,p}(B_1(0))$ and $\frac14<s<s'<r'<r<1$ with $r',\ s'$ depending on $v,\ w$ and μ, such that*

$$
z=v \quad on \quad B_{s'}\ , \quad z=w \quad on \quad B_1\setminus B_{r'}, \qquad (6)
$$

$$
\frac{r-s}{m}\ge r'-s'\ge\frac{r-s}{3m}
$$

and

$$
\begin{aligned}
&\left(\int_{B_{r'}\setminus B_{s'}}|D^kz|^2\right)^{\frac12}+\mu\left(\int_{B_{r'}\setminus B_{s'}}|D^kz|^q\right)^{\frac1q}\\
&\qquad\le C\frac{(r-s)^\rho}{m^\rho}\Bigg[⨍_{B_r\setminus B_s}\left(1+\sum_{l=0}^k|D^lv|^2+\sum_{l=0}^k|D^lw|^2+\frac{m^2}{(r-s)^2}\sum_{l=1}^{k-1}|D^l(v-w)|^2\right)\\
&\qquad\qquad+\mu^p⨍_{B_r\setminus B_s}\left(1+\sum_{l=0}^k|D^lv|^p+\sum_{l=1}^k|D^lw|^p+\frac{m^p}{(r-s)^p}\sum_{l=0}^{k-1}|D^l(v-w)|^p\right)\Bigg]^{\frac12} \qquad (7)
\end{aligned}
$$

where $C=C(n,p,q)>0$ and $\rho=\rho(p,q,n)>0$.

Proof. As in Lemma 2.4 in [10], choose $m \in IN$ and set

$$f = 1 + \sum_{l=0}^{k} |D^l v|^2 + \sum_{l=0}^{k} |D^l w|^2 + \tfrac{m^2}{(r-s)^2} \sum_{l=0}^{k-1} |D^l (v-w)|^2$$
$$+\mu^p (1 + \sum_{l=0}^{k} |D^l v|^p + \sum_{l=0}^{k} |D^l w|^p + \tfrac{m^p}{(r-s)^p} \sum_{l=0}^{k-1} |D^l (v-w)|^p).$$

We may find $h \in \{1, ..., m\}$ such that

$$\int_{B_{s+\frac{h(r-s)}{m}} \setminus B_{s+\frac{(h-1)(r-s)}{m}}} f \mathrm{d}x \leq \frac{1}{m} \int_{B_r \setminus B_s} f \mathrm{d}x.$$

Set, for $t \in \left[s + \frac{(h-1)(r-s)}{m}, s + \frac{h(r-s)}{m}\right]$,

$$\psi(t) = \int_{B_t \setminus B_s} f \mathrm{d}x$$

which is a continuous increasing function. By Lemma 2.3, there exists $[s', r'] \subset \left[s + \frac{(h-1)(r-s)}{m}, s + \frac{h(r-s)}{m}\right]$ such that

$$\frac{r-s}{m} \geq r' - s' \geq \frac{r-s}{3m}$$

and

$$\int_{B_t \setminus B_{s'}} f \mathrm{d}x \leq 3\frac{(t-s')m}{r-s} \int_{B_{s+\frac{h(r-s)}{m}} \setminus B_{s+\frac{(h-1)(r-s)}{m}}} f \mathrm{d}x \leq 3\frac{t-s'}{r-s} \int_{B_r \setminus B_s} f \mathrm{d}x, \tag{8}$$

$$\int_{B_{r'} \setminus B_t} f \mathrm{d}x \leq 3\frac{r'-t}{r-s} \int_{B_r \setminus B_s} f \mathrm{d}x \tag{9}$$

for all $t \in (s', r')$. Set

$$u = \begin{cases} v(x) & \text{if} \quad x \in B_{s'} \\ \frac{(r'-|x|)v(x) + (|x|-s')w(x)}{r'-s'} & \text{if} \quad x \in B_{r'} \setminus B_{s'} \\ w(x) & \text{if} \quad x \in B_1 \setminus B_{r'}. \end{cases}$$

A direct computation shows that

$$\sum_{l=0}^{k} |D^l u|^2 + \mu^q \left(\sum_{l=0}^{k} |D^l u|^p \right) \leq Cf.$$

If we apply Lemma 2.2 to the function u, we then find $z \in W^{k,p}(B_1)$ satisfying (6). Moreover, from (8) and (9), using (5), one readily cheks that

$$\left(\int_{B_{r'} \setminus B_{s'}} |D^k z|^2 \right)^{\frac{1}{2}} + \mu \left(\int_{B_{r'} \setminus B_{s'}} |D^k z|^q \right)^{\frac{1}{q}}$$
$$\leq c(r'-s')^\rho \left\{ \frac{|B_{r'} \setminus B_{s'}|^{\frac{1}{2}}}{(r'-s')^{\frac{1}{2}}} \left(\fint_{B_r \setminus B_s} f \right)^{\frac{1}{2}} + \frac{|B_{r'} \setminus B_{s'}|^{\frac{1}{p}}}{(r'-s')^{\frac{1}{p}}} \left(\fint_{B_r \setminus B_s} f \right)^{\frac{1}{p}} \right\}$$
$$\leq c(r'-s')^\rho \left\{ \left(\fint_{B_r \setminus B_s} f \right)^{\frac{1}{2}} + \left(\fint_{B_r \setminus B_s} f \right)^{\frac{1}{p}} \right\},$$

from which (7) follows. □

3. The decay estimate

As usual, to get the partial regularity result stated in Theorem 2.1, we need a decay estimate for the excess function $U(x_0, r)$ defined as follows

$$U(x_0, r) = \fint_{B_{r(x_0)}} [|D^k u - (D^k u)_{x_0,r}|^2 + |D^k u - (D^k u)_{x_0,r}|^p] \mathrm{d}y,$$

which measures how the k-order derivatives are far from being constant in the ball $B_{r(x_0)}$. The desired decay estimate is established in the next proposition.

Proposition 3.1. *Fix $M > 0$. There exists a constant $C_M > 0$ such that for every $0 < \tau < \frac{1}{4}$, there exists $\epsilon = \epsilon(\tau, M)$ such that, if*

$$|(D^k u)_{x_0,r}| \leq M \qquad \text{and} \qquad U(x_0, r) \leq \epsilon$$

then

$$U(x_0, \tau r) \leq C_M \tau^2 U(x_0, r).$$

Proof. Fix M and τ. We shall determine C_M later. We argue by contradiction assuming that there exists a sequence $B_{r_h}(x_h)$ satisfying

$$B_{r_h}(x_h) \subset \Omega, \qquad |(D^k u)_{x_h,r_h}| \leq M, \qquad \lim_h U(x_h, r_h) = 0,$$

but

$$U(x_h, \tau r_h) > C_M \tau^2 U(x_h, r_h). \tag{10}$$

Set

$$A_h = (D^k u)_{x_h,r_h} \qquad \lambda_h^2 = U(x_h, r_h)$$

and let P the polynomial such that

$$\int_{B_{r_h}(x_h)} D^l(u - P) = 0 \qquad l = 0, \ldots, k.$$

Step 1. Blow up. We rescale the function u in each $B_{r_h}(x_h)$ to obtain a sequence of functions on $B_1(0)$. Set

$$v_h(y) = \frac{1}{\lambda_h r_h^k}[u(x_h + r_h y) - P(x_h + r_h y)],$$

then

$$D^k v_h(y) = \frac{1}{\lambda_h}[D^k u(x_h + r_h y) - A_h].$$

Clearly we have

$$(D^l v_h)_{0,1} = 0 \qquad l = 0, \ldots, k.$$

Moreover,

$$\frac{U(x_h, r_h)}{\lambda_h^2} = \fint_{B_1} [|D^k v_h|^2 + \lambda_h^{p-2} |D^k v_h|^p] \mathrm{d}y = 1. \tag{11}$$

Then, passing possibly to a subsequence, we may suppose that

$$v_h \rightharpoonup v \qquad \text{weakly} \quad \text{in} \quad W^{k,2}(B_1; \mathbb{R}^N) \tag{12}$$

and, since $\forall h \ \ |A_h| \le M$,

$$A_h \to A. \tag{13}$$

Step 2. v solves a linear system. Now we show that

$$\int_{B_1(0)} \frac{\partial^2 f}{\partial \xi^i_\alpha \partial \xi^j_\beta}(A) D_\beta v^j D_\alpha \phi^i \mathrm{d}y = 0 \qquad\qquad \forall \phi \in C^k_0(B_1; I\!R^N). \tag{14}$$

Since we assume $q-1 \le p$ we can write the usual Euler-Lagrange system for u (see Rem. 2.1). Then, rescaling in each $B_{r_h}(x_h)$, we get for any $\phi \in C^k_0(B_1; I\!R^N)$ and any $1 \le i \le N$

$$\int_{B_1(0)} \frac{\partial f}{\partial \xi^i_\alpha}(A_h + \lambda_h D^k v_h) D_\alpha \phi^i \mathrm{d}y = 0$$

where $|\alpha| = k$. Then

$$\frac{1}{\lambda_h}\int_{B_1(0)} [\frac{\partial f}{\partial \xi^i_\alpha}(A_h + \lambda_h D^k v_h) - \frac{\partial f}{\partial \xi^i_\alpha}(A_h)] D_\alpha \phi^i \mathrm{d}y = 0. \tag{15}$$

Let us split

$$B_1 = E^+_h \cup E^-_h = \{y \in B_1 : \lambda_h |D^k v_h(y)| > 1\} \cup \{y \in B_1 : \lambda_h |D^k v_h(y)| \le 1\}$$

then, by (11), we get

$$|E^+_h| \le \int_{E^+_h} \lambda_h^2 |D^k v_h|^2 \mathrm{d}y \le \lambda_h^2 \int_{B_1(0)} |D^k v_h|^2 \mathrm{d}y \le c\lambda_h^2. \tag{16}$$

Now, by (H4) and Hölder's inequality, we observe that

$$\begin{aligned}
&\frac{1}{\lambda_h}\Big| \int_{E^+_h} [Df(A_h + \lambda_h D^k v_h) - Df(A_h)] D\phi \mathrm{d}y\Big| \\
&\qquad \le \frac{c}{\lambda_h}|E^+_h| + c\lambda_h^{q-2}\int_{E^+_h} |D^k v_h|^{q-1}\mathrm{d}y \\
&\qquad\qquad \le c\lambda_h + c\left(\int_{E^+_h} \lambda_h^{p-2}|D^k v_h|^p \mathrm{d}y\right)^{\frac{q-1}{p}} \lambda_h^{\frac{2q-p-2}{p}} |E^+_h|^{\frac{p-q+1}{p}} \le c\lambda_h
\end{aligned}$$

where we used again the assumption $q-1 \le p$.

From this it follows that

$$\lim_h \frac{1}{\lambda_h}\int_{E^+_h} [Df(A_h + \lambda_h D^k v_h) - Df(A_h)] D\phi \mathrm{d}y = 0. \tag{17}$$

On E^-_h we have

$$\begin{aligned}
\frac{1}{\lambda_h}\int_{E^-_h} [Df(A_h + \lambda_h D^k v_h) - Df(A_h)] D\phi \ \mathrm{d}y &= \int_{E^-_h}\int_0^1 D^2 f(A_h + s\lambda_h D^k v_h) D^k v_h D\phi \ \mathrm{d}s \ \mathrm{d}y \\
&= \int_{E^-_h}\int_0^1 [D^2 f(A_h + s\lambda_h D^k v_h) - D^2 f(A_h)] D^k v_h D\phi \ \mathrm{d}s \ \mathrm{d}y \\
&+ \int_{E^-_h} D^2 f(A_h) D^k v_h D\phi \ \mathrm{d}y.
\end{aligned}$$

Note that (16) ensures that $\chi_{E_h^-} \to \chi_{B_1}$ in $L^r(B_1)$ for all $r < \infty$ and by (11) we have, passing possibly to a subsequence,

$$\lambda_h D^k v_h(y) \to 0 \qquad a.e. \qquad \text{in} \quad B_1.$$

Then, by (12), (13) and the uniform continuity of $D^2 f$ on bounded sets, we get

$$\lim_h \frac{1}{\lambda_h} \int_{E_h^-} [Df(A_h + \lambda_h D^k v_h) - Df(A_h)] D\phi \, \mathrm{d}y = \int_{B_1} D^2 f(A) D^k v D\phi \, \mathrm{d}y.$$

Collecting (15), (17) and the above equality, we obtain that v satisfies system (14), which is linear and elliptic with constant coefficients by (H3). By standard regularity results (see [12]), we have for any $0 < \tau < 1$

$$\fint_{B_\tau} |D^k v - (D^k v)_\tau|^2 \mathrm{d}y \le c\tau^2 \fint_{B_1} |D^k v - (D^k v)_1|^2 \mathrm{d}y \le c\tau^2. \tag{18}$$

Moreover we have

$$v \in C^\infty(B_1; I\!R^N) \tag{19}$$

and

$$\lambda_h^{\frac{p-2}{p}} (v_h - v) \rightharpoonup 0 \quad \text{weakly in } W^{k,p}_{\mathrm{loc}}(B_1; I\!R^N).$$

Step 3. Upper bound. We set

$$f_h(\xi) = \frac{1}{\lambda_h^2} [f(A_h + \lambda_h \xi) - f(A_h) - \lambda_h Df(A_h)\xi]$$

and, for every $r < 1$, we consider

$$I_{h,r}(w) = \int_{B_r} f_h(D^k w) \mathrm{d}y.$$

Note that, by the strong ellipticity assumption (H3), it follows that $f_h(\xi) \ge 0$, for any ξ, and remember that v_h is a local minimizer for each $I_{h,r}$. Fix $\frac{1}{4} < s < 1$. Passing to a subsequence we may always assume that

$$\lim_h [I_{h,s}(v_h) - I_{h,s}(v)]$$

exists.

We shall prove that

$$\lim_h [I_{h,s}(v_h) - I_{h,s}(v)] \le 0. \tag{20}$$

Consider $r > s$ and fix $m \in I\!N$. Observe that, since $v \in W^{k,p}(B_1)$ and $v_h \in W^{k,p}(B_1)$, Lemma 2.4, with $\mu = \lambda_h^{\frac{q-2}{q}}$, implies that there exist $z_h \in W^{k,p}(B_1)$ and $\frac{1}{4} < s < s_h < r_h < r < 1$ such that

$$z_h = v \quad \text{on} \quad B_{s_h} \qquad z_h = v_h \quad \text{on} \quad B_1 \setminus B_{r_h}$$

and

$$\left(\int_{B_{r_h}\setminus B_{s_h}} |D^k z_h|^2\right)^{\frac{1}{2}} + \lambda_h^{\frac{q-2}{q}} \left(\int_{B_{r_h}\setminus B_{s_h}} |D^k z_h|^q\right)^{\frac{1}{q}}$$
$$\leq C\frac{(r-s)^\rho}{m^\rho}\Big[\fint_{B_r\setminus B_s} (1+\sum_{l=0}^{k}|D^l v|^2+\sum_{l=0}^{k}|D^l v_h|^2+\frac{m^2}{(r-s)^2}\sum_{l=1}^{k-1}|D^l(v-v_h)|^2)$$
$$+\lambda_h^{\frac{q-2}{q}p}\fint_{B_r\setminus B_s}(1+\sum_{l=0}^{k}|D^l v|^p+\sum_{l=0}^{k}|D^l v_h|^p+\frac{m^p}{(r-s)^p}\sum_{l=1}^{k-1}|D^l(v-v_h)|^p)\Big]^{\frac{1}{2}}. \quad (21)$$

Since by (19), $D^k v$ is locally bounded on B_1 we get

$$\begin{aligned} I_{h,s}(v_h)-I_{h,s}(v) &\leq I_{h,r_h}(v_h)-I_{h,r_h}(v)+I_{h,r_h}(v)-I_{h,s}(v) \\ &= I_{h,r_h}(v_h)-I_{h,r_h}(v)+\int_{B_{r_h}\setminus B_s} f_h(D^k v) \\ &\leq I_{h,r_h}(z_h)-I_{h,r_h}(v)+c(r-s) \\ &\leq \int_{B_{r_h}\setminus B_{s_h}} [f_h(D^k z_h)-f_h(D^k v)]+c(r-s) \end{aligned} \quad (22)$$

where we used the minimality of v_h.

As $|f_h(\xi)|\leq c(|\xi|^2+\lambda_h^{q-2}|\xi|^q)$, we get by (21), using the fact that $\frac{r-s}{m}<1$ and that the quantity on square brackets is greater or equal than 1,

$$\begin{aligned} I_{h,r_h}(z_h)-I_{h,r_h}(v) &\leq c\int_{B_{r_h}\setminus B_{s_h}} |D^k z_h|^2+\lambda_h^{q-2}|D^k z_h|^q \\ &\leq c\frac{(r-s)^{2\rho}}{m^{2\rho}}\Big[\fint_{B_r\setminus B_s}(1+\sum_{l=0}^{k}|D^l v|^2+\sum_{l=0}^{k}|D^l v_h|^2+\frac{m^2}{(r-s)^2}\sum_{l=1}^{k-1}|D^l(v-v_h)|^2)\Big]^{\frac{q}{2}} \\ &+ c\frac{(r-s)^{2\rho}}{m^{2\rho}}\Big[\lambda_h^{\frac{q-2}{q}p}\fint_{B_r\setminus B_s}(1+\sum_{l=0}^{k}|D^l v|^p+\sum_{l=0}^{k}|D^l v_h|^p+\frac{m^p}{(r-s)^p}\sum_{l=1}^{k-1}|D^l(v-v_h)|^p)\Big]^{\frac{q}{2}} \\ &= J_{h,1}+J_{h,2}. \end{aligned}$$

Since $D^l v_h \to D^l v$ strongly in $L^2(B_1;\mathbb{R}^N)$ for every $l<k$, we have, using (11)

$$\limsup_{h\to\infty} J_{h,1}\leq cm^{-2\rho}.$$

Moreover, since for $l=0,\dots,k$

$$\lambda_h^{\frac{p(q-2)}{q}}\int_{B_1}|D^l v_h|^p \leq c\lambda_h^{\frac{2(q-p)}{q}}\lambda_h^{p-2}\int_{B_1}|D^k v_h|^p\leq c\lambda_h^{\frac{2(q-p)}{q}}$$

and

$$\lambda_h^{\frac{p(q-2)}{q}}\int_{B_1}|D^l(v_h-v)|^p\leq c\lambda_h^{\frac{p(q-2)}{q}}\int_{B_1}|D^k v_h|^p\leq c\lambda_h^{\frac{2(q-p)}{q}}$$

we have

$$\lim_h J_{h,2}=0.$$

Hence we conclude letting first $m \to \infty$ and then $r \to s$ in (22).

Step 4. Lower bound. We shall prove that, for *a.e.* $\frac{1}{4} < r < \frac{1}{2}$, if $t < r$ then

$$\limsup_h \int_{B_t} |D^k v - D^k v_h|^2 (1 + \lambda_h^{p-2} |D^k v - D^k v_h|^{p-2}) \le \lim_h [I_{h,r}(v_h) - I_{h,r}(v)].$$

For any Borel set $A \subset B_1$, let us define

$$\mu_h(A) = \int_A \sum_{l=0}^k |D^l v_h|^2 \, \mathrm{d}x.$$

Passing possibly to a subsequence, since $\mu_h(B_1) \le c$, we may suppose

$$\mu_h \rightharpoonup \mu \quad \text{weakly} * \text{ in the sense of measures,}$$

where μ is a Borel measure over B_1, with finite total variation. Then for *a.e.* $r < 1$

$$\mu(\partial B_r) = 0$$

and let us choose such a radius r. Consider $\frac{1}{4} < t < s < r$, also such that $\mu(\partial B_s) = 0$, and fix $m \in IN$. Observe that, as $v_h \in W^{k,p}(B_1)$ Lemma 2.4 implies that there exist $z_h \in W^{k,p}(B_1)$ and $\frac{1}{4} < s < s_h < r_h < r < 1$ such that

$$z_h = v_h \quad \text{on} \quad B_{s_h} \qquad z_h = v_h \quad \text{on} \quad B_1 \setminus B_{r_h}$$
$$r_h - s_h \ge \frac{r-s}{3m}$$

and

$$\left(\int_{B_{r_h} \setminus B_{s_h}} |D^k z_h|^2 \right)^{\frac{1}{2}} + \lambda_h^{\frac{q-2}{q}} \left(\int_{B_{r_h} \setminus B_{s_h}} |D^k z_h|^q \right)^{\frac{1}{q}}$$
$$\le C \frac{(r-s)^\rho}{m^\rho} \Big[\fint_{B_r \setminus B_s} \left(1 + \sum_{l=0}^k |D^l v_h|^2 \right) + \lambda_h^{\frac{(q-2)p}{q}} \fint_{B_r \setminus B_s} (1 + \sum_{l=0}^k |D^l v_h|^p) \Big]^{\frac{1}{2}}. \quad (23)$$

Passing possibly to a subsequence, we may suppose that

$$z_h \rightharpoonup v_{r,s} \qquad \text{weakly} \quad \text{in} \quad W^{k,2}(B_1).$$

and

$$v_{r,s} = v \quad \text{in} \quad (B_1 \setminus B_r) \cup B_s.$$

Moreover, from (23) and the interpolation inequality with $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{q}$, we deduce that

$$\begin{aligned} \lambda_h^{p-2} \int_{B_1} |D^k z_h|^p &\le c\lambda_h^{p-2} \left(\int_{B_1} |D^k z_h|^2 \right)^{\frac{\theta p}{2}} \left(\int_{B_1} |D^k z_h|^q \right)^{\frac{(1-\theta)p}{2}} \\ &\le c\lambda_h^{p-2} \left(\int_{B_1} |D^k z_h|^q \right)^{\frac{(1-\theta)p}{2}} \\ &\le c\lambda_h^{p-2} (c\lambda_h^{2-q})^{\frac{p-2}{q-2}} \le c, \end{aligned} \quad (24)$$

since $\theta = \frac{p-2}{q-2}\frac{q}{p}$.

Consider $\zeta_h \in C_0^\infty(B_{r_h})$ such that $0 \le \zeta_h \le 1$, $\zeta_h = 1$ on B_{s_h} and $|D^l \zeta_h| \le \frac{C}{(r_h - s_h)^l}$, for $l = 0, \dots, k$, and set

$$\psi_h^\epsilon = \zeta_h(z_h - v_{r,s}^\epsilon),$$

where $v_{r,s}^\epsilon = \rho_\epsilon \star v_{r,s}$, and ρ_ϵ is the usual sequence of mollifiers. Now, setting $v^\epsilon = \rho_\epsilon \star v$, we observe that

$$\begin{aligned} I_{h,r_h}(v_h) - I_{h,r_h}(v^\epsilon) &= I_{h,r_h}(v_h) - I_{h,r_h}(z_h) + I_{h,r_h}(z_h) - I_{h,r_h}(v_{r,s}^\epsilon + \psi_h^\epsilon) \\ &\quad + I_{h,r_h}(\psi_h^\epsilon + v_{r,s}^\epsilon) - I_{h,r_h}(v_{r,s}^\epsilon) - I_{h,r_h}(\psi_h^\epsilon) + I_{h,r_h}(v_{r,s}^\epsilon) - I_{h,r_h}(v^\epsilon) \\ &\quad + I_{h,r_h}(\psi_h^\epsilon) \\ &= R_{h,1} + R_{h,2} + R_{h,3} + R_{h,4} + R_{h,5}. \end{aligned} \tag{25}$$

To bound $R_{h,1}$ we observe that

$$I_{h,r_h}(v_h) - I_{h,r_h}(z_h) = \int_{B_{r_h}\setminus B_{s_h}} f_h(D^k v_h) - \int_{B_{r_h}\setminus B_{s_h}} f_h(D^k z_h) \ge - \int_{B_{r_h}\setminus B_{s_h}} f_h(D^k z_h)$$

on the other hand we have

$$\begin{aligned} \int_{B_{r_h}\setminus B_{s_h}} f_h(D^k z_h) &\le \int_{B_{r_h}\setminus B_{s_h}} |D^k z_h|^2 + \lambda_h^{q-2}|D^k z_h|^q \\ &\le cm^{-2\rho}\left[⨍_{B_r\setminus B_s} (1 + \sum_{l=0}^k |D^l v_h|^2) + \lambda_h^{\frac{q-2}{q}p} ⨍_{B_r\setminus B_s} (1 + \sum_{l=0}^k |D^l v_h|^p)\right]^{\frac{q}{2}} \end{aligned}$$

and then arguing as we did in Step 3 to bound $J_{h,1}$ we get

$$\limsup_h \int_{B_{r_h}\setminus B_{s_h}} f_h(D^k z_h) \le Cm^{-2\rho}$$

hence, letting $h \to \infty$ we get

$$\liminf_h R_{h,1} \ge -Cm^{-2\rho}. \tag{26}$$

We obtain that

$$\begin{aligned} R_{h,2} &= \int_{B_{r_h}\setminus B_{s_h}} f_h(D^k z_h) - f_h(D^k \psi_h^\epsilon + D^k v_{r,s}^\epsilon) \\ &\ge -c \int_{B_{r_h}\setminus B_{s_h}} |D^k\psi_h^\epsilon + D^k v_{r,s}^\epsilon|^2 + \lambda_h^{q-2}|D^k\psi_h^\epsilon + D^k v_{r,s}^\epsilon|^q \\ &\ge -c \int_{B_{r_h}\setminus B_{s_h}} |D^k z_h|^2 + \lambda_h^{q-2}|D^k z_h|^q + |D^k v_{r,s}^\epsilon|^2 + \lambda_h^{q-2}|D^k v_{r,s}^\epsilon|^q \\ &- c \int_{B_{r_h}\setminus B_{s_h}} (\sum_{l=0}^{k-1} \frac{m^{2(k-l)}}{(r-s)^{2(k-l)}}|D^l(z_h - v_{r,s}^\epsilon)|^2 + \lambda_h^{q-2}\sum_{l=0}^{k-1}\frac{m^{q(k-l)}}{(r-s)^{q(k-l)}}|D^l(z_h - v_{r,s}^\epsilon)|^q) \\ &= -S_{h,1} - S_{h,2} \end{aligned}$$

where we used the bound $r_h - s_h \ge \frac{r-s}{3m}$. Denoting by P_l the polynomial of degree $k-1$ such that

$$\int_{B_1} (D^l(P_l - z_h)) = 0,$$

for $l < k$, and setting

$$p^* = \begin{cases} \frac{np}{n-lp} & \text{if } p < \frac{n}{l} \\ r > p & \text{if } p \geq \frac{n}{l}, \end{cases}$$

since $q < p^*$, we get by (23), for every $l = 0, \ldots, k-1$

$$\begin{aligned} \int_{B_1} \lambda_h^{q-2} |D^l z_h|^q &\leq c\lambda_h^{q-2} \left\{ \int_{B_1} |D^l(z_h - P_l)|^q + |D^l(P_l)|^q \right\} \\ &\leq c\lambda_h^{q-2} \left\{ \left(\int_{B_1} |D^l(z_h - P_l)|^{p^*} \right)^{\frac{q}{p^*}} + \left(\int_{B_1} |D^l(P_l)|^{p^*} \right)^{\frac{q}{p^*}} \right\} \\ &\leq c\lambda_h^{q-2} \left(\int_{B_1} |D^k z_h|^p \right)^{\frac{q}{p}} \\ &\leq c\lambda_h^{\frac{2(q-p)}{p}} \left(\lambda_h^{p-2} \int_{B_1} |D^k z_h|^p \right)^{\frac{q}{p}}. \end{aligned}$$

Therefore, using (24), we obtain

$$\limsup_{h\to\infty} S_{h,2} \leq c \sum_{l=0}^{k-1} \frac{m^{2(k-l)}}{(r-s)^{2(k-l)}} \int_{B_{\frac{1}{2}}} |D^l(v_{r,s} - v^\epsilon_{r,s})|^2 .$$

To bound $S_{h,1}$, observe that for every h

$$\begin{aligned} \int_{B_{r_h}\setminus B_{s_h}} |D^k v^\epsilon_{r,s}|^2 &\leq c \int_{B_r\setminus B_s} |D^k v_{r,s}|^2 + c \int_{B_{\frac{1}{2}}} |D^k v_{r,s} - D^k v^\epsilon_{r,s}|^2 \\ &\leq \liminf_j c \int_{B_r\setminus B_s} |D^k z_j|^2 + c \int_{B_{\frac{1}{2}}} |D^k v_{r,s} - D^k v^\epsilon_{r,s}|^2 \\ &= c \liminf_j \int_{(B_r\setminus B_s)\setminus(B_{r_j}\setminus B_{s_j})} |D^k v_j|^2 \\ &\quad + c \limsup_j \int_{B_{r_j}\setminus B_{s_j}} |D^k z_j|^2 + c \int_{B_{\frac{1}{2}}} |D^k v_{r,s} - D^k v^\epsilon_{r,s}|^2. \end{aligned}$$

We control the second integral as usual using Lemma 2.4, while the first is less or equal than $c\mu(B_r \setminus B_s)$.

Moreover we can estimate

$$\int_{B_{r_h}\setminus B_{s_h}} |D^k z_h|^2 + \lambda_h^{q-2} |D^k z_h|^q$$

as we did in Step 3 to bound $J_{h,1}$. Hence

$$\begin{aligned} &\liminf\nolimits_h R_{h,2} \geq -cm^{-2\rho} - c\mu(B_r \setminus B_s) \\ &-c \textstyle\int_{B_{\frac{1}{2}}} |D^k v_{r,s} - D^k v^\epsilon_{r,s}|^2 - \sum_{l=0}^{k-1} \frac{m^{2(k-l)}}{(r-s)^{2(k-l)}} \int_{B_{\frac{1}{2}}} |D^l(v_{r,s} - v^\epsilon_{r,s})|^2. \end{aligned} \tag{27}$$

To bound $R_{h,3}$ we observe that

$$f_h(A+B) - f_h(A) - f_h(B) = \int_0^1 \int_0^1 D^2 f_h(sA+tB)AB\mathrm{d}s\,\mathrm{d}t$$

and, by the definition of f_h,

$$D^2 f_h(sD^k v^\epsilon_{r,s} + tD^k\psi^\epsilon_h) = D^2 f(A_h + s\lambda_h D^k v^\epsilon_{r,s} + t\lambda_h D^k\psi^\epsilon_h)$$

is bounded and converges to $D^2 f(A)$ *a.e.* Since

$$R_{h,3} = \int_{B_{r_h}} \mathrm{d}x \int_{[0,1]\times[0,1]} D^2 f(A_h + s\lambda_h D^k v^\epsilon_{r,s} + t\lambda_h D^k\psi^\epsilon_h) D^k v^\epsilon_{r,s} D^k \psi^\epsilon_h \mathrm{d}s\,\mathrm{d}t$$

and we may suppose that $\psi^\epsilon_h \rightharpoonup \psi^\epsilon$ weakly in $W^{k,2}(B_1)$, and arguing as in the proof of (27), we have

$$\int_{B_1} |D^k\psi^\epsilon|^2 \le \sum_{l=0}^{k-1} \frac{m^{2(k-l)}}{(r-s)^{2(k-l)}} \int_{B_{\frac{1}{2}}} |D^l(v_{r,s} - v^\epsilon_{r,s})|^2 + c\int_{B_{\frac{1}{2}}} |D^k v_{r,s} - D^k v^\epsilon_{r,s}|^2.$$

Then we get easily

$$\limsup_h |R_{h,3}| \le c(M)||D^k v^\epsilon_{r,s}||_{L^2(B_{\frac{1}{2}})}||D^k\psi^\epsilon||_{L^2(B_{\frac{1}{2}})}. \tag{28}$$

To bound $R_{h,4}$ we observe that

$$R_{h,4} = \int_{B_{r_h}\setminus B_s} [f_h(D^k v^\epsilon_{r,s}) - f_h(D^k v^\epsilon)] \ge -\int_{B_{r_h}\setminus B_{s-\epsilon}} f_h(D^k v^\epsilon) \ge -c|B_r \setminus B_{s-\epsilon}|.$$

Then

$$\liminf_h R_{h,4} \ge -c|B_r \setminus B_{s-\epsilon}|. \tag{29}$$

Moreover (H3) implies

$$|R_{h,5}| = I_{h,r_h}(\psi^\epsilon_h) = \int_{B_{r_h}} f_h(D^k\psi^\epsilon_h) \ge \gamma \int_{B_t} (1 + \lambda_h^{p-2}|D^k v^\epsilon - D^k v_h|^{p-2})|D^k v^\epsilon - D^k v_h|^2 \tag{30}$$

for ϵ small enough.

Passing to a subsequence we may suppose that

$$\limsup_h R_{h,5} = \lim_h R_{h,5}.$$

Therefore returning to (25), from (26), (27), (28), (29) and (30) we get

$$\begin{aligned}
&\liminf_h [I_{h,r}(v_h) - I_{h,r}(v^\epsilon)] \\
&\qquad \ge \gamma \limsup_h \int_{B_s} (1 + \lambda_h^{p-2}|D^k v^\epsilon - D^k v_h|^{p-2})|D^k v^\epsilon - D^k v_h|^2 - c|B_r\setminus B_{s-\epsilon}| - c\mu(B_r\setminus B_s) \\
&\qquad\qquad - c||D^k v^\epsilon_{r,s}||_{L^2(B_{\frac{1}{2}})}||D^k\psi^\epsilon||_{L^2(B_{\frac{1}{2}})} - cm^{-2\rho} - \int_{B_{\frac{1}{2}}} |Dv_{r,s} - Dv^\epsilon_{r,s}|^2 \\
&\qquad\qquad\qquad - c\sum_{l=0}^{k-1} \frac{m^{2(k-l)}}{(r-s)^{2(k-l)}} \int_{B_{\frac{1}{2}}} |D^l(v_{r,s} - v^\epsilon_{r,s})|^2.
\end{aligned}$$

Passing to the limit as $\epsilon \to 0^+$ we get easily

$$\begin{aligned}\liminf_h [I_{h,r}(v_h) - I_{h,r}(v)]&\\ \geq \gamma \limsup_h \int_{B_s} (1+\lambda_h^{p-2}|D^k v - D^k v_h|^{p-2})|D^k v - D^k v_h|^2 - c|B_r \setminus B_s| - c\mu(B_r \setminus B_s) - cm^{-2\rho}&\end{aligned}$$

then passing to the limit as $m \to \infty$ and $s \to r$ we get

$$\limsup_h \int_{B_r} |D^k v - D^k v_h|^2 (1+\lambda_h^{p-2}|D^k v - D^k v_h|^{p-2}) \leq \lim_h [I_{h,r}(v_h) - I_{h,r}(v)].$$

Step 5. Conclusion. From the two previous steps we conclude that, for any B_τ, with $0 < \tau < \frac{1}{4}$

$$\lim_h \int_{B_\tau} |D^k v - D^k v_h|^2 (1+\lambda_h^{p-2}|D^k v - D^k v_h|^p) = 0.$$

Now, from this equality and by (18) we get

$$\begin{aligned}\lim_h \frac{U(x_h, \tau r_h)}{\lambda_h^2} &= \lim_h \frac{1}{\lambda_h^2} ⨍_{B_{\tau r_h}(x_h)} (|D^k u - (D^k u)_{\tau r_h}|^2 + |D^k u - (D^k u)_{\tau r_h}|^p)\mathrm{d}x\\ &= \lim_h ⨍_{B_\tau} (|D^k u - (D^k u)_\tau|^2 + \lambda_h^{p-2}|D^k u - (D^k u)_\tau|^p)\mathrm{d}y\\ &= ⨍_{B_\tau} (|D^k v - (D^k v)_\tau|^2)\mathrm{d}y\\ &\leq C_M^* \tau^2\end{aligned}$$

which contradicts (10) if we choose $C_M = 2C_M^*$. □

The proof of Theorem 2.1 follows by Proposition 3.1 by a standard iteration argument, see [12].

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