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HAMILTON-JACOBI EQUATIONS FOR CONTROL PROBLEMS OF PARABOLIC EQUATIONS

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Abstract. We study Hamilton-Jacobi equations related to the boundary (or internal) control of semilinear parabolic equations, including the case of a control acting in a nonlinear boundary condition, or the case of a nonlinearity of Burgers' type in 2D. To deal with a control acting in a boundary condition a fractional power $(-A)^{\beta}$ – where $(A, D(A))$ is an unbounded operator in a Hilbert space X – is contained in the Hamiltonian functional appearing in the Hamilton-Jacobi equation. This situation has already been studied in the literature. But, due to the nonlinear term in the state equation, the same fractional power $(-A)^{\beta}$ appears in another nonlinear term whose behavior is different from the one of the Hamiltonian functional. We also consider cost functionals which are not bounded in bounded subsets in X, but only in bounded subsets in a space $Y \hookrightarrow X$. To treat these new difficulties, we show that the value function of control problems we consider is equal in bounded sets in Y to the unique viscosity solution of some Hamilton-Jacobi-Bellman equation. We look for viscosity solutions in classes of functions which are Hölder continuous with respect to the time variable.

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1. Introduction

In this paper we study the uniqueness and existence of viscosity solution of the Hamilton-Jacobi-Bellman equation

$$\begin{aligned} -\frac{\partial v}{\partial t}(t,x) - \left(D_x v(t,x) \mid Ax\right)_X + \left(D_x v(t,x) \mid (-A)^{\beta} F(t,\Lambda x)\right)_X \\ +H\left(t,x,(-A)^{\beta} D_x v(t,x)\right) = 0 \qquad \text{in } (0,T)\times X, \\ v(T,x) = g(x) \qquad \text{in } X. \end{aligned} \tag{1.1}$$

In this setting X is a real Hilbert equipped with the inner product $(\cdot \mid \cdot)_X$ and the norm $|\cdot|_X$, A is an unbounded operator with domain $D(A)$ in X, it is supposed to be self-adjoint and strictly dissipative in X, $(-A)^{\beta}$ is the β-fractional power of $(-A)$, and $0 \le \beta \le \frac{1}{2}$, Λ is a bounded linear operator from $D((-A)^{\alpha})$ into X_0 with $0 \le \alpha < \frac{1}{2}$, X_0 is another real Hilbert space equipped with the inner product $(\cdot \mid \cdot)_{X_0}$ and the norm $|\cdot|_{X_0}$,

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$F \in C([0,T] \times X_0; X)$, the Hamiltonian functional H is continuous in $[0,T] \times X \times X$, and the mapping g is Lipschitz continuous in X. Precise assumptions are stated in Section 2.

Equation (1.1) is related to optimal control problems of semilinear parabolic equations (including in particular the case where the control acts in a nonlinear boundary condition). More precisely, for all $t \in [0,T]$ and $x \in X$, consider an optimal control problem of the form

$$(\mathcal{P}_{t,x}) \qquad \min \Big\{ J(t,u,y) \mid u \in \mathcal{M}(t,T;U) \text{ and } (y,u) \text{ is solution of equation (1.2)} \Big\},$$

where the cost functional J is defined by

$$J(t,y,u) = \int_t^T L(r, y(r), u(r))\, \mathrm{d}r + g(y(T)),$$

and the state equation is

$$y' = Ay + (-A)^{\beta} \left[Bu - F(\cdot, \Lambda y) \right], \qquad y(t) = x. \tag{1.2}$$

The control space $\mathcal{M}(t,T;U)$ is a set of bounded measurable functions with values in U, and U is a bounded subset in X_Γ, X_Γ is a Banach space, $B \in \mathcal{L}(X_\Gamma, X)$. In Section 3 we prove that equation (1.1) admits at most one viscosity solution. In Section 4, we prove that the value function of problem $(\mathcal{P}_{t,x})$ is the unique viscosity solution of equation (1.1), when H is defined by

$$H(t,x,p) = \sup_{u \in U} \left[-(p \mid Bu)_X - L(t,x,u) \right].$$

Applications are discussed in Section 5. Before presenting what is new in the present paper, observe that by setting

$$\mathbf{H}(t,x,(-A)^{\beta} D_x v(t,x)) = H(t,x,(-A)^{\beta} D_x v(t,x)) + \Big((-A)^{\beta} D_x v(t,x) \mid F(t,\Lambda x) \Big)_X,$$

equation (1.1) can be written in the form

$$-\frac{\partial v}{\partial t}(t,x) - \Big(D_x v(t,x) \mid Ax \Big)_X + \mathbf{H}\Big(t,x,(-A)^{\beta} D_x v(t,x)\Big) = 0 \qquad \text{in } (0,T) \times X. \tag{1.3}$$

Equation (1.3) seems to be simpler to handle than equation (1.1). However assumptions on $F(t,\Lambda x)$ and on $H(t,x,p)$ are different and we cannot simplify the presentation of the paper by considering equation (1.3) (see *e.g.* the estimates involving H and F in the proof of Th. 3.5).

During the eighties and the nineties several fundamental advances have been made in the study of Hamilton-Jacobi equation in infinite dimension. These equations were first studied by Barbu and Da Prato (see *e.g.* [2]), mainly in classes of convex functions. The method of viscosity solutions has been extended to infinite dimension by Crandall and Lions in a series of papers [10–14]. All these papers correspond to the case when $\beta = 0$. Other contributions are due to Cannarsa and Frankowska [5], Ishii [20], Soner [27], Tataru [28,29], Crandall and Lions [15, 16], Cannarsa and Tessitore [6–9] in order to deal with boundary controls. In particular equations of the form (1.3) with $0 < \beta < \frac{1}{2}$ are studied in [6], [8] to treat Neumann boundary controls. The case of Dirichlet controls is considered in [7, 9], it corresponds to the situation when $\frac{1}{2} < \beta < 1$ and has to be studied independently. More recently the case of the Navier-Stokes equations has been studied in [18, 26].

The main motivation of the present paper is to characterize the value function of control problems governed by semilinear parabolic equations, including the case of equations with a nonlinear boundary condition, or the case of nonlinearity of Burgers' type in two dimension, and with cost functionals whose growth is quadratic or even higher than quadratic. For example we study the case of partial differential equations with nonlinear boundary conditions of the form:

$$\frac{\partial y}{\partial t} - \Delta y + y = f \quad \text{in }]t,T[\times \Omega, \qquad \frac{\partial y}{\partial n} + \widehat{h}(y) = u \quad \text{on }]t,T[\times \Gamma, \qquad y(t) = x \quad \text{in } \Omega, \tag{1.4}$$

with cost functionals of the type

$$\widehat{J}(t,y,u)=\int_t^T \widehat{L}(r,y(r),u(r))\,\mathrm{d}r+\widehat{g}(y(T)),$$

where $\widehat{h}$ is any regular nondecreasing function obeying $\widehat{h}(0)=0$, and where $\widehat{L}$ and $\widehat{g}$ may be quadratic cost functionals. Many thermal processes lead to the kind of model corresponding to equation (1.4) (see [23]). The papers mentioned above do not include this model in their possible applications. If the initial condition x belong to $X=L^2(\Omega)$, equation (1.4) is well posed and it admits a unique weak solution belonging to $C([0,T];X)$ (the solution also belongs to $L^2(0,T;H^1(\Omega))$). We can write equation (1.4) in the form

$$y'=Ay+(-A)^{\beta}\left[Bu-\widehat{F}\,(\cdot,\Lambda y)\,\right],\qquad y\,(t)=x, \tag{1.5}$$

by defining Λ as the trace mapping on Γ:

$$\Lambda\ :\ y\longmapsto y|_{\Gamma}.$$

In this example Λ is bounded from $H^{2\alpha}(\Omega)=D((-A)^{\alpha})$ into $X_0=L^2(\Gamma)$ for all $\frac{1}{4}<\alpha<\frac{1}{2}$, $D(A)=\{y\in H^2(\Omega)\mid \frac{\partial y}{\partial n}=0\}$, $Ay=\Delta y$, and we have to take $\frac{1}{4}<\beta<\frac{1}{2}$. For a parabolic equation with a nonlinearity of Burgers' type we can take $\beta=\frac{1}{2}$. Let us denote by $y_{t,x,u}$ the solution to equation (1.5). To characterize the value function $\widehat{v}(t,x)$ of the problem

$$(\widehat{\mathcal{P}}_{t,x})\qquad\qquad \min\left\{\widehat{J}\,(t,u,y)\mid u\in\mathcal{M}(t,T;U) \text{ and } (y,u) \text{ is solution of equation (1.5)}\right\},$$

we have to study the dependence of $y_{t,x,u}$ and of $\Lambda y_{t,x,u}$ with respect to t and to x. Due to the nonlinear term in equation (1.5), we can prove continuity properties for $y_{\cdot,x,u}$ and $\Lambda y_{\cdot,x,u}$ and Lipschitz properties for $y_{t,\cdot,u}$ and $\Lambda y_{t,\cdot,u}$ when the initial condition x stays in bounded subsets in Y, for a space $Y\hookrightarrow X$, but these properties are not true if we consider only bounded subsets in X. Therefore it is natural to study the properties of the value function $\widehat{v}(t,x)$ when x remains in bounded subsets of Y, and to look for solutions to equation (1.4) in a space of the type $C([0,T];Y)$ or at least $L^{\infty}_w(0,T;Y)$ (the space of bounded and weakly measurable functions from $(0,T)$ into Y).

Another difficulty comes from the cost functional. In the literature on Hamilton-Jacobi-Bellman equations, it is often assumed that the cost functionals either are bounded or satisfy a linear growth condition [6, 8, 18, 20]. Thus the case of quadratic cost functionals is not treated in these papers.

To overcome the two difficulties mentioned above, the one coming from the nonlinearity in the state equation and the other one due to the growth condition of the cost functional, we suggest to proceed as follows. First, we show that, for an initial condition in $B_Y(M_0)$ (the ball in Y centered at the origin and with radius M_0), the solution y of equation (1.2) satisfies $y(\cdot)\in B_Y(R_T)$ in (t,T) for some $R_T=R(M_0,T)$ which can be explicitly estimated independently of $t\in(0,T)$. Next, we associate with the mappings $\widehat{L}(t,\cdot,u)$, $\widehat{g}$ and $\widehat{F}(t,\Lambda\cdot)$, other mappings $L(t,\cdot,u)$, g and $F(t,\Lambda\cdot)$ which are identical to the previous ones in the ball $B_Y(R_T)$, but which satisfies some global boundedness and Lipschitz properties. Let us consider the problem $(\mathcal{P}_{t,x})$ – the one introduced at the beginning of the introduction – defined with $L(t,\cdot,u)$, g and $F(t,\Lambda\cdot)$. We are able to show that value function $v(t,x)$ of problem $(\mathcal{P}_{t,x})$ obeys $\widehat{v}(t,x)=v(t,x)$ for $t\in(0,T)$ and $x\in B_Y(R_T)$. We show that v is the unique viscosity solution of the Hamilton-Jacobi equation (1.1). Thus $\widehat{v}$ is not the viscosity solution to equation (1.1), but it is equal to the viscosity solution of equation (1.1) in bounded sets in $(0,T)\times Y$.

Sections 2, 3 and 4 are devoted to the study of equation (1.2), equation (1.1), and the value function of problem $(\mathcal{P}_{t,x})$. In these sections, only the mappings $L(t,\cdot,u)$, g and $F(t,\Lambda\cdot)$ intervene. The assumptions are precisely stated in Section 2. The definition of the mappings $L(t,\cdot,u)$, g and $F(t,\Lambda\cdot)$ from $\widehat{L}(t,\cdot,u)$, $\widehat{g}$ and $\widehat{F}(t,\Lambda\cdot)$ is treated in examples of Section 5 by using projection operators. Three examples are considered. The first one is a control problem for the state equation (1.4), and the two others correspond to problems for a two dimensional scalar equation of Burgers' type. The interest of the third example is to show that the method

using a projection operator in the cost functional and the state equation is flexible enough to involve different kind of projections adapted to the nonlinearity and to the functionals we have to deal with.

Let us finally mention that the definition of viscosity solutions that we take is not totally standard. Indeed we consider viscosity solutions which are Hölder continuous with respect to the time variable. This Hölder continuity condition, which is a new argument in the definition of viscosity solutions – see Definition 3.2 – plays a major role in the proof of uniqueness to estimate the nonlinear term F. A preliminary version of the present paper corresponds to a part of the Ph.D. thesis by the first author [17].

2. Preliminaries on the evolution equation

In this section we want to study properties of solutions of the evolution equation

$$y' = Ay + (-A)^{\beta}\left[Bu - F(\cdot, \Lambda y)\right] \quad \text{in } (t,T), \qquad y(t) = x, \tag{2.1}$$

where $t \in [0,T)$.

2.1. **Assumptions**

Throughout the paper we make the following assumptions.

(i) The unbounded operator A, with domain $D(A)$ in X, is a closed and densely defined selfadjoint operator in X, such that $(Ax \mid x)_X \leq -\omega |x|_X^2$ for all $x \in D(A)$, where $\omega > 0$.

(ii) $B \in \mathcal{L}(X_\Gamma, X)$.

(iii) The linear operator Λ is bounded from $D((-A)^{\alpha})$ into X_0 for some $\alpha \in [0, \frac{1}{2}[$, that is:

$$|\Lambda x|_{X_0} \leq C_\alpha \left|(-A)^{\alpha} x\right|_X \quad \text{for all } x \in D((-A)^{\alpha}). \tag{2.2}$$

The exponent $\beta \in [0, \frac{1}{2}]$ is given fixed.

(iv) F is a continuous mapping from $[0,T] \times X_0$ into X, which satisfies:

$$|F(t,x) - F(t,y)|_X \leq K_F |x-y|_{X_0}, \text{ and } |F(t,x)|_X \leq M_F, \tag{2.3}$$

for all $t \in [0,T]$, and all x, $y \in X_0$. Moreover, there exists $\eta_1 \in]0,1]$ such that:

$$|F(t,x) - F(s,x)|_X \leq M_{1,F}\left(1 + |x|_{X_0}\right)|t-s|^{\eta_1}. \tag{2.4}$$

In addition, we assume that either $\beta < \frac{1}{2}$, or $\beta = \frac{1}{2}$ and

$$\begin{aligned} &D((-A)^{\frac{1}{2}}) \hookrightarrow X_0, \\ &|F(t,x)|_{D((-A)^{\beta_0})} \leq M\left(\beta_0, |x|_{D((-A)^{\frac{1}{2}})}\right) \quad \text{for all } t \in [0,T] \text{ and all } x \in D((-A)^{\frac{1}{2}}), \\ &(-A)^{\beta_0} B \in \mathcal{L}(X_\Gamma, X), \quad \text{for some } 0 < \beta_0 < \beta = \tfrac{1}{2}, \end{aligned} \tag{2.5}$$

where $M\left(\beta_0, |x|_{D((-A)^{\frac{1}{2}})}\right) > 0$ only depends on β_0 and $|x|_{D((-A)^{\frac{1}{2}})}$.

(v) The control u belongs to $\mathcal{M}(t,T;U)$, the space of measurable functions from (t,T) into U, where U is a nonempty, bounded and closed subset of X_Γ, such that

$$|u|_{X_\Gamma} \leq M_U \quad \text{for all } u \in U. \tag{2.6}$$

We now state assumptions needed in Section 3 to study equation (1.1).

(vi) The mapping $g \in C(X)$ is Lipschitz continuous and bounded in X, *i.e.*:

$$|g(x) - g(y)| \leq K_g |x-y|_X \quad \text{and} \quad |g(x)| \leq M_g, \quad \text{for all } x, y \in X.$$

(vii) The Hamiltonian functional H satisfies

$$|H(t,x,p) - H(s,y,q)| \leq K_H \left(|t-s|^{\eta_2} + |x-y|_X + |p-q|_X\right). \tag{2.7}$$

In Section 4, we make the following additional assumption.

(viii) The Hamiltonian functional $H : [0,T] \times X \times X \to \mathbb{R}$ is defined by:

$$H(t,x,p) = \sup_{u \in U} \left[-(p \mid Bu)_X - L(t,x,u) \right], \tag{2.8}$$

where the functional $L \in C([0,T] \times X \times U)$ satisfies:

$$|L(t,x,u) - L(s,y,u)| \leq K_L \left(|t-s|^{\eta_2} + |x-y|_X\right) \quad \text{and} \quad |L(t,x,u)| \leq M_L,$$

for all $t, s \in [0,T]$, all $x, y \in X$, and all $u \in U$, with $0 < \eta_2 \leq 1$.

Observe that if H is defined by (2.8) and if L satisfies the estimate stated in (**viii**), then

$$|H(t,x,p) - H(s,y,q)| \leq K_H \left(|t-s|^{\eta_2} + |x-y|_X + |p-q|_X\right),$$

with $K_H = \max(K_L, \|B\| M_U)$. Thus assumption (**vii**) is automatically satisfied in that case.

Due to assumption (**i**), $(A, D(A))$ is the infinitesimal generator of a strongly continuous analytic semigroup of contractions on X which satisfies

$$\|e^{tA}\|_{\mathcal{L}(X)} \leq e^{-\omega t}. \tag{2.9}$$

Moreover (see [19], Th. 1.4.3, Chap. 1 and [3], Prop. 5.1, Chap. 1), for all $\delta \geq 0$, there exists a constant M_δ such that, for all $t > 0$:

$$\left\| (-A)^\delta e^{At} \right\|_{\mathcal{L}(X)} \leq M_\delta t^{-\delta}. \tag{2.10}$$

If $0 < \delta \leq 1$, and $x \in D((-A)^\delta)$, we have:

$$\left| \left(e^{At} - I\right) x \right|_X \leq \frac{1}{\delta} M_{1-\delta}\, t^\delta \left| (-A)^\delta x \right|_X. \tag{2.11}$$

Besides, for all $\delta < \gamma$ and all $x \in D((-A)^\gamma)$, one has:

$$\left| (-A)^\delta x \right|_X \leq M_{\delta,\gamma} \left| (-A)^\gamma x \right|_X^{\frac{\delta}{\gamma}} |x|_X^{1-\frac{\delta}{\gamma}}. \tag{2.12}$$

With Young's inequality the last estimate implies that, for all $\delta \in \left]0, \frac{1}{2}\right[$, and all $\sigma > 0$, there exists a constant $C_{\delta,\sigma}$ such that:

$$\left| (-A)^\delta x \right|_X \leq \sigma \left| (-A)^{\frac{1}{2}} x \right|_X + C_{\delta,\sigma} |x|_X \quad \text{for } x \in D((-A)^{\frac{1}{2}}). \tag{2.13}$$

2.2. **Properties and regularities of mild solutions of equation (2.1)**

Theorem 2.1. *For all $x \in X$ and all $u \in \mathcal{M}(t,T;U)$, equation (2.1) admits a unique mild solution $y_{t,x,u}$ in $L^1(t,T;D((-A)^\alpha))$, it obeys:*

$$y_{t,x,u}(s) = \mathrm{e}^{(s-t)A}x + (-A)^\beta \int_t^s \mathrm{e}^{(s-r)A}\big[Bu(r) - F(r,\Lambda y_{t,x,u}(r))\big]\,\mathrm{d}r, \tag{2.14}$$

for all $s \in [t,T]$. Moreover $y_{t,x,u}$ belongs to $C([t,T];X)$ and satisfies the estimate

$$\|y_{t,x,u}\|_{C([t,T];X)} \le C(1 + |x|_X + \|u\|_{L^\infty(t,T;U)}).$$

Proof. Let $t_1 \in (t,T]$ be such that $C_\alpha K_F M_{\alpha+\beta}\frac{(t_1-t)^{1-(\alpha+\beta)}}{1-(\alpha+\beta)} \le 1/2$. Let us set $E = L^1(t,t_1;D((-A)^\alpha))$, and let us show that the mapping

$$y \longmapsto (\Psi y)(s) = \mathrm{e}^{(s-t)A}x + (-A)^\beta \int_t^s \mathrm{e}^{(s-r)A}\left[Bu(r) - F(r,\Lambda y(r))\right]\mathrm{d}r,$$

is a contraction in E. First we have:

$$\begin{aligned}
\int_t^{t_1} |(-A)^\alpha(\Psi y)(s)|\,\mathrm{d}s &\le \int_t^{t_1}\left|(-A)^\alpha \mathrm{e}^{(s-t)A}x\right|\mathrm{d}s + \int_t^{t_1}\int_t^s \left|(-A)^{\alpha+\beta}\mathrm{e}^{(s-r)A}\left[Bu(r) - F(r,\Lambda y(r))\right]\right|\mathrm{d}r\,\mathrm{d}s\\
&\le M_\alpha |x|_X \frac{(t_1-t)^{1-\alpha}}{1-\alpha} + M_{\alpha+\beta}\int_t^{t_1}\int_t^s \frac{1}{(s-r)^{\alpha+\beta}}\left[\|B\|M_U + M_F\right]\mathrm{d}r\,\mathrm{d}s\\
&\le M_\alpha |x|_X \frac{(t_1-t)^{1-\alpha}}{1-\alpha} + M_{\alpha+\beta}\left[\|B\|M_U + M_F\right]\frac{(t_1-t)^{2-(\alpha+\beta)}}{[1-(\alpha+\beta)][2-(\alpha+\beta)]}.
\end{aligned}$$

Thus, if $y \in E$, Ψy belongs to E. Moreover if y_1, $y_2 \in E$, we can write

$$\begin{aligned}
&\int_t^{t_1} |(-A)^\alpha(\Psi y_1)(s) - (-A)^\alpha(\Psi y_2)(s)|_X\,\mathrm{d}s\\
&\qquad\le \int_t^{t_1}\int_t^s \left|(-A)^{\alpha+\beta}\mathrm{e}^{(s-r)A}\left[F(r,\Lambda y_1(r)) - F(r,\Lambda y_2(r))\right]\right|_X \mathrm{d}r\mathrm{d}s\\
&\qquad\le \int_t^{t_1}\int_t^s \frac{M_{\alpha+\beta}}{(s-r)^{\alpha+\beta}}K_F\,|\Lambda y_1(r) - \Lambda y_2(r)|_{X_0}\,\mathrm{d}r\,\mathrm{d}s\\
&\qquad= K_F\int_t^{t_1}|\Lambda y_1(r) - \Lambda y_2(r)|_{X_0}\int_r^{t_1}\frac{M_{\alpha+\beta}}{(s-r)^{\alpha+\beta}}\mathrm{d}s\,\mathrm{d}r\\
&\qquad\le C_\alpha K_F M_{\alpha+\beta}\int_t^{t_1}|(-A)^\alpha y_1(r) - (-A)^\alpha y_2(r)|_X\frac{(t_1-r)^{1-(\alpha+\beta)}}{1-(\alpha+\beta)}\mathrm{d}r\\
&\qquad\le C_\alpha K_F M_{\alpha+\beta}\frac{(t_1-t)^{1-(\alpha+\beta)}}{1-(\alpha+\beta)}\int_t^{t_1}|(-A)^\alpha y_1(r) - (-A)^\alpha y_2(r)|_X\,\mathrm{d}r.
\end{aligned}$$

Thus Ψ is a contraction in E, and it admits a unique fixed point in E, which is the unique solution y in E to equation (2.1). In addition y belongs to $C([t,t_1];X)$ and formula (2.14) is satisfied for all $s \in [t,t_1]$. We can repeat this process on the interval $[t_1,2t_1]$, and step by step, we prove that equation (2.1) admits a unique solution in $L^1(t,T;D((-A)^\alpha))$, which belongs to $C([t,T];X)$ and satisfies formula (2.14). □

Proposition 2.2. *Assume that $x \in D\left((-A)^{\alpha}\right)$. Then the solution $y_{t,x,u}$ of (2.1) satisfies:*

$$|\Lambda y_{t,x,u}(s) - \Lambda x|_{X_0} \to 0 \quad \text{uniformly with respect to } u \in \mathcal{M}(t,T;U) \quad \text{when} \quad s \searrow t. \tag{2.15}$$

Proof. Let x be in $D\left((-A)^{\alpha}\right)$. With inequality (2.2) we have

$$|\Lambda(y_{t,x,u}(s) - x)|_{X_0} \leq C_{\alpha} \left|(-A)^{\alpha}\left(y_{t,x,u}(s) - x\right)\right|_X.$$

Due to (2.14), we can write

$$\begin{aligned} \left|(-A)^{\alpha}\left(y_{t,x,u}(s) - x\right)\right|_X \leq & \left|(-A)^{\alpha}\left(\mathrm{e}^{(s-t)A}x - x\right)\right|_X \\ & + \left|(-A)^{\beta+\alpha}\int_t^s \mathrm{e}^{(s-r)A}\left[Bu(r) - F\left(r, \Lambda y_{t,x,u}(r)\right)\right]\mathrm{d}r\right|_X. \end{aligned} \tag{2.16}$$

We can estimate the two terms in the right hand side of (2.16) as follows:

$$\left|(-A)^{\alpha}\left(\mathrm{e}^{(s-t)A}x - x\right)\right|_X = \left|\left(\mathrm{e}^{(s-t)A} - I\right)(-A)^{\alpha}x\right|_X, \tag{2.17}$$

and

$$\left|(-A)^{\beta+\alpha}\int_t^s \mathrm{e}^{(s-r)A}\left[Bu(r) - F\left(r, \Lambda y_{t,x,u}(r)\right)\right]\mathrm{d}r\right|_X \leq M_{\alpha+\beta}\frac{(s-t)^{1-(\alpha+\beta)}}{1-(\alpha+\beta)}\left(M_F + \|B\| M_U\right). \tag{2.18}$$

The two terms (2.17) and (2.18) go to 0 uniformly with respect to $u \in \mathcal{M}(t,T;U)$ when $s \searrow t$, because $(\alpha+\beta) < 1$. □

Proposition 2.3. *Let $y_{t,x,u}$ be the weak solution of (2.1). There exists a constant $C_1(\beta)$, independent of u, such that*

$$\left|y_{t,x,u}(s) - \mathrm{e}^{(s-t)A}x\right|_X \leq C_1(\beta)(s-t)^{1-\beta} \text{ for all } x \in X. \tag{2.19}$$

Proof. Let $y_{t,x,u}$ be the weak solution of equation (2.1). With the integral formulation (2.14), we have

$$\left|y_{t,x,u}(s) - \mathrm{e}^{(s-t)A}x\right|_X \leq \left|(-A)^{\beta}\int_t^s \mathrm{e}^{(s-r)A}\left[Bu(r) - F\left(r, \Lambda y(r)\right)\right]\mathrm{d}r\right|_X.$$

From (2.3) and (2.10) it follows that

$$\begin{aligned} \left|(-A)^{\beta}\int_t^s \mathrm{e}^{(s-r)A}\left[Bu(r) - F\left(r, \Lambda y(r)\right)\right]\mathrm{d}r\right|_X &\leq \int_t^s \frac{M_{\beta}}{(s-r)^{\beta}}\left[\|B\|M_U + M_F\right]\mathrm{d}r \\ &\leq M_{\beta}\left[\|B\|M_U + M_F\right]\frac{(s-t)^{1-\beta}}{1-\beta}. \end{aligned}$$

The proof is complete. □

Proposition 2.4. *We assume that $\beta = \frac{1}{2}$ and that the corresponding additional conditions of assumption* (**iv**) *is satisfied. There exists a constant $C_1\left(\beta_0, |x|_{D((-A)^{\frac{1}{2}})}\right)$, independent of u, such that*

$$\left|y_{t,x,u}(s) - \mathrm{e}^{(s-t)A}x\right|_X \leq C_1\left(\beta_0, |x|_{D((-A)^{\frac{1}{2}})}\right)(s-t)^{\frac{1}{2}+\beta_0} \text{ for all } x \in D((-A)^{\frac{1}{2}}). \tag{2.20}$$

Proof. Assume that $x \in D((-A)^{\frac{1}{2}})$. Let $y_{t,x,u}$ be the weak solution of equation (2.1). By using the integral formulation (2.14), we have

$$\left|y_{t,x,u}(s) - \mathrm{e}^{(s-t)A}x\right|_X \leq \left|(-A)^{\beta-\beta_0}\int_t^s \mathrm{e}^{(s-r)A}\left[(-A)^{\beta_0}Bu(r) - (-A)^{\beta_0}F(r,\Lambda y(r))\right]\mathrm{d}r\right|_X.$$

With (2.5) and (2.10) we have

$$\begin{aligned}\left|(-A)^{\beta-\beta_0}\int_t^s \mathrm{e}^{(s-r)A}\left[(-A)^{\beta_0}Bu(r) - (-A)^{\beta_0}F(r,\Lambda y(r))\right]\mathrm{d}r\right|_X \\ \leq \int_t^s \frac{M_{\beta-\beta_0}}{(s-r)^{\beta-\beta_0}}\left[\|(-A)^{\beta_0}B\|M_U + M\left(\beta_0, |x|_{D((-A)^{\frac{1}{2}})}\right)\right]\mathrm{d}r \\ \leq M_{\beta-\beta_0}\left[\|(-A)^{\beta_0}B\|M_U + M\left(\beta_0, |x|_{D((-A)^{\frac{1}{2}})}\right)\right]\frac{(s-t)^{1-\beta+\beta_0}}{1-\beta+\beta_0}.\end{aligned}$$

The proof is complete. □

To prove the other propositions, we need the following theorem.

Theorem 2.5 [1] Th. 3.3.1, Chap. 2)**.** *Let δ and γ be in $[0,1[$ and $\epsilon > 0$. If the mapping*

$$t \in J \subset \mathbb{R}^+ \mapsto t^\delta u(t),$$

belongs to $L^\infty_{loc}(J;\mathbb{R})$, and if there exist two positive constants a, b such that

$$u(t) \leq a\,t^{-\delta} + b\int_0^t (t-\tau)^{-\gamma}u(\tau)\,\mathrm{d}\tau, \text{ for } \quad a.e.\ t \in J^* = J\backslash\{0\},$$

then there exits a positive constant $c := c(\delta,\gamma,\epsilon)$ independent of a and b such that

$$u(t) \leq a\,t^{-\delta}\left(1 + c\,b\,t^{1-\gamma}\mathrm{e}^{(1+\epsilon)k(\gamma,b)t}\right) \text{ for } \ a.e.\ t \in J^*,$$

where $k(\gamma,b) := (\Gamma(1-\gamma)\,b)^{1/(1-\gamma)}$.

Proposition 2.6. *Let x and x_0 be in X, $u \in \mathcal{M}(t,T;U)$, and let $y_{t,x,u}$ and $y_{t,x_0,u}$ be the corresponding solutions to equation (2.1). Then, for all $\theta \in [0,1-\alpha[$, there exists a constant $C_2(\alpha,\beta,\theta)$ such that:*

$$|\Lambda y_{t,x,u}(r) - \Lambda y_{t,x_0,u}(r)|_{X_0} \leq \frac{C_2(\alpha,\beta,\theta)}{(r-t)^{\alpha+\theta}}\left|(-A)^{-\theta}(x-x_0)\right|_X, \tag{2.21}$$

for all $r \in (t,T]$. (The constant $C_2(\alpha,\beta,\theta)$ is explicitly given in (2.24).)

Proof. Using the integral formulation (2.14) for $y_{t,x,u}$ and $y_{t,x_0,u}$, and (2.2), we obtain

$$\begin{aligned}|\Lambda y_{t,x,u}(r) - \Lambda y_{t,x_0,u}(r)|_{X_0} \leq C_\alpha\left|(-A)^{\alpha+\theta}\mathrm{e}^{(r-t)A}(-A)^{-\theta}(x-x_0)\right|_X \\ + C_\alpha\left|(-A)^{\beta+\alpha}\int_t^r \mathrm{e}^{(r-s)A}\left[F(s,\Lambda y_{t,x,u}(s)) - F(s,\Lambda y_{t,x_0,u}(s))\right]\mathrm{d}s\right|_X.\end{aligned} \tag{2.22}$$

Setting $\theta = 0$ in this estimate, we first obtain

$$|\Lambda y_{t,x,u}(r) - \Lambda y_{t,x_0,u}(r)|_{X_0} \leq \frac{C_\alpha M_\alpha}{(r-t)^\alpha}|x-x_0|_X + \frac{2C_\alpha M_{\alpha+\beta}M_F}{1-(\alpha+\beta)}(r-t)^{1-(\alpha+\beta)}.$$

Multiplying both sides by $(r-t)^{\alpha+\theta}$ we have

$$(r-t)^{\alpha+\theta}\left|\Lambda y_{t,x,u}(r)-\Lambda y_{t,x_0,u}(r)\right|_{X_0} \leq (T-t)^{\theta} C_\alpha M_\alpha |x-x_0|_X + \frac{2C_\alpha M_{\alpha+\beta} M_F}{1-(\alpha+\beta)}(T-t)^{1+\theta-\beta} \in L^\infty(t,T;\mathbb{R}). \quad (2.23)$$

Next with (2.22) we write

$$\left|\Lambda y_{t,x,u}(r)-\Lambda y_{t,x_0,u}(r)\right|_{X_0} \leq \frac{C_\alpha M_{\alpha+\theta}}{(r-t)^{\alpha+\theta}}\left|(-A)^{-\theta}(x-x_0)\right|_X + \int_t^r \frac{C_\alpha M_{\beta+\alpha}}{(r-s)^{\beta+\alpha}} K_F \left|\Lambda y_{t,x,u}(s)-\Lambda y_{t,x_0,u}(s)\right|_{X_0} \mathrm{d}s.$$

Since the function $r \mapsto (r-t)^{\alpha+\theta}\left|\Lambda y_{t,x,u}(r)-\Lambda y_{t,x_0,u}(r)\right|_{X_0}$ belongs to $L^\infty(t,T)$, we can use Theorem 2.5 with for example $\varepsilon = 1$, and we obtain (2.21) by setting

$$C_2(\alpha,\beta,\theta) = C_\alpha M_{\alpha+\theta}\left(1 + C_{\beta+\alpha,\alpha+\theta}\left(c\, C_\alpha M_\beta K_F\right) T^{1-\beta+\alpha} \mathrm{e}^{2\,k(\beta+\alpha, K_F C_\alpha M_{\beta+\alpha})T}\right) \quad (2.24)$$

where c is the constant appearing in Theorem 2.5. □

Proposition 2.7. *Let x and x_0 be in X, $u \in \mathcal{M}(t,T;U)$, and let $y_{t,x,u}$ and $y_{t,x_0,u}$ be the corresponding solutions to equation* (2.1)*. Then, for all $\theta \in [0, 1-\alpha[$, there exists a constant $C_3(\alpha,\beta,\theta)$ such that*

$$|y_{t,x,u}(r) - y_{t,x_0,u}(r)|_X \leq \left(\frac{M_\theta}{(r-t)^\theta} + \frac{C_3(\alpha,\beta,\theta)}{1-(\alpha+\theta)}(r-t)^{1-(\alpha+\beta+\theta)}\right)\left|(-A)^{-\theta}(x-x_0)\right|_X. \quad (2.25)$$

(*The constant $C_3(\alpha,\beta,\theta)$ is explicitly given in* (2.27).)

Proof. With Proposition 2.6, we have:

$$\begin{aligned}|y_{t,x,u}(r) - y_{t,x_0,u}(r)|_X &\leq \left|(-A)^{\theta}\mathrm{e}^{(r-t)A}(-A)^{-\theta}(x-x_0)\right|_X + \int_t^r \frac{M_\beta}{(r-s)^\beta} K_F \left|\Lambda y_{t,x,u}(s)-\Lambda y_{t,x_0,u}(s)\right|_{X_0}\mathrm{d}s \\ &\leq \frac{M_\theta}{(r-t)^\theta}\left|(-A)^{-\theta}(x-x_0)\right|_X + M_\beta K_F C_2(\alpha,\beta,\theta)\left|(-A)^{-\theta}(x-x_0)\right|_X \int_t^r \frac{1}{(r-s)^\beta}\frac{1}{(s-t)^{\alpha+\theta}}\mathrm{d}s \\ &\leq \left(\frac{M_\theta}{(r-t)^\theta} + M_\beta K_F\, C_2(\alpha,\beta,\theta)\int_t^r \frac{1}{(r-s)^\beta}\frac{1}{(s-t)^{\alpha+\theta}}\mathrm{d}s\right)\left|(-A)^{-\theta}(x-x_0)\right|_X.\end{aligned}$$

By using the integral formula of the beta function we have:

$$\int_t^r \frac{1}{(r-s)^\beta (s-t)^{\alpha+\theta}}\mathrm{d}s = \frac{\Gamma(1-\beta)\,\Gamma(1-(\alpha+\theta))}{\Gamma(2-(\alpha+\theta+\beta))}(r-t)^{1-(\alpha+\theta+\beta)}. \quad (2.26)$$

By setting

$$C_3(\alpha,\beta,\theta) = \frac{\Gamma(1-\beta)\,\Gamma(1-(\alpha+\theta))}{\Gamma(2-(\alpha+\theta+\beta))} M_\beta K_F\, C_2(\alpha,\beta,\theta), \quad (2.27)$$

and

$$C_4(\alpha,\beta,\theta,t;r) = \left(\frac{M_\theta}{(r-t)^\theta} + \frac{C_3(\alpha,\beta,\theta)}{1-(\alpha+\theta)}(r-t)^{1-(\alpha+\beta+\theta)}\right), \quad (2.28)$$

we obtain

$$|y_{t,x,u}(r) - y_{t,x_0,u}(r)|_X \leq C_4(\alpha,\beta,\theta,t;r) \left|(-A)^{-\theta}(x-x_0)\right|_X. \qquad \square$$

Proposition 2.8. *Let x be in X, s, $t \in [0,T)$, and $u \in \mathcal{M}(\min(t,s),T;U)$. Let us denote by $y_{t,x,u}$ and $y_{s,x,u}$ the solutions of equation (2.1) respectively corresponding to the initial data (t,x) and (s,x). Then there exist a constant $C_5(\alpha,\beta)$ and a continuous mapping $\bar{a}(t,s,x)$ (independent of α) such that, for all $r \in]\max(s,t),T]$, we have:*

$$|\Lambda y_{t,x,u}(r) - \Lambda y_{s,x,u}(r)|_{X_0} \leq \frac{C_5(\alpha,\beta)}{(r-\max(t,s))^{\alpha}} \bar{a}(t,s,x). \tag{2.29}$$

The function $s \mapsto \bar{a}(t,s,x)$ goes to 0 when s goes to t, for all fixed $x \in X$. (The constant $C_5(\alpha,\beta)$ and the mapping $\bar{a}$ are explicitly defined in (2.34) *and* (2.35).)

Proposition 2.9. *With the same assumptions and notation as in the previous proposition, there exists a constant $C_6(\alpha,\beta)$ such that, for all $r \in [\max(s,t),T]$, we have:*

$$|y_{t,x,u}(r) - y_{s,x,u}(r)|_X \leq C_6(\alpha,\beta)\,\bar{a}(t,s,x). \tag{2.30}$$

(*The constant $C_6(\alpha,\beta)$ is explicitly defined in* (2.36).)

Proof of Proposition 2.8. Consider the case where $s < t$. The case $t < s$ can be treated in a similar way. Let be $r > t > s$, with estimate (2.2) and with (2.14), we have

$$|\Lambda y_{t,x,u}(r) - \Lambda y_{s,x,u}(r)|_{X_0} \leq C_\alpha \left|(-A)^{\alpha} e^{(r-t)A}\left(e^{|t-s|A}x - x\right)\right|_X \tag{2.31}$$

$$+ C_\alpha \left|\int_s^t (-A)^{\alpha+\beta} e^{(r-\sigma)A}\left[Bu(\sigma) - F(\sigma,\Lambda y_{s,x,u}(\sigma))\right] d\sigma\right|_X \tag{2.32}$$

$$+ C_\alpha \left|\int_t^r (-A)^{\alpha+\beta} e^{(r-\sigma)A}\left[F(\sigma,\Lambda y_{t,x,u}(\sigma)) - F(\sigma,\Lambda y_{s,x,u}(\sigma))\right] d\sigma\right|_X. \tag{2.33}$$

Now we can write

$$(2.31) \leq \frac{C_\alpha M_\alpha}{(r-t)^{\alpha}} \left|\left(e^{|t-s|A} - I\right)x\right|_X.$$

Since $(r-t)^{\alpha} \leq (r-\sigma)^{\alpha}$ for all $\sigma \in (s,t)$, we have

$$(2.32) \leq C_\alpha M_{\alpha+\beta}(M_F + \|B\|M_U)\int_s^t \frac{d\sigma}{(r-\sigma)^{\alpha+\beta}} \leq \frac{C_\alpha M_{\alpha+\beta}(M_F + \|B\|M_U)}{(r-t)^{\alpha}} \int_s^t \frac{d\sigma}{(r-\sigma)^{\beta}}.$$

Similarly, for all $\sigma \in (s,t)$, we have $(r-\sigma)^{\beta} \geq (t-\sigma)^{\beta}$. Then

$$\int_s^t \frac{d\sigma}{(r-\sigma)^{\beta}} \leq \int_s^t \frac{d\sigma}{(t-\sigma)^{\beta}} = \frac{(t-s)^{1-\beta}}{1-\beta},$$

and therefore we obtain

$$(2.32) \leq \frac{C_\alpha M_{\alpha+\beta}(M_F + \|B\|M_U)}{(r-t)^{\alpha}(1-\beta)} |t-s|^{1-\beta}.$$

The last term can be estimated as follows

$$(2.33) \leq \int_t^r \frac{C_\alpha M_{\alpha+\beta} K_F}{(r-\sigma)^{\alpha+\beta}} |\Lambda y_{t,x,u}(\sigma) - \Lambda y_{s,x,u}(\sigma)|_{X_0} d\sigma$$

$$\leq \frac{C_\alpha M_{\alpha+\beta} K_F}{(r-t)^{\alpha}} \int_t^r \frac{1}{(r-\sigma)^{\beta}} |\Lambda y_{t,x,u}(\sigma) - \Lambda y_{s,x,u}(\sigma)|_{X_0} d\sigma.$$

From the estimates obtained for (2.31), (2.32), (2.33), we deduce that the function $r \mapsto (r-t)^{\alpha} \left|\Lambda y_{t,x,u}(r) - \Lambda y_{s,x,u}(r)\right|_{X_0}$ belongs to $L^{\infty}(t,T)$. Applying Theorem 2.5, we obtain

$$\left|\Lambda y_{t,x,u}(r) - \Lambda y_{s,x,u}(r)\right|_{X_0} \leq \frac{C_5(\alpha,\beta)}{(r-t)^{\alpha}}\, \bar{a}(t,s,x),$$

with

$$C_5(\alpha,\beta) = 2C_{\alpha} \max\left(\tfrac{M_{\alpha}}{1-\alpha}, \tfrac{M_{\alpha+\beta}(M_F+\|B\|M_U)}{1-\beta}\right)\left(1 + c\, C_{\alpha} M_{\beta+\alpha} K_F T^{(1-\beta)} \mathrm{e}^{CT}\right), \tag{2.34}$$

where c and C are given in Theorem 2.5, and

$$\bar{a}(t,s,x) = \left|\mathrm{e}^{|t-s|A}x - x\right|_X + |t-s|^{1-\beta}. \tag{2.35}$$

The function $s \mapsto \bar{a}(t,s,x)$ goes to 0 when s goes to t, for all x fixed in X. □

Proof of Proposition 2.9. Consider the case where $0 \leq s < t$. We have:

$$\begin{aligned}\left|y_{t,x,u}(r) - y_{s,x,u}(r)\right|_X &\\ \leq \left|\mathrm{e}^{|t-s|A}x - x\right|_X &+ \frac{M_{\beta}(M_F + \|B\|M_U)}{1-\beta}(t-s)^{1-\beta} + C_5(\alpha,\beta)K_F\, \bar{a}(t,s,x) \int_t^r \frac{M_{\beta}}{(r-\sigma)^{\beta}} \frac{1}{(\sigma-t)^{\alpha}} \mathrm{d}\sigma.\end{aligned}$$

From (2.26) with $\theta = 0$, it yields:

$$\int_t^r \frac{1}{(r-\sigma)^{\beta}} \frac{1}{(\sigma-t)^{\alpha}} \mathrm{d}\sigma \leq \frac{\Gamma(1-\beta)\Gamma(1-\alpha)}{\Gamma(2-(\alpha+\beta))} T^{1-(\alpha+\beta)} \leq 4T^{1-(\alpha+\beta)}.$$

Hence

$$\left|y_{t,x,u}(r) - y_{s,x,u}(r)\right|_X \leq \left(1 + \frac{M_{\beta}(M_F + \|B\|M_U)}{1-\beta} + C_5(\alpha,\beta)K_F 4T^{1-(\alpha+\beta)}\right)\bar{a}(t,s,x). \tag{2.36}$$

The proof is complete. □

Proposition 2.10. *Let x and x_0 be in X, $t \in [0,T)$, and $u \in \mathcal{M}(t,T;U)$. Let us denote by $y_{t,x,u}$ and $y_{t,x_0,u}$ the solutions of equation (2.1) respectively corresponding to the initial data (t,x) and (t,x_0). Then, for all $r \in [t,T]$ and all $s \in]t,T]$, we have:*

$$\left|\Lambda y_{t,x,u}(s) - \Lambda y_{t,x_0,u}(s)\right|_{X_0} \leq \frac{C_7(\alpha,\beta)}{(s-t)^{\alpha}} |x - x_0|_X, \tag{2.37}$$

and

$$\left|y_{t,x,u}(r) - y_{t,x_0,u}(r)\right|_X \leq C_8(\alpha,\beta)\, |x - x_0|_X. \tag{2.38}$$

Proof. The function $w = y_{t,x,u} - y_{t,x_0,u}$ satisfies

$$\begin{aligned}\left|\Lambda w(s)\right|_{X_0} &\leq \frac{M_{\alpha}}{(s-t)^{\alpha}} |x - x_0|_X + \left|(-A)^{\beta+\alpha} \int_t^s \mathrm{e}^{(s-r)A}\left[F(r, \Lambda y_{t,x,u}(r)) - F(r, \Lambda y_{,x_0,u}(r))\right] \mathrm{d}r\right|_X \\ &\leq \frac{M_{\alpha}}{(s-t)^{\alpha}} |x - x_0|_X + M_{\alpha+\beta}K_F \int_t^s \frac{1}{(s-r)^{\alpha+\beta}} \left|\Lambda w(r)\right|_{X_0} \mathrm{d}r.\end{aligned}$$

Estimate (2.37) now follows from Theorem 2.5. We can also obtain the estimate

$$|w(r)|_X \leq |x - x_0|_X + K_F \int_t^r \frac{1}{(r-s)^\beta} \frac{C_\alpha}{(s-t)^\alpha} |x - x_0|_X \, \mathrm{d}s.$$

By the same calculation as in the proof of Proposition 2.7, we have

$$|w(r)|_X \leq \left[1 + K_F C_\alpha T^{1-(\alpha+\beta)}\right] |x - x_0|_X,$$

and (2.38) is established. □

3. Viscosity solutions and uniqueness result

In this section we study the uniqueness of solution to equation (1.1). It is well known that, by a change of variable in time, the terminal value problem (1.1) is equivalent the Hamilton-Jacobi-Bellman equation

$$\begin{aligned} &\frac{\partial v}{\partial t}(t,x) - \left(D_x v(t,x) \mid Ax\right)_X + \left((-A)^\beta D_x v(t,x) \mid F(t,\Lambda x)\right)_X \\ &\qquad + H\left(t, x, (-A)^\beta D_x v(t,x)\right) = 0, \quad \forall (t,x) \in]0,T] \times X, \\ &v(0,x) = g(x). \end{aligned} \tag{3.1}$$

Let $C_A^1(]0,T[\times X)$ be the set of all functions Φ (called test functions) satisfying the following conditions:

(α) $\Phi \in C^1(]0,T[\times X)$.

(β) $D_x\Phi(\cdot,x)$ is constant (in t) and $D_x\Phi(t,\cdot)$ is Lipschitz on X, *i.e.*:

$$|D_x\Phi(t,x) - D_x\Phi(t,y)|_X \leq K_\Phi |x-y|_X.$$

(γ) For all $\theta \in [0, 1-\alpha[$, $D_x\Phi(t,x)$ belongs to $D((-A)^\theta)$ if and only if $x \in D((-A)^\theta)$.

(δ) The mapping $x \mapsto D_x\Phi(t,x)$ is continuous from $D((-A)^{\frac{1}{2}})$ into itself.

Remark 3.1. Since $D_x\Phi(t,x)$ does not depend on t, from the last condition we can infer that the mapping $(t,x) \mapsto D_x\Phi(t,x)$ is continuous from $[0,T] \times D((-A)^{\frac{1}{2}})$ into $D((-A)^{\frac{1}{2}})$.

Definition 3.2. Consider functions w satisfying:

(i) $w \in C([0,T] \times X)$ and $|w(t,x)| \leq M_w$, for all $(t,x) \in [0,T] \times X$.

(ii) $|w(t,x) - w(t,y)| \leq K_w |x-y|_X$ for all $t \in [0,T]$, and all x, $y \in X$.

(iii) $|w(t,x) - w(t,y)| \leq C_{t,\theta} \left|(-A)^{-\theta}(x-y)\right|_X$, for all $t \in]0,T]$, all x, $y \in X$, and all $\theta \in [0, 1-\alpha[$, where the constant $C_{t,\theta}$ is bounded on all compact subset of $]0,T]$.

(iv) $w(\cdot,x)$ is Hölder continuous in time of exponent $0 < \eta \leq 1$, for all $x \in D((-A)^{\frac{1}{2}})$. More precisely there exists a constant $M_{1,w}$ such that:

$$|w(t,x) - w(s,x)| \leq M_{1,w} \left(1 + \left|(-A)^{\frac{1}{2}} x\right|_X\right) |t-s|^\eta.$$

We say that a function w satisfying (i)-(iv) is a viscosity subsolution of (3.1) on $[0,T]$ if, for every $\Phi \in C_A^1(]0,T[\times X)$, the conditions (α_1) and (β_1) are satisfied, where:

$$\begin{aligned} &\frac{\partial \Phi}{\partial t}(t,x) + H\left(t, x, (-A)^\beta D_x\Phi(t,x)\right) \\ &\qquad + \left((-A)^{\frac{1}{2}} D_x\Phi(t,x) \mid (-A)^{\frac{1}{2}} x\right)_X + \left((-A)^\beta D_x\Phi(t,x) \mid F(t,\Lambda x)\right)_X \leq 0 \\ &\text{for all } (t,x) \in \left(]0,T[\times D((-A)^{\frac{1}{2}})\right) \cap \arg\max(w - \Phi), \end{aligned} \tag{α_1}$$

$$(\beta_1) \qquad \lim_{t\searrow 0}\sup_{x\in X}\left[w(t,x)-g\left(\mathrm{e}^{tA}x\right)\right]^{+}=0.$$

(Recall that $[f]^{+}=\max(f,0)$ and $[f]^{-}=\min(f,0)$.)

We say that a function w satisfying (i)-(iv) is a viscosity supersolution of (3.1) on $[0,T]$ if, for every $\Phi\in C_A^1(]0,T[\times X)$, the two conditions (α_2) and (β_2) are satisfied, where:

$$(\alpha_2) \qquad \begin{aligned} &\frac{\partial\Phi}{\partial t}(t,x)+H\left(t,x,(-A)^{\beta}D_x\Phi(t,x)\right)\\ &\qquad+\left((-A)^{\frac12}D_x\Phi(t,x)\mid(-A)^{\frac12}x\right)_X+\left((-A)^{\beta}D_x\Phi(t,x)\mid F(t,\Lambda x)\right)_X\geq 0\\ &\text{for all } (t,x)\in\left(]0,T[\times D((-A)^{\frac12})\right)\cap\arg\min(w-\Phi), \end{aligned}$$

$$(\beta_2) \qquad \lim_{t\searrow 0}\sup_{x\in X}\left[w(t,x)-g\left(\mathrm{e}^{tA}x\right)\right]^{-}=0.$$

Finally, w is a viscosity solution of (3.1) if it is both a subsolution and a supersolution of equation (3.1).

Remark 3.3. If (t,x) belongs to $]0,T[\times D((-A)^{\frac12})$, then Λx is well defined and $F(t,\Lambda x)$ is meaningful.

Remark 3.4. If in place of equation (3.1) we consider equation (1.1), the conditions (α_1), (α_2), (β_1), and (β_2) have to be modified accordingly (see Sect. 4).

Theorem 3.5. *Assume that* (**i**) – (**vii**) *of Section 2 hold. Let w be a viscosity subsolution and v be a viscosity supersolution of the Hamilton-Jacobi-Bellman equation (3.1). Then*

$$w(t,x)\leq v(t,x) \qquad \text{for all } (t,x)\in[0,T]\times X. \tag{3.2}$$

Before proving this theorem let us state a useful lemma.

Lemma 3.6. *Assume that φ and $\psi\in C_A^1(]0,T[\times X)$, and let w and v be two continuous functions in $[0,T]\times X$. If*

$$(t_0,x_0)\in\operatorname{argmax}_{\mathcal{O}}(w-\varphi) \quad \text{and} \quad (s_0,y_0)\in\operatorname{argmin}_{\mathcal{O}}(v-\psi), \tag{3.3}$$

where $\mathcal{O}$ is an open set of $]0,T[\times X$, then

$$D_x\varphi(t_0,x_0)\subset D_x^{+}w(t_0,x_0) \quad \text{and} \quad D_x\psi(s_0,y_0)\subset D_x^{-}v(s_0,y_0). \tag{3.4}$$

Proof. We establish the result only for the function φ. Due to (3.3), for all $x\in X$, we have:

$$w(t_0,x)-w(t_0,x_0)-[\varphi(t_0,x)-\varphi(t_0,x_0)]\leq 0.$$

With the condition (β) in the definition of $C_A^1(]0,T[\times X)$, we have:

$$|\varphi(t_0,x)-\varphi(t_0,x_0)-(D_x\varphi(t_0,x_0)\mid x-x_0)_X|\leq K_{\Phi}|x-x_0|^2.$$

Combining this estimate with the previous inequality, we obtain:

$$\operatorname{limsup}_{|x-x_0|\to 0}\frac{w(t_0,x)-w(t_0,x_0)-(D_x\varphi(t_0,x_0)\mid x-x_0)_X}{|(x-x_0)|}\leq 0. \qquad \square$$

Let us recall Young's inequality. For all p, $q>1$ such that $\frac1p+\frac1q=1$, we have:

$$ab\leq\frac{\lambda^p}{p}a^p+\frac{1}{q\lambda^q}b^q \text{ for all } \lambda>0, \text{ and all } a,\ b\geq 0. \tag{3.5}$$

Proof of Theorem 3.5. We are going to use the same kind of proof as in [6]. The proof is divided in five steps.
Step 1. Since $w, v \in C([0,T] \times X)$, it is enough to prove that

$$w_\sigma(t,x) \leq v_\sigma(t,x) \text{ for all } (t,x) \in]0,T[\times X \text{ and all } \sigma > 0, \tag{3.6}$$

where

$$w_\sigma(t,x) = w(t,x) - \frac{\sigma}{T-t} \text{ and } v_\sigma(t,x) = v(t,x) + \frac{\sigma}{T-t}.$$

As w is a subsolution of (3.1), then w_σ is a subsolution of

$$\begin{aligned} &\frac{\partial w_\sigma}{\partial t}(t,x) - (D_x w_\sigma(t,x) \mid Ax)_X + \left((-A)^\beta D_x w_\sigma(t,x) \mid F(t,\Lambda x)\right)_X \\ &\qquad + H\left(t,x,(-A)^\beta D_x w_\sigma(t,x)\right) = -\frac{\sigma}{(T-t)^2} \leq -\frac{\sigma}{T^2}, \\ &w_\sigma(0,x) = g(x) - \frac{\sigma}{T}. \end{aligned} \tag{3.7}$$

Similarly, v_σ is a supersolution of

$$\begin{aligned} &\frac{\partial v_\sigma}{\partial t}(t,x) - (D_x v_\sigma(t,x) \mid Ax)_X + \left((-A)^\beta D_x v_\sigma(t,x) \mid F(t,\Lambda x)\right)_X \\ &\qquad + H\left(t,x,(-A)^\beta D_x v_\sigma(t,x)\right) = \frac{\sigma}{(T-t)^2} \geq \frac{\sigma}{T^2}, \\ &v_\sigma(0,x) = g(x) + \frac{\sigma}{T}. \end{aligned} \tag{3.8}$$

Step 2. Let $0 < \eta \leq 1$ be an exponent such that $v(\cdot,x)$ and $w(\cdot,x)$ be Hölder continuous of exponent η (Condition (iv) in Def. 3.2). We set

$$\bar{\eta} = \min(\eta, \eta_1, \eta_2). \tag{3.9}$$

Let ε and μ be in $]0,1]$. For all (t,x) and all (s,y) in $]0,T[\times X$, we define

$$\Phi_{\varepsilon,\mu}(t,s,x,y) = w_\sigma(t,x) - v_\sigma(s,y) - \frac{1}{2\varepsilon}\left((-A)^{-1}(x-y) \mid (x-y)\right)_X - \frac{(t-s)^2}{2\varepsilon^{\frac{2}{\bar{\eta}}}} - \frac{\mu}{2}\left(|x|_X^2 + |y|_X^2\right).$$

Let τ be a small parameter satisfying $0 < \tau < \frac{T}{2}$, and set $Q_\tau = [\tau, T-\tau] \times X$. From condition (iii) in Definition 3.2, v and w are weakly continuous in Q_τ. Thus v_σ and w_σ are weakly continuous in Q_τ. Moreover the mapping

$$(x,y) \mapsto \frac{1}{2\varepsilon}\left((-A)^{-1}(x-y) \mid (x-y)\right)_X + \frac{\mu}{2}|x|_X^2 + \frac{\mu}{2}|y|_X^2$$

is convex, and continuous (for the strong topology of $X \times X$). Therefore $\Phi_{\varepsilon,\mu}$ is weakly lower semicontinuous in Q_τ^2. Besides for all couples $(t,x),(s,y) \in Q_\tau$

$$\Phi_{\varepsilon,\mu}(t,s,x,y) \leq M_w + M_v - \frac{\mu}{2}\left(|x|_X^2 + |y|_X^2\right) \to -\infty \text{ when } \max(|x|_X, |y|_X) \to +\infty.$$

Thus there exists $(t_{\varepsilon,\mu}, x_{\varepsilon,\mu}, s_{\varepsilon,\mu}, y_{\varepsilon,\mu}) \in Q_\tau^2$ such that

$$\Phi_{\varepsilon,\mu}(t_{\varepsilon,\mu}, s_{\varepsilon,\mu}, x_{\varepsilon,\mu}, y_{\varepsilon,\mu}) = \max_{Q_\tau \times Q_\tau} \Phi_{\varepsilon,\mu}.$$

Let us verify that if τ is small enough $(0 < \tau < \tau_\sigma)$, then $t_{\varepsilon,\mu}, s_{\varepsilon,\mu} < T - \tau$. Indeed, from the inequality

$$w_\sigma(t,x) - v_\sigma(s,y) \leq M_w + M_v - \frac{\sigma}{T-t} - \frac{\sigma}{T-s},$$

it follows that

$$\lim_{\max(s,t)\to T}(w_\sigma(t,x)-v_\sigma(s,y))=-\infty \qquad \text{uniformly w.r. to } x \text{ and } y.$$

Hence we have shown that there exists $(t_{\varepsilon,\mu},x_{\varepsilon,\mu},s_{\varepsilon,\mu},y_{\varepsilon,\mu})\in\left([\tau,T-\tau[\times X\right)^2$ such that

$$\Phi_{\varepsilon,\mu}(t_{\varepsilon,\mu},s_{\varepsilon,\mu},x_{\varepsilon,\mu},y_{\varepsilon,\mu})=\max_{Q_\tau\times Q_\tau}\Phi_{\varepsilon,\mu}.$$

Step 3. We are going to obtain some a priori estimates on $t_{\varepsilon,\mu},s_{\varepsilon,\mu},x_{\varepsilon,\mu},y_{\varepsilon,\mu}$. Since

$$\Phi_{\varepsilon,\mu}(t_{\varepsilon,\mu},t_{\varepsilon,\mu},x_{\varepsilon,\mu},x_{\varepsilon,\mu})+\Phi_{\varepsilon,\mu}(s_{\varepsilon,\mu},s_{\varepsilon,\mu},y_{\varepsilon,\mu},y_{\varepsilon,\mu})\le 2\Phi_{\varepsilon,\mu}(t_{\varepsilon,\mu},s_{\varepsilon,\mu},x_{\varepsilon,\mu},y_{\varepsilon,\mu}),$$

then

$$\begin{aligned}w_\sigma(t_{\varepsilon,\mu},x_{\varepsilon,\mu})-v_\sigma(t_{\varepsilon,\mu},x_{\varepsilon,\mu})+w_\sigma(s_{\varepsilon,\mu},y_{\varepsilon,\mu})-v_\sigma(s_{\varepsilon,\mu},y_{\varepsilon,\mu})\\ \le 2\left(w_\sigma(t_{\varepsilon,\mu},x_{\varepsilon,\mu})-v_\sigma(s_{\varepsilon,\mu},y_{\varepsilon,\mu})\right)-\frac{1}{\varepsilon}\left((-A)^{-1}(x_{\varepsilon,\mu}-y_{\varepsilon,\mu})\mid(x_{\varepsilon,\mu}-y_{\varepsilon,\mu})\right)_X-\frac{(t_{\varepsilon,\mu}-s_{\varepsilon,\mu})^2}{\varepsilon^{\frac{2}{\eta}}}.\end{aligned}$$

Consequently we obtain

$$\begin{aligned}\frac{1}{\varepsilon}\left((-A)^{-1}(x_{\varepsilon,\mu}-y_{\varepsilon,\mu})\mid(x_{\varepsilon,\mu}-y_{\varepsilon,\mu})\right)_X+\frac{(t_{\varepsilon,\mu}-s_{\varepsilon,\mu})^2}{\varepsilon^{\frac{2}{\eta}}}\\ \le w_\sigma(t_{\varepsilon,\mu},x_{\varepsilon,\mu})-w_\sigma(s_{\varepsilon,\mu},y_{\varepsilon,\mu})+v_\sigma(t_{\varepsilon,\mu},x_{\varepsilon,\mu})-v_\sigma(s_{\varepsilon,\mu},y_{\varepsilon,\mu})\\ \le 2(M_w+M_v)=C.\end{aligned}\tag{3.10}$$

We deduce that

$$|t_{\varepsilon,\mu}-s_{\varepsilon,\mu}|\le C\varepsilon^{\frac{1}{\eta}}\tag{3.11}$$

$$\left|(-A)^{-\frac{1}{2}}(x_{\varepsilon,\mu}-y_{\varepsilon,\mu})\right|_X\le C\sqrt{\varepsilon}.\tag{3.12}$$

Now let us show that

$$\lim_{\mu\searrow 0}\mu\left(\left|x_{\varepsilon(\mu),\mu}\right|_X^2+\left|y_{\varepsilon(\mu),\mu}\right|_X^2\right)=0,\tag{3.13}$$

where $0<\varepsilon(\mu)<1$ is any function of μ. For all $x\in X$, we have:

$$\Phi_{\varepsilon,\mu}(t_{\varepsilon,\mu},t_{\varepsilon,\mu},x,x)\le\Phi_{\varepsilon,\mu}(t_{\varepsilon,\mu},s_{\varepsilon,\mu},x_{\varepsilon,\mu},y_{\varepsilon,\mu}),$$

i.e.

$$\begin{aligned}w_\sigma(t_{\varepsilon,\mu},x)-v_\sigma(t_{\varepsilon,\mu},x)-\mu|x|_X^2\le w_\sigma(t_{\varepsilon,\mu},x_{\varepsilon,\mu})-v_\sigma(s_{\varepsilon,\mu},y_{\varepsilon,\mu})-\frac{1}{2\varepsilon}\left((-A)^{-1}(x_{\varepsilon,\mu}-y_{\varepsilon,\mu})\mid(x_{\varepsilon,\mu}-y_{\varepsilon,\mu})\right)_X\\ -\frac{(t_{\varepsilon,\mu}-s_{\varepsilon,\mu})^2}{\varepsilon^{\frac{2}{\eta}}}-\frac{\mu}{2}\left(|x_{\varepsilon,\mu}|_X^2+|y_{\varepsilon,\mu}|_X^2\right).\end{aligned}$$

By taking $x=0$, we deduce that

$$\frac{\mu}{2}\left(|x_{\varepsilon,\mu}|_X^2+|y_{\varepsilon,\mu}|_X^2\right)\le 2(M_w+M_v).$$

Thus, $\frac{\mu}{2}\left(|x_{\varepsilon,\mu}|_X^2+|y_{\varepsilon,\mu}|_X^2\right)$ is bounded independently of $\varepsilon\in]0,1]$ and $\mu\in]0,1]$. We are going to use this property to prove (3.13).

We define now Ψ_ε on $Q_\tau \times Q_\tau$ by:

$$\Psi_\varepsilon(t,s,x,y) = w_\sigma(t,x) - v_\sigma(s,y) + \frac{1}{2\varepsilon}\left(A^{-1}(x-y) \mid (x-y)\right)_X - \frac{(t-s)^2}{2\varepsilon^{\frac{2}{\eta}}}.$$

The function Ψ_ε is bounded from above (independently of ε):

$$\Psi_\varepsilon \leq M_w + M_v,$$

and upper semicontinuous. So, for every $\delta > 0$, there exists $(t_{\varepsilon,\delta}, x_{\varepsilon,\delta}, s_{\varepsilon,\delta}, y_{\varepsilon,\delta}) \in Q_\tau^2$ such that

$$\Psi_\varepsilon(t_{\varepsilon,\delta}, s_{\varepsilon,\delta}, x_{\varepsilon,\delta}, y_{\varepsilon,\delta}) \geq \Psi_\varepsilon(t,s,x,y) - \delta \quad \forall\,(t,x),(s,y) \in Q_\tau.$$

The point $(t_{\varepsilon,\delta}, s_{\varepsilon,\delta}, x_{\varepsilon,\delta}, y_{\varepsilon,\delta})$ corresponds to a supremum (and *a priori* not a maximum), because we have not proved that $\Psi_\varepsilon \to -\infty$ when $\max(|x|_X, |y|_X) \to \infty$. Starting from the inequality

$$\Phi_{\varepsilon,\mu}(t_{\varepsilon,\mu}, s_{\varepsilon,\mu}, x_{\varepsilon,\mu}, y_{\varepsilon,\mu}) \geq \Phi_{\varepsilon,\mu}(t_{\varepsilon,\delta}, s_{\varepsilon,\delta}, x_{\varepsilon,\delta}, y_{\varepsilon,\delta})$$

it follows that

$$\begin{aligned}\Psi_\varepsilon(t_{\varepsilon,\mu}, s_{\varepsilon,\mu}, x_{\varepsilon,\mu}, y_{\varepsilon,\mu}) - \frac{\mu}{2}\left(|x_{\varepsilon,\mu}|_X^2 + |y_{\varepsilon,\mu}|_X^2\right) &\geq \Psi_\varepsilon(t_{\varepsilon,\delta}, s_{\varepsilon,\delta}, x_{\varepsilon,\delta}, x_{\varepsilon,\delta}) - \frac{\mu}{2}\left(|x_{\varepsilon,\delta}|_X^2 + |y_{\varepsilon,\delta}|_X^2\right) \\ &\geq \Psi_\varepsilon(t_{\varepsilon,\mu}, s_{\varepsilon,\mu}, x_{\varepsilon,\mu}, y_{\varepsilon,\mu}) - \delta - \frac{\mu}{2}\left(|x_{\varepsilon,\delta}|_X^2 + |y_{\varepsilon,\delta}|_X^2\right).\end{aligned}$$

Hence we have

$$\frac{\mu}{2}\left(|x_{\varepsilon,\mu}|_X^2 + |y_{\varepsilon,\mu}|_X^2\right) \leq \frac{\mu}{2}\left(|x_{\varepsilon,\delta}|_X^2 + |y_{\varepsilon,\delta}|_X^2\right) + \delta \qquad \text{for all } \delta > 0.$$

This inequality is satisfied for all $\varepsilon > 0$. In particular if $0 < \varepsilon(\mu) < 1$ is a function of μ, we can write

$$\frac{\mu}{2}\left(\left|x_{\varepsilon(\mu),\mu}\right|_X^2 + \left|y_{\varepsilon(\mu),\mu}\right|_X^2\right) \leq \frac{\mu}{2}\left(\left|x_{\varepsilon(\mu),\delta}\right|_X^2 + \left|y_{\varepsilon(\mu),\delta}\right|_X^2\right) + \delta \leq \mu\frac{2}{\delta}(M_w + M_v) + \delta.$$

Thus

$$\limsup_{\mu \searrow 0} \frac{\mu}{2}\left(\left|x_{\varepsilon(\mu),\mu}\right|_X^2 + \left|y_{\varepsilon(\mu),\mu}\right|_X^2\right) = \delta.$$

We take the limit when $\delta \to 0$ to obtain (3.13).

Let us prove that

$$\frac{1}{2\varepsilon}\left|(-A)^{-\frac{1}{2}}(x_{\varepsilon,\mu} - y_{\varepsilon,\mu})\right|_X^2 \leq \frac{|x_{\varepsilon,\mu} - y_{\varepsilon,\mu}|_X^2}{8\varepsilon\lambda} + 2M_v^2\varepsilon\lambda + \frac{\mu}{2}|x_{\varepsilon,\mu}|_X^2, \tag{3.14}$$

for all constant $\lambda > 0$. From the inequality

$$\Phi_{\varepsilon,\mu}(t_{\varepsilon,\mu}, s_{\varepsilon,\mu}, x_{\varepsilon,\mu}, x_{\varepsilon,\mu}) \leq \Phi_{\varepsilon,\mu}(t_{\varepsilon,\mu}, s_{\varepsilon,\mu}, x_{\varepsilon,\mu}, y_{\varepsilon,\mu}),$$

we deduce

$$\frac{1}{2\varepsilon}\left((-A)^{-1}(x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) \mid (x_{\varepsilon,\mu} - y_{\varepsilon,\mu})\right)_X \leq v_\sigma(s_{\varepsilon,\mu}, x_{\varepsilon,\mu}) - v_\sigma(s_{\varepsilon,\mu}, y_{\varepsilon,\mu}) + \frac{\mu}{2}|x_{\varepsilon,\mu}|_X^2.$$

With (ii) in Definition 3.2 we have:

$$\frac{1}{\varepsilon}\left|(-A)^{-\frac{1}{2}}\left(x_{\varepsilon,\mu}-y_{\varepsilon,\mu}\right)\right|_X^2 \leq 2M_v\left|x_{\varepsilon,\mu}-y_{\varepsilon,\mu}\right|+\mu|x_{\varepsilon,\mu}|_X^2.$$

Estimate (3.14) follows from Young's inequality.

Step 4. We are going to show that $\min(t_{\varepsilon,\mu}, s_{\varepsilon,\mu}) = \tau$. If it is not true, then $t_{\varepsilon,\mu} > \tau$ and $s_{\varepsilon,\mu} > \tau$. Let φ and ψ be two mappings defined by:

$$\varphi(t,x) = v_\sigma(s_{\varepsilon,\mu}, y_{\varepsilon,\mu}) + \frac{1}{2\varepsilon}\left((-A)^{-1}(x-y_{\varepsilon,\mu}) \mid (x-y_{\varepsilon,\mu})\right)_X + \frac{(t-s_{\varepsilon,\mu})^2}{2\varepsilon^{\frac{2}{\eta}}} + \frac{\mu}{2}\left(|x|_X^2 + |y_{\varepsilon,\mu}|_X^2\right)$$

$$\psi(t,y) = w_\sigma(t_{\varepsilon,\mu}, x_{\varepsilon,\mu}) - \frac{1}{2\varepsilon}\left((-A)^{-1}(x_{\varepsilon,\mu}-y) \mid (x_{\varepsilon,\mu}-y)\right)_X - \frac{(t_{\varepsilon,\mu}-s)^2}{2\varepsilon^{\frac{2}{\eta}}} - \frac{\mu}{2}\left(|x_{\varepsilon,\mu}|_X^2 + |y|_X^2\right).$$

The mappings φ, ψ belong to $C_A^1(]0,T[\times X)$. Indeed

(i) $\varphi \in C^1(]0,T[\times X)$ and $D_x\varphi(t,x) = \frac{1}{\varepsilon}(-A)^{-1}(x-y_{\varepsilon,\mu}) + \mu x$.

(ii) $D_x\varphi(\cdot,x)$ is constant in t, and $x \mapsto D_x\varphi(t,x)$ is Lipschitz from X into X because $(-A)^{-1}$ is a linear and continuous operator from X into X:

$$|D_x\varphi(t,x) - D_x\varphi(t,y)|_X^2 \leq 2\left(\frac{1}{\varepsilon}\left|A^{-1}(x-y)\right|_X^2 + \mu|x-y|_X^2\right) \leq 2\left(\mu + \frac{C}{\varepsilon}\right)|x-y|_X^2.$$

(iii) It is clear that for all $\theta \in [0, 1-\alpha[$, $D_x\varphi(t,x) \in D((-A)^\theta) \Leftrightarrow x \in D((-A)^\theta)$.

(iv) Moreover the mapping $x \mapsto D_x\varphi(t,x)$ is continuous from $D((-A)^{\frac{1}{2}})$ into itself.

The mappings φ, ψ have been chosen to satisfy:

$$(t_{\varepsilon,\mu}, x_{\varepsilon,\mu}) \in \arg\max_{Q_\tau}(w_\sigma - \varphi), \quad \text{and} \quad (s_{\varepsilon,\mu}, y_{\varepsilon,\mu}) \in \arg\min_{Q_\tau}(v_\sigma - \psi). \tag{3.15}$$

Since v_σ and w_σ satisfy (iii) in Definition 3.2, with [5, Cor. 3.4], $D_x^+ w_\sigma(t_{\varepsilon,\mu}, x_{\varepsilon,\mu})$ and $D_x^- v_\sigma(s_{\varepsilon,\mu}, y_{\varepsilon,\mu})$ are included in $D((-A)^\theta)$ for all $\theta \in \left[0, \frac{3}{4}\right[$. Due to Lemma 3.6, $D_x\varphi(t_{\varepsilon,\mu}, x_{\varepsilon,\mu}) \subset D_x^+ w_\sigma(t_{\varepsilon,\mu}, x_{\varepsilon,\mu})$ and $D_x\psi(s_{\varepsilon,\mu}, y_{\varepsilon,\mu}) \subset D_x^- v_\sigma(s_{\varepsilon,\mu}, y_{\varepsilon,\mu})$. Hence, we have

$$D_x\varphi(t_{\varepsilon,\mu}, x_{\varepsilon,\mu}),\ D_x\psi(s_{\varepsilon,\mu}, y_{\varepsilon,\mu}) \in D((-A)^\theta) \text{ for all } \theta \in [0, 1-\alpha[.$$

Therefore $x_{\varepsilon,\mu}$ and $y_{\varepsilon,\mu}$ belong to $D\left((-A)^\theta\right)$ for all $\theta \in [0, 1-\alpha[$, and in particular $x_{\varepsilon,\mu}, y_{\varepsilon,\mu} \in D((-A)^{\frac{1}{2}})$. Since w_σ is a viscosity subsolution of (3.7), and $(t_{\varepsilon,\mu}, x_{\varepsilon,\mu})$ belongs to $\left(]0,T[\times D((-A)^{\frac{1}{2}})\right) \cap \arg\max(w_\sigma - \varphi)$, we have:

$$\begin{aligned}
\frac{t_{\varepsilon,\mu}-s_{\varepsilon,\mu}}{\varepsilon^{\frac{2}{\eta}}} + H\left(t_{\varepsilon,\mu}, x_{\varepsilon,\mu}, (-A)^\beta\left(\frac{1}{\varepsilon}(-A)^{-1}(x_{\varepsilon,\mu}-y_{\varepsilon,\mu}) + \mu x_{\varepsilon,\mu}\right)\right) \\
+ \left((-A)^{\frac{1}{2}}\left(\frac{1}{\varepsilon}(-A)^{-1}(x_{\varepsilon,\mu}-y_{\varepsilon,\mu}) + \mu x_{\varepsilon,\mu}\right) \mid (-A)^{\frac{1}{2}} x_{\varepsilon,\mu}\right)_X \\
+ \left((-A)^\beta\left(\frac{1}{\varepsilon}(-A)^{-1}(x_{\varepsilon,\mu}-y_{\varepsilon,\mu}) + \mu x_{\varepsilon,\mu}\right) \mid F(t_{\varepsilon,\mu}, \Lambda x_{\varepsilon,\mu})\right)_X \\
\leq -\frac{\sigma}{T^2}.
\end{aligned}$$

In the same way, since v_σ is a viscosity supersolution of (3.8), with (3.15) we have:

$$\begin{aligned}\frac{t_{\varepsilon,\mu}-s_{\varepsilon,\mu}}{\varepsilon^{\frac{2}{\eta}}}+H\left(s_{\varepsilon,\mu},y_{\varepsilon,\mu},(-A)^{\beta}\left(\frac{1}{\varepsilon}(-A)^{-1}\left(x_{\varepsilon,\mu}-y_{\varepsilon,\mu}\right)-\mu y_{\varepsilon,\mu}\right)\right)\\ +\left((-A)^{\frac{1}{2}}\left(\frac{1}{\varepsilon}\left(-A\right)^{-1}\left(x_{\varepsilon,\mu}-y_{\varepsilon,\mu}\right)-\mu y_{\varepsilon,\mu}\right)\mid(-A)^{\frac{1}{2}}y_{\varepsilon,\mu}\right)_X\\ +\left((-A)^{\beta}\left(\frac{1}{\varepsilon}\left(-A\right)^{-1}\left(x_{\varepsilon,\mu}-y_{\varepsilon,\mu}\right)+\mu y_{\varepsilon,\mu}\right)\mid F\left(s_{\varepsilon,\mu},\Lambda y_{\varepsilon,\mu}\right)\right)_X\\ \geq\frac{\sigma}{T^2}.\end{aligned}$$

Substracting the previous two inequalities we obtain

$$\begin{aligned}H\left(t_{\varepsilon,\mu},x_{\varepsilon,\mu},(-A)^{\beta}\left(\frac{1}{\varepsilon}(-A)^{-1}\left(x_{\varepsilon,\mu}-y_{\varepsilon,\mu}\right)+\mu x_{\varepsilon,\mu}\right)\right)\\ -H\left(s_{\varepsilon,\mu},y_{\varepsilon,\mu},(-A)^{\beta}\left(\frac{1}{\varepsilon}(-A)^{-1}\left(x_{\varepsilon,\mu}-y_{\varepsilon,\mu}\right)-\mu y_{\varepsilon,\mu}\right)\right)\\ +\left((-A)^{\frac{1}{2}}\left(\frac{1}{\varepsilon}\left(-A\right)^{-1}\left(x_{\varepsilon,\mu}-y_{\varepsilon,\mu}\right)\right)\mid(-A)^{\frac{1}{2}}\left(x_{\varepsilon,\mu}-y_{\varepsilon,\mu}\right)\right)_X\\ +\mu\left|(-A)^{\frac{1}{2}}x_{\varepsilon,\mu}\right|_X^2+\mu\left|(-A)^{\frac{1}{2}}y_{\varepsilon,\mu}\right|_X^2\\ +\left((-A)^{\beta}\left(\frac{1}{\varepsilon}\left(-A\right)^{-1}\left(x_{\varepsilon,\mu}-y_{\varepsilon,\mu}\right)\right)\mid F\left(t_{\varepsilon,\mu},\Lambda x_{\varepsilon,\mu}\right)-F\left(s_{\varepsilon,\mu},\Lambda y_{\varepsilon,\mu}\right)\right)_X\\ +\mu\left((-A)^{\beta}x_{\varepsilon,\mu}\mid F\left(t_{\varepsilon,\mu},\Lambda x_{\varepsilon,\mu}\right)\right)_X-\mu\left((-A)^{\beta}y_{\varepsilon,\mu}\mid F\left(s_{\varepsilon,\mu},\Lambda y_{\varepsilon,\mu}\right)\right)_X\\ \leq-\frac{2\sigma}{T^2}.\end{aligned}$$

Thus we have

$$\mu\left|(-A)^{\frac{1}{2}}x_{\varepsilon,\mu}\right|_X^2+\mu\left|(-A)^{\frac{1}{2}}y_{\varepsilon,\mu}\right|_X^2+\frac{1}{\varepsilon}\left|x_{\varepsilon,\mu}-y_{\varepsilon,\mu}\right|_X^2 \tag{3.16}$$

$$\leq-\frac{2\sigma}{T^2}+K_H\left(\left|t_{\varepsilon,\mu}-s_{\varepsilon,\mu}\right|^{\eta_2}+\left|x_{\varepsilon,\mu}-y_{\varepsilon,\mu}\right|_X+\mu\left|(-A)^{\beta}\left(x_{\varepsilon,\mu}+y_{\varepsilon,\mu}\right)\right|_X\right) \tag{3.17}$$

$$+\frac{1}{\varepsilon}\left|(-A)^{\beta-1}\left(x_{\varepsilon,\mu}-y_{\varepsilon,\mu}\right)\right|_X\left|F\left(t_{\varepsilon,\mu},\Lambda x_{\varepsilon,\mu}\right)-F\left(s_{\varepsilon,\mu},\Lambda y_{\varepsilon,\mu}\right)\right|_X \tag{3.18}$$

$$+\mu\left|(-A)^{\beta}x_{\varepsilon,\mu}\right|_X\left|F\left(t_{\varepsilon,\mu},\Lambda x_{\varepsilon,\mu}\right)\right|_X+\mu\left|(-A)^{\beta}y_{\varepsilon,\mu}\right|_X\left|F\left(s_{\varepsilon,\mu},\Lambda y_{\varepsilon,\mu}\right)\right|_X. \tag{3.19}$$

Estimates of (3.17)–(3.19):

Estimate of (3.17). With (3.11) and Young's inequality we can write

$$\left|t_{\varepsilon,\mu}-s_{\varepsilon,\mu}\right|^{\eta_2}\leq C\varepsilon^{\frac{\eta_2}{\eta}}\leq C\varepsilon\qquad\text{and}\qquad K_H\left|x_{\varepsilon,\mu}-y_{\varepsilon,\mu}\right|_X\leq 2\varepsilon K_H^2+\frac{\left|x_{\varepsilon,\mu}-y_{\varepsilon,\mu}\right|_X^2}{8\varepsilon}. \tag{3.20}$$

For all $\sigma > 0$, with (2.13), we have:

$$\begin{aligned}\mu K_H \left|(-A)^{\beta}\left(x_{\varepsilon,\mu}+y_{\varepsilon,\mu}\right)\right|_X &\le \mu K_H\left(\left|(-A)^{\beta}x_{\varepsilon,\mu}\right|_X+\left|(-A)^{\beta}y_{\varepsilon,\mu}\right|_X\right)\\ &\le \mu K_H\left[\sigma\left(\left|(-A)^{\frac12}x_{\varepsilon,\mu}\right|_X+\left|(-A)^{\frac12}y_{\varepsilon,\mu}\right|_X\right)+C_{\beta,\sigma}\left(|x_{\varepsilon,\mu}|_X+|y_{\varepsilon,\mu}|_X\right)\right]\\ &\le \mu K_H\left[\frac12\sigma\left(\left|(-A)^{\frac12}x_{\varepsilon,\mu}\right|_X^2+\left|(-A)^{\frac12}y_{\varepsilon,\mu}\right|_X^2+2\right)+C_{\beta,\sigma}\left(|x_{\varepsilon,\mu}|_X+|y_{\varepsilon,\mu}|_X\right)\right].\end{aligned}$$

Choosing $\sigma=\frac{1}{K_H}$ we have:

$$\mu K_H \left|(-A)^{\beta}\left(x_{\varepsilon,\mu}+y_{\varepsilon,\mu}\right)\right|_X \le \mu\left[\frac12\left|(-A)^{\frac12}x_{\varepsilon,\mu}\right|_X^2+\frac12\left|(-A)^{\frac12}y_{\varepsilon,\mu}\right|_X^2+1+K_HC_{\beta,1/K_H}\left(|x_{\varepsilon,\mu}|_X+|y_{\varepsilon,\mu}|_X\right)\right]. \tag{3.21}$$

With (3.20) and (3.21), we have:

$$\begin{aligned}(3.17) \le\ & K_HC\varepsilon+2\varepsilon K_H^2+\frac{|x_{\varepsilon,\mu}-y_{\varepsilon,\mu}|_X^2}{8\varepsilon}\\ &+\mu\left[\frac12\left|(-A)^{\frac12}x_{\varepsilon,\mu}\right|_X^2+\frac12\left|(-A)^{\frac12}y_{\varepsilon,\mu}\right|_X^2+1+K_HC_{\beta,1/K_H}\left(|x_{\varepsilon,\mu}|_X+|y_{\varepsilon,\mu}|_X\right)\right].\end{aligned}$$

Estimate of (3.18). We first write

$$(3.18) \le \frac{1}{\varepsilon}\left\|(-A)^{\beta-\frac12}\right\|\left|(-A)^{-\frac12}\left(x_{\varepsilon,\mu}-y_{\varepsilon,\mu}\right)\right|_X\left|F\left(t_{\varepsilon,\mu},\Lambda x_{\varepsilon,\mu}\right)-F\left(s_{\varepsilon,\mu},\Lambda y_{\varepsilon,\mu}\right)\right|_X. \tag{3.22}$$

From (2.2) and (2.4), it yields

$$\begin{aligned}\left|F\left(t_{\varepsilon,\mu},\Lambda x_{\varepsilon,\mu}\right)-F\left(s_{\varepsilon,\mu},\Lambda y_{\varepsilon,\mu}\right)\right|_X &\le M_{1,F}\left(1+|\Lambda x_{\varepsilon,\mu}|_{X_0}\right)|t_{\varepsilon,\mu}-s_{\varepsilon,\mu}|^{\eta_1}+K_F\left|\Lambda x_{\varepsilon,\mu}-\Lambda y_{\varepsilon,\mu}\right|_{X_0}\\ &\le M_{1,F}\left(1+C_{1/2}\left|(-A)^{\frac12}x_{\varepsilon,\mu}\right|_X\right)|t_{\varepsilon,\mu}-s_{\varepsilon,\mu}|^{\eta_1}\\ &\quad+K_FC_\alpha\left|(-A)^{\alpha}\left(x_{\varepsilon,\mu}-y_{\varepsilon,\mu}\right)\right|_X.\end{aligned}$$

Then, choosing $\alpha_0>0$ such that $\alpha+\alpha_0<\frac12$, we obtain

$$(3.18) \le C\frac{\left|(-A)^{-\frac12}\left(x_{\varepsilon,\mu}-y_{\varepsilon,\mu}\right)\right|_X}{\varepsilon^{1-\alpha_0}}\frac{\left|(-A)^{\alpha}\left(x_{\varepsilon,\mu}-y_{\varepsilon,\mu}\right)\right|_X}{\varepsilon^{\alpha_0}} \tag{3.23}$$

$$+C\frac{\left|(-A)^{-\frac12}\left(x_{\varepsilon,\mu}-y_{\varepsilon,\mu}\right)\right|_X}{\varepsilon^{1-\alpha_0}}\left(1+\left|(-A)^{\frac12}x_{\varepsilon,\mu}\right|_X\right)\frac{|t_{\varepsilon,\mu}-s_{\varepsilon,\mu}|^{\eta_1}}{\varepsilon^{\alpha_0}}. \tag{3.24}$$

Estimate of (3.23). We first estimate the factor $\left|(-A)^{-\frac12}\left(x_{\varepsilon,\mu}-y_{\varepsilon,\mu}\right)\right|_X/\varepsilon^{1-\alpha_0}$. From inequality (3.10) it follows that

$$\frac{1}{\varepsilon}\left|(-A)^{-\frac12}\left(x_{\varepsilon,\mu}-y_{\varepsilon,\mu}\right)\right|_X^2 \le w\left(t_{\varepsilon,\mu},x_{\varepsilon,\mu}\right)-w\left(s_{\varepsilon,\mu},y_{\varepsilon,\mu}\right)+v\left(t_{\varepsilon,\mu},x_{\varepsilon,\mu}\right)-v\left(s_{\varepsilon,\mu},y_{\varepsilon,\mu}\right).$$

As $t_{\varepsilon,\mu}$, $s_{\varepsilon,\mu}\in[\tau,T-\tau]$ and $[\tau,T-\tau]$ is compact, with properties (**iii**) and (**iv**) in Definition 3.2, setting $C(v,w)=M_{1,v}+M_{1,w}$ and $C(\tau)=\sup_{t\in[\tau,T]}C_{t,1/2,v}+C_{t,1/2,w}$, we have

$$\frac{1}{\varepsilon}\left|(-A)^{-\frac12}\left(x_{\varepsilon,\mu}-y_{\varepsilon,\mu}\right)\right|_X^2 \le 2C(v,w)\left(1+\left|(-A)^{\frac12}x_{\varepsilon,\mu}\right|_X\right)|t_{\varepsilon,\mu}-s_{\varepsilon,\mu}|^{\eta}+C(\tau)\left|(-A)^{-\frac12}\left(x_{\varepsilon,\mu}-y_{\varepsilon,\mu}\right)\right|_X.$$

As

$$C(\tau)\left|(-A)^{-\frac{1}{2}}(x_{\varepsilon,\mu}-y_{\varepsilon,\mu})\right|_X \leq \frac{\left|(-A)^{-\frac{1}{2}}(x_{\varepsilon,\mu}-y_{\varepsilon,\mu})\right|_X^2}{2\varepsilon}+\frac{\varepsilon}{2}C(\tau)^2,$$

then

$$\frac{1}{\varepsilon}\left|(-A)^{-\frac{1}{2}}(x_{\varepsilon,\mu}-y_{\varepsilon,\mu})\right|_X^2 \leq 4C(v,w)\left(1+\left|(-A)^{\frac{1}{2}}x_{\varepsilon,\mu}\right|_X\right)|t_{\varepsilon,\mu}-s_{\varepsilon,\mu}|^{\eta}+\varepsilon C(\tau)^2.$$

Hence, we obtain:

$$\frac{1}{\varepsilon^{1-\alpha_0}}\left|(-A)^{-\frac{1}{2}}(x_{\varepsilon,\mu}-y_{\varepsilon,\mu})\right|_X \leq 2\sqrt{C(v,w)\left(1+\left|(-A)^{\frac{1}{2}}x_{\varepsilon,\mu}\right|_X\right)}\frac{|t_{\varepsilon,\mu}-s_{\varepsilon,\mu}|^{\frac{\eta}{2}}}{\varepsilon^{\frac{1}{2}-\alpha_0}}+\varepsilon^{\alpha_0}C(\tau). \tag{3.25}$$

We now estimate the factor $|(-A)^{\alpha}(x_{\varepsilon,\mu}-y_{\varepsilon,\mu})|_X/\varepsilon^{\alpha_0}$. With (2.12), we have:

$$\frac{1}{\varepsilon^{\alpha_0}}|(-A)^{\alpha}(x_{\varepsilon,\mu}-y_{\varepsilon,\mu})|_X \leq M_{\alpha,1/2}\left|(-A)^{\frac{1}{2}}(x_{\varepsilon,\mu}-y_{\varepsilon,\mu})\right|_X^{2\alpha}\frac{|x_{\varepsilon,\mu}-y_{\varepsilon,\mu}|_X^{1-2\alpha}}{\varepsilon^{\frac{1}{2}-\alpha_0}}.$$

Applying Young's inequality (3.5) to the left hand side and taking $q=1/(1-2\alpha)$, $p=1/2\alpha$, $q\lambda^q=\varepsilon^{\frac{1}{2}-\frac{\alpha_0}{1-2\alpha}}$, then $\frac{\lambda^p}{p}=C\varepsilon^{(1-2\alpha-2\alpha_0)/4\alpha}$ and we obtain:

$$\frac{1}{\varepsilon^{\alpha_0}}|(-A)^{\alpha}(x_{\varepsilon,\mu}-y_{\varepsilon,\mu})|_X \leq \frac{|x_{\varepsilon,\mu}-y_{\varepsilon,\mu}|_X}{\varepsilon^{1/2}}+C\varepsilon^{(1-2\alpha-2\alpha_0)/4\alpha}\left|(-A)^{\frac{1}{2}}(x_{\varepsilon,\mu}-y_{\varepsilon,\mu})\right|_X. \tag{3.26}$$

The exponent of $(1-2\alpha-2\alpha_0)/4\alpha$ of ε is positive because $\alpha+\alpha_0<\frac{1}{2}$.

With (3.25), (3.26), and with Young's inequality we can write

$$(3.23) \leq \left[C\left(1+\left|(-A)^{\frac{1}{2}}x_{\varepsilon,\mu}\right|_X\right)\frac{|t_{\varepsilon,\mu}-s_{\varepsilon,\mu}|^{\eta}}{\varepsilon^{1-2\alpha_0}}+\varepsilon^{2\alpha_0}C(\tau)\right] + \frac{|x_{\varepsilon,\mu}-y_{\varepsilon,\mu}|_X^2}{8\varepsilon}+C\varepsilon^{\frac{(1-2\alpha-2\alpha_0)}{2\alpha}}\left|(-A)^{\frac{1}{2}}(x_{\varepsilon,\mu}-y_{\varepsilon,\mu})\right|_X^2.$$

With (3.11) one has

$$\frac{|t_{\varepsilon,\mu}-s_{\varepsilon,\mu}|^{\eta}}{\varepsilon^{1-2\alpha_0}} \leq C\frac{\varepsilon^{\frac{\eta}{\eta}}}{\varepsilon^{1-2\alpha_0}} \leq C\varepsilon^{2\alpha_0},$$

which gives

$$(3.23) \leq C\left(\varepsilon^{2\alpha_0}+\varepsilon^{\frac{(1-2\alpha-2\alpha_0)}{2\alpha}}\right)\left(1+\left|(-A)^{\frac{1}{2}}x_{\varepsilon,\mu}\right|_X^2+\left|(-A)^{\frac{1}{2}}y_{\varepsilon,\mu}\right|_X^2\right)+\varepsilon^{2\alpha_0}C(\tau)+\frac{|x_{\varepsilon,\mu}-y_{\varepsilon,\mu}|_X^2}{8\varepsilon}. \tag{3.27}$$

Estimate of (3.24). We finally estimate (3.24) with (3.25), (3.26) and (3.11), and we have

$$\begin{aligned}(3.24) &\leq C(\tau)\left(1+\left|(-A)^{\frac{1}{2}}x_{\varepsilon,\mu}\right|_X\right)^{3/2}\left(\frac{|t_{\varepsilon,\mu}-s_{\varepsilon,\mu}|^{\frac{\eta}{2}+\eta_1}}{\varepsilon^{\frac{1}{2}}}+|t_{\varepsilon,\mu}-s_{\varepsilon,\mu}|^{\eta_1}\right)\\ &\leq C(\tau)\varepsilon\left(1+\left|(-A)^{\frac{1}{2}}x_{\varepsilon,\mu}\right|_X^2\right).\end{aligned} \tag{3.28}$$

We complete the estimate of (3.18) with (3.27) and (3.28):

$$(3.18) \le C\left(\varepsilon + \varepsilon^{2\alpha_0} + \varepsilon^{(1-2\alpha-2\alpha_0)/2\alpha}\right)\left(1 + \left|(-A)^{\frac12} x_{\varepsilon,\mu}\right|_X^2 + \left|(-A)^{\frac12} y_{\varepsilon,\mu}\right|_X^2\right) + C(\tau)\varepsilon^{2\alpha_0} + \frac{|x_{\varepsilon,\mu} - y_{\varepsilon,\mu}|_X^2}{8\varepsilon}. \tag{3.29}$$

Estimate of (3.19). Since F is bounded, as in (3.17), and with (3.21) we obtain

$$(3.19) \le \mu\left[\frac18\left|(-A)^{\frac12} x_{\varepsilon,\mu}\right|_X^2 + \frac12 + \frac18\left|(-A)^{\frac12} y_{\varepsilon,\mu}\right|_X^2 + \left(|x_{\varepsilon,\mu}|_X^2 + |y_{\varepsilon,\mu}|_X^2\right) + C\right].$$

End of step 4

Collecting the different estimates of the terms in the right hand side of (3.16) we obtain:

$$\left(\mu - \tfrac{\mu}{2} - C\left(\varepsilon^{2\alpha_0} + \varepsilon^{(1-2\alpha-2\alpha_0)/2\alpha}\right) - \tfrac{\mu}{8}\right)\left[\left|(-A)^{\frac12} x_{\varepsilon,\mu}\right|_X^2 + \left|(-A)^{\frac12} y_{\varepsilon,\mu}\right|_X^2\right] + \tfrac{1}{\varepsilon}\left(1 - \tfrac18 - \tfrac14 - \tfrac18\right)|x_{\varepsilon,\mu} - y_{\varepsilon,\mu}|_X^2 \le -\tfrac{2\sigma}{T^2} + \mu C\left(1 + |x_{\varepsilon,\mu}|_X^2 + |y_{\varepsilon,\mu}|_X^2\right) + C(\tau)\left[\varepsilon^{\frac{1}{\bar\eta}} + \varepsilon^{2\alpha_0}\right]. \tag{3.30}$$

We take $\varepsilon(\mu)$ small enough to have $C\left(\varepsilon^{2\alpha_0} + \varepsilon^{(1-2\alpha-2\alpha_0)/2\alpha}\right) < \frac{3\mu}{8}$, and we take the limit when μ tends to zero. We obtain the contradiction $0 \le -\frac{2\sigma}{T^2}$. Thus the equality $\min(t_{\varepsilon,\mu}, s_{\varepsilon,\mu}) = \tau$ is established.

Step 5. We are going to conclude with the initial data. We argue by contradiction. If (3.6) does not hold, then there exists $(t_0, x_0) \in\,]0, T[\, \times X$ such that

$$0 < w_\sigma(t_0, x_0) - v_\sigma(t_0, x_0) = \delta_0. \tag{3.31}$$

We choose τ and ε small enough to have $(t_0, x_0) \in Q_\tau$, and

$$\left[w(t,x) - g\left(\mathrm{e}^{tA}x\right)\right]^+ \le \frac{\delta_0}{8} \qquad \text{and} \qquad \left[v(t,x) - g\left(\mathrm{e}^{tA}x\right)\right]^- \le \frac{\delta_0}{8},$$

for all $x \in X$ and all $t \in [0, \tau + C\varepsilon^{1/\bar\eta}]$, where C is the constant in (3.11). One has

$$\Phi_{\varepsilon,\mu}(t_0, t_0, x_0, x_0) \le \Phi_{\varepsilon,\mu}(t_{\varepsilon,\mu}, s_{\varepsilon,\mu}, x_{\varepsilon,\mu}, y_{\varepsilon,\mu}) \le w_\sigma(t_{\varepsilon,\mu}, x_{\varepsilon,\mu}) - v_\sigma(s_{\varepsilon,\mu}, y_{\varepsilon,\mu}),$$

i.e.

$$\begin{aligned} w_\sigma(t_0, x_0) - v_\sigma(t_0, x_0) - \mu|x_0|_X^2 &\le w_\sigma(t_{\varepsilon,\mu}, x_{\varepsilon,\mu}) - v_\sigma(s_{\varepsilon,\mu}, y_{\varepsilon,\mu}) \\ &= \left[w_\sigma(t_{\varepsilon,\mu}, x_{\varepsilon,\mu}) + \frac{\sigma}{T - t_{\varepsilon,\mu}} - g\left(\mathrm{e}^{t_{\varepsilon,\mu}A}x_{\varepsilon,\mu}\right)\right] + g\left(\mathrm{e}^{t_{\varepsilon,\mu}A}x_{\varepsilon,\mu}\right) - g\left(\mathrm{e}^{s_{\varepsilon,\mu}A}y_{\varepsilon,\mu}\right) \\ &\quad + \left[g\left(\mathrm{e}^{s_{\varepsilon,\mu}A}y_{\varepsilon,\mu}\right) + \frac{\sigma}{T - s_{\varepsilon,\mu}} - v_\sigma(s_{\varepsilon,\mu}, y_{\varepsilon,\mu})\right] - \left[\frac{\sigma}{T - s_{\varepsilon,\mu}} + \frac{\sigma}{T - t_{\varepsilon,\mu}}\right] \\ &\le \left[w(t_{\varepsilon,\mu}, x_{\varepsilon,\mu}) - g\left(\mathrm{e}^{t_{\varepsilon,\mu}A}x_{\varepsilon,\mu}\right)\right]^+ + K_g\left|\mathrm{e}^{t_{\varepsilon,\mu}A}x_{\varepsilon,\mu} - \mathrm{e}^{s_{\varepsilon,\mu}A}y_{\varepsilon,\mu}\right| \\ &\quad + \left[v(s_{\varepsilon,\mu}, y_{\varepsilon,\mu}) - g\left(\mathrm{e}^{s_{\varepsilon,\mu}A}y_{\varepsilon,\mu}\right)\right]^-. \end{aligned} \tag{3.32}$$

Since $0 < \tau = \min(t_{\varepsilon,\mu}, s_{\varepsilon,\mu})$ and $|t_{\varepsilon,\mu} - s_{\varepsilon,\mu}| \le C\varepsilon^{1/\bar\eta}$, we have

$$\left[w(t_{\varepsilon,\mu}, x_{\varepsilon,\mu}) - g\left(\mathrm{e}^{t_{\varepsilon,\mu}A}x_{\varepsilon,\mu}\right)\right]^+ \le \frac{\delta_0}{8} \qquad \text{and} \qquad \left[v(s_{\varepsilon,\mu}, y_{\varepsilon,\mu}) - g\left(\mathrm{e}^{s_{\varepsilon,\mu}A}y_{\varepsilon,\mu}\right)\right]^- \le \frac{\delta_0}{8}.$$

Still using $\tau = \min(t_{\varepsilon,\mu}, s_{\varepsilon,\mu})$, with (2.11), we can write

$$\begin{aligned}
\left|\mathrm{e}^{t_{\varepsilon,\mu}A}x_{\varepsilon,\mu} - \mathrm{e}^{s_{\varepsilon,\mu}A}y_{\varepsilon,\mu}\right|_X &\leq \left|\left(\mathrm{e}^{t_{\varepsilon,\mu}A} - \mathrm{e}^{s_{\varepsilon,\mu}A}\right)x_{\varepsilon,\mu}\right|_X + \left|\mathrm{e}^{s_{\varepsilon,\mu}A}\left(x_{\varepsilon,\mu} - y_{\varepsilon,\mu}\right)\right|_X \\
&\leq \left|\left(\mathrm{e}^{|t_{\varepsilon,\mu}-s_{\varepsilon,\mu}|A} - I\right)\mathrm{e}^{\tau A}x_{\varepsilon,\mu}\right|_X + \left|(-A)^{\frac{1}{2}}\mathrm{e}^{s_{\varepsilon,\mu}A}(-A)^{-\frac{1}{2}}\left(x_{\varepsilon,\mu} - y_{\varepsilon,\mu}\right)\right|_X \\
&\leq M_0\left|t_{\varepsilon,\mu} - s_{\varepsilon,\mu}\right|\left|A\mathrm{e}^{\tau A}x_{\varepsilon,\mu}\right|_X + \frac{C}{\sqrt{\tau}}\left|(-A)^{-\frac{1}{2}}\left(x_{\varepsilon,\mu} - y_{\varepsilon,\mu}\right)\right|_X \\
&\leq \frac{C}{\tau}\left|x_{\varepsilon,\mu}\right|_X\left|t_{\varepsilon,\mu} - s_{\varepsilon,\mu}\right| + \frac{C}{\sqrt{\tau}}\left|(-A)^{-\frac{1}{2}}\left(x_{\varepsilon,\mu} - y_{\varepsilon,\mu}\right)\right|_X \\
&\leq \frac{C}{\tau}|x_{\varepsilon,\mu}|_X\,\varepsilon^{\frac{1}{\bar{\eta}}} + \frac{C}{\sqrt{\tau}}\sqrt{\varepsilon} \leq \frac{C}{\tau}\frac{\varepsilon^{\frac{1}{\bar{\eta}}}}{\mu^{1/2}} + \frac{C}{\sqrt{\tau}}\sqrt{\varepsilon}.
\end{aligned}$$

(The last inequality is obtained with (3.12).) We choose $\mu \leq \frac{\delta_0}{4K_g}$, and next ε such that

$$\frac{C}{\tau}\frac{\varepsilon^{\frac{1}{\bar{\eta}}}}{\mu^{1/2}} + \frac{C}{\sqrt{\tau}}\sqrt{\varepsilon} \leq \mu\,.$$

We obtain:

$$\left|\mathrm{e}^{t_{\varepsilon,\mu}A}x_{\varepsilon,\mu} - \mathrm{e}^{s_{\varepsilon,\mu}A}y_{\varepsilon,\mu}\right|_X \leq \mu \leq \frac{\delta_0}{4K_g}\cdot$$

Then with (3.32), (3.31) and this inequality we have:

$$\delta_0 - \mu\left|x_0\right|_X^2 = w_\sigma\left(t_0, x_0\right) - v_\sigma\left(t_0, x_0\right) - \mu\left|x_0\right|_X^2 \leq \frac{\delta_0}{8} + K_g\left(\frac{\delta_0}{4K_g}\right) + \frac{\delta_0}{8} = \frac{\delta_0}{2}\cdot$$

By passing to the limit when μ tends to zero, we obtain a contradiction. Therefore we have proved that

$$w\left(t,x\right) \leq v\left(t,x\right) \quad \text{for all } \left(t,x\right) \in Q_\tau. \qquad \square$$

4. Properties of the value function and existence results

For all $t \in [0,T]$ and $x \in X$, we consider the optimal control problem

$$(\mathcal{P}_{t,x}) \qquad \min\Big\{J\left(t,y,u\right) \mid u \in \mathcal{M}(t,T;U) \text{ and } (y,u) \text{ is solution of equation (2.1)}\Big\},$$

where the cost functional J is defined by

$$J(t,y,u) = \int_t^T L(r,y(r),u(r))\,\mathrm{d}r + g(y(T)).$$

We assume that assumptions (**i**)–(**viii**) of Section 2 are satisfied. Let $v(t,x)$ be the value function of problem $(\mathcal{P}_{t,x})$, that is

$$v(t,x) = \inf_{u\in\mathcal{M}(t,T;U)} J(t,y_{t,x,u},u).$$

In the following it will be convenient to use the notation

$$I_{t,x}(u) = J(t,y_{t,x,u},u).$$

4.1. **Properties of the value function**

Proposition 4.1. *For all x, $x_0 \in X$, and all $t \in [0,T]$, the value function v satisfies:*

$$|v(t,x) - v(t,x_0)| \leq K_v |x - x_0|_X, \tag{4.1}$$

with K_v independent of t.

Proof. With estimate (2.38), we have

$$\begin{aligned}
v(t,x) - v(t,x_0) &= \inf_{u \in \mathcal{M}(t,T;U)} I_{t,x}(u) - \inf_{u \in \mathcal{M}(t,T;U)} I_{t,x_0}(u) \\
&\leq \sup_{u \in \mathcal{M}(t,T;U)} \left(I_{t,x}(u) - I_{t,x_0}(u) \right) \\
&\leq \sup_{u \in \mathcal{M}(t,T;U)} \left\{ \int_t^T |L(\cdot, y_{t,x,u}, u) - L(\cdot, y_{t,x_0,u}, u)| \, \mathrm{d}r + |g(y_{t,x,u}(T)) - g(y_{t,x_0,u}(T))| \right\} \\
&\leq \sup_{u \in \mathcal{M}(t,T;U)} \left\{ \int_t^T K_L |y_{t,x,u}(r) - y_{t,x_0,u}(r)|_X \, \mathrm{d}r + K_g |y_{t,x,u}(T) - y_{t,x_0,u}(T)|_X \right\} \qquad (4.2) \\
&\leq \sup_{u \in \mathcal{M}(t,T;U)} (K_L T + K_g) \|y_{t,x,u} - y_{t,x_0,u}\|_{L^\infty(0,T;X)} \\
&\leq (K_L T + K_g) \, C_8(\alpha,\beta) |x - x_0|_X. \qquad (4.3)
\end{aligned}$$

By permuting x and x_0, we obtain estimate (4.1) with $K_v = (K_L T + K_g) C_8(\alpha,\beta)$. □

Proposition 4.2. *The value function v is continuous and bounded in $[0,T] \times X$.*

Proof. Let us show that v is bounded. As L and g are bounded, we have:

$$v(t,x) = \inf_{u \in \mathcal{M}(t,T;U)} I_{t,x}(u) \leq M_g + T M_L = M_v.$$

Moreover,

$$v(t,x) \geq - \sup_{u \in \mathcal{M}(t,T;U)} |I_{t,x}(u(\cdot))| \geq -(M_g + T M_L) = -M_v.$$

Hence we have

$$|v(t,x)| \leq M_v \quad \text{for all } (t,x) \in [0,T] \times X.$$

Let $(t,x) \in [0,T] \times X$ be fixed, and first show that the function $t \mapsto v(t,x)$ is continuous. Let $0 \leq s < t$, we have:

$$\begin{aligned}
v(t,x) - v(s,x) &\leq \sup_{u \in \mathcal{M}(t,T;U)} \left(\int_t^T |L(\cdot, y_{t,x,u}, u) - L(\cdot, y_{s,x,u}, u)| \, \mathrm{d}r + |g(y_{t,x,u}(T)) - g(y_{s,x,u}(T))| \right) \\
&\quad + \sup_{u \in \mathcal{M}(t,T;U)} \int_s^t |L(\cdot, y_{s,x,u}, u)| \, \mathrm{d}r \\
&\leq \sup_{u \in \mathcal{M}(t,T;U)} (K_L T + K_g) \|y_{t,x,u} - y_{s,x,u}\|_{L^\infty(t,T;X)} + |t - s| M_L.
\end{aligned}$$

By permuting s and t, and with Proposition 2.9 we have

$$|v(t,x) - v(s,x)| \leq C_6(\alpha,\beta) (K_L T + K_g) \bar{a}(t,s,x) + |t - s| M_L. \tag{4.4}$$

Hence with (4.1) we obtain

$$|v(t,x)-v(s,x_0)| \le [C_6(\alpha,\beta)(K_L T+K_g)\,\bar{a}(t,s,x)+K_v\,|x-x_0|_X]+|t-s|\,M_L.$$

The proof is complete. □

Proposition 4.3. *For all $t\in[0,T[$, and all x, $x_0\in X$, the value function satisfies*

$$|v(t,x)-v(t,x_0)| \le C(\alpha,\beta,\theta,t)\left|(-A)^{-\theta}(x-x_0)\right|_X \quad \text{for all } \theta\in[0,1-\alpha[\,. \tag{4.5}$$

The constant $C(\alpha,\beta,\theta,t)$ is explicitly given in (4.7), it blows up when $t\to T$ and when $\alpha+\theta\to 1$, but it stays bounded on all compact subset of $[0,T[$.

Proof. We have

$$v(t,x)-v(t,x_0) \le \sup_{u\in\mathcal{M}(t,T;U)}\left\{\int_t^T K_L\,|y_{t,x,u}(r)-y_{t,x_0,u}(r)|_X\,\mathrm{d}r+K_g\,|y_{t,x,u}(T)-y_{t,x_0,u}(T)|_X\right\}.$$

With estimate (2.25), we obtain

$$v(t,x)-v(t,x_0) \le \left\{\int_t^T K_L C_4(\alpha,\beta,\theta,t;r)\mathrm{d}r+K_g C_4(\alpha,\beta,\theta,t;T)\right\}\left|(-A)^{-\theta}(x-x_0)\right|_X,$$

where $C_4(\alpha,\beta,\theta,t;r)$ is given in (2.28). As $1-(\alpha+\beta+\theta)>-1$, then function $r\mapsto(r-t)^{1-(\alpha+\beta+\theta)}$ is integrable over (t,T). So, one has

$$\begin{aligned}\int_t^T K_L C_4(\alpha,\beta,\theta,t;r)(r)\,\mathrm{d}r &= K_L\int_t^T\left(\frac{M_\theta}{(r-t)^\theta}+\frac{C_3(\alpha,\beta,\theta)}{1-(\alpha+\theta)}(r-t)^{1-(\alpha+\beta+\theta)}\right)\mathrm{d}r\\ &= K_L\left[\frac{M_\theta}{1-\theta}(T-t)^{1-\theta}+\frac{C_3(\alpha,\beta,\theta)}{[2-(\alpha+\beta+\theta)]\,[1-(\alpha+\theta)]}(T-t)^{2-(\alpha+\beta+\theta)}\right].\end{aligned} \tag{4.6}$$

Setting

$$C(\alpha,\beta,\theta,t)=(4.6)+K_g C_4(\alpha,\beta,\theta,t;T), \tag{4.7}$$

we can write

$$v(t,x)-v(t,x_0)\le C(\alpha,\beta,\theta,t)\left|(-A)^{-\theta}(x-x_0)\right|_X.$$

And permuting x and x_0 we obtain (4.5). □

Proposition 4.4. *For all $x\in D((-A)^{\frac12})$ and all s, $t\in[0,T]$ there exists a constant C, independent of x, t and s, such that*

$$|v(t,x)-v(s,x)|\le C\left(1+\left|(-A)^{\frac12}x\right|_X\right)|t-s|^{\frac12}. \tag{4.8}$$

Proof. Let us recall (4.4):

$$|v(t,x)-v(s,x)|\le C_6(\alpha,\beta)\,(K_L T+K_g)\,\bar{a}(t,s,x)+|t-s|\,M_L,$$

where $\bar{a}(t,s,x)=\left|\mathrm{e}^{|t-s|A}x-x\right|_X+|t-s|^{1-\beta}$. With (2.11), if $x\in D((-A)^{\frac12})$, we get:

$$\left|\left(\mathrm{e}^{|t-s|A}-I\right)x\right|_X\le C\,|t-s|^{\frac12}\left|(-A)^{\frac12}x\right|_X.$$

Since $\beta \leq 1/2$, we have

$$\bar{a}(t,s,x) \leq C(\beta,T)\left(1+\left|(-A)^{\frac{1}{2}}x\right|_X\right)|t-s|^{\frac{1}{2}}.$$

Hence

$$|v(t,x)-v(s,x)| \leq \left[C_6(\alpha,\beta)\left(K_L T+K_g+C(\beta,T)\left(1+\left|(-A)^{\frac{1}{2}}x\right|_X\right)\right)+M_L T^{\frac{1}{2}}\right]|t-s|^{\frac{1}{2}}.$$

The proof is complete. □

4.2. **Existence results**

Theorem 4.5. *Assume that assumptions* **(i)**–**(viii)** *of Section 2 hold. Then the value function v is a viscosity solution of the Hamilton-Jacobi-Bellman equation (1.1) in the sense of Definition 3.2.*

The proof is split into three steps:

Step 1. We show that v satisfies the condition (α_1) in the definition of subsolutions.
Step 2. We show that v satisfies condition (α_2) in the definition of supersolutions.
Step 3. We show that v satisfies both terminal conditions (β_1) and (β_2).

Proof. We only treat the case $0 \leq \beta < \frac{1}{2}$. The case $\beta = \frac{1}{2}$ can be treated with obvious modifications by using estimate (2.20) in place of (2.19).
Step 1. Let $\Phi \in C_A^1(]0,T[\times X)$, and $(t,x) \in \left(]0,T[\times D((-A)^{\frac{1}{2}})\right) \cap \arg\max_{]0,T[\times X}(v-\Phi)$. Let $u(\cdot) = u \in U$ be a constant control. For all $s > t$ we have

$$v(t,x)-\Phi(t,x) = \max(v-\Phi) \geq v(s,y_{t,x,u}(s))-\Phi(s,y_{t,x,u}(s)). \tag{4.9}$$

By the dynamic programming principle it yields

$$v(t,x) \leq \int_t^s L(r,y_{t,x,\tilde{u}}(r),u)\,\mathrm{d}r + v(s,y_{t,x,\tilde{u}}(s)), \quad \text{for all } \tilde{u} \in U.$$

This inequality holds true in particular for u. From (4.9) we deduce

$$\Phi(t,x)-\Phi(s,y_{t,x,u}(s)) \leq v(t,x)-v(s,y_{t,x,u}(s)) \leq \int_t^s L(r,y_{t,x,u}(r),u)\,\mathrm{d}r,$$

hence

$$\Phi(t,x)-\Phi(s,x)+\Phi(s,x)-\Phi(s,y_{t,x,u}(s))-\int_t^s L(r,y_{t,x,u}(r),u)\,\mathrm{d}r \leq 0. \tag{4.10}$$

With assumption **(viii)** of Section 2 satisfied by L and a classical calculation we obtain:

$$\lim_{s\searrow t}\left\{\frac{\Phi(t,x)-\Phi(s,x)}{s-t}-\frac{1}{s-t}\int_t^s L(r,y_{t,x,u}(r),u)\,\mathrm{d}r\right\} = -\frac{\partial\Phi}{\partial t}(t,x)-L(t,x,u). \tag{4.11}$$

On the other hand,

$$\frac{\Phi(s,x)-\Phi(s,y_{t,x,u}(s))}{s-t} = \frac{\Phi(s,x)-\Phi\left(s,\mathrm{e}^{(s-t)A}x\right)}{s-t}+\frac{\Phi\left(s,\mathrm{e}^{(s-t)A}x\right)-\Phi(s,y_{t,x,u}(s))}{s-t}. \tag{4.12}$$

So,

$$\begin{aligned}\frac{\Phi\left(s,\mathrm{e}^{(s-t)A}x\right)-\Phi\left(s,y_{t,x,u}\left(s\right)\right)}{s-t} &= \frac{1}{s-t}\left(D_x\Phi\left(s,\xi\left(s\right)\right)\mid \mathrm{e}^{(s-t)A}x-y_{t,x,u}\left(s\right)\right)_X\\ &= \frac{1}{s-t}\left(D_x\Phi\big(s,\mathrm{e}^{(s-t)A}x\big)\mid \mathrm{e}^{(s-t)A}x-y_{t,x,u}\left(s\right)\right)_X \qquad (4.13)\\ &\quad+\frac{1}{s-t}\left(D_x\Phi\big(s,\xi\left(s\right)\big)-D_x\Phi\big(s,\mathrm{e}^{(s-t)A}x\big)\mid \mathrm{e}^{(s-t)A}x-y_{t,x,u}\left(s\right)\right)_X,\end{aligned}$$

where

$$\xi\left(s\right)=\lambda\left(s\right)\mathrm{e}^{(s-t)A}x+\left(1-\lambda\left(s\right)\right)y_{t,x,u}\left(s\right),$$

for a function $\lambda:[t,T]\to\left]0,1\right[$. We denote by K_Φ the Lipschitz constant of $D_x\Phi\left(t,\cdot\right)$ in X (condition (β) in the definition of $C^1_A(]0,T[\times X)$). With estimate (2.19), we have

$$\begin{aligned}&\left|\frac{1}{s-t}\left(D_x\Phi\big(s,\xi\left(s\right)\big)-D_x\Phi\big(s,\mathrm{e}^{(s-t)A}x\big)\mid \mathrm{e}^{(s-t)A}x-y_{t,x,u}\left(s\right)\right)_X\right|\\ &\qquad\le\frac{1}{s-t}K_\Phi\left|\xi\left(s\right)-\mathrm{e}^{(s-t)A}x\right|_X\left|\mathrm{e}^{(s-t)A}x-y_{t,x,u}\left(s\right)\right|_X\\ &\qquad=\frac{1}{s-t}K_\Phi\left(1-\lambda\left(s\right)\right)\left|\mathrm{e}^{(s-t)A}x-y_{t,x,u}\left(s\right)\right|_X^2\\ &\qquad\le\frac{1}{s-t}K_\Phi\left|\mathrm{e}^{(s-t)A}x-y_{t,x,u}\left(s\right)\right|_X^2\\ &\qquad\le C_1(\beta)K_\Phi\frac{|s-t|^{2(1-\beta)}}{s-t}=C|s-t|^{1-2\beta}\to 0\text{ when }s\searrow t,\text{ as }\beta<1/2, \qquad (4.14)\end{aligned}$$

and C is independent of u (C depends only on M_U). In the case when $\beta=\frac{1}{2}$, using (2.20) we obtain $C_1\left(\beta,|x|_{D((-A)^{\frac{1}{2}})}\right)K_\Phi|s-t|^{2\beta_0}$ in place of $C|s-t|^{1-2\beta}$. For the term (4.13), we have

$$\begin{aligned}&\frac{1}{s-t}\left(D_x\Phi\big(s,\mathrm{e}^{(s-t)A}x\big)\mid \mathrm{e}^{(s-t)A}x-y_{t,x,u}\left(s\right)\right)_X\\ &\qquad=\left((-A)^\beta D_x\Phi\big(s,\mathrm{e}^{(s-t)A}x\big)\mid\frac{-1}{s-t}\int_t^s\mathrm{e}^{(s-r)A}\left[Bu-F\left(r,\Lambda y_{t,x,u}\left(r\right)\right)\right]\mathrm{d}r\right)_X. \qquad (4.15)\end{aligned}$$

We know that $\mathrm{e}^{(s-t)A}x$ tends to x in $D((-A)^{\frac{1}{2}})$ when $s\searrow t$. Thus using condition (δ) in the definition of $C^1_A(]0,T[\times X)$ and Proposition 2.2, we can pass to the limit in (4.15), when $s\searrow t$, and we obtain

$$\lim_{s\searrow t}\frac{\Phi\left(s,\mathrm{e}^{(s-t)A}x\right)-\Phi\left(s,y_{t,x,u}\left(s\right)\right)}{s-t}=-\left(\left(-A\right)^\beta D_x\Phi\left(t,x\right)\mid Bu-F\left(t,\Lambda x\right)\right)_X. \qquad (4.16)$$

Moreover

$$\frac{\Phi\left(s,x\right)-\Phi\left(s,\mathrm{e}^{(s-t)A}x\right)}{s-t}=\left(D_x\Phi\left(s,\eta\left(s\right)\right)\mid\frac{x-\mathrm{e}^{(s-t)A}x}{s-t}\right)_X, \qquad (4.17)$$

where

$$\eta\left(s\right)=\lambda\left(s\right)\mathrm{e}^{(s-t)A}x+\left(1-\lambda\left(s\right)\right)x, \qquad (4.18)$$

for a fonction $\lambda : [t,T] \to]0,1[$. We remark that $\eta(s) \in D((-A)^{\frac{1}{2}})$ as $x \in D((-A)^{\frac{1}{2}})$. Thus $D_x\Phi(s,\eta(s)) \in D((-A)^{\frac{1}{2}})$ due to (γ) in the definition of $C_A^1(]0,T[\times X)$. The right hand side of (4.17) can be written as follows

$$\begin{aligned}\left(D_x\Phi(s,\eta(s)) \mid \frac{-1}{s-t}\int_0^{s-t} Ae^{rA}x\mathrm{d}r\right)_X &= \frac{1}{s-t}\int_0^{s-t}\left((-A)^{\frac{1}{2}} D_x\Phi(s,\eta(s)) \mid (-A)^{\frac{1}{2}} e^{rA}x\right)_X \mathrm{d}r \\ &\to \left((-A)^{\frac{1}{2}} D_x\Phi(t,x) \mid (-A)^{\frac{1}{2}} x\right)_X \quad \text{when } s \searrow t, \end{aligned} \tag{4.19}$$

because the mapping $(s,x) \mapsto D_x\Phi(s,x)$ is continuous from $[0,T]\times D((-A)^{\frac{1}{2}})$ into $D((-A)^{\frac{1}{2}})$. Hence with (4.10), (4.11), (4.12), (4.16) and (4.19), we conclude that

$$\begin{aligned}0 \geq -\frac{\partial\Phi}{\partial t}(t,x) - L(t,x,u) + \left((-A)^{\beta} D_x\Phi(t,x) \mid F(t,\Lambda x)\right)_X \\ - \left((-A)^{\beta} D_x\Phi(t,x) \mid Bu\right)_X + \left((-A)^{\frac{1}{2}} D_x\Phi(t,x) \mid (-A)^{\frac{1}{2}} x\right)_X.\end{aligned}$$

By passing to the supremum with respect to $u \in U$ we have:

$$\begin{aligned}0 \geq -\frac{\partial\Phi}{\partial t}(t,x) + \left((-A)^{\frac{1}{2}} D_x\Phi(t,x) \mid (-A)^{\frac{1}{2}} x\right)_X \\ + \left((-A)^{\beta} D_x\Phi(t,x) \mid F(t,\Lambda x)\right)_X + H\left(t, x(-A)^{\beta} D_x\Phi(t,x)\right).\end{aligned}$$

Thus the condition (α_1) in the definition of subsolutions of equation (1.1) is satisfied.

Step 2. Let Φ be in $C_A^1(]0,T[\times X)$, and $(t,x) \in]0,T[\times D((-A)^{\frac{1}{2}}) \cap \arg\min(v-\Phi)$. For all $u(\cdot) \in \mathcal{M}(0,T;U)$ we have

$$v(t,x) - v(s,y_{t,x,u}(s)) \leq \Phi(t,x) - \Phi(s,y_{t,x,u}(s)). \tag{4.20}$$

Thanks to the dynamic programming principle, for all $\varepsilon > 0$, there exists a control $u_\varepsilon(\cdot)$, $\varepsilon-$optimal, such that

$$\varepsilon(s-t) + v(t,x) \geq \int_t^s L(r, y_{t,x,u_\varepsilon}(r), u_\varepsilon(r))\,\mathrm{d}r + v(s, y_{t,x,u_\varepsilon}(s)). \tag{4.21}$$

Setting $u = u_\varepsilon$ in (4.20), and substracting (4.21), we obtain:

$$\Phi(t,x) - \Phi(s,y_{t,x,u_\varepsilon}(s)) + \varepsilon(s-t) - \int_t^s L(r, y_{t,x,u_\varepsilon}(r), u_\varepsilon(r))\,\mathrm{d}r \geq 0,$$

that is:

$$\frac{\Phi(t,x) - \Phi(s,y_{t,x,u_\varepsilon}(s))}{s-t} + \varepsilon - \frac{1}{s-t}\int_t^s L(r, y_{t,x,u_\varepsilon}(r), u_\varepsilon(r))\,\mathrm{d}r \geq 0. \tag{4.22}$$

There exist a function $\eta_1 : [0,T-t] \mapsto \mathbb{R}$, a function $\lambda_2 : [t,T] \mapsto [0,1]$, and a function $\lambda_3 : [t,T] \mapsto [0,1]$ such that

$$\begin{aligned}\frac{\Phi(t,x) - \Phi(s,y_{t,x,u_\varepsilon}(s))}{s-t} &= \frac{\Phi(t,x) - \Phi(s,x)}{s-t} + \frac{\Phi(s,x) - \Phi\left(s, e^{(s-t)A}x\right)}{s-t} + \frac{\Phi\left(s, e^{(s-t)A}x\right) - \Phi(s,y_{t,x,u_\varepsilon}(s))}{s-t} \\ &= -\frac{\partial\Phi}{\partial t}(t,x) + \eta_1(s-t) + \left(D_x\Phi(s,\eta_2(s)) \mid \frac{x - e^{(s-t)A}x}{s-t}\right)_X \\ &\quad + \frac{1}{s-t}\left(D_x\Phi\big(s, e^{(s-t)A}x\big) \mid e^{(s-t)A}x - y_{t,x,u_\varepsilon}(s)\right)_X \\ &\quad + \frac{1}{s-t}\left(D_x\Phi(s,\eta_3(s)) - D_x\Phi\big(s, e^{(s-t)A}x\big) \mid e^{(s-t)A}x - y_{t,x,u_\varepsilon}(s)\right)_X\end{aligned} \tag{4.23}$$

where

$$\eta_1(s-t) \to 0 \quad \text{when} \quad s \searrow t\,, \qquad \eta_2(s) = \lambda_2(s)\mathrm{e}^{(s-t)A}x + (1-\lambda_2(s))x\,,$$
$$\eta_3(s) = \lambda_3(s)\mathrm{e}^{(s-t)A}x + (1-\lambda_3(s))y_{t,x,u_\varepsilon}(s)\,.$$

As in (4.14) we have

$$\frac{1}{s-t}\left|\Big(D_x\Phi(s,\eta_3(s)) - D_x\Phi\big(s,\mathrm{e}^{(s-t)A}x\big) \mid \mathrm{e}^{(s-t)A}x - y_{t,x,u_\varepsilon}(s)\Big)_X\right| \le C_1(\beta)K_\Phi\,|s-t|^{1-2\beta} \to 0 \text{ when } s \searrow t.$$

In (4.19) we have shown that

$$\left(D_x\Phi(s,\eta_2(s)) \mid \frac{x-\mathrm{e}^{(s-t)A}x}{s-t}\right)_X = \Big((-A)^{\frac12} D_x\Phi(t,x) \mid (-A)^{\frac12} x\Big)_X + \eta_4(s-t),$$

where $\eta_4(s-t) \to 0$ when $s \searrow t$.

Due to the definition of H, we can use the inequality

$$-\Big((-A)^\beta \mathrm{e}^{(s-r)A} D_x\Phi\left(s,\mathrm{e}^{(s-t)A}x\right) \mid Bu_\varepsilon(r)\Big) - L\left(r, y_{t,x,u_\varepsilon}(r), u_\varepsilon(r)\right) \le H\left(r, y_{t,x,u_\varepsilon}(r), (-A)^\beta \mathrm{e}^{(s-r)A} D_x\Phi\left(s,\mathrm{e}^{(s-t)A}x\right)\right).$$

Thus

$$\begin{aligned}
&\frac{1}{s-t}\Big(D_x\Phi\big(s,\mathrm{e}^{(s-t)A}x\big) \mid \mathrm{e}^{(s-t)A}x - y_{t,x,u_\varepsilon}(s)\Big)\\
&\quad= \left(D_x\Phi\big(s,\mathrm{e}^{(s-t)A}x\big) \mid \frac{-1}{s-t}\int_t^s (-A)^\beta \mathrm{e}^{(s-r)A}\left[Bu_\varepsilon(r) - F\left(r,\Lambda y_{t,x,u_\varepsilon}(r)\right)\right]\mathrm{d}r\right)\\
&\quad\le \frac{1}{s-t}\int_t^s \Big[H\left(r, y_{t,x,u_\varepsilon}(r), (-A)^\beta \mathrm{e}^{(s-r)A} D_x\Phi\big(s,\mathrm{e}^{(s-t)A}x\big)\right) + L\left(r, y_{t,x,u_\varepsilon}(r), u_\varepsilon(r)\right)\Big]\mathrm{d}r\\
&\qquad+\frac{1}{s-t}\int_t^s \Big((-A)^\beta \mathrm{e}^{(s-r)A} D_x\Phi\big(s,\mathrm{e}^{(s-t)A}x\big) \mid F\left(r,\Lambda y_{t,x,u_\varepsilon}(r)\right)\Big)\mathrm{d}r.
\end{aligned}$$

By collecting together all the terms in (4.22), we obtain

$$\begin{aligned}
\varepsilon + \frac{1}{s-t}\int_t^s H\left(r, y_{t,x,u_\varepsilon}(r), (-A)^\beta \mathrm{e}^{(s-r)A} D_x\Phi\big(s,\mathrm{e}^{(s-t)A}x\big)\right) &\\
-\frac{\partial\Phi}{\partial t}(t,x) + \eta_1(s-t) + \eta_4(s-t) + \Big((-A)^{\frac12} D_x\Phi(t,x) \mid (-A)^{\frac12} x\Big)_X &\\
+\frac{1}{s-t}\int_t^s \Big((-A)^\beta \mathrm{e}^{(s-r)A} D_x\Phi\big(s,\mathrm{e}^{(s-t)A}x\big) \mid F\left(r,\Lambda y_{t,x,u_\varepsilon}(r)\right)\Big)_X &\\
\ge -C_1(\beta)K_\Phi\,|s-t|^{1-2\beta}\,. &
\end{aligned} \tag{4.24}$$

(If $\beta = \frac12$, we have to replace the last line by $-C_1\left(\beta, |x|_{D((-A)^{\frac12})}\right)K_\Phi\,|s-t|^{1-2\beta_0}$.) With Proposition 2.2 we have

$$\Lambda y_{t,x,u_\varepsilon}(r) \to \Lambda x \text{ uniformly w.r. to } u_\varepsilon \text{ when } r \to t\,.$$

By passing to the limit in the previous inequality when $s \to t$, and after when $\varepsilon \to 0$, we obtain:

$$\begin{aligned}
0 \le -\frac{\partial\Phi}{\partial t}(t,x) + \Big((-A)^{\frac12} D_x\Phi(t,x) \mid (-A)^{\frac12} x\Big)_X &\\
+H\left(t, x(-A)^\beta D_x\Phi(t,x)\right) + \Big((-A)^\beta D_x\Phi(t,x) \mid F(t,\Lambda x)\Big)_X.&
\end{aligned}$$

Step 3. Let us show that

$$\lim_{t\searrow 0}\sup_{x\in X}\left|v\left(T-t,x\right)-g\left(\mathrm{e}^{tA}x\right)\right|=0.$$

From the definition of $v(T-t,x)$, it follows that

$$v\left(T-t,x\right)-g\left(\mathrm{e}^{tA}x\right)\leq\int_{T-t}^{T}L\left(r,y_{T-t,x,u}\left(r\right),u\left(r\right)\right)\mathrm{d}r+K_g\left|y_{T-t,x,u}\left(T\right)-\mathrm{e}^{tA}x\right|_X,$$

for all $u\in\mathcal{M}(T-t,T;U)$. Due to estimate (2.19), there exists a constant C independent of t and u such that

$$\left|y_{T-t,x,u}\left(T\right)-\mathrm{e}^{tA}x\right|\leq Ct^{1-\beta}.$$

Then we have

$$v\left(T-t,x\right)-g\left(\mathrm{e}^{tA}x\right)\leq\int_{T-t}^{T}L\left(r,y_{T-t,x,u}\left(r\right),u\left(r\right)\right)\mathrm{d}r+K_gCt^{1-\beta}$$

for all $u\left(\cdot\right)\in\mathcal{M}(T-t,T;U)$. Since L is bounded we obtain

$$\lim_{t\searrow 0}\sup_{x\in X}\left[v\left(T-t,x\right)-g\left(\mathrm{e}^{tA}x\right)\right]\leq 0. \tag{4.25}$$

For the opposite inequality, we choose a control $u_\delta\left(\cdot\right)\in\mathcal{M}(T-t,T;U)$, δ-optimal, such that

$$v\left(T-t,x\right)+\delta>\int_{T-t}^{T}L\left(r,y_\delta\left(r\right),u_\delta\left(r\right)\right)\mathrm{d}r+g\left(y_\delta\left(T\right)\right),$$

where $y_\delta=y_{T-t,x,u_\delta}$. As L is bounded, for t small enough we have

$$v\left(T-t,x\right)+\delta>-\delta+g\left(y_\delta\left(T\right)\right).$$

We can write

$$g\left(y_\delta\left(T\right)\right)\geq-\left|g\left(y_\delta\left(T\right)\right)-g\left(\mathrm{e}^{tA}x\right)\right|+g\left(\mathrm{e}^{tA}x\right),$$

and

$$v\left(T-t,x\right)-g\left(\mathrm{e}^{tA}x\right)>-2\delta-K_g\left|y_\delta\left(T\right)-\mathrm{e}^{tA}x\right|_X\geq-2\delta-K_gCt^{1-\beta}.$$

Hence

$$\lim_{t\searrow 0}\sup_{x\in X}\left[v\left(T-t,x\right)-g\left(\mathrm{e}^{tA}x\right)\right]\geq-2\delta. \tag{4.26}$$

We can pass to the limit when δ tends to zero, and we have shown that conditions (β_1) and (β_2) of Definition 3.2 are satisfied by v. □

5. Examples of optimal control problems

In this section we study the value function of problems of the form

$(\widehat{\mathcal{P}}_{t,x})$ $$\min\left\{\widehat{J}\left(t,u,y\right)\mid u\in\mathcal{M}(t,T;U)\text{ and }(y,u)\text{ is solution of equation (5.1)}\right\}.$$

where

$$y'=Ay+\left(-A\right)^{\beta}\left[Bu-\widehat{F}\left(\cdot,\Lambda y\right)\right]\quad\text{in }(t,T),\qquad y\left(t\right)=x, \tag{5.1}$$

and

$$\widehat{J}(t,y,u)=\int_t^T\widehat{L}(r,y(r),u(r))\,\mathrm{d}r+\widehat{g}(y(T)).$$

We do not assume that $\widehat{F}$, $\widehat{L}$ and $\widehat{g}$ obeys the assumptions of Section 2. But we consider examples such that, for all $u \in \mathcal{M}(t,T;U)$, and all $x \in Y$ – where Y is a suitably chosen Banach space – equation (5.1) admits a unique solution which satisfies

$$\|y\|_{L^\infty_w(0,T;Y)} \le R(M_0,T) \quad \text{and} \quad \|\Lambda y\|_{L^\infty_w(0,T;Y_0)} \le R(M_0,T) \qquad \text{if } |x|_Y \le M_0,$$

for all $M_0 > 0$, where $R(M_0,T)$ is a function of M_0 and T, Y_0 is a subspace of X_0 such that Λ is continuous from $D((-A)^\alpha) \cap Y$ into Y_0. We denote by $\widehat{v}(t,x)$ the value function of problem $(\widehat{\mathcal{P}}_{t,x})$. We introduce a projection operator P_{M_0} from X on the ball $B_Y(R(M_0,T))$ in Y, centered at the origin and with radius $R(M_0,T)$, and a projection operator $P^0_{M_0}$ from X_0 on the ball $B_{Y_0}(R(M_0,T))$ in Y_0, centered at the origin and with radius $R(M_0,T)$. We set

$$F_{M_0}(\cdot,\Lambda y) = \widehat{F}(\cdot,P^0_{M_0}\Lambda y), \quad L_{M_0}(r,y,u) = \widehat{L}(r,P_{M_0}y,u), \quad \text{and} \quad g_{M_0}(y) = \widehat{g}(P_{M_0}y).$$

In the different examples we verify that F_{M_0}, L_{M_0}, and g_{M_0} obeys the assumptions of Section 2 (with constants depending on M_0). We denote by $(\mathcal{P}^{M_0}_{t,x})$ the problem $(\mathcal{P}_{t,x})$ of Section 1, corresponding to F_{M_0}, L_{M_0}, and g_{M_0}, and by $v_{M_0}(t,x)$ its value function. We verify that

$$\widehat{v}(t,x) = v_{M_0}(t,x) \quad \text{for all } x \in Y \quad \text{such that } |x|_Y \le M_0.$$

Due to Theorems 3.5 and 4.5, we know that v_{M_0} is the unique viscosity solution of equation (1.1) corresponding to F_{M_0}, L_{M_0}, and g_{M_0}. Due to the definition of problem $(\mathcal{P}^{M_0}_{t,x})$, it is obvious that

$$v_{M_1}(t,x) = v_{M_0}(t,x) \quad \text{for all } x \in Y \quad \text{such that } |x|_Y \le M_0,$$

if $M_1 \ge M_0$. Thus, to characterize $\widehat{v}(t,x)$ when $x \in Y$, it is enough to characterize $v_{M_0}(t,x)$ for all M_0. And $v_{M_0}(t,x)$ is characterized as the unique viscosity solution of equation (1.1) corresponding to F_{M_0}, L_{M_0}, and g_{M_0}.

In the following Ω is a bounded open subset in $\mathbb{R}^N$, with a regular boundary Γ, and we set $\Omega_{t,T} =]t,T[\times\Omega$ and $\Sigma_{t,T} =]t,T[\times\Gamma$.

5.1. **State equation of example 1**

Consider the equation

$$\frac{\partial y}{\partial t} - \Delta y + y = f \quad \text{in } \Omega_{t,T}, \quad \frac{\partial y}{\partial n} + \widehat{h}(y) = u \quad \text{on } \Sigma_{t,T}, \quad y(t) = x_0 \quad \text{in } \Omega, \tag{5.2}$$

where $\widehat{h}$ is a regular nondecreasing function satisfying $h(y) = 0$ (*e.g.* the well known 'Stefan-Boltzmann radiation condition' corresponds to $\widehat{h}(y) = k_r |y|^3 y + k_c y$, k_r is the radiation coefficient and k_c the convection coefficient [23]). We make the following assumptions.

$(\mathcal{A}_1)$ U is a closed bounded convex and nonempty subset in $L^q(\Gamma)$ for some $q \ge 2$, and it obeys the condition

$$q > N-1, \qquad |u|_{L^2(\Gamma)} \le M_U \quad \text{and} \quad |u|_{L^q(\Gamma)} \le M_U \quad \text{for all } u \in U.$$

$(\mathcal{A}_2)$ The function f belongs to $C^{0,\eta_1}([0,T];L^p(\Omega))$ for some $p \ge 2$, and it obeys the condition

$$p \ge N/2, \quad \|f\|_{C([0,T];L^p(\Omega))} \le M_f, \quad \text{and} \quad \|f\|_{C([0,T];L^2(\Omega))} \le M_f.$$

Let be $X = L^2(\Omega)$, $X_\Gamma = L^2(\Gamma)$, $X_0 = L^2(\Gamma)$, and let us define the unbounded operator A in X by

$$D(A) = \left\{ x \in H^2(\Omega) \mid \frac{\partial x}{\partial n} = 0 \ \text{ on } \ \Gamma \right\} \qquad \text{and} \qquad Ax = \Delta x - x \ \text{ for all } \ x \in D(A).$$

Assumption (**i**) of Section 2 is clearly satisfied. We define the Neumann operator $N \in \mathcal{L}(X_\Gamma; X)$ by $Nu = z$, where z is the solution of the boundary value problem

$$-\Delta z + z = 0 \quad \text{in } \Omega, \qquad \frac{\partial z}{\partial n} = u \quad \text{on } \Gamma.$$

The operator N is also bounded from X_Γ into $H^{3/2}(\Omega)$, and from X_Γ into $D((-A)^\alpha)$ for all $0 < \alpha < \frac{3}{4}$ (see [21]). With the extrapolation method, the semigroup $(\mathrm{e}^{tA})_{t\geq 0}$ can be extended to $(D(A^*))' = (D(A))'$. Denoting by $(\mathrm{e}^{t\widehat{A}})_{t\geq 0}$ the corresponding semigroup, $(\widehat{A}, D(\widehat{A}))$, its infinitesimal generator, is an unbounded operator in $(D(A^*))'$ with domain $D(\widehat{A}) = X$. Thus the operator $(-\widehat{A})N$ is bounded from X_Γ into $(D(A^*))'$. The operator Λ is the trace operator. It satisfies assumption **(iii)** of Section 2 for all $\alpha \in]\frac{1}{4}, \frac{1}{2}[$. It is well known – see *e.g.* [4] – that equation (5.2) can be rewritten in the form

$$y' = \widehat{A}y + f + (-\widehat{A})N\left[u - \widehat{h}(\Lambda y)\right] \qquad \text{in } (t,T), \qquad y(t) = x_0. \tag{5.3}$$

We define the operator B by $B = (-\widehat{A})^{1-\beta}N$ for some β given fixed in $]1/4, 1/2[$. Due to the regularizing properties of N, mentioned above, we can also verify that $B \in \mathcal{L}(X_\Gamma, X)$. We set $\widehat{F}(t,y) = (-A)^{-\beta}f(t) + B\widehat{h}(y)$. Equation (5.3) is nothing else than

$$y' = \widehat{A}y + (-\widehat{A})^\beta\left[Bu - \widehat{F}(\Lambda y)\right] \qquad \text{in } (t,T), \qquad y(t) = x_0. \tag{5.4}$$

As in [24], Theorem 3.1, we can prove the following result.

Theorem 5.1. *For all $x_0 \in L^\infty(\Omega)$ and all $u \in \mathcal{M}(t,T;U)$, equation (5.2) admits a unique weak solution in $C([t,T]; L^2(\Omega))$. This solution belongs to $C_b(]t,T] \times \overline{\Omega})$ and it satisfies the estimate*

$$\|y\|_{L^\infty(\Omega_{t,T})} + \|y\|_{L^\infty(\Sigma_{t,T})} \leq |x_0|_{L^\infty(\Omega)} + C(p,\tilde{p},q,\tilde{q},\Omega,T)\Big(\|f\|_{L^{\tilde{p}}(t,T;L^p(\Omega))} + \|u\|_{L^{\tilde{q}}(t,T;L^q(\Gamma))}\Big), \tag{5.5}$$

where the exponents $1 < \tilde{p} < \infty$ and $1 < \tilde{q} < \infty$ obeys

$$\frac{N}{2p} + \frac{1}{\tilde{p}} < 1 \qquad \textit{and} \qquad \frac{N-1}{2q} + \frac{1}{\tilde{q}} < \frac{1}{2},$$

and the constant $C(p,\tilde{p},q,\tilde{q},\Omega,T)$ depends on N, p, $\tilde{p}$, q, $\tilde{q}$, Ω, T, but is independent of t.

From Theorem 5.1, and assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, we deduce

$$\|y\|_{L^\infty(\Omega_{t,T})} \leq |x_0|_{L^\infty(\Omega)} + C(p,\tilde{p},q,\tilde{q},\Omega,T)\Big((T-t)^{1/\tilde{p}}M_f + (T-t)^{1/\tilde{q}}M_U\Big).$$

5.2. **Setting of the control problem for example 1**

For all $t \in [0,T[$, and all $x_0 \in L^\infty(\Omega)$, we study the following control problem

$$(\widehat{\mathcal{P}}_{t,x_0}) \qquad \min\Big\{\widehat{J}(t,u,y) \mid u \in \mathcal{M}(t,T;U) \text{ and } (y,u) \text{ is solution of equation (5.2)}\Big\},$$

where the cost function $\widehat{J}$ is of the form

$$\widehat{J}(t,u,y)=\int_t^T\int_\Omega \widehat{G}(s,\xi,y(s,\xi))\mathrm{d}\xi\mathrm{d}s+\int_t^T\int_\Gamma \widehat{K}(s,\sigma,u(s,\sigma))\mathrm{d}\sigma\mathrm{d}s+\int_\Omega \widehat{k}(\xi,y(T,\xi))\mathrm{d}\xi. \tag{5.6}$$

We make the following assumptions on the data of problem $(\widehat{\mathcal{P}}_{t,x_0})$.

$(\mathcal{A}_3)$ For all $(t,y)\in[0,T]\times\mathbb{R}$, $\widehat{G}(t,\cdot,y)$ is measurable in Ω. For almost all $\xi\in\Omega$, $\widehat{G}(\cdot,\xi,\cdot)$ is continuous in $[0,T]\times\mathbb{R}$, and we have the estimates:

$$C_{\widehat{G}}\le\widehat{G}(t,\xi,y)\le\widehat{G}_1(\xi)\times\eta(|y|),$$
$$\left|\widehat{G}(t,\xi,y)-\widehat{G}(s,\xi,z)\right|\le\widehat{G}_2(\xi)\big(\eta(|y|)+\eta(|z|)\big)\Big(|t-s|^{\eta_2}+|y-z|\Big),$$

where $\widehat{G}_1\in L^1(\Omega)$, $\widehat{G}_2\in L^2(\Omega)$, and η an increasing function from $\mathbb{R}^+$ in $\mathbb{R}^+$.

$(\mathcal{A}_4)$ For all $(t,u)\in\mathbb{R}^2$, $\widehat{K}(t,\cdot,u)$ is measurable in Γ. For *a.e.* $(t,\sigma)\in\Sigma$, $\widehat{K}(t,\sigma,\cdot)$ is convex. For *a.e.* $\sigma\in\Gamma$, $\widehat{K}(\cdot,\sigma,\cdot)$ is continuous in $\mathbb{R}^2$ and we have the estimates:

$$C_{\widehat{K}}\le\widehat{K}(t,\sigma,u)\le\widehat{K}_1(\sigma)+c_0|u|^q\quad\text{and}\quad\left|\widehat{K}(t,\sigma,u)-\widehat{K}(s,\sigma,u)\right|\le\left(\widehat{K}_1(\sigma)+c_0|u|^q\right)|t-s|^{\eta_2},$$

where $\widehat{K}_1\in L^1(\Gamma)$.

$(\mathcal{A}_5)$ For all $y\in\mathbb{R}$, $\widehat{k}(\cdot,y)$ is measurable in Ω. For *a.e.* $\xi\in\Omega$, $\widehat{k}(\xi,\cdot)$ is continuous in $\mathbb{R}$ and we have the estimates:

$$C_{\widehat{k}}\le\widehat{k}(\xi,y)\le\widehat{k}_1(\xi)\times\eta(|y|)\quad\text{and}\quad\left|\widehat{k}(\xi,y)-\widehat{k}(\xi,z)\right|\le\widehat{k}_2(\xi)\big(\eta(|y|)+\eta(|z|)\big)|y-z|,$$

where $\widehat{k}_1\in L^1(\Omega)$, $\widehat{k}_2\in L^2(\Omega)$, and η as in $(\mathcal{A}_1)$.

The value function of problem $(\widehat{\mathcal{P}}_{t,x_0})$ is defined by

$$\widehat{v}(t,x_0)=\inf_{\mathcal{M}(t,T;U)}\widehat{J}(t,\widehat{y}_{t,x_0,u},u), \tag{5.7}$$

where $\widehat{y}_{t,x_0,u}$ is the solution of equation (5.2).

5.3. Existence of solution to $(\widehat{\mathcal{P}}_{t,x_0})$

We have the following theorem [23, Th. 6.1].

Theorem 5.2. *For all $t\in[0,T[$, all $x_0\in L^\infty(\Omega)$, problem $(\widehat{\mathcal{P}}_{t,x_0})$ admits at least one solution.*

By setting

$$\widehat{L}(t,y,u)=\int_\Omega\widehat{G}(t,\xi,y(\xi))\mathrm{d}\xi+\int_\Gamma\widehat{K}(t,\sigma,u(\sigma))\mathrm{d}\sigma,\quad\widehat{g}(y)=\int_\Omega k(\xi,y(\xi))\mathrm{d}\xi,$$

we notice that problem $(\widehat{\mathcal{P}}_{t,x_0})$ is an optimal control problem of the form of problems studied in Section 4. However $\widehat{L}$, $\widehat{g}$, and $\widehat{F}$ do not satisfy assumptions of Section 2. Thus we cannot apply Theorem 4.5 to the value function $\widehat{v}$, and we cannot claim that $\widehat{v}$ is the viscosity solution to Hamilton-Jacobi-Bellman equation corresponding to $\widehat{L}$, $\widehat{g}$, and $\widehat{F}$. To overcome this drawback, we introduce in the next section a family of problems whose value function is locally equal to $\widehat{v}$ and is the unique viscosity solution to a Hamilton-Jacobi-Bellman equation.

5.4. Definition of a problem $(\mathcal{P}_{t,x_0})$ equivalent to $(\widehat{\mathcal{P}}_{t,x_0})$

For all $M_0 > 0$, we set $R(M_0,T) = M_0 + C(p,\tilde{p},q,\tilde{q},\Omega,T)(T^{1/\tilde{q}}M_U + T^{1/\tilde{p}}M_f)$, where $C(p,\tilde{p},q,\tilde{q},\Omega,T)$ is the constant appearing in (5.5). We set

$$T_{R(M_0,T)}(y) = \min\big(R(M_0,T), \max(-R(M_0,T), y)\big) \qquad \text{for all } y \in \mathbb{R}.$$

In this example $Y = L^\infty(\Omega)$, $Y_0 = L^\infty(\Gamma)$, P_{M_0} (respectively $P^0_{M_0}$) is the projection operator from $L^2(\Omega)$ (respectively $L^2(\Gamma)$) into the ball in $L^\infty(\Omega)$ (respectively $L^\infty(\Gamma)$), centered at the origin and with radius $R(M_0,T)$, defined by

$$P_{M_0}y(\xi) = T_{R(M_0,T)}(y(\xi)) \quad \text{for } a.e.\ \xi \in \Omega \quad (\text{resp. } P^0_{M_0}y(\xi) = T_{R(M_0,T)}(y(\xi)) \quad \text{for } a.e.\ \xi \in \Gamma).$$

We set

$$G(t,\xi,y(\xi)) = \widehat{G}(t,\xi,P_{M_0}y(\xi)) \quad \text{and} \quad k\left(\xi,y(\xi)\right) = \widehat{k}(\xi,P_{M_0}y(\xi)) \qquad \text{for all } y \in L^2(\Omega),$$

and $h\left(y(\,\xi)\right) = \widehat{h}(P^0_{M_0}y(\xi)))$ for all $y \in L^2(\Gamma)$. The mappings h, G, and k clearly depend on M_0. We have not noticed this dependence in order not to load the notation. We set

$$L\left(t,y,u\right) = \int_\Omega G\left(t,\xi,y\left(\xi\right)\right)\mathrm{d}\xi + \int_\Gamma \widehat{K}\left(t,\sigma,u(\sigma)\right)\mathrm{d}\sigma, \quad g\left(y\right) = \int_\Omega k(\xi,y(\xi))\mathrm{d}\xi, \quad \text{and} \quad F(y) = h(y).$$

Proposition 5.3. *The mappings L, g and F satisfy the assumptions of Section 2.*

Proof. With $(\mathcal{A}_3)$, $(\mathcal{A}_4)$, and the definition of G, for all $(t,y,u) \in [0,T] \times L^2\left(\Omega\right) \times L^2\left(\Gamma\right)$, we have:

$$\begin{aligned}\left|L\left(t,y,u\right)\right| &\leq \int_\Omega \left|G\left(t,\xi,y(\xi)\right)\right|\mathrm{d}\xi + \int_\Gamma \left|\widehat{K}\left(t,\sigma,u(\sigma)\right)\right|\mathrm{d}\sigma \\ &\leq \eta(R(M_0,T))|\widehat{G}_1|_{L^1(\Omega)} + |\widehat{K}_1|_{L^1(\Gamma)} + c_0M_U^q = M_L\,.\end{aligned}$$

With $(\mathcal{A}_3)$ and Cauchy-Schwarz inequality, we have

$$\begin{aligned}\left|L\left(t,y,u\right) - L\left(s,z,u\right)\right| &= \left|\int_\Omega \left(G\left(s,\xi,y(\xi)\right) - G\left(t,\xi,z(\xi)\right)\right)\mathrm{d}\xi\right| + \left|\int_\Gamma \left(\widehat{K}\left(t,\sigma,u(\sigma)\right)\mathrm{d}\sigma - \widehat{K}\left(s,\sigma,u(\sigma)\right)\right)\mathrm{d}\sigma\right| \\ &\leq 2\eta(R(M_0,T))\big(|\widehat{G}_2|_{L^2(\Omega)} + |\widehat{G}_2|_{L^1(\Omega)}\big)\big(|y-z|_{L^2(\Omega)} + |t-s|^{\eta_2}\big) + \big(|\widehat{K}_1|_{L^1(\Gamma)} + c_0M_U^q\big)|t-s|^{\eta_2},\end{aligned}$$

that is

$$\left|L\left(t,y,u\right) - L\left(s,z,u\right)\right| \leq K_L\big(|y-z|_{L^2(\Omega)} + |t-s|^{\eta_2}\big).$$

The estimates for g can be obtained in a similar way with $(\mathcal{A}_5)$. The estimates for F directly follows from the definition of h. □

For all $t \in [0,T[$, and all $x_0 \in L^2(\Omega)$, we consider the optimal control problem

$$(\mathcal{P}_{t,x_0}) \qquad \min\big\{J\left(t,u,y\right) \mid u \in \mathcal{M}(t,T;U) \text{ and } (y,u) \text{ is solution of equation (5.8)}\big\},$$

where the cost function J is defined by

$$J(t,u,y) = \int_t^T L\left(s,y(s),u(s)\right)\mathrm{d}s + g(y(T))\,,$$

and the state equation is

$$\frac{\partial y}{\partial t} - \Delta y + y = f \quad \text{in } Q_{t,T}, \qquad \frac{\partial y}{\partial n} + h(\Lambda y) = u \quad \text{on } \Sigma_{t,T}, \qquad y(t) = x_0 \quad \text{in } \Omega. \tag{5.8}$$

Equation (5.8) can be written in the form

$$y' = \widehat{A}y + (-\widehat{A})^{\beta}\left[Bu - F(\Lambda y)\right] \qquad \text{in } (t,T), \qquad y(t) = x_0,$$

where $\widehat{F}(t,y) = (-A)^{-\beta}f(t) + Bh(y)$ satisfies assumption **(iv)** of Section 2. The value function of problem $(\mathcal{P}_{t,x_0})$ is defined by

$$v(t,x_0) = \inf_{u\in\mathcal{M}(t,T;U)} J(t, y_{t,x_0,u}, u)\,, \tag{5.9}$$

where $y_{t,x_0,u}$ is solution of equation (5.8). From Theorems 3.5 and 4.5, we deduce that v is the unique viscosity solution of equation (1.1) corresponding to F, L, and g.

Theorem 5.4. *Let t be in $[0,T[$ and assume that $|x_0|_{L^\infty(\Omega)} \le M_0$. A pair (y,u) is a solution of problem $(\widehat{\mathcal{P}}_{t,x_0})$ if and only if it is a solution of $(\mathcal{P}_{t,x_0})$.*

Proof. If $|x_0|_{L^\infty(\Omega)} \le M_0$, with Theorem 5.1 we can easily verify that, for all $u \in \mathcal{M}(t,T;U)$, the solution $\widehat{y}_{t,x_0,u}$ of equation (5.2) obeys $\|\widehat{y}_{t,x_0,u}\|_{L^\infty(\Omega_{t,T})} \le R(M_0,T)$. Thus $\widehat{F}(\widehat{y}_{t,x_0,u}) = F(\widehat{y}_{t,x_0,u})$, and $\widehat{y}_{t,x_0,u}$ is also the solution of (5.8). That is $\widehat{y}_{t,x_0,u} = y_{t,x_0,u}$. Thus we do not distinguish $\widehat{y}_{t,x_0,u}$ and $y_{t,x_0,u}$ up to the end of the proof. Assume that $(y_{t,x_0,\bar{u}}, \bar{u})$ is a solution of $(\widehat{\mathcal{P}}_{t,x_0})$, then for all $u \in \mathcal{M}(t,T;U)$, we have

$$J(t, y_{t,x_0,\bar{u}}, \bar{u}) = \widehat{J}(t, y_{t,x_0,\bar{u}}, \bar{u}) \le \widehat{J}(t, y_{t,x_0,u}, u) = J(t, y_{t,x_0,u}, u),$$

that is $(y_{t,x_0,\bar{u}}, \bar{u})$ is a solution of $(\mathcal{P}_{t,x_0})$. We prove that any solution of $(\mathcal{P}_{t,x_0})$ is a solution of $(\widehat{\mathcal{P}}_{t,x_0})$ in a similar way. □

Corollary 5.5. *For all $t \in [0,T[$ and all x_0 satisfying $|x_0|_{L^\infty(\Omega)} \le M_0$, we have $\widehat{v}(t,x_0) = v(t,x_0)$.*

5.5. **State equation of example 2**

Consider the following Burgers type equation in 2-D:

$$\frac{\partial y}{\partial t} - \Delta y + \partial_{x_1}(y^2) = u\chi_\omega \quad \text{in } \Omega_{t,T}, \quad y = 0 \quad \text{on } \Sigma_{t,T}, \qquad y(t) = x_0 \quad \text{in } \Omega. \tag{5.10}$$

In this example, ω is an open subset in Ω, χ_ω is the characteristic function of ω, and $(\mathcal{A}_1)$ is replaced by

$(\mathcal{A}_1')$ U is a closed bounded convex and nonempty subset in $L^{10}(\omega)$ and it obeys the condition

$$|u|_{L^{10}(\omega)} \le M_U \quad \text{for all } u \in U.$$

Set $X = L^2(\Omega)$, $X_\Gamma = L^2(\omega)$, $X_0 = L^2(\Omega)$, $\alpha = 0$, and let Λ be the identity in X. We now define the unbounded operator A in X by

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega) \qquad \text{and} \qquad Ax = \Delta x \ \text{ for all } \ x \in D(A)\,.$$

Equation (5.10) can be rewritten in the form

$$y' = \widehat{A}y + (-A)^{\frac{1}{2}}\left[Bu - \widehat{F}(\Lambda y)\right] \qquad \text{in } (t,T), \qquad y(t) = x_0, \tag{5.11}$$

where

$$\widehat{F}(y) = 2(-A)^{-\frac{1}{2}}\big(y\partial_{x_1} y\big) \qquad \text{and} \qquad Bu = (-A)^{-\frac{1}{2}}(u\chi_\omega)\,.$$

In this example we take $\beta = \frac{1}{2}$.

Theorem 5.6. *For all $x_0 \in L^\infty(\Omega)$ and all $u \in \mathcal{M}(t,T;U)$, equation (5.10) admits a unique weak solution in $C([t,T];L^2(\Omega))$. This solution belongs to $L^\infty(]t,T[\times\Omega)$ and it satisfies the estimate*

$$\|y\|_{L^\infty(\Omega_{t,T})} \le C(\Omega,T)(|x_0|_{L^\infty(\Omega)} + |x_0|^3_{L^\infty(\Omega)} + \|u\|_{L^{10}((t,T)\times\omega)} + \|u\|^3_{L^{10}((t,T)\times\omega)}). \tag{5.12}$$

Proof. It is well known that equation (5.10) admits a unique weak solution in $C([t,T];L^2(\Omega)) \cap L^2(t,T;H^1_0(\Omega))$ and that

$$\|y\|_{C([t,T];L^2(\Omega))} + \|y\|_{L^2(t,T;H^1_0(\Omega))} \le 2|x_0|_{L^2(\Omega)} + C(\Omega)\|u\|_{L^2((t,T)\times\omega)}.$$

If we multiply equation (5.10) by $|y|^{2p-2}y$, and if we integrate over $(t,\tau)\times\Omega$, after integration by parts, we formally obtain:

$$\frac{1}{2p}\int_\Omega |y(\tau)|^{2p} + \int_t^\tau \int_\Omega (2p-1)|\nabla y|^2|y|^{2p-2} = \frac{1}{2p}\int_\Omega |x_0|^{2p} + \int_t^\tau \int_\omega u|y|^{2p-2}y.$$

This identity leads to the estimate:

$$\|y\|_{L^\infty(t,T;L^{2p}(\Omega))} \le C(\Omega,p)\big(|x_0|_{L^{2p}(\Omega)} + \|u\|_{L^{2p}((t,T)\times\omega)}\big) \quad \text{for all } 1 \le p \le 5.$$

This formal estimate can be justified (see [22], Th. 5). Thus we have

$$\|y^2\|_{L^\infty(t,T;L^p(\Omega))} \le C\big(|x_0|^2_{L^{2p}(\Omega)} + \|u\|^2_{L^{2p}((t,T)\times\omega)}\big).$$

Passing the term $\partial_{x_1}(y^2)$ in the right hand side of the equation and using regularity results for the heat equation, we obtain:

$$\begin{aligned}\|y\|_{L^{\tilde p}(t,T;W^{1,p}(\Omega))} &\le C_1(\Omega,p,\tilde p)\big(|x_0|_{L^p(\Omega)} + \|u\|_{L^p((t,T)\times\omega)} + \|y^2\|_{L^{\tilde p}(t,T;L^p(\Omega))}\big)\\ &\le C_2(\Omega,p,\tilde p)\big(|x_0|_{L^p(\Omega)} + \|u\|_{L^p((t,T)\times\omega)} + |x_0|^2_{L^{2p}(\Omega)} + \|u\|^2_{L^{2p}((t,T)\times\omega)}\big),\end{aligned}$$

for all $1 < \tilde p < 2$ and all $1 \le p \le 5$. In addition we have:

$$\|y\,\partial_{x_1} y\|_{L^{\tilde p}(t,T;L^{p/2}(\Omega))} \le \|y\|_{L^\infty(t,T;L^p(\Omega))}\|\partial_{x_1} y\|_{L^{\tilde p}(t,T;L^p(\Omega))}.$$

Using this estimate and regularity results for the heat equation we can write:

$$\|y\|_{L^\infty(\Omega_{t,T})} \le |x_0|_{L^\infty(\Omega)} + C(\Omega,p,\tilde p)\big(\|u\|_{L^p((t,T)\times\omega)} + \|y\,\partial_{x_1} y\|_{L^{\tilde p}(t,T;L^{p/2}(\Omega))}\big),$$

provided that $\frac{2}{p} + \frac{1}{\tilde p} < 1$. Choosing $p = 5$, $\tilde p = \frac{15}{8}$, and combining the previous estimates we obtain the desired result. □

From Theorem 5.6, and assumption $(\mathcal{A}'_1)$, we deduce

$$\|y\|_{L^\infty(\Omega_{t,T})} \le C(\Omega,T)\Big(|x_0|_{L^\infty(\Omega)} + |x_0|^3_{L^\infty(\Omega)} + (T-t)^{1/10}M_U + (T-t)^{3/10}M_U^3\Big).$$

5.6. **Setting of the control problem for example 2**

For all $t \in [0,T[$, and all $x_0 \in L^\infty(\Omega)$, we study the following control problem

$$(\widehat{\mathcal{P}}_{t,x_0}) \qquad \min\Big\{\widehat{J}(t,u,y) \mid u \in \mathcal{M}(t,T;U) \text{ and } (y,u) \text{ is solution of equation (5.10)}\Big\},$$

where the cost function $\widehat{J}$ is

$$\widehat{J}(t,u,y)=\int_t^T\int_\Omega \widehat{G}(s,\xi,y(s,\xi))\mathrm{d}\xi\mathrm{d}s+\int_t^T\int_\omega \widehat{K}(s,\xi,u(s,\xi))\mathrm{d}\xi\mathrm{d}s+\int_\Omega \widehat{k}(\xi,y(T,\xi))\mathrm{d}\xi.$$

We assume that $\widehat{G}$ obeys $(\mathcal{A}_3)$, $\widehat{k}$ obeys $(\mathcal{A}_5)$, and that $\widehat{K}$ obeys

$(\mathcal{A}_4')$ For all $(t,u)\in\mathbb{R}^2$, $\widehat{K}(t,\cdot,u)$ is measurable in ω. For *a.e.* $(t,\xi)\in\mathbb{R}\times\omega$, $\widehat{K}(t,\xi,\cdot)$ is convex. For *a.e.* $\xi\in\omega$, $\widehat{K}(\cdot,\xi,\cdot)$ is continuous in $\mathbb{R}^2$ and we have the estimates:

$$C_{\widehat{K}}\le\widehat{K}(t,\xi,u)\le\widehat{K}_1(\xi)+c_0|u|^q \quad\text{and}\quad \left|\widehat{K}(t,\xi,u)-\widehat{K}(s,\xi,u)\right|\le\left(\widehat{K}_1(\xi)+c_0|u|^q\right)|t-s|^{\eta_2},$$

where $q=10$, $\widehat{K}_1\in L^1(\omega)$.

For all $M_0>0$, we set

$$R(M_0,T)=C(\Omega,T)\Big(M_0+M_0^3+T^{1/10}M_U+T^{3/10}M_U^3\Big),$$

and we define the truncated problem in a similar way as in example 1, with obvious modifications. More precisely, Y, P_{M_0}, G, k, and g are defined as in Example 1,

$$L(t,y,u)=\int_\Omega G(t,\xi,y(\xi))\,\mathrm{d}\xi+\int_\omega \widehat{K}(t,\xi,u(\xi))\,\mathrm{d}\xi,\quad\text{and}\quad F(y)=2(-A)^{-\frac12}\big(P_{M_0}y\,\partial_{x_1}P_{M_0}y\big),$$

for all $y\in L^2(\Omega)$. We can take any β_0 in $(0,1/2)$. For example setting $\beta_0=1/4$, we have

$$|F(y)|_{D((-A)^{\frac14})}\le C|\big(P_{M_0}y\big)^2|_{D((-A)^{\frac14})}\le C|y|^2_{D((-A)^{\frac12})}\qquad\text{for all } y\in D((-A)^{\frac12})=H_0^1(\Omega),$$

and the additional condition in assumption (iv) when $\beta=\frac12$ is satisfied. The other conditions in assumption (iv) are also satisfied. We can define $(\mathcal{P}_{t,x_0})$ similarly as in Example 1. In particular the state equation for $(\mathcal{P}_{t,x_0})$ is

$$y'=Ay+(-A)^{\frac12}[Bu-F(\Lambda y)]\qquad\text{in } (t,T),\qquad y(t)=x_0.$$

Denoting by $\widehat{v}(t,x_0)$ the value function of problem $(\widehat{\mathcal{P}}_{t,x_0})$, and by $v(t,x_0)$ the value function of problem $(\mathcal{P}_{t,x_0})$, as in example 1, we can prove the following Theorem.

Theorem 5.7. *The value function $v(t,x_0)$ is the unique viscosity solution of the Hamilton-Jacobi-Bellman equation* (1.1) *associated with $(\mathcal{P}_{t,x_0})$. For all $t\in[0,T[$ and all $x_0\in X$ satisfying $|x_0|_{L^\infty(\Omega)}\le M_0$, we have $\widehat{v}(t,x_0)=v(t,x_0)$.*

5.7. **Example 3**

We consider the same equation as in Example 2, and now $(\mathcal{A}_1)$ is replaced by

$(\mathcal{A}_1'')$ U is a closed bounded convex and nonempty subset in $L^8(\omega)$ and it obeys the condition

$$|u|_{L^8(\omega)}\le M_U\quad\text{for all } u\in U.$$

Set $X=L^2(\Omega)$, $X_\Gamma=L^2(\omega)$, $X_0=H_0^{3/4}(\Omega)$, $\alpha=\frac38$, and let Λ be the identity in X_0 (thus Λ is considered as an unbounded operator in X). We define the unbounded operator $(A,D(A))$ in X as in Example 2. Equation (5.10) can be rewritten in the form

$$y'=Ay+(-A)^{\frac{\varepsilon'}{2}}[Bu-\widehat{F}(y)]\qquad\text{in } (t,T),\qquad y(t)=x_0,\tag{5.13}$$

where

$$\widehat{F}(y) = 2(-A)^{-\frac{\varepsilon'}{2}}\big(y\partial_{x_1} y\big) \qquad \text{and} \qquad Bu = (-A)^{-\frac{\varepsilon'}{2}}(u\chi_\omega), \quad \text{with } 0 \le \varepsilon' \le 1.$$

In this example, we take $\beta = \frac{\varepsilon'}{2}$ (below we choose $\varepsilon' = 5/8$).

Theorem 5.8. *For all $x_0 \in H_0^{3/4}(\Omega)$ and all $u \in \mathcal{M}(t,T;U)$, equation (5.10) admits a unique weak solution in $C([t,T];H_0^{3/4}(\Omega))$, and it satisfies the estimate*

$$\|y\|_{C([t,T];H_0^{3/4}(\Omega))} \le C(\Omega,T)(|x_0|_{H_0^{3/4}(\Omega)} + |x_0|^3_{H_0^{3/4}(\Omega)} + \|u\|_{L^8((t,T)\times\omega)} + \|u\|^3_{L^8((t,T)\times\omega)}\big). \tag{5.14}$$

Proof. Observe that $H_0^{3/4}(\Omega) \hookrightarrow L^8(\Omega)$. As in Example 2, we have

$$\|y^2\|_{L^\infty(t,T;L^p(\Omega))} \le C\big(|x_0|^2_{L^{2p}(\Omega)} + \|u\|^2_{L^{2p}((t,T)\times\omega)}\big),$$
$$\|y\|_{L^{\tilde p}(t,T;W^{1,p}(\Omega))} \le C_2(\Omega,p,\tilde p)\big(|x_0|_{L^p(\Omega)} + \|u\|_{L^{2p}((t,T)\times\omega)} + |x_0|^2_{L^{2p}(\Omega)} + \|u\|^2_{L^{2p}((t,T)\times\omega)}\big),$$

for all $1 < \tilde p < 2$ and all $1 < p \le 4$, and

$$\|y\,\partial_{x_1} y\|_{L^{\tilde p}(t,T;L^{p/2}(\Omega))} \le \|y\|_{L^\infty(t,T;L^p(\Omega))}\|\partial_{x_1} y\|_{L^{\tilde p}(t,T;L^p(\Omega))} \qquad \text{if in addition } 2 \le p \le 4.$$

Using this estimate and regularity results for the heat equation we can write:

$$\|y\|_{C([t,T];H_0^{3/4}(\Omega))} \le C(\Omega,\tilde p)\big(|x_0|_{H_0^{3/4}(\Omega)} + \|u\|_{L^2((t,T)\times\omega)} + \|y\,\partial_{x_1} y\|_{L^{\tilde p}(t,T;L^2(\Omega))}\big), \tag{5.15}$$

if $\frac{1}{\tilde p} < \frac{5}{8}$. Thus, choosing $p = 4$, $\tilde p = \frac{16}{9}$, and combining the above estimates, we obtain:

$$\|y\|_{C([t,T];H_0^{3/4}(\Omega))} \le C(\Omega,T)\big(|x_0|_{H_0^{3/4}(\Omega)} + |x_0|^3_{H_0^{3/4}(\Omega)} + \|u\|_{L^8((t,T)\times\omega)} + \|u\|^3_{L^8((t,T)\times\omega)}\big). \tag{5.16}$$

□

From Theorem 5.8, and assumptions $(\mathcal{A}_1'')$ and $(\mathcal{A}_5)$, we deduce

$$\|y\|_{C([t,T];H_0^{3/4}(\Omega))} \le C(\Omega,T)(|x_0|_{H_0^{3/4}(\Omega)} + |x_0|^3_{H_0^{3/4}(\Omega)} + (T-t)^{1/8}M_U + (T-t)^{3/8}M_U^3\big).$$

Now we set

$$R(M_0,T) = C(\Omega,T)\big(M_0 + M_0^3 + T^{1/8}M_U + T^{3/8}M_U^3\big).$$

Let us denote by $P^0_{M_0}$ the orthogonal projection in $H_0^{3/4}(\Omega)$ on the ball centered at zero and with radius $R(M_0,T)$. Let us set

$$F(y) = 2(-A)^{-\frac{\varepsilon'}{2}}\left(P^0_{M_0} y\,\partial_{x_1}\left(P^0_{M_0} y\right)\right) \qquad \text{for all } y \in H_0^{3/4}(\Omega).$$

Let us show that F is bounded and Lipschitz from $H_0^{3/4}(\Omega)$ into X. We have

$$|F(y)|_{L^2(\Omega)} = 2|P^0_{M_0} y\,\partial_{x_1}\left(P^0_{M_0} y\right)|_{H^{-\varepsilon'}(\Omega)} = 2|P^0_{M_0} y\,\partial_{x_1}\left(P^0_{M_0} y\right)|_{B^{-\varepsilon'}_{2,2}(\Omega)}.$$

Using the product estimate [25], page 171

$$B^{s_1}_{p_1,q_1}(\Omega)\cdot B^{s_2}_{p_2,q_2}(\Omega) \hookrightarrow B^{s_1}_{p,q}(\Omega),$$

with

$$s_1 = -\varepsilon' = -\frac{5}{8}, \quad s_2 = \varepsilon = \frac{3}{4}, \quad p_1 = \frac{32}{11}, \quad p_2 = p = 2, \quad q_1 = q_2 = q = 2,$$

we have

$$|F(y)|_{L^2(\Omega)} \leq C|P^0_{M_0}y|_{B^{\varepsilon}_{2,2}(\Omega)}|\partial_{x_1}\left(P^0_{M_0}y\right)|_{B^{-\varepsilon'}_{p_1,2}(\Omega)}.$$

Recall that

$$B^{\varepsilon}_{2,2}(\Omega) = H^{\varepsilon}(\Omega), \qquad B^{-\varepsilon'}_{2,2}(\Omega) = H^{-\varepsilon'}(\Omega), \quad \text{and} \quad B^{-\varepsilon'}_{p_1,2}(\Omega) = \left(B^{\varepsilon'}_{p_1',2}(\Omega)\right)',$$

where $p_1' = p_1/(p_1-1)$. Moreover

$$B^{\varepsilon'}_{p_1',2}(\Omega) \hookrightarrow B^{\varepsilon''}_{p_1',p_1'}(\Omega) = W^{\varepsilon'',p_1'}(\Omega) \hookrightarrow H^{\hat{\varepsilon}}(\Omega),$$

for all $\hat{\varepsilon} < \varepsilon'' < \varepsilon'$, and if $2 = \frac{2p_1'}{2-(\varepsilon''-\hat{\varepsilon})p_1'}$. We choose $\hat{\varepsilon} = \frac{1}{4}$ and $\varepsilon'' = \frac{9}{16}$, we have $H^{-\hat{\varepsilon}}(\Omega) \hookrightarrow B^{-\varepsilon'}_{p_1,2}(\Omega)$, and we obtain

$$|F(y)|_{L^2(\Omega)} \leq C|P^0_{M_0}y|_{H^{3/4}_0(\Omega)}|\partial_{x_1}\left(P^0_{M_0}y\right)|_{H^{-\hat{\varepsilon}}(\Omega)} \leq C|P^0_{M_0}y|_{H^{3/4}_0(\Omega)}|P^0_{M_0}y|_{H^{3/4}_0(\Omega)}.$$

Thus F is bounded from X_0 into X. Let us verify that F is Lipschitz. We have

$$\begin{aligned}
|F(y)-F(z)|_{L^2(\Omega)} &\leq 2|P^0_{M_0}y\,\partial_{x_1}\left(P^0_{M_0}y\right) - P^0_{M_0}z\,\partial_{x_1}\left(P^0_{M_0}z\right)|_{H^{-\varepsilon'}(\Omega)} \\
&\leq 2|P^0_{M_0}y\,\partial_{x_1}\left(P^0_{M_0}y - P^0_{M_0}z\right)|_{H^{-\varepsilon'}(\Omega)} + 2|P^0_{M_0}z\,\partial_{x_1}\left(P^0_{M_0}y - P^0_{M_0}z\right)|_{H^{-\varepsilon'}(\Omega)} \\
&\leq C\left(|P^0_{M_0}y|_{H^{3/4}_0(\Omega)} + |P^0_{M_0}y|_{H^{3/4}_0(\Omega)}\right)|P^0_{M_0}y - P^0_{M_0}z|_{H^{3/4}_0(\Omega)} \\
&\leq C\left(|P^0_{M_0}y|_{H^{3/4}_0(\Omega)} + |P^0_{M_0}y|_{H^{3/4}_0(\Omega)}\right)|y-z|_{H^{3/4}_0(\Omega)}.
\end{aligned}$$

To obtain the last inequality we have used the Lipschitz continuity of $P^0_{M_0}$ from $H^{3/4}_0(\Omega)$ into itself. We define $(\widehat{\mathcal{P}}_{t,x_0})$ as in Example 2, and we assume that $\widehat{G}$ obeys $(\mathcal{A}_3'')$, $\widehat{k}$ obeys $(\mathcal{A}_5'')$, and $\widehat{K}$ obeys $(\mathcal{A}_4'')$, where $(\mathcal{A}_3'')$, $(\mathcal{A}_5'')$, and $(\mathcal{A}_4'')$ respectively correspond to $(\mathcal{A}_3)$, $(\mathcal{A}_5)$, and $(\mathcal{A}_4')$, where $\eta(|y|) = |y|^r$, $1 \leq r \leq 4$, $\widehat{G}_1 \in L^{\rho_1}(\Omega)$, $\widehat{G}_2 \in L^{\rho_2}(\Omega)$, $\widehat{k}_1 \in L^{\rho_1}(\Omega)$, $\widehat{k}_2 \in L^{\rho_2}(\Omega)$, and $q = 8$, with $\frac{r}{8} + \frac{1}{\rho_1} = 1$ and $\frac{r}{8} + \frac{1}{\rho_2} = \frac{1}{2}$. We define $(\widehat{\mathcal{P}}_{t,x_0})$ as in Example 2. Next we define $(\mathcal{P}_{t,x_0})$ with the state equation

$$y' = Ay + (-A)^{\frac{\varepsilon'}{2}}\left[Bu - F(\Lambda y)\right] \qquad \text{in } (t,T), \qquad y(t) = x_0,$$

and with G, k, g, and L defined as in example 2, but where P_{M_0} is now the projection in $L^2(\Omega)$ on the ball in $H^{3/4}_0(\Omega)$ centered at zero and with radius $R(M_0,T)$. Recall that P_{M_0} is Lipschitz continuous from $L^2(\Omega)$ into itself. Since the embedding from $H^{3/4}_0(\Omega)$ into $L^8(\Omega)$ is continuous, with assumptions $(\mathcal{A}_3'')$, $(\mathcal{A}_5'')$, and $(\mathcal{A}_4'')$, we easily verify that assumptions of Section 2 are satisfied by L, g, and F. Denoting by $\widehat{v}(t,x_0)$ the value function of problem $(\widehat{\mathcal{P}}_{t,x_0})$, and by $v(t,x_0)$ the value function of problem $(\mathcal{P}_{t,x_0})$, we can prove the following Theorem.

Theorem 5.9. *The value function $v(t,x_0)$ is the unique viscosity solution of the Hamilton-Jacobi-Bellman equation* (1.1) *associated with $(\mathcal{P}_{t,x_0})$. For all $t \in [0,T[$ and all x_0 satisfying $|x_0|_{H^{3/4}_0(\Omega)} \leq M_0$, we have $\widehat{v}(t,x_0) = v(t,x_0)$.*

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