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TOPOLOGY AND GEOMETRY OF NONTRIVIAL RANK-ONE CONVEX HULLS FOR TWO-BY-TWO MATRICES

Carl-Friedrich Kreiner[1] and Johannes Zimmer[2]

Abstract. Continuing earlier work by Székelyhidi, we describe the topological and geometric structure of so-called T_4-configurations which are the most prominent examples of nontrivial rank-one convex hulls. It turns out that the structure of T_4-configurations in $\mathbb{R}^{2\times 2}$ is very rich; in particular, their collection is open as a subset of $(\mathbb{R}^{2\times 2})^4$. Moreover a previously purely algebraic criterion is given a geometric interpretation. As a consequence, we sketch an improved algorithm to detect T_4-configurations.

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1. Introduction

T_4-configurations (see Def. 2.2 below) are the focus of several recent investigations [9, 11, 18]. This interest can be explained by their importance for a variety of different fields. Firstly, rank-one convex hulls of sets and rank-one convex envelopes of functions are intrinsically important notions in the calculus of variations [4, 13]. Secondly, the rank-one convex envelope of a nonconvex microscopic energy function of a material serves as a model for its macroscopic energy, as an approximation of the quasiconvex envelope. The quest for a reliable method for the computation of these hulls and envelopes highlights the relevance of rank-one convexity for engineering [2]. Thirdly, rank-one convex hulls are, in connection with convex integration, important tools for the regularity theory of elliptic systems. Müller and Šverák develop these techniques to obtain Lipschitz continuous (weak) solution to elliptic systems which are nowhere C^1 [14]. The use of T_4-configurations as a basis of counterexamples to regularity apparently goes back to Scheffer [16].

Here, we show that T_4-configurations are not as exotic objects as one would expect at first sight. Namely, we show that they form an open set in the set of quadruples of matrices in $\mathbb{R}^{2\times 2}$ (Prop. 2.5) (see also [9], Prop. 4.26). We remark that this is no longer true in higher space dimensions. This is another manifestation of the observation that the case of $\mathbb{R}^{2\times 2}$ is the most interesting one. Rank-one convexity is always implied by quasiconvexity, which is the central notion in the calculus of variations. However, the converse is wrong in higher space dimensions [17]. In $\mathbb{R}^{2\times 2}$, equality of rank-one convexity and quasiconvexity is a long-standing open problem.

In Section 3, we give a purely geometric characterization of the different types of T_4-configurations as well as so-called degenerate T_4-configurations (Th. 3.8). In Section 4, we explore the topological structure of the set of

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[1] Max Planck Institute for Mathematics in the Sciences, Inselstr. 22, 04 103 Leipzig, Germany.
[2] University of Bath, Department of Mathematical Sciences, Claverton Down, Bath BA2 7AY, UK; zimmer@maths.bath.ac.uk

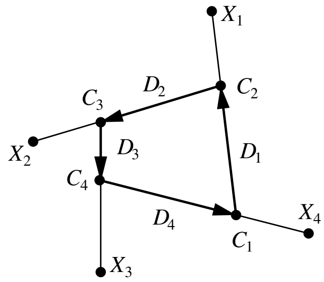

FIGURE 1. A T_4-configuration, projected to $\mathbb{R}^2$.

T_4-configurations in $\mathbb{R}^{2\times 2}$. The different connected components are described as well as their boundaries. We close in Section 5 with an algorithmic method for the efficient detection of T_4-configurations. There, we improve the algebraic methods presented in [10, 11] for the particular case of $\mathbb{R}^{2\times 2}$. In this case, some time-consuming (semi-)algebraic tests can be replaced by simple linear algebra. In this section as well as the previous ones, we exploit deep ideas presented in [18]. The methods described in Section 5 can be used to augment and improve previous algorithms for the computation of rank-one convex hulls [1, 2, 5].

2. T_4-CONFIGURATIONS ARE OPEN IN $\mathbb{R}^{2\times 2}$

For the reader's convenience, we recall the definition of the rank-one convex hull. A detailed discussion can be found in [4, 8, 9, 12].

Definition 2.1.
(a) A function $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ is called *rank-one convex* if for all fixed $A, X \in \mathbb{R}^{m\times n}$ with $\operatorname{rank}(X) = 1$ the scalar function $t \mapsto f(A + tX)$ is convex in the usual sense.
(b) For a compact set $K \subset \mathbb{R}^{m\times n}$, its *rank-one convex hull* is defined as

$$K^{rc} := \{Y \in \mathbb{R}^{m\times n} \mid f(Y) \leq \sup_{X\in K} f(X)\ \forall\, f\colon \mathbb{R}^{m\times n} \to \mathbb{R} \text{ rank-one convex}\}.$$

Since obviously every convex function is rank-one convex, the rank-one convex hull K^{rc} of a compact set K is contained in the usual convex hull, which will be denoted K^{co}.

Our main object of study will be T_4-configurations. They are the most prominent example of sets with a non-trivial rank-one convex hull.

Definition 2.2. A set $K = \{X_1, \ldots, X_4\} \subset \mathbb{R}^{m\times n}$ with $\operatorname{rank}(X_j - X_k) \geq 2$ for $j \neq k$ is called a *T_4-configuration* if a permutation σ of $\{1, \ldots, 4\}$ exists (with $\sigma(1) = 1$), and rank-one matrices $D_1, \ldots, D_4 \in \mathbb{R}^{m\times n}$, positive scalars $\kappa_1, \ldots, \kappa_4$, and matrices $C_1, \ldots, C_4 \in \mathbb{R}^{m\times n}$ such that the relations

$$C_{j+1} - C_j = D_j, \qquad X_{\sigma(j)} - C_{j+1} = \kappa_j D_j \tag{1}$$

hold, where the index j is counted modulo 4 (see Fig. 1).

This definition is taken from [10, 11]. The difference to similar definitions in [9], Definition 7, and [18], Definition 1, is only that T_4-configurations are considered here as sets, rather than tuples.

It is crucial for the present investigations that T_4-configurations in $\mathbb{R}^{2\times 2}$ can be characterized differently, due to a result of Székelyhidi [18] (Th. 2.3 below). In this characterization, every set $K := \{X_1, \ldots, X_4\} \subset \mathbb{R}^{2\times 2}$ is considered as an edge-colored graph with vertices $X_1, \ldots, X_4$. The color of the edge joining X_j and X_k is determined by the sign of $\det(X_j - X_k)$. As $\det(X_j - X_k) = \det(X_k - X_j)$ in $\mathbb{R}^{2\times 2}$, this sign is well-defined. To simplify the presentation, we will use dashed and solid lines instead of colors. The edge joining X_j and X_k

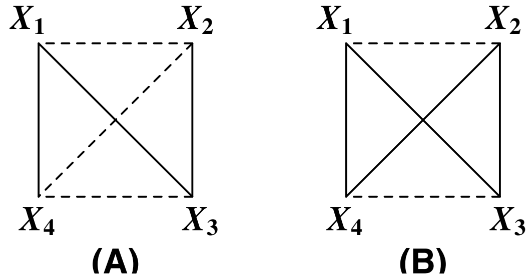

FIGURE 2. Possible sign configurations.

is *solid* if $\det(X_j - X_k) > 0$, and *dashed* if $\det(X_j - X_k) < 0$. Two vertices X_j and X_k are not joined by an edge if $\det(X_j - X_k) = 0$. The resulting graph is called the *sign diagram* of $X_1, \dots, X_4$.

We remark that Definition 2.2 is invariant under permutation of X_j as well as under multiplication (from the left, say) of all X_j by $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Therefore, it is natural to say that two sign diagrams are *equivalent* if they can be obtained from each other by renumbering the nodes or exchanging the positive (solid) and negative (dashed) edges.

For example, the sign diagram associated to classical T_4-configuration by Tartar [19]

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad X_2 = \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}, \quad X_3 = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}, \quad X_4 = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

is shown as sign diagram (A) in Figure 2.

Now we have the notation for the important theorem [18], Theorem 2, mentioned above.

Theorem 2.3 (Székelyhidi, 2003)**.** *Let* $K = \{X_1, \dots, X_4\} \subset \mathbb{R}^{2\times 2}$ *with* $\det(X_j - X_k) \neq 0$ *for* $j \neq k$*. Then there are exactly three cases:*

(1) *If the sign diagram is, up to equivalence, as in (A) of Figure 2, then there exists exactly one permutation* σ *of* $\{1, \dots, 4\}$ *with* $\sigma(1) = 1$ *such that* (1) *has a solution.*
Such a set K *will be called a* ***simple*** T_4***-configuration.***

(2) *If the sign diagram is, up to equivalence, as in (B) of Figure 2, then exactly one of the following holds:*
   (a) *There exists a* $P \in K^{co}$ *with* $\det(X_j - P) < 0$ *for all* $j = 1, \dots, 4$*. In this case,* (1) *has a solution for all permutations* σ *of* $\{1, \dots, 4\}$*.*
   K *will be called a* ***sixfold*** T_4***-configuration.***
   (b) *There exists a* $P \in K^{co}$ *with* $\det(X_j - P) = 0$ *for all* $j = 1, \dots, 4$*. In this case,* $K^{rc} = K^{co} \cap \{Y \mid \det(Y - P) = 0\}$*.*
   K *will be called a* ***degenerate*** T_4***-configuration****.*
   (c) *There exists a* $P \in K^{co}$ *with* $\det(X_j - P) > 0$ *for all* $j = 1, \dots, 4$*. In this case, the rank-one convex hull is trivial,* $K^{rc} = K$*.*

(3) *For all other sign diagrams, the rank-one convex hull is trivial,* $K^{rc} = K$*.*

Examples of sixfold and degenerate T_4-configurations can be found in Example 4.4 below. We learned about their existence from Bernd Kirchheim [9].

The importance of T_4-configurations in the case of $\mathbb{R}^{2\times 2}$ becomes evident in the following theorem [18], Theorem 1.

Theorem 2.4 (Székelyhidi, 2003)**.** *Let* $K \subset \mathbb{R}^{2\times 2}$ *be a compact set such that* $\operatorname{rank}(X - Y) = 2$ *for all* $X, Y \in K$*,* $X \neq Y$*. Then* K *contains a simple, a sixfold or a degenerate* T_4*-configuration.*

We should stress that, in this paper, T_4-*configuration* means either a simple T_4-configuration or a sixfold T_4-configuration. We first state that T_4-configurations are not as special as they might appear to be at first glance (see [9], Prop. 4.26 for an alternative, elementary proof).

Proposition 2.5. *The set*

$$\mathcal{T}_4(\mathbb{R}^{2\times 2}) := \{(X_1,\dots,X_4)\in(\mathbb{R}^{2\times 2})^4 \;\big|\; K=\{X_1,\dots,X_4\} \textit{ is a } T_4\textit{-configuration}\}$$

is open in $(\mathbb{R}^{2\times 2})^4$.

Proof. This follows easily from Theorem 2.3 and the continuity of the determinant. Namely, every set corresponding to a given sign diagram is open. For the set with sign diagram (A), this finishes the proof. For sets with sign diagram (B), we observe $P \in K^{co}$ is also an open condition (see also the Lem. 3.6, which is independent from this proposition). □

As this proof relies on Székelyhidi's deep Theorem 2.3, we wish to outline an alternative, elementary proof. Without loss of generality, we can assume that the T_4-configurations we consider are already suitably numbered. Thus, $\{X_1,\dots,X_4\}$ is a T_4-configuration if and only if there exist matrices $C_1,\dots,C_4$ and $\lambda_1,\dots,\lambda_4\in(0,1)$ such that

$$\det(X_j - C_j) = 0, \tag{2}$$

$$C_{j+1} = \lambda_j C_j + (1-\lambda_j)X_j \tag{3}$$

for $j\in\{1,\dots,4\}$, indices counted modulo 4. We regard the system of these equations as an implicit function $f\colon \mathbb{R}^{36}\to\mathbb{R}^{20}$,

$$f(X_1,\dots,X_4,C_1,\dots,C_4,\lambda_1,\dots,\lambda_4)=0.$$

The derivative Df is invertible with respect to the 20 variables C_j and λ_j for $j\in\{1,\dots,4\}$. Hence the assertion follows from the Implicit Function Theorem.

We close this section by remarking that Proposition 2.5 can not be generalized to space dimensions higher than $\mathbb{R}^{2\times 2}$.

Proposition 2.6. *Let* $m,n\in\mathbb{N}$, *and* $m\ge 3$ *or* $n\ge 3$. *The set*

$$\mathcal{T}_4(\mathbb{R}^{m\times n}) := \{(X_1,\dots,X_4)\in(\mathbb{R}^{m\times n})^4 \;\big|\; K=\{X_1,\dots,X_4\} \textit{ is a } T_4\textit{-configuration}\}$$

has empty interior (with respect to the natural topology on $(\mathbb{R}^{m\times n})^4$*).*

Proof. This follows easily from a dimension argument. Condition (1) may be rewritten as

$$X_j = C_j + \mu_j(C_{j+1}-C_j),\quad \mu_j>1 \qquad \text{for } j\in\{1,\dots,4\} \text{ counted modulo 4,} \tag{4}$$

$$\text{all } 2\times 2\text{-minors of } (C_{j+1}-C_j) \text{ vanish} \qquad \text{for } j\in\{1,\dots,4\} \text{ counted modulo 4.} \tag{5}$$

In $\mathbb{R}^{2\times 3}$, there are 28 unknowns, namely $C_1,\dots,C_4\in\mathbb{R}^{2\times 3}$ and $\mu_1,\dots,\mu_4\in\mathbb{R}$. Condition (5) describes a variety in $\mathbb{R}^{28}$ in the variables $C_1,\dots,C_4,\mu_1,\dots,\mu_4$. Its dimension equals at most four times that of the variety of the rank-one matrices in $\mathbb{R}^{2\times 3}$, *i.e.*, $4\cdot 4=16$, plus 4 for the independent parameters $\mu_1,\dots,\mu_4$. (A computation shows that the dimension equals indeed no less than 20, [6].) Every point in this variety with $\mu_j>1$ determines at most one T_4-configuration $(X_1,\dots,X_4)$ *via* equation (4). Hence the set of four-tuples $(X_1,\dots,X_4)$ forming a T_4-configuration is contained in the differentiable image of an affine variety and therefore by Sard's Theorem and the characterization of singular points of a variety (*e.g.*, [7], Sect. 14) a 20-dimensional surface. The general statement for arbitrary m,n follows by embedding $\mathbb{R}^{2\times 3}$ into $\mathbb{R}^{m\times n}$. □

3. Geometric structure of configurations of type (B)

We now turn to the structure of the sets with sign diagram of type (B). We identify occasionally $\mathbb{R}^{2\times 2}$ with $\mathbb{R}^4$. In particular, $\mathbb{R}^{2\times 2}$ will be equipped with the standard inner product for matrices $\langle\cdot,\cdot\rangle$, *i.e.*, the Euclidean inner product in $\mathbb{R}^4$.

Lemma 3.1. *Let $\mathcal{H}$ be a three-dimensional linear subspace of $\mathbb{R}^{2\times 2}$ with normal N. For a given point $P \in \mathcal{H}$, let $\mathscr{R}_1(P) \subset \mathbb{R}^{2\times 2}$ be the* ***rank-one cone centered at*** *$P \in \mathcal{H}$,*

$$\mathscr{R}_1(P) := \left\{ Y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \in \mathbb{R}^{2\times 2} \,\middle|\, \det(Y - P) = 0 \right\}.$$

Then there exists a linear isomorphism $T_{\mathcal{H}} \colon \mathbb{R}^3 \to \mathcal{H}$ such that the following holds:

(1) *If $\det(N) \neq 0$, then $\mathcal{H} \cap \mathscr{R}_1(P)$ is the image of the double cone*

$$C := \{(z_1, z_2, z_3) \in \mathbb{R}^3 \mid z_1^2 + z_2^2 - z_3^2 = 0\}$$

under the affine transformation $(z_1, z_2, z_3) \mapsto T_{\mathcal{H}}(z_1, z_2, z_3) + P$.

(2) *If $\det(N) = 0$, then $\mathcal{H} \cap \mathscr{R}_1(P)$ is the image of the pair of planes*

$$E := \{(z_1, z_2, z_3) \in \mathbb{R}^3 \mid z_1^2 - z_2^2 = 0\}$$

under the affine transformation $(z_1, z_2, z_3) \mapsto T_{\mathcal{H}}(z_1, z_2, z_3) + P$.

Moreover, $T_{\mathcal{H}}$ does not depend on the choice of $P \in \mathcal{H}$.

Remark 3.2. The structure and shape of $\mathscr{R}_1(P) = P + \mathscr{R}_1(0)$ is independent of the choice of P. The same applies to the structure and shape of $\mathscr{R}_1(P) \cap \mathcal{H}$ for $P \in \mathcal{H}$. We will use these facts frequently in the subsequent proofs.

Proof. As stated in the previous remark, we may assume $P = 0$. The definition of N and $\mathscr{R}_1(0)$ read

$$\begin{aligned} n_1 y_1 + n_2 y_2 + n_3 y_3 + n_4 y_4 &= 0, \\ y_1 y_4 - y_2 y_3 &= 0. \end{aligned}$$

A substitution of y_4 in the second equation yields for $n_4 \neq 0$

$$n_1 y_1^2 + n_2 y_1 y_2 + n_3 y_1 y_3 + n_4 y_2 y_3 = 0 \tag{6}$$

or, equivalently,

$$\frac{1}{2} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}^T \begin{pmatrix} 2n_1 & n_2 & n_3 \\ n_2 & 0 & n_4 \\ n_3 & n_4 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 0. \tag{7}$$

Denote the symmetric matrix in (7) by S. Then

$$\mathcal{H} \cap \mathscr{R}_1(0) = \left\{ \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \,\middle|\, (y_1, y_2, y_3) \text{ solves (7) and } y_4 = -\frac{1}{n_4} \sum_{j=1}^{3} n_j y_j \right\}.$$

We now classify the geometric object associated with the quadric defined by equation (7). The characteristic polynomial χ_S of S reads

$$\chi_S(t) = t^3 - 2n_1 t^2 - (n_2^2 + n_3^2 + n_4^2)t + 2n_4(n_1 n_4 - n_2 n_3),$$

and its derivative χ_S' has one negative and one positive root. By Rolle's Theorem, S has at least one positive eigenvalue λ_1 and at least one negative eigenvalue λ_3. Therefore the quadric in question is either the union of two intersecting planes or a double cone, depending on whether 0 is an eigenvalue of S or not [15], p. 102. By the structure of the characteristic polynomial, $\chi_S(0) = 0$ if and only if $\det(N) = 0$. The linear isomorphism $T_{\mathcal{H}}$

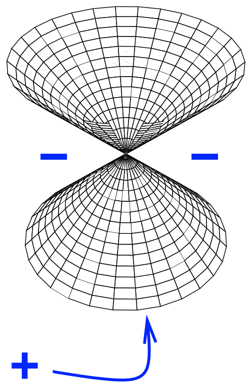

FIGURE 3. Separation of determinant sign regions by $\mathcal{H} \cap \mathscr{R}_1(0)$.

is easily constructed such that the canonical basis vectors of $\mathbb{R}^3$ are mapped to the principal axes of S, using the obvious scaling and the relation $\sum_{j=1}^4 n_j y_j = 0$ in order to ensure $T_{\mathcal{H}}(\mathbb{R}^3) = \mathcal{H}$.

The case $n_4 = 0$ is treated similarly. $\square$

Corollary 3.3. (a) *Let $E \subset \mathbb{R}^{2\times 2}$ be a two-dimensional affine plane and $X_0 \in E$ arbitrary. Then $\mathscr{J} := E \cap \mathscr{R}_1(X_0)$ is exactly one of the following four objects:*

(1) *$\mathscr{J} = E$, and $\det(X - Y) = 0$ for all $X, Y \in E$.*
(2) *$\mathscr{J} = \{0\}$, and $\det(X - Y)$ has the same sign for all $X, Y \in E$, $X \neq Y$.*
(3) *$\mathscr{J}$ is a line $L = X_0 + \operatorname{span}\{W\}$, and $\det(X - Y)$ has the same sign for all $X, Y \in E$ with $X - Y \notin \operatorname{span}\{W\}$.*
(4) *$\mathscr{J}$ consists of two lines $L_j = X_0 + \operatorname{span}\{W_j\}$ for $j = 1, 2$ intersecting at X_0, and for $X = X_0 + \lambda W_1 + \mu W_2 \in E$ we have $\det(X - X_0) = \lambda\mu\, c$ with a constant $c = c(W_1, W_2) \neq 0$.*

For fixed E, the case does not depend on the choice of $X_0 \in E$.

(b) *Let $\mathcal{H}$ be a three-dimensional subspace of $\mathbb{R}^{2\times 2}$ such that $\det(N) \neq 0$ for its normal N. Then either of the open sets $\{X \in \mathcal{H} \mid \det(X) > 0\}$ or $\{X \in \mathcal{H} \mid \det(X) < 0\}$ has two connected components while the other one is connected (cf. Fig. 3).*

Proof.

(a) Due to translation invariance we may assume $X_0 = 0$. Let $E = \operatorname{span}\{V_1, V_2\}$. If E is a rank-one plane (this happens iff $\det(V_1) = \det(V_2) = \det(V_1 + V_2) = 0$), case 1 clearly applies. For the other cases, let us choose a vector V_3 which is linearly independent of V_1 and V_2. We know from Lemma 3.1 that, for a generic choice of V_3, the intersection $\mathscr{R}_1(0) \cap \operatorname{span}\{V_1, V_2, V_3\}$ is a double cone. The conic sections which can occur as the intersection of this double cone and the plane E are exactly the remaining cases 2, 3, and 4 since the double cone and E intersect in 0. In case 2, the continuity of the determinant implies $(-1)^c \det(X) > 0$ for all $X \in E \setminus \{0\}$ with a fixed $c \in \mathbb{Z}_2$. The more general statement follows from the trivial observations $\det(X - Y) = \det((X_0 + X - Y) - X_0)$ and $X_0 + X + Y \in E$. A similar argument shows case 3. The respective claim in case 4 follows from an easy computation, $c = \det(W_1 + W_2)$.

(b) This follows from the continuity of the determinant, Lemma 3.1, and (a), or from a symmetry argument. $\square$

We recall from elementary analytic geometry that the zero set of

$$\frac{z_1^2}{a^2} + \frac{z_2^2}{b^2} - \frac{z_3^2}{c^2} = d$$

is a one-sheeted (connected) hyperboloid for $d > 0$, a double cone for $d = 0$, and a two-sheeted hyperboloid for $d < 0$. The following lemma is an easy consequence of the fact that double cones and one-sheeted hyperboloids are ruled varieties.

Lemma 3.4. *Let $\mathcal{H}$ be a three-dimensional linear subspace of $\mathbb{R}^{2\times 2}$ with normal N such that* $\det(N) \neq 0$*. Let $T_{\mathcal{H}} \colon \mathbb{R}^3 \to \mathcal{H}$ be the linear isomorphism of Lemma 3.1 with $T_{\mathcal{H}}(C) = \mathcal{H} \cap \mathscr{R}_1(0)$. Then the surface of the image of the one-sheeted hyperboloid $H_d := \{(z_1, z_2, z_3) \in \mathbb{R}^3 \mid z_1^2 + z_2^2 - z_3^2 = d > 0\}$ under $T_{\mathcal{H}}$ consists of rank-one lines.*

Proof. We use the same notation as in the proof of Lemma 3.1 and demonstrate the proof for the case $n_4 \neq 0$. In analogy to the proof of Lemma 3.1, where the case $d = 0$ is treated, one can see that for $d > 0$, a point in $T_{\mathcal{H}}(H_d)$ satisfies the relation

$$\frac{1}{2}(n_1 y_1^2 + n_2 y_1 y_2 + n_3 y_1 y_3 + n_4 y_2 y_3) = d.$$

Consequently, a line $l(t) = c + tr$ for $c \in T_{\mathcal{H}}(C)$, $r \in \mathcal{H}$ fixed, $t \in \mathbb{R}$, lies in the surface of the $T_{\mathcal{H}}(H_d)$, if and only if the quadratic polynomial

$$n_1(c_1 + tr_1)^2 + n_2(c_1 + tr_1)(c_2 + tr_2) + n_3(c_1 + tr_1)(c_3 + tr_3) + n_4(c_2 + tr_2)(c_3 + tr_3) - 2d \tag{8}$$

vanishes for all $t \in \mathbb{R}$. The constant term is zero because $c \in T_{\mathcal{H}}$. The coefficient of t^2 equals

$$n_1 r_1^2 + n_2 r_1 r_2 + n_3 r_1 r_3 + n_4 r_2 r_3,$$

hence (8) can be the zero polynomial only if $r \in T_{\mathcal{H}}(C) = \mathcal{H} \cap \mathscr{R}_1(0)$, *cf.* (6). The lemma follows now from the fact that every point on the surface of a one-sheeted hyperboloid lies on two lines belonging to two one-parameter families. The case $n_4 = 0$ is treated similarly. □

We now proceed to identify T_4-configurations with sets on hyperboloids in the three-dimensional affine subspace they span. These hyperboloids belong to the one-parameter family of quadrics which is induced by the double cone $\mathcal{H} \cap \mathscr{R}_1(P)$ $(P \in \mathcal{H})$. The following proposition prepares the ground for Theorem 3.8 by excluding some special cases (compare [3], Lem. 4 for a similar statement).

Proposition 3.5. *Let $K = \{X_1, \dots, X_4\} \subset \mathbb{R}^{2\times 2}$ be a set with sign diagram* (B) *(in particular, without rank-one connections). Then exactly one of the following three possibilities holds.*

(1) *K is contained in a two-dimensional affine subspace. In this case $K^{rc} = K$ holds.*
(2) *K is contained in a unique three-dimensional affine subspace $\mathcal{H}$ with normal N such that* $\det(N) = 0$*. In this case $K^{rc} = K$.*
(3) *K is contained in a unique three-dimensional affine subspace $\mathcal{H}$ with normal N such that* $\det(N) \neq 0$*. Furthermore, a unique $P \in \mathcal{H}$ and $d \in \mathbb{R}$ exist such that $K \subset P + T_{\mathcal{H}}(H_d)$ with $T_{\mathcal{H}}$ as in Lemma 3.1, where $H_d := \{(z_1, z_2, z_3) \in \mathbb{R}^3 \mid z_1^2 + z_2^2 - z_3^2 = d\}$. In this case,* $\det(P - X_j)$ *has the same value for all $X_j \in K$ for $j \in \{1, \dots, 4\}$.*

Proof.
Step 1. We show first that, if K is contained in a two-dimensional affine subspace, then its rank-one convex hull is trivial. Without loss of generality we may assume that the affine span of K contains the zero matrix. The possible conic section $\mathscr{J}$, say, of the rank-one cone $\mathscr{R}_1(0)$ and a two-dimensional subspace E are given by Corollary 3.3. If $\mathscr{J} = E$, then the elements of $K \subset E$ are pairwise rank-one connected. For $\mathscr{J} = \{0\}$, the elements of $E \setminus \{0\}$ have the same determinant (Cor. 3.3), which is not possible since K has sign diagram (B). The same holds true if $\mathscr{J}$ is a line.

Therefore, we only need to study the case of two intersecting lines $\operatorname{span}\{W_1\}$ and $\operatorname{span}\{W_2\}$. Figure 4 shows how X_1X_2 and X_3X_4 can be separated. We use some ideas of Matoušek and Plecháč [12], Section 5. For a

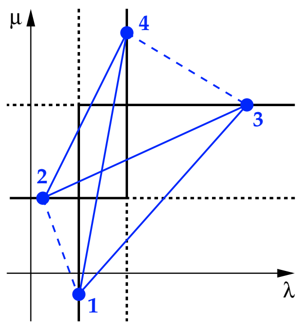

FIGURE 4. The matrices $X_1, \ldots, X_4$ can be separated as indicated in the figure. This follows from the fact that they have sign diagram (B) and $\det(\lambda W_1 + \mu_2 W_2) = \lambda\mu \det(W_1 + W_2)$, as shown in Corollary 3.3.

quadruple $X_1, \ldots, X_4$ with sign diagram (B), we can assume without loss of generality that the elements are labeled such that

$$\begin{aligned} \det(X_1 - X_2) &< 0, && (9)\\ \det(X_3 - X_4) &< 0, && (10)\\ \det(X_j - X_k) &> 0 \text{ for } (j,k) \in \{(1,3),(1,4),(2,3),(2,4)\}. && (11) \end{aligned}$$

Let $X_j := \lambda_j W_j 1 + \mu_j W_2$ and suppose $\det(W_1 + W_2) > 0$. Without loss of generality, we can assume $\lambda_1 > \lambda_2$. This implies $\mu_1 < \mu_2$ (see inequality (9)). Likewise, we assume $\lambda_3 > \lambda_4$ and $\lambda_4 > \lambda_1$, which implies $\mu_3 < \mu_4$, $\lambda_3 > \lambda_2$ and therefore $\mu_3 > \mu_2$. Now every point in the convex hull of $X_1, \ldots, X_4$ lies in one of the following open quadrants $\{\lambda < \lambda_1, \mu < \mu_2\}$, $\{\lambda > \lambda_1, \mu < \mu_3\}$, $\{\lambda < \lambda_4, \mu > \mu_2\}$, $\{\lambda > \lambda_4, \mu > \mu_3\}$, except for the union L of the line segments $[\lambda_2, \lambda_1] \times \{\mu_2\}$, $\{\lambda_1\} \times [\mu_1, \mu_2]$, $[\lambda_4, \lambda_3] \times \{\mu_3\}$, $\{\lambda_4\} \times [\mu_3, \mu_4]$. Rank-one convexity on E reduces to separate convexity in (λ, μ). Hence $K^{rc} \subset L$, and we may conclude $K = K^{rc}$ (*cf.* [10], Th. 3.11).

Step 2. Now suppose K is not contained in a plane. Let $\mathcal{H}$ be the unique three-dimensional affine subspace spanned by K, and denote its normal by N. We consider first the case $\det(N) = 0$. Without loss of generality we may assume $X_4 = 0$.

By Lemma 3.1, $\det(N) = 0$ implies that there exist rank-one matrices $V_1, V_2, V_3 \in \mathcal{H}$ such that $\mathcal{H} \cap \mathscr{R}_1(0) = \operatorname{span}\{V_1, V_3\} \cup \operatorname{span}\{V_2, V_3\}$. Since therefore $\operatorname{rank}(V_1 + V_3) = \operatorname{rank}(V_2 + V_3) = 1$, we obtain

$$\det(\lambda V_1 + \mu V_2 + \nu V_3) = \lambda\mu \det(V_1 + V_2).$$

Since the right-hand side is independent of V_3, all sign relations of K are invariant under the canonical projection to $\operatorname{span}\{V_1, V_2\} \subset \mathcal{H}$. The arguments from the planar case carry over, and we conclude that necessarily $K = K^{rc}$ whenever $\det(N) = 0$.

Step 3. We finally consider the general case $\det(N) \neq 0$. For a given matrix, let $(\xi_1, \ldots, \xi_4)$ be coordinates with respect to the basis $\{X_1, X_2, X_3, N\}$ of $\mathbb{R}^{2\times 2}$. Then the conditions for $\sum_{j=1}^{3} \xi_j X_j + \xi_4 N \in \mathbb{R}^{2\times 2}$ to lie in $\mathcal{H} \cap \mathscr{R}_1(0)$ read

$$\xi_4 = 0 \quad \text{and} \quad \det(\xi_1 X_1 + \xi_2 X_2 + \xi_3 X_3) = 0.$$

Hence by a straightforward computation, using the relation $a_1 b_4 + a_4 b_1 - a_2 b_3 - a_3 b_2 = \det(A) + \det(B) - \det(A - B)$,

$$\xi_4 = 0, \qquad \frac{1}{2}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}^T \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = 0, \tag{12}$$

with

$$S_{jj} = 2\det(X_j), \qquad S_{jk} = \det(X_j) + \det(X_k) - \det(X_j - X_k) \text{ for } j \neq k. \tag{13}$$

For fixed $p = (p_1, p_2, p_3) \in \mathbb{R}^3$ and fixed $d \in \mathbb{R}$, let us define

$$\mathcal{Y}(p,d) := \left\{ y \in \mathbb{R}^3 \;\middle|\; \tfrac{1}{2}(y-p)^T S(y-p) = d \right\}. \tag{14}$$

We remark that $\mathcal{Y}(p,d)$ is a cone (for $d = 0$) or a hyperboloid, centered at p; the hyperboloid is one-sheeted if $d > 0$ and two-sheeted if $d < 0$. The claim is that there exist $p \in \mathbb{R}^3$ and $d \neq 0$ such that

$$K \subset \left\{ y_1 X_1 + y_2 X_2 + y_3 X_3 \;\middle|\; y \in \mathcal{Y}(p,d) \right\} \subset \mathcal{H}.$$

This is equivalent to $(1,0,0), (0,1,0), (0,0,1), (0,0,0) \in \mathcal{Y}(p,d)$, *i.e.*, for these four vectors v_j

$$\frac{1}{2}(v_j - p)^T S(v_j - p) = d.$$

Spelling this out, with the new variable $q := d - \frac{1}{2}p^T Sp$, we obtain the linear system

$$\left(\begin{array}{ccc|c} & & & 1 \\ & S & & 1 \\ & & & 1 \\ \hline 0 & 0 & 0 & 1 \end{array} \right) \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ q \end{pmatrix} = \begin{pmatrix} \det(X_1) \\ \det(X_2) \\ \det(X_3) \\ 0 \end{pmatrix}. \tag{15}$$

An explicit computation shows $\det(S) = c\det(N)$ for some nonzero constant $c \in \mathbb{R}$, [6]. Since $\det(N) \neq 0$ this implies that (15) has a unique solution. In particular, $d = \frac{1}{2}p^T Sp$ and $p^T = S^{-1}(\det(X_1), \det(X_2), \det(X_3))^T$. The entries of the vector $\det(S)p$ are polynomials in the entries of X_1, X_2, X_3, hence the same holds for the entries of $P := p_1X_1 + p_2X_2 + p_3X_3$ and therefore for $\det(P - X_j)$. The claim that $\det(P - X_j)$ has the same value for all $X_j \in K$ follows from comparing the respective polynomials [6] (see [3] for a different argument). $\square$

Lemma 3.6. *Let $K = \{X_1, \ldots, X_4\}$ be a T_4-configuration with sign diagram* (B) *(compare Fig. 2) and P the center of the associated quadric from Proposition 3.5. Then*

$$P \in \operatorname{int} K^{co}.$$

Proof. We first show that $P \notin \partial K^{co}$. Assume the opposite, *i.e.*, that P is a convex combination of three elements X_1, X_2, X_3 of K. Then there is a (two-dimensional) plane $E \subset \mathcal{H}$ such that $P, X_1, X_2, X_3 \in E$. By inspecting the different cases of Corollary 3.3, we find that $E \cap \mathscr{R}_1(P)$ necessarily consists of two intersecting lines $P + \operatorname{span}\{W_1\}$ and $P + \operatorname{span}\{W_2\}$. By Proposition 3.5, $\det(P - X_j)$ has the same sign for all $j = 1, \ldots 4$, and from Theorem 2.3 we conclude that $\det(P - X_j) < 0$. We may assume without loss of generality (replacing W_j by $-W_j$ if necessary) that $\det(\lambda_1 W_1 + \lambda_2 W_2 - P) < 0$ for $\lambda_1\lambda_2 < 0$ and $\det(\lambda_1 W_1 + \lambda_2 W_2 - P) > 0$ for $\lambda_1\lambda_2 > 0$. Since $P \in \{X_1, X_2, X_3\}^{co}$, we can assure (by relabeling if necessary) that $X_1 = P + \lambda_1^{(1)} W_1 + \lambda_2^{(1)} W_2$ with $\lambda_1^{(1)}, \lambda_2^{(1)} < 0$, while for $j = 2, 3$, we have $X_j = P + \lambda_1^{(j)} W_1 + \lambda_2^{(j)} W_2$ with $\lambda_1^{(j)}, \lambda_2^{(j)} > 0$ (or *vice versa*, compare Fig. 5). In any case, $\det(X_1 - X_2) < 0$ and $\det(X_1 - X_3) < 0$. This contradicts the assumption that K has sign diagram (B).

Suppose now $P \notin K^{co}$. Theorem 2.3 implies that there exists a $Q \in K^{co}$ such that $\det(Q - X_j) < 0$ for all $X_j \in K$. We know from Proposition 3.5 that $\det(P - X_j) < 0$ for all $X_j \in K$. The set $\mathcal{Y} := \{Y \in \mathbb{R}^{2\times 2} \mid \det(Y - X_j) < 0 \ \forall\ X_j \in K\}$ is path-connected. Hence, a curve $\gamma\colon [0,1] \to \mathcal{Y}$ exists such that $\gamma(0) = P$ and $\gamma(1) = Q$. We conclude that there exists a $Y_0 \in \gamma([0,1]) \cap \partial K^{co}$. However, the first part of this proof carries over to Y_0 instead of P. This rules out the existence of $Y_0 \in \partial K^{co}$, and we arrive at a contradiction. $\square$

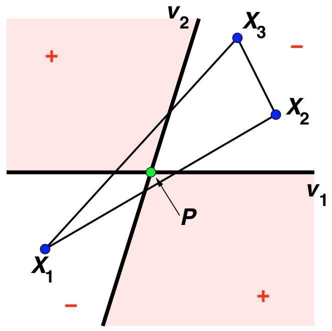

FIGURE 5. Illustration to the proof of Lemma 3.6.

Proposition 3.7. *Let $K = \{X_1, \dots, X_4\}$ be a sixfold or a degenerate T_4-configuration. In the situation of Proposition 3.5, let $P = 0$ and $A \in \mathcal{H}$ be such that* $\operatorname{span}\{A\}$ *is the axis of rotation of the quadric $T_{\mathcal{H}}(H_d)$.*

Then the two sets $K \cap \{Y \mid \langle Y, A\rangle_{\mathbb{R}^4} > 0\}$ and $K \cap \{Y \mid \langle Y, A\rangle_{\mathbb{R}^4} > 0\}$ contain two points each.

Proof. By Corollary 3.3 (b), either $\{Y \in \mathcal{H} \mid \det(Y) < 0\}$ or $\{Y \in \mathcal{H} \mid \det(Y) < 0\}$ contains exactly two connected components. To fix the notation, we discuss here the case where $\mathcal{H}$ is such that $\{Y \mid \det(Y) > 0\}$ has two connected components. With the notation of Proposition 3.5, we will have to distinguish the cases $d > 0$ (H_d being a one-sheeted hyperboloid), $d = 0$ (cone), and $d = 0$ (two-sheeted hyperboloid).

Case A: $d > 0$. We have $\det(X) < 0$ for all $X \in T_{\mathcal{H}}(H_d)$. Since by Lemma 3.6 $P = 0 \in K^{co}$, the sign diagram will (after renumbering $X_1, \dots, X_4$ if necessary) due to Theorem 2.3 (2) be as in Figure 2.

Step A-1. Assume first that $K \cap \{Y \mid \langle Y, A\rangle_{\mathbb{R}^4} > 0\}$ contains at least three points X_1, X_2, X_3. Then these three points span a two-dimensional affine plane E. We investigate the possible cases of Corollary 3.3 (a). Obviously, case 1 cannot occur. In case 2, $\det(X_j - X_k)$ has the same sign for all $1 \le j < k \le 3$, contradicting sign diagram (B). Case 3 cannot occur for the same reason.

In case 4, we have to consider four subcases. Let $Z \in E$ be the best approximation of 0 in E.

- If $\lambda Z \in T_{\mathcal{H}}(H_d)$ with $0 < \lambda < 1$, then we are in the situation of Figure 6(a). X_1, X_2, X_3 are on the top branch of the hyperbola $E \cap T_{\mathcal{H}}(H_d)$, hence $\det(X_j - X_k) < 0$ for $1 \le j < k \le 3$. This contradicts the assumption that we have a sign diagram of type (B).
- If $Z \in T_{\mathcal{H}}(H_d)$ (Fig. 6(b)), then $E \cap T_{\mathcal{H}}(H_d)$ consists of two intersecting rank-one lines (compare Lem. 3.4), and at least two of the three matrices X_1, X_2, X_3 must be rank-one connected.
- Suppose that $\lambda Z \in T_{\mathcal{H}}(H_d)$ with $\lambda > 1$ and that all three X_j lie on the same branch of the hyperbola $E \cap T_{\mathcal{H}}(H_d)$. We are then in the situation of Figure 6(c). Then $\det(X_j - X_k) > 0$ for $1 \le j < k \le 3$, which is impossible for a sign diagram of type (B).
- Suppose that $\lambda Z \in T_{\mathcal{H}}(H_d)$ with $\lambda > 1$ and that only X_1 and X_2 lie on the same branch of the hyperbola $E \cap T_{\mathcal{H}}(H_d)$. We are then in the situation of Figure 6(d). Since $\det(A - B) < 0$ whenever A and B lie on distinct branches of this hyperbola, we find $\det(X_2 - X_1) < 0$, $\det(X_3 - X_1) < 0$. Together with $\det(X_3) < 0$ this is a contradiction to the criterion in Theorem 2.3 (2).

Step A-2. Assume now that $X_1 \in \{Y \mid \langle Y, A\rangle_{\mathbb{R}^4} = 0\}$. That is, at least one point lies on the horizontal symmetry plane of the hyperboloid $T_{\mathcal{H}}(H_d)$. The presence of sign diagram (B) means that for exactly two matrices $X_2, X_3 \in K$, $\det(X_1 - X_2)$ and $\det(X_1 - X_3)$ have the same sign. As above in **A-1**, we conclude $\det(X_1 - X_j) > 0$ for $j = 2, 3$ from the criterion in Theorem 2.3 (2). The matrices $X_1, \dots, X_3$ span a two-dimensional affine hyperplane E. The cases 1–3 of Corollary 3.3 can be eliminated as above. In case 4, we

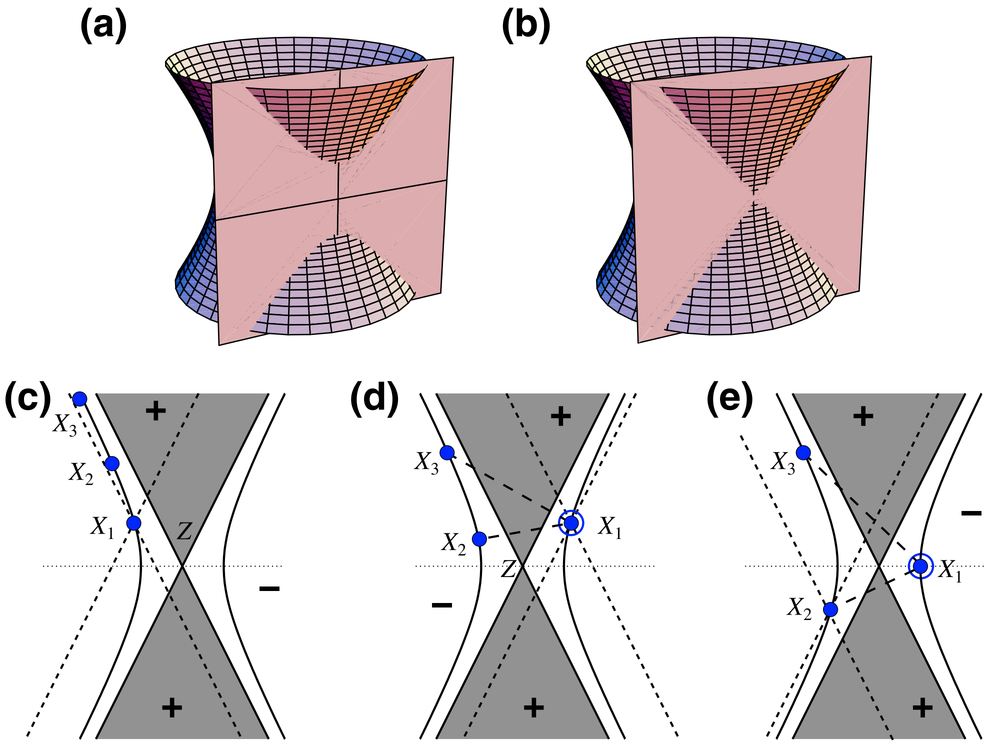

FIGURE 6. Illustration to the proof of Proposition 3.7.

have to consider only two subcases, since the situation of Figure 6a cannot occur. Let Z be as above the best approximation of 0 in E.

- If $Z = X_1$ then $E \cap T_{\mathcal{H}}(H_d)$ consists of two intersecting rank-one lines, hence X_1 is rank-one connected to X_2, X_3.
- If $\lambda Z \in T_{\mathcal{H}}(H_d)$ for some $\lambda > 1$ then $E \cap T_{\mathcal{H}}(H_d)$ is a hyperbola, and because of $\det(X_1 - X_j) > 0$ ($j = 2, 3$) these three matrices lie on the same branch of the hyperbola. However, it follows that $\det(X_2 - X_3) > 0$, which is a contradiction to sign diagram (B), compare Figure 6e.

Case B: $d = 0$ (cone). This means $\det(X_j) = 0$ for all $X_j \in K$. Clearly $\langle X_j, A\rangle_{\mathbb{R}^4} \neq 0$ for all $X_j \in K$ since $\{Y \in \mathcal{H} \mid \langle Y, A\rangle_{\mathbb{R}^4} = 0\} \cap T_{\mathcal{H}}(H_d) = \{0\}$.

Assume $K \cap \{Y \mid \langle Y, A\rangle_{\mathbb{R}^4} > 0\}$ contains at least three points X_1, X_2, X_3. Then K^{co} contains $P = 0$ [if and] only if $\{X_1, X_2, X_3\}^{co} \cap \operatorname{span}\{X_4\} \neq \emptyset$. However, the fact that X_1, X_2, X_3 lie on a cone implies $X_j \in \operatorname{span}\{X_4\}$ for some $j \in \{1, 2, 3\}$. This contradicts the fact that $\det(X_j - X_k) \neq 0$ for $j \neq k$.

Case C: $d < 0$ (two-sheeted hyperboloid). (We will see in Theorem 3.8 that this case cannot occur.) In this case we have $\det(X_j) > 0$ for all $X_j \in K$. If $K \subset \{Y \in \mathcal{H} \mid \langle Y, A\rangle_{\mathbb{R}^4} > 0\}$ then clearly $P = 0 \notin K^{co}$, contradicting Proposition 3.6. Suppose $X_1, X_2, X_3 \in K \cap \{Y \in \mathcal{H} \mid \langle Y, A\rangle_{\mathbb{R}^4} > 0\}$ and $\langle X_4, A\rangle_{\mathbb{R}^4} < 0$. Then $\det(X_4 - X_j) > 0$ for $j = 1, 2, 3$, contradicting sign diagram (B). Note furthermore $T_{\mathcal{H}}(H_d) \cap \{Y \in \mathcal{H} \mid \langle Y, A\rangle_{\mathbb{R}^4} = 0\} = \emptyset$. $\square$

Theorem 3.8. *Let $K = \{X_1, \dots, X_4\} \subset \mathbb{R}^{2\times 2}$ be a set with sign diagram* (B) *that is not contained in a plane. Denote by $\mathcal{H}$ the three-dimensional affine subspace containing K. The following are equivalent:*

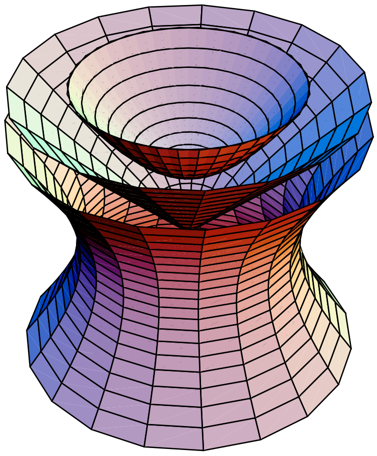

FIGURE 7. One-sheeted, two-sheeted hyperboloid and cone with similar parameters.

(a) *K is a sixfold T_4-configuration (degenerate T_4-configuration, respectively).*
(b) *There exist unique $P \in \mathcal{H}$ and $d > 0$ ($d = 0$, respectively) such that $P \in K^{co}$ and $K \subset P + T_{\mathcal{H}}(H_d)$ with $T_{\mathcal{H}}$ from Lemma 3.1 and with the one-sheeted hyperboloid $H_d := \{(z_1, z_2, z_3) \in \mathbb{R}^3 \mid z_1^2 + z_2^2 - z_3^2 = d\}$. (Note that the surface of the hyperboloid $T_{\mathcal{H}}(H_d)$ consists of rank-one lines.)*

Proof. By Corollary 3.3 (b), either $\{Y \in \mathcal{H} \mid \det(Y) < 0\}$ or $\{Y \in \mathcal{H} \mid \det(Y) < 0\}$ contains exactly two connected components. We consider here the case where $\mathcal{H}$ is such that $\{Y \in \mathcal{H} \mid \det(Y) > 0\}$ has two connected components; the rest follows by multiplication by $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Assume for a contradiction that the quadric H_d from Proposition 3.5 is a two-sheeted hyperboloid, *i.e.*, $d < 0$. By Proposition 3.5, we have $\det(X_j - P) =: c > 0$ for all $X_j \in K$. Define $f\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ by $f(X) := \max\{c - \det(X - P), 0\}$. This function f is rank-one convex and nonnegative, and $K \subset f^{-1}(0)$, hence $K^{rc} \subset f^{-1}(0) \cap \mathcal{H}$. But this set has two connected components, and by Proposition 3.7, each contains two elements of K. In particular, K cannot be a sixfold or degenerate T_4-configuration.

Now let K be a sixfold T_4-configuration. Clearly $d \neq 0$, since $d = 0$ means $\det(P - X_j) = 0$ for all $X_j \in K$, hence K is possibly a degenerate, but not a sixfold T_4. We conclude $d > 0$. The claim $P \in K^{co}$ has been proven in Lemma 3.6, the rest of (a)⇒(b) for T_4-configurations follows from Proposition 3.5.

A degenerate T_4-configuration is characterized by the existence of a $Y \in K^{co}$ with $K \subset \mathscr{R}_1(Y)$. Since necessarily $Y \in \mathcal{H}$ and $\mathscr{R}_1(Y) = (Y - P) + \mathscr{R}_1(P)$ we conclude $Y = P$ and $d = 0$. This shows (a)⇒(b) for degenerate T_4-configurations.

The implication (b)⇒(a) follows in both cases from Theorem 2.3. □

4. Topological structure in $(\mathbb{R}^{2\times 2})^4$

In this section, we investigate the number of connected components of $\mathcal{T}_4(\mathbb{R}^{2\times 2})$ in $(\mathbb{R}^{2\times 2})^4$ and their boundaries.

We start with a technical lemma about sign diagrams.

Lemma 4.1. *For $j, k \in \{1, \ldots, 4\}$, let $\lozenge_{jk} \in \{<, >\}$. To represent a given sign diagram, we define*

$$\mathcal{D}(\lozenge_{jk})_{jk} := \left\{(X_1, \ldots, X_4) \,\middle|\, \det(X_j - X_k) \ \lozenge_{jk} \ 0 \ \forall\ j, k \in \{1, \ldots, 4\}\right\}.$$

Then $\mathcal{D}(\lozenge_{jk})_{jk}$ is connected as a subset of $(\mathbb{R}^{2\times 2})^4$.

Proof. We first remark that we can without loss of generality assume $X_1 = 0$. Indeed, by setting $\mathcal{D}_0 := \mathcal{D}(\lozenge_{jk})_{jk} \cap \{(0, X_2, \ldots, X_4) \,|\, 0, X_2, \ldots, X_4 \in \mathbb{R}^{2\times 2}\}$, we recover $\mathcal{D}(\lozenge_{ij})_{jk}$ as

$$\mathcal{D}(\lozenge_{jk})_{jk} = \bigcup_{X_1 \in \mathbb{R}^{2\times 2}} \mathcal{D}_0 + (X_1, X_1, X_1, X_1). \tag{16}$$

To consider the differences $X_j - X_k$, we introduce the map

$$F\colon \begin{cases} \mathcal{D}_0 \to (\mathbb{R}^{2\times 2})^6 \\ F\colon (0, X_2, \ldots, X_4) \longmapsto (X_2, X_3 - X_2, X_4 - X_3, X_4, X_3, X_4 - X_2), \end{cases}$$

which is continuous and injective. The inverse map $F^{-1}\colon F(\mathcal{D}_0) \to \mathcal{D}_0$ is continuous as well, since $F^{-1}(M_1, \ldots, M_6) = (0, M_1, M_5, M_4)$. Thus, F is a homeomorphism.

Given that $\mathrm{GL}^{\pm}(2, \mathbb{R}) := \{M \in \mathbb{R}^{2\times 2} \,|\, \det M \gtrless 0\}$ are connected, the same holds for $F(\mathcal{D}_0)$. Hence $\mathcal{D}_0$ is connected as preimage under a homeomorphism. The claim follows now from (16). □

Proposition 4.2. *$\mathcal{T}_4(\mathbb{R}^{2\times 2})$ is the union of 12 connected components of simple T_4-configurations and 6 connected components of sixfold T_4-configurations.*

Proof. As there are six connections in every sign diagram, there are obviously $2^6 = 64$ different sign diagrams. Elementary bookkeeping shows that they constitute 6 equivalence classes. These classes are shown in the left and the middle panel of Figure 8. The equivalence class of (A) comprises 12 sign diagrams, and every simple T_4-configuration corresponds to one of these. We conclude from Lemma 4.1 that $\mathcal{T}_4(\mathbb{R}^{2\times 2})$ contains 12 connected components of simple T_4-configurations.

The equivalence class of (B) consists of 6 sign diagrams. Since the sign diagram (B) describes, according to Proposition 2.3, sixfold T_4-configurations as well as degenerate T_4-configurations and sets with trivial rank-one convex hull, there is no direct correspondence between the sign diagram (B) and sixfold T_4-configurations. The claim for diagrams of type (B) follows from the geometric characterization of the (B)-components stated in Theorem 3.8. □

Proposition 4.3. *The boundary of the* (A)*-components consists of quadruples* $(X_1, \ldots, X_4)$ *with at least one rank-one connection. That is to say, for every quadruple belonging to the boundary of an* (A)*-component, indices $j \neq k$ exist with* $\det(X_j - X_k) = 0$.

The boundary of the (B)*-components consists of tuples with rank-one connections and of the set of all degenerate T_4-configurations.*

Proof. Obviously, the set of all tuples corresponding to a specific sign diagram is bounded by tuples without sign diagram, *i.e.*, tuples with at least one rank-one connection. This finishes the proof for the (A)-components. The statement for the (B)-components again follows from the geometric picture given in Theorem 3.8. □

Example 4.4. We now give examples for different kinds of boundaries. This will show that all the boundaries listed in Proposition 4.3 are indeed nonempty.

(1) *The boundary of an* (A)*-component and the set of quadruples with trivial rank-one convex hull:* The four-tuple of

$$X_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 1 & 2 \\ 1-\varepsilon & 2 \end{pmatrix}, X_3 = \begin{pmatrix} 5 & 1 \\ 2 & 1 \end{pmatrix}, X_4 = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix},$$

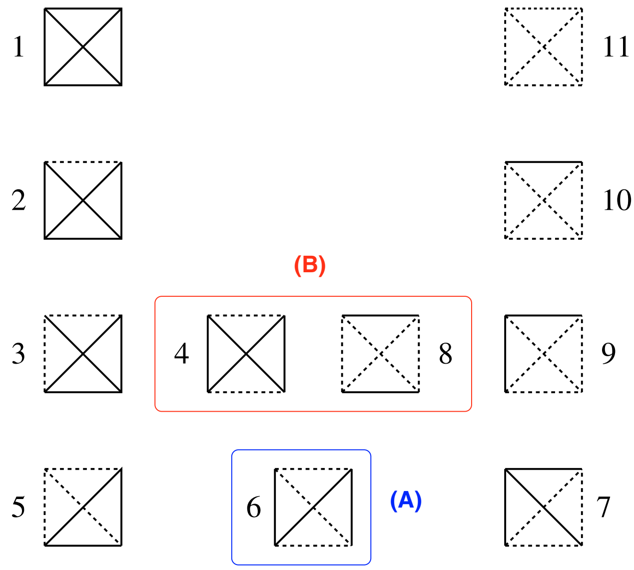

FIGURE 8. The left and the middle panel exhibit the different equivalence classes of sign diagrams. The sign diagrams in right panel are generated from the ones in the left panel and the same row by inverting all the edges. Therefore, they are another representative of the same equivalence class as the corresponding diagram in the left panel. The numbering is referred to in Section 5.

is a simple T_4-configuration for $\varepsilon > 0$ and therefore belongs to an (A)-component (note this T_4-configuration is non-planar, unlike the classical Tartar example). For $\varepsilon = 0$, we have $\det(X_1 - X_2) = 0$. For $\varepsilon < 0$, the sign diagram is given by the bottom picture in the left panel in Figure 8. Hence, for $\varepsilon < 0$, case 3 of Theorem 2.3 applies.

(2) *The boundary of an* (A)*-component and a* (B)*-component:* The four-tuple of

$$X_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, X_3 = \begin{pmatrix} 5 & 1 \\ 2 & 1 \end{pmatrix}, X_4 = \begin{pmatrix} 1 & -1 \\ -\varepsilon & -1 \end{pmatrix},$$

is a simple T_4-configuration for $\varepsilon > 0$ and contains a rank-one connection for $\varepsilon = 0$, namely $\det(X_2 - X_4) = 0$. For $\varepsilon < 0$, we find a sign diagram of type (B), and Theorem 2.3 shows that the rank-one convex hull is trivial, as case 2 applies.

(3) *The boundary of an* (A)*-component and a* (B)*-component of sixfold* T_4*-configurations:* The four-tuple of

$$X_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 2 & 0 \\ 2 & -5 \end{pmatrix}, X_3 = \begin{pmatrix} -1 & 5 \\ -2 & 0 \end{pmatrix}, X_4 = \begin{pmatrix} 1 & 1 \\ -5+\varepsilon & -5 \end{pmatrix},$$

is, as the previous example, a simple T_4-configuration for $\varepsilon > 0$ and has a rank-one connection ($\det(X_1 - X_4) = 0$) for $\varepsilon = 0$. For $\varepsilon < 0$, it belongs to the sign diagram (B). One can check that the tuple is a sixfold T_4-configuration for $\varepsilon < 0$.

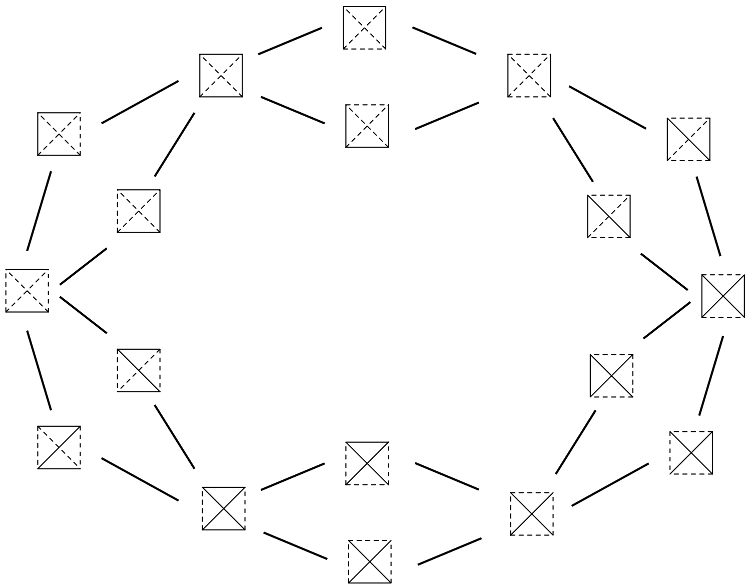

FIGURE 9. The geometric arrangement of the connected components of (A) and (B), marked with their sign diagram. Components linked by a line share a common boundary.

(4) *A degenerate T_4-configuration as a boundary point of a* (B)*-component of sixfold T_4-configurations:* The set

$$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \varepsilon & -1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} \right\},$$

belongs to the sign diagram type (B) for small ε. For $\varepsilon < 0$, we find a sixfold T_4-configuration, for $\varepsilon = 0$ a degenerate T_4-configuration, and for $\varepsilon > 0$ the rank-one convex hull is trivial. This example was found while performing experiments as described in Section 5.

We note for completeness that the set of all degenerate T_4-configurations is not open.

Corollary 4.5. *The set*

$$\mathcal{K}(\mathbb{R}^{2\times2}) := \{(X_1, \dots, X_4) \in (\mathbb{R}^{2\times2})^4 \mid \{X_1, \dots, X_4\} \text{ is a degenerate } T_4\text{-configuration}\}$$

has no interior points with respect to the natural topology of $(\mathbb{R}^{2\times2})^4$.

Proof. This follows immediately from Theorem 3.8 and Lemma 3.6. □

We close this section with a somewhat surprising, though simple, result. This answers a question posed by Daniel Faraco (personal communication). The question is whether the different connected components of type (A) and type (B) are arranged in a ring-like structure. The answer is positive.

Proposition 4.6. *The six connected components of type* (B) *and and the twelve connected components of type* (B) *are arranged in a ring-like structure as shown in Figure 9. Neighboring components, i.e., components with a common boundary, are linked by a line. For every component of type* (B), *there are exactly four neighboring components of type* (A).

Proof. These statements are easily seen by simple bookkeeping. To see that linked components do share a common boundary, one can use the previous examples and symmetry arguments. □

5. Detection of T_4-configurations

One application of the results presented in Section 4, in combination with [18], is an algorithm to answer the following question: Given $k \geq 4$ matrices $X_1, \ldots, X_k \in \mathbb{R}^{m\times n}$ without rank-one connections (*i.e.*, $\operatorname{rank}(X_i - X_j) \geq 2$ for $i \neq j$), do they form a T_k-configuration? This problem was posed in [9], Section 8. In [11], an efficient algorithm is presented for the general case $k \geq 4$ and arbitrary $m, n \geq 2$, based on algebraic geometry. For special case $k = 4$ and $m = n = 2$, we now present a substantially improved algorithm. However, unlike the algorithm in [11], it does not generalize to higher space dimensions or general T_k-configurations.

The algorithm can be formulated as follows.

Algorithm 5.1.

Input: *A set* $K := \{X_1, \ldots, X_4\} \subset \mathbb{R}^{2\times 2}$ *without rank-one connections.*
Procedure: (1) *Compute the sign diagram for* K.
(2) *If the sign diagram is neither of type* (A) *nor of type* (B)*, then* K *is not a* T_4*-configuration.*
(3) *If the sign diagram is of type* (A)*, then* K *is a simple* T_4*-configuration.*
(4) *If the sign diagram is of type* (B)*, then*
  (a) *Solve the linear system given by Equation* (15) *for* $p = (p_1, p_2, p_3)^T$ *and determine* d *via* $d = \frac{1}{2}p^T S p$.
  (b) *If* $p \notin \{(1,0,0), (0,1,0), (0,0,1), (0,0,0)\}^{co}$*, then* K *is not a* T_4*-configuration. Otherwise, the following possibilities exist:*
    – *If* $d > 0$*, then* K *is a sixfold* T_4*-configuration.*
    – *If* $d = 0$*, then* K *is a degenerate* T_4*-configuration.*
    – *If* $d < 0$*, then* K *is not a* T_4*-configuration.*

Output: T_4*-configuration are detected in Steps (3) or (4). Other configurations are rejected in one of the steps (2) to (4).*

Proof. The correctness of the algorithm follows immediately from [18] and Theorem 3.8. The condition $p \in \{(1,0,0), (0,1,0), (0,0,1), (0,0,0)\}^{co}$ is equivalent to $P \in K^{co}$. □

We use Algorithm 5.1 to investigate the frequency of T_4-configurations in $\left(\mathbb{R}^{2\times 2}\right)^4$. More precisely, we are interested in the measure of the set of T_4-configurations in the unit cube of $\left(\mathbb{R}^{2\times 2}\right)^4$ with respect to the $\|\,.\,\|_\infty$ norm. We use a Monte Carlo method by applying Algorithm 5.1 to one billion randomly chosen points in $\left(\mathbb{R}^{2\times 2}\right)^4$. For an implementation in MATLAB on an Intel Pentium 4 processor with 2.66 GHz CPU speed and 1GB memory, 100 000 tests took about 20 seconds. This compares to 8–10 seconds for 1 test with the algorithm in [11].

The results are recorded in Table 1. Their accuracy can be checked by exploiting the symmetry of the sign diagrams. Namely, sign diagrams that arise from each other by inversion of edges (*e.g.*, 1 and 11, or 4 and 8), have equal probability. Indeed, Table 1 shows that their frequency agrees to a high level of accuracy.

The following observations can be made. Firstly, T_4-configurations are not as exotic as one might assume at a first glance. Indeed, about 8.92 percent of all configurations were found to be a T_4. Secondly, the vast majority of T_4-configurations is simple (98.0%). Thirdly, T_4-configurations are rare among those configurations with sign diagram (B) (3.2%).

TABLE 1. Results of the Monte Carlo computation. The numbering of the sign diagrams corresponds to Figure 8.

#	Comment	# Random Points	Frequency (%)
0	With rank-one connection	0	0.00
1		48 034 773	4.80
2		131 975 267	13.20
3		169 470 978	16.95
4	Sign diagram (B)	28 375 290	2.84
	thereof sixfold T_4-config.	909 395	0.09
	thereof degen. T_4-config.	0	0.00
5		78 443 346	7.84
6	Sign diagram (A) (simple T_4)	87 362 467	8.74
7		78 445 984	7.84
8	Sign diagram (B)	28 382 134	2.84
	thereof sixfold T_4-config.	907 839	0.09
	thereof degen. T_4-config.	0	0.00
9		169 486 545	16.95
10		131 990 504	13.20
11		48 032 712	4.80
	Sum	1 000 000 000	
	Total # of simple T_4	87 362 467	8.74
	Total # of sixfold T_4	1 817 234	0.18
	Total # of degenerate T_4	0	0.00
	Total # of T_4	89 179 701	8.92

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