

ON THE CIRCLE CRITERION FOR BOUNDARY CONTROL SYSTEMS IN FACTOR FORM: LYAPUNOV STABILITY AND LUR'E EQUATIONS

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Abstract. A Lur'e feedback control system consisting of a linear, infinite-dimensional system of boundary control in factor form and a nonlinear static sector type controller is considered. A criterion of absolute strong asymptotic stability of the null equilibrium is obtained using a quadratic form Lyapunov functional. The construction of such a functional is reduced to solving a Lur'e system of equations. A sufficient strict circle criterion of solvability of the latter is found, which is based on results by Oostveen and Curtain [*Automatica* **34** (1998) 953–967]. All the results are illustrated in detail by an electrical transmission line example of the distortionless loaded \mathfrak{RLCG} -type. The paper uses extensively the philosophy of reciprocal systems with bounded generating operators as recently studied and used by Curtain in (2003) [*Syst. Control Lett.* **49** (2003) 81–89; *SIAM J. Control Optim.* **42** (2003) 1671–1702].

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1. INTRODUCTION

This paper uses some results of abstract linear systems in factor form, obtained by the authors in earlier papers [14, 16, 17], and surveyed and sharpened in Section 2; these systems are related but not identical to well-posed linear systems (formerly called Weiss–Salamon abstract linear systems) [37, 41], see [14], Section 4.5, [16], Section 7. More precisely, the paper mainly considers SISO systems of boundary control in factor form [16], described by

$$\left\{ \begin{array}{l} \dot{x}(t) = A[x(t) + du(t)] \\ y = c^\# x \end{array} \right\}, \quad (1.1)$$

where it is assumed that $A : (D(A) \subset H) \rightarrow H$ generates a linear exponentially stable (EXS), C_0 -semigroup $\{S(t)\}_{t \geq 0}$ on a Hilbert space H with a scalar product $\langle \cdot, \cdot \rangle_H$, $d \in H$ is a factor control vector, $u \in L^2(0, \infty)$ is a scalar control function, y is a scalar output defined by an A -bounded linear observation functional $c^\#$ (bounded on D_A , *i.e.* the space $D(A)$ equipped with the graph norm of A , here equivalent to $\|x\|_A := \|Ax\|_H$). The restriction of $c^\#$ to $D(A)$ is representable as $c^\#x = \langle h, Ax \rangle_H \quad \forall x \in D(A)$ for some $h \in H$, or shortly $c^\#|_{D(A)} = h^*A$. A precise connection to well-posed linear systems needs more assumptions, and will be provided at the end of Section 2 in Remark 2.3. It must however be stressed that the authors do not follow the approach in [41] or [39]: given a well-posed linear system (the four maps: state to state, input to state,

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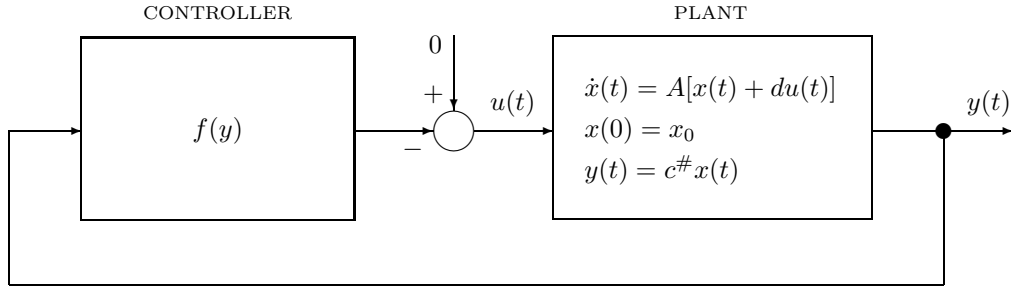


FIGURE 1.1. The Lur'e control system.

state to output, and input-output map) then one gets generating operators A, B, C and results. Here one uses sufficient assumptions permitting the use of the four former maps of a well-posed linear system and results, as they are created by these assumptions, and nothing more. Moreover system (1.1) is also related to hyperbolic boundary control systems as in [21, 23, 35], see Remark 2.2.

Section 2 on preliminaries starts by recalling in Section 2.1 the definition of (infinite-time) admissibility of an unbounded observation operator, and the Lyapunov equation solvability criterion of admissibility, see Lemma 2.1. We discuss bounded observation or control operators, the latter being handled by duality. In Section 2.2 the general admissibility of Section 2.1 is reduced to that of the observation functional of SISO open-loop system (1.1), and it is shown that modulo a unitary transformation on $L^2(0, \infty)$, admissibility can be reduced to admissibility involving bounded system operators. In Appendix A one takes into account that system (1.1) has an exponentially stable C_0 -semigroup $\{S(t)\}_{t \geq 0}$, whence its generator A has an inverse $A^{-1} \in \mathbf{L}(\mathbf{H})$, which generates the semigroup $\{e^{tA^{-1}}\}_{t \geq 0}$. Its strong asymptotic stability is investigated in this Appendix and turns out to be a reasonable assumption; specialized equivalent criteria indicate applicability in examples. In Section 2.3 classical and weak solutions of the state differential equation of (1.1) *i.e.*

$$\dot{x}(t) = A[x(t) + du(t)], \quad x(0) = x_0, \tag{1.2}$$

are derived, based on a generalization of Phillip's theorem [36], Corollary 2.11, p. 109. In particular, it is shown that if the control belongs to $D(R^2)$ (*i.e.* the domain of the square of the generator R of the right-shift semigroup on $L^2(0, \infty)$) and the initial state x_0 is in $D(A)$ (*i.e.* the domain of the state system operator A), then (1.2) has an explicit (unique) classical solution, see equation (2.4) in Lemma 2.8 below. This result is used next in Section 2.4 to explain the input-output map of (1.1) originally defined for $x_0 \in D(A)$ and $u \in D(R^2)$ and then extended to any initial state in \mathbf{H} and any control in $L^2(0, \infty)$. This is achieved by assuming compatibility condition (2.6) *i.e.* $d \in D(c^\#)$, admissibility of the observation functional of (1.1) and requiring that \hat{g} (the Laplace transform of its impulse response g) is a bounded analytic function on Π^+ (the right complex half-plane). See Lemma 2.10 for the extended input-output map \overline{F} and the system transfer function \hat{g} , (this lemma abbreviates some results of [16]). Section 2 ends with Section 2.5 where the notion of admissible factor control vector d of (1.1) is defined (using duality with respect to an unbounded observation functional). This property is sufficient for differential equation (1.2) to have (a unique) weak solution for any initial state in \mathbf{H} and any control in $L^2(0, \infty)$, see Lemma 2.11.

The results of Section 2 combined with the input-output approach using *passivity* or *contractivity* lead in, respectively, [13, 17] to a circle criterion for the nonlinear Lur'e type feedback system described by Figure 1.1¹, consisting of a linear infinite-dimensional system of boundary control in factor form followed by a nonlinear static sector type controller in the loop. Such a system can be modelled by the nonlinear abstract differential equation

$$\dot{x}(t) = A \{x(t) - df [c^\# x(t)]\}$$

¹ For reasons of traditional conformity, we have, as in [40], Section 5.6, Theorem 37 Case (iii), p. 227, reintroduced the standard sign inversion node between the controller and the plant.

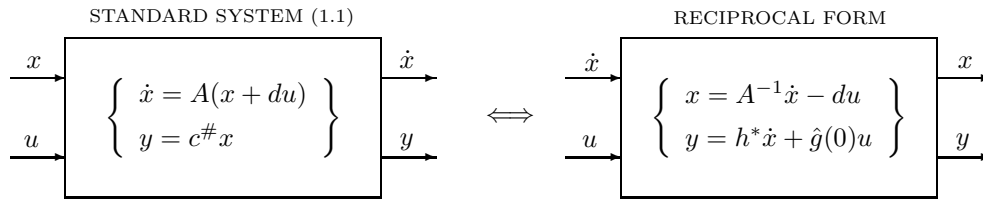


FIGURE 1.2. The original system and its reciprocal form.

where $A : (D(A) \subset H) \rightarrow H$ generates a linear C_0 -semigroup on H , $d \in H$ is a factor control vector, $f : \mathbb{R} \rightarrow \mathbb{R}$ and $c^\#$ is A -bounded linear functional (bounded on D_A). It is commonly known that in the finite-dimensional SISO case the circle criterion can also be derived using a quadratic Lyapunov functional $V(x) = x^T \mathcal{H}x$, $\mathcal{H} = \mathcal{H}^T \in \mathbf{L}(\mathbb{R}^n)$. Usually its derivative $\dot{V} = \dot{x}^T \mathcal{H}x + x^T \mathcal{H}\dot{x}$ along solutions of the Lur'e system $\dot{x} = Ax - bf(c^T x)$ is being represented as a quadratic form with respect to an extended vector $[x \ f]^T$. The sector conditions imposed on f can be expressed as an extended quadratic form too. From the requirement $\dot{V} \leq 0$ one gets the so-called Lur'e resolving system of equations. Thus determination of an unknown matrix $\mathcal{H} > 0$ reduces to solving of the latter. This standard reduction leads to a variety of solvability results commonly known as the Kalman-Popov lemma and the Yacubovič frequency-domain theorem. Global asymptotic stability of the origin is usually deduced either from LaSalle's invariance principle or by showing that control and/or output are $L^2(0, \infty)$ -functions. Since global asymptotic stability is independent of a particular form of a nonlinearity satisfying the sector conditions it was named absolute stability². E. Noldus [28–30] was probably the first who noticed that if A is invertible then one can write $x = A^{-1}\dot{x} - A^{-1}bf(c^T x)$ instead of \dot{x} when calculating \dot{V} and representing it as a quadratic form of $[\dot{x} \ f]^T$. An important observation is that this fits the philosophy of reciprocal systems as recently studied by Curtain [6, 7]. Its essence here is to use the so-called *reciprocal system* expressed in terms of bounded operators exclusively instead of the original system which contains unbounded operators. The reciprocal system arises from the standard one by expressing x and y in terms of \dot{x} and u as explained in the reciprocal form of Figure 1.2. The reciprocal system is then obtained by exchanging x and $-\dot{x}$ giving the reciprocal system parameters $[A^{-1}, d, -h^*, \hat{g}(0)]$.

The present paper explains how Lyapunov state space theory together with the abstract results of Section 2 can give similar stability conditions. An absolute stability criterion (main Th. 3.1) is derived in Section 3. It is obtained by using a quadratic form Lyapunov functional $V(x) = \langle x, \mathcal{H}x \rangle_H$, $0 \leq \mathcal{H} = \mathcal{H}^* \in \mathbf{L}(H)$. A delicate procedure of evaluating the derivative of the quadratic form along the system state trajectories is studied. This procedure consists of two stages. First, we approximate weak state trajectories by classical ones and apply the method of Noldus to get a Lur'e type system of equations containing only bounded operators. Next, using a necessary condition of solvability of the Lur'e equations (3.1) and the Phragmén-Lindelöf principle, we extend the validity of the formula expressing \dot{V} to all weak state trajectories. Finally the results of Section 2 enable us to prove global strong asymptotic stability of the null equilibrium in Theorem 3.1. An important consequence is that stability depends on the solvability of the Lur'e equations (3.1) (or equivalently (3.2)).

The aim of Section 4 is to discuss the solvability of these equations. The main difficulty is due to the fact that the open-loop system control- and/or observation operators are unbounded, which are mathematically difficult. However, it turns out that Noldus's method together with: 1) the philosophy of reciprocal systems, and 2) the proof of the Riccati results [32], Theorem 19 and Corollary 20, of Oostveen and Curtain is useful in our context. We start in Section 4.1 by giving appropriate spectral factorization results based on Szegő's factorization theorem and Devinatz-Shinbrot contribution. The problem of realization of the spectral factor is formulated next in Section 4.2. An adaptation of the results of Oostveen and Curtain (with in particular Lem. 4.3) is essential in Section 4.3 which deals with the solvability of the Lur'e system (3.1). The main result of

² The results that can be obtained using the LaSalle invariance principle differ slightly from those one can get using $L^2(0, \infty)$ -theory. As a rule the latter give slightly restricted absolute stability sectors.

Section 4.3 is Theorem 4.1 which provides a method for solving (3.1) or (3.2). It proceeds, modulo adaptation involving the transfer function mapping $\hat{g}(s) \mapsto \hat{g}(s^{-1}) - \hat{g}(0)$, as in the spectral factorization method for solving the Riccati equation of Callier and Winkin [4]. This requires:

- finding the spectral factor $\phi \in H^\infty(\Pi^+)$ such that $1/\phi \in H^\infty(\Pi^+)$ and $\phi(0) = \sqrt{\delta}$;
- determining the vector $\mathcal{G} \in H$ from the realization identity (4.4);
- solving the Lyapunov operator equation to determine \mathcal{H} by backsubstitution of \mathcal{G} into (3.1).

Many other Lur'e system results are available such as [31], Theorem 4, p. 570, [24], Theorem 3, p. 902, [26], [2], Theorem 2.1, p. 179, [33], Theorem 3, p. 740, [3], Theorem 3.1, [34], Theorem 2 and [35], Theorem 3, p. 482. However they do not fit our context.

Section 5 presents an exhaustive illustration of the results for the example of a loaded distortionless electric \mathfrak{RLC} -transmission line for which we prove the global strong asymptotic stability. After verifying that the linear part has all the properties needed to apply our main result we establish the existence of a (unique) weak solution to the closed-loop system equations for any initial condition. Here we took advantage of the fact that system equations can be reduced to a functional difference equation of the delayed-type such that the method of steps is applicable. The coercive frequency-domain inequality of the circle-type (4.3) is obtained from its weak version (4.18) by a small restriction of the sector (k_1, k_2) . Spectral factorization, realization identities and appropriate Lyapunov operator equations are analyzed in detail in Section 5.1 – the case of nonpositive b and in Section 5.2 – the case of positive b . In the latter case the sector guaranteeing absolute stability turns to be essentially smaller than the Hurwitz sector.

Related, although different absolute stability results have been proved in [3, 25]. A discussion and some prospects for further investigations, including the idea of using a Popov method for improving the absolute stability conditions, are presented in the concluding Section 6.

2. PRELIMINARIES

2.1. Admissibility of unbounded output operators: general case

We start by recalling the notion of admissibility of observation operators in the context of output trajectories in $L^2(0, \infty)$. To do this consider, in a Hilbert space H with scalar product $\langle \cdot, \cdot \rangle_H$, the homogeneous observation system

$$\left\{ \begin{array}{l} \dot{x}(t) = Ax(t) \\ x(0) = x_0 \\ y(t) = Cx(t) \end{array} \right\}, \quad t \geq 0.$$

We assume that $A : (D(A) \subset H) \rightarrow H$ generates a linear C_0 -semigroup $\{S(t)\}_{t \geq 0}$ on H and $C \in \mathbf{L}(D_A, Y)$ is an observation (output) operator, where D_A stands for the space $D(A)$ equipped with the graph norm and Y is another Hilbert space with a scalar product $\langle \cdot, \cdot \rangle_Y$.

Definition 2.1. The observation operator $C \in \mathbf{L}(D_A, Y)$ is called *infinite-time admissible* or shortly *admissible* if there exists $\gamma > 0$ such that

$$\int_0^\infty \|CS(t)x\|_Y^2 dt \leq \gamma \|x\|_H^2 \quad \forall x \in D(A),$$

i.e., the *observability map*

$$P : (D(A) \subset H) \ni x \mapsto CS(\cdot)x \in L^2(0, \infty; Y)$$

is defined and bounded on $D(A)$ in the H -norm.

If the observation operator C is admissible then, as $D(A)$ is dense in H , P has (by standard operator theory) a bounded extension (closure) denoted by \overline{P} which is defined on all of H , *i.e.* $\overline{P} \in \mathbf{L}(H, L^2(0, \infty; Y))$.

The following characterization of admissibility is known [11], Theorems 3 and 4.

Lemma 2.1. $C \in \mathbf{L}(D_A, Y)$ is admissible iff there exists a bounded self-adjoint nonnegative solution $\mathcal{H} \in \mathbf{L}(H)$ of the Lyapunov operator equation

$$\langle Ax_1, \mathcal{H}x_2 \rangle_H + \langle x_1, \mathcal{H}Ax_2 \rangle_H = -\langle Cx_1, Cx_2 \rangle_Y \quad \forall x_1, x_2 \in D(A).$$

Moreover, this solution is unique if additionally the semigroup $\{S(t)\}_{t \geq 0}$ is strongly asymptotically stable (**AS**), i.e. for every $x_0 \in H$, $\lim_{t \rightarrow \infty} S(t)x_0 = 0$.

Standard arguments involving continuity, the fact that $D(A)$ is dense in H , and the closed graph theorem lead to the following lemma, see also [32, 41].

Lemma 2.2. An operator $C \in \mathbf{L}(H, Y)$ is admissible iff $CS(\cdot)x \in L^2(0, \infty; Y)$ for any $x \in H$.

In addition to Lemma 2.2 observe that if $C \in \mathbf{L}(H, Y)$ is admissible then the adjoint of P is given by

$$P^*y = \int_0^\infty S^*(t)C^*y(t)dt, \quad y \in L^2(0, \infty; Y).$$

Consequently admissibility for bounded control operators can be introduced by using adjoint operators. An operator $B \in \mathbf{L}(U, H)$, where U is a Hilbert space with scalar product $\langle \cdot, \cdot \rangle_U$, is said to be an (*infinite-time*) *admissible* control operator if its adjoint $B^* \in \mathbf{L}(H, U)$ is an admissible observation operator with respect to the adjoint semigroup. This leads to the following result.

Lemma 2.3. An operator $B \in \mathbf{L}(U, H)$ is admissible iff $B^*S^*(\cdot)x \in L^2(0, \infty; U)$ for any $x \in H$, or equivalently the reachability operator Q given by

$$Qu = \int_0^\infty S(t)Bu(t)dt$$

is in $\mathbf{L}(L^2(0, \infty; U), H)$.

2.2. Admissibility of the observation functional of system (1.1) of boundary control in factor form

Recall the generators of the *left-* and *right-shift* C_0 -semigroup on $L^2(0, \infty)$, i.e. respectively L and $R = L^*$ given by

$$Lf = f', \quad D(L) = W^{1,2}(0, \infty) := \{f \in L^2(0, \infty) : f \text{ absolutely continuous, } f' \in L^2(0, \infty)\}$$

$$Rf = -f', \quad D(R) = \{f \in W^{1,2}(0, \infty) : f(0) = 0\}.$$

Define two operators:

$$\begin{aligned} V &\in \mathbf{L}(H, L^2(0, \infty)), & (Vx)(t) &:= h^*S(t)x \\ W &\in \mathbf{L}(L^2(0, \infty), H), & Wu &:= \int_0^\infty S(t)du(t)dt. \end{aligned}$$

With these notations and definitions, Definition 2.1 gets the following equivalent form.

Lemma 2.4. The observation functional $c^\#$ is admissible iff the observability operator

$$P = VA, \quad D(P) = D(A)$$

has a bounded continuous extension onto H denoted by \overline{P} .

We shall also need the next lemma.

Lemma 2.5. *If $c^\#$ is admissible then \overline{P} , the closure of P has the form*

$$\text{Range}(V) \subset D(L), \quad \overline{P} = LV.$$

*In particular for all $x_0 \in H$, $(\overline{P}x_0)(t) = \frac{d}{dt} [h^*S(t)x_0] \in L^2(0, \infty)$ with Laplace transform $(\widehat{\overline{P}x_0})(s) = c^\#(sI - A)^{-1}x_0 \in H^2(\Pi^+)$.*

In what follows $\Pi^+ := \{s \in \mathbb{C} : \text{Re } s > 0\}$ denotes the open right-half complex plane, $H^\infty(\Pi^+)$ is the Banach space of analytic functions f on Π^+ , equipped with the norm $\|f\|_{H^\infty(\Pi^+)} = \sup_{s \in \Pi^+} |f(s)|$ and $H^2(\Pi^+)$ is the Hardy space of functions f analytic on Π^+ such that $\sup_{\sigma > 0} \int_{-\infty}^{\infty} |f(\sigma + j\omega)|^2 d\omega < \infty$, where $f(j\omega) := \lim_{\sigma \rightarrow 0^+} f(\sigma + j\omega)$ exists for almost all $\omega \in \mathbb{R}$. The space $H^2(\Pi^+)$ is unitarily isomorphic to $L^2(0, \infty)$ by the normalized Laplace transform, to be more precise by

$$\langle f, g \rangle_{L^2(0, \infty)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(j\omega) \overline{\hat{g}(j\omega)} d\omega$$

where \hat{f}, \hat{g} are the Laplace transform of f and g , respectively. Moreover as in [20], p. 134, we shall frequently use the unitary operator $U \in \mathbf{L}(H^2(\Pi^+))$ given by

$$(U\varphi)(s) := (1/s)\varphi(1/s),$$

which for the $j\omega$ -axis $H^2(\Pi^+)$ -norm corresponds to the change of variable $\omega \mapsto -\omega^{-1}$.

The utility of this transformation appears by taking $\varphi(s) = (\widehat{\overline{P}x_0})(s) = c^\#(sI - A)^{-1}x_0$, $x_0 \in H$. Then $U\varphi(s) = -h^*(sI - A^{-1})^{-1}x_0$. Hence we have

Lemma 2.6. *$c^\# = h^*A$ is admissible for the semigroup $\{S(t)\}_{t \geq 0}$ iff $-h^*$ is admissible for the semigroup $\{e^{tA^{-1}}\}_{t \geq 0}$.*

Observe that one ends up with an observation system with bounded system operators, viz. $A^{-1} \in \mathbf{L}(H)$ and $h^* \in \mathbf{L}(H, \mathbb{R})$. Moreover the asymptotic stability of the semigroup generated by A^{-1} holds under light conditions, see Appendix A.

2.3. Classical and weak state trajectories of system (1.1)

The following weakened version of Phillips' theorem is important.

Lemma 2.7. *If $u \in D(L)$ then $v(t) := \int_0^t S(t - \tau)du(\tau)d\tau$ is a classical (i.e. C^1) solution of*

$$\begin{cases} \dot{v}(t) = Av(t) + du(t) \\ v(0) = 0 \end{cases}.$$

An important step of the proof given in [13], Appendix A, is to show that

$$\dot{v}(t) = S(t)du(0) + \int_0^t S(t - \tau)d\dot{u}(\tau)d\tau, \quad t \geq 0. \tag{2.1}$$

Now in our boundary control system in factor form (1.1), we are considering the state differential equation (1.2), where $x(\cdot)$ is a weak solution iff it is a continuous H -valued function of t such that for all $w \in D(A^*)$,

$t \mapsto \langle x(t), w \rangle$ is absolutely continuous and for almost all t and all $w \in D(A^*)$ we have

$$\frac{d}{dt} \langle x(t), w \rangle_{\mathbb{H}} = \langle x(t) + du(t), A^*w \rangle_{\mathbb{H}}.$$

Closely related is the $(\frac{d}{dt}, A^{-1})$ -reversed differential equation

$$\frac{d}{dt} [A^{-1}x(t)] = A[A^{-1}x(t)] + u(t)d = x(t) + u(t)d \tag{2.2}$$

arising from (1.2) by applying A^{-1} to both sides and by commuting the time derivative and A^{-1} . Indeed we have the following results.

Lemma 2.8. *There holds:*

a) *If $x_0 \in \mathbb{H}$ and $u \in D(L)$, then the weak solution of (1.2) coincides with the classical solution of (2.2), and is given by*

$$x(t) = S(t)x_0 + A \int_0^t S(t - \tau)du(\tau)d\tau = S(t)x_0 + Av(t). \tag{2.3}$$

b) *If $x_0 \in D(A)$ and $u \in D(R^2)$, then the classical solution of (1.2) is given by*

$$x(t) = S(t)x_0 + A \int_0^t S(t - \tau)du(\tau)d\tau = S(t)x_0 + \int_0^t S(t - \tau)d\dot{u}(\tau)d\tau - du(t) \tag{2.4}$$

with $\int_0^t S(t - \tau)d\dot{u}(\tau)d\tau \in D(A)$ and

$$\dot{x}(t) = A[x(t) + du(t)] = AS(t)x_0 + \int_0^t S(t - \tau)d\ddot{u}(\tau)d\tau - d\dot{u}(t). \tag{2.5}$$

For the proof, see Appendix B.

2.4. Input-output map and transfer function of system (1.1)

If $x_0 \in D(A), u \in D(R^2)$ and moreover the *compatibility condition*

$$d \in D(c^\#) \tag{2.6}$$

holds, then by $c^\#|_{D(A)} = h^*A$, and Lemma 2.8: $x(t) \in D(c^\#)$ for every $t \geq 0$ and the output given by

$$y(t) = c^\#x(t) = (Px_0)(t) + \int_0^t h^*S(t - \tau)d\ddot{u}(\tau)d\tau - h^*d\dot{u}(t) - c^\#du(t) = h^*\dot{x}(t) - c^\#du(t),$$

is a continuous function of $t \geq 0$.

Let $c^\#$ be admissible. Then the validity of the formula for the output $y(\cdot) \in L^2(0, \infty)$ extends to all $x_0 \in \mathbb{H}$ and $u \in D(R)$. Indeed, this is clear for the *homogeneous part* of the output, because $h^*S(\cdot)d \in D(L)$, whence

$$\begin{aligned} y(t) = c^\#x(t) &= (\overline{P}x_0)(t) + \int_0^t (\overline{P}d)(t - \tau)\dot{u}(\tau)d\tau - c^\#du(t) \\ &= \underbrace{(\overline{P}x_0)(t)}_{\text{homogeneous part}} + \underbrace{(\overline{P}d \star \dot{u})(t) - c^\#du(t)}_{\text{nonhomogeneous part}} \\ &= (\overline{P}x_0)(t) + \frac{d}{dt} \left[\int_0^t (\overline{P}d)(t - \tau)u(\tau)d\tau \right] - c^\#du(t). \end{aligned}$$

Since the homogeneous part of the output is fully determined by the observability map we turn our attention to the nonhomogeneous part, *i.e.* to the input-output map. Its characterization is explained by the next two lemmas [16], see the operator \overline{F} in Lemma 2.10 below as well as the associated transfer function \hat{g} , which corresponds to a causal operator.

Definition 2.2. The operator $H \in \mathbf{L}(\mathbf{L}^2(0, \infty))$ is called *causal* or *nonanticipative* if

$$(Hu_T)_T = (Hu)_T \quad \forall u \in \mathbf{L}^2(0, \infty)$$

where u_T denotes the *truncation* of u at time $T > 0$, $u_T(t) = \begin{cases} u(t) & \text{if } t < T \\ 0 & \text{otherwise} \end{cases}$.

Lemma 2.9. *If the compatibility condition (2.6) holds, then the function*

$$\hat{g}(s) := sc^\#(sI - A)^{-1}d - c^\#d = sh^*A(sI - A)^{-1}d - c^\#d \tag{2.7}$$

is well-defined and analytic on $[\mathbb{C} \setminus \sigma(A)] \supset \Pi^+$. If in addition to (2.6), $c^\#$ is admissible then:

- (i) $\hat{g}(s) = s(\widehat{\overline{P}d})(s) - c^\#d$ with $\widehat{\overline{P}d} \in \mathbf{H}^\infty(\Pi^+) \cap \mathbf{H}^2(\Pi^+)$.
- (ii) *The convolution operator K with kernel $\overline{P}d$, *i.e.*, $Ku := \overline{P}d \star u$ belongs to $\mathbf{L}(\mathbf{L}^2(0, \infty))$ and it maps the domain of R into itself.*

Definition 2.2 and Lemma 2.9 leads to the following results, see [16], Theorem 4.1, Corollary 4.1, Theorem 4.2.

Lemma 2.10. *If (2.6) holds, $c^\#$ is admissible and*

$$\hat{g} \in \mathbf{H}^\infty(\Pi^+) \tag{2.8}$$

then, the input-output operator F ,

$$F = -KR - c^\#dI, \quad D(F) = D(R)$$

is bounded and its closure $\overline{F} \in \mathbf{L}(\mathbf{L}^2(0, \infty))$ is causal and given by

$$\text{Range}(K) \subset D(R), \quad \overline{F} = -RK - c^\#dI.$$

Moreover \hat{g} is then the transfer function of the system (1.1), and \overline{F} is a convolution operator in the sense of distributions given by

$$\overline{F}u = g \star u, \quad u \in \mathbf{L}^2(0, \infty),$$

with impulse response g given by

$$g := \mathcal{D}(\overline{P}d) - c^\#d\delta_0,$$

with Laplace transform \hat{g} (here \mathcal{D} denotes the distributional derivative, and δ_0 stands for the Dirac distribution at zero).

Finally if in addition

$$c^\# \subset c^\#_L, \quad c^\#_L x_0 := \lim_{h \rightarrow 0^+} \frac{1}{h} c^\# \int_0^h S(\sigma)x_0 d\sigma, \quad D(c^\#_L) = \left\{ x_0 \in \mathbf{H} : \exists \lim_{h \rightarrow 0^+} \frac{1}{h} c^\# \int_0^h S(\sigma)x_0 d\sigma \right\}$$

*being the Lebesgue extension of $c^\#$, then $c^\#d = (\overline{P}d)(0+)$ *i.e.* the Lebesgue value given by*

$$(\overline{P}d)(0+) := \lim_{t \rightarrow 0^+} \frac{1}{t} \int_0^t (\overline{P}d)(\tau) d\tau = \lim_{s \rightarrow \infty, s \in \mathbb{R}} s(\widehat{\overline{P}d})(s),$$

and $\lim_{s \rightarrow \infty, s \in \mathbb{R}} \hat{g}(s) = 0$.

2.5. Admissible factor control vector of system (1.1)

Definition 2.3. The factor control vector $d \in H$ of (1.1) is called *admissible* if

$$\text{Range}(W) \subset D(A).$$

Remark 2.1. By Definition 2.3, $W \in \mathbf{L}(\mathbf{L}^2(0, \infty), H)$ because the semigroup $\{S(t)\}_{t \geq 0}$ is **EXS**. Moreover, the reachability operator Q satisfies $Q := AW \in \mathbf{L}(\mathbf{L}^2(0, \infty), H)$.

Introducing $\mathcal{R}_t \in \mathbf{L}(\mathbf{L}^2(0, \infty))$ – the reflection operator at $t > 0$,

$$(\mathcal{R}_t u)(\tau) := \begin{cases} u(t - \tau), & \tau \in [0, t) \\ 0, & \tau \geq t \end{cases}, \quad \|\mathcal{R}_t\|_{\mathbf{L}(\mathbf{L}^2(0, \infty))} \leq 1$$

we can formulate the following Lemma, [14], Theorem 4.2, [17], which for system (1.1) is a result similar to Oostveen and Curtain [32], Lemma 12.

Lemma 2.11. Let $d \in H$ be an admissible factor control vector and let $u \in \mathbf{L}^2(0, \infty)$. Then, with $x(t)$ the reachable state at t , the function

$$x(t) := Q\mathcal{R}_t u = A \int_0^t S(t - \tau) du(\tau) d\tau,$$

is a weak solution of $\dot{x}(t) = A[x(t) + du(t)]$ for $x(0) = 0$. Actually, the operator $u \mapsto Q\mathcal{R}_{(\cdot)} u$ belongs to $\mathbf{L}(\mathbf{L}^2(0, \infty), \text{BUC}_0([0, \infty); H))$ where $\text{BUC}_0([0, \infty); H)$ stands for the Banach space of bounded uniformly strongly continuous (in the norm of H) functions defined on $[0, \infty)$, taking values in H and vanishing at infinity.

Remark 2.2. The infinite-time admissibility of the factor control vector d is related to finite-time hyperbolic regularity (i.e. finite-time admissibility of the control operator) in [35], [21], Section 7.4, [23].

Remark 2.3. Under the assumptions of [16], Section 7, i.e.: 1) the semigroup generated by A is **EXS**; 2) $d \in D(c^\#)$; 3) $c^\#$ is admissible; 4) d is an admissible factor control vector; 5) the transfer function $\hat{g} \in H^\infty(\Pi^+)$; and 6) $c^\# \subset c_L^\#$, then the conclusions of [16], Theorems 7.1 and 7.2 hold, i.e. by 1)÷5) one gets a well-posed linear system (four maps well defined on $\mathbf{L}^2(0, \infty), H, \mathbf{L}^2(0, \infty)$) which is regular by 6); this well-posed linear system has then generating operators $A, B = Ad, C = h^*A$, and $D = 0$, and dynamical properties given as in [16], Theorem 7.2, where especially the state differential equation has to be solved weakly in H .

As stated in the introduction, henceforth we shall use assumptions that are needed for the results below. In particular we shall never use $c^\# \subset c_L^\#$ needed for regularity.

3. MAIN RESULT: ASYMPTOTIC STABILITY OF THE LUR'E FEEDBACK SYSTEM

Consider the Lur'e feedback control system in Figure 1.1, which consists of a linear part described by (1.1), and a scalar static controller nonlinearity $f : \mathbb{R} \rightarrow \mathbb{R}$.

Remark 3.1. In [13, 17] the sign inversion node between controller and plant is absent in the feedback loop of Figure 1.1, while in [9, 25] this node has to be put in front of the controller. To recover the results for the former case, replace $f(y)$ by $-f(y)$ and k_1 and k_2 below by respectively $-k_2$ and $-k_1$; for the latter case replace x by $-x, y$ by $-y$, while leaving k_1 and k_2 below invariant. The first case is traditionally preferred in the synthesis of electronic circuits, while the second one appears in negative unity feedback systems.

The aim here is to get sufficient conditions for global strong asymptotic stability for the Lur'e feedback system. For this purpose we assume:

(A1) The linear part of the feedback system from u to y is our boundary control system in factor form (1.1), where:

- (H1)** A generates an **EXS** semigroup $\{S(t)\}_{t \geq 0}$ on \mathbb{H} ;
- (H2)** the compatibility condition (2.6) holds;
- (H3)** the observation functional $c^\#$ is admissible, $c^\#|_{D(A)} = h^*A$;
- (H4)** the transfer function \hat{g} , defined by (2.7), satisfies (2.8).

Hence, for any $x_0 \in \mathbb{H}$, the input-output equation in $L^2(0, T)$ for any $T > 0$ is given by

$$y_T = (\overline{P}x_0)_T + (\overline{F}u)_T = (\overline{P}x_0)_T + (\overline{F}u_T)_T.$$

The last equality holds by the causality of \overline{F} .

(A2) There exist constants k_1 and $k_2 > k_1$ such that with

$$\delta := (1 - k_1 c^\# d)(1 - k_2 c^\# d) \geq 0, \quad q := k_1 k_2, \quad e := -\frac{k_1 + k_2}{2} + k_1 k_2 c^\# d,$$

the Lur'e system

$$\left\{ \begin{array}{l} (A^{-1})^* \mathcal{H} + \mathcal{H}A^{-1} - qhh^* = -\mathcal{G}\mathcal{G}^* \\ -\mathcal{H}d + eh = -\sqrt{\delta}\mathcal{G} \end{array} \right\} \quad (3.1)$$

has a solution $(\mathcal{H}, \mathcal{G})$, $\mathcal{G} \in \mathbb{H}$, $\mathcal{H} \in \mathbf{L}(\mathbb{H})$, $\mathcal{H} = \mathcal{H}^* \geq 0$, or so does the equivalent system

$$\left\{ \begin{array}{l} \langle Ax, \mathcal{H}x \rangle_{\mathbb{H}} + \langle x, \mathcal{H}Ax \rangle_{\mathbb{H}} = q(h^*Ax)^2 - (\mathcal{G}^*Ax)^2 \quad \forall x \in D(A) \\ -\mathcal{H}d + eh = -\sqrt{\delta}\mathcal{G} \end{array} \right\}. \quad (3.2)$$

(A3) The factor control vector $d \in \mathbb{H}$ is admissible.

Next for sufficiently small $\varepsilon > 0$, we define the sector:

$$\begin{aligned} \mathcal{S}_\varepsilon := & \left\{ f \in C(\mathbb{R}) : -\infty < k_1 < \frac{1}{2} \left[k_1 + k_2 - \sqrt{(k_2 - k_1)^2 - 4\varepsilon} \right] \leq \frac{f(y)}{y} \right. \\ & \left. \leq \frac{1}{2} \left[k_1 + k_2 + \sqrt{(k_2 - k_1)^2 - 4\varepsilon} \right] < k_2 < \infty \quad \forall y \in \mathbb{R} \setminus \{0\}, f(0) = 0 \right\}. \end{aligned}$$

In the sequel we shall also use the limiting sector

$$\mathcal{S}_0 := \lim_{\varepsilon \rightarrow 0^+} \mathcal{S}_\varepsilon = \left\{ f \in C(\mathbb{R}) : -\infty < k_1 \leq \frac{f(y)}{y} \leq k_2 < \infty \quad \forall y \in \mathbb{R} \setminus \{0\}, f(0) = 0 \right\}.$$

We assume moreover

(A4) The given Lur'e feedback system of Figure 1.1 is such that for $f \in \mathcal{S}_0$, for any $x_0 \in \mathbb{H}$, the truncated output y_T belongs to $L^2(0, T)$ for any $T > 0$, *i.e.* the closed-loop fixed point output equation

$$y_T = (\overline{P}x_0)_T - (\overline{F}f(y_T))_T$$

has a unique solution $y_T \in L^2(0, T)$ for all $T > 0$.

One can use standard existence theory based on the *Banach fixed point theorem* or the *Leray-Schauder alternative* to ensure **(A4)** [13]. However the results obtained in this way are rather restrictive as they usually involve strong conditions imposed on f and/or \overline{F} . For this reason **(A4)** is assumed here, knowing that in a particular application often more useful structural information is available: see *e.g.* the example of Section 5.

Lemma 3.1. *Let assumptions **(A1)**÷**(A2)** hold. Then*

- a) \mathcal{G}^*A is an admissible observation functional with respect to the semigroup generated by A .
- b) The function $s \mapsto s\mathcal{G}^*A(sI - A)^{-1}d$ belongs to $H^\infty(\Pi^+)$.

Proof. a) Consider the first equation of (3.2) under **(A2)**. Inserting $x = S(t)x_0$ with $t \geq 0$, $x_0 \in D(A)$ into this equation gives

$$\frac{d}{dt} \langle S(t)x_0, \mathcal{H}S(t)x_0 \rangle_{\mathbb{H}} = q [h^*AS(t)x_0]^2 - [\mathcal{G}^*AS(t)x_0]^2$$

where by **EXS** the two last terms decay exponentially. Thus integration over $[0, \infty)$ yields

$$\langle x_0, \mathcal{H}x_0 \rangle_{\mathbb{H}} = q \|Px_0\|_{L^2(0, \infty)}^2 - \|\mathcal{G}^*AS(\cdot)x_0\|_{L^2(0, \infty)}^2,$$

whence by the admissibility of $c^\#$

$$\|\mathcal{G}^*AS(\cdot)x_0\|_{L^2(0, \infty)}^2 \leq \left[\|\mathcal{H}\|_{\mathbb{L}(\mathbb{H})} + |q| \|\overline{P}\|_{\mathbb{L}(\mathbb{H}, L^2(0, \infty))}^2 \right] \|x_0\|_{\mathbb{H}}^2 \quad \forall x_0 \in D(A).$$

The results follows by Definition 2.1.

- b) After complexifying the state space \mathbb{H} we get from the first equation of (3.1):

$$2 \operatorname{Re} \langle A^{-1}x, \mathcal{H}x \rangle_{\mathbb{H}} = q |h^*x|^2 - |\mathcal{G}^*x|^2 \quad \forall x \in \mathbb{H}. \quad (3.3)$$

By **EXS** $j\mathbb{R} \cap \sigma(A) = \emptyset$, whence (3.3) holds for $x = (j\omega I - A^{-1})^{-1}d$ with $\omega \in \mathbb{R} \setminus \{0\}$. Thus

$$-2 \operatorname{Re} \langle d, \mathcal{H}(j\omega I - A^{-1})^{-1}d \rangle_{\mathbb{H}} = q |h^*(j\omega I - A^{-1})^{-1}d|^2 - |\mathcal{G}^*(j\omega I - A^{-1})^{-1}d|^2, \quad \omega \neq 0.$$

Now $\langle d, \mathcal{H}(j\omega I - A^{-1})^{-1}d \rangle_{\mathbb{H}}$ can be eliminated using the second equation of (3.1), whence

$$|\sqrt{\delta} - \mathcal{G}^*(j\omega I - A^{-1})^{-1}d|^2 = \delta + q |h^*(j\omega I - A^{-1})^{-1}d|^2 + 2e \operatorname{Re} [h^*(j\omega I - A^{-1})^{-1}d].$$

But

$$\begin{aligned} -h^*(zI - A^{-1})^{-1}d &= z^{-1}h^*A(z^{-1}I - A)^{-1}d = sh^*A(sI - A)^{-1}d|_{s=z^{-1}} = \hat{g}(z^{-1}) + c^\#d, \\ -\mathcal{G}^*(zI - A^{-1})^{-1}d &= z^{-1}\mathcal{G}^*A(z^{-1}I - A)^{-1}d = s\mathcal{G}^*A(sI - A)^{-1}d|_{s=z^{-1}} \end{aligned}$$

and therefore

$$|\sqrt{\delta} + j\omega\mathcal{G}^*A(j\omega I - A)^{-1}d|^2 = \delta + q |\hat{g}(j\omega) + c^\#d|^2 - 2e \operatorname{Re} [\hat{g}(j\omega) + c^\#d] \quad \forall \omega \in \mathbb{R},$$

whence $j\omega \mapsto j\omega\mathcal{G}^*A(j\omega I - A)^{-1}d \in L^\infty(j\mathbb{R})$ by (2.8). Now observe that $\Pi^+ \ni s \mapsto s\mathcal{G}^*A(sI - A)^{-1}d$ is analytic. Hence all the assumptions of the *Phragmén-Lindelöf theorem* [44], Theorem 10, p. 80, are met with an opening angle $\alpha = \pi/2$, *i.e.* here the *maximum modulus principle* holds on $\overline{\Pi^+}$. Thus $s \mapsto s\mathcal{G}^*A(sI - A)^{-1}d$ belongs to $H^\infty(\Pi^+)$. \square

Theorem 3.1. *Let assumptions **(A1)**÷**(A4)** hold. Let f belong to \mathcal{S}_ε . Then the origin of the space \mathbb{H} is globally strongly asymptotically stable (**GSAS**).*

Proof. By **(A4)** the control $u_T := -f(y_T)$ is in $L^2(0, \infty)$. For reasons of simplicity from now on we write u_T as u and we shall omit truncation for all related time functions. The absence of truncation will be maintained as long as it is not important, *i.e.* until we hit (3.4). Now as $D(R^2)$ is dense in $L^2(0, \infty)$, u can be approximated by a sequence $\{u_n\}_{n \in \mathbb{N}} \subset D(R^2)$ such that $u_n \rightarrow u$ in $L^2(0, \infty)$. As $D(A)$ is dense in \mathbb{H} the initial state x_0 can also be approximated by a sequence $\{x_{0n}\}_{n \in \mathbb{N}} \subset D(A)$ such that $x_{0n} \rightarrow x_0$ in \mathbb{H} . By Lemma 2.8 the sequence $\{x_n\}_{n \in \mathbb{N}}$ of corresponding classical solutions of (1.2) is given by,

$$\begin{aligned} x_n(t) &= S(t)x_{0n} + A \int_0^t S(t-\tau) du_n(\tau) d\tau \\ &= S(t)x_{0n} + \int_0^t S(t-\tau) d\dot{u}_n(\tau) d\tau - du_n(t), \quad t \geq 0. \end{aligned}$$

By **EXS** and the convolution result that $L^2(0, \infty) \star L^2(0, \infty) \subset \text{BUC}_0[0, \infty)$, the sequence $\{x_n\}_{n \in \mathbb{N}}$ is in $\text{BUC}_0([0, \infty); \mathbb{H})$. Now the sequence $\{x_n\}_{n \in \mathbb{N}}$ converges in $\text{BUC}_0([0, \infty); \mathbb{H})$ to a weak solution x of (1.2),

$$x(t) = S(t)x_0 + A \int_0^t S(t-\tau) du(\tau) d\tau, \quad t \geq 0,$$

which is in $\text{BUC}_0([0, \infty); \mathbb{H})$ by **EXS**, **(A3)**, and Lemma 2.11. For convergence, observe that $\text{BUC}_0([0, \infty); \mathbb{H})$ is a closed subspace of the Banach space $\text{BUC}([0, \infty); \mathbb{H})$ and

$$\begin{aligned} \|x(t) - x_n(t)\|_{\mathbb{H}} &\leq \left\| S(t)(x_0 - x_{0n}) + A \int_0^t S(t-\tau) d[u(\tau) - u_n(\tau)] d\tau \right\|_{\mathbb{H}} \\ &\leq \|x_0 - x_{0n}\|_{\mathbb{H}} \sup_{t \geq 0} \|S(t)\|_{\mathbf{L}(\mathbb{H})} + \|Q\|_{\mathbf{L}(L^2(0, \infty); \mathbb{H})} \|u - u_n\|_{L^2(0, \infty)}, \end{aligned}$$

where we used **EXS**, **(A3)**, Lemma 2.11 and the fact that $\|\mathcal{R}_t\|_{\mathbf{L}(L^2(0, \infty))} \leq 1$.

Recall that classical solutions satisfy

$$\dot{x}_n(t) = A[x_n(t) + du_n(t)] \iff A^{-1}\dot{x}_n(t) = x_n(t) + du_n(t)$$

while for the limit weak solution there holds

$$\frac{d}{dt} \langle x(t), w \rangle_{\mathbb{H}} = \langle x(t) + du(t), A^*w \rangle_{\mathbb{H}} \quad \forall w \in D(A^*) \iff \frac{d}{dt} [A^{-1}x(t)] = x(t) + du(t).$$

The objective is to get the quadratic form $V(x) = x^* \mathcal{H} x$ as a Lyapunov functional for weak solutions. To do this consider the function $t \mapsto V[x_n(t)]$. It is clearly continuously differentiable and its derivative along classical solutions reads as

$$\begin{aligned} \dot{V}[x_n(t)] &= \dot{x}_n^* \mathcal{H} x_n + x_n^* \mathcal{H} \dot{x}_n = \dot{x}_n^* \mathcal{H} (A^{-1}\dot{x}_n - du_n) + (A^{-1}\dot{x}_n - du_n)^* \mathcal{H} \dot{x}_n \\ &= \begin{bmatrix} \dot{x}_n \\ u_n \end{bmatrix}^* \begin{bmatrix} \mathcal{H} A^{-1} + (A^{-1})^* \mathcal{H} & -\mathcal{H} d \\ -d^* \mathcal{H} & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_n \\ u_n \end{bmatrix}. \end{aligned}$$

Moreover x_n and u_n leads to the output $y_n = c^\# x_n = h^* \dot{x}_n - c^\# du_n$, whence

$$[k_1 y_n + u_n][k_2 y_n + u_n] = [k_1 h^* \dot{x}_n + (1 - k_1 c^\# d) u_n][k_2 h^* \dot{x}_n + (1 - k_2 c^\# d) u_n],$$

which by subtracting and adding gives

$$\begin{aligned} \dot{V}[x_n(t)] = & \begin{bmatrix} \dot{x}_n \\ u_n \end{bmatrix}^* \begin{bmatrix} \mathcal{H}A^{-1} + (A^{-1})^* \mathcal{H} - qhh^* & -\mathcal{H}d + eh \\ -d^* \mathcal{H} + eh^* & -\delta \end{bmatrix} \begin{bmatrix} \dot{x}_n \\ u_n \end{bmatrix} \\ & + (k_1 y_n + u_n)(k_2 y_n + u_n). \end{aligned}$$

Hence **(A2)** and $f \in \mathcal{S}_0$ give

$$\dot{V}[x_n(t)] = - \left[\mathcal{G}^* \dot{x}_n + \sqrt{\delta} u_n \right]^2 + (k_1 y_n + u_n)(k_2 y_n + u_n) \leq 0.$$

Integrating both sides from 0 to t we get

$$\begin{aligned} \underbrace{V[x_n(t)]}_{\textcircled{1}} - \underbrace{V(x_{0n})}_{\textcircled{2}} = & - \underbrace{\int_0^t \left[\mathcal{G}^* \dot{x}_n(\tau) + \sqrt{\delta} u_n(\tau) \right]^2 d\tau}_{\textcircled{3}} \\ & + \underbrace{\int_0^t [k_1 y_n(\tau) + u_n(\tau)][k_2 y_n(\tau) + u_n(\tau)] d\tau}_{\textcircled{4}}. \end{aligned}$$

Now we examine the convergence of all terms. Since V is continuous and x_{0n} tends to x_0 as $n \rightarrow \infty$ then $\textcircled{2}$ tends to $V(x_0)$. $\textcircled{1}$ tends to $V[x(t)]$ in $\text{BUC}_0[0, \infty)$ because x_n tends to x in $\text{BUC}_0([0, \infty); \mathbf{H})$. Indeed for any $t \geq 0$

$$\begin{aligned} |x^*(t)\mathcal{H}x(t) - x_n^*(t)\mathcal{H}x_n(t)| & \leq |x^*(t)\mathcal{H}[x(t) - x_n(t)]| + |[x(t) - x_n(t)]^* \mathcal{H}x_n(t)| \\ & \leq \|\mathcal{H}\|_{\mathbf{L}(\mathbf{H})} [\|x\|_{\text{BUC}_0([0, \infty); \mathbf{H})} + \sup_{n \in \mathbb{N}} \|x_n\|_{\text{BUC}_0([0, \infty); \mathbf{H})}] \|x(t) - x_n(t)\|_{\mathbf{H}}. \end{aligned}$$

To examine the convergence of terms $\textcircled{4}$ and $\textcircled{3}$ observe that if $\varphi_n \rightarrow \varphi$ in $L^1(0, \infty)$ as $n \rightarrow \infty$ then $\int_0^t \varphi_n(\tau) d\tau \rightarrow \int_0^t \varphi(\tau) d\tau$ in $\text{BUC}[0, \infty)$ which follows from the estimate:

$$\sup_{t \geq 0} \left| \int_0^t \varphi_n(\tau) d\tau - \int_0^t \varphi(\tau) d\tau \right| \leq \sup_{t \geq 0} \int_0^t |\varphi(\tau) - \varphi_n(\tau)| d\tau = \|\varphi_n - \varphi\|_{L^1(0, \infty)}.$$

Since

$$(k_1 y_n + u_n)(k_2 y_n + u_n) = [k_1 (\overline{P}x_{0n} + \overline{F}u_n) + u_n] [k_2 (\overline{P}x_{0n} + \overline{F}u_n) + u_n]$$

and $\overline{P} \in \mathbf{L}(\mathbf{H}, L^2(0, \infty))$, $\overline{F} \in \mathbf{L}(L^2(0, \infty))$ then

$$(k_1 y_n + u_n)(k_2 y_n + u_n) \longrightarrow [k_1 (\overline{P}x_0 + \overline{F}u) + u] [k_2 (\overline{P}x_0 + \overline{F}u) + u] \quad \text{as } n \rightarrow \infty$$

in $L^1(0, \infty)$, and consequently $\textcircled{4}$ tends to

$$\int_0^t \{k_1 [(\overline{P}x_0)(\tau) + (\overline{F}u)(\tau)] + u(\tau)\} \{k_2 [(\overline{P}x_0)(\tau) + (\overline{F}u)(\tau)] + u(\tau)\} d\tau$$

in $\text{BUC}[0, \infty)$. As regards $\textcircled{3}$ we can prove as well that it converges in $\text{BUC}[0, \infty)$ because $\mathcal{G}^* \dot{x}_n + \sqrt{\delta} u_n$ converges in $L^2(0, \infty)$. To see this, notice that by (2.5)

$$\mathcal{G}^* \dot{x}_n(t) = \mathcal{G}^* AS(t)x_{0n} + \int_0^t \mathcal{G}^* S(t-\tau) d\ddot{u}_n(\tau) d\tau - \mathcal{G}^* \dot{u}_n(t).$$

Then by Lemma 3.1, \mathcal{G}^*A is an admissible observation functional with respect to the semigroup generated by A . Hence with $\overline{P}_{\mathcal{G}}$ denoting the extended observability map associated with \mathcal{G}^*A , we get using Lemma 2.5 with $c^\#|_{D(A)} = h^*A$ replaced by \mathcal{G}^*A ,

$$\mathcal{G}^* \dot{x}_n(t) = (\overline{P}_{\mathcal{G}} x_{0n})(t) + \int_0^t (\overline{P}_{\mathcal{G}} d)(t-\tau) \dot{u}_n(\tau) d\tau = (\overline{P}_{\mathcal{G}} x_{0n})(t) + (\overline{P}_{\mathcal{G}} d \star \dot{u}_n)(t).$$

Consider now the construction of the input-output operator in Lemmas 2.9 and 2.10, with $c^\#|_{D(A)} = h^*A$ replaced by \mathcal{G}^*A , and recall that, by Lemma 3.1, the function $s \mapsto s\mathcal{G}^*A(sI - A)^{-1}d$ belongs to $H^\infty(\Pi^+)$. Then (without any compatibility condition) it follows, with $K_{\mathcal{G}}u = \overline{P}_{\mathcal{G}}d \star u$, that the operator $-K_{\mathcal{G}}R$ extends to a bounded, everywhere defined, operator on $L^2(0, \infty)$, which is precisely $-RK_{\mathcal{G}}$.

Hence we conclude that

$$\mathcal{G}^* \dot{x}_n + \sqrt{\delta}u_n \longrightarrow \overline{P}_{\mathcal{G}}x_0 - RK_{\mathcal{G}}u + \sqrt{\delta}u \quad \text{as } n \rightarrow \infty$$

in $L^2(0, \infty)$. Finally we obtain

$$\begin{aligned} V[x(t)] - V(x_0) &= - \int_0^t \left[(\overline{P}_{\mathcal{G}}x_0)(\tau) - (RK_{\mathcal{G}}u)(\tau) + \sqrt{\delta}u(\tau) \right]^2 d\tau \\ &\quad + \int_0^t [k_1y(\tau) + u(\tau)][k_2y(\tau) + u(\tau)] d\tau \quad \forall t \in [0, T]. \end{aligned} \quad (3.4)$$

Thus by **(A1)**÷**(A4)**, V is a Lyapunov functional for weak solutions. Now let $f \in \mathcal{S}_\varepsilon$. Then

$$\dot{V} = - \left[\overline{P}_{\mathcal{G}}x_0 - RK_{\mathcal{G}}u + \sqrt{\delta}u \right]^2 + (k_1y + u)(k_2y + u) \leq -\varepsilon y^2 \quad \forall t \in [0, T]. \quad (3.5)$$

This is because $\left\{ \frac{1}{2} [k_1 + k_2 - \sqrt{(k_2 - k_1)^2 - 4\varepsilon}]y + u \right\} \left\{ \frac{1}{2} [k_1 + k_2 + \sqrt{(k_2 - k_1)^2 - 4\varepsilon}]y + u \right\} = (k_1y + u)(k_2y + u) + \varepsilon y^2$.

Integrating both sides of (3.5) from 0 to t and using $\mathcal{H} \geq 0$ we obtain,

$$-V(x_0) \leq V[x(t, x_0)] - V(x_0) \leq -\varepsilon \int_0^t y^2(\tau) d\tau \quad \forall t \in [0, T]$$

whence

$$\|\mathcal{H}\|_{\mathbf{L}(\mathbf{H})} \|x_0\|_{\mathbf{H}}^2 \geq V(x_0) \geq \varepsilon \int_0^t y^2(\tau) d\tau \quad \forall t \in [0, T].$$

This yields

$$\|y\|_{\mathbf{L}^2(0, T)} \leq \sqrt{\frac{1}{\varepsilon} \|\mathcal{H}\|_{\mathbf{L}(\mathbf{H})} \|x_0\|_{\mathbf{H}}} \quad \forall T > 0,$$

and thus

$$\|y\|_{\mathbf{L}^2(0, \infty)} \leq \sqrt{\frac{1}{\varepsilon} \|\mathcal{H}\|_{\mathbf{L}(\mathbf{H})} \|x_0\|_{\mathbf{H}}}.$$

Since $f \in \mathcal{S}_0$

$$\int_0^\infty u^2(t) dt = \int_0^\infty f^2[y(t)] dt = \int_0^\infty y^2(t) \frac{f^2[y(t)]}{y^2(t)} dt \leq \max\{k_1^2, k_2^2\} \|y\|_{\mathbf{L}^2(0, \infty)}^2,$$

whence

$$\|u\|_{L^2(0,\infty)} \leq \max\{|k_1|, |k_2|\} \sqrt{\frac{1}{\varepsilon} \|\mathcal{H}\|_{\mathbf{L}(\mathbf{H})}} \|x_0\|_{\mathbf{H}}. \tag{3.6}$$

Hence there holds that $y, u \in L^2(0, \infty)$.

Since, by **(A3)**, $d \in \mathbf{H}$ is an admissible factor control vector, then by Lemma 2.11

$$x(t) = S(t)x_0 + Q\mathcal{R}_t u \quad t \geq 0$$

where $Q \in \mathbf{L}(L^2(0, \infty), \mathbf{H})$ is the reachability map of Remark 2.1 and \mathcal{R}_t stands for the reflection operator at $t > 0$. Now by the second assertion of Lemma 2.11, **EXS** and (3.6)

$$\begin{aligned} \|x(t)\|_{\mathbf{H}} &\leq \|S(t)x_0\|_{\mathbf{H}} + \sup_{t \geq 0} \|Q\mathcal{R}_t u\|_{\mathbf{H}} \\ &\leq \left[M + \|Q\mathcal{R}_{(\cdot)}\|_{\mathbf{L}(L^2(0,\infty), \text{BUC}_0([0,\infty); \mathbf{H}))} \max\{|k_1|, |k_2|\} \sqrt{\frac{1}{\varepsilon} \|\mathcal{H}\|_{\mathbf{L}(\mathbf{H})}} \right] \|x_0\|_{\mathbf{H}}. \end{aligned} \tag{3.7}$$

The stability in the sense of Lyapunov of the null equilibrium easily follows from (3.7).

Finally the null equilibrium is globally strongly attracting, because $\{S(t)\}_{t \geq 0}$ is **EXS** and $t \mapsto Q\mathcal{R}_t u \in \text{BUC}_0([0, \infty); \mathbf{H})$ for any $u \in L^2(0, \infty)$. □

4. SUFFICIENT CRITERION FOR SOLVABILITY OF THE LUR'E SYSTEM OF EQUATIONS

In this section we shall get sufficient conditions for checking **(A2)**, *i.e.* for the solvability of the Lur'e system of equations (3.1) or equivalently (3.2) with respect to the pair $(\mathcal{H}, \mathcal{G})$. Guided by the reciprocal system philosophy, our inspiration stems from the Oostveen and Curtain Riccati results in [32], modulo adaptation to the case where d is not supposed to be admissible *i.e.* **(A3)** does not hold as an intellectual challenge motivated by “parabolic regularity” as in [21, 22]. Recall here that in [21], Section 6, a whole variety of linear controlled abstract parabolic systems without finite-time admissibility of the control operator is discussed; however the authors of [21] consider some LQ-problems with mainly bounded observation operators, so their context is different from that of the present paper.

It should be emphasized that in control theory there is a big tradition to separate the problem of solvability of (3.1) or (3.2) from the problem at hand which gives rise to considering the Lur'e system of equations. This is because a whole variety of control tasks leads to such equations. Hence though main Theorem 4.1 of this section plays an auxiliary role for Theorem 3.1 it is independently important to know that the admissibility of d is not essential for the solvability of (3.1) or (3.2).

The method for getting our main result Theorem 4.1 is as in the spectral factorization method for solving the Riccati equation of Callier and Winkin [4], modulo the transfer function mapping $\hat{g}(s) \mapsto \hat{g}(s^{-1}) - \hat{g}(0)$. Spectral factorization is handled first. Some other preliminary results follow next, and finally we get our result.

4.1. Spectral factorization

The following result is important in our context. Equation (4.1) below is called a *spectral factorization equation*, where π is called a *spectral density function* and ϕ is called a *spectral factor*.

Lemma 4.1. *Let $\omega \mapsto \pi(j\omega)$ be a real-valued, nonnegative function on the $j\omega$ -axis such that π belongs to $L^\infty(\mathbb{R})$ and $\pi(j\omega) = \pi(-j\omega)$. Let in addition π be coercive, *i.e.* there exists an $\varepsilon > 0$ such that $\pi(j\omega) \geq \varepsilon$ for all $\omega \in \mathbb{R}$. Then:*

- (i) *There exists a function $\phi \in H^\infty(\Pi^+)$ such that*

$$\pi(j\omega) = \phi(j\omega)\phi(-j\omega) = |\phi(j\omega)|^2, \tag{4.1}$$

and $1/\phi$ is as well in $H^\infty(\Pi^+)$. Moreover $\phi(s)$ can be chosen to be real, i.e. it satisfies $\phi(s) = \overline{\phi(\bar{s})}$, meaning that its Taylor expansion around a positive number has real coefficients or that $\phi(s)$ takes real values for real arguments; furthermore such $\phi(s)$ is unique modulo a ± 1 factor.

(ii) If moreover $\pi(j\omega)$ has an analytic extension in a domain containing a full neighbourhood of $s = 0$ which is para-Hermitian self-adjoint (i.e. $\pi(s) = \pi(-s)$), then

$$\left(s \mapsto \frac{\phi(s) - \phi(0)}{s} \right) \in H^\infty(\Pi^+) \cap H^2(\Pi^+)$$

and the factor $\phi(s)$ of assertion (i) is unique by the normalization condition $\phi(0) = \sqrt{\pi(0)}$.

Remark 4.1. Part (i) is well-known. It is traditionally first obtained on the unit circle of the complex z -plane and then solved on the imaginary axis of the complex s -plane by using a linear fractional transformation $z = (s - 1)^{-1}(s + 1)$ which maps bijectively the closed right-half plane onto the closed unit disc. Results are associated with G. Szegő, see especially [20], two theorems, p. 53; Chapter 8, [19], Section 1.14.

Proof of Lemma 4.1, (ii). Note that, since the spectral density function has a para-Hermitian self-adjoint analytic extension in a domain containing a full neighbourhood of $s = 0$, then we have there the factorization

$$\pi(s) = \phi(s)\phi(-s),$$

with $\phi(s)$ analytic at $s = 0$ (this can be seen by considering the successive self-adjoint polynomial approximations and their factorizations of the Taylor expansion π near zero). This jointly with $\phi \in H^\infty(\Pi^+)$ leads to the fact that the function $s \mapsto \frac{\phi(s) - \phi(0)}{s}$ is analytic and bounded in a full neighbourhood of $s = 0$ and finally is in $H^\infty(\Pi^+) \cap H^2(\Pi^+)$. Due to the analyticity of $\phi(s)$ at $s = 0$ a spectral factorization of statement (i) can be made unique by the normalization condition $\phi(0) = \sqrt{\pi(0)} > 0$. \square

4.2. State-feedback realization problem

Let us assume that **(A1)** holds, or equivalently **(H1)** \div **(H4)** are satisfied. Assume additionally that

(H5) There exist $k_1, k_2, k_1 < k_2$ such that the Popov function

$$\begin{aligned} \pi(j\omega) &:= 1 + (k_1 + k_2) \operatorname{Re}[\hat{g}(j\omega)] + k_1 k_2 |\hat{g}(j\omega)|^2 \\ &= \delta - 2e \operatorname{Re}[\hat{g}(j\omega) + c^\# d] + q |\hat{g}(j\omega) + c^\# d|^2, \quad \omega \in \mathbb{R}, \end{aligned} \quad (4.2)$$

satisfies the coercivity condition³

$$\pi(j\omega) \geq \varepsilon > 0 \quad \forall \omega \in \mathbb{R}. \quad (4.3)$$

Note that as $\hat{g} \in H^\infty(\Pi^+)$ and $\hat{g}(s) = \overline{\hat{g}(\bar{s})}$ one gets $\pi \in L^\infty(\mathbb{R})$, $\pi(j\omega) = \pi(-j\omega)$. It follows from Lemma 4.1 that the spectral factorization problem (4.1) with the Popov spectral density function π has a solution ϕ such that $1/\phi \in H^\infty(\Pi^+)$. Furthermore, as $\hat{g}(s) + c^\# d = s(\widehat{Pd})(s)$, it follows by Lemma 2.9 and **EXS** that the Popov function has a para-Hermitian self-adjoint analytic extension in a domain containing a full neighbourhood of $s = 0$ which reads

$$\begin{aligned} \pi(s) &:= 1 + \frac{(k_1 + k_2)}{2} [\hat{g}(s) + \hat{g}(-s)] + k_1 k_2 \hat{g}(s) \hat{g}(-s) \\ &= \delta - es \left[(\widehat{Pd})(s) - (\widehat{Pd})(-s) \right] - qs^2 (\widehat{Pd})(s) (\widehat{Pd})(-s). \end{aligned}$$

³ If $k_1 k_2 < 0$ then the frequency-domain inequality (4.3) means geometrically that the plot of the transfer function $\hat{g}(j\omega)$ is located inside the circle with centre at $-(k_1^{-1} + k_2^{-1})/2$ and radius $(k_2^{-1} - k_1^{-1})/2$. In particular, this yields $\hat{g} \in H^\infty(\Pi^+)$.

Hence $\pi(0) = \delta > 0$ and again by Lemma 4.1 the spectral factorization problem is uniquely solvable by adding the requirement $\phi(0) = \sqrt{\pi(0)} = \sqrt{\delta}$ and

$$\left(s \mapsto \frac{\phi(s) - \phi(0)}{s} = \frac{\phi(s) - \sqrt{\delta}}{s} \right) \in H^\infty(\Pi^+) \cap H^2(\Pi^+).$$

Henceforth given **(H1)**÷**(H5)**, we call *realization problem* that of finding a $\mathcal{G} \in \mathbf{H}$ satisfying the identity:

$$\frac{\phi(s) - \sqrt{\delta}}{s} = \mathcal{G}^* A(sI - A)^{-1} d \quad \forall s \in \Pi^+, \tag{4.4}$$

where $\phi \in H^\infty(\Pi^+)$ is that spectral factor of the Popov density function π which satisfies $1/\phi \in H^\infty(\Pi^+)$ and ϕ is analytic at $s = 0$ with $\phi(0) = \sqrt{\delta}$ (the outer normalized spectral factor). The realization equation (4.4) is equivalent to

$$\phi(s^{-1}) = \sqrt{\delta} - \mathcal{G}^*(sI - A^{-1})^{-1} d \quad \forall s \in \Pi^+. \tag{4.5}$$

This will turn out to be a realization of the spectral factor of the Popov function in the proof of Theorem 4.1 due to the Oostveen and Curtain Lemma 4.3: in that proof it is seen that \mathcal{G}^* is proportional to a state-feedback operator dictated by a solution of a Riccati equation.

Lemma 4.2. *If the pair (A^{-1}, d) is approximately reachable i.e. $\overline{\text{Span}\{A^{-n}d\}_{n=0}^\infty} = \mathbf{H}$ then the realization problem (4.4), or its equivalent form (4.5) has at most one solution.*

Proof. Indeed, if there were two solutions \mathcal{G}_1 and \mathcal{G}_2 then we would have

$$[\mathcal{G}_1 - \mathcal{G}_2]^*(sI - A^{-1})^{-1} d = 0 \quad \forall s \in \Pi^+$$

and by approximate reachability: $\mathcal{G}_1 = \mathcal{G}_2$. □

4.3. Main auxiliary result: sufficient criterion using a strict circle inequality

The proof of the Riccati results [32], Theorem 19 and Corollary 20, of Oostveen and Curtain contains the lemma below, where the admissibility of the *bounded* observation and control operators C and B is as in Lemmas 2.2 and 2.3⁴. Other infinite-dimensional Riccati results exist, e.g. [38, 42, 43], but their application in the proof of Theorem 4.1 is not obvious.

Lemma 4.3. *Let $A : (D(A) \subset \mathbf{H}) \rightarrow \mathbf{H}$ generate an **AS** linear C_0 -semigroup on \mathbf{H} , let $B \in \mathbf{L}(U, \mathbf{H})$, let $C \in \mathbf{L}(\mathbf{H}, Y)$ be an admissible observation operator, let the transfer function*

$$G(s) := C(sI - A)^{-1} B \tag{4.6}$$

belong to $H^\infty(\Pi^+, \mathbf{L}(U, Y))$ and $Q \in \mathbf{L}(Y)$, $Q = Q^$, $N \in \mathbf{L}(Y, U)$, $R \in \mathbf{L}(U)$, $R = R^* \geq \eta I > 0$ such that the Popov function*

$$\Pi(j\omega) := R + NG(j\omega) + [NG(j\omega)]^* + [G(j\omega)]^* Q G(j\omega), \quad \omega \in \mathbb{R} \tag{4.7}$$

is in $L^\infty(\mathbb{R}, \mathbf{L}(U))$. Assume moreover that the Popov function is coercive i.e.

$$\Pi(j\omega) \geq \varepsilon I > 0 \quad \forall \omega \in \mathbb{R}. \tag{4.8}$$

Then the operator Riccati equation:

$$A^* X x + X A x - (B^* X + N C)^* R^{-1} (B^* X + N C) x + C^* Q C x = 0 \quad \forall x \in D(A) \tag{4.9}$$

⁴ For the case that the Popov function is nonnegative but not coercive, see [5] as a complement of information.

has a self-adjoint bounded solution $X = X^*$, $X \in \mathbf{L}(\mathbf{H})$,

$$X = \Psi^* \mathcal{T} \Psi, \quad \mathcal{T} := Q - (Q\mathbb{F} + N^*)\mathcal{R}^{-1}(Q\mathbb{F} + N^*)^*, \tag{4.10}$$

where Ψ and \mathbb{F} are, respectively, the extended observability map and the extended input-output map associated with the system triple (A, B, C) and

$$\mathcal{R} := R + N\mathbb{F} + \mathbb{F}^*N + \mathbb{F}Q\mathbb{F}^*$$

is the Toeplitz operator with the Popov function Π as its symbol, and such that with

$$F_X := -R^{-1}(B^*X + NC) \in \mathbf{L}(\mathbf{H}, \mathbf{U})$$

there holds: $W, W^{-1} \in \mathbf{H}^\infty(\Pi^+, \mathbf{L}(\mathbf{U}))$, where

$$W(s) := I - F_X(sI - A)^{-1}B,$$

and

$$\Pi(j\omega) = W(j\omega)^*RW(j\omega) \quad \forall \omega \in \mathbb{R}. \tag{4.11}$$

Remark 4.2. Here F_X is a state-feedback operator and $W(s)$ is the control loop return difference induced by $u = F_X x$. To prove that $W \in \mathbf{H}^\infty(\Pi^+, \mathbf{L}(\mathbf{U}))$, one needs according to the proof of [32], Theorem 19, pp. 961–962, to revisit the proof of [32], Lemma 18. The arguments in the proof of the latter use only the fact that B is *finite-time admissible* (which is the case as B is bounded) whence one can guarantee that the spectral factorization W in (4.11) has a realization (A, B, C_W, D_W) with bounded operators B, C_W, D_W resulting in a well-defined extended output equation

$$y = \Psi_W x_0 + \mathbb{F}_W u$$

where $\Psi_W \in \mathbf{L}(\mathbf{H}, \mathbf{L}^2(0, \infty))$ and $\mathbb{F}_W \in \mathbf{L}(\mathbf{L}^2(0, \infty))$ with

$$\Psi_W x_0 = (\mathbb{F}_W^*)^{-1}(\mathbb{F}^*Q + N)\Psi x_0$$

and

$$(\widehat{\mathbb{F}_W u})(j\omega) = W(j\omega)\hat{u}(j\omega).$$

Using this result in the proof of [32], Theorem 19, it turns out that $C_W = -F_X$ and $D_W = I$ and that $W \in \mathbf{H}^\infty(\Pi^+, \mathbf{L}(\mathbf{U}))$. Thus here the solution of the Riccati equation is stabilizing in the sense that the latter property holds, *i.e. the control loop return difference stabilizing property*. Finally if the pair (A, B) is reachable then such solution is unique.

We have not assumed that B is admissible, because in the context of Theorem 4.1 this would require that d is admissible, which we do not want to assume. If B is admissible, then X is a *unique strongly stabilizing solution* [32], where in particular $A + BF_X$ is the generator of an **AS** semigroup obtained by the state-feedback $u = F_X x$.

Theorem 4.1. *Let assumptions (H1)÷(H5) hold. Moreover assume that:*

(H6) *The operator $A : (D(A) \subset \mathbf{H}) \longrightarrow \mathbf{H}$ is such that the semigroup generated by A^{-1} is **AS**;*

Then:

- (i) *The system (3.1) has a solution $(\mathcal{H}, \mathcal{G})$, $\mathcal{H} \in \mathbf{L}(\mathbf{H})$, $\mathcal{H} = \mathcal{H}^* \geq 0$, $\mathcal{G} \in \mathbf{H}$, where in particular: \mathcal{G} is the solution of the realization equation (4.4), where ϕ is the spectral factor of the Popov function π (given by (4.2)) such that $\phi(0) = \sqrt{\delta}$, and both ϕ and $1/\phi$ are in $\mathbf{H}^\infty(\Pi^+)$;*
- (ii) *Assume that the pair (A^{-1}, d) is approximately reachable. Then this \mathcal{G} can be obtained by solving the realization problem (4.4) or its equivalent form (4.5), while \mathcal{H} can then be determined by solving the first (i.e. Lyapunov) equation of (3.1).*

Remark 4.3. Assertion (ii) of Theorem 4.1 is important in that it facilitates finding a solution $(\mathcal{H}, \mathcal{G})$ of the Lur'e system (3.1). Indeed as stated it avoids solving the open-loop Riccati operator equation. See also [4] for ideas on using this way of proceeding for solving a Riccati equation when the pair (A^{-1}, d) is not approximately reachable.

Remark 4.4. The following comments are in order before the proof of Theorem 4.1. The ideas of using (3.1) instead of (3.2) and reducing the problem of solvability of (3.1) to the solvability of the Riccati equation (4.9) are copied from our reports [15], [18], and were inspired by the reciprocal system approach. The latter system, see the introduction, is expressed in terms of bounded operators exclusively, whereas the original system contains unbounded operators. The notion of reciprocal system was introduced by Curtain who also developed it as an analytic tool for regular well-posed linear infinite-dimensional control systems [6, 7]. Recall that any SISO regular well-posed linear system (1.1) with $0 \in \rho(A)$ instead of A being the generator of an **EXS** C_0 -semigroup, can be written as a reciprocal system as in the introduction. However in our system (1.1) d is not assumed to be admissible, and hence does not correspond to a well-posed linear system as in [6, 7].

Proof. Ad (i). Consider Lemma 4.3. Set U and Y equal to \mathbb{R} and replace the triples (A, B, C) and (Q, N, R) respectively by $(A^{-1}, d, -h^*)^5$ and $(q, -e, \delta)$, where by (4.3) $\pi(0) = \delta > 0$. Then:

- ❶ The semigroup $\{e^{tA^{-1}}\}_{t \geq 0}$ is **AS** by assumption **(H6)**;
- ❷ The admissibility of h^* with respect to the semigroup $\{e^{tA^{-1}}\}_{t \geq 0}$ follows by assumption **(H3)** and by Lemma 2.6;
- ❸ The transfer function (4.6) gives

$$G(s) = -h^*(sI - A^{-1})^{-1}d = s^{-1}h^*A(s^{-1}I - A)^{-1}d, \tag{4.12}$$

whence

$$G(s) = \hat{g}(s^{-1}) + c^\#d = \hat{g}(s^{-1}) - \hat{g}(0), \tag{4.13}$$

where \hat{g} is the transfer function in (2.7). The transfer function described in (4.12) and (4.13) is in $H^\infty(\Pi^+)$ due to **(H4)**;

- ❹ The Popov function (4.7) reads

$$\Pi(j\omega) = \delta - 2e \operatorname{Re}[G(j\omega)] + q|G(j\omega)|^2 \quad \forall \omega \in \mathbb{R},$$

such that by (4.13) and (4.2)

$$\Pi(j\omega) = \pi((j\omega)^{-1}) \quad \forall \omega \in \mathbb{R} \setminus \{0\}. \tag{4.14}$$

The Popov function Π satisfies the coercivity condition (4.8) by (4.14) and **(H5)**.

Now all assumptions of Lemma 4.3 are valid and applying the latter gives that the Riccati operator equation (4.9), which reads here as

$$(A^{-1})^*X + XA^{-1} - \frac{1}{\delta}(Xd + eh)(Xd + eh)^* + qhh^* = 0, \tag{4.15}$$

has a solution $X = X^* \in \mathbf{L}(\mathbf{H})$. The symbol of the Toeplitz operator \mathcal{T} , defined in (4.10), reads with $U = Y = \mathbb{R}$ as

$$\begin{aligned} \mathcal{T}(j\omega) &= Q - [QG(j\omega) + N^*] \Pi^{-1}(j\omega) [QG(j\omega) + N^*]^* = -(N^2 - QR) \Pi^{-1}(j\omega) \\ &= -(e^2 - q\delta) \Pi^{-1}(j\omega) = -\frac{1}{4}(k_2 - k_1)^2 \Pi^{-1}(j\omega) \quad \forall \omega \in \mathbb{R}, \end{aligned}$$

⁵ The triple $(A^{-1}, d, -h^*)$ is suggested by the reciprocal system recipe of the introduction, and is consistent with [15, 18], and [6], Definition 4, [7], Definition 3.1.

whence by (4.10) $X \leq 0$. This solution is such that with

$$F_X = -\frac{1}{\delta}(d^*X + eh^*) \tag{4.16}$$

there holds: $W, 1/W \in H^\infty(\Pi^+)$, $\Pi(j\omega) = \frac{1}{\delta} |W(j\omega)|^2$ for all $\omega \in \mathbb{R}$, where

$$W(s)\sqrt{\delta} = 1 - F_X(sI - A^{-1})^{-1}d. \tag{4.17}$$

Hence by (4.15), (4.16) the pair $(\mathcal{H}, \mathcal{G})$, $\mathcal{H} := -X \geq 0$, $\mathcal{G} := \sqrt{\delta}F_X^*$ is a solution of (3.1)⁶. Next the function $\phi(s) := \sqrt{\delta}W(s^{-1})$ is in $H^\infty(\Pi^+)$ jointly with $1/\phi$ and by (4.14) ϕ satisfies (4.1). As $A^{-1} \in \mathbf{L}(\mathbf{H})$, W is analytic at $\{\infty\}$ and takes the value 1 at $\{\infty\}$, *i.e.* $\lim_{|s| \rightarrow \infty} W(s) = 1$, whence ϕ is analytic at 0 and $\lim_{s \rightarrow 0} \phi(s) =: \phi(0) = \sqrt{\delta}$.

Finally it follows from (4.17) and (4.16) that \mathcal{G} satisfies the realization equation (4.5).

Ad **(ii)**. By **(i)** and Lemma 4.2 the realization equation (4.5) has a unique solution (uniquely determined by the spectral factor ϕ),

$$\mathcal{G} := -\frac{1}{\sqrt{\delta}}(-\mathcal{H}d + eh),$$

where \mathcal{H} is a solution of the Riccati operator equation

$$(A^{-1})^*\mathcal{H} + \mathcal{H}A^{-1} + \frac{1}{\delta}(-\mathcal{H}d + eh)(-\mathcal{H}d + eh)^* - qhh^* = 0.$$

Hence we conclude that the second element in the pair $(\mathcal{H}, \mathcal{G})$ being a solution of (3.1) can be determined by solving the realization problem, while the first element can then be determined by solving the first (*i.e.* Lyapunov) equation of (3.1). □

Observe that upon identifying $CA^{-1} = -h^*$, $A^{-1}B = d$, $Q = q$, $N = -e$ and $R = \delta$, our Riccati equation (4.15) coincides with that of [7], Formulae (4.27) and (4.28), Furthermore [7], (4.24), then agrees with our realization identity (4.5). However there are some differences. Since our results have stability as objective, $X \leq 0$ was paramount, while in main result [7], Theorem 4.5, it is absent. Moreover our Theorem 4.1 was proved under assumption **(H6)**, but it was not required that d is admissible; and in [7], Theorem 4.5, **(H6)** is not required but the admissibility of d is implicitly required, as this follows from the assumption that the system is well-posed (observe that finite- and infinite-time admissibility coincide when A generates an **EXS** semigroup).

We are now ready for an example in which the function π , given by (4.2), will first be tested for the condition

$$\pi(j\omega) \geq 0 \quad \forall \omega \in \mathbb{R}, \tag{4.18}$$

which is weaker than the coercivity condition (4.3).

5. EXAMPLE: DISTORTIONLESS LOADED **RLCG**-TRANSMISSION LINE

In this section we discuss an electrical transmission line as a plant in Figure 1.1 illustrating hereby the results of the previous sections.

⁶ Let $\Pi(j\omega) = |M(j\omega)|^2$, where both M and $1/M$ are in $H^\infty(\Pi^+)$. Then

$$\langle x, \mathcal{H}x \rangle_{\mathbf{H}} = -\langle \Psi x, T\Psi x \rangle_{L^2(0, \infty)} = \left\| \frac{1}{2}(k_2 - k_1)M^{-1}(j\omega)(\widehat{\Psi}x)(j\omega) \right\|_{H^2(\Pi^+)}^2 \quad \forall x \in \mathbf{H}$$

displays how spectral factorization defines \mathcal{H} .

The *distortionless* transmission line is a $\Re\mathcal{L}\mathcal{C}\mathcal{E}$ line for which $\alpha := \Re/\mathcal{L} = \mathcal{G}/\mathcal{C}$. Following [16], Section 5.1, consider such line loaded by a resistance $R_0 > 0$. Recall that with $w(t) \in \mathbb{R}^2$, the system dynamics can be described by

$$\begin{cases} w(t) = C_S w(t-r) + u(t)b_0 \\ y(t) = c_0^T w(t-r) \end{cases} \tag{5.1}$$

where

$$C_S = \begin{bmatrix} 0 & 1 \\ -b & 0 \end{bmatrix}, \quad b = \frac{\kappa}{\rho^2}, \quad \kappa = \frac{R_0 - z}{R_0 + z}, \quad z = \sqrt{\frac{\mathcal{L}}{\mathcal{C}}}, \quad \rho = e^{\alpha r},$$

$$b_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c_0 = \begin{bmatrix} 0 \\ a \end{bmatrix}, \quad a = \frac{1 + \kappa}{\rho} > 0.$$

By using the Hilbert space $\mathbf{H} = L^2(-r, 0) \oplus L^2(-r, 0)$ with $r = \sqrt{\mathcal{L}\mathcal{C}}$ equipped with the standard scalar product, one can convert its dynamics into an abstract model in factor form as in (1.1). More precisely:

- The state space operator A takes the form

$$Ax = x', \quad D(A) = \left\{ x \in W^{1,2}(-r, 0) \oplus W^{1,2}(-r, 0) : x(0) = C_S x(-r) \right\}$$

and generates a C_0 -semigroup $\{S(t)\}_{t \geq 0}$ on \mathbf{H} (or even a C_0 -group if $\det C_S \neq 0$). This semigroup is **EXS** iff $|\lambda(C_S)| < 1$ or equivalently $|b| < 1$ [10], pp. 148–154], which is the case⁷. Thus assumption **(H1)** holds.

- The observation functional $c^\#$ is given by

$$c^\# x = c_0^T x(-r), \quad D(c^\#) = \left\{ x \in \mathbf{H} : c_0^T x \text{ is right-continuous at } -r \right\},$$

and reads on $D(A)$ as

$$c^\#|_{D(A)} = h^* A, \quad h = \vartheta \begin{bmatrix} b\mathbf{1} \\ -\mathbf{1} \end{bmatrix} \in \mathbf{H}, \quad \vartheta := \frac{a}{1+b},$$

where $\mathbf{1}$ denotes the constant function taking the value 1 on $[-r, 0]$. The admissibility of $c^\#$ was implicitly discussed in [12], p. 363, and proved in [16]. Thus assumption **(H3)** holds.

- The factor control vector is identified as

$$d = \frac{-1}{1+b} d_0, \quad d_0 = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \in \mathbf{H},$$

where d is admissible [16], p. 20, whence assumption **(A3)** holds. By the proof presented therein the pair (A^{-1}, d) is exactly (hence approximately) reachable.

The compatibility condition (2.6) holds with $c^\# d = -\vartheta$ and by (2.7) the transfer function reads

$$\hat{g}(s) = \frac{ae^{-sr}}{1 + be^{-2sr}}.$$

⁷ An alternative proof follows by applying Datko's theorem on **EXS** see e.g. [8], Theorem 5.1.3, p. 217, upon noting that the operator $(\mathcal{F}x)(\theta) := [rD + \theta I]x(\theta)$, $\theta \in [-r, 0]$, where D denotes a unique solution to the discrete matrix Lyapunov equation $C_S^T D C_S - D = -I$, belongs to $\mathbf{L}(\mathbf{H})$ is self-adjoint and nonnegative, and solves the Lyapunov operator equation

$$\langle x, \mathcal{F}Ax \rangle_{\mathbf{H}} + \langle Ax, \mathcal{F}x \rangle_{\mathbf{H}} = -\|x\|_{\mathbf{H}}^2 \quad \forall x \in D(A).$$

This is confirmed by applying the Laplace transform directly to (5.1). Moreover,

$$\|\hat{g}\|_{\mathbf{H}^\infty(\Pi^+)} = \frac{a}{1-|b|}$$

and thus (2.8) is satisfied. See [16] for more information. In particular assumptions **(A1)** i.e. **(H1)**÷**(H4)** hold.

(5.1) with nonlinear feedback $u = -f(y)$ reduces to the delay-difference equation

$$w(t) = C_S w(t-r) - b_0 f[c_0^T w(t-r)]$$

which can be solved iteratively by the method of steps. With $f \in S_0$, the operator $y \mapsto f(y)$ maps continuously $L^2(-r, 0)$ into itself and therefore for any $T > 0$ the output y is in $L^2(0, T)$ with

$$y(nr+r+\theta) = c_0^T C_S^n w(\theta) - \sum_{k=0}^{n-1} c_0^T C_S^{n-1-k} b_0 f[y(kr+r+\theta)], \quad n \in \mathbb{N}, n \geq 2, \theta \in [-r, 0],$$

where $y(r+\theta) = c_0^T w(\theta)$, $\theta \in [-r, 0]$. Observing that

$$c_0^T C_S^{2k} b_0 = a(-b)^k, \quad c_0^T C_S^{2k+1} b_0 = 0, \quad k = 0, 1, 2, \dots,$$

and recalling the explicit formulae for \bar{P} [16], pp. 16–17, and $\bar{F}u = g \star u$ (where g is given by [16]), Formula (5.18), leads by elementary manipulations to the conclusion that the closed-loop fixed point output equation

$$y_T = (\bar{P}x_0)_T - (\bar{F}f(y_T))_T$$

has a unique solution $y_T \in L^2(0, T)$ for all $T > 0$. Thus **(A4)** holds.

For comparison purposes the closed-loop semigroup generator corresponding to the linear feedback $f(y) = \mu y$ takes the form

$$A_\mu x = x', \quad D(A_\mu) = \left\{ x \in W^{1,2}(-r, 0) \oplus W^{1,2}(-r, 0) : x(0) = [C_S - \mu b_0 c_0^T] x(-r) \right\}.$$

Indeed, $D(A_\mu)$ consists of these x for which $x - \mu dc^\# x \in D(A)$. The latter holds if $x \in W^{1,2}(-r, 0) \oplus W^{1,2}(-r, 0)$ and $x(0) - \mu dc^\# x = C_S [x(-r) - \mu dc^\# x]$, or equivalently, if $x(0) = [C_S - \mu b_0 c_0^T] x(-r)$. The semigroup generated on $\mathbf{H} = L^2(-r, 0) \oplus L^2(-r, 0)$ by A_μ is **EXS** iff all eigenvalues of the matrix $C_S - \mu b_0 c_0^T = \begin{bmatrix} 0 & 1 \\ -b & -a\mu \end{bmatrix}$, are in the open unit disk [10]. This is the case if

$$|\mu| < \frac{1+b}{a}. \quad (5.2)$$

Stability condition (5.2) yields the *Hurwitz sector* which has to be compared with a sector (k_1, k_2) generated by the frequency-domain inequality (4.3). It is clear that by (5.2) the upper bound for k_2 is $\frac{1+b}{a}$ and the lower bound for k_1 is $-\frac{1+b}{a}$. Now we check assumption **(A2)**. This will be done separately for $b \leq 0$ and for $b > 0$.

5.1. The case of nonpositive b

Substituting $k_2 = -k_1 = \frac{1+b}{a}$ into (4.2) gives

$$\pi(j\omega) = 1 - \left(\frac{1+b}{a} \right)^2 |\hat{g}(j\omega)|^2 = \frac{-4b \sin^2 \omega r}{(1-b)^2 + 4b \cos^2 \omega r} \geq 0 \quad \forall \omega \in \mathbb{R}$$

and therefore the Hurwitz sector (5.2) agrees with the sector implied by (4.18).

TABLE 1. Lower and upper bounds of the sector $\mathcal{S}_{2\nu}$.

Case	k_1	k_2
$b \leq 0$	$-\sqrt{\left(\frac{1+b}{a}\right)^2 - \nu}$	$\sqrt{\left(\frac{1+b}{a}\right)^2 - \nu}$
$b > 0$	$\frac{4b}{a(1+b)} - \sqrt{\frac{(1-b)^4}{a^2(1+b)^2} - \nu}$	$\frac{4b}{a(1+b)} + \sqrt{\frac{(1-b)^4}{a^2(1+b)^2} - \nu}$

TABLE 2. Constants of the Lur'e system (3.2).

Case	q	e	δ
$b \leq 0$	$-\left(\frac{1+b}{a}\right)^2 + \nu$	$\vartheta \left(\frac{1+b}{a}\right)^2 - \nu\vartheta$	$\nu\vartheta^2$
$b > 0$	$\frac{-b^2 + 6b - 1}{a^2} + \nu$	$\frac{b^2 - 10b + 1}{a(1+b)} - \nu\vartheta$	$\frac{16b}{(1+b)^2} + \nu\vartheta^2$

TABLE 3. Constants determining the spectral factor $\phi(s) = \frac{\alpha + \beta e^{-sr} + \gamma e^{-2sr}}{1 + b e^{-2sr}}$.

Case	α	β	γ
$b \leq 0$	$\frac{a\sqrt{\nu} + \sqrt{a^2\nu - 4b}}{2}$	0	$\frac{a\sqrt{\nu} - \sqrt{a^2\nu - 4b}}{2}$
$b > 0$	$\frac{2b}{\beta} + \sqrt{\left(\frac{2b}{\beta}\right)^2 - b}$	$\frac{\sqrt{16b + a^2\nu} - \sqrt{a^2\nu}}{2}$	$\frac{2b}{\beta} - \sqrt{\left(\frac{2b}{\beta}\right)^2 - b}$

To have (4.3) satisfied we replace $k_2 = -k_1 = \frac{1+b}{a}$ by $k_2 = -k_1 = \sqrt{\left(\frac{1+b}{a}\right)^2 - \nu}$ with sufficiently small $\nu > 0$ getting

$$\pi(j\omega) = \frac{-4b \sin^2 \omega r}{(1-b)^2 + 4b \cos^2 \omega r} + \nu |\hat{g}(j\omega)|^2 \geq \nu \inf_{\omega \in \mathbb{R}} |\hat{g}(j\omega)|^2 = \frac{\nu a^2}{(1+|b|)^2} := \eta > 0 \quad \forall \omega \in \mathbb{R},$$

whence **(H5)** holds. Finally **(H6)** is valid by Corollary A.1 because A is dissipative. To see this note that, as $|b| < 1$ ($\iff |\lambda(C_S)| < 1$), $C_S^T C_S - I = \text{diag}\{b^2 - 1, 0\} \leq 0$, whence

$$\langle Ax, x \rangle_{\mathbb{H}} + \langle x, Ax \rangle_{\mathbb{H}} = x^T(-r) [C_S^T C_S - I] x(-r) \leq 0.$$

Now all assumptions of Theorem 4.1 are met and by the latter the Lur'e system (3.1) with k_1, k_2, q, e and δ given in Tables 1 and 2 has a solution $(\mathcal{H}, \mathcal{G})$, $\mathcal{H} \in \mathbf{L}(\mathbb{H})$, $\mathcal{H} = \mathcal{H}^* \geq 0$, whence **(A2)** is met.

By Theorem 3.1 the origin of \mathbb{H} is **GSAS** for any $f \in \mathcal{S}_{2\nu}$. This agrees with the result in [17], Section 4.1, modulo $\varepsilon = 2\nu$.

Here an explicit solution $(\mathcal{H}, \mathcal{G})$ of the equivalent Lur'e system (3.2) is obtained by the method of Theorem 4.1/(ii), for which the details can be found in [18]. The spectral factorization problem admits a simple spectral factor ϕ , $1/\phi \in H^\infty(\Pi^+)$, $\phi(0) = \sqrt{\delta}$ of the form presented in Table 3. The solution \mathcal{G} of the realization

TABLE 4. Solutions of the realization problem (4.4) and the Lur'e system (3.2).

Case	$\mathcal{G} \in \mathbf{H}$	$(\mathcal{H}x)(\theta) = Hx(\theta), H \in \mathbf{L}(\mathbb{R}^2)$
$b < 0$	$\frac{b\alpha - \gamma}{1+b} \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix}$	$H = \text{diag}\{\gamma^2 - b^2, 1 - \alpha^2\} > 0$
$b > 0$	$\begin{bmatrix} \frac{b(\alpha + \beta) - \gamma}{1+b} \mathbf{1} \\ \frac{b\alpha - (\beta + \gamma)}{1+b} \mathbf{1} \end{bmatrix}$	$H = \begin{vmatrix} \gamma^2 - b^2 & \frac{\beta(\gamma - b\alpha)}{1+b} \\ \frac{\beta(\gamma - b\alpha)}{1+b} & 1 - \alpha^2 \end{vmatrix} > 0$

equation (4.4) is unique by the approximate reachability of the pair (A^{-1}, d) . It is given in Table 4, which contains also the corresponding solution \mathcal{H} of the resulting Lyapunov equation in (3.2).

5.2. The case of positive b

The Hurwitz sector (5.2) is essentially larger than the sector implied by (4.18), because for $k_2 = \frac{1+b}{a}$ we cannot take $k_1 = -\frac{1+b}{a}$ to have the latter satisfied. An another choice of k_1, k_2 has to be proposed. Assuming $k_2 = \frac{1+b}{a}$ we search for the minimal admissible value of k_1 for which (4.18) holds. Since

$$\begin{aligned} \pi(j\omega) &= 1 + (k_1 + k_2) \text{Re}[\hat{g}(j\omega)] + k_1 k_2 |\hat{g}(j\omega)|^2 \\ &= \frac{(1+b)^2 \cos^2 \omega r + (1-b)^2 \sin^2 \omega r + (1+b)^2 \cos \omega r + k_1 a(1+b) \cos \omega r + k_1 a(1+b)}{(1-b)^2 + 4b \cos^2 \omega r} \end{aligned}$$

then, treating the numerator as a polynomial in $\cos \omega r$, we give k_1 its minimal admissible value for which the frequency domain inequality (4.18) holds, *viz.* $k_1 = -\frac{1+b}{a} + \frac{8b}{a(1+b)} = -\frac{b^2-6b+1}{a(1+b)}$, whence

$$\pi(j\omega) = \frac{4b(1 + \cos \omega r)^2}{(1-b)^2 + 4b \cos^2 \omega r} \geq 0.$$

For meeting (4.3), we replace $k_2 = \frac{1+b}{a}$ and $k_1 = -\frac{1+b}{a} + \frac{8b}{a(1+b)}$ successively by $k_{1,2} = \frac{4b}{a(1+b)} \mp \sqrt{\frac{(1-b)^4}{a^2(1+b)^2} - \nu}$, with $\nu > 0$ sufficiently small giving

$$\pi(j\omega) = \frac{4b(1 + \cos \omega r)^2}{(1-b)^2 + 4b \cos^2 \omega r} + \nu |\hat{g}(j\omega)|^2 \geq \nu \inf_{\omega \in \mathbb{R}} |\hat{g}(j\omega)|^2 = \frac{\nu a^2}{(1+|b|)^2} := \eta > 0 \quad \forall \omega \in \mathbb{R}.$$

Thus **(H5)** holds. **(H6)** holds as the method of Section 5.1 for checking **(H6)** does not depend on the sign of b . Thus all assumptions of Theorem 4.1 are met and by its assertion the Lur'e system (3.1) with parameters k_1, k_2, q, e and δ given in Tables 1 and 2 has a solution $(\mathcal{H}, \mathcal{G}), \mathcal{H} \in \mathbf{L}(\mathbf{H}), \mathcal{H} = \mathcal{H}^* \geq 0$, whence **(A2)** holds. By Theorem 3.1 the origin of \mathbf{H} is **GSAS** for any $f \in \mathcal{S}_{2\nu}$. This agrees with the result in [17], Section 4.1, modulo $\varepsilon = 2\nu$. Again an explicit solution $(\mathcal{H}, \mathcal{G})$ of the equivalent Lur'e system (3.2) is obtained by the method of Theorem 4.1/(ii), with details in [18]. The simple spectral factor $\phi, 1/\phi \in \mathbf{H}^\infty(\Pi^+), \phi(0) = \sqrt{\delta}$ is given in Table 3. The solution $(\mathcal{H}, \mathcal{G})$ of the equivalent Lur'e system (3.2) is given in Table 4; here again the solution \mathcal{G} of the realization equation (4.4) is unique by the approximate reachability of the pair (A^{-1}, d) .

6. DISCUSSION AND CONCLUSIONS

The most important results of this paper are:

- A criterion for the absolute global strong asymptotic stability presented in Section 3 based on quadratic Lyapunov functionals *viz.* Theorem 3.1, whose assumptions however require to check the solvability of the Lur’e system (3.1).
- Solvability results for this Lur’e system in Section 4, leading to Theorem 4.1. The criterion of Section 3 jointly with those of Section 4 lead to results similar to those of the input-output approach [17].
- A detailed presentation of an example of electrical transmission-line, illustrating the results of previous sections, in Section 5. The discussion shows that this paper’s stability criteria are checkable.

A class of first-order hyperbolic systems on a *higher-dimensional* spatial domain but with *bounded* observation is discussed in [21], Section 7.4.

In [25] a circle criterion has been derived for a nonlinear feedback system having in its feedback loop, an integrator and a sector nonlinearity in front of an infinite-dimensional well-posed linear plant. Due to the smoothing action of the integrator, the results of [25] are not comparable with those of the present paper.

Moreover observe that, contrary to the system of Section 5.1, the absolute stability conditions generated by the circle criterion for the system of Section 5.2 are, in the case where $b < 0$, significantly more conservative than the Hurwitz sector condition. It is known that for finite-dimensional autonomous *continuous* Lur’e systems *Popov’s method* leads to considerably better stability conditions than the circle criterion. It is less known that a generalization of Popov’s method to finite-dimensional autonomous *discrete* Lur’e systems is possible only by *further restricting the class of admissible nonlinearities*. This causes one to expect some difficulties to get an appropriate Popov type stability criterion for the system described by

$$\left\{ \begin{array}{l} A^{-1}\dot{x}(t) = x(t) - df[y(t)] \quad x_0 \in H \\ y(t) = (c^\#x)(t) \end{array} \right\} \tag{6.1}$$

which is sufficiently general to handle discrete-time systems, as can be seen by noting that (5.1) is an equivalent model giving the essentially discrete-time dynamics of the electrical distortionless loaded \mathfrak{RLC} -transmission line (see [12], p. 365, for details). An additional observation is that the input-output approach for finite-dimensional feedback systems is usually based on some smoothness assumptions imposed on the system output. Thus an other difficulty for obtaining a generalization of Popov’s method will be that one has to examine some *differentiability properties of the system output*. This is mainly why in [3] a Popov like criterion was obtained by the Lyapunov method for an infinite-dimensional Lur’e system of *indirect control*, *i.e.*

$$\left\{ \begin{array}{l} \dot{x}(t) = A\{x(t) - df[\sigma(t)]\} \\ \dot{\sigma}(t) = \langle q, x(t) \rangle_H + \rho f[\sigma(t)] \end{array} \right\}. \tag{6.2}$$

Regarding the variable σ as the system output one can readily notice that here the output is differentiable. This is in contrast to (6.1) where the output y is generally not differentiable.

Recently a fairly general result concerning Popov’s criterion for (6.2) has been obtained using the input-output approach [9], Sections 3 and 5.

Finally our results can be applied to a class of systems with an unstable plant. If there exists $\mu \in \mathbb{R}$ such that

$$1 - \mu c^\# d = 1 + \mu \hat{g}(0) \neq 0 \tag{6.3}$$

then, adding and subtracting $\mu dc^\#x$, one can write the closed-loop dynamics as

$$\dot{x}(t) = A \{ x(t) - df [c^\#x(t)] \} = A_\mu \left\{ x - \frac{1}{1 - \mu c^\# d} d f_\mu [c^\#x(t)] \right\}$$

with

$$A_\mu x := A [x - \mu d c^\# x], \quad D(A_\mu) = \{x \in D(c^\#) : x - \mu d c^\# x \in D(A)\} \quad (6.4)$$

and

$$f_\mu(y) := f(y) - \mu y.$$

The new transfer function is

$$\hat{g}_\mu(s) = \frac{1}{1 - \mu c^\# d} [c^\# (sI - A_\mu)^{-1} d - c^\# d] = \frac{\hat{g}(s)}{1 + \mu \hat{g}(s)}.$$

Now one can try to apply the whole theory to the new Lur'e system which is possible only if A_μ generates an **EXS** semigroup *i.e.* when the original linear part is exponentially stabilizable by constant output feedback $u = -\mu y$.

APPENDIX A. ON STRONG ASYMPTOTIC STABILITY OF THE SEMIGROUP $\{e^{tA^{-1}}\}_{t \geq 0}$

In the sequel $\sigma(\cdot)$, $\sigma_P(\cdot)$, $\sigma_C(\cdot)$ will respectively denote the spectrum, the point (*i.e.* discrete) spectrum and the continuous spectrum of an operator. We shall need the following result.

Lemma A.1. *Let $A : (D(A) \subset \mathbb{H}) \rightarrow \mathbb{H}$ be the generator of an **EXS** C_0 -semigroup $\{S(t)\}_{t \geq 0}$ on \mathbb{H} . Then the semigroup $\{e^{tA^{-1}}\}_{t \geq 0}$, generated by $A^{-1} \in \mathbf{L}(\mathbb{H})$, is **(AS)**, *i.e.* for every $x_0 \in \mathbb{H}$, $\lim_{t \rightarrow \infty} e^{tA^{-1}} x_0 = 0$, if and only if $\{e^{tA^{-1}}\}_{t \geq 0}$ is uniformly bounded.*

Proof. If the semigroup $\{S(t)\}_{t \geq 0}$ is **EXS**, then $\sigma_P(A^{-1}) \cap j\mathbb{R} = \emptyset$, $\sigma_P((A^*)^{-1}) \cap j\mathbb{R} = \emptyset$ and $0 \in \sigma_C(A^{-1})$ is the only possible point of the spectrum of A^{-1} on $j\mathbb{R}$. This together with the assumption that $\{e^{tA^{-1}}\}_{t \geq 0}$ is uniformly bounded gives that the semigroup $\{e^{tA^{-1}}\}_{t \geq 0}$ is **AS** by [1], Stability Theorem, p. 837, see also [27].

Conversely, if $\{e^{tA^{-1}}\}_{t \geq 0}$ is **AS**, then for all $x \in \mathbb{H}$, $\sup_{t \geq 0} \|e^{tA^{-1}} x\|_{\mathbb{H}} < \infty$, such that by the uniform boundedness principle $\sup_{t \geq 0} \|e^{tA^{-1}}\|_{\mathbf{L}(\mathbb{H})} < \infty$, whence $\{e^{tA^{-1}}\}_{t \geq 0}$ is uniformly bounded. \square

Corollary A.1. *Let $A : (D(A) \subset \mathbb{H}) \rightarrow \mathbb{H}$ be the generator an **EXS** C_0 -semigroup $\{S(t)\}_{t \geq 0}$ on \mathbb{H} . Then the semigroup $\{e^{tA^{-1}}\}_{t \geq 0}$ is **AS** if the operator inequality*

$$\langle Ax, \mathcal{X}x \rangle_{\mathbb{H}} + \langle \mathcal{X}x, Ax \rangle_{\mathbb{H}} \leq 0 \quad \forall x \in D(A) \quad (\text{A.1})$$

has a bounded self-adjoint solution $\mathcal{X} = \mathcal{X}^ \in \mathbf{L}(\mathbb{H})$ which is coercive, *i.e.* $\mathcal{X} \geq \varepsilon I$ for some $\varepsilon > 0$.*

Proof. Representing \mathcal{X} as $\mathcal{X} = \mathfrak{T}^* \mathfrak{T}$ one can see that $\mathfrak{T} A^{-1} \mathfrak{T}^{-1}$ is a *dissipative operator*, whence A^{-1} is similar to a dissipative operator. Consequently A^{-1} generates a bounded semigroup and the assertion follows from Lemma A.1. \square

Corollary A.2. *Let $A : (D(A) \subset \mathbb{H}) \rightarrow \mathbb{H}$ be the generator an **EXS** C_0 -semigroup $\{S(t)\}_{t \geq 0}$ on \mathbb{H} . Let \mathbb{H} have a Riesz basis of eigenvectors of A . Then the semigroup $\{e^{tA^{-1}}\}_{t \geq 0}$ is **AS**.*

Proof. The eigenvalues of A^{-1} are reciprocals of eigenvalues of A and both semigroups and $\{S(t)\}_{t \geq 0}$ satisfy the *spectrum determined growth assumption*. Hence, the first one is bounded as the second one is **EXS**. Now we apply Lemma A.1. \square

B. PROOF OF LEMMA 2.8

Proof. **a)** If $u \in D(L) = W^{1,2}[0, \infty)$ then, for the first statement observe that with $x(\cdot) + du(\cdot)$ continuous and $z = A^*w, w \in D(A^*)$

$$\begin{aligned} \langle x(t) - x_0, w \rangle &= \int_0^t \langle x(\tau) + du(\tau), A^*w \rangle d\tau \quad \forall w \in D(A^*) \\ \iff \langle x(t) - x_0, [A^{-1}]^*z \rangle &= \int_0^t \langle x(\tau) + du(\tau), z \rangle d\tau \quad \forall z \in H \\ \iff \langle [A^{-1}][x(t) - x_0], z \rangle &= \int_0^t \langle x(\tau) + du(\tau), z \rangle d\tau \quad \forall z \in H \\ \iff \langle [A^{-1}][x(t) - x_0], z \rangle &= \left\langle \int_0^t [x(\tau) + du(\tau)]d\tau, z \right\rangle \quad \forall z \in H \\ \iff [A^{-1}][x(t) - x_0] &= \int_0^t [x(\tau) + du(\tau)]d\tau. \end{aligned}$$

For the second statement note that by Lemma 2.7, (2.3) is a classical solution of (2.2).

b) If $x_0 \in D(A)$ and $u \in D(R^2)$, then $x(\cdot)$ given by (2.3) reduces to (2.4), which defines a *classical* solution of (1.2). Indeed, then

$$x(t) + du(t) = S(t)x_0 + \int_0^t S(t - \tau)d\dot{u}(\tau)d\tau \in D(A) \quad \forall t \geq 0.$$

Here the first component is clearly in $D(A)$ and

$$AS(t)x_0 = S(t)Ax_0 = \dot{S}(t)x_0.$$

For the second one we have by Lemma 2.7 $\int_0^t S(t - \tau)d\dot{u}(\tau)d\tau \in D(A)$ with $\dot{u} \in D(R) \subset D(L)$ replacing u , and by (2.1)

$$\begin{aligned} A \int_0^t S(t - \tau)d\dot{u}(\tau)d\tau &= \frac{d}{dt} \left[\int_0^t S(t - \tau)d\dot{u}(\tau)d\tau \right] - d\dot{u}(t) \\ &= \int_0^t S(t - \tau)d\ddot{u}(\tau)d\tau - d\dot{u}(t), \end{aligned} \tag{B.1}$$

whence

$$\dot{x}(t) = A[x(t) + du(t)] = AS(t)x_0 + \int_0^t S(t - \tau)d\ddot{u}(\tau)d\tau - d\dot{u}(t) \tag{B.2}$$

where the right-hand side of (B.2) is continuous in $t \geq 0$. □

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