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THE PSEUDO-p-LAPLACE EIGENVALUE PROBLEM AND VISCOSITY SOLUTIONS AS $p \to \infty$

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Abstract. We consider the pseudo-p-Laplacian, an anisotropic version of the p-Laplacian operator for $p \neq 2$. We study relevant properties of its first eigenfunction for finite p and the limit problem as $p \to \infty$.

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1. Introduction

Imagine a nonlinear elastic membrane, fixed on a boundary $\partial\Omega$ of a plane domain Ω. If $u(x)$ denotes its vertical displacement, and if its deformation energy is given by $\int_\Omega |\nabla u|^p \, \mathrm{d}x$, then a minimizer of the Rayleigh quotient

$$\frac{\displaystyle\int_\Omega |\nabla u|^p \, \mathrm{d}x}{\displaystyle\int_\Omega |u|^p \, \mathrm{d}x}$$

on $W_0^{1,p}(\Omega)$ satisfies the Euler–Lagrange equation

$$-\Delta_p u = \lambda_p \, |u|^{p-2} u \quad \text{in } \Omega. \tag{1.1}$$

Here $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the well-known p-Laplace operator. This eigenvalue problem has been extensively studied in the literature. A somewhat surprising recent result is that (as $p \to \infty$) the limit equation reads

$$\min \{ |\nabla u| - \Lambda_\infty u, \ -\Delta_\infty u \} = 0. \tag{1.2}$$

Here $\Delta_\infty u = \sum_{i,j} u_{x_i} u_{x_j} u_{x_i x_j}$, $\Lambda_\infty = \lim_{p\to\infty} \Lambda_p$ and $\Lambda_p = \lambda_p^{1/p}$ (see [19, 26]). Although the function $\mathrm{d}(x, \partial\Omega)$ minimizes $||\nabla u||_\infty / ||u||_\infty$, it is not always a viscosity solution of (1.2), see [26].

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If the membrane is woven out of elastic strings in a rectangular fashion, then its deformation energy is given by

$$\int_\Omega \sum_i \left| \frac{\partial u}{\partial x_i} \right|^p \, \mathrm{d}x. \tag{1.3}$$

A minimizer of

$$\frac{\displaystyle\int_\Omega \sum_i \left| \frac{\partial u}{\partial x_i} \right|^p \, \mathrm{d}x}{\displaystyle\int_\Omega |u|^p \, \mathrm{d}x}$$

on $W_0^{1,p}(\Omega)$, if it exists, will satisfy the equation

$$-\tilde{\Delta}_p u = \tilde{\lambda}_p \, |u|^{p-2} u \tag{1.4}$$

with

$$\tilde{\Delta}_p u = \sum_i \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) \tag{1.5}$$

as the **pseudo-p-Laplacian operator**. This operator has been around for while and is treated for instance in [37] (pp. 106 and 155) or [45, 46]. The physical interpretation of the associated energy is our invention. It is the purpose of this paper to investigate ground state solutions of (1.4) in any dimension n. Considerable attention is also given to the limit equation (as $p \to \infty$)

$$\min \left\{ \max_k \, |u_{x_k}| - \tilde{\Lambda}_\infty u, \; -\tilde{\Delta}_\infty u \right\} = 0. \tag{1.6}$$

Here $\tilde{\Lambda}_p = \tilde{\lambda}_p^{1/p}$, $\tilde{\Lambda}_\infty = \lim \tilde{\Lambda}_p$ and $\tilde{\Delta}_\infty u = \sum_{j \in I(\nabla u(x))} |u_{x_j}|^2 u_{x_j x_j}$ with

$$I(\xi) = \left\{ k \in \mathbb{N} \mid 1 \le k \le n, \; \max_{j=1,\dots,n} |\xi_j| = |\xi_k| \right\} \quad \text{for} \quad \xi \in \mathbb{R}^n. \tag{1.7}$$

It is well-known, that the infinite-Laplacian operator Δ_∞ is closely related to finding a minimal Lipschitz extension of a given function $\phi \in C^{0,1}(\partial\Omega)$ into Ω. We shall give a related geometric interpretation of the anisotropic operator $\tilde{\Delta}_\infty$.

In our treatment we were inspired by analogous results on the torsion problem

$$-\Delta_p u = 1 \tag{1.8}$$

which has the limit equation (as $p \to \infty$)

$$\min \{ \, |\nabla u| - 1, \; -\Delta_\infty u \, \} = 0 \tag{1.9}$$

and $\mathrm{d}(x, \partial\Omega)$ as a solution to the limit problem. (see [10, 28] and (6.2) in [22].)

The corresponding pseudo torsion problem

$$-\tilde{\Delta}_p u = 1 \tag{1.10}$$

has limit equation (as $p \to \infty$)

$$\min \left\{ \max_k \, |u_{x_k}| - 1, \; - \sum_{j \in I(\nabla u(x))} |u_{x_j}|^2 u_{x_j x_j} \right\} = 0 \tag{1.11}$$

with $I(\nabla u)$ as in (1.7) (see (6.3) in [22]).

Our paper is organized as follows. In Section 2 we prove the existence, uniqueness and regularity of weak and viscosity solutions. In Section 3 we derive the limit equation for $p \to \infty$. In Section 4 we provide some instructive examples. Section 5 deals with $\tilde{\Delta}_\infty$, its geometric interpretation and minimal Lipschitz extensions. Section 6 is dedicated to a concavity result and Section 7 addresses symmetry questions for symmetric domains.

For later reference let us list some facts in the case that $n = 1$, in which both p-Laplace and pseudo-p-Laplace coincide, because then $\Delta_p u = \tilde{\Delta}_p u = (|u'|^{p-2}u')' = (p-1)|u'|^{p-2}u''$ for C^2–functions. These are taken from [36], see also [18]. Let $(a, b) \subset \mathbb{R}$ and

$$\mu_p = \inf_{W_0^{1,p}((a,b))} \frac{||v'||_p^p}{||v||_p^p}. \tag{1.12}$$

Then the minimizing function w_p solves

$$(p-1)|w_p'|^{p-2}w_p'' + \mu_p|w_p|^{p-2}w_p = 0 \quad \text{in } (a, b), \tag{1.13}$$

with $w_p(a) = w_p(b) = 0$. Moreover, w_p is of class $C^{1,\alpha}$ and

$$\mu_p = (p-1)\left[\frac{2}{b-a}\int_0^1 \frac{\mathrm{d}t}{(1-t^p)^{1/p}}\right]^p. \tag{1.14}$$

A straightforward calculation gives

$$\mu_p^{1/p} = \frac{2\pi(p-1)^{1/p}}{(b-a)p\sin(\frac{\pi}{p})}\ , \tag{1.15}$$

and $\mu_p^{1/p} = \mu_q^{1/q}$ if $1/p + 1/q = 1$.

2. Existence, uniqueness and regularity of solutions

If we minimize the functional

$$J_p(v) = \int_\Omega \sum_{j=1}^n \left|\frac{\partial v}{\partial x_j}\right|^p \mathrm{d}x \quad \text{on} \quad K := \{\ v \in W_0^{1,p}(\Omega) \mid ||v||_{L^p(\Omega)} = 1\ \}, \tag{2.1}$$

then *via* standard arguments (see [39]) a minimizer u_p exists for every $p > 1$ and it is a weak solution to the equation (1.4), *i.e.*

$$\int_\Omega \sum_{j=1}^n \left|\frac{\partial u_p}{\partial x_j}\right|^{p-2} \frac{\partial u_p}{\partial x_j} \cdot \frac{\partial v}{\partial x_j}\ \mathrm{d}x = \tilde{\lambda}_p \int_\Omega |u_p|^{p-2}u_p \cdot v\ \mathrm{d}x \tag{2.2}$$

for any $v \in W_0^{1,p}(\Omega)$. Here $\tilde{\lambda}_p = J_p(u_p)$. Note that $\tilde{\Lambda}_p := \tilde{\lambda}_p^{1/p}$ is the minimum of the Rayleigh quotient

$$R_p(v) := \frac{\left(\int_\Omega \sum_{j=1}^n \left|\frac{\partial v}{\partial x_j}\right|^p \mathrm{d}x\right)^{1/p}}{||v||_p} \tag{2.3}$$

on $W_0^{1,p}(\Omega) \setminus \{0\}$. Without loss of generality we may assume that u_p is nonnegative. Otherwise we can replace it by its modulus. Moreover any nonnegative weak solution of (2.2) is necessarily bounded, as can be shown by adapting the arguments from [35]. The nonnegativity and boundedness of u_p are helpful in deriving its positivity everywhere, which follows from a Harnack-type inequality due to Trudinger [43].

If $p > n$, then u_p is Hölder continuous because of the Sobolev-embedding theorem. But even for general $p > 1$, one can show its Hölder continuity by a different argument of Sakaguchi, see Lemma 2.3 below. Since Sakaguchi's proof requires uniqueness of u_p, let us first prove uniqueness.

Lemma 2.1. *The positive minimizer of (2.1) is unique.*

A proof of this lemma was given in [4] (Th. 5), but only under the additional smoothness assumption $u_p \in C^1(\Omega)$. Our proof does not need this assumption and follows [29] (Prop. 4) and [9], see also [1,3,12,16,40] for related results on the p-Laplace operator. We simply observe that for positive functions v the functional $J_p(v)$ is convex in v^p. In fact under the substitution $w = v^p$ the side constraint in K is linear, and the functional transforms into

$$J_p(v) = p^{-p} \int_\Omega w^{1-p} \sum_{j=1}^n \left| \frac{\partial w}{\partial x_j} \right|^p \, \mathrm{d}x =: p^{-p} \sum_{j=1}^n E_j(w),$$

with

$$E_j(w) = \int_\Omega w^{1-p} \left| \frac{\partial w}{\partial x_j} \right|^p \, \mathrm{d}x.$$

Note that the integrand of E_j can be written as $h(w,y) = w^{1-p} y^p$ with y standing for $|\partial w/\partial x_j|$. This function of two variables is convex, since the trace of its Hessian $D^2 h$ is positive and the determinant of its Hessian vanishes. In fact,

$$\operatorname{trace} D^2 h = (p-1)p w^{-p-1} y^p + p(p-1) w^{1-p} y^{p-2} > 0$$

and

$$\det D^2 h = p^2 (p-1)^2 \left[w^{-p-1} y^p \; w^{1-p} y^{p-2} - w^{-2p} y^{2p-2} \right] = 0.$$

Let us now show how uniqueness of a minimizer u follows from this convexity property. If there are two solutions u and U of (2.1), then for $t \in [0,1]$ the test function $u_t = \eta^{1/p}$ with $\eta := t u^p + (1-t) U^p$ is admissible in (2.1), because $\int_\Omega u_t^p \, \mathrm{d}x = t \int_\Omega u^p \, \mathrm{d}x + (1-t) \int_\Omega U^p \, \mathrm{d}x = 1$.

Now we calculate $\nabla u_t = \eta^{-1+1/p} [t u^{p-1} \nabla u + (1-t) U^{p-1} \nabla U]$, so that

$$\begin{aligned}
\left| \frac{\partial u_t}{\partial x_j} \right|^p &= \eta^{1-p} |t u^{p-1} u_{x_j} + (1-t) U^{p-1} U_{x_j}|^p \\
&= \eta \left| \frac{t u^p}{\eta} \frac{u_{x_j}}{u} + \frac{(1-t) U^p}{\eta} \frac{U_{x_j}}{U} \right|^p \\
&= \eta \left| s(x) \frac{u_{x_j}}{u} + (1 - s(x)) \frac{U_{x_j}}{U} \right|^p \text{ with } s(x) := \frac{t u^p}{t u^p + (1-t) U^p} \in (0,1) \\
&\le \eta \left[s(x) \left| \frac{u_{x_j}}{u} \right| + (1 - s(x)) \left| \frac{U_{x_j}}{U} \right| \right]^p \\
&\le \eta \left[s(x) \left| \frac{u_{x_j}}{u} \right|^p + (1 - s(x)) \left| \frac{U_{x_j}}{U} \right|^p \right] \\
&= t\, u^p \left| \frac{u_{x_j}}{u} \right|^p + (1-t)\, U^p \left| \frac{U_{x_j}}{U} \right|^p \\
&= \; t\, |u_{x_j}|^p + (1-t)\, |U_{x_j}|^p
\end{aligned}$$

and

$$\int_\Omega \left| \frac{\partial}{\partial x_j} u_t \right|^p \, \mathrm{d}x \le t \int_\Omega \left| \frac{\partial}{\partial x_j} u \right|^p \, \mathrm{d}x + (1-t) \int_\Omega \left| \frac{\partial}{\partial x_j} U \right|^p \, \mathrm{d}x. \tag{2.4}$$

This shows that $J_p(u_t) \le t J_p(u) + (1-t) J_p(U)$. Because u and U are both solutions of (2.1), so is u_t. Therefore equality must hold in (2.4) for every $j = 1, \ldots, n$, *i.e.*

$$\frac{\nabla u}{u} = \frac{\nabla U}{U} \quad \text{a.e. in } \Omega. \tag{2.5}$$

But (2.5) implies that $\nabla(u/U) = 0$ a.e. in Ω, so that $u = (\mathit{const.}) \cdot U$. Finally the norm constraint in (2.1) implies that $u = U$. This completes the proof of Lemma 2.1.

Remark 2.2. Let us remark in passing that this convexity argument can be used to prove uniqueness for positive solutions to a more general class of problems, namely

$$\tilde{\Delta}_p u + f(x,u) = 0 \qquad \text{in } \Omega, \tag{2.6}$$

with $u = 0$ on $\partial\Omega$, provided $f : \Omega \times [0,\infty)$ satisfies the hypotheses

(a) for a.e. $x \in \Omega$ the map $r^{1-p} f(x,r)$ is strictly decreasing in $r \in [0,\infty)$;

(b) There exists $c > 0$ with $f(x,r) \le c(r^{p-1} + 1)$ for a.e. $x \in \Omega$ and $r \in [0,\infty)$.

To prove uniqueness for problem (2.6), one has to observe that solutions are critical points of a functional

$$H_p(v) := \int_\Omega \left[\frac{1}{p} \Big(\sum_j |v_{x_j}|^p \Big) - F(x,v) \right] \, \mathrm{d}x$$

with $F(x,v) := \int_0^v f(x,|s|)\,\mathrm{d}s$. Because of (b), the functional H_p is well defined on $W_0^{1,p}(\Omega)$. By definition it is even in v and its first part convex in v^p. The second part $-\int F(x,v)\,\mathrm{d}x$ is even strictly convex in v^p due to (a). Hence H_p can have at most one positive critical point. Corresponding results for the p-Laplacian were stated in [12] in case $p = 2$ and in [16] for general p, but under an additional assumption on f.

Lemma 2.3. *The nonnegative minimizer of (2.1) is Hölder-continuous.*

Let us give two proofs. For the first proof we note that u_p minimizes the functional $J_p(v) - \tilde{\lambda}_p \int_\Omega v^p\,\mathrm{d}x$ on $W_0^{1,p}(\Omega)$ and we observe that the norms $(\sum_{i=1}^n |\xi_i|^p)^{1/p}$ and $(\sum_{i=1}^n |\xi_i|^2)^{p/2}$ are equivalent, and refer to Theorems 2.1 and 3.1 in [20].

For the second proof we follow ideas from [40] and fix $p \in (1,\infty)$. The unique solution u of the degenerate equation (2.1) is approximated by u_ε, where u_ε is minimizes

$$\int_\Omega \sum_{j=1}^n \left(\varepsilon v^2 + \frac{\partial v}{\partial x_j}^2 \right)^{p/2} \mathrm{d}x \quad \text{on} \quad K := \{\ v \in W_0^{1,p}(\Omega) \mid ||v||_{L^p(\Omega)} = 1\ \}\cdot \tag{2.7}$$

The Euler–Lagrange equation associated to (2.7) reads

$$-\sum_{j=1}^n [(\varepsilon u^2 + u_{x_j}^2)^{(p-2)/2}\ u_{x_j}]_{x_j} = \lambda |u|^{p-2} u - \varepsilon \sum_{j=1}^n (\varepsilon u^2 + u_{x_j}^2)^{(p-2)/2}\ u \tag{2.8}$$

and has the advantage of being nondegenerate elliptic. In contrast to (1.4) this equation (2.8) satisfies structural assumptions which lead to *a priori* estimates independent of $\varepsilon > 0$. To be specific, the set $\{u_\varepsilon\}$ is uniformly bounded in $W_0^{1,p}(\Omega)$ as $\varepsilon \to 0$ and has a weakly convergent subsequence converging to the minimizer u of (2.1). From Lemma 9.6 in [21] (p. 213f) or Theorem 7.1 in [31] (p. 286f) applied to (2.8) we have a uniform bound of u_ε in $L^\infty(\Omega)$. This allows us to apply Theorem 1.1 in [31] (p. 251) and to conclude that u_ε has a bound in $C^\alpha(\overline{\Omega})$ uniformly in ε for some $\alpha \in (0,1)$ depending on p. So these functions u_ε converge uniformly by the Ascoli–Arzelà theorem, because they are uniformly bounded and uniformly Hölder continuous. Moreover they converge to the unique (positive normalized) solution of (1.4) which is therefore in $C^\alpha(\overline{\Omega})$ as well. This ends the proof of Lemma 2.3.

Example 2.4. In cartesian coordinates the operator $\tilde{\Delta}_p$ is separable. So an Ansatz of the form $u(z,y) = v(z)w(y)$ with nonnegative v and w and $z = (x_1,\dots,x_j)$, $y = (x_{j+1}\dots,x_n)$ gives

$$\tilde{\Delta}_p u = w^{p-1}(y)\tilde{\Delta}_p u(z) + u^{p-1}(z)\tilde{\Delta}_p w(y). \tag{2.9}$$

As a consequence of this the first eigenfunction u_p and eigenvalue λ_p on a cube $C := (a,b)^n$ (or square) is given by

$$u_p(x) = \prod_{j=1}^{n} w_p(x_j) \qquad \text{and} \quad \tilde{\lambda}_p(C) = n\mu_p, \tag{2.10}$$

where w_p and μ_p are defined in (1.13) and (1.14).

The next item will be viscosity solutions. As in [26] we plan to show that every weak solution is a viscosity solution. For every $z \in \mathbb{R}$, $q \in \mathbb{R}^n$ and for every real symmetric $n \times n$ matrix X we consider the equation

$$F_p(z,q,X) = -(p-1)\sum_{j=1}^{n} \left[|q_j|^{p-2}X_{jj}\right] - \tilde{\lambda}_p|z|^{p-2}z = 0. \tag{2.11}$$

For $p \geq 2$ the function F_p is continuous, while for $p \in (1,2)$ F_p is singular at every q in a Cartesian plane $\{q_k = 0\}$. In this respect the pseudo-Laplace operator is more singular than the p-Laplace operator, for which the corresponding F_p is singular only at one point $q = 0$. The upper and lower semicontinuous envelopes F_p^* and F_{p*} of F_p coincide with F_p for $p \geq 2$ and are obviously modified to $+\infty$ and $-\infty$ on $N := \{(z,q,X) \mid \min_k |q_k| = 0\}$.

Definition 2.5. We call $u \in C(\Omega)$ a viscosity subsolution (resp. supersolution) of (2.11) if

$$F_{p*}(\phi(x), \nabla\phi(x), D^2\phi(x)) \leq 0 \;\; \big(\text{resp. } F_p^*(\phi(x), \nabla\phi(x), D^2\phi(x)) \geq 0\big) \tag{2.12}$$

for every $\phi \in C^2(\Omega)$ with $u - \phi$ attaining a local maximum (resp. minimum) zero at x. We call u a viscosity solution of (2.11) if it is both a viscosity subsolution and a viscosity supersolution of (2.11).

Lemma 2.6. *For $p \geq 2$ every (weak) solution of (2.2) is a viscosity solution of (2.11).*

For the proof we check first if u is a viscosity subsolution. Without loss of generality fix $x_0 \in \Omega$ and choose $\phi \in C^2(\Omega)$ such that $u(x_0) = \phi(x_0)$ and $u(x) < \phi(x)$ for $x \neq x_0$. We want to show that

$$-(p-1)\sum_{k=1}^{n} \left[\left|\frac{\partial\phi}{\partial x_k}(x_0)\right|^{p-2} \frac{\partial^2\phi}{\partial x_k^2}(x_0)\right] - \tilde{\lambda}_p\, |\phi(x_0)|^{p-2}\, \phi(x_0) \leq 0, \tag{2.13}$$

and argue by contradiction. Otherwise there exists a small neighborhood of $B_r(x_0)$, in which (2.13) is violated. Set $M = \sup\{\phi(x) - u(x) \mid x \in \partial B_r(x_0)\}$ and $\Phi = \phi - M/2$. Then $\Phi > u$ on $\partial B_r(x_0)$, $\Phi(x_0) < u(x_0)$ and

$$-\sum_{k=1}^{n} \left(|\Phi_{x_k}|^{p-2}\Phi_{x_k}\right)_{x_k} > \tilde{\lambda}_p\, |\phi(x)|^{p-2}\, \phi(x) \qquad \text{in } B_r(x_0). \tag{2.14}$$

If we multiply (2.14) by $(u-\Phi)^+$ and integrate by parts, we obtain

$$\int_{\{u>\Phi\}} \sum_{k=1}^{n} |\Phi_{x_k}|^{p-2}\Phi_{x_k}(u-\Phi)_{x_k}\, \mathrm{d}x > \tilde{\lambda}_p \int_{\{u>\Phi\}} |\phi|^{p-2}\phi(u-\Phi)\, \mathrm{d}x. \tag{2.15}$$

Now we exploit the fact that u is a weak solution of (2.2) and pick $v = (u-\Phi)^+$, extended by zero outside $B_r(x_0)$, as a test function in (2.2). Then

$$\int_{\{u>\Phi\}} \sum_{k=1}^{n} |u_{x_k}|^{p-2}u_{x_k}(u-\Phi)_{x_k}\, \mathrm{d}x = \tilde{\lambda}_p \int_{\{u>\Phi\}} |u|^{p-2}u(u-\Phi)\, \mathrm{d}x. \tag{2.16}$$

Subtracting (2.15) from (2.16) we obtain

$$\int_{\{u>\Phi\}} \sum_{k=1}^{n} \left(|u_{x_k}|^{p-2}u_{x_k} - |\Phi_{x_k}|^{p-2}\Phi_{x_k}\right)\ (u-\Phi)_{x_k}\ \mathrm{d}x < \tilde{\lambda}_p \int_{\{u>\Phi\}} (|u|^{p-2}u - |\phi|^{p-2}\phi)(u-\Phi)\ \mathrm{d}x. \qquad (2.17)$$

But the right hand side of (2.17) is nonpositive while the left hand side is nonnegative. Therefore $\{\ u(x) > \Phi(x)\} = \emptyset$, a contradiction to $\Phi(x_0) < u(x_0)$. This proves that u is a viscosity subsolution. The proof that u is also a viscosity supersolution is left as an exercise to the reader.

Remark 2.7. If $p \in (1,2)$ we need to modify F_p. If at least one of the $\phi_{x_k}(x_0)$ vanishes, the left hand side of of (2.13) would become $+\infty$; a problem that we cannot resolve so easily. In the context of the p-Laplace operator, Ohmuma and Sato [38] have circumvented this difficulty by changing the differential equation into $-|\nabla u|(\Delta_p u - \Lambda_p |u|^{p-2}u) = 0$. In order to change (2.11) correspondingly, we could replace it by

$$G_p(z,q,X) = \min_k\{|q_k|\} \left(-(p-1)\sum_{j=1}^{n} \left[|q_j|^{p-2}X_{jj}\right] - \tilde{\Lambda}_p |z|^{p-2}z \right). \qquad (2.18)$$

Then G_p is continuous in its arguments. However, a word of caution is in order. If one multiplies a differential equation (even with an everywhere positive smooth factor), the set of its viscosity solutions can change dramatically, see [30] (p. 243f). A better known example for this effect are the (practically identical) eikonal equations $|\nabla u| = 1$ and $-|\nabla u| = -1$, whose viscosity solutions are quite different, see [14].

Now we will prove local Lipschitz regularity, when $p \geq 2$, of viscosity (super)solutions to the pseudo-p-Laplacian equation $-\tilde{\Delta}_p u = 0$. This result is obtained by means of a local version of the Harnack inequality. We show essentially that the signed L^1-distance function

$$\begin{aligned} \delta(x) &= \mathrm{dist}_1(x, \partial Q_r(\xi)) \\ &= \inf\left\{ \sum_{j=1}^{n} |x_j - y_j| \mid y \in \partial Q_r(\xi) \right\} \\ &= r - \sum_{j=1}^{n} |x_j - \xi_j|, \end{aligned} \qquad (2.19)$$

where

$$Q_r(\xi) = \left\{ x \in \mathbb{R}^n\ :\ \sum_{j=1}^{n} |x_j - \xi_j| < r \right\}$$

is a diamond shaped "ball" of radius r with center in ξ, acts locally as a barrier for viscosity solutions of $-\tilde{\Delta}_p u \geq 0$. The key point in the proof of Theorem 2.9 is the following lemma.

Lemma 2.8. *Let us consider $\delta_\alpha(x) = r - \sum_{j=1}^{n} |x_j - \xi_j|^{\alpha_j}$ where $\alpha = (\alpha_j)_{j=1}^n$ is a vector in $\mathbb{R}^n$ such that $0 < \alpha_j < 1$ for every $j = 1, ..., n$. Then, for every $x \in Q_r(\xi)\backslash A$, with*

$$A := \{\ x \in Q_r(\xi)\ :\ x_j = \xi_j \text{ for some } j\ \} \cup \partial Q_r(\xi)\ ,$$

we have

$$\tilde{\Delta}_p \delta_\alpha(x) = (p-1)\sum_{j=1}^{n} \alpha_j^{p-1}(1-\alpha_j)|x_j - \xi_j|^{(\alpha_j - 1)(p-1)-1} > 0. \qquad (2.20)$$

For the proof we just have to compute

$$\begin{aligned} D_{x_j}\delta_\alpha(x) &= -\alpha_j|x_j-\xi_j|^{\alpha_j-1}\left(\frac{x_j-\xi_j}{|x_j-\xi_j|}\right) \\ D_{x_jx_j}\delta_\alpha(x) &= \alpha_j(1-\alpha_j)|x_j-\xi_j|^{\alpha_j-2}, \end{aligned}$$

from which the claim follows.

Theorem 2.9. *Let $Q_r(\xi)\subset\Omega$ and $\delta(x)$ as in (2.19). If $u\geq 0$ is a viscosity solution of $-\tilde{\Delta}_p u(x)\geq 0$ and $u(\xi)>0$, then (i) and (ii) hold for every $x\in Q_r(\xi)$*

(i) $u(x)\geq u(\xi)\dfrac{\delta(x)}{\delta(\xi)}$;

(ii) $u(x)-u(\xi)\geq -\left(\sum_{j=1}^n|x_j-\xi_j|\right)/k$, *with* $k=\delta(\xi)/u(\xi)=r/u(\xi)$.

To prove this theorem take $0<c<k$ and define

$$\omega(x):=\frac{c}{r}u(x)-\frac{\delta(x)}{r}\cdot$$

We have

$$\begin{cases}\omega(\xi) = \dfrac{c}{r}u(\xi)-\dfrac{\delta(\xi)}{r}<\dfrac{k}{r}u(\xi)-1=0\\ \omega(x)\geq 0 \quad \text{on } \partial Q_r(\xi),\end{cases}$$

thus the function $\omega(x)$ attains its negative minimum in the interior of $Q_r(\xi)$. We claim that this minimum is attained in ξ. In fact, otherwise suppose that

$$\min_{Q_r(\xi)}\omega(x)=\omega(x_c)<\omega(\xi).$$

Let us define for $0<\alpha_j<1$, $j=1,...,n$,

$$\omega_\alpha(x)=\frac{c}{r}u(x)-\frac{\delta_\alpha(x)}{r}$$

and set $\overline{\alpha}=\inf\{\ \alpha_j\mid j=1,\dots,n\ \}$. For $x\in Q_r(\xi)$, choosing $\overline{\alpha}$ sufficiently close to 1, the following inequality holds after a lenghty but straightforward calculation

$$\left|\sum_{j=1}^n\left(|x_j-\xi_j|^{\alpha_j}-|x_j-\xi_j|\right)\right|\leq \text{const. }(1-\overline{\alpha}). \tag{2.21}$$

Thus the function ω_α is close to ω and satisfies for a suitable choice of $\overline{\alpha}<1$

$$\begin{cases}\omega_\alpha(\xi) = \dfrac{c}{r}u(\xi)-\dfrac{\delta_\alpha(\xi)}{r}<0\\ \omega_\alpha(x)\geq 0 \qquad \text{on } \partial Q_r(\xi),\end{cases}$$

so it attains a strict negative minimum in a point $x_{c,\alpha}$ in the interior of $Q_r(\xi)$. Using (2.21) and modifying α (if necessary) we can assume also

$$|\omega_\alpha(x_c)-\omega(x_c)|<\omega(\xi)-\omega(x_c).$$

The last inequality implies obviously that $x_{c,\alpha} \neq \xi$. Now we can assume that $x_{c,\alpha} \notin A$, where A is the set previously defined on which δ_α is not C^2. If not, by a continuity argument we can consider another α, nearer 1, for which $x_{c,\alpha} \notin A$. Also the function

$$\frac{r}{c}\omega_\alpha(x) = u(x) - \frac{\delta_\alpha(x)}{c}$$

assumes its negative minimum at $x_{c,\alpha} \notin A$. The function $u(x)$ is a viscosity solution of $-\tilde{\Delta}_p u(x) \geq 0$, and the function $\varphi(x) = (\delta_\alpha(x) + ru(x_{c,\alpha}))/c$ is of class C^2 in a neighbor of $x_{c,\alpha}$ and by construction is a suitable test function for u in $x_{c,\alpha}$. This implies

$$-\tilde{\Delta}_p [\varphi(x)] = -\frac{1}{c}\tilde{\Delta}_p [\delta_\alpha(x)] \geq 0.$$

But this last inequality is clearly in contrast with Lemma 2.8. This means that the minimum of ω_α is attained in ξ, as claimed. Then also ω attains its minimum in ξ, so

$$\omega(x) = \frac{c}{r}u(x) - \frac{\delta(x)}{r} \geq \frac{c}{r}u(\xi) - \frac{\delta(\xi)}{r} = \frac{c}{r}u(\xi) - 1.$$

From the previous inequality, letting $c \to k$,we obtain,

$$ku(x) - \delta(x) \geq ku(\xi) - \delta(\xi) \tag{2.22}$$

which gives in turn

$$ku(x) \geq \delta(x).$$

This last inequality, when we replace k with his expression $\delta(\xi)/u(\xi)$, implies (i).

It follows from inequality (2.22)

$$ku(x) - ku(\xi) \geq \delta(x) - \delta(\xi) = -\sum_{j=1}^{n} |x_j - \xi_j|$$

which is exactly (ii).

Remark 2.10. By means of (i) in Theorem 2.9 it is not difficult to show that the set $T = \{x \in \Omega \,:\, u(x) > 0\}$, where u is the viscosity (super)solution of the pseudo-p-Laplacian, is open and closed in Ω. Thanks to Lemma 2.6, we got a new proof of the strict positivity of u_p, the first eigenfunction of the pseudo-p-Laplacian operator.

Remark 2.11. As in [11] inequality (ii) in Theorem 2.9 gives us local Lipschitz continuity of every viscosity solution to $-\tilde{\Delta}_p u(x) \geq 0$. Consider $y \in \Omega$ and $0 < r \leq \delta(y)$. Let x be a point lying in $Q_{r/4}(y)$. From (ii) in Theorem 2.9 we have (for every $x \in Q_r(y)$)

$$-\frac{u(y)}{r}\left(\sum_{i=1}^{n} |x_i - y_i|\right) \leq u(x) - u(y).$$

Now $y \in Q_{r/2}(x) \subseteq Q_r(y)$, and proceeding as before but changing the rule of x and y we get (for every $y \in Q_{r/2}(x)$)

$$-\frac{u(x)}{r/2}\left(\sum_{i=1}^{n} |x_i - y_i|\right) \leq u(y) - u(x).$$

Putting together those two inequalities we obtain

$$-\frac{u(y)}{r}\left(\sum_{i=1}^{n} |x_i - y_i|\right) \leq u(x) - u(y) \leq \frac{2u(x)}{r}\left(\sum_{i=1}^{n} |x_i - y_i|\right).$$

By applying (i) in Theorem 2.9 to $Q_{r/2}$ and observing that $\sum_{i=1}^n |x_i - y_i| < r/4$ we get

$$\frac{u(x)}{2} \leq u(y).$$

Then we have, for every $x \in Q_{r/4}(y)$

$$-\frac{u(y)}{2}\left(\sum_{i=1}^n |x_i - y_i|\right) \leq u(x) - u(y) \leq \frac{4u(y)}{r}\left(\sum_{i=1}^n |x_i - y_i|\right)$$

and so local Lipschitz continuity follows

$$|u(x) - u(y)| \leq \frac{4}{r}\left(\sup_{t\in Q_r(y)} u(t)\right)\left(\sum_{i=1}^n |x_i - y_i|\right).$$

We can extend this property to any nonnegative viscosity eigenfunction and in particular to the weak first eigenfunction by means of Lemma 2.6. Let us remark that the result must be local because if for a non-negative u we have $-\tilde{\Delta}_p u \geq 0$, then for every $\Lambda > 0$ also Λu verifies the same inequality. Moreover, for $p = 2$ and plane domains with reentrant corners the Lipschitz-constant blows up as ξ approaches such a corner.

3. The limit eigenvalue equation for $p \to \infty$

In this chapter we study the sequence $\{\tilde{\Lambda}_p\}$, $\{u_p\}$ of normalized eigenvalues and eigenfunctions as $p \to \infty$. In particular we will derive the equation which is satisfied by the cluster points u_∞ of u_p. Let us consider a bounded domain $\Omega \subset \mathbb{R}^n$. The L^1-distance function to the boundary $\delta(x)$ introduced in the previous chapter is Lipschitz continuous, satisfies $\max_k |\delta_{x_k}(x)| = 1$ almost everywhere in Ω and it is equal to zero on the boundary of Ω. We have then for every $\varphi \in W_0^{1,\infty}(\Omega)$ and $y \in \partial\Omega$

$$|\varphi(x)| = |\varphi(x) - \varphi(y)| \leq \max_k ||\varphi_{x_k}||_\infty \delta_1(x)$$

which implies

$$\frac{1}{||\delta_1||_\infty} \leq \frac{||\max_k |\varphi_{x_k}|\,||_\infty}{||\varphi||_\infty}. \tag{3.1}$$

Now let us define

$$\tilde{\Lambda}_\infty := \frac{||\max_k |\delta_{x_k}|\,||_\infty}{||\delta||_\infty}. \tag{3.2}$$

Therefore $\tilde{\Lambda}_\infty$ is a geometric quantity related to Ω. It is the inverse of the radius of the largest L^1 ball inside Ω. We can prove the following Lemma 3.1, which explains the analytic meaning of $\tilde{\Lambda}_\infty$.

Lemma 3.1. *The following limit holds*

$$\left(\lim_{p\to\infty} \tilde{\lambda}_p^{1/p} =\right) \lim_{p\to\infty} \tilde{\Lambda}_p = \tilde{\Lambda}_\infty,$$

where $\tilde{\Lambda}_p = R_p(u_p)$.

From the definition of the Rayleigh quotient (see (2.3)) and $\delta(x)$ we get

$$\tilde{\Lambda}_p \leq \frac{1}{||\delta||_p}$$

which implies

$$\limsup_{p\to\infty} \tilde\Lambda_p \le \tilde\Lambda_\infty.$$

In order to obtain the opposite inequality, we observe that $||\nabla u_p||_p \le C < \infty$ uniformly in p, because $\delta(x)$ can be used as a test function in any of the Rayleigh quotients. But then (see also [10] and [26]) Hölder's inequality allows us to conclude that $||\nabla u_p||_m \le C < \infty$ for $p > m > n$. We can thus select a subsequence (still denoted by $\{u_p\}$) converging strongly in C^α and weakly in $W^{1,m}$ to a cluster point u_∞ of the original sequence. This function u_∞ is a viscosity supersolution of $-\tilde\Delta_\infty u = 0$, and so by means of Theorem 5.4 (see Rem. 5.5) we know that $u_\infty > 0$ in Ω. From the lower semicontinuity of the Rayleigh quotient we get now

$$\frac{\left(\sum_{i=1}^n \int_\Omega |u_{\infty,x_i}|^q\right)^{1/q}}{||u_\infty||_q} \le \liminf_{p\to\infty} \frac{\left(\sum_{i=1}^n \int_\Omega |u_{p,x_i}|^q\right)^{1/q}}{||u_p||_q}.$$

Multiplying and dividing the last inequality by $||u_p||_p$, we get by Hölder's inequality that for $p > q$ we have

$$\frac{\left(\sum_{i=1}^n \int_\Omega |u_{\infty,x_i}|^q\right)^{1/q}}{||u_\infty||_q} \le \liminf_{p\to\infty} \left(\tilde\Lambda_p \frac{||u_p||_p}{||u_p||_q}\right).$$

By taking first the limit in p and next the limit in q and using (3.1) we conclude that $\tilde\Lambda_\infty \le \liminf_{p\to\infty} \tilde\Lambda_p$, which completes the proof of the lemma.

Now we derive the limit equation, which the cluster points of the sequence u_p must satisfy.

Theorem 3.2. *Every cluster point u_∞ of the sequence $\{u_p\}$ is a viscosity solution of the equation*

$$F_\infty(u, \nabla u, D^2u) = \min\left\{\max_k\ |u_{x_k}| - \tilde\Lambda_\infty u,\ -\tilde\Delta_\infty u\right\} = 0.$$

We show first the result for viscosity supersolutions. We consider a subsequence $\{u_p\}$ converging uniformly in Ω to a function u_∞. Let us fix a point $\xi \in \Omega$ and a function $\varphi \in C^2$ such that $u_\infty(\xi) = \varphi(\xi)$ and $u_\infty(x) > \varphi(x)$ for $x \neq \xi$. Also fix $B_R(\xi) \subset\subset \Omega$. If $0 < r < R$ we have

$$\inf\{u_\infty(x) - \varphi(x) \mid x \in B_R(\xi)\backslash B_r(\xi)\} > 0.$$

The sequence $\{u_p\}$ converges uniformly, so for sufficiently large p we have

$$\inf\{u_p(x) - \varphi(x) \mid x \in B_R(\xi)\backslash B_r(\xi)\} > u_p(\xi) - \varphi(\xi).$$

For those p we have

$$\inf\{u_p(x) - \varphi(x) \mid x \in B_R(\xi)\} = u_p(x_p) - \varphi(x_p)$$

with $x_p \in B_r(\xi)$, and obviously $x_p \to \xi$ when $p \to \infty$. The function u_p is a viscosity solution of (2.11), therefore according to (2.12)

$$-\sum_{j=1}^n (p-1)|\varphi_{x_j}(x_p)|^{p-2}\varphi_{x_jx_j}(x_p) \ge \tilde\Lambda_p^p|\varphi(x_p)|^{p-2}\varphi(x_p). \tag{3.3}$$

Now $u_\infty(\xi) > 0$, but then also $\varphi(x_p) > 0$ for sufficiently large p and by (3.3) $\max_k |\varphi_{x_k}(x_p)| \neq 0$ for large p. Dividing both members of (3.3) by the term $(p-1)\,[\max_k |\varphi_{x_k}(x_p)|]^{p-4}$ we obtain (all the functions are evaluated in x_p)

$$-\sum_{j=1}^n \left(\frac{|\varphi_{x_j}|}{\max_k |\varphi_{x_k}|}\right)^{p-4} |\varphi_{x_j}|^2\varphi_{x_jx_j} \ge \left(\frac{\tilde\Lambda_p\varphi}{\max_k |\varphi_{x_k}|}\right)^{p-4} \frac{\tilde\Lambda_p^4\varphi^3}{p-1}. \tag{3.4}$$

Let us take the limit for $p \to \infty$ in (3.4). We obtain the following necessary condition:

$$\frac{\tilde{\Lambda}_\infty \varphi(\xi)}{\max_k |\varphi_{x_k}(\xi)|} \le 1, \tag{3.5}$$

and taking in account (3.5), letting $p \to \infty$ in (3.4) we obtain

$$-\tilde{\Delta}_\infty \varphi(\xi) = -\sum_{j \in I(D\varphi(\xi))} |\varphi_{x_j}(\xi)|^2 \varphi_{x_j x_j}(\xi) \ge 0. \tag{3.6}$$

Inequalities (3.5) and (3.6) must hold together, and therefore the cluster points u_∞ of the sequence u_p must satisfy, in the viscosity sense, the following equation

$$\min\left\{ \max_k |\varphi_{x_k}(\xi)| - \tilde{\Lambda}_\infty \varphi(\xi),\ -\tilde{\Delta}_\infty \varphi(\xi) \right\} \ge 0. \tag{3.7}$$

This shows that u_∞ is a viscosity supersolution of

$$F_\infty(u, \nabla u, D^2 u) = \min\left\{ \max_k |u_{x_k}| - \tilde{\Lambda}_\infty u,\ -\tilde{\Delta}_\infty u \right\} = 0.$$

Let us run the proof for subsolutions. Fix a point $\xi \in \Omega$ and a function $\varphi \in C^2$ such that $u_\infty(\xi) = \varphi(\xi)$ and $u_\infty(x) < \varphi(x)$ for $x \ne \xi$. We have to show that

$$\min\left\{ \max_k |\varphi_{x_k}(\xi)| - \tilde{\Lambda}_\infty \varphi(\xi),\ -\tilde{\Delta}_\infty \varphi(\xi) \right\} \le 0.$$

Clearly if $\Big(\max_k |\varphi_{x_k}(\xi)| - \tilde{\Lambda}_\infty \varphi(\xi)\Big) \le 0$, then there is nothing to prove. Therefore we assume $(\max_k |\varphi_{x_k}(\xi)| - \tilde{\Lambda}_\infty \varphi(\xi)\Big) > 0$, *i.e.*

$$\frac{\tilde{\Lambda}_\infty \varphi(\xi)}{\max_k\ |\varphi_{x_k}(\xi)|} < 1. \tag{3.8}$$

As in the supersolution case, repeating step by step the proof but reversing the inequality between left and right member, we get (the functions are all evaluated in x_p, which is now the maximum point of $u_p(x) - \varphi(x)$)

$$-\sum_{j=1}^{n} \left(\frac{|\varphi_{x_j}|}{\max_k |\varphi_{x_k}|} \right)^{p-4} |\varphi_{x_j}|^2 \varphi_{x_j x_j} \le \left(\frac{\tilde{\Lambda}_p \varphi}{\max_k |\varphi_{x_k}|} \right)^{p-4} \frac{\tilde{\Lambda}_p^4 \varphi^3}{p-1}.$$

Letting $p \to \infty$ and taking into account (3.8) we get

$$-\tilde{\Delta}_\infty \varphi(\xi) \le 0,$$

which ends the proof.

We do not know how to prove uniqueness of solutions to the Dirichlet problem for $F_\infty(u, \nabla u, D^2 u) = 0$, but as in [26], we are able to obtain a comparison result. In the setting of viscosity solutions given in [14], the function F_∞ is degenerate elliptic but not proper. Therefore the standard theory cannot be applied directly. The strict positivity of u_p for $1 < p \le \infty$ allows us to consider in place of $F_\infty(u, \nabla u, D^2 u) = 0$ a new equation satisfied by $w_\infty = \log u_\infty$ (see [26, 40]). Let us write

$$G_\infty(\nabla w, D^2 w) = 0, \tag{3.9}$$

where

$$G_\infty(\nabla w,\ D^2w) = \min\left\{ \max_k\ |w_{x_k}| - \tilde{\Lambda}_\infty,\ -\tilde{\Delta}_\infty w - \sum_{j\in I(\nabla w)} |w_{x_j}|^4 \right\}$$

and the set $I(\nabla w)$ is defined as before. We claim that if u is a viscosity supersolution (subsolution) of $F_\infty(u,\ \nabla u,\ D^2u) = 0$, then $w = \log u$ is a viscosity supersolution (subsolution) of $G_\infty(\nabla w,\ D^2w) = 0$. Let us take $\xi \in \Omega$ and $\varphi \in C^2$ such that $\varphi(\xi) = w(\xi)$ and $\varphi(\xi) < w(x)$ for $x \neq \xi$. The function $\theta(x) = e^{\varphi(x)}$ is a good test function for u in ξ. Then we have

$$\min\left\{ \max_k\ |\theta_{x_k}(\xi)| - \tilde{\Lambda}_\infty\theta(\xi),\ -\tilde{\Delta}_\infty\theta(\xi) \right\} \geq 0.$$

We write the last inequality in terms of $\varphi(x)$ as

$$\min\left\{ e^{\varphi}\left(\max_k\ |\varphi_{x_k}| - \tilde{\Lambda}_\infty\right)(\xi),\ -\mathrm{e}^{3\varphi}\left(\tilde{\Delta}_\infty\varphi(\xi) + \sum_{j\in I(D\varphi)} |\varphi_{x_j}|^4\right)(\xi) \right\} \geq 0,$$

and the claim follows. The proof for subsolutions is symmetric.

Now we can study $G_\infty(\nabla w, D^2w) = 0$, which (in contrast to $F_\infty = 0$) is now proper.

Theorem 3.3. *Let Ω be a bounded domain, u be a uniformly continuous viscosity subsolution and v be a uniformly continuous viscosity supersolution of (3.9) in Ω. Then the following equality holds:*

$$\sup_{x\in\overline{\Omega}}(u(x) - v(x)) = \sup_{x\in\partial\Omega}(u(x) - v(x)). \tag{3.10}$$

There is no loss of generality if we assume $u, v \geq 0$. Otherwise we add constants to u and v. We proceed by contradiction. Suppose that (3.10) is false, then

$$\sup_{x\in\overline{\Omega}}(u(x) - v(x)) > \sup_{x\in\partial\Omega}(u(x) - v(x)). \tag{3.11}$$

To obtain a contradiction, we construct a new supersolution w having the following properties:

(i) $||v - w||_\infty$ is small enough to preserve the inequality (3.11);
(ii) w is a *strict* supersolution of (3.9).

With those properties in mind, we introduce the following function (see [26])

$$f(z) = \frac{1}{\alpha}\log\left(1 + A\left(\mathrm{e}^{\alpha z} - 1\right)\right),$$

where $\alpha, A > 1$. This function was shown to satisfy a) through d) in [26]:

a) $f'(z) > 1$ for every $z \geq 0$;
b) f_A is invertible and $(f_A)^{-1} = (f_{A^{-1}})$ for every $z \geq 0$;
c) $1 - [f'(z)]^{-1} + [f'(z)]^{-2}f''(z) < 0$ for every $z \geq 0$;
d) $0 < f(z) - 1 < (A-1)/\alpha$ for every $z \geq 0$.

We define $w = f(v)$. Taking A sufficiently close to 1, property (i) holds easily. Let us check (ii). Let $\xi \in \Omega$ and $\varphi \in C^2$ such that $\varphi(\xi) = w(\xi)$ and $\varphi(x) \leq w(x)$ for $x \neq \xi$. Set $\theta = f^{-1}(\varphi)$. The function f^{-1} is monotone increasing, and so θ is a good test function for v at ξ. But v is a supersolution of (3.9), therefore

$$\min\left\{ \max_k\ |\theta_{x_k}| - \tilde{\Lambda}_\infty,\ -\tilde{\Delta}_\infty\theta(\xi) - \sum_{j\in I(D\theta(\xi))} |\theta_{x_j}(\xi)|^4 \right\} \geq 0. \tag{3.12}$$

It follows from (3.12) that

$$\max_k\ |\theta_{x_k}(P)| - \tilde{\Lambda}_\infty \geq 0, \tag{3.13}$$

$$-\tilde{\Delta}_\infty\theta(\xi)\ -\sum_{j\in I(D\theta(\xi))}|\theta_{x_j}(\xi)|^4 \geq 0. \tag{3.14}$$

But if we write explicitly

$$\begin{aligned}\theta_{x_j} &= [f'(\theta)]^{-1}\varphi_{x_j}\\ \theta_{x_jx_j} &= [f'(\theta)]^{-1}\varphi_{x_jx_j} - [f'(\theta)]^{-3}f''\varphi_{x_j}^2\end{aligned}$$

we get from (3.13)

$$\max_k\ |\theta_{x_k}(\xi)| \geq f'(\theta(\xi))\tilde{\Lambda}_\infty \tag{3.15}$$

$$\max_k\ |\theta_{x_k}(\xi)| - \tilde{\Lambda}_\infty \geq [f'(\theta(\xi)) - 1]\tilde{\Lambda}_\infty. \tag{3.16}$$

With some calculus we obtain from (3.14)

$$-\tilde{\Delta}_\infty\varphi(\xi)\ -\sum_{j\in I(D\varphi(\xi))}|\varphi_{x_j}(\xi)|^4 \geq -\left[1-\frac{1}{f'}+\frac{f''}{(f')^2}\right](\theta(\xi))\sum_{j\in I(D\varphi(\xi))}|\varphi_{x_j}(\xi)|^4.$$

Using c), (3.15) and $\theta(\xi) = v(\xi)$ we get from the previous inequality

$$-\tilde{\Delta}_\infty\varphi(\xi) - \sum_{j\in I(D\varphi(\xi))}|\varphi_{x_j}(\xi)|^4 \geq -\left[1-\frac{1}{f'}+\frac{f''}{f'^2}\right](v(\xi))(f'(v(\xi)))^4\tilde{\Lambda}_\infty^4. \tag{3.17}$$

From (3.16) and (3.17) we obtain

$$\min\left\{\max_k\ |\varphi_{x_k}(\xi)| - \tilde{\Lambda}_\infty\,, -\tilde{\Delta}_\infty\theta(\xi)\ -\sum_{j\in I(D\theta(\xi))}|\theta_{x_j}(\xi)|^4\right\} \geq \rho(\xi) > 0, \tag{3.18}$$

where we have defined

$$\rho(x){=}\min\left\{[f'(\theta(x)){-}1]\tilde{\Lambda}_\infty,\ -\left[1{-}\frac{1}{f'(v(x))}{+}\frac{f''(v(x))}{(f'(v(x)))^2}\right](f'(v(x)))^4\tilde{\Lambda}_\infty^4\right\}.$$

Inequality (3.18) and properties a) and c) tell us that w is a strict supersolution.

Now the contradiction follows easily by standard techniques for viscosity solutions, see [14]. Let us sketch the conclusion. We consider (x_t, y_t) a minimum point of the function

$$u(x) - w(y) - \frac{t}{2}|x-y|^2$$

in $\overline{\Omega}\times\overline{\Omega}$. Up to a subsequence, we have that

$$x_t \to \xi \quad \text{and} \quad y_t \to \xi,$$

where $\xi \in \overline{\Omega}$ is a maximum point of $(u-w)$ in $\overline{\Omega}$. But inequality (3.11) holds, so ξ lies in the interior. We apply the max principle for semicontinuous function (see Chap. 3 in [14] for this results and for the definition of the semijets $\overline{J}^{2,+}(u(x_t))$ and $\overline{J}^{2,-}(w(x_t)))$, which ensure the existence of real symmetric matrices $X_t,\ Y_t$ such that

$$\begin{aligned}(t(x_t-y_t);\ X_t) &\in \overline{J}^{2,+}(u(x_t))\\ (t(x_t-y_t);\ Y_t) &\in \overline{J}^{2,-}(w(x_t))\\ (X_t\nu,\nu)-(Y_t\mu,\mu) &\geq 3t|\nu-\mu|^2.\end{aligned}$$

Now u is a subsolution of $G_\infty = 0$, so

$$G_\infty\left(t(x_t-y_t);\ X_t\right) \leq 0. \tag{3.19}$$

Since w is a strict supersolution of $G_\infty = 0$, we get from (3.18)

$$G_\infty\left(t(x_t-y_t);\ Y_t\right) \geq \rho(x_t) > 0. \tag{3.20}$$

Now (3.19) and (3.20) give after some calculation

$$\rho(x_t) \leq 0,$$

which is obviously a contradiction. This completes the proof.

Remark 3.4. Theorem 3.3 also holds when one of the functions takes the value $-\infty$ on the whole boundary.

A useful application of Theorem 3.3 is the following characterization of $\tilde{\Lambda}_\infty$.

Theorem 3.5. *Let Ω be a bounded convex domain. If u is a continuous positive solution in Ω of*

$$F_\infty(u,\ \nabla u,\ D^2u) = \min\left\{\max_k |u_{x_k}| - \Lambda u,\ -\tilde{\Delta}_\infty u\right\} = 0, \tag{3.21}$$

with zero boundary value, then $\Lambda = \tilde{\Lambda}_\infty$.

Let us observe that, if $\Lambda \leq 0$, then equation (3.21) reduces to $-\tilde{\Delta}_\infty u = 0$ with zero Dirichlet boundary conditions, whose only solution (see Rem. 5.3) is $u = 0$ on the whole Ω. Therefore necessarily $\Lambda > 0$.

In the next step we show that $\Lambda \leq \tilde{\Lambda}_\infty$. Let us fix a point $\xi \in \Omega$ such that

$$\mathrm{dist}_1(\xi, \partial\Omega) = \left[\tilde{\Lambda}_\infty\right]^{-1}.$$

We can assume that $\xi = 0$. If, *ex absurdum*, $\Lambda > \tilde{\Lambda}_\infty$, then the rhombus

$$R_{1/\Lambda}(\xi) = \left\{x \in \mathbb{R}^n\ :\ \sum_{j=1}^n |x_j - \xi_j| < 1/\Lambda\right\}$$

is strictly contained in Ω, and more precisely $\partial\Omega \cap \partial R_{1/\Lambda}(\xi) = \emptyset$. Let $\delta(x) = dist_1(\xi, \partial R_{1/\Lambda}(\xi))$. But $C\delta(x)$ and $u(x)$ are solution of (3.21) in the rhombus $R_{1/\Lambda}(\xi)$ for any positive constant C. The comparison principle (observe that $u > 0$ on the boundary of the rhombus) in Theorem 3.3 gives us

$$\log C\delta(x) \leq \log u(x) \qquad \text{in } R_{1/\Lambda}(\xi),$$

which is clearly a contradiction for large values of C.Thus $0 < \Lambda \leq \tilde{\Lambda}_\infty$.

Let us finally show that $\Lambda = \tilde{\Lambda}_\infty$ by contradiction. Suppose $\Lambda < \tilde{\Lambda}_\infty$. Introduce $\Omega_\varepsilon = \Omega + R_\varepsilon(0)$, and for small value of ε we have $\tilde{\Lambda}_\infty(\Omega_\varepsilon) > \Lambda$. Now consider a rombus $Q_{1/\Lambda}$ such that $\Omega_\varepsilon \cap Q_{1/\Lambda} = \emptyset$, and connect with a narrow tube this rhombus with the domain Ω_ε. Let Ω_Λ the set obtained joining Ω_ε with $Q_{1/\Lambda}$ and with the narrow tube. We have, by construction, that

$$\max_{y\in\Omega_\Lambda} \mathrm{dist}_1(x,y)=\Lambda$$

and $\partial\overline{\Omega}\cup\partial\Omega_\Lambda=\emptyset$. Let u_Λ a positive $\infty-$eigenfunction on Ω_Λ: again, by the comparison principle (remark that $u_\Lambda>0$ on $\partial\Omega$) we get for $C>0$

$$\log Cu_\Lambda(x)\geq \log u(x)$$

for $x\in\Omega$. Letting $C\to 0^+$ we arrive at a contradiction, which completes the proof of the theorem.

Remark 3.6. We can prove (see [26]) Theorem 3.5 in more general domain Ω satisfying the property $\partial\Omega=\partial\overline{\Omega}$ (*i.e.* non-punctured domain). Juutinen [25] obtained a proof of this result for a general domain Ω, but his argument is related to a comparison result proved in [24] for the ∞-Laplacian that actually, in the case of ∞-pseudoLaplacian, we don't know if it holds.

4. Some examples

In order to better understand the limiting case as $p\to\infty$ we shall now study special domains $\Omega\subset\mathbb{R}^2$. In this section u_p is a weak first eigenfuctions for the pseudo-p-Laplace operator, and a point in Ω has Cartesian components x and y.

Example 4.1. Consider the square $S:=\{(x,y)\mid \max\{|x|,|y|\}<1\}$.

We know that the function distance to the boundary (see (2.19) for the definition of dist_1)

$$\delta_S(x,y)=\mathrm{dist}_1((x,y),\partial S)=\min\{1-|x|,1-|y|\}\qquad \forall (x,y)\in S \tag{4.1}$$

is a minimizer for the Rayleigh quotient R_∞ defined in (3.1). Nevertheless we claim that δ_S is *not* a genuine ∞-eigenfunction for S (we adopt here a definition given in [26], where "genuine" is equivalent to be a viscosity solution of the limit Eq. (1.6)). To this end we will show that δ_S does not solve (1.6) in the viscosity sense, which in this case becomes (observe that $\tilde\Lambda_\infty=1/||\delta_S||_\infty=1$)

$$\min\Big\{\max\{|u_x|,|u_y|\}\ -u,\ -\tilde\Delta_\infty u\Big\}=0. \tag{4.2}$$

It is not difficult to verify that $\delta_S(x,y)$ is a viscosity supersolution of (4.2). Also we note that the term $[\max\{|u_x|,|u_y|\}\ -u]$ acts only in the origin. We show that, along the ridge (the set where the function δ_S is not C^1: in this example it is $S\cap\{(x,y)\mid |x|=|y|\}$), δ_S is not a viscosity subsolution of (4.2). Let us consider the point $(1/3,1/3)$ and the following function of class C^2

$$\varphi(x,y)=\frac13-\frac15\left(x-\frac13\right)-\frac45\left(y-\frac13\right)+4\left(x-\frac13\right)^2-\left(y-\frac13\right)^2.$$

It is simple to verify that

(i) $\varphi(1/3,1/3)=2/3=\delta_S(1/3,1/3)$;

(ii) $\varphi(x,y)>\delta_S(x,y)$ for $(x,y)\neq(1/3,1/3)$.

Elementary calculations show that

$$\min\left\{\max\left\{\left|\varphi_x\left(\frac13,\frac13\right)\right|,\left|\varphi_y\left(\frac13,\frac13\right)\right|\right\}-\varphi\left(\frac13,\frac13\right),\ -\tilde\Delta_\infty\varphi\left(\frac13,\frac13\right)\right\}>0,$$

and this proves our claim. But for the set S we are able to compute explicitly a "genuine" viscosity solution of (4.2) (which is another minimizer of the Rayleigh quotient).

Proposition 4.2. *A "genuine" solution of (4.2) in S is given by*

$$u_\infty(x,y) = (1-|x|)(1-|y|). \tag{4.3}$$

In Section 2, Example 2.4 we have shown how to construct the first eigenfuction u_p for the pseudo-p-Laplacian on the square. If we compute the uniform limit as $p \to \infty$ of this sequence of functions we find that the unique limit point is the function u_∞ given in (4.3). According to our analysis u_∞ must solve the limit equation (4.2) in the viscosity sense , and therefore it is the only "genuine" (in the sense above) first eigenfuction for the pseudo-∞-Laplacian. Let us prove this ad hoc. In points where u_∞ is of class C^2, the thesis follows by a simple computation (observe that the second derivatives of u_∞ w.r.t x and w.r.t y are identically 0). The term $[\max\{|u_x|,|u_y|\} - u]$ is active only in the origin, as before.

The function u_∞ is a viscosity supersolution (in points where u_∞ is not C^1, the set of test functions is empty).

Let us proceed with the check for viscosity subsolutions. We have only to test points where u_∞ is not C^1, namely points along the coordinate axes inside the square. For simplicity we do the computations just for the point $(1/3,0)$. Let us consider a function $\varphi(x,y)$ of class C^2 such that

(i) $\varphi(1/3,0) = u_\infty(1/3,0)$;
(ii) $\varphi(x,y) > u_\infty(x,y)$ for every $(x,y) \neq (1/3,0)$.

Clearly for such functions we can construct the tangent plane in $(1/3,0)$. The possible tangent planes for φ in $(1/3,0)$ are

$$p(x,y) = 1 - x + \lambda y, \qquad |\lambda| \leq \frac{2}{3}.$$

Then, for every admissible λ, we have

$$\max\{|\varphi_x|(1/3,0), |\varphi_y|(1/3,0)\} = |\varphi_x(1/3,0)| = 1.$$

But if we consider the restriction $\{(x,y) \in S \mid y=0\}$, we observe that $\varphi_{xx}(1/3,0) \geq 0$, so the thesis follows because we have $-\tilde{\Delta}_\infty \varphi(1/3,0) \leq 0$.

Example 4.3. Let us consider the rhombus $R := \{(x,y) \mid |x|+|y| < 1\}$. We have evidently $\tilde{\Lambda}_\infty = 1$. Let us introduce the function distance to the boundary

$$\delta_R(x,y) = \mathrm{dist}_1((x,y),\partial R) = 1 - |x| - |y| \qquad \forall (x,y) \in R. \tag{4.4}$$

The ridge (of δ_R) is the intersection of R with the coordinate axes and, as before, δ_R is a minimizer of the Rayleigh quotient.

Proposition 4.4. *The function δ_R defined in (4.4) is a "genuine" eigenfunction of the pseudo-∞-Laplacian on R.*

We verify that δ_R satisfies equation (4.2) in the viscosity sense. First of all we observe that in the point $(0,0)$ (and only in this point) the term $[\max\{|u_x|,|u_y|\} - u]$ is active. Clearly δ_R is a viscosity supersolution of (4.2), because in any regular point (outside the ridge) the function δ_R is of class C^2, while on the ridge the set of admissible test functions is empty.

It remains to show that δ_R is a viscosity subsolution. Again we need to verify this fact only for points $(x_0,y_0) \neq (0,0)$ which lie on the ridge. In order to simplify the computations we fix $(x_0,y_0) = (1/3,0)$. Let us consider a function $\varphi(x,y)$ of class C^2 such that

(i) $\varphi(1/3,0) = \delta_R(1/3,0)$;
(ii) $\varphi(x,y) > \delta_R(x,y)$ for $(x,y) \neq (1/3,0)$.

Let us prove that $-\tilde{\Delta}_\infty \varphi(1/3,0) \leq 0$.

This function φ has a tangent plane in $(1/3, 0)$, and this plane must contain (see (i) and (ii)) the straight line through $(0, 0, 1)$ and $(1, 0, 0)$. The equation of such planes is given by

$$p(x, y) = 1 - x - \alpha y$$

where $|\alpha| \leq 1$ (see (ii)). When $|\alpha| < 1$, then

$$\max\{|\varphi_x(1/3, 0)|, |\varphi_y(1/3, 0)|\} = |\varphi_x(1/3, 0)| = 1.$$

But condition (ii) implies also

$$\varphi_{xx}(1/3, 0) \geq 0$$

and then we get $-\tilde{\Delta}_\infty \varphi(1/3, 0) = -\varphi_{xx}(1/3, 0) \leq 0$, as required. When $|\alpha| = 1$, we have

$$|\varphi_x(1/3, 0)| = |\varphi_y(1/3, 0)|\} = 1.$$

Condition (ii) implies

$$\varphi_{xx}(1/3, 0) \geq 0 \quad \text{and} \quad \varphi_{yy}(1/3, 0) \geq 0.$$

As before we obtain $-\tilde{\Delta}_\infty \varphi(1/3, 0) \leq 0$, and then the thesis follows.

Remark 4.5. We do not know if this function is the unique "genuine" viscosity solution.

Remark 4.6. The viscosity solution in the rhombus is a linear function, while in the square we found a quadratic viscosity solution. The rhombus is a "special" domain for our distance function and it seems to be the only domain for which the function L^1-distance to the boundary is a genuine ∞-eigenfunction.

Example 4.7. Let us consider the disk of radius 1 centered in the origin $D = \{(x, y) \mid x^2 + y^2 < 1\}$. As in the first example, we will show that the function distance to the boundary

$$\delta_D(x, y) = \sqrt{1 - [\min\{|x|, |y|\}]^2} - \max\{|x|, |y|\}$$

is not a "genuine" eigenfunction, *i.e.* it does not solve, in the sense of viscosity, equation (4.2) inside D (see Rem. 4.6 in this regard). In particular, δ_D fails the subsolution test. In fact, if we consider the point on the ridge $(3/5, 3/5)$ together with the function

$$\varphi(x, y) = \frac{1}{5} - \frac{11}{12}\left(x - \frac{3}{5}\right) - \frac{5}{6}\left(y - \frac{3}{5}\right) - \frac{1}{2}\left(x - \frac{3}{5}\right)^2$$

we can show that δ_D fails to be a viscosity subsolution of (4.2) in the above point.

We do not know any "genuine" ∞-eigenfunction for the disk.

5. A geometric interpretation of $\tilde{\Delta}_\infty$

In this chapter we give a geometrical meaning to the limit for $p \to \infty$ for solutions to the Dirichlet problem

$$\begin{cases} -\tilde{\Delta}_p u = -\sum_{j=1}^n \left(|u_{x_j}|^{p-2} u_{x_j}\right)_{x_j} = 0 & \text{on } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{5.1}$$

where $\Omega \subseteq \mathbb{R}^n$ is a bounded domain and $g \in \mathrm{Lip}(\partial\Omega)$ (we will define later this function space). This is inspired by Jensen [24], where a similar discussion is given for the p-Laplacian operator. In [5] Aronsson introduced the definition of a *Minimal Lipschitz Extension* (briefly **MLE**), that is a function $u \in W^{1,\infty}(\Omega)$ such that

$$||\nabla u||_{L^\infty(\Omega)} \leq ||\nabla w||_{L^\infty(\Omega)}, \qquad \forall w \text{ s.t. } (u - w) \in W_0^{1,\infty}(\Omega). \tag{5.2}$$

When the domain has a sufficiently regular boundary, we can say that u is a **MLE** of g *into* Ω, where $g = u|_{\partial\Omega}$. Such an extension exists but is obviously not unique. But Aronsson provided also the definition of *Absolutely Minimizing Lipschitz Extension* (briefly **AMLE**), that is a function $u \in W^{1,\infty}(\Omega)$ such that for every $D \subset\subset \Omega$

$$||\nabla u||_{L^\infty(D)} \leq ||\nabla w||_{L^\infty(D)}, \qquad \forall w \text{ s.t. } (u-w) \in W_0^{1,\infty}(D). \tag{5.3}$$

Now the uniqueness of **AMLE** becomes an interesting problem. Aronsson proved in [5] that an **AMLE** of class $C^2 \cap \mathrm{Lip}(\Omega)$ is unique, but this is not enough because in [6] he constructed an **AMLE** of class $C^{4/3}$ but not C^2. This means that the class C^2 of "classical" solutions is in general too small to solve the **AMLE** problem. A natural question is the following: what is the Euler equation of (5.3)? The approach in [5] was to consider the minimal p-harmonic extension, *i.e.* a function $u \in W^{1,p}(\Omega)$ such that

$$||\nabla u||_{L^p(\Omega)} \leq ||\nabla w||_{L^p(\Omega)}, \qquad \forall w \text{ s.t. } (u-w) \in W_0^{1,\infty}(\Omega), \tag{5.4}$$

or, equivalently (for finite p !), for every $D \subset\subset \Omega$

$$||\nabla u||_{L^p(D)} \leq ||\nabla w||_{L^p(D)}, \qquad \forall w \text{ s.t. } (u-w) \in W_0^{1,p}(D). \tag{5.5}$$

The inequality in (5.5) can be read as absolutely minimal p-harmonic extension, but for finite p the solutions of (5.4) and (5.5) are the same. Now the Euler equation of (5.5) is given by

$$\begin{cases} -\Delta_p u = -\sum_{j=1}^n \left(|\nabla u|^{p-2}\nabla u\right)_{x_j} = 0 & \text{on } \Omega \\ \hfill u = g & \text{on } \partial\Omega. \end{cases} \tag{5.6}$$

If we (formally) expand the derivatives and (formally) divide both members by $(p-2)|\nabla u|^{p-4}$; after sending (formally) $p \to \infty$ in (5.6) we get

$$\Delta_\infty u = \sum_{i,j=1}^n \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j} = 0. \tag{5.7}$$

The limit equation (5.7) can then be interpreted as the Euler equation of (5.3), as (5.3) is the limit for $p \to \infty$ of (5.5). The operator Δ_∞ is called the ∞-*Laplacian.* Jensen [24] showed that given a function $g \in \mathrm{Lip}(\partial\Omega)$

(i) there exists an **AMLE** of g into Ω;
(ii) every **AMLE** on Ω is a solution of (5.7);
(iii) the viscosity solution of (5.7) with Dirichlet datum g is unique, and in this sense the **AMLE** is uniquely determined.

Observe that the definition of Lipschitz function depends on the metric that we consider in $\mathbb{R}^n$. A real valued function is Lipschitz continuous with Lipschitz constant L if $|u(x) - u(y)| \leq L|x-y|$. Here $|x-y|$ is the Euclidean distance and $L = ||\nabla u||_\infty$, but other distances are conceivable. If we introduce the following (cab driver's) distance function

$$d(x,y) = \sum_{i=1}^n |x_i - y_i|,$$

the Lipschitz constant in $|u(x) - u(y)| \leq L\ d(x,y)$ is given by the number $L = \sup_x \max_i |u_{x_i}(x)|$, *i.e.* the formal limit as $p \to \infty$ of

$$|||\nabla u|||_{L^p(\Omega)} = \left(\sum_{i=1}^n \int_\Omega |\partial u/\partial x_i|^p \,\mathrm{d}x\right)^{1/p}, \tag{5.8}$$

an equivalent L^p norm of ∇u.

For later reference, we define also the (pseudo-distance) function

$$d_\Omega(x,y) = \liminf_{(\xi,\eta)\to(x,y)} \inf\left\{ \int_0^1 \sum_{i=1}^n \left|\frac{\mathrm{d}\beta_i}{\mathrm{d}t}\right| \, \mathrm{d}t \ : \ \beta \in C^1([0,1],\Omega),\ \beta(0)=\xi \ \ \beta(1)=\eta \right\} \tag{5.9}$$

(this function is not a distance, in general, because it does not verify the triangle inequality in a set like $\{(x_1,x_2) \mid x_1^2+x_2^2<1\}\backslash\{[-1/2,1/2]\times\{0\}\}$) and the function space

$$\mathrm{Lip}(\partial\Omega) = \left\{ g \in C(\partial\Omega) \mid \sup_{x,y\in\partial\Omega} \left(\frac{|g(x)-g(y)|}{d_\Omega(x,y)}\right) < \infty \right\}. \tag{5.10}$$

The Euler equation related to the variational problem in (5.5), with $||\cdot||$ replaced by $|||\cdot|||$ from (5.8), is exactly the one in (5.1). Let us expand the derivative (formally) in (5.1) obtaining

$$-(p-1)\sum_{i=1}^n |u_{x_i}|^{p-2} u_{x_i x_i} = 0. \tag{5.11}$$

Dividing both members of (5.11) (formally) by $(p-1)[\max_k |u_{x_k}|]^{p-4}$ and letting (formally) $p\to\infty$ we obtain

$$-\tilde{\Delta}_\infty u(x) = -\sum_{j\in I(\nabla u(x))} |u_{x_j}(x)|^2 u_{x_j x_j}(x) = 0 \tag{5.12}$$

where $I(\xi) = \{\ k\in\mathbb{N} \mid 1\le k\le n,\ \max_{j=1,\dots,n}|\xi_j| = |\xi_k|\ \}$ for $\xi\in\mathbb{R}^n$. (see also [22], Eq. (6.3)) and $\tilde{\Delta}_\infty$ is the pseudo-∞-Laplacian operator. Equation (5.12) can therefore be interpreted as the Euler equation related to the **AMLE** problem. In fact we will now show the following:

if $u\in C^2\cap W^{1,\infty}$ satisfies (5.3) with $||\cdot||$ replaced by $|||\cdot|||$ and

$$|||\nabla u|||_{L^\infty(D)} = \max_{k=1,\dots,n} ||u_{x_k}||_{L^\infty(D)}$$

for every $D \subset\subset \Omega$ (i.e. if u is an **AMLE**), then u solves (5.12).

We fix a point $\xi\in\Omega$, and we can suppose $\xi=0$. Then we take $B(0,\varepsilon)\subset\Omega$ and we define

$$w(x) = u(x) + \frac{\gamma}{2}\varepsilon^2 - \frac{\gamma}{2}|x|^2.$$

Taylor expansions yield $(i,j=1,\dots,n)$

$$\begin{cases} u(x) = u(\xi) + p_i x_i + \dfrac{1}{2}\mu_{ij}x_i x_j + o(|x|^2) \\ w(x) = u(\xi) + \dfrac{\gamma}{2}\varepsilon^2 + p_i x_i + \dfrac{1}{2}(\mu_{ij}-\delta_{ij}\gamma)x_i x_j + o(|x|^2). \end{cases}$$

Calculus gives

$$\begin{cases} ||u_{x_j}||_{L^\infty(B(0,\varepsilon))} = |p_j| + \varepsilon\sqrt{\sum_{i=1}^n \mu_{ij}^2} + o(\varepsilon) \\ ||w_{x_j}||_{L^\infty(B(0,\varepsilon))} = |p_j| + \varepsilon\sqrt{\sum_{i=1}^n (\mu_{ij}-\delta_{ij}\gamma)^2} + o(\varepsilon). \end{cases}$$

Now u is an **AMLE**, and observing that for sufficiently small ε we have $I(\nabla u(B(0,\varepsilon)) = I(\nabla w(B(0,\varepsilon)) = I(\nabla u(0)$, (5.3) translates into

$$\begin{aligned} 0 \ &\leq \ \frac{1}{\#I(\nabla u(0))}\left(\sum_{j\in I(\nabla u(0))}||w_{x_j}||_{L^\infty(B(0,\varepsilon))} - \sum_{j\in I(\nabla u(0))}||u_{x_j}||_{L^\infty(B(0,\varepsilon))}\right)\\ &\leq \ \frac{\varepsilon}{\#I(\nabla u(0))}\sum_{j\in I(\nabla u(0))}\left(\sqrt{\sum_{i=1}^{n}(\mu_{ij}-\delta_{ij}\gamma)^2}-\sqrt{\sum_{i=1}^{n}\mu_{ij}^2}\right)+o(\varepsilon)\\ &=: \varepsilon f(\gamma)+o(\varepsilon). \end{aligned}$$

Dividing this inequality by ε and letting $\varepsilon \to 0$, we obtain $f(\gamma) \geq 0$ for every real γ. But u is an **AMLE** $(0 = f(0) \leq f(\gamma))$, and the function f is of class C^1, therefore we have

$$\frac{\mathrm{d}f}{\mathrm{d}\gamma}(0) = 0.$$

The last equality translates into

$$-\sum_{j\in I(\nabla u(0))} u_{x_jx_j} = 0,$$

but this equation is exactly equation (5.12), apart from a constant factor. Let us make some remarks.

Remark 5.1. The limit equation depends strongly on the metric (*i.e.* on the definition of a Lipschitz function): with the Euclidean metric we obtain the ∞-Laplacian operator while under our choice we get the so-called pseudo-∞-Laplacian.

Remark 5.2. In the points where the gradient is identically zero, equation (5.12) does not make sense. This observation is not new. In fact Aronsson proved [5] (see also [24]):

Let $u \in C^2(\Omega)$ be a nonconstant solution of (5.7), then $|\nabla u| > 0$ in Ω.

Points where the gradient vanishes are points where there is a loss of regularity: in equation (5.12) this fact become more apparent than in equation (5.7). See also the work of Crandall *et al.* [13] in this regard.

Remark 5.3. The Dirichlet problem for equation (5.12) has a unique viscosity solution. This follows from a result of Barles and Busca [8], which generalizes Jensen's uniqueness theorem.

From the proof of Lemma 2.8 we have

$$\tilde{\Delta}_\infty \delta_\alpha(x) = \sum_{j\in I(D\delta_\alpha)} \alpha_j^3(1-\alpha_j)|x_j-\xi_j|^{3\alpha_j-4} > 0, \qquad \forall\ x\in Q_r(\xi)\backslash A. \tag{5.13}$$

Via inequality (5.13) we have the following result (see [11] for the corresponding result on the ∞-Laplacian case).

Theorem 5.4. *Under the assumptionss of Theorem 2.9, if* $u \geq 0$ *is a viscosity solution of* $-\tilde{\Delta}_\infty u(x) \geq 0$ *and* $u(\xi) > 0$*, then*

(i) $u(x) \geq u(\xi)\delta(x)/\delta(\xi)$ *for every* $x \in Q_r(\xi)$*, where as before* δ *is the distance to the boundary of* $Q_r(\xi)$*;*

(ii) $u(x) - u(\xi) \geq -\left(\sum_{j=1}^n |x_j - \xi_j|\right)/k$*, for every* $x \in Q_r(\xi)$*, where* $k = \delta(\xi)/u(\xi)$*.*

We omit the proof, which is essentally the same as that of Theorem 2.9.

Remark 5.5. Remark 2.10 and Remark 2.11 hold word by word and therefore nonnegative viscosity (super)solutions of (5.12) are locally Lipschitz continuous and are either trivial or strictly positive in Ω.

6. Concavity

Let us now return to the case of finite $p \in (1,\infty)$ and to the unique weak solutions of (1.4). Sakaguchi proved that $u^{(p-1)/p}$ is concave if u solves (1.8), and Ishibashi and Koike proved it for solutions of (1.10). Moreover, Sakaguchi proved that the solutions of (1.1) are all log-concave, *i.e.* $\log u$ is always concave. It is the purpose of this section to prove log-concavity for solutions of (1.4).

Theorem 6.1. *If Ω is convex, then the solution of* (1.4) *(and thus also of* (1.6)*) is log-concave.*

For the proof we follow ideas from [40], see also [27], and fix $p \in (1,\infty)$. The normalized solution u (with $||u||_p = 1$) of the degenerate equation (1.4) is approximated by u_ε, where u_ε is normalized and solves

$$-\sum_{j=1}^{n}[(\varepsilon u^2 + u_{x_j}^2)^{(p-2)/2}\ u_{x_j}]_{x_j} = \Lambda|u|^{p-2}u - \varepsilon\sum_{j=1}^{n}(\varepsilon u^2 + u_{x_j}^2)^{(p-2)/2}\ u. \tag{6.1}$$

As explained in the proof of Lemma 2.3 above, these functions u_ε converge uniformly by the Ascoli–Arzelà theorem, because they are uniformly bounded and uniformly Hölder continuous. Moreover they converge to the unique (positive normalized) solution of (1.4). Since log-concavity is preserved under uniform convergence it suffices to prove log-concavity of the solutions to the regularized equation (6.1).

To this end we set $v_\varepsilon = \log u_\varepsilon$ and notice that v_ε satisfies the regular elliptic equation

$$-\sum_{j=1}^{n}[(\varepsilon + v_{x_j}^2)^{(p-2)/2}\ v_{x_j}]_{x_j} = \Lambda + \sum_{j=1}^{n}\left[(\varepsilon + v_{x_j}^2)^{(p-2)/2}\left((p-1)v_{x_j}^2 - \varepsilon\right)\right], \tag{6.2}$$

whose coefficients and right hand side depend only on ∇v. Now one has to observe that v_ε is a classical solution, because (6.2) is nondegenerate elliptic and fortunately there are the necessary estimates from [32] or [44]. Therefore a theorem of Korevaar applies as in the case of the usual p-Laplace operator, see [40] for more details.

Remark 6.2. For the benefit of the reader let us remark in passing, that the corresponding proof of Theorem 6.1 in [22] contains several typographical errors (which do not affect the validity of the theorem). In fact, the assumption $p > n$ is not needed and can be replaced by $p > 1$. Moreover, setting $I_1 = \emptyset$ in [22], equation (5.5), this equation must read

$$-\sum_{k\in I_2}(A(v_{x_k})v_{x_k})_{x_k} = \hat{G}_\varepsilon(v, Dv)$$

with

$$A(r) = \left(\varepsilon + \left(\frac{p-1}{p}\right)^2 r^2\right)^{(p-2)/2} \qquad \text{for} \quad r \in \mathbb{R}$$

and

$$\hat{G}_\varepsilon(r,q) = \frac{1}{v}\frac{p-1}{p^2}\left\{1 + \sum_{k\in I_2}\left(\frac{p^2}{p-1}q_k^2 - \varepsilon\right)A(q_k)\right\} \qquad \text{for} \quad (r,q) \in \mathbb{R}^{1+n}.$$

7. Symmetry

In the present section we remain in the case of finite $p \in (1,\infty)$ and with the (unique) weak solutions.

Remark 7.1. Mohammed Moussa from Kenitra City in Morocco has asked one of us if the first eigenfunction on a ball is radially symmetric. The answer to this question is negative, because any radial positive eigenfunction $u(r)$ would have to satisfy $u'(r) \le 0$ and

$$-(p-1)\left(\frac{|u'(r)|}{r}\right)^{p-2}\sum_i\left[\frac{u''(r)}{r^2}{x_i}^2 + \frac{u'}{r} - \frac{u'(r)}{r^3}{x_i}^2\right]|x_i|^{p-2} = \lambda_p u^{p-1}\ ,$$

or after a partial separation of variables

$$\sum_i \left[\left(\frac{u'(r)}{r} \right)' |x_i|^p + u'(r)\ |x_i|^{p-2} \right] = g(r) := -\frac{\lambda_p}{p-1}\ u^{p-1}\ r^{p-1}\ |u'(r)|^{2-p}\ .$$

Let us show that for $p \neq 2$ this leads to a contradiction. In fact, for the choices $x = (r, 0, \ldots, 0)$ and $x = (r\sin\theta, r\cos\theta, 0, \ldots, 0)$ we obtain two equations for two unknowns $(u'/r)'$ and u'. Elementary algebra gives now

$$u'(r) = -g(r)r^{2-p}\ \frac{1 - |\sin\theta|^p - |\cos\theta|^p}{|\cos\theta|^2\ |\sin\theta|^{p-2} + |\sin\theta|^2\ |\cos\theta|^{p-2}} = -g(r)r^{2-p}\ h(\theta)$$

with $h(\theta)$ independent of r. Now numerical evidence suggests that for most p the factor h does indeed depend on θ, in which case we have a contradiction. But even if h happens to be constant h_p, which is the case for $p = 6$, we arrive at the absurd fact that

$$|u'(r)|^{p-1} = -\frac{h_p \lambda_p}{p-1}\ u^{p-1}\ r \quad \text{or} \quad u'(r) = c(p)\ u\ r^{1/(p-1)}\ .$$

An integration of this last equation shows that its solution cannot have compact support.

Since the first eigenfunction on a ball is never radially symmetric, does it have any symmetries at all? The pseudo-p-Laplacian operator is not invariant under rotations, but only invariant under reflections in cartesian directions (exchanging x_i by $-x_i$) or in diagonal directions (permuting x_i with x_j). Due to their uniqueness this must be reflected in its first eigenfunctions. Therefore on a disk any positive p-eigenfunction has level sets which are convex (due to Sect. 6) and symmetric (due to its uniqueness). In other words, on a ball in $\mathbb{R}^n$ with center in the origin the function u_p is Steiner-symmetric with respect to every cartesian plane $x_i = 0$ or diagonal plane $x_i = x_j$.

Some of this statement prevails if Ω is less symmetric and if one employs rearrangement arguments as in [27].

Theorem 7.2. *Let $\Omega \subset \mathbb{R}^n$ be Steiner symmetric with respect to a cartesian plane $x_i = 0$ or a diagonal plane $x_i = x_j$, and let $u \in W_0^{1,p}(\Omega)$ and u^* its Steiner-symmetrization. Then for every $i = 1, \ldots, n$*

$$\int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^p\ \mathrm{d}x \geq \int_\Omega \left| \frac{\partial u^*}{\partial x_i} \right|^p\ \mathrm{d}x. \tag{7.1}$$

This theorem implies that eigenfunctions on special Steiner-symmetric domains must be Steiner symmetric, because they minimize the corresponding Rayleigh quotients. For cartesian planes the theorem is a special case of Theorem 2.31 and Corollary 2.32 in [27]. Note that it does not deal with symmetries in other than cartesian directions.

Remark 7.3. One may wonder what happens to expressions like $\int\ |u_{x_i}|^p\ \mathrm{d}x$ under Schwarz symmetrization. Let us dispel any hopes about analogues to Theorem 7.2 for this kind of symmetrization by providing a counterexample that we learned from Talenti [41]. Let $f : \mathbb{R} \to \mathbb{R}_+$ be a positive, smooth, decreasing and rapidly decaying function, α a positive parameter and u defined in $\mathbb{R}^2$ by $u(x,y) = f(\alpha^{-2}x^2 + \alpha^2 y^2)$. Then a calculation shows

$$\int_{\mathbb{R}^2} \left| \frac{\partial u}{\partial x} \right|^p\ \mathrm{d}x\mathrm{d}y = \alpha^{-p} \cdot 4^p \cdot \frac{[\Gamma((p+1)/2)]^2}{\Gamma(p+1)} \cdot \int_0^\infty\ t^{p/2} |f'(t)|^p\ \mathrm{d}t,$$

so that

$$\int_{\mathbb{R}^2} \left| \frac{\partial u}{\partial x} \right|^p\ \mathrm{d}x\mathrm{d}y$$

can be made arbitrarily small by taking α large enough, while the corresponding integral for the Schwarz-symmetrization u^* of u is the one that is attained for $\alpha = 1$.

Symmetrization methods are also used to prove Faber–Krahn type results for the p-Laplacian operator. These state that among all domains of given volume the first eigenvalue $\lambda_p(\Omega)$ is minimized by a ball of same volume. What about $\tilde{\lambda}_p(\Omega)$ as a function of the domain? After all, even for the function u in Remark 7.3 the Rayleigh quotient (2.3) is minimal at $\alpha = 1$, *i.e.* in the radially symmetric case. This question is answered as follows

Theorem 7.4. *Among all domains of given volume, the ℓ_q ball (given by $\{\ x \in \mathbb{R}^n\ |\sum_j |x_j|^q \leq c\}$ with suitable c) minimizes $\tilde{\lambda}_p(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$.*

This theorem is consequence of a special case of a fundamental result on convex rearrangement, in which level sets of u are replaced by sets of equal volume and prescribed convex shape K, see Theorem 3.1 and the example (with $p = 4$) on p. 287 in [2]. Under such rearrangement, and K is homothetic to the dual of the unit ball ℓ_p, the numerator in the Rayleigh quotient is shown to decrease, while the denominator stays invariant.

In terms of plane domains, as $p \to \infty$ or as the strings in our woven membrane become more and more elastic, K must approach the shape of a rhombus if $\lambda_p(K)$ is minimal among all domains of given area. On the other hand for $p \to 1$ the membrane with smallest fundamental eigenvalue attains the shape of a square.

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Notes added in proofs

Last week Thierry Champion from Toulon kindly pointed out to us that in passing from (3.4) to (3.6) we had jumped too fast to the conclusion. This gap in the argument can be bridged by applying the (somewhat lengthy but apparetnly unavoidable) reasoning in [22], Section 3. The same applies to the two formulas after (3.8) of our paper.

Since the acceptance of this manuscript some of its results have been generalized to a more general class of operators, see M. Belloni, V. Ferone, B. Kawohl, Isoperimetric inequalities, Wolf shape and related questions for strongly nonlinear elliptic operators, *Journ. Appl. Math. Phys. (ZAMP),* to appear.

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