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FEEDBACK STABILIZATION OF NAVIER–STOKES EQUATIONS

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Abstract. One proves that the steady-state solutions to Navier–Stokes equations with internal controllers are locally exponentially stabilizable by linear feedback controllers provided by a LQ control problem associated with the linearized equation.

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1. Introduction

Consider the controlled Navier–Stokes system with the non-slip Dirichlet boundary conditions

$$\begin{aligned}
&y_t(x,t) - \nu\Delta y(x,t) + (y\cdot\nabla)y(x,t) = m(x)u(x,t) + f_0(x) + \nabla p(x,t), \quad (x,t)\in Q\\
&(\nabla\cdot y)(x,t) = 0, \qquad \forall\,(x,t)\in Q = \Omega\times(0,\infty)\\
&y = 0, \qquad \text{on } \Sigma = \partial\Omega\times(0,\infty)\\
&y(x,0) = y_0(x), \qquad x\in\Omega.
\end{aligned} \tag{1.1}$$

These equations govern the motion of viscous incompressible flows in a domain Ω of R^d, $d = 2$ or $d = 3$ where $y = (y_1, ..., y_d)$ is the state, the velocity field $u = (u_1, u_2, ..., u_d)$ is the control input and p is the unknown pressure. Here m is the characteristic function of an open subset ω of Ω, and $f_0, y_0 \in (L^2(\Omega))^d$, $\nabla\cdot y_0 = 0$ are given vector fields.

Denote by n the normal to $\partial\Omega$ and set

$$H = \{y\in(L^2(\Omega))^d,\ \nabla\cdot y = 0,\ y\cdot n = 0 \text{ on } \partial\Omega\},\ V = \{y\in(H_0^1(\Omega))^d,\ \nabla\cdot y = 0\}\cdot$$

Denote by $P : (L^2(\Omega))^d \longrightarrow H$ the Leray projector and set

$$b(y,z,w) = \sum_{i,j=1}^{d}\int_\Omega y_i D_i z_j w_j \,\mathrm{d}x. \tag{1.2}$$

Define the operator $B : V \longrightarrow V'$ by

$$(By,w) = b(y,y,w),\ \forall\, y,w\in V. \tag{1.3}$$

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Then we may rewrite equation (1.1) as

$$\begin{aligned}&\frac{\mathrm{d}y}{\mathrm{d}t}(t)+\nu Ay(t)+By(t)=P(mu)+Pf_0, \quad t\in[0,\infty)\\&y(0)=y_0,\end{aligned}\tag{1.1$'$}$$

where $A\in L(V,V')$ (the Stokes operator) is defined by

$$(Ay,w)=\sum_{i=1}^{d}\int_{\Omega}\nabla y_i\cdot\nabla w_i\mathrm{d}x,\ \forall\, y,w\in V.\tag{1.4}$$

Let (y_e,p_e) be a steady-state solution to (1.1), *i.e.*,

$$\begin{aligned}&-\nu\Delta y_e+(y_e\cdot\nabla)y_e=\nabla p_e+f_0(x) && \text{in } \Omega\\&\nabla\cdot y_e=0 && \text{in } \Omega\\&y_e=0 && \text{on } \partial\Omega.\end{aligned}\tag{1.5}$$

Throughout this paper we shall assume that *the boundary $\partial\Omega$ is a finite union of $d-1$ dimensional C^∞-connected manifolds diffeomorphic with $S_d^r=\{x\in R^d,\ |x|=r\}$.*

It is well known that for $d=2,3$ always there is a steady-state solution and for small viscosity constant ν this solution might be instable. However, by some recent results in [14, 15] (see also [3]) if (y_e,p_e) and y_0 are sufficiently smooth, for instance if

$$(y_e,p_e)\in((H^3(\Omega))^d)\cap V)\times H^1(\Omega),\ y_0\in(H^2(\Omega))^d\cap V\tag{1.6}$$

and $\|y_0-y_e\|_{(H^2(\Omega))^d}\le\eta$ is sufficiently small then for each $T>0$ there are

$$u\in H^1(0,T;(L^2(\Omega))^d),\ y\in L^\infty(0,T;(H^2(\Omega))^d\cap V)\cap H^1(0,T;H)\tag{1.7}$$

and $p\in L^2(0,T;H^1(\Omega))$ satisfying (1.1) and such that $y(x,T)\equiv y_e(x)$. (In 2-D similar exact controllability results were previously obtained in [11,13].) In particular, this implies that there is a controller u which stabilizes the steady-state solution y_e.

Here we shall use a different approach to stabilization inspired by Liapunov stability theory for finite dimensional systems. One must recall that a key element in stabilization of nonlinear ordinary differential systems is the linear feedback controller stabilizing the linearized system, usually, provided by an algebraic Riccati equation associated with an infinite horizon LQ problem. However in the case of infinite dimensional systems with unbounded nonlinearities, as is the case here, the situation is more complicated and our goal is to show (see Th. 1 below) that this approach still works with an appropriate LQ problem. This will allow to solve the local exponential stabilization problem for the Navier–Stokes using the solution of an appropriate algebraic Riccati equation associated with the linearized Stokes equation. As seen below the technique is applicable to a larger class of nonlinear evolution equations and in particular to parabolic semilinear equations.

For recent results on stabilization of fluid flows we refer to the works [1, 6, 7, 9, 12] and the references given there.

Here and throughout in the sequel $H^k(\Omega)$ and $H^1(0,T;X)$ (X is a Hilbert space) are usual Sobolev spaces on Ω and $(0,T)$, respectively. We shall denote by the same symbol $|\cdot|$ the norm of H and of $(L^2(\Omega))^d$. We shall denote by $\|\cdot\|$ the norm of V and by $(\cdot,\cdot)$ the pairing between V, V' (the dual space of V) and, respectively, the scalar product of H. Finally $|\cdot|_s$ is the norm of the Sobolev space $(H^s(\Omega))^d$.

2. Stabilization of the linearized equation

Substituting y by $y+y_e$ and p by $p+p_e$ into equation (1.1) we are lead to the null stabilization of the equation

$$\begin{array}{ll} y_t-\nu\Delta y+(y\cdot\nabla)y+(y_e\cdot\nabla)y+(y\cdot\nabla)y_e=mu+\nabla p & \text{in } Q\\ \nabla\cdot y=0 & \text{in } Q\\ y=0 & \text{on } \Sigma\\ y(x,0)=y_0(x)-y_e(x)=y^0(x), & x\in\Omega. \end{array} \tag{2.1}$$

Equivalently,

$$\begin{array}{l} \dfrac{\mathrm{d}y}{\mathrm{d}t}(t)+\nu Ay(t)+A_0y(t)+By(t)=P(mu),\ t\geq 0\\ y(0)=y^0 \end{array} \tag{2.2}$$

where B,A are given by (1.3, 1.4) and $A_0\in L(V,H)$ is defined by

$$(A_0y,w)=b(y_e,y,w)+b(y,y_e,w),\ \forall\, w\in H. \tag{2.3}$$

Consider the linearized system

$$\begin{array}{l} \dfrac{\mathrm{d}y}{\mathrm{d}t}(t)+\nu Ay(t)+A_0y(t)=P(mu)(t),\ t\geq 0\\ y(0)=y^0 \end{array} \tag{2.4}$$

and the corresponding LQ(linear quadratic) optimal control problem

$$\varphi(y^0)=\mathrm{Min}\left\{\frac{1}{2}\int_0^\infty\left(\left|A^{\frac{3}{4}}y(t)\right|^2+|u(t)|^2\right)\mathrm{d}t;\ \text{subject to (2.4)}\right\}\cdot \tag{2.5}$$

We shall denote by $D(\varphi)$ the set of all $y^0\in H$ such that $\varphi(y^0)<\infty$ and note that for each $y^0\in D(A^{\frac{1}{4}})$ the linear Stokes equation (2.4) is exactly null controllable on each interval $[0,T]$. More precisely, there are $u\in L^2(0,T;(L^2(\Omega))^d)$, $y\in L^2(0,T;D(A^{\frac{3}{4}}))$ satisfying (2.4) and such that $y(T)\equiv 0$. Here is the argument. Let $y^0\in D(A^{\frac{1}{4}})$ and $0<T_0<T$. Then clearly equation (2.4) with $u=0$ has a unique strong solution

$$y\in L^2\left(0,T_0;D\left(A^{\frac{3}{4}}\right)\right),t^{\frac{1}{2}}Ay\in L^2(0,T_0;H).$$

Indeed by (2.4) we get the following *a priori* estimates

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left|A^{\frac{1}{4}}y\right|^2+2\nu\left|A^{\frac{3}{4}}y\right|^2&=-2b\left(y_e,y,A^{\frac{1}{2}}y\right)-2b\left(y,y_e,A^{\frac{1}{2}}y\right)+2\left(mu,A^{\frac{1}{2}}y\right)\\ &\leq C\left|A^{\frac{1}{2}}y\right|(|y_e|_2\|y\|+|u|),\ \text{a.e. } t>0. \end{aligned}$$

This yields

$$\left|A^{\frac{1}{4}}y(t)\right|^2+\int_0^t\left|A^{\frac{3}{4}}y\right|^2\mathrm{d}s\leq C\left(\int_0^t|u|^2\mathrm{d}s+\left|A^{\frac{1}{4}}y^0\right|^2\right)\cdot$$

Next if multiply (2.4) by $tAy(t)$ and use the latter estimate we obtain after a similar calculation that

$$\int_0^t s|Ay(s)|^2\mathrm{d}s\leq C\left(\int_0^t|u|^2\mathrm{d}s+\left|A^{\frac{1}{4}}y^0\right|^2\right)$$

which implies the desired result by standard argument. Now, if multiply the equation by tA^2y we find that $t|Ay(t)|^2 \in C((0,T_0]$ and therefore $y(T_0) \in (H^2(\Omega))^d$. Finally, by the controllability results established in [14,15] (see also Lem. 3.1 in [3]) we know that there is a solution (y,u) to (2.4) on (T_0,T) such that

$$u \in H^1\left(T_0,T;(L^2(\Omega))^d\right), y \in L^2(T_0,T;D(A)), y(T)=0.$$

In particular, this implies that $\varphi(y^0)<\infty, \forall y^0 \in D(A^{\frac{1}{4}})$ and more precisely, we have

$$\varphi\left(y^0\right) \leq C\left|A^{\frac{1}{4}}y^0\right|^2, \forall y^0 \in D\left(A^{\frac{1}{4}}\right). \tag{2.6}$$

Moreover, we have

$$\varphi(y^0) \geq C\left|A^{\frac{1}{4}}y^0\right|^2. \tag{2.7}$$

Indeed, it is easily seen that for each $y^0 \in D(\varphi)$ problem (2.5) has a unique solution $(y^*,u^*) \in L^2(R^+;D(A^{\frac{3}{4}}))$ $\times L^2(R^+;(L^2(\Omega))^d)$. Moreover, as easily seen by (2.4), $y^* \in C_w(R^+,D(A^{\frac{1}{4}}))$. (Here C_w denotes the space of weakly continuous functions.) If multiply equation (2.4) where $y=y^*, u=u^*$ by $A^{\frac{1}{2}}y^*$ and integrate on $(0,\infty)$ we obtain

$$\begin{aligned}\frac{1}{2}\left|A^{\frac{1}{4}}y^0\right|^2 &\leq \int_0^\infty \left(\nu\left(Ay^*,A^{\frac{1}{2}}y^*\right)+\left(A_0y^*,A^{\frac{1}{2}}y^*\right)+|u^*|\left|A^{\frac{1}{2}}y^*\right|\right)\mathrm{d}t\\ &\leq C\int_0^\infty\left(\left|A^{\frac{3}{4}}y^*\right|^2+|u^*|^2\right)\mathrm{d}t = C\varphi\left(y^0\right)\end{aligned}$$

because (see *e.g.* [10,17]) we have

$$\left|\left(A_0y,A^{\frac{1}{2}}y\right)\right| \leq \left|b\left(y,y_e,A^{\frac{1}{2}}y\right)\right|+\left|b\left(y_e,y,A^{\frac{1}{2}}y\right)\right| \leq C\|y\|\left|A^{\frac{1}{2}}y\right||y_e|_2 \leq C\|y\|^2|y_e|_2.$$

By (2.7) we may infer therefore that $D(\varphi)=D(A^{\frac{1}{4}})=W$. The space W is endowed with the graph norm $|y|_W = |A^{\frac{1}{4}}y|$. Here and everywhere in the sequel A^s, $s\in(0,1)$, is the fractional power of the Stokes operator A and $A^s=A^{[s]}A^{s-[s]}$ for $s\geq 1$. We refer to [10] for definition and basic properties. Here we recall only that

$$V=D\left(A^{\frac{1}{2}}\right), D(A^s)\subset (H^{2s})^d\cap H, \forall s\geq 0$$

and $(H^m(\Omega))^d\cap V\subset D\left(A^{\frac{m}{2}}\right)$ for all positive integers m.

Since the function φ is quadratic we may infer that there is a linear self-adjoint operator $R: H\to H$ with the domain $D(R)$ such that

$$\frac{1}{2}\left(Ry^0,y^0\right)=\varphi\left(y^0\right), \forall y^0\in D(R)\subset W.$$

Moreover, $R\in L(W,W')$ and the latter equality extends to all of W.

Proposition 1. *Let $d=2,3$. Then the optimal control u^* is expressed as*

$$u^*(t)=-mRy^*(t), \forall t>0. \tag{2.8}$$

Moreover, $V\subset D(R)$, i.e.,

$$|Ry|\leq C\|y\|, \forall y\in V \tag{2.9}$$

and there are $\omega_i>0, i=1,2$ such that

$$\omega_1\|y\|_W^2 \leq (Ry,y) \leq \omega_2\|y\|_W^2, \forall y\in W. \tag{2.10}$$

The operator R is a solution to the algebraic Riccati equation

$$(\nu Ay + A_0 y, Ry) + \frac{1}{2}|mRy|^2 = \frac{1}{2}\left|A^{\frac{3}{4}}y\right|^2,\ \forall\, y \in D(A). \tag{2.11}$$

Proof. Estimate (2.10) follows by (2.6) and (2.7). Since the quadratic cost functional (2.5) is unbounded on H the conclusions of Proposition 1 are not directly implied by the general theory of infinite dimensional LQ control problems (see *e.g.* [8,16]) and so it requires a direct treatment briefly presented below.

By the dynamic programming principle (see *e.g.* [2]) it follows that for each $T > 0$, (y^*, u^*) is the solution to optimal control problem

$$\text{Min}\left\{\frac{1}{2}\int_0^T \left(\left|A^{\frac{3}{4}}y(t)\right|^2 + |u(t)|^2\right)\mathrm{d}t + \varphi(y(T)); (y,u) \text{ subject to (2.4)}\right\}. \tag{2.5$'$}$$

Thus by the maximum principle we have that (see [2])

$$u^*(t) = mq^T(t),\ \forall\, t \in [0,T) \tag{2.12}$$

where $q^T \in L^2(0,T;H) \cap C_w([0,T];V')$ is the solution to the adjoint equation

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}q^T - (\nu A + A_0)^* q^T = A^{\frac{3}{2}}y^*,\ t \in (0,T)\\ &q^T(T) = -Ry^*(T).\end{aligned} \tag{2.13}$$

(For existence in (2.13) we use the fact that $q^T(T) \in W' \subset V'$ and apply the standard existence theory for linear evolution equations.)

By (2.12) and the unique continuous property for the Stokes equation

$$q_t - (\nu A + A_0)^* q = 0 \text{ in } Q;\ q = 0 \text{ on } \Sigma$$

(which is a consequence of the Carleman inequality established in [14,15] for the Stokes equation) it follows that $q^T = q^{T'}$ on $(0,T)$ for $0 < T < T'$. Hence $q^T = q$ is independent of T and so (2.12, 2.13) extend to all of R^+. Moreover, we have

$$Ry^0 = -q^T(0). \tag{2.14}$$

Indeed for all $y^0, z^0 \in D\left(A^{\frac{1}{4}}\right)$ we have by (2.5)$'$ that

$$\begin{aligned}\varphi(y^0) - \varphi(z^0) \le &\int_0^T \left(A^{\frac{3}{4}}y^*(t), A^{\frac{3}{4}}(y^*(t) - z^*(t)) + (u^*(t), u^*(t) - v^*(t))\right)\mathrm{d}t\\ &+ (Ry^*(T), y^*(T) - z^*(T))\end{aligned}$$

where (z^*, v^*) is the optimal pair corresponding to z^0. On the other hand, by (2.12) and (2.13) we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(q^T(t), y^*(t) - z^*(t)\right) = \left(A^{\frac{3}{4}}y^*(t), A^{\frac{3}{4}}(y^*(t) - z^*(t)) + (u^*(t), u^*(t) - v^*(t))\right).$$

Integrating on $(0,T)$ and substituting into the above inequality we obtain that

$$\varphi\left(y^0\right) - \varphi\left(z^0\right) \le -\left(q^T(0), y^0 - z^0\right)$$

which implies (2.14) as desired.

By (2.12) we infer that

$$q(t) = -Ry^*(t), \ \forall\, t \geq 0 \tag{2.15}$$

and this implies (2.8) as claimed.

By (2.4) we have also that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|y^*(t)\|^2 + 2\nu|Ay^*(t)|^2 + 2b(y_e, y^*(t), Ay^*(t)) \\ + 2b(y^*(t), y_e, Ay^*(t)) \leq 2|mq(t)||Ay^*(t)|, \ \text{a.e. } t \in (0,T).$$

Since as noticed earlier

$$|b(y_e, y, Ay)| + |b(y, y_e, Ay)| \leq C|y_e|_2\|y\||Ay|$$

we find the estimate

$$\|y^*(t)\|^2 + \int_0^t |Ay^*(t)|^2 \mathrm{d}t \leq C_T\left\|y^0\right\|^2, \forall t \in (0,T). \tag{2.16}$$

On the other hand, coming back to equation (2.13) and substituting $z = A^{-\frac{1}{2}}q$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}z - \nu Az - A^{-\frac{1}{2}}A_0^*A^{\frac{1}{2}}z = Ay^*. \tag{2.17}$$

Noticing that

$$\left|\left(A^{-\frac{1}{2}}A_0^*A^{\frac{1}{2}}z, Az\right)\right| \leq \left|b\left(y_e, A^{\frac{1}{2}}z, A^{\frac{1}{2}}z\right)\right| + \left|b\left(A^{\frac{1}{2}}z, y_e, A^{\frac{1}{2}}z\right)\right| \leq C|Az|\|z\||y_e|_2$$

and recalling that $Ay^* \in L^2(0,T;H)$ we see by (2.17) that $z \in C_w([0,T);V)$. Hence $q \in C([0,T);H)$ and so $-q(0) = Ry^0 \in H$. Then (2.9) follows by the closed graph theorem.

Next by $(2.5)'$ we have

$$\varphi(y^*(t)) = \frac{1}{2}\int_t^\infty \left(\left|A^{\frac{3}{4}}y^*\right|^2 + |u^*|^2\right)\mathrm{d}s, \ \forall\, t \geq 0 \tag{2.18}$$

and therefore

$$\left(Ry^*(t), \frac{\mathrm{d}y^*}{\mathrm{d}t}(t)\right) + \frac{1}{2}\left|A^{\frac{3}{4}}y^*(t)\right|^2 + \frac{1}{2}|mRy^*(t)|^2 = 0, \ \text{a.e. } t > 0.$$

Since $|P(mRy)| \leq C\|y\|, \forall y \in V$ we see that the operator $\nu A + A_0 + P(mR)$ with the domain $D(A)$ generates a C_0- semigroup on H (this is just the flow $y_0 \to y^*(t)$). This implies that

$$Ay^*, A_0y^*, P(mRy^*) \in C([0,\infty);H)$$

and in virtue of (2.18) and (2.8) this yields

$$-(Ry^*(t), \nu Ay^*(t) + A_0y^*(t)) - \frac{1}{2}|mRy^*(t)|^2 + \frac{1}{2}\left|A^{\frac{3}{4}}y^*(t)\right|^2 = 0, \ \forall\, t \geq 0$$

thereby completing the proof. □

Remark 1. It is easily seen that the equation (2.11) has a unique self-adjoint solution R satisfying conditions (2.9) and (2.10). This is an immediate consequence of the fact that any such a solution stabilizes system (2.4) *via* feedback law (2.8). The general problem of uniqueness in equation (2.11) with bounded right hand side is studied in [8] but the arguments extend in our case too.

3. Stabilization of the Navier–Stokes equation

Theorem 1 below is the main result of this paper.

Theorem 1. *Let $d = 2, 3$ and let R be the operator defined in Proposition 1. Let (y_e, p_e) $((H^3(\Omega))^d \cap V) \times H^1(\Omega)$ be a steady–state solution to equation (1.1). Then the feedback controller*

$$u = -mR(y - y_e) \tag{3.1}$$

exponentially stabilizes y_e in a neighbourhood $\mathcal{V} = \{y_0 \in W;\ \|y_0 - y_e\|_W < \rho\}$ of y_e. More precisely, for each $y_0 \in \mathcal{V}$ there is a weak solution $y \in L^\infty_{\rm loc}(R^+; H) \cap L^2_{\rm loc}(R^+; V)$ to closed loop system

$$\begin{aligned}&\frac{\mathrm{d}y}{\mathrm{d}t} + \nu Ay + By + P(mR(y - y_e)) = Pf_0,\ t > 0\\ &y(0) = y_0\end{aligned} \tag{3.2}$$

such that

$$\begin{aligned}&\int_0^\infty \left|A^{\frac{3}{4}}(y(t) - y_e)\right|^2 \mathrm{d}t \le C\|y_0 - y_e\|_W^2\\ &|y(t) - y_e| \le C\mathrm{e}^{-\gamma t}\|y_0 - y_e\|_W,\ \forall\, y_0 \in \mathcal{V}\end{aligned} \tag{3.3}$$

for some $\gamma > 0$.

Proof. As seen earlier we may reduce the problem to that of stability of the null solution to corresponding closed loop system (2.2), *i.e.*,

$$\begin{aligned}&\frac{\mathrm{d}y}{\mathrm{d}t} + \nu Ay + A_0 y + By + P(mRy) = 0,\ t > 0\\ &y(0) = y^0.\end{aligned} \tag{3.4}$$

We consider the approximating equation

$$\begin{aligned}&\frac{\mathrm{d}y_N}{\mathrm{d}t} + \nu Ay_N + A_0 y_N + B_N y_N + P(mRy_N) = 0,\\ &y_N(0) = y^0,\end{aligned} \tag{3.5}$$

where

$$B_N y = By \text{ if } \|y\| \le N,\ B_N y = \frac{N^2}{\|y\|^2} By \text{ if } \|y\| > N.$$

We note that

$$(B_N y - B_N z, y - z) \ge -\varepsilon\|y - z\|^2 - C_{N,\varepsilon}|y - z|^2, \forall y, z \in V$$

and by (2.9) we have

$$|P(mRy)| \le C\|y\|, \forall y \in V.$$

Then arguing as in [4, 6] it follows that the operator $\mathcal{A}_N = \nu A + A_0 + B_N + P(mR)$ with the domain $D(A)$ is m quasi-accretive in H (*i.e.*, $\lambda I + \mathcal{A}_N$ is m-accretive in H for some $\lambda > 0$). Thus for each $y^0 \in D(A)$ equation (3.5) has a unique solution $y_N \in W^{1,\infty}_{\rm loc}(R^+; H) \cap L^\infty_{\rm loc}(R^+; D(A))$. (If $y^0 \in V$ then $y_N \in W^{1,2}_{\rm loc}(R^+; H)$ $\cap L^2_{\rm loc}(R^N; D(A)) \cap C(R^+; V)$.) Also the following estimate holds

$$|y_N(t)|^2 + \int_0^t \left(\|y_N'\|_{V'}^{\frac{4}{3}} + \|y_N\|^2\right) \mathrm{d}s \le C_T,\ \forall\, t \in (0, T), \tag{3.6}$$

where $y_N' = \frac{\mathrm{d}y_N}{\mathrm{d}t}$ and C_T is independent of N.

This implies that there is a subsequence $N \to \infty$ such that on each finite interval $(0, T)$,

$$\begin{gathered} y_N \longrightarrow y \text{ weak star in } L^\infty(0,T;H), \text{ weakly in } L^2(0,T;V) \\ \text{strongly in } L^2(0,T;H) \end{gathered} \tag{3.7}$$

where $y \in L^2_{\mathrm{loc}}(R^+;V) \cap C_w(R^+;H)$, $\dfrac{\mathrm{d}y}{\mathrm{d}t} \in L^{\frac{3}{4}}_{\mathrm{loc}}(R^+;V')$ is a weak solution to equation (3.4). (See *e.g.* [10,17] for the definition of the weak solution.)

Now we multiply equation (3.5) by Ry_N and use equation (2.11) to obtain after some calculation that

$$\frac{\mathrm{d}}{\mathrm{d}t}(Ry_N(t), y_N(t)) + |mRy_N(t)|^2 + \left|A^{\frac{3}{4}} y_N(t)\right|^2 = -2(B_N y_N(t), Ry_N(t)), \text{ a.e. } t > 0. \tag{3.8}$$

On the other hand, recalling that (see *e.g.* [10,17])

$$|b(y,z,w)| \leq C|y|_{m_1}|z|_{m_2+1}|w|_{m_3}, m_1 + m_2 + m_3 \geq \frac{d}{2}$$

it follows by Proposition 1 that

$$\begin{aligned} |(B_N y_N, Ry_N)| &\leq \inf\left(1, \frac{N^2}{\|y_N\|^2}\right)|b(y_N, y_N, Ry_N)| \\ &\leq C|y_N|_1|y_N|_{\frac{3}{2}}|Ry_N| \leq C\|y_N\|\left|A^{\frac{3}{4}}y_N\right||Ry_N| \\ &\leq \left|A^{\frac{3}{4}}y_N\right|\|y_N\|^2 \leq C\left|A^{\frac{3}{4}}y_N\right|^2 (Ry_N, y_N)^{\frac{1}{2}} \end{aligned} \tag{3.9}$$

because by interpolation we have

$$\|y\|^2 \leq \left|A^{\frac{3}{4}}y\right|\left|A^{\frac{1}{4}}y\right| \leq C\left|A^{\frac{3}{4}}y\right|(Ry,y)^{\frac{1}{2}}.$$

We set

$$E = \left\{y^0 \in W;\ (Ry^0, y^0) < \rho\right\}\cdot$$

Then by (3.8) and (3.9) we see that for ρ sufficiently small and independent of N and $y^0 \in E$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(Ry_N(t), y_N(t)) + \frac{1}{2}\left|A^{\frac{3}{4}}y_N(t)\right|^2 \leq 0, \text{ a.e. } t > 0.$$

By (2.10) this yields

$$\frac{\mathrm{d}}{\mathrm{d}t}(Ry_N(t), y_N(t)) + \gamma(Ry_N(t), y_N(t)) \leq 0, \text{ a.e. } t > 0$$

for some positive constant γ independent of N and

$$\int_0^\infty \left|A^{\frac{3}{4}}y_N(t)\right|^2 \mathrm{d}t \leq 2(Ry^0, y^0).$$

Then again using (2.10) this yields

$$|y_N(t)| \leq \|y_N(t)\|_W \leq C\left\|y^0\right\|_W \mathrm{e}^{-\gamma t}, \ \forall\, t \geq 0$$

for some $\gamma > 0$ and $C > 0$ independent of N. Then recalling that in virtue of (3.7) we may assume that $\{y_N\}$ is strongly convergent in H a.e. on R^+, letting N tend to $+\infty$ it follows by (3.7) that

$$|y(t)| \leq C\|y^0\|_W \mathrm{e}^{-\gamma t}, \ \forall t \geq 0$$

and so (3.3) follows for $\mathcal{V} = E + y_e$. This completes the proof. □

We note that if $d = 2$ then the solution y to closed loop system (3.2) is a strong solution and unique for each $y_0 \in \mathcal{V}$.

Remark 2. The same linearization technique can be used to solve the local H^∞ problem for the Navier–Stokes equation with exogeneous disturbances. This problem was solved in [5] via differential game approach.

Remark 3. It is readily seen that under conditions of Theorem 1 the feedback controller $u = -\psi(mR(y - y_e))$ is asimptotically stable for all continuous mappings $\psi : H \to H$ such that

$$(\psi(z), z) \geq \frac{1+\delta}{2}|z|^2, \forall z \in H \tag{3.10}$$

for some $\delta > 0$. This follows as above by multiplying the closed loop equation (2.4) with the feedback control $u = -\psi(mRy)$ by Ry and using (2.11) and (3.10).

This can be seen as a robustness property of the feedback controller (3.1) with respect to static perturbations in the input.

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