ESAIM: Control, Optimisation and Calculus of Variations August 2002, Vol. 7, 597–614
URL: `http://www.emath.fr/cocv/` DOI: 10.1051/cocv:2002068

A POSITIVE SOLUTION FOR AN ASYMPTOTICALLY LINEAR ELLIPTIC PROBLEM ON $\mathbb{R}^N$ AUTONOMOUS AT INFINITY *

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Abstract. In this paper we establish the existence of a positive solution for an asymptotically linear elliptic problem on $\mathbb{R}^N$. The main difficulties to overcome are the lack of *a priori* bounds for Palais–Smale sequences and a lack of compactness as the domain is unbounded. For the first one we make use of techniques introduced by Lions in his work on concentration compactness. For the second we show how the fact that the "Problem at infinity" is autonomous, in contrast to just periodic, can be used in order to regain compactness.

Mathematics Subject Classification. 35J60, 58E05.

Received July 9, 2001. Revised January 3, 2002.

1. Introduction

In this paper we study the existence of a positive solution $u \in H^1(\mathbb{R}^N)$ for an equation of the form

$$-\Delta u + V(x)u = f(u), \qquad x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 2$ and we assume on the potential $V \in C(\mathbb{R}^N, \mathbb{R})$

(V1) there exists $\alpha > 0$ such that $V(x) \geq \alpha$ for all $x \in \mathbb{R}^N$;
(V2) $\lim_{|x|\to\infty} V(x) = V(\infty) \in (0, \infty)$

and on the nonlinear term $f \in C(\mathbb{R}^+, \mathbb{R})$

(f1) $f(s)s^{-1} \to 0$ as $s \to 0+$;
(f2) There is $a \in]0, \infty[$ such that $f(s)s^{-1} \to a$ as $s \to +\infty$ and

$$a > \inf \sigma(-\Delta + V(x)),$$

where $\sigma(-\Delta + V(x))$ denotes the spectrum of the self-adjoint operator $-\Delta + V(x) : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$.

Keywords and phrases: Elliptic equations, asymptotically linear problems in $\mathbb{R}^N$, lack of compactness.

* *The second author is partially supported by Waseda University Grant for Special Research Project 2001A-098.*

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The main features of problem (1.1) is that the nonlinearity is asymptotically linear and that the associated problem at "infinity" is autonomous. Our main results are the following:

Theorem 1.1. *Assume that (V1), (V2), (f1), (f2) hold and that for $F(s) = \int_0^s f(t)\mathrm{d}t$,*

(f3) there exists $\delta > 0$ such that $2F(s)s^{-2} \leq V(\infty) - \delta$ for all $s \in \mathbb{R}^+$.

Then (1.1) has a positive solution.

Theorem 1.2. *Assume that (V1), (V2) hold with*

(V3) $V(x) \leq V(\infty)$ for all $x \in \mathbb{R}^N$

and that (f1), (f2) hold with

(f4) Defining $G : \mathbb{R}^+ \to \mathbb{R}$ by $G(s) = \frac{1}{2}f(s)s - F(s)$,

(i) *$G(s) \geq 0$ for all $s \geq 0$;*

(ii) *there exists $\delta > 0$ such that*

$$2F(s)s^{-2} \geq V(\infty) - \delta \Longrightarrow G(s) \geq \delta.$$

Then (1.1) has a positive solution.

Remark 1.3. 1) The condition $G(s) \geq 0, \forall s \geq 0$ implies that $2F(s)s^{-2}$ is a non-decreasing function. Thus as a special case (f3) holds under the conditions: $G(s) \geq 0, \forall s \geq 0$ and $a < V(\infty)$.

2) Condition (f4) holds if $f(s)s^{-1}$ is a non-decreasing function of $s \geq 0$. In fact, if $f(s)s^{-1}$ is non-decreasing, condition (f1) implies $G(s) > 0$, $\forall\, s > 0$ and $G(s)$ is a non-decreasing function of $s \geq 0$.

3) Under the setting of Theorem 1.2, we can also show the existence of a least energy solution. See Theorem 4.5 in Section 4.

Theorems 1.1 and 1.2 will be proved by a variational approach. Because we look for positive solutions, we may assume without restriction that $f(s) = 0$ for all $s \leq 0$. We associate with (1.1) the functional $I : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x - \int_{\mathbb{R}^N} F(u)\,\mathrm{d}x.$$

We shall work on $H^1(\mathbb{R}^N) \equiv H$ with the norm

$$||u||^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\mathrm{d}x$$

which, because of (V1), is equivalent to the standard $H^1(\mathbb{R}^N)$ norm. We also use the notation:

$$||u||_p = \left(\int_{\mathbb{R}^N} |u|^p\,\mathrm{d}x\right)^{1/p} \quad \text{for all } p \in (1, \infty).$$

Under conditions (f1) and (f2) we are able to prove, this is the contents of Lemmas 2.1 and 2.3, that I has a Mountain Pass geometry (MP geometry for short). Namely setting

$$\Gamma = \{\gamma \in C([0,1], H), \gamma(0) = 0 \text{ and } I(\gamma(1)) < 0\},$$

we have $\Gamma \neq \emptyset$ and

$$c \equiv \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)) > 0.$$

The value $c \in \mathbb{R}$ is called the Mountain Pass level (MP level for short) for I. Ekeland's principle implies that there exists a Cerami sequence at the MP level c, namely a sequence $\{u_n\} \subset H$ such that

$$I(u_n) \to c \text{ and } ||I'(u_n)||_{H^{-1}}(1 + ||u_n||) \to 0 \quad \text{as } n \to \infty.$$

At this point to get an existence result it clearly suffices to show that $\{u_n\}$ is bounded and then that $\{u_n\}$ has a convergent subsequence whose limit is a non-trivial critical point of I. These two steps constitute the heart of the proofs of Theorems 1.1 and 1.2.

The difficulty to prove that $\{u_n\}$ is bounded is linked to the fact that we are considering an asymptotically linear problem. To get boundedness of $\{u_n\}$, in most works the following superlinear condition is assumed:

$$\exists \mu > 2 : \ 0 < \mu F(s) \leq f(s)s \quad \text{for all } s > 0. \tag{1.2}$$

This condition is introduced by Ambrosetti and Rabinowitz [1]. We remark that (1.2) implies $\liminf_{s\to\infty} f(s)/s^{\mu-1} > 0$ and thus our equation does not satisfy (1.2).

There are very few works on asymptotically linear problems on unbounded domains. We believe that the first result is due to Stuart and Zhou [11]. They study a problem of the type of (1.1) assuming that it has a radial symmetry. Thanks to this assumption, the problem is somehow set in $\mathbb{R}$ and possesses a stronger compactness.

More closely related to the present paper is the work of the first author [7] in which a problem of the form

$$-\Delta u + Ku = f(x, u), \ x \in \mathbb{R}^N$$

is studied, where $K > 0$ is a constant and $f(x, s)$ is asymptotically linear in s and periodic in $x \in \mathbb{R}^N$. Subsequently, taking advantages of some techniques introduced in [7], an extended study of radially symmetric problems on $\mathbb{R}^N$ was done in [12]. Finally we wish to mention [13] where a first order Hamiltonian system with an asymptotically linear part is studied.

As in [7] our proof of the boundedness of $\{u_n\}$ relies on the work of Lions [8] on the concentration compactness principle. It is however more delicate now because of the spectral structure of (1.1) and the fact that (1.1) is not enjoying a translation invariance. The argument roughly goes as follows. We assume, by contradiction, that $\|u_n\| \to \infty$. Then setting $w_n = u_n\|u_n\|^{-1}$ there is a subsequence of $\{w_n\}$ with $w_n \rightharpoonup w \geq 0$ in H satisfying the alternative:

1. (non-vanishing) there exist $\alpha > 0$, $R < \infty$ and $\{y_n\} \subset \mathbb{R}^N$ such that

$$\lim_{n\to\infty} \int_{y_n + B_R} w_n^2 \, \mathrm{d}x \geq \alpha > 0.$$

2. (vanishing) for any $R > 0$

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{y+B_R} w_n^2 \, \mathrm{d}x = 0.$$

Here we use the notation : $B_R = \{x \in \mathbb{R}^N; \, |x| \leq R\}$.

We prove that neither of the two cases can occur and this gives us the desired contradiction. To show that non-vanishing (1) cannot occur, we need to distinguish the cases $\{y_n\}$ is bounded and $|y_n| \to \infty$. When $\{y_n\}$ is bounded, we show that $w \neq 0$ and satisfies the equation

$$-\Delta w + V(x)w = aw, \ x \in \mathbb{R}^N.$$

Namely $w \geq 0$ is an eigenvector associated to an eigenvalue strictly above the infimum of the spectrum of $-\Delta + V(x)$. We show, by a spectral argument, that this is impossible. When $|y_n| \to \infty$, we show that the sequence $\tilde{w}_n(\cdot) = w_n(\cdot - y_n)$ weakly converge to a function $\tilde{w} \neq 0$ satisfying

$$-\Delta\tilde{w} + V(\infty)\tilde{w} = a\tilde{w}.$$

Here again we get a contradiction since the operator $-\Delta$ has no eigenfunction in $H^1(\mathbb{R}^N)$. Next we show that either (f3) or (f4) permit us to rule out the vanishing (2). This is quite straightforward when (f3) holds. When (f4) is satisfied, we need a more delicate argument. On one hand, since $\{u_n\}$ is a Cerami sequence we have that

$$\int_{\mathbb{R}^N} G(u_n)\mathrm{d}x$$

is bounded uniformly. On the other hand, condition (f4) will imply that these integrals must go to $+\infty$ as $n \to \infty$.

Concerning the second difficulty, namely to prove that our bounded sequence $\{u_n\}$ converges to a non-trivial critical point of I, we need to show a strict inequality between the "energy" of our problem and the one of the associated "problem at infinity". To be more precise, let $\tilde{I} : H \to \mathbb{R}$ be the functional defined by

$$\tilde{I}(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(\infty)u^2)\,\mathrm{d}x - \int_{\mathbb{R}^N} F(u)\,\mathrm{d}x$$

and set

$$\tilde{m} = \inf\left\{\tilde{I}(u);\ u \neq 0 \text{ and } \tilde{I}'(u) = 0\right\}\cdot$$

We set $\tilde{m} = \infty$ if $\tilde{I}$ has no non-trivial critical points. We shall prove that $u_n \rightharpoonup u \neq 0$ with $I'(u) = 0$ if

$$c < \tilde{m}. \tag{1.3}$$

If $\tilde{I}$ has no non-trivial critical points, (1.3) is trivially satisfied. We will see that it takes place if (f3) holds. When $\tilde{I}$ has non-trivial critical points, the following fact is important to show (1.3):

$$\tilde{m} \geq \inf_{\gamma\in\tilde{\Gamma}} \max_{t\in[0,1]} \tilde{I}(\gamma(t)),$$

where $\tilde{\Gamma} = \{\gamma \in C([0,1], H);\ \gamma(0) = 0,\ \tilde{I}(\gamma(1)) < 0\}$. These types of estimates are usually shown under the assumption that $s \to f(s)s^{-1}$ is non-decreasing. Under this condition we can use the "natural constraint":

$$M = \left\{u \in H \setminus \{0\};\ \tilde{I}'(u)u = 0\right\}\cdot$$

M is somehow radially homeomorphic to the unit sphere and it is then easy to see that, in addition,

$$\tilde{m} = \inf_{u\in M} \tilde{I}(u) = \inf_{u\in M} \max_{t>0} \tilde{I}(tu).$$

For results in that direction we mention, for example [10]. Also in [9] an existence result, Theorem 4.27, comparable to our relies, although indirectly, on the presence of a smooth natural constraint.

In contrast to these works, we do not use the monotonicity of $f(s)/s$ but instead we take advantage of dilation $t \to u(x/t)$ properties. Key to our approach is the use of results on autonomous problems established by Berestycki and Lions [2] when $N \geq 3$ and Berestycki *et al.* [3] when $N = 2$. In particular we use their result saying that, under an almost necessary condition on h, any autonomous problem of the form

$$-\Delta u = h(u)$$

possesses a least energy solution satisfying Pohozaev identity. Roughly speaking we use the fact that the problem at infinity is autonomous to show the existence of a path $\gamma \in \Gamma$ satisfying $\max_{t\in[0,1]} I(\gamma(t)) < \tilde{m}$.

We remark that the approach we present here works for general nonlinearities f, not simply for the asymptotically linear ones. Also we believe it could be proved fruitful in other situations as, for example, singular perturbation problems.

Notation. Throughout the article the letter C will denote various positive constants whose exact value may change from line to line but are not essential to the analysis of the problem. Also if we take a subsequence of a sequence $\{u_n\}$ we shall denote it again $\{u_n\}$.

2. A Mountain Pass Geometry for I

In this section we prove that, under the assumptions (V1), (f1), (f2), I possesses a MP geometry. Since $I(0) = 0$, it is a consequence of the two following results:

Lemma 2.1. *Assume that (V1), (f1), (f2) hold. Then $I(u) = \frac{1}{2}||u||^2 + o(\|u\|^2)$, $I'(u)u = ||u||^2 + o(||u||^2)$ as $u \to 0$ in H.*

Proof. We fix a $p \in]2, \frac{2N}{N-2}[$. For any $\varepsilon > 0$ it follows from (f1) and (f2) that there exists a $C_\varepsilon > 0$ such that

$$|f(s)| \leq \varepsilon|s| + C_\varepsilon|s|^{p-1} \quad \text{for all } s \in \mathbb{R} \tag{2.1}$$

or also

$$|F(s)| \leq \frac{\varepsilon}{2}|s|^2 + \frac{C_\varepsilon}{p}|s|^p \quad \text{for all } s \in \mathbb{R}. \tag{2.2}$$

Hence, for any $u \in H$, we have

$$\left|\int_{\mathbb{R}^N} F(u)\,\mathrm{d}x\right| \leq \frac{\varepsilon}{2}\|u\|_2^2 + \frac{C_\varepsilon}{p}\|u\|_p^p.$$

Recalling the Sobolev embedding $H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$, this implies that $\int_{\mathbb{R}^N} F(u)\,\mathrm{d}x = o(\|u\|^2)$ as $u \to 0$ in H. In a similar way, we have $\int_{\mathbb{R}^N} f(u)u\,\mathrm{d}x = o(||u||^2)$ as $u \to 0$ in H. This gives the conclusion of Lemma 2.1. ☐

Corollary 2.2. *Under the assumptions (V1), (f1), (f2) there exists $\rho_0 > 0$ such that*

(i) *for any non-trivial critical point u of I,*

$$||u|| \geq \rho_0;$$

(ii) *for any Palais-Smale sequence $\{u_n\}$ at level $b \neq 0$,*

$$\liminf_{n\to\infty} ||u_n|| \geq \rho_0.$$

Lemma 2.3. *Assume that (V1), (f1), (f2) hold. Then we can find a $v \in H$, $v \neq 0$ satisfying $I(v) < 0$.*

Proof. The operator $-\Delta + V(x)$ being self-adjoint, the infimum of its spectrum can be characterized as (see [4], Prop. VI.9)

$$\inf \sigma(-\Delta + V(x)) = \inf_{u\in H:||u||_2=1} \langle -\Delta u + V(x)u, u\rangle = \inf_{u\in H:||u||_2=1} ||u||^2 \tag{2.3}$$

where $\langle\cdot,\cdot\rangle$ is the scalar product of $L^2(\mathbb{R}^N)$. Since by assumption $\inf\sigma(-\Delta+V(x)) < a$, we thus can find $\tilde{u} \in H$ such that $||\tilde{u}||_2 = 1$ and $||\tilde{u}||^2 < a$. Replacing if necessary $\tilde{u}$ by $|\tilde{u}|$, we can suppose without restriction that $\tilde{u} \geq 0$ a.e. on $\mathbb{R}^N$. To prove the lemma, it suffices to show that

$$\lim_{t\to+\infty} \frac{I(t\tilde{u})}{t^2} < 0. \tag{2.4}$$

First, let us establish that

$$\lim_{t\to+\infty}\int_{\mathbb{R}^N}\frac{F(t\tilde{u})}{t^2}\,\mathrm{d}x=\frac{1}{2}a. \tag{2.5}$$

To prove (2.5), it is convenient to separate the cases $\tilde{u}(x)>0$ and $\tilde{u}(x)=0$ (without loss of generality we can assume that $\tilde{u}$ is defined everywhere on $\mathbb{R}^N$). Let x be such that $\tilde{u}(x)>0$. We have, by (f2),

$$\lim_{t\to+\infty}\frac{F(t\tilde{u}(x))}{t^2}=\lim_{t\to+\infty}\frac{F(t\tilde{u}(x))}{(t\tilde{u}(x))^2}(\tilde{u}(x))^2=\frac{1}{2}a(\tilde{u}(x))^2. \tag{2.6}$$

Now let x be such that $\tilde{u}(x)=0$. Then

$$\frac{F(t\tilde{u}(x))}{t^2}=0=\frac{1}{2}a(\tilde{u}(x))^2 \quad \text{for all } t>0. \tag{2.7}$$

Thus combining (2.6) and (2.7) we obtain that

$$\lim_{t\to+\infty}\frac{F(t\tilde{u}(x))}{t^2}=\frac{1}{2}a(\tilde{u}(x))^2 \qquad \text{a.e. on } \mathbb{R}^N. \tag{2.8}$$

On the other hand, by (f1), (f2), there exists a constant $C>0$ such that

$$0\le\frac{f(s)}{s}\le C \quad \text{for all } s\in\mathbb{R} \tag{2.9}$$

and thus

$$0\le\frac{F(s)}{s^2}\le\frac{C}{2} \quad \text{for all } s\in\mathbb{R}. \tag{2.10}$$

Consequently

$$0\le\frac{F(t\tilde{u}(x))}{t^2}\le\frac{C}{2}(\tilde{u}(x))^2 \text{ a.e. on } \mathbb{R}^N. \tag{2.11}$$

Now, (2.8) and (2.11) allow us to apply Lebesgue's theorem to get

$$\lim_{t\to+\infty}\int_{\mathbb{R}^N}\frac{F(t\tilde{u}(x))}{t^2}\,\mathrm{d}x=\frac{1}{2}a\int_{\mathbb{R}^N}(\tilde{u}(x))^2\,\mathrm{d}x=\frac{1}{2}a, \tag{2.12}$$

that is (2.5). Then, we easily deduce that

$$\lim_{t\to+\infty}\frac{I(t\tilde{u})}{t^2}=\frac{1}{2}||\tilde{u}||^2-\lim_{t\to+\infty}\int_{\mathbb{R}^N}\frac{F(t\tilde{u}(x))}{t^2}\,\mathrm{d}x=\frac{1}{2}(||\tilde{u}||^2-a)<0$$

and the lemma is proved. □

Remark 2.4. As we shall see in Section 4, if there exists $s_0>0$ such that $2F(s_0)s_0^{-2}>V(\infty)$, Lemma 2.3 can be proved in a simpler way.

Since the functional I has a MP geometry, we deduce (see [6]) the existence of a Cerami sequence at the MP level c, namely of a $\{u_n\}\subset H$ such that

$$I(u_n)\to c \text{ and } ||I'(u_n)||_{H^{-1}}(1+||u_n||)\to 0 \quad \text{as } n\to\infty.$$

3. Boundedness of $\{u_n\}$

In this section we establish that $\{u_n\}$ is bounded. Seeking a contradiction, we assume, throughout this section, that a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) satisfies $\|u_n\| \to \infty$ and we set $w_n = u_n\|u_n\|^{-1}$. Clearly, $\{w_n\}$ is bounded and, up to a subsequence, satisfies the alternative:

1. (non-vanishing) there exist $\alpha > 0$, $R < \infty$ and $\{y_n\} \subset \mathbb{R}^N$ such that

$$\lim_{n\to\infty} \int_{y_n+B_R} w_n^2 \,\mathrm{d}x \geq \alpha > 0.$$

2. (vanishing) for all $R > 0$

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{y+B_R} w_n^2 \,\mathrm{d}x = 0.$$

We prove that none of the two cases can occur, getting so the desired contradiction.

Lemma 3.1. *Assume that (V1), (V2), (f1), (f2) and either (f3) or (f4) hold. Then the vanishing of $\{w_n\}$ is impossible.*

Proof. We develop a contradiction argument assuming that $\{w_n\}$ vanishes. Then we immediately get, using (V2), that

$$\int_{\mathbb{R}^N} (V(x) - V(\infty))w_n^2 \,\mathrm{d}x \to 0 \quad \text{as } n \to \infty.$$

Thus

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla w_n|^2 + V(\infty)w_n^2 \,\mathrm{d}x = \lim_{n\to\infty} ||w_n||^2 = 1$$

and, in particular,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} w_n^2 \,\mathrm{d}x \leq \frac{1}{V(\infty)}\cdot \tag{3.1}$$

Also, since $I(u_n) \to c$, we have that $I(u_n)||u_n||^{-2} \to 0$ which can be written as

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{F(u_n)}{u_n^2} w_n^2 \,\mathrm{d}x = \frac{1}{2}\cdot \tag{3.2}$$

Combining (3.1) and (3.2), we immediately get a contradiction when (f3) holds. Now assuming that (f4) holds, we set for $\delta > 0$ given in (f4)

$$\Omega_n = \left\{ x \in \mathbb{R}^N : \frac{F(u_n(x))}{u_n(x)^2} \leq \frac{1}{2}(V(\infty) - \delta) \right\}\cdot$$

Then, for all $n \in \mathbb{N}$,

$$\int_{\Omega_n} \frac{F(u_n)}{u_n^2} w_n^2 \,\mathrm{d}x \leq \frac{1}{2}(V(\infty) - \delta) \int_{\Omega_n} w_n^2 \,\mathrm{d}x$$

and, passing to the limit, we deduce using (3.1), (3.2) that necessarily

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N\setminus\Omega_n} \frac{F(u_n)}{u_n^2} w_n^2 \,\mathrm{d}x \geq \frac{\delta}{2V(\infty)} > 0. \tag{3.3}$$

We claim that

$$\limsup_{n\to\infty} |\mathbb{R}^N \backslash \Omega_n| = \infty. \tag{3.4}$$

To check this we assume by contradiction that $\limsup_{n\to\infty} |\mathbb{R}^N\backslash\Omega_n| < \infty$. On one hand, using (2.10) it follows that

$$\int_{\mathbb{R}^N\backslash\Omega_n} \frac{F(u_n)}{u_n^2} w_n^2 \,\mathrm{d}x \le C \int_{\mathbb{R}^N\backslash\Omega_n} w_n^2 \,\mathrm{d}x. \tag{3.5}$$

On the other hand we recall that any bounded vanishing sequence $\{v_n\} \subset H$ satisfies $v_n \to 0$ strongly in $L^q(\mathbb{R}^N)$ for any $q \in]2, \frac{2N}{N-2}[$ (a proof of this result is given in Lem. 2.18 of [5] and is a special case of Lem. I.1 of [8]). Thus, using Hölder inequality, we get under the assumption $\limsup_{n\to\infty} |\mathbb{R}^N\backslash\Omega_n| < \infty$ that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N\backslash\Omega_n} w_n^2 \,\mathrm{d}x = 0.$$

Then (3.5) contradicts (3.3) and this proves (3.4). Now observe that since, $G(s) \ge 0,\ \forall s \in \mathbb{R}$

$$\int_{\mathbb{R}^N} G(u_n)\,\mathrm{d}x \ge \int_{\mathbb{R}^N\backslash\Omega_n} G(u_n)\,\mathrm{d}x \ge \delta\, |\mathbb{R}^N\backslash\Omega_n|.$$

Thus $\limsup_{n\to\infty} \int_{\mathbb{R}^N} G(u_n)\,\mathrm{d}x = +\infty$. But, because $\{u_n\}$ is a Cerami sequence,

$$\int_{\mathbb{R}^N} G(u_n)\,\mathrm{d}x = I(u_n) - \frac{1}{2} I'(u_n)u_n \to c.$$

This contradiction proves the lemma. □

We now have to prove that the non-vanishing of $\{w_n\}$ can not occur either. Still by contradiction we assume that $\{w_n\}$ is non-vanishing. At this point it is convenient to distinguish the cases $\{y_n\}$ bounded or unbounded.

Lemma 3.2. *Assume that (V1), (V2), (f1), (f2) hold and that $\{y_n\}$ is bounded. Then the non-vanishing of $\{w_n\}$ is impossible.*

Proof. The sequence $\{w_n\}$ being bounded, up to a subsequence, $w_n \rightharpoonup w \in H$.

Step 1. *The weak limit w is non-negative.*

Since $\{w_n\}$ is non-vanishing and $\{y_n\}$ bounded, we have $w \ne 0$. Also since $\{u_n\}$ is a Cerami sequence

$$-\Delta u_n + V(x)u_n = f(u_n) + \varepsilon_n \text{ in } H^{-1}(\mathbb{R}^N)$$

with $\varepsilon_n \to 0$ in $H^{-1}(\mathbb{R}^N)$. Thus

$$-\Delta w_n + V(x)w_n = \frac{f(u_n)}{u_n} w_n + \frac{\varepsilon_n}{||u_n||}\cdot \tag{3.6}$$

Multiplying (3.6) by $w_n^- \equiv \max\{-w_n, 0\}$ and integrating, we get

$$\int_{\mathbb{R}^N} (|\nabla w_n^-|^2 + V(x)|w_n^-|^2)\,\mathrm{d}x = \int_{\mathbb{R}^N} \frac{\varepsilon_n}{||u_n||} w_n^- \,\mathrm{d}x,$$

that is

$$||w_n^-||^2 = \int_{\mathbb{R}^N} \frac{\varepsilon_n}{||u_n||} w_n^- \, \mathrm{d}x\cdot \tag{3.7}$$

Since $\{w_n^-\}$ is also bounded, we have

$$\int_{\mathbb{R}^N} \frac{\varepsilon_n}{||u_n||} w_n^- \, \mathrm{d}x \to 0.$$

Therefore, by (3.7), $||w_n^-|| \to 0$. We deduce that $w_n \to w = w^+$ a.e. on $\mathbb{R}^N$.

Step 2. *w is an eigenvector of $-\Delta + V(x)$ associated to the eigenvalue a.*

To prove that $-\Delta w + V(x)w = aw$ it is sufficient to show that for any $\varphi \in C_0^\infty(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} [\nabla w \nabla \varphi + V(x) w \varphi] \, \mathrm{d}x = a \int_{\mathbb{R}^N} w \varphi \, \mathrm{d}x. \tag{3.8}$$

Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be arbitrary but fixed. Multiplying (3.6) by φ and integrating, we obtain

$$\int_{\mathbb{R}^N} [\nabla w_n \nabla \varphi + V(x) w_n \varphi] \, \mathrm{d}x = \int_{\mathbb{R}^N} \frac{f(u_n)}{u_n} w_n \varphi \, \mathrm{d}x + \int_{\mathbb{R}^N} \frac{\varepsilon_n}{||u_n||} \varphi \, \mathrm{d}x. \tag{3.9}$$

Clearly

$$\int_{\mathbb{R}^N} \frac{\varepsilon_n}{||u_n||} \varphi \, \mathrm{d}x \to 0 \tag{3.10}$$

and also since $\{w_n\}$ converges weakly to w in H,

$$\int_{\mathbb{R}^N} [\nabla w_n \nabla \varphi + V(x) w_n \varphi] \, \mathrm{d}x \to \int_{\mathbb{R}^N} [\nabla w \nabla \varphi + V(x) w \varphi] \, \mathrm{d}x. \tag{3.11}$$

We claim that

$$\int_{\mathbb{R}^N} \frac{f(u_n)}{u_n} w_n \varphi \, \mathrm{d}x \to a \int_{\mathbb{R}^N} w \varphi \, \mathrm{d}x. \tag{3.12}$$

If (3.12) holds, then combining (3.9)–(3.12), we get (3.8) and the proof of Step 2 is completed. Thus let us prove (3.12). First, we shall establish that

$$\frac{f(u_n)}{u_n} w_n \to aw, \quad \text{a.e. in } \mathbb{R}^N. \tag{3.13}$$

In this aim, it is convenient to distinguish the two cases $w(x) = 0$ and $w(x) \neq 0$ (without restriction we can suppose that w is defined everywhere on $\mathbb{R}^N$). Let $x \in \mathbb{R}^N$ be such that $w(x) = 0$. By (2.9),

$$0 \le \left| \frac{f(u_n(x))}{u_n(x)} w_n(x) \right| \le C |w_n(x)|,$$

and since $w_n(x) \to w(x) = 0$, we get

$$\frac{f(u_n(x))}{u_n(x)} w_n(x) \to 0 = aw(x).$$

Let $x \in \mathbb{R}^N$ be such that $w(x) \neq 0$. Then necessarily $u_n(x) \to +\infty$. Thus by (f2)

$$\frac{f(u_n(x))}{u_n(x)} \to a.$$

Consequently, also in this case,

$$\frac{f(u_n(x))}{u_n(x)} w_n(x) \to aw(x)$$

and this proves (3.13).

Denote by $\Omega \subset \mathbb{R}^N$ a compact set such that supp $\varphi \subset \Omega$. The compactness of the Sobolev embedding $H^1(\Omega) \hookrightarrow L^1(\Omega)$ implies $w_n \to w$ strongly in $L^1(\Omega)$. Thus in particular, after taking a subsequence if necessary, there exists $h \in L^1(\Omega)$ such that

$$|w_n(x)| \leq h(x) \text{ a.e. } x \in \Omega$$

(see Th. IV.9 in [4]) and using again (2.9) we have for all $n \in \mathbb{N}$,

$$\left| \frac{f(u_n)}{u_n} w_n \right| \leq C|w_n| \leq Ch, \text{ a.e. } x \in \Omega. \tag{3.14}$$

Now (3.13) and (3.14) allows to apply Lebesgue's theorem and we get

$$\int_{\mathbb{R}^N} \frac{f(u_n)}{u_n} w_n \varphi \, \mathrm{d}x = \int_\Omega \frac{f(u_n)}{u_n} w_n \varphi \, \mathrm{d}x \to a \int_\Omega w\varphi \, \mathrm{d}x = a \int_{\mathbb{R}^N} w\varphi \, \mathrm{d}x,$$

namely (3.12). This ends Step 2.

Step 3. When $a > \inf \sigma(-\Delta + V(x))$, the operator $-\Delta + V(x)$ has no non-negative eigenvector associated to the eigenvalue a.

Seeking a contradiction, suppose that $u \in H$ is non-negative and satisfies

$$-\Delta u + V(x)u = au \quad \text{in } \mathbb{R}^N.$$

First, we fix a constant A such that

$$\inf \sigma(-\Delta + V(x)) < A < a.$$

By the variational characterization of $\inf \sigma(-\Delta + V(x))$, there exists $v \in H$ satisfying

$$\frac{||v||^2}{||v||_2^2} < A.$$

Thus, since $C_0^\infty(\mathbb{R}^N)$ is dense in H, we may assume $v \in C_0^\infty(\mathbb{R}^N)$. Now, let $R > 0$ be such that *supp* $v \subset B_R$ and consider the Dirichlet problem for $-\Delta + V(x)$ on B_R. Denote by l the infimum of the spectrum of $-\Delta + V(x)$ on B_R. By construction,

$$l \leq \frac{||v||^2}{||v||_2^2} < A < a. \tag{3.15}$$

On the other hand, l is an eigenvalue of $-\Delta + V(x)$ associated to an eigenvector $v_R > 0$ on B_R. Note that, moreover, we have $\frac{\partial v_R}{\partial n} \leq 0$. Then, integrating twice by parts.

$$\begin{aligned} l\langle u, v_R\rangle_{B_R} &= \langle u, (-\Delta + V(x))v_R\rangle_{B_R} \\ &= \int_{B_R} \nabla u \nabla v_R \,\mathrm{d}x - \int_{\partial B_R} \frac{\partial v_R}{\partial n} u \,\mathrm{d}\sigma + \int_{B_R} V(x) u v_R \,\mathrm{d}x \\ &\geq \int_{B_R} \nabla u \nabla v_R \,\mathrm{d}x + \int_{B_R} V(x) u v_R \,\mathrm{d}x \\ &= \int_{B_R} (-\Delta u) v_R \,\mathrm{d}x - \int_{\partial B_R} \frac{\partial u}{\partial n} v_R \,\mathrm{d}\sigma + \int_{B_R} V(x) u v_R \,\mathrm{d}x \\ &= \int_{B_R} (-\Delta u) v_R \,\mathrm{d}x + \int_{B_R} V(x) u v_R \,\mathrm{d}x \\ &= \langle (-\Delta + V(x))u, v_R\rangle_{B_R} = a\langle u, v_R\rangle_{B_R}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle_{B_R}$ denotes the scalar product of $L^2(B_R)$. But since $u \geq 0$ and $v_R > 0$, we have $< u, v_R >_{B_R} > 0$, and thus the above calculation shows that $l \geq a$ in contradiction with (3.15). This completes Step 3.

Combining Steps 1, 2 and 3, Lemma 3.2 is proved. □

Lemma 3.3. *Assume that (V1), (V2), (f1), (f2) hold and that, up to a subsequence, $|y_n| \to \infty$. Then the non-vanishing of $\{w_n\}$ is impossible.*

Proof. Setting $\tilde{u}_n(x) = u_n(x + y_n)$ and $\tilde{w}_n(x) = w(x + y_n)$ we have, from (3.6), that,

$$-\Delta \tilde{w}_n + V(x + y_n)\tilde{w}_n = \frac{f(\tilde{u}_n)}{\tilde{u}_n}\tilde{w}_n + \frac{\tilde{\varepsilon}_n}{||u_n||} \tag{3.16}$$

with $\tilde{\varepsilon}_n \to 0$ in $H^{-1}(\mathbb{R}^N)$. Up to a subsequence, $\tilde{w}_n \rightharpoonup \tilde{w}$ weakly in H and by construction $\tilde{w} \neq 0$. We shall prove that $\tilde{w}$ satisfies

$$-\Delta \tilde{w} + V(\infty)\tilde{w} = a\tilde{w} \tag{3.17}$$

and, since $-\Delta$ has no eigenvector on $\mathbb{R}^N$, this will provide us the desired contradiction. To prove (3.17), it is sufficient to establish that for any $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} [\nabla \tilde{w} \nabla \varphi + V(\infty)\tilde{w}\varphi] \,\mathrm{d}x = a \int_{\mathbb{R}^N} \tilde{w}\varphi \,\mathrm{d}x.$$

Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be arbitrary but fixed. Multiplying (3.16) by φ and integrating, we obtain

$$\int_{\mathbb{R}^N} [\nabla \tilde{w}_n \nabla \varphi + V(x + y_n)\tilde{w}_n\varphi] \,\mathrm{d}x = \int_{\mathbb{R}^N} \frac{f(\tilde{u}_n)}{\tilde{u}_n}\tilde{w}_n\varphi \,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{\tilde{\varepsilon}_n}{||u_n||}\varphi \,\mathrm{d}x.$$

We clearly have

$$\int_{\mathbb{R}^N} \frac{\tilde{\varepsilon}_n}{||u_n||}\varphi \,\mathrm{d}x \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} \nabla \tilde{w}_n \nabla \varphi \,\mathrm{d}x \to \int_{\mathbb{R}^N} \nabla \tilde{w} \nabla \varphi \,\mathrm{d}x.$$

Next we show $\int_{\mathbb{R}^N} V(x + y_n)\tilde{w}_n\varphi \,\mathrm{d}x \to \int_{\mathbb{R}^N} V(\infty)\tilde{w}_n\varphi \,\mathrm{d}x$. Denote by $\Omega \subset \mathbb{R}^N$ a compact set such that $\operatorname{supp}\varphi \subset \Omega$. By (V2), we see $V(x + y_n) \to V(\infty)$ uniformly on Ω. Thus

$$\int_\Omega (V(x + y_n) - V(\infty))\tilde{w}_n\varphi \,\mathrm{d}x \to 0.$$

Therefore

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} V(x+y_n)\tilde{w}_n\varphi\,\mathrm{d}x = \lim_{n\to\infty} V(\infty)\int_{\mathbb{R}^N}\tilde{w}_n\varphi\,\mathrm{d}x = V(\infty)\int_{\mathbb{R}^N}\tilde{w}\varphi\,\mathrm{d}x.$$

At this point to end the proof we just need to show that

$$\int_{\mathbb{R}^N}\frac{f(\tilde{u}_n)}{\tilde{u}_n}\tilde{w}_n\varphi\,\mathrm{d}x \to a\int_{\mathbb{R}^N}\tilde{w}\varphi\,\mathrm{d}x.$$

This is done exactly as in Lemma 3.2. □

Remark 3.4. The proof that $\{u_n\}$ is bounded carries on to any Cerami sequence *i.e.* to any sequence $\{u_n\}\subset H$ such that $\{I(u_n)\}$ is bounded and $||I'(u_n)||_{H^{-1}}(1+||u_n||)\to 0$.

4. A non-trivial critical point for I

In this section we prove that, up to a subsequence, our bounded Cerami sequence $\{u_n\}$ converges weakly to a non-trivial critical point of I. For this we shall study autonomous problems and make use of classical results due to Berestycki and Lions [2] when $N\geq 3$ and Berestycki *et al.* [3] when $N=2$. For the convenience of readers, we state their results here. We remark that we do not state them in their full generality. We consider the equation

$$-\Delta u = h(u) \quad \text{in } \mathbb{R}^N \tag{4.1}$$

where it is assumed on h that

(h0) $h:\mathbb{R}\to\mathbb{R}$ is continuous and odd;
(h1) there exists a limit $\lim_{s\to 0+} h(s)/s \in]-\infty, 0[$;
(h2) when $N\geq 3$, it holds that $\lim_{s\to\infty}|h(s)|s^{-\frac{N+2}{N-2}}=0$.
When $N=2$, for any $\alpha>0$ there exists a $C_\alpha>0$ such that $|h(s)|\leq C_\alpha e^{\alpha s^2}$ for all $s\in\mathbb{R}$.

Associated to (4.1) is the functional $J:\, H\to\mathbb{R}$ given by

$$J(u)=\int_{\mathbb{R}^N}\frac{1}{2}|\nabla u|^2 - H(u)\,\mathrm{d}x,$$

where $H(u)=\int_0^u h(s)\,\mathrm{d}s$. We say that a solution w of (4.1) is a least energy solution if $J(w)=m$, where

$$m=\inf\{J(u); u\neq 0 \text{ and } J'(u)=0\}\cdot$$

Proposition 4.1. *([2, 3]) Assume (h0)–(h2). Then J is well defined and we have*

(i) *(4.1) has a non-trivial solution if and only if $H(s_0)>0$ for some $s_0>0$;*
(ii) *if $H(s_0)>0$ for some $s_0>0$, then $m>0$ and there exists a least energy solution w of (4.1) which satisfies $w>0$ on $\mathbb{R}^N$ and, as any critical point of J, the Pohozaev identity:*

$$(N-2)\int_{\mathbb{R}^N}|\nabla w|^2\mathrm{d}x = 2N\int_{\mathbb{R}^N} H(w)\,\mathrm{d}x.$$

The key to our compactness result is the following observation, which complements the results of Proposition 4.1:

Proposition 4.2. *Assume (h0)–(h2) and $H(s_0) > 0$ for some $s_0 > 0$. Let v be a critical point of (4.1) with $v(x) > 0$ for all $x \in \mathbb{R}^N$. Then, there exists a path $\gamma \in C([0,1], H)$ such that $\gamma(t)(x) > 0$ for all $x \in \mathbb{R}^N$, $t \in (0,1]$, $\gamma(0) = 0$, $J(\gamma(1)) < 0$, $v \in \gamma([0,1])$ and*

$$\max_{t\in[0,1]} J(\gamma(t)) = J(v).$$

In particular, for the least energy solution w obtained in Proposition 4.1, we have

$$\inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)) \leq J(w).$$

The proof of Proposition 4.2 will be given in the Appendix.

Now we return to our problem (1.1) and we consider the associated "problem at infinity", *i.e.*, the autonomous problem:

$$-\Delta u + V(\infty)u = f(u) \quad \text{in } \mathbb{R}^N. \tag{4.2}$$

Since any solution of (4.2) is non-negative, we can regard it as a solution of (4.1) with

$$h(s) = \begin{cases} -V(\infty)s + f(s), & \text{for } s \geq 0, \\ -h(-s), & \text{for } s < 0. \end{cases}$$

Thus we can observe that a least energy solution for (4.2) – which we may assume positive – is also a least energy solution of (4.2) and the converse is also true.

Remark 4.3. Applying Proposition 4.1(i) to $h(s) = -V(\infty)s + f(s)$, we see that under condition (f3), $\tilde{I}$ has no non-trivial critical points.

Lemma 4.4. *Assume that (V1), (V2), (f1), (f2) hold. Let $\{u_n\} \subset H$ be a bounded PS sequence for I at the level c. Then, up to a subsequence, $u_n \rightharpoonup u \neq 0$ with $I'(u) = 0$ if either one of the following conditions hold:*

(i) *(f3) is satisfied;*
(ii) *(f4) holds and*

$$V(x) \leq V(\infty) \quad \textit{for all } x \in \mathbb{R}^N \textit{ and } V(x) \not\equiv V(\infty). \tag{4.3}$$

Proof. Since $\{u_n\}$ is bounded in H, we can assume that, up to a subsequence, $u_n \rightharpoonup u$. Let us prove that $I'(u) = 0$. Noting that $C_0^\infty(\mathbb{R}^N)$ is dense in H, it suffices to check that $I'(u)\varphi = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. Let $(\cdot,\cdot)$ denote the inner product on H associated to our chosen norm. Then

$$I'(u_n)\varphi - I'(u)\varphi = (u_n - u, \varphi) - \int_{\mathbb{R}^N} (f(u_n) - f(u))\,\varphi \,\mathrm{d}x \to 0$$

since $u_n \rightharpoonup u$ weakly in H and strongly in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$ for $q \in [2, \frac{2N}{N-2}[$. Thus recalling that $I'(u_n) \to 0$ we indeed have that $I'(u) = 0$. At this point if $u \neq 0$ the lemma is proved. Thus we assume that $u = 0$. We claim that in this case $\{u_n\}$ is also a PS sequence for $\tilde{I}$ at the level c. Indeed, as $n \to \infty$,

$$\tilde{I}(u_n) - I(u_n) = \int_{\mathbb{R}^N} (V(\infty) - V(x))u_n^2 \,\mathrm{d}x \to 0$$

since $V(x) \to V(\infty)$ as $|x| \to \infty$ and $u_n \to 0$ in $L^2_{\text{loc}}(\mathbb{R}^N)$. Also, for the same reasons, we have

$$\sup_{||v|| \le 1} |(\tilde{I}'(u_n) - I'(u_n), v)| = \sup_{||v|| \le 1} \left| \int_{\mathbb{R}^N} (V(\infty) - V(x)) u_n v \, \mathrm{d}x \right| \to 0.$$

Next we claim that $\{u_n\}$ does not vanish. Indeed on one hand by (2.1) we have for any $u \in H$

$$\int_{\mathbb{R}^N} |f(u)u| \, \mathrm{d}x \le \varepsilon ||u||_2^2 + C_\varepsilon ||u||_p^p$$

and thus, if $\{u_n\}$ vanishes, we get that

$$\int_{\mathbb{R}^N} f(u_n) u_n \, \mathrm{d}x \to 0.$$

On the other hand, since $I'(u_n)u_n \to 0$, we have

$$||u_n||^2 - \int_{\mathbb{R}^N} f(u_n) u_n \, \mathrm{d}x \to 0.$$

So if we assume that $\{u_n\}$ vanishes, we arrive at the conclusion that $||u_n|| \to 0$ in a contradiction with Corollary 2.2(ii).

Thus $\{u_n\}$ is a non-vanishing sequence. Namely there exist $\alpha > 0$, $R > 0$ and $\{y_n\} \subset \mathbb{R}^N$ such that

$$\lim_{n \to \infty} \int_{y_n + B_R} u_n^2 \, \mathrm{d}x \ge \alpha > 0.$$

Let $\tilde{u_n}(x) = u_n(x + y_n)$. Since $\{u_n\}$ is a PS sequence for $\tilde{I}$, this is also the case of $\{\tilde{u}_n\}$. Arguing as in the case of $\{u_n\}$ we get that $\tilde{u}_n \rightharpoonup \tilde{u}$, up to a subsequence, with $\tilde{I}'(\tilde{u}) = 0$. Since $\{\tilde{u}_n\}$ is non-vanishing we also have that $\tilde{u} \neq 0$.

At this point if (i) is satisfied, $\tilde{I}$ has no non-trivial critical points (see Rem. 4.3) and we obtain a contradiction. If (ii) holds, we have that $G(s) \ge 0$ for all $s \in \mathbb{R}$ and we get from Fatou's lemma that

$$c = \limsup_{n \to \infty} \left[\tilde{I}(\tilde{u}_n) - \frac{1}{2} \tilde{I}'(\tilde{u}_n) \tilde{u}_n \right] = \lim_{n \to \infty} \int_{\mathbb{R}^N} G(\tilde{u}_n) \, \mathrm{d}x \ge \int_{\mathbb{R}^N} G(\tilde{u}) \, \mathrm{d}x = \tilde{I}(\tilde{u}) - \frac{1}{2} \tilde{I}'(\tilde{u}) \tilde{u} = \tilde{I}(\tilde{u}).$$

Namely $\tilde{u} \neq 0$ is a critical point of $\tilde{I}$ satisfying $\tilde{I}(\tilde{u}) \le c$. By the strong maximum principle $\tilde{u} > 0$ on $\mathbb{R}^N$. Then, by Proposition 4.2, we can find a path $\gamma(t) \in C([0,1], H)$ such that $\gamma(t)(x) > 0$, $\forall\, x \in \mathbb{R}^N$, $\forall\, t \in (0,1]$, $\gamma(0) = 0$, $\tilde{I}(\gamma(1)) < 0$, $\tilde{u} \in \gamma([0,1])$ and

$$\max_{t \in [0,1]} \tilde{I}(\gamma(t)) = \tilde{I}(\tilde{u}).$$

Since we have assumed (4.3), we have

$$I(\gamma(t)) < \tilde{I}(\gamma(t)) \quad \text{for all } t \in (0,1].$$

Thus

$$c \le \max_{t \in [0,1]} I(\gamma(t)) < \max_{t \in [0,1]} \tilde{I}(\gamma(t)) \le c.$$

This is a contradiction. $\square$

End of the proofs of Theorems 1.1 and 1.2. Except for a special case $V(x) \equiv V(\infty)$ in Theorem 1.2, Theorems 1.1 and 1.2 follow from Lemma 4.4. Thus we consider the case $V(x) \equiv V(\infty)$.

Since, in this case, $\sigma(-\Delta+V(\infty)) = [V(\infty), \infty)$, we have $a > V(\infty)$ from (f2). Thus $H(s) = F(s) - \frac{1}{2}V(\infty)s^2$ satisfies $H(s_0) > 0$ for sufficiently large $s_0 > 0$ and we deduce the existence of a non-trivial critical point from Proposition 4.1. □

At the end of this section, we show the existence of a least energy solution in the setting of Theorem 1.2.

Theorem 4.5. *Under the assumptions of Theorem 1.2, (1.1) has a least energy solution. More precisely, there exists a solution $w \in H$ such that $I(w) = m$, where*

$$m = \inf\{I(u);\, u \neq 0 \text{ and } I'(u) = 0\} \cdot$$

Proof. First we observe that $m = \inf\{I(u);\, u \neq 0 \text{ and } I'(u) = 0\}$ satisfies

$$0 \leq m \leq c,$$

where c is the MP level for I. In fact, by (f4)(i), we have for any critical point u of I

$$I(u) = I(u) - \frac{1}{2}I'(u)u = \int_{\mathbb{R}^N} G(u)\,\mathrm{d}x \geq 0$$

and thus $m \geq 0$. On the other hand, in Lemma 4.4 we obtained a non-trivial critical point u of I as a weak limit of a bounded PS sequence $\{u_n\}$ for I at level c. Thus again from (f4)(i) and Fatou's lemma we have

$$I(u) = I(u) - \frac{1}{2}I'(u)u = \int_{\mathbb{R}^N} G(u)\,\mathrm{d}x \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} G(u_n)\,\mathrm{d}x = \liminf_{n\to\infty} I(u_n) - \frac{1}{2}I'(u_n)u_n = c.$$

Therefore we have $m \in [0, c]$.

Next let $\{v_n\}$ be a sequence of non-trivial critical points of I satisfying

$$I(v_n) \to m \in [0, c].$$

By Corollary 2.2(i), we have $\liminf_{n\to\infty} ||v_n|| \geq \rho_0 > 0$. Repeating the arguments in the previous sections, we can see that $\{v_n\}$ is bounded and v_n weakly converges to a function $w \neq 0$. It is easy to see that w is a non-trivial critical point of I and thus $I(w) \geq m$. On the other hand, using (f4)(i) and Fatou's lemma again, we can see that

$$I(w) \leq \liminf_{n\to\infty} I(v_n) = m.$$

Therefore we have $I(w) = m$. □

5. Appendix: Proof of Proposition 4.2

In this appendix we give a proof of Proposition 4.2. In the proof the following scale change plays an important role. For a non-trivial critical point $v \in H^1(\mathbb{R}^N)$ of J, we set

$$v_t(x) = v(x/t) \quad \text{for } t > 0.$$

For any $t > 0$, v_t has the following properties which can be checked through direct calculation.

Lemma 5.1. (i) $||\nabla v_t||_2^2 = t^{N-2}||\nabla v||_2^2$.

(ii) *For any continuous function F satisfying* $\limsup_{s\to 0+} |F(s)|/s^2 < \infty$.

$$\int_{\mathbb{R}^N} F(v_t)\,\mathrm{d}x = t^N \int_{\mathbb{R}^N} F(v)\,\mathrm{d}x.$$

(iii) $||v_t||_q^q = t^N ||v||_q^q$ *for all* $q \in [2,\infty)$.

Proof of Proposition 4.2 for $N \geq 3$. By the Pohozaev identity, we have $\int_{\mathbb{R}^N} H(v)\,\mathrm{d}x = \frac{N-2}{2N}||\nabla v||_2^2$. Thus

$$J(v_t) = \frac{t^{N-2}}{2}||\nabla v||_2^2 - t^N \int_{\mathbb{R}^N} H(v)\,\mathrm{d}x = \left(\frac{1}{2}t^{N-2} - \frac{N-2}{2N}t^N\right)||\nabla v||_2^2.$$

We can see easily that

(i) $\max_{t>0} J(v_t) = J(v)$;
(ii) $J(v_t) \to -\infty$ as $t \to \infty$;
(iii) $||v_t||^2_{H^1(\mathbb{R}^N)} = ||\nabla v_t||_2^2 + ||v_t||_2^2 = t^{N-2}||\nabla v||_2^2 + t^N||v||_2^2 \to 0$ as $t \to 0$.

We choose $L > 1$ such that $J(v_L) < 0$ and set $\gamma(t) = v_{Lt}$ for $t \in (0,1]$, $\gamma(0) = 0$. This is the desired path. □

Next we deal with the case $N = 2$. First we observe

Lemma 5.2. *Assume* $N = 2$. *Then, for any* $t > 0$,

(i) $||\nabla v_t||_2^2 = ||\nabla v||_2^2$;
(ii) *For any continuous function* $F(s)$ *satisfying* $\limsup_{s\to 0+} F(s)/s^2 < \infty$.

$$\int_{\mathbb{R}^N} F(v_t)\,\mathrm{d}x = t^2 \int_{\mathbb{R}^N} F(v)\,\mathrm{d}x.$$

(iii) $\int_{\mathbb{R}^2} H(v_t)\,\mathrm{d}x = 0$;
(iv) $J(v_t) = J(v)$;
(v) $\int_{\mathbb{R}^2} h(v_t)v_t\,\mathrm{d}x = t^2||\nabla v||_2^2$.

Proof. (i)–(ii) are special cases of (i)-(ii) of Lemma 5.1. (iii) is a consequence of Pohozaev identity and (ii). (iv) is obtained directly from (i) and (iii). To see (v), we remark that v is a solution of (4.1). Thus

$$||\nabla v||_2^2 = \int_{\mathbb{R}^2} h(v)v\,\mathrm{d}x$$

and, as a special case of (ii), we have

$$\int_{\mathbb{R}^2} h(v_t)v_t\,\mathrm{d}x = t^2 \int_{\mathbb{R}^2} h(v)v\,\mathrm{d}x = t^2||\nabla v||_2^2.$$

□

Proof of Proposition 4.2 for $N = 2$. When $N = 2$, construction of a path $\gamma(t)$ is rather complicated. First we join 0 and v.

Step 1. A path γ joining v to 0.

Recalling the conditions (h1), (h2), we can find constants α, $C > 0$ such that

$$|h(s)| \leq C\mathrm{e}^{\alpha s^2}|s| \quad \text{for all } s \in \mathbb{R}.$$

Now we compute $\frac{\mathrm{d}}{\mathrm{d}\theta}J(\theta v_t)$, assuming that $\theta \in [0,1]$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}J(\theta v_t) &= \theta\, ||\nabla v_t||_2^2 - \int_{\mathbb{R}^2} h(\theta v_t)v_t \,\mathrm{d}x \geq \theta\, ||\nabla v_t||_2^2 - \theta C \int_{\mathbb{R}^2} \mathrm{e}^{\alpha\theta v_t^2} v_t^2 \,\mathrm{d}x \\ &\geq \theta\, ||\nabla v_t||_2^2 - \theta C \int_{\mathbb{R}^2} \mathrm{e}^{\alpha v_t^2} v_t^2 \,\mathrm{d}x = \theta \left(||\nabla v||_2^2 - Ct^2 \int_{\mathbb{R}^2} \mathrm{e}^{\alpha v^2} v^2 \,\mathrm{d}x \right).\end{aligned}$$

We choose $t_0 \in (0,1)$ sufficiently small so that

$$||\nabla v||_2^2 - Ct_0^2 \int_{\mathbb{R}^2} \mathrm{e}^{\alpha v^2} v^2 \,\mathrm{d}x > 0.$$

For such a t_0, we have

$$\frac{\mathrm{d}}{\mathrm{d}\theta}J(\theta v_{t_0}) \geq 0, \quad \forall\, \theta \in [0,1].$$

Thus first we join v and v_{t_0} along the curve $t \mapsto v_t$ and next v_{t_0} and 0 along a line $\theta \mapsto \theta v_{t_0}$, we can easily see that this is a desired path.

Step 2. A path joining v and ∞.

We fix $t_1 > 1$ and first compute $\frac{\mathrm{d}}{\mathrm{d}\theta}\big|_{\theta=1} J(\theta v_{t_1})$ and $\frac{\mathrm{d}}{\mathrm{d}\theta}\big|_{\theta=1} \int_{\mathbb{R}^2} H(\theta v_{t_1})\,\mathrm{d}x$.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}\bigg|_{\theta=1} J(\theta v_{t_1}) &= ||\nabla v_{t_1}||_2^2 - \int_{\mathbb{R}^2} h(v_{t_1})v_{t_1}\,\mathrm{d}x = ||\nabla v||_2^2 - {t_1}^2||\nabla v||_2^2 < 0, \\ \frac{\mathrm{d}}{\mathrm{d}\theta}\bigg|_{\theta=1} \int_{\mathbb{R}^2} H(\theta v_{t_1})\,\mathrm{d}x &= \int_{\mathbb{R}^2} h(v_{t_1})v_{t_1}\,\mathrm{d}x = {t_1}^2||\nabla v||_2^2 > 0.\end{aligned}$$

Thus for a $\theta_1 \in (1,\infty)$ sufficiently close to 1, we have

$$\begin{aligned} J(\theta v_{t_1}) &\leq J(v_{t_1}) = J(v) \quad \forall\, \theta \in [1,\theta_1], \\ \int_{\mathbb{R}^2} H(\theta_1 v_{t_1})\,\mathrm{d}x &> \int_{\mathbb{R}^2} H(v_{t_1})\,\mathrm{d}x = 0.\end{aligned}$$

Next we consider $(\theta_1 v_{t_1})_t = \theta_1 v_{t_1 t}$ for $t \geq 1$. We have

$$J(\theta_1 v_{t_1 t}) = \frac{\theta_1^2}{2}||\nabla v_{t_1}||_2^2 - t^2 \int_{\mathbb{R}^2} H(\theta_1 v_{t_1})\,\mathrm{d}x.$$

Thus $J(\theta_1 v_{t_1 t})$ is a decreasing function of t and $J(\theta_1 v_{t_1 t}) \to -\infty$ as $t \to \infty$. Therefore we join v and v_{t_1} along a curve $t \mapsto v_t$, next join v_{t_1} to $\theta_1 v_{t_1}$ along a line $\theta \mapsto \theta v_{t_1}$, finally join $\theta_1 v_{t_1}$ to $\theta_1 v_{t_1 t_2}$ ($t_2 \gg 1$) along a curve $t \mapsto \theta_1 v_{t_1 t}$. We can easily see that this is a desired path. ☐

The first author thanks Professor C.A. Stuart for providing him a simpler proof of Step 3 of Lemma 3.1. The authors also thank Professor Z.Q. Wang for pointing to them the interest to show the existence of a least energy solution in Theorem 4.5.

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