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AN EXAMPLE IN THE GRADIENT THEORY OF PHASE TRANSITIONS

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Abstract. We prove by giving an example that when $n \geq 3$ the asymptotic behavior of functionals $\int_\Omega \varepsilon|\nabla^2 u|^2 + (1-|\nabla u|^2)^2/\varepsilon$ is quite different with respect to the planar case. In particular we show that the one-dimensional ansatz due to Aviles and Giga in the planar case (see [2]) is no longer true in higher dimensions.

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1. Introduction

This paper is devoted to the study of the asymptotic behavior of functionals

$$F_\varepsilon^\Omega(u) := \int_\Omega \left(\varepsilon|\nabla^2 u|^2 + \frac{(1-|\nabla u|^2)^2}{\varepsilon}\right) \qquad \Omega \subset \mathbf{R}^n \tag{1}$$

as $\varepsilon \downarrow 0$, where u maps Ω into $\mathbf{R}$. This problem was raised by Aviles and Giga in [2] in connection with the mathematical theory of liquid crystals and more recently by Gioia and Ortiz in [9] for modeling the behavior of thin film blisters. Recently many authors have studied the planar case giving strong evidences that, as conjectured by Aviles and Giga in [2], the sequence (F_ε) Γ-converge (in the strong topology of $W^{1,3}$: see [1] for a discussion of such a choice and a rigorous setting) to the functional

$$F_\infty^\Omega(u) := \begin{cases} \dfrac{1}{3}\displaystyle\int_{J_{\nabla u}} |\nabla u^+ - \nabla u^-|^3 \mathrm{d}\mathcal{H}^{n-1} & \text{if } |\nabla u| = 1,\ u \in W^{1,\infty} \\ +\infty & \text{otherwise.} \end{cases}$$

Here $J_{\nabla u}$ denotes the set of points where ∇u has a jump and $|\nabla u^+ - \nabla u^-|$ is the amount of this jump. Of course the first line of the previous definition makes sense only for particular choices of u, such as piecewise C^1. For a rigorous setting the reader should think about a suitable function space S which contains piecewise C^1 functions and on which we can give a precise meaning to the above integral (for example a natural choice would be $\{u|\nabla u \in BV\}$; however this space turns out not to be the natural one: we refer again to [1] for a discussion of this topic).

Partial results in proving Aviles and Giga's conjecture (*i.e.* compactness of minimizers of F_ε^Ω, estimates from below on $F_\varepsilon^\Omega(u_\varepsilon)$ and a suitable weak formulation for the problem of minimizing F subject to some boundary conditions) can be found in [1, 3, 5–8].

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In their first work Aviles and Giga based their conjecture on the following ansatz (which they made in the case $n = 2$):

Conjecture 1.1. *Let us choose a map $w : \Omega \to \mathbf{R}$ (with $\Omega \subset \mathbf{R}^n$ bounded open set containing 0) such that:*

(a) *w is Lipschitz and satisfies the eikonal equation $|\nabla w| = 1$;*
(b) *∇w is constant in $\{x_1 < 0\}$ and in $\{x_1 > 0\}$.*

Let us define $E := \inf\{\liminf_\varepsilon F_\varepsilon^\Omega(u_\varepsilon) : \|u_\varepsilon - w\|_{W^{1,3}} \to 0\}$. Then there exists a family of functions w_ε such that:

(i) *the component of ∇w_ε perpendicular to $(1, 0, \ldots, 0)$ is constant;*
(ii) *$w_\varepsilon \to w$ in $W^{1,3}$;*
(iii) *$\lim F_\varepsilon^\Omega(w_\varepsilon) = E$.*

This ansatz has been proved by Jin and Kohn in [8] for $n = 2$. It reduces the problem of finding E to a one dimensional problem in the calculus of variations which can be explicitly solved. This analysis leads to the result $E = F_\infty^\Omega(w)$, which means that at w the Γ-limit of F_ε^Ω exists and coincides with $F_\infty^\Omega(w)$. With a standard cut and paste argument (see [4]) it can be proved that the same happens for every w which is piecewise affine. In the next section we will prove the following theorem:

Theorem 1.2. *Let u be the function $u(x_1, x_2, x_3) = |x_3|$ and C the cylinder $\{|x_1|^2 + |x_2|^2 < 1\}$. Then there exists (u_k) such that:*

(a) *every u_k is piecewise affine (being the union of a finite number of affine pieces) and satisfies the eikonal equation;*
(b) *$\lim_k F_\infty^C(u_k) < F_\infty^C(u)$;*
(c) *$u_k \to u$ strongly in $W^{1,p}$ for every $p < \infty$.*

The proof can be easily generalized to every $n \geq 3$. As an easy corollary we get that the one-dimensional ansatz fails for $n \geq 3$. Moreover this failure means that F cannot be the Γ-limit of F_ε^Ω for $n \geq 3$.

Corollary 1.3. *The one-dimensional ansatz is not true for $n \geq 3$.*

Proof. As already observed, being every u_k piecewise affine, there is a family of functions $u_{k,\varepsilon}$ such that $u_{k,\varepsilon}$ converge to u_k in $W^{1,p}$ (for every $p < \infty$) and $\lim_\varepsilon F_\varepsilon^C(u_{k,\varepsilon}) = F_\infty^C(u_k)$. A standard diagonal argument gives a sequence $(u_{k,\varepsilon(k)})$ strongly converging to u in $W^{1,p}$ such that $\lim_k F_{\varepsilon(k)}^C(u_{k,\varepsilon(k)}) < F_\infty^C(u)$. ☐

2. The example

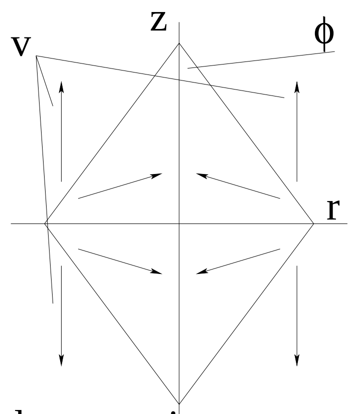
In this section we prove Theorem 1.2. First of all we recall the following fact:

(Curl) If $v : \mathbf{R}^n \to \mathbf{R}^n$ is a piecewise constant vector field, then v is a gradient if and only if for every hyperplane of discontinuity π the right trace and the left trace of v have same component parallel to π.

The building block of the construction of Theorem 1.2 is the following vector field, depending on a parameter $\phi \in (0, \pi/2)$. First of all we fix in $\mathbf{R}^3$ a system of cylindrical coordinates (r, θ, z) and then we call A the cone given by $\{z > 0, r < 1, (1 - r) > z \tan\phi\}$ and A' the reflection of A with respect to the plane $\{z = 0\}$. Hence we put

$$
\begin{aligned}
v(r,\theta,z) &= (0,0,1) && \text{if } z > 0 \text{ and } (r,\theta,z) \notin A\\
v(r,\theta,z) &= (\sin(2\phi), \theta+\pi, \cos(2\phi)) && \text{if } z > 0 \text{ and } z \in A\\
v(r,\theta,z) &= (0,0,-1) && \text{if } z < 0 \text{ and } (r,\theta,z) \notin A'\\
v(r,\theta,z) &= (\sin(2\phi), \theta+\pi, -\cos(2\phi)) && \text{if } z < 0 \text{ and } z \in A'.
\end{aligned}
$$

It is easy to see that v maps every plane $\{\theta = \alpha\} \cup \{\theta = \alpha + \pi\}$ into itself. Moreover the restrictions of v to these planes all look like as in the following picture



Lemma 2.1. *The vector field v is the gradient of a function w. Moreover there is a sequence of piecewise affine functions* w_k *such that:*

(a) $w_k \to w$ *strongly in* $W^{1,p}_{loc}$ *for every p;*
(b) $F^{\Omega}_{\infty}(w_k) \to F^{\Omega}_{\infty}(w)$ *for every open set* $\Omega \subset\subset \mathbf{R}^3$.

Proof. We consider the restriction of v to the plane $P := \{\theta = 0\} \cup \{\theta = \pi\}$. As already noticed v maps this plane into itself. Moreover its restriction to it satisfies condition (Curl), hence on P v is the gradient of a scalar function w. Moreover we can find such a w so that it is identically zero on the line $\{z = 0\} \cap P$. Hence w is symmetric with respect to the z axis and so we can extend w to the whole three-dimensional space so to build a cylindrically symmetric function. It is easy to check that the gradient of such a function is equal to v.

We call this function w as well and we will prove that it satisfies conditions (a) and (b) written above.

(a) Our goal is approximating v with piecewise constant gradient fields. First of all we do it in the upper half-space $\{z > 0\}$. For every n we take a regular n-agon B_n which is inscribed to the circle of radius 1 and lies on the plane $\{z = 0\}$. The vertices of this n-agon are given by $V_i := (1, 2i\pi/n, 0)$.

Hence we construct the pyramid A^n with vertex $V := (0, 0, \cot\phi)$ and base B_n. In the pyramid we identify n different regions $A^n_1, \dots, A^n_n$, where every A^n_i is given by the tetrahedron with vertices $(0, 0, 0)$, V, V_i, V_{i+1}. After this we put v_n equal to $(0, 0, 1)$ outside A^n and in every A^n_i we put

$$v_n(r, \theta, z) \equiv (\sin 2\phi, \pi + (2i + 1)\pi/n, \cos 2\phi).$$

It is easy to see that v_n satisfies condition (Curl), hence it is the gradient of some function w_n. Moreover we can choose w_n in such a way that it is identically 0 on $\{z = 0\}$. Then we extend w_n to the lower half space $\{z < 0\}$ just by imposing $w_n(r, \theta, -z) = w_n(r, \theta, z)$. It is not difficult to see that ∇w_n converges strongly to ∇w in L^p_{loc} for every p.

(b) Now we check that the previous construction satisfies also the second condition of the lemma. We fix an open set $\Omega \subset\subset \mathbf{R}^3$ and we observe that both w_k and w satisfy the eikonal equation in Ω. Moreover we call L^n_i the triangle with vertices V, V_i, V_{i+1} and L^n the union of L^n_i (so L^n is the "lateral surface" of the pyramid A^n). Finally we denote by L the lateral surface of the cone A, *i.e.* the set $\{(1 - r) = z \tan\phi\}$.

(i) The amount of jump of v_n (*i.e.* $|v^+_n - v^-_n|$) on L^n is constant and equal to the value of $|v^+ - v^-|$ on L. Moreover the area of L^n is converging to the area of L. The same happens on the symmetric sets in the lower half-space $\{z < 0\}$.

(ii) Let us call B the base of the cone. The right and left traces of v_n coincides with those of v on $B_n \cup (\{z = 0\} \setminus B)$. Moreover the area of $B \setminus B_n$ is converging to zero.
(iii) The vector fields v_n are discontinuous also on the triangles T_i^n joining V, $(0,0,0)$ and V_i (and on the symmetric triangles lying on $\{z < 0\}$). The amount of jump of v_n on each of these triangles is given by

$$|v_n^+ - v_n^-| = 2\sin(\pi/n).$$

Moreover the area of everyone is given by $(\cot\phi)/2$. Hence

$$\int_{\cup_i T_i^n} |v_n^+ - v_n^-|^3 \mathrm{d}\mathcal{H}^2 = 4n\cot\phi\sin^3\pi/n.$$

The right hand side goes to zero as $n \to \infty$ and this completes the proof.

□

Proof of Theorem 1.2. First of all we pass from the cartesian coordinates of the statement to the cylindrical coordinates (r, θ, z) given by $x_3 = z$, $x_1 = r\cos\theta$, $x_2 = r\sin\theta$ (and sometimes we will denote the elements of $\mathbf{R}^3$ with (y, z), where $y \in \mathbf{R}^2$ and $z \in \mathbf{R}$).

We take w as in the previous lemma. First of all let us compute $F_\infty^C(w)$ where C is the cylinder $\{r < 1\}$. As in the previous proof we call L the lateral surface of the cone, that is the set $\{r - 1 = z\tan\phi\}$. The value of $|\nabla w^+ - \nabla w^-|$ on the surface L is given by $2\sin\phi$ and the area of L is given by $\pi/\sin\phi$: the same happens for the symmetric of L lying on the half-space $\{z < 0\}$. On the base of the cylinder we have $|\nabla w^+ - \nabla w^-| = 2|\cos 2\phi|$. Hence

$$a(\phi) := F_\infty^C(u) - F_\infty^C(w) = \frac{\pi}{3}[8 - 8\cos^3 2\phi - 16\sin^2\phi]$$

and it can be easily checked that for ϕ close enough to zero, $a(\phi)$ is positive.

Therefore let us fix an α for which $a(\alpha) > 0$ and let us agree that w is constructed as in the previous lemma by choosing $\phi = \alpha$. Given $\rho > 0$ and $x \in \mathbf{R}^2$ we define $w_{x,\rho}$ in the cylinder $C_{x,\rho} := \{(y, z) : |y - x| \le \rho\} \subset \mathbf{R}^3$ as $w_{x,\rho}(y, z) = \rho w((y - x)/\rho, z/\rho)$. It is easy to see that

$$F_\infty^{C_{x,\rho}}(u) - F_\infty^{C_{x,\rho}}(w_{x,\rho}) = a(\alpha)\rho^2. \tag{2}$$

Let us fix ε and take ρ such that $\rho\cot\alpha < \varepsilon$. Thanks to Besicovitch Covering lemma we can cover $\mathcal{H}^2$ almost all $D := \{z = 0, r \le 1\}$ with a disjoint countable family of closed discs D_i such that every D_i has radius $r_i < \rho$, center x_i and is contained in D. We construct u_ε by putting $u_\varepsilon \equiv w_{x_i,\rho_i}$ in the cylinder C_{x_i,ρ_i}.

Since ∇u_ε coincides with ∇u in $\{z \ge \varepsilon\}$ and satisfies the eikonal equation, it is easy to see that $u_\varepsilon \to u$ locally in the strong topology of $W^{1,p}$. Moreover equation (2) implies that

$$F_\infty^C(u) - F_\infty^C(u_\varepsilon) = \sum_i a(\alpha)r_i^2 = a(\alpha).$$

At this point, using the previous lemma we can approximate the function u_ε in the cylinders C_{x_i,ρ_i} with piecewise affine functions in such a way that their traces coincide with the trace of u_ε on the boundary of C_{x_i,ρ_i}. Using standard diagonal arguments for every ε we can find a sequence of piecewise affine functions u_ε^k which converge in $W^{1,p}$ to u_ε and such that $F_\infty^C(u_\varepsilon^k) \to F_\infty^C(u_\varepsilon)$. Moreover, again using diagonal arguments, we can construct the sequence u_ε^k so that each one is a finite union of affine pieces.

Finally, one last diagonal argument, gives a sequence $\tilde{u}_k$ suh that:
(a) $\tilde{u}_k$ is a finite union of affine pieces;
(b) $\lim_k F_\infty^C(\tilde{u}_k) < F_\infty^C(u)$;
(c) $\tilde{u}_k \to u$ strongly in $W^{1,p}$ for every $p < \infty$.

□

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