ESAIM: COCV 24 (2018) 1789–1813
https://doi.org/10.1051/cocv/2017082

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

DYNAMIC BOUNDARY CONTROL GAMES WITH NETWORKS OF STRINGS

Martin Gugat[1,*] and Sonja Steffensen[2]

Abstract. Consider a star-shaped network of strings. Each string is governed by the wave equation. At each boundary node of the network there is a player that performs Dirichlet boundary control action and in this way influences the system state. At the central node, the states are coupled by algebraic conditions in such a way that the energy is conserved. We consider the corresponding antagonistic game where each player minimizes a certain quadratic objective function that is given by the sum of a control cost and a tracking term for the final state. We prove that under suitable assumptions a unique Nash equilibrium exists and give an explicit representation of the equilibrium strategies.

Mathematics Subject Classification. 49N70, 91A23

Received March 29, 2017. Accepted December 14, 2017.

1. Introduction

Since 1950, when J. Nash introduced the Nash equilibrium for finite-dimensional, noncooperative games [21], game theoretic models have been used to mathematically describe and analyse the strategic behaviour of a (finite) number of decision makers in various fields of applications such as economics, management, political sciences, engineering sciences, sociology and others (see *e.g.* [5, 8, 16, 17, 22, 23]). The standard Nash game models the following situation: assume that there exists a finite number $N \in \mathbb{N}$ of decision makers, so-called players, each of which possesses a set of strategies u_i and aims to minimize his/her own cost function J_i. In the standard setting, the players' sets of strategies are independent, whereas the cost functions in general depend on all players' strategies. Furthermore, each player i observes the strategies of the other players and chooses his/her optimal strategy under the assumption that the others will not change their chosen strategy. In the noncooperative Nash game it is assumed that the players do not cooperate with each other, *i.e.* make agreements about the choice of their strategies.

A multistrategy vector $u = (u_1, \ldots, u_N)$ is then called a *Nash equilibrium* if there exists no incentive for any player to change his/her strategy unilaterally, *i.e.* a Nash equilibrium is given by a vector $u^* = (u_1^*, \ldots, u_N^*)$ where each u_i^* solves the minimization problem

$$\min_{u_i \in U_i} J_i(u_i, u_{-i}^*)$$

Keywords and phrases: Vibrating string, boundary control, network, Nash equilibrium, game, pipeline network, gas transport.

[1] Friedrich-Alexander Universität Erlangen-Nürnberg (FAU), Department Mathematik, Cauerstr. 11, 91058 Erlangen, Germany.
[2] RWTH Aachen University, Templergraben 55, 52056 Aachen, Germany.

[*] Corresponding author: martin.gugat@fau.de

where u_{-i} denotes the multistrategy vector of the rival players of player i.

In contrast to their static counterparts, dynamic (or differential) games are time-dependent games in the sense that the chosen strategies may vary at any instant of time, *i.e.* the sets of strategies are subsets of suitably chosen function spaces and the cost functions are given by a cost functionals that are often given by integrals.

While Nash equilibria for Nash games governed by ordinary differential equations have already been studied thoroughly, see for example [2] and the references therein, only recently Nash equilibria for networked hyperbolic systems of partial differential equations have been considered. For traffic flow modelled by scalar conservation laws, the existence of Nash equilibria has been studied in [3]. In this paper we study Nash equilibria for a star–shaped network of strings with N rays and N players, where each player influences the system state in the network through a Dirichlet control at one end of one of the rays.

We study a pure-strategy Nash equilibrium, where for given strategies of the other players, each player cannot improve the value of her/his objective function. The objective functions are given by sums of squared L^2-norms with a term for the control cost and terms of tracking-type. We show that for the corresponding game a unique Nash equilibrium exists. We determine an explicit representation of the equilibrium strategies. Our objective function is an integral where the integrand is a quadratic function. General quadratic games with ordinary differential equations have been considered in [9], where also the definition of a Nash equilibrium is provided. Optimal boundary control problems with the wave equation for one player only and objective functions of the same type have already been considered, see for example [6, 10, 11, 18].

Fluid networks appear in many engineering applications (see [24]) such as gas pipelines. A simple linear model for the dynamics in a network of horizontal pipelines without friction for the ith pipe $i \in \{1, \dots, N\}$ is given by

$$\begin{cases} \rho_t^{(i)} + q_x^{(i)} = 0 \\ q_t^{(i)} + a^2\, \rho_x^{(i)} = 0 \end{cases} \tag{1.1}$$

(see [1]). Here $\rho^{(i)}$ denotes the gas density, $q^{(i)}$ the flow rate of the gas and $a > 0$ is the sound speed. This system implies that $\rho^{(i)}$ satisfies the wave equation

$$\rho_{tt}^{(i)} = a^2\, \rho_{xx}^{(i)}$$

along the pipe. The following coupling conditions are used to model the flow through a junction of N pipes where for all adjacent pipes $x = 0$ is the end of the junction (see [24]): Continuity of the density, that is

$$\rho^{(i)}(t,\, 0) = \rho^{(j)}(t,\, 0)$$

for all $i, j \in \{1, \dots, N\}$ and all t in the time interval $[0,\, T]$ and conservation of mass, which leads similar to Kirchhoff's law to the equation $\sum_{k=1}^{N} q^{(k)}(t,\, 0) = 0$. In terms of the densities, this yields the node conditions

$$\rho^{(i)}(t,\, 0) = \rho^{(j)}(t,\, 0), \quad \text{for all } i, j \in \{1, \dots, N\}; \quad \sum_{k=1}^{N} \rho_x^{(k)}(t,\, 0) = 0.$$

With the system dynamics for the densities defined by the wave equation and these node conditions, we consider a game with a star-shaped graph where each player is located at a boundary node. The application that we have in mind is a gas pipeline network where the players represent the gas market participants, *i.e.* producers or consumers.

In Nash games the preference of each player is modeled by her/his objective functional. Moreover, each player aims to minimize her/his own object functional only. Similarly to the objective functional for an optimal Dirichlet boundary control problem (for a single player) used in [19] we consider for each player i the objective functional

$$J_i(u) = \frac{\gamma_i}{2} \int_0^T u_i(\tau)^2 \, \mathrm{d}\tau + \frac{1}{2} \sum_{j=1}^N \left[\int_0^1 \left(\rho^{(j)}(T, x) - \rho^{(j)}_{\mathcal{D},i}(x)\right)^2 + \left(q^{(j)}(T, x) - q^{(j)}_{\mathcal{D},i}(x)\right)^2 \mathrm{d}x\right], \tag{1.2}$$

that consists of a quadratic costs for the ith player's control u_i and a tracking type term. In this term we assume the lengths of the pipelines to be normalized to one. Furthermore, $\rho^{(j)}_{\mathcal{D},\,i}(x)$ denotes the density profile that is desired by player i in pipe j at the terminal time T. Moreover, $q^{(j)}_{\mathcal{D},i}(x)$ denotes the profile of the flow rate in pipe j that is desired by player i at terminal time T. Here $q(T, \cdot)$ is determined by (1.1) and in such a way that $J_i(u)$ is minimized. In the application $\rho^{(j)}_{\mathcal{D},\,i}(x)$ and $q^{(j)}_{\mathcal{D},\,i}(x)$ might be the desired initial states for the next time period. Next, we give a detailed formal definition of the considered system. Instead of ρ in the sequel we will use the notation w for the system state.

2. Definitions and notation

Let a time $T \geq 4$ be given. Let $N \geq 3$ be a natural number and consider N strings of length 1. We use the notation $\mathcal{D}'((0,\,1))$ for the distributions on the interval $(0, 1)$ (see [26]) and

$$H^{-1}(0,1) = \{Y \in \mathcal{D}'((0,\,1)) : \text{ there is } f \in L^2(0,1) \text{ such that } f' = Y\}.$$

Define the set $\Omega = (0,\,T) \times (0,\,1)$. Define the set of initial states

$$B = \left\{(y_0^{(i)}, y_1^{(i)})_{i=1}^N : \; y_0^{(i)} \in L^2(0,\,1),\; y_1^{(i)} \in H^{-1}(0,\,1),\; i \in \{1, \ldots, N\}\right\}.$$

For $\left(y_0^{(i)}, y_1^{(i)}\right)_{i=1}^N \in B$ and $u_i \in L^2(0,\,T)$ for $i \in \{1, \ldots, N\}$ we consider the system $(\mathbf{S})$ that is defined by the equations (2.1)–(2.6) below:

$$w^{(i)}(0,x) = y_0^{(i)}(x), \qquad x \in (0,\,1), \qquad i \in \{1, \ldots, N\}, \tag{2.1}$$
$$w_t^{(i)}(0,x) = y_1^{(i)}(x), \qquad x \in (0,\,1), \qquad i \in \{1, \ldots, N\}, \tag{2.2}$$
$$w_{tt}^{(i)}(t,x) = w_{xx}^{(i)}(t,x), \qquad (t,x) \in \Omega, \qquad i \in \{1, \ldots, N\}, \tag{2.3}$$
$$w^{(i)}(t,0) = w^{(j)}(t,0), \qquad t \in (0,\,T), \qquad i,j \in \{1, \ldots, N\}, \tag{2.4}$$
$$0 = w_x^{(1)}(t,0) + w_x^{(2)}(t,0) + \cdots + w_x^{(N)}(t,0), \quad t \in (0,\,T), \tag{2.5}$$
$$w^{(i)}(t,\,1) = u_i(t), \qquad t \in (0,\,T), \qquad i \in \{1, \ldots, N\}. \tag{2.6}$$

System $(\mathbf{S})$ is a star-shaped network of vibrating strings with Dirichlet boundary control action at the boundary nodes. The exact controllability of networks of vibrating strings is studied in [25]. An overview on the control of networks of vibrating strings can be found in [4]. The observation and control of vibrations in tree-shaped networks of strings is studied in [7].

We consider dynamic games with N players that control the system by their strategies u_i, $i \in \{1, \ldots, N\}$. For $i \in \{1, \ldots, N\}$ we assume that the strategy of the player i is given by a control function $u_i \in L^2(0,\,T)$. Each player has the goal to minimize his/her own cost functional that is denoted by J_i. In order to define these objective functionals J_i, let numbers $\gamma_i > 0$ be given that serve as the weighting factors of the control costs in J_i. For i and $j \in \{1, \ldots, N\}$ let states that are desired by player i for the terminal time T for the jth string be given by the desired position $g^{(j)}_{\mathcal{D},i}(\cdot) \in L^2(0,\,1)$ and the antiderivatives of the desired velocity $h^{(j)}_{\mathcal{D},i}(\cdot) \in L^2(0,\,1)$.

We consider the objective functionals

$$J_i(u) = \frac{\gamma_i}{2}\int_0^T u_i(\tau)^2\,\mathrm{d}\tau + \frac{1}{2}\sum_{j=1}^N \left[\int_0^1 \left(w^{(j)}(T,\,x) - g_{\mathcal{D},i}^{(j)}(x)\right)^2 + \left(V_i^{(j)}(x) - h_{\mathcal{D},i}^{(j)}(x)\right)^2 \mathrm{d}x\right] \tag{2.7}$$

where for $i \in \{1, \ldots, N\}$ the $V_i^{(j)}(x) \in L^2(0,\,1)$ are antiderivatives of $w_t^{(j)}(T,\,x)$ in the sense that

$$\left(V_i^{(j)}\right)_x(x) = w_t^{(j)}(T,\,x) \tag{2.8}$$

and such that the equation

$$\int_0^1 V_i^{(j)}(x)\,\mathrm{d}x = \int_0^1 h_{\mathcal{D},i}^{(j)}(x)\,\mathrm{d}x \tag{2.9}$$

holds (see [11]). By (2.8) and (2.9), the $V_i^{(j)}$ are uniquely determined. The value of the objective function J_i depends on the system state on all N strings that is determined by all the controls $u = (u_1, \ldots, u_N)$ although player i decides only on the choice of u_i.

This paper has the following structure. In Section 3 we present our main result in Theorem 3.1. In Section 4 we study the well-posedness of System (**S**) that is defined by (2.1)–(2.6). An explicit representation of the equilibrium strategies is given in at the end of Section 7 in Theorem 7.3. In the representation, we need some matrices that are discussed at the beginning of the section. For the proof of the main result, a transformation of the objective function is essential that is given in Section 5. In Section 6 the system that consists of each player's necessary optimality conditions is stated. It allows to derive an explicit representation of the Nash equilibrium using two intermediate Lemmas. This yields the proof of Theorem 7.3. Then Theorem 7.3 implies Theorem 3.1.

3. Main result: the unique Nash equilibrium

In this section we present our main result about the game with N players and system (2.1)–(2.6) where the aim of each player $i \in \{1, \ldots, N\}$ is to minimize the objective function J_i as defined in (2.7). The boundary control game with the wave equation has a unique Nash-equilibrium. In fact we can determine the optimal strategies explicitly. This is stated in Theorem 3.1.

Theorem 3.1. *Assume that $\gamma_i = \gamma_j$ for all $i,\, j \in \{1, \ldots, N\}$. Then there exists a unique Nash equilibrium for system* (**S**) *that is defined by (2.1)–(2.6) with the objective functions J_i given in (2.7). The optimal strategies are* 4*-periodic. An explicit representation of the optimal strategies is given in Theorem 7.3.*

Remark 3.2. In Theorem 3.1, we have assumed that the weights γ_i are all equal. Hence there exists a number $\gamma > 0$ such that $\gamma_i = \gamma$ for all $i \in \{1, \ldots, N\}$. In this paper, we focus on this special case since in this case the analysis is more explicit and less technical than in the general case where the weights are allowed to differ. However, in the future we want to extend the analysis to this general case. We expect that in the general case it is sufficient to assume that the γ_i are sufficiently large or that for all $i,\, j \in \{1, 2, \ldots, N\}$, the distance $|\gamma_i - \gamma_j|$ is sufficiently small.

Assume that

$$T = 4\,K + \mathit{\Delta}$$

with $K \in \{1, 2, 3, \ldots\}$ and $\mathit{\Delta} \in [0,\,4)$. Let $i \in \{1, \ldots, N\}$, be given. Since the optimal strategies are 4–periodic, it is sufficient to have explicit representations for the restrictions $u_i|_{(0,\,4)}$ to determine the optimal strategies. In fact, in Theorem 7.3 such an explicit representation is given.

4. Well-posedness

In this section we study the well-posedness of the system (2.1)–(2.6). As in [12], define the orthogonal symmetric $N \times N$ reverberation matrix

$$A = \frac{N-2}{N} \begin{pmatrix} 1 & \frac{2}{2-N} & \frac{2}{2-N} & \cdots & \frac{2}{2-N} \\ \frac{2}{2-N} & 1 & \frac{2}{2-N} & \cdots & \frac{2}{2-N} \\ \vdots & & \ddots & & \vdots \\ \frac{2}{2-N} & \cdots & \frac{2}{2-N} & 1 & \frac{2}{2-N} \\ \frac{2}{2-N} & \frac{2}{2-N} & \cdots & \frac{2}{2-N} & 1 \end{pmatrix}. \tag{4.1}$$

In this section we define the d'Alembert solution in the sense of characteristics for system ($\mathbf{S}$) analogously to Theorem 1 in [13].

Theorem 4.1 (Well-posedness of ($\mathbf{S}$))**.** *Let the initial state* $\left(y_0^{(i)},\, y_1^{(i)}\right)_{i=1}^N \in B$ *be given. For* $x \in (0,\,1)$ *and* $i \in \{1,\dots,N\}$ *define the functions* α_i, β_i *by*

$$\alpha_i(x) = \frac{1}{2}\, y_0^{(i)}(x) + \frac{1}{2} \int_0^x y_1^{(i)}(s)\, ds,\ x \in (0,\,1), \tag{4.2}$$

$$\beta_i(x) = \frac{1}{2}\, y_0^{(i)}(x) - \frac{1}{2} \int_0^x y_1^{(i)}(s)\, ds,\ x \in (0,\,1). \tag{4.3}$$

For $t \in (1,\,2)$ *let*

$$\alpha_i(t) = u_i(t-1) - \beta_i\,(2-t). \tag{4.4}$$

Together with (4.2) this yields the values of $\alpha_i(t)$ *for* $t \in (0,\,2)$.

For $t \in (0,\,2)$ *let the equation*

$$\begin{pmatrix} \beta_1(-t) \\ \beta_2(-t) \\ \vdots \\ \beta_N(-t) \end{pmatrix} = -A \begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \\ \vdots \\ \alpha_N(t) \end{pmatrix} \tag{4.5}$$

define the values of $\beta_i(t)$ *for* $t \in (-2,\,0)$. *For* $t \in (2,\,4)$ *equation (4.4) yields the values of* $\alpha_i(t)$. *Hence for* $t \in (0,\,4)$ *the values of* $\alpha_i(t)$ *are well-defined. Now (4.5) defines the values of* $\beta_i(t)$ *for* $t \in (-4,\,0)$.

By repeating the process we define the functions $\alpha_i(t)$, $\beta_i(t)$ *inductively and obtain functions* $\alpha_i \in L^2(0,\,T+1)$ *and* $\beta_i \in L^2(-T,\,1)$. *Then* $(w^{(i)})_{i=1}^N$ *given by*

$$w^{(i)}(t,x) = \alpha_i(x+t) + \beta_i(x-t) \tag{4.6}$$

is the unique solution of ($\mathbf{S}$) *in the sense described below.*

The functions $w^{(i)}$ *are in* $L^2(\Omega)$. *Given the family of test functions*

$$\mathcal{T} = \{\varphi \in C^2(\mathbb{R}^2) : \text{There exists a set } Q = [t_1, t_2] \times [x_1, x_2] \subset \Omega$$
$$\text{such that the support of } \varphi \text{ is contained in the interior of } Q\},$$

the function $w^{(i)}$ satisfies the wave equation (2.3) in the following weak sense:

$$\int_{\Omega} w^{(i)}(t,x)\,\varphi_{tt}(t,x)\,\mathrm{d}(t,x) = \int_{\Omega_i} w^{(i)}(t,x)\,\varphi_{xx}(t,x)\,\mathrm{d}(t,x) \quad \textit{for all} \;\; \varphi \in \mathcal{T}.$$

The functions $w^{(i)}$ satisfy (2.1) for almost every $x \in (0,\,1)$, (2.4) and (2.6) for almost every $t \in [0,\,T]$ and (2.2) and (2.5) in the sense of distributions.

Proof. Due to (4.2) and (4.3), it is easy to check that $(w^{(i)})_{i=1}^N$ satisfies the initial conditions (2.1) and (2.2). Similar as in the proof of Theorem 1 in [13], integration by parts shows that $v^{(i)}$ satisfies the wave equation in the weak sense given in Theorem 4.1. To be precise, for all $\varphi \in \mathcal{T}$ and $i \in \{1, \dots, N\}$ we have

$$\int_{\Omega} \varphi_{xx}(t,\,x)\,\alpha(x+t)\,\mathrm{d}(t,x) = \int_{\Omega} \varphi_{tt}(t,\,x)\,\alpha(x+t)\,\mathrm{d}(t,x).$$

The corresponding equation with $\beta(x-t)$ follows analogously.

Writing the boundary conditions (2.6) in terms of α_i and β_i yields equation (4.4). By definition of $\beta_i(\cdot)$, for almost every $t > 0$ we have (4.5). Hence we have

$$\begin{pmatrix} w^{(1)}(t,0) \\ w^{(2)}(t,0) \\ \vdots \\ w^{(N)}(t,0) \end{pmatrix} = \begin{pmatrix} \alpha_1(t) + \beta_1(-t) \\ \alpha_2(t) + \beta_2(-t) \\ \vdots \\ \alpha_N(t) + \beta_N(-t) \end{pmatrix} = (I - A) \begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \\ \vdots \\ \alpha_N(t) \end{pmatrix}$$
$$= \begin{pmatrix} \frac{2}{N} & \frac{2}{N} & \cdots & \frac{2}{N} \\ \vdots & \vdots & & \vdots \\ \frac{2}{N} & \frac{2}{N} & \cdots & \frac{2}{N} \end{pmatrix} \begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \\ \vdots \\ \alpha_N(t) \end{pmatrix},$$

which implies the equation $w^{(i)}(t,0) = w^{(j)}(t,0)$ for $t \in (0,\,T)$ almost everywhere and $i,j \in \{1,\dots,N\}$. Hence (2.4) holds. Moreover, we have

$$\begin{pmatrix} w_x^{(1)}(t,0) \\ w_x^{(2)}(t,0) \\ \vdots \\ w_x^{(N)}(t,0) \end{pmatrix} = \begin{pmatrix} \alpha_1'(t) + \beta_1'(-t) \\ \alpha_2'(t) + \beta_2'(-t) \\ \vdots \\ \alpha_N'(t) + \beta_N'(-t) \end{pmatrix} = (I + A) \begin{pmatrix} \alpha_1'(t) \\ \alpha_2'(t) \\ \vdots \\ \alpha_N'(t) \end{pmatrix}$$
$$= \frac{N-2}{N} \begin{pmatrix} \frac{2N-2}{N-2} & \frac{2}{2-N} & \cdots & \frac{2}{2-N} \\ \frac{2}{2-N} & \ddots & & \vdots \\ \vdots & & \ddots & \frac{2}{2-N} \\ \frac{2}{2-N} & \cdots & \frac{2}{2-N} & \frac{2N-2}{N-2} \end{pmatrix} \begin{pmatrix} \alpha_1'(t) \\ \alpha_2'(t) \\ \vdots \\ \alpha_N'(t) \end{pmatrix}.$$

This implies the equation $w_x^{(1)}(t,0) + w_x^{(2)}(t,0) + \cdots + w_x^{(N)}(t,0) = 0$ in the sense of distributions on $(0,\, T)$, hence (2.5) holds. The uniqueness of the solution follows from the conservation of the energy

$$E(t) = \frac{1}{2} \sum_{j=1}^{N} \int_0^1 \left(w_t^{(j)}(t,\, x)\right)^2 + \left(w_x^{(j)}(t,\, x)\right)^2 \,\mathrm{d}x.$$

Thus we have proved Theorem 4.1. □

In Lemma 4.4, we present a recursion for the α_i that is independent of the β_i. In order to write the representation in a compact form, we introduce some auxiliary functions in the next lemma.

Lemma 4.2. *Recall that* $T = 4\,K + \varDelta$ *with* $K \in \{1, 2, 3, 4, \ldots\}$ *and* $\varDelta \in [0,\, 4)$. *For* $x \in (0, 1)$ *such that* $\varDelta + x \notin \mathbb{N}$, *there exist unique numbers* $\kappa_+(x) \in \{K,\, K+1\}$, $m_+(x) \in \{0,\, 1\}$, $\sigma_+(x) \in \{-1,\, 1\}$ *and* $z_+(x) \in (0, 1)$ *such that*

$$4\,K + \varDelta + x = 4\,\kappa_+(x) + 2\,m_+(x) + \sigma_+(x)\, z_+(x). \tag{4.7}$$

Moreover, for all $x \in (0, 1)$, *such that* $\varDelta - x \notin \mathbb{Z}$, *there exist unique numbers* $\kappa_-(x) \in \{K,\, K+1\}$, $m_-(x) \in \{0,\, 1\}$, $\sigma_-(x) \in \{-1,\, 1\}$ *and* $z_-(x) \in (0, 1)$ *such that*

$$4\,K + \varDelta - x = 4\,\kappa_-(x) + 2m_-(x) + \sigma_-(x)\, z_-(x). \tag{4.8}$$

For $x \in (0, 1)$ *such that* $\varDelta + x \notin \mathbb{N}$ *(*$\varDelta - x \notin \mathbb{Z}$ *respectively) we define the auxiliary functions*

$$\mathcal{C}_+(x) = 1 + \sigma_+(x)\, z_+(x),\ \mathcal{C}_-(x) = 1 + \sigma_-(x)\, z_-(x).$$

Then for these $x \in (0,\, 1)$ *there is a unique* $j \in \{0,\, 1,\, 2\}$ *such that*

$$\mathcal{C}_+(x) = 1 + \varDelta + x - 2\,j \in [0, 2)$$

and there is a unique $k \in \{0,\, 1,\, 2\}$ *such that*

$$\mathcal{C}_-(x) = 1 + \varDelta - x - 2\,k \in [0, 2).$$

We have

$$\bigcup_{x \in (0,\,1):\, \varDelta + x \notin \mathbb{N}} \mathcal{C}_+(x) \cup \bigcup_{x \in (0,\,1):\, \varDelta - x \notin \mathbb{Z}} \mathcal{C}_-(x) = (0,\, 1) \cup (1,\, 2). \tag{4.9}$$

Hence for all $u \in L^2(0,\, T)$ *that we extend by the definition* $u(t) = 0$ *for* $t > T$ *we have*

$$\int_0^T u^2(s)\, ds = \sum_{j=0}^{\infty} \int_0^1 u(\mathcal{C}_+(x) + 4j)^2 + u(\mathcal{C}_+(x) + 4j + 2)^2 + u(\mathcal{C}_-(x) + 4j)^2 + u(\mathcal{C}_-(x) + 4j + 2)^2 \,\mathrm{d}x. \tag{4.10}$$

If $\mathcal{C}_+(x) + 4\,\kappa_+(x) \le T$, *we have* $m_+(x) = 1$. *If* $x \in [0,\, 1)$ *and* $\mathcal{C}_-(x) + 4\,\kappa_-(x) \le T$, *we have* $m_-(x) = 1$.

Proof. Consider (4.7). There are five cases:

1. $\varDelta + x \in (0,\, 1)$. Then we have $\kappa_+(x) = K$, $m_+(x) = 0$, $\sigma_+(x) = 1$, $z_+(x) = \varDelta + x$, $\mathcal{C}_+(x) = 1 + \varDelta + x$.

2. $\Delta + x \in (1, 2)$. Then we have $\kappa_+(x) = K$, $m_+(x) = 1$, $\sigma_+(x) = -1$, $z_+(x) = 2 - \Delta - x$, $\mathcal{C}_+(x) = 1 + \Delta + x - 2$.
3. $\Delta + x \in (2, 3)$. Then we have $\kappa_+(x) = K$, $m_+(x) = 1$, $\sigma_+(x) = 1$, $z_+(x) = \Delta + x - 2$, $\mathcal{C}_+(x) = 1 + \Delta + x - 2$.
4. $\Delta + x \in (3, 4)$. Then we have $\kappa_+(x) = K + 1$, $m_+(x) = 0$, $\sigma_+(x) = -1$, $z_+(x) = 4 - \Delta - x$, $\mathcal{C}_+(x) = 1 + \Delta + x - 4$.
5. $\Delta + x \in (4, 5)$. Then we have $\kappa_+(x) = K + 1$, $m_+(x) = 0$, $\sigma_+(x) = 1$, $z_+(x) = \Delta + x - 4$, $\mathcal{C}_+(x) = 1 + \Delta + x - 4$.

Consider (4.8). There are five cases:

1. $\Delta - x \in (-1, 0)$. Then we have $\kappa_-(x) = K$, $m_-(x) = 0$, $\sigma_-(x) = -1$, $z_-(x) = x - \Delta$, $\mathcal{C}_-(x) = 1 + \Delta - x$.
2. $\Delta - x \in (0, 1)$. Then we have $\kappa_-(x) = K$, $m_-(x) = 0$, $\sigma_-(x) = 1$, $z_-(x) = \Delta - x$, $\mathcal{C}_-(x) = 1 + \Delta - x$.
3. $\Delta - x \in (1, 2)$. Then we have $\kappa_+(x) = K$, $m_-(x) = 1$, $\sigma_-(x) = -1$, $z_-(x) = 2 - \Delta + x$, $\mathcal{C}_-(x) = 1 + \Delta - x - 2$.
4. $\Delta - x \in (2, 3)$. Then we have $\kappa_-(x) = K$, $m_-(x) = 1$, $\sigma_-(x) = 1$, $z_-(x) = \Delta - x - 2$, $\mathcal{C}_-(x) = 1 + \Delta - x - 2$.
5. $\Delta - x \in (3, 4)$. Then we have $\kappa_-(x) = K + 1$, $m_-(x) = 0$, $\sigma_-(x) = -1$, $z_-(x) = 4 - \Delta + x$, $\mathcal{C}_-(x) = 1 + \Delta - x - 4$.

Let $z \in (0, 1) \cup (1, 2)$ be given. Then there exists $j \in \{1, 2, 3\}$ such that $z - \Delta - 1 + 2j \in (-1, 0) \cup (0, 1)$. Then there exists a point $x \in (0, 1)$ such that $z = 1 + \Delta - 2j \pm x$, and (4.9) follows.

If $\mathcal{C}_+(x) + 4\kappa_+(x) \leq T$, the definition of $\mathcal{C}_+(x)$ implies $T \geq 4\kappa_+(x) + \sigma_+(x)\, z_+(x) + 2\, m_+(x) + 1 - 2\, m_+(x)$. By (4.7), this implies $4K + \Delta + x + 1 - 2\, m_+(x) \leq T = 4K + \Delta$. Thus we have $2\, m_+(x) \geq x + 1$ which implies $m_+(x) = 1$.

If $\mathcal{C}_-(x) + 4\kappa_-(x) \leq T$, the definition of $\mathcal{C}_-(x)$ implies $T \geq 4\kappa_-(x) + \sigma_-(x)\, z_-(x) + 2\, m_-(x) + 1 - 2\, m_-(x)$. By (4.8), this implies $4K + \Delta - x + 1 - 2\, m_-(x) \leq T = 4K + \Delta$. Thus we have $2\, m_+(x) \geq 1 - x$ which implies $m_-(x) = 1$. □

Remark 4.3. If $\Delta = 0$, we have $\mathcal{C}_\pm(x) = 1 \pm x$, $\kappa_\pm(x) = K$ and $m_\pm(x) = 0$.

Now we consider the case $\Delta \in (0, 1)$: Then for $x \in (0, 1 - \Delta)$, we have $\mathcal{C}_+(x) = 1 + x + \Delta$ and for $x \in (1 - \Delta, 1)$ we have $\mathcal{C}_+(x) = x + \Delta - 1$. In both cases, we have $\kappa_+(x) = K$.

If $\Delta = 1$, we have $\mathcal{C}_+(x) = x$ and $\mathcal{C}_-(x) = 2 - x$.

Now we consider the case $\Delta \in (1, 2)$: Then for all $x \in (0, 1)$, we have $\mathcal{C}_+(x) = x + \Delta - 1$ and $\kappa_+(x) = K$.

If $\Delta = 2$, we have $\mathcal{C}_\pm(x) = 1 \pm x$.

Lemma 4.4. *For $t > T$, let $u(t) = 0$. For $t \in (0, 1)$, define $\alpha(-t) = -A\,\beta(t)$ with β as defined in (4.3). Due to (4.4) and (4.5) we have for $t \in (0, T)$*

$$\alpha(t+1) = u(t) - \beta(1-t) = u(t) + A\,\alpha(t-1) \quad \textit{and} \quad \alpha(t+2) = u(t+1) + A\,\alpha(t). \tag{4.11}$$

This implies that for $x \in (0, 1)$ almost everywhere and $m \in \{0, 1\}$ we have

$$\alpha(4K + 2m \pm x) = A^m \alpha(\pm x) + m\, u(4K + 1 \pm x) + \sum_{j=0}^{K-1} A^{1-m}\, u(1 \pm x + 4j) + A^m u(1 \pm x + 4j + 2) \tag{4.12}$$

which in turn implies

$$A\,\alpha(4K - x) = -\beta(x) + \sum_{j=0}^{K-1} u(1 - x + 4j) + A\, u(1 - x + 4j + 2). \tag{4.13}$$

Define $\mathcal{B}_\pm(x) = \mathcal{C}_\pm(x) - 1$. *Then we have the equation* $\alpha(T \pm x)$

$$= A^{m_\pm(x)}\alpha(\mathcal{B}_\pm(x)) + m_\pm(x)\, u(\mathcal{C}_\pm(x) + 4\,\kappa_\pm(x)) + \sum_{j=0}^{\kappa_\pm(x)-1} A^{1-m_\pm(x)}\, u(\mathcal{C}_\pm(x) + 4\,j) + A^{m_\pm(x)} u(\mathcal{C}_\pm(x) + 4\,j + 2). \tag{4.14}$$

Hence $A^{m_\pm(x)}\,\alpha(T \pm x)$

$$= \alpha(\mathcal{B}_\pm(x)) + m_\pm(x)\, A^{m_\pm(x)}\, u(\mathcal{C}_\pm(x) + 4\,\kappa_\pm(x)) + \sum_{j=0}^{\kappa_\pm(x)-1} A\, u(\mathcal{C}_\pm(x) + 4\,j) + u(\mathcal{C}_\pm(x) + 4\,j + 2). \tag{4.15}$$

Proof. Equation (4.11) implies

$$\alpha(t+3) = u(t+2) + A\alpha(t+1) = u(t+2) + Au(t) + \alpha(t-1).$$

By induction for $k \in \{1, 2, 3, \ldots\}$ this yields

$$\alpha(t + 4k - 1) = \alpha(t-1) + \sum_{j=0}^{k-1} A\, u(t + 4\,j) + u(t + 4\,j + 2). \tag{4.16}$$

For $t = 1 + x$ this yields

$$\alpha(x + 4\,k) = \alpha(x) + \sum_{j=0}^{k-1} A\, u(1 + x + 4\,j) + u(1 + x + 4\,j + 2)$$

and for $t = 1 - x$ we obtain from (4.16)

$$\alpha(-x + 4\,k) = \alpha(-x) + \sum_{j=0}^{k-1} A\, u(1 - x + 4\,j) + u(1 - x + 4\,j + 2).$$

For $k = K$ this yields (4.12) for $m = 0$. The case $m = 1$ follows with (4.11). Lemma 4.2 and (4.12) imply (4.14). ☐

The proof of the main result is based upon the transformation of the objective functions J_i that is presented in the next section. The fact that the objective function is quadratic allows us to give an explicit expansion around the equilibrium strategies $v_i^{\pm,\,(j),*}(x)$.

5. Transformation of the objective functions

In this section we express the objective functionals J_i (for $i \in \{1, 2, \ldots, N\}$) of the players that have been defined in (2.7) in terms of the traveling waves. For the transformation we will need the following definitions.

For $x \in (0,\, 1)$ almost everywhere, define the functions

$$b_{+,\,i}^{(j)}(x) = -\, g_{\mathcal{D},i}^{(j)}(x) - h_{\mathcal{D},i}^{(j)}(x), \tag{5.1}$$

$$b_{-,\,i}^{(j)}(x) = \quad g_{\mathcal{D},i}^{(j)}(x) - h_{\mathcal{D},i}^{(j)}(x) \tag{5.2}$$

and let

$$b_{+,\,i}(x) = \left(b^{(j)}_{+,\,i}(x)\right)_{j=1}^{N}, \quad b_{-,\,i}(x) = \left(b^{(j)}_{-,\,i}(x)\right)_{j=1}^{N}. \tag{5.3}$$

Let A_i denote the ith column of the matrix $A = (a_{ij})_{i,j=1}^N$ defined in (4.1) and e_i denote the ith column of the identity matrix I. For $i \in \{1, \ldots, N\}$ define the numbers

$$\bar{I}_i = \sum_{j=1}^{N} \int_0^1 h^{(j)}_{\mathcal{D},i}(x) \, - h^{(j)}_{\mathcal{D},j}(x)\,\mathrm{d}x\, a_{ij}. \tag{5.4}$$

Let $i \in \{1, \ldots, N\}$, $k \in \{0, \ldots, 2\,K\}$ and $x \in (0,\,1)$ be given.
If $\mathcal{C}_+(x) + 2k \in (0,\,T)$, $(\mathcal{C}_-(x) + 2k \in (0,\,T)$ respectively) define

$$v_i^{+,(k)}(x) = u_i(\mathcal{C}_+(x) + 2k),\ v_i^{-,(k)}(x) = u_i(\mathcal{C}_-(x) + 2k). \tag{5.5}$$

Since the optimal strategies are 4–periodic, it is sufficient to have explicit representations for

$$v_i^{+,(0)}(x) = u_i(\mathcal{C}_+(x)),\ v_i^{-,(0)}(x) = u_i(\mathcal{C}_-(x)), \tag{5.6}$$
$$v_i^{+,(1)}(x) = u_i(\mathcal{C}_+(x) + 2),\ v_i^{-,(1)}(x) = u_i(\mathcal{C}_-(x) + 2) \tag{5.7}$$

for $x \in (0,\,1)$ almost everywhere and $i \in \{1, \ldots, N\}$ to determine the optimal strategies.

Now, we express the tracking type parts of the objective functional in terms of the travelling waves. Due to (2.8) and (4.6) we have for $j \in \{1, 2, \ldots, N\}$

$$V_i^{(j)}(x) = \alpha_j(x+T) - \beta_j(x-T) + r_i^{(j)} \tag{5.8}$$

where the real constant $r_i^{(j)}$ is chosen such that (2.9) holds, that is

$$r_i^{(j)} = \int_0^1 h^{(j)}_{\mathcal{D},i}(x) \, - \alpha_j(x+T) + \beta_j(x-T)\,\mathrm{d}x. \tag{5.9}$$

The choice from (5.9) gives the minimal value of J_i with respect to the constants $r_i^{(j)}$ for $j \in \{1, \ldots, N\}$. Due to (5.9), for all $j \in \{1, \ldots, N\}$ we have

$$r_j^{(j)} = \int_0^1 h^{(j)}_{\mathcal{D},j}(x) \, - \alpha_j(x+T) + \beta_j(x-T)\,\mathrm{d}x. \tag{5.10}$$

The difference of (5.9) and (5.10) yields

$$r_i^{(j)} - r_j^{(j)} = \int_0^1 h^{(j)}_{\mathcal{D},i}(x) \, - h^{(j)}_{\mathcal{D},j}(x)\,\mathrm{d}x.$$

Hence the values of $r_i^{(j)}$ for $j \neq i$ are given in terms of the $r_j^{(j)}$ by $r_i^{(j)} = r_j^{(j)} + \int_0^1 h^{(j)}_{\mathcal{D},i}(x) \, - h^{(j)}_{\mathcal{D},j}(x)\,\mathrm{d}x$.

Due to the definition (2.7) of J_i, the traveling waves representation (4.6) of the state and (5.8) for $i \in \{1, 2, \ldots, N\}$ we have

$$J_i(u) = \frac{\gamma_i}{2} \int_0^T u_i(\tau)^2 \,\mathrm{d}\tau + \frac{1}{2} \sum_{j=1}^{N} \int_0^1 \left(\alpha_j(x+T) + \beta_j(x-T) - g_{\mathcal{D},i}^{(j)}(x) \right)^2 \mathrm{d}x$$
$$+ \frac{1}{2} \sum_{j=1}^{N} \int_0^1 \left(\alpha_j(x+T) - \beta_j(x-T) + r_i^{(j)} - h_{\mathcal{D},i}^{(j)}(x) \right)^2 \mathrm{d}x. \tag{5.11}$$

This yields the equation

$$J_i(u) = \frac{\gamma_i}{2} \int_0^T u_i(\tau)^2 \,\mathrm{d}\tau + \sum_{j=1}^{N} \int_0^1 \alpha_j(x+T)^2 + \beta_j(x-T)^2 + \frac{1}{2} \left(r_i^{(j)} \right)^2 - \alpha_j(x+T) \left(g_{\mathcal{D},i}^{(j)}(x) + h_{\mathcal{D},i}^{(j)}(x) - r_i^{(j)} \right)$$
$$- \beta_j(x-T) \left(g_{\mathcal{D},i}^{(j)}(x) - h_{\mathcal{D},i}^{(j)}(x) + r_i^{(j)} \right) - r_i^{(j)} h_{\mathcal{D},i}^{(j)}(x) + \frac{1}{2} g_{\mathcal{D},i}^{(j)}(x)^2 + \frac{1}{2} h_{\mathcal{D},i}^{(j)}(x)^2 \,\mathrm{d}x.$$

In order to write J_i in a more compact form, for $x \in (0, 1)$ almost everywhere we define

$$d_i(x) = \frac{1}{2} \sum_{j=1}^{N} \left(g_{\mathcal{D},i}^{(j)}(x) \right)^2 + \left(h_{\mathcal{D},i}^{(j)}(x) \right)^2 \tag{5.12}$$

and the vectors

$$r_i = \left(r_i^{(j)} \right)_{j=1}^{N}, \quad h_i(x) = \left(h_{\mathcal{D},i}^{(j)}(x) \right)_{j=1}^{N}.$$

With the notation $b_{\pm,\,i}(x)$ defined in (5.3) and $\|(z_1, \ldots, z_N)\|_2 = \sqrt{\sum_{j=1}^{N} (z_j)^2}$, due to (5.1) and (5.2) we have

$$J_i(u) = \frac{\gamma_i}{2} \|u_i\|_{L^2(0,T)}^2 + \frac{1}{2} \|r_i\|_2^2 + \int_0^1 \|\alpha(x+T)\|_2^2 + \|\beta(x-T)\|_2^2 \,\mathrm{d}x$$
$$+ \int_0^1 (b_{+,i}(x) + r_i)^T \alpha(x+T) - (b_{-,i}(x) + r_i)^T \beta(x-T) - r_i^T h_i(x) + d_i(x) \,\mathrm{d}x.$$

Due to (4.5), this yields

$$J_i(u) = \frac{\gamma_i}{2} \|u_i\|_{L^2(0,T)}^2 + \frac{1}{2} \|r_i\|_2^2 + \int_0^1 \|\alpha(x+T)\|_2^2 + \|A\,\alpha(T-x)\|_2^2 \,\mathrm{d}x$$
$$+ \int_0^1 (b_{+,i}(x) + r_i)^T \alpha(x+T) + (b_{-,i}(x) + r_i)^T A\,\alpha(T-x) - r_i^T h_i(x) + d_i(x) \,\mathrm{d}x. \tag{5.13}$$

In order to simplify the transformation of the objective function, we extend the functions $u_i \in L^2(0,T)$ by zero for $t > T$, that is we define $u_i(t) = 0$ for $t > T$. This allows us to extend the definition (5.5) of the $v_i^{\pm,(k)}(x)$ to all $k \in \{0, 1, 2, 3, \ldots\}$. Note that still only a finite number of $v_i^{\pm,(k)}(x)$ are not zero. Since the sum over j is finite, equation (4.10) yields

$$\|u_i\|^2_{L^2(0,T)} = \int_0^1 \sum_{j=0}^{\infty} \sum_{m=0}^{1} u_i(\mathcal{C}_+(x) + 4\,j + 2\,m)^2 + u_i(\mathcal{C}_-(x) + 4\,j + 2\,m)^2 \,\mathrm{d}x. \tag{5.14}$$

We introduce the linear operator

$$L_\pm\, u(x) = m_\pm(x)\, A^{m_\pm(x)}\, u(\mathcal{C}_\pm(x) + 4\,\kappa_\pm(x)) + \sum_{j=0}^{\kappa_\pm(x)-1} A\, u(\mathcal{C}_\pm(x) + 4\,j) + u(\mathcal{C}_\pm(x) + 4\,j + 2).$$

Due to (4.15), we have $A^{m_\pm(x)}\,\alpha(T \pm x) = \alpha(\mathcal{B}_\pm(x)) + L_\pm\, u(x)$. For $x \in (0,1)$ almost everywhere, define the integrand

$$\begin{aligned}\mathcal{G}_i(x,\, u) = {} & \frac{\gamma_i}{2}\left(\sum_{j=0}^{\infty}\sum_{m=0}^{1}\left[u_i(\mathcal{C}_+(x) + 4\,j + 2\,m)^2 + u_i(\mathcal{C}_-(x) + 4\,j + 2\,m)^2\right]\right) + \frac{1}{2}\,\|r_i\|_2^2 - r_i^T\, h_i(x) + d_i(x)\\ & + \|\alpha(\mathcal{B}_+(x)) + L_+\, u(x)\|_2^2 + \|\alpha(\mathcal{B}_-(x)) + L_-\, u(x)\|_2^2 + (b_{+,i}(x) + r_i)^T\, A^{m_+(x)}\, \left(\alpha(\mathcal{B}_+(x)) + L_+\, u(x)\right)\\ & + (b_{-,i}(x) + r_i)^T\, A^{1+m_-(x)}\, \left(\alpha(\mathcal{B}_-(x)) + L_-(x)\right).\end{aligned} \tag{5.15}$$

Due to the representation (5.13) of the objective function J_i and the representation (5.14) of $\|u_i\|^2_{L^2(0,T)}$ we have

$$J_i(u) = \int_0^1 \mathcal{G}_i(x,\, u)\,\mathrm{d}x. \tag{5.16}$$

The next step in our analysis is to express the J_i in terms of the controls $v_j^{\pm,(k)}(x)$ defined in (5.5). With a slight abuse of notation, we write $J_i(v^\pm)$ for the corresponding transformed objective function. For $x \in (0,1)$ almost everywhere, the corresponding integrand is a quadratic function of the vectors $v_+^{(j)}(x) = \left(v_k^{+,(j)}(x)\right)_{k=1}^N$ and $v_-^{(j)}(x) = \left(v_k^{-,(j)}(x)\right)_{k=1}^N$, namely

$$\begin{aligned}\mathcal{H}_i(x,\, v^\pm(x)) = {} & \frac{\gamma_i}{2}\sum_{j=0}^{\infty}\left[\left(v_i^{+,(2j)}(x)\right)^2 + \left(v_i^{+,(2j+1)}(x)\right)^2 + \left(v_i^{-,(2j)}(x)\right)^2 + \left(v_i^{-,(2j+1)}(x)\right)^2\right] + \frac{1}{2}\,\|r_i\|_2^2 - r_i^T\, h_i(x) + d_i(x)\\ & + \left\|\alpha(\mathcal{B}_+(x)) + m_+(x)A^{m_+(x)}\, v_+^{(2\kappa_+(x))}(x) + sum_{j=0}^{\kappa_+(x)-1} A\, v_+^{(2j)}(x) + v_+^{(2j+1)}(x)\right\|_2^2\\ & + \left\|\alpha(\mathcal{B}_-(x)) + m_-(x)A^{m_-(x)}\, v_-^{(2\kappa_-(x))}(x) + \sum_{j=0}^{\kappa_-(x)-1} A\, v_-^{(2j)}(x) + v_-^{(2j+1)}(x)\right\|_2^2\\ & + (b_{+,i}(x) + r_i)^T\, A^{m_+(x)}\left(\alpha(\mathcal{B}_+(x)) + m_+(x)A^{m_+(x)}\, v_+^{(2\kappa_+(x))}(x) + \sum_{j=0}^{\kappa_+(x)-1} A\, v_+^{(2j)}(x) + v_+^{(2j+1)}(x)\right)\\ & + (b_{-,i}(x) + r_i)^T\, A^{1+m_-(x)}\left(\alpha(\mathcal{B}_-(x)) + m_-(x)A^{m_-(x)}\, v_-^{(2\kappa_-(x))}(x) + \sum_{j=0}^{\kappa_-(x)-1} A\, v_-^{(2j)}(x) + v_-^{(2j+1)}(x)\right).\end{aligned} \tag{5.17}$$

Then for the objective function J_i as a function of the controls $v_k^{\pm,(j)}(x)$ we have $J_i(v^{\pm}) = \int\limits_0^1 \mathcal{H}_i(x,\, v^{\pm}(x))\,\mathrm{d}x$.

6. Minimization of the objective functions

For all $i \in \{1,\ldots,N\}$, in the integrands $\mathcal{H}_i(x,\, v^{\pm}(x))$ of $J_i(v^{\pm})$ as given in (5.17), the vectors $v_+^{(j)}(x)$ and $v_-^{(j)}(x)$ for different values of $x \in (0,1)$ are not coupled. For $x \in (0,\,1)$ almost everywhere the quadratic function $\mathcal{H}_i(x,\,\cdot)$ is bounded below by zero. This can be seen from (5.14) and (5.11). Thus for $x \in (0,1)$ pointwise almost everywhere, the matrices in the corresponding quadratic forms are positive semi-definite. Due to the first part of $\mathcal{H}_i(x,\, v^{\pm}(x))$ where γ_i appears as a factor, the quadratic forms are positive definite. Hence we can determine the choices of the $v_i^{\pm,(j)}(x)$ for which $\mathcal{H}_i(x,\, v^{\pm}(x))$ is minimal. In this way we obtain unique optimal controls $v_i^{+,(j)}(x)$, $v_i^{-,(j)}(x)$ as a function of a given vector $r_i \in \mathbb{R}^N$ for $x \in (0,\,1)$ almost everywhere. Then we determine the corresponding choice of the r_i for which the integral objective function J_i is minimal which is equivalent to the system of linear equations (5.9). For a Nash equilibrium, this linear equation must be satisfied for all $i \in \{1,\ldots,n\}$ simultaneously. If the γ_i are sufficiently large or all equal, the corresponding matrix is regular. Thus we obtain the equilibrium strategies, where for $i \in \{1,\ldots,N\}$ player i solves the optimal control problem

$$\min_{\left(v_i^{\pm,(j)}(x)\right)_{j=0}^{2K-1}} \min_{r_i\in\mathbb{R}^N} \int_0^1 \mathcal{H}_i(x,\, v^{\pm}(x))\,\mathrm{d}x = \min_{r_i\in\mathbb{R}^N} \min_{\left(v_i^{\pm,(j)}(x)\right)_{j=0}^{2K-1}} \int_0^1 \mathcal{H}_i(x,\, v^{\pm}(x))\,\mathrm{d}x.$$

We start by considering the inner level problem

$$\min_{\left(v_i^{\pm,(j)}(x)\right)_{j=0}^{2K-1}} \int_0^1 \mathcal{H}_i(x,\, v^{\pm}(x))\,\mathrm{d}x$$

for fixed vectors r_j, $j \in \{1,\ldots,N\}$. The approach to this problem is the pointwise minimization of the integrand for $x \in (0,1)$ almost everywhere that is summarized in the following lemma. For further details on necessary (and sufficient) conditions in the context of optimal control see *e.g.* the monographs [15, 20, 27].

Lemma 6.1. *Let natural numbers $i \in \{1,\ldots,N\}$ and N_1 be given. For $x \in (0,1)$ almost everywhere and $z \in \mathbb{R}^{N_1}$ let a quadratic functional*

$$\mathcal{H}_i(x,\, z) = z^T M_i(x)\, z + m_i(x)^T z + s_i(x)$$

be given where the $N_1 \times N_1$ matrices $M_i(x)$ are symmetric and positive definite and $m_i(x) \in \mathbb{R}^{N_1}$. For $x \in (0,1)$ almost everywhere, define

$$z_i^*(x) = \arg\min \mathcal{H}_i(x,\,\cdot).$$

If $\mathcal{H}_i(\cdot,\, z^*(\cdot)) \in L^1(0,1))$, we have

$$\min_{z(x)\in(L^2(0,1))^{N_1}} \int_0^1 \mathcal{H}_i(x,\, z(x))\,\mathrm{d}x = \int_0^1 \mathcal{H}_i(x,\, z_i^*(x))\,\mathrm{d}x.$$

Since the integrand of J_i is quadratic and strictly convex, the necessary and sufficient optimality conditions for the minimization of $\mathcal{H}_i(x, v^{\pm}(x))$ with respect to $v_i^{+,(j)}(x)$ and $v_i^{-,(j)}(x)$ are given by the system of linear equations

$$\frac{\partial}{\partial v_i^{+,(m)}(x)} \mathcal{H}_i(x,\, v^{\pm}(x)) = 0,\ \frac{\partial}{\partial v_i^{-,(m)}(x)} \mathcal{H}_i(x,\, v^{\pm}(x)) = 0,\ m \in \{0, \ldots, 2\,K-1\}. \tag{6.1}$$

In the sequel we use the notation

$$\partial_{v_i^{\pm,(m)}} \mathcal{H}_i(x,\, v^{\pm}(x)) = \frac{\partial}{\partial v_i^{\pm,(m)}(x)}\, \mathcal{H}_i(x,\, v^{\pm}(x)).$$

To continue our analysis, we study the structure of the equilibrium strategies. It turns out that the equilibrium strategies are periodic with period 4 on the time interval $(0,\, T)$.

Lemma 6.2. *Let $i \in \{1, \ldots, N\}$ be given. Every strategy u_i of player i that satisfy the optimality system (6.1) for given strategies u_j $(j \neq i)$ is 4–periodic on the time interval $(0,\, T)$. Let A_i denote the ith column of the matrix A defined in (4.1).*

If for all $i \in \{1, \ldots, N\}$ the strategies u_i satisfy the optimality system (6.1) simultaneously, then for $x \in (0,\, 1)$ almost everywhere and $l \in \{0, 1, \ldots, \kappa_+(x) - 1\}$ such that $\mathcal{C}_+(x) + 4\,l + 2 \in (0,\, T)$ for every solution of (6.1) we have with $D = \operatorname{diag}(\gamma_i)_{i=1}^N$,

$$\begin{aligned} v_+^{(2l+1)}(x) = {} & -D^{-1}\, \left((b_{+,i}(x) + r_i)^T\ A^{m_+(x)} e_i \right)_{i=1}^N - 2\, D^{-1}\, \alpha(\mathcal{B}_+(x)) \\ & -2\, \kappa_+(x)\, D^{-1}\, v_+^{(1)}(x) - 2\, (\kappa_+(x) + m_+(x))\, D^{-1}\, A\, v_+^{(0)}(x) \end{aligned} \tag{6.2}$$

and for $x \in (0,\, 1)$ almost everywhere and $l \in \{0, 1, \ldots, \kappa_+(x)\}$ such that $\mathcal{C}_+(x) + 4\,l \in (0,\, T)$

$$\begin{aligned} v_+^{(2l)}(x) = {} & -D^{-1}\, \left((b_{+,i}(x) + r_i)^T\ A^{m_+(x)}\, A_i \right)_{i=1}^N - 2\, D^{-1}\, A\, \alpha(\mathcal{B}_+(x)) \\ & -2\, (\kappa_+(x) + m_+(x))\, D^{-1}\, v_+^{(0)}(x) - 2\, \kappa_+(x)\, D^{-1}\, A\, v_+^{(1)}(x). \end{aligned} \tag{6.3}$$

For $x \in (0,\, 1)$ almost everywhere and $l \in \{0, 1, \ldots, \kappa_-(x)\}$ such that $\mathcal{C}_-(x) + 4\,l \in (0,\, T)$ we have

$$\begin{aligned} = {} & -D^{-1}\, \left((b_{-,i}(x) + r_i)^T\, A_i^{m_-(x)} \right)_{i=1}^N - 2\, D^{-1}\, \alpha(\mathcal{B}_-(x)) \\ & -2\, (\kappa_-(x) + m_-(x))\, D^{-1}\, v_-^{(0)}(x) - 2\, \kappa_-(x)\, D^{-1}\, A\, v_-^{(1)}(x), \end{aligned} \tag{6.4}$$

and for $x \in (0,\, 1)$ almost everywhere and $l \in \{0, 1, \ldots, \kappa_-(x) - 1\}$ such that $\mathcal{C}_-(x) + 4\,l + 2 \in (0,\, T)$ we have

$$\begin{aligned} v_-^{(2l+1)}(x) = {} & -D^{-1}\, \left((b_{-,i}(x) + r_i)^T\ A^{m_-(x)}\, A_i \right)_{i=1}^N - 2\, D^{-1}\, \alpha(\mathcal{B}_-(x)) \\ & -2\, \kappa_-(x)\, D^{-1}\, v_-^{(1)}(x) - 2\, (\kappa_-(x) + m_-(x))\, D^{-1}\, A\, v_-^{(0)}(x). \end{aligned} \tag{6.5}$$

Proof. Let $A_i^{m_+(x)}$ denote the ith column of $A^{m_\pm(x)}$. We use the Kronnecker notation $\delta_l^k = \begin{cases} 1 & \text{if} \quad l = k, \\ 0 & \text{if} \quad l \neq k. \end{cases}$

First we consider the case that $m = 2\,l$ is even. Then for $x \in (0,\, 1)$ almost everywhere and

$$v_i^{+,(m)}(x) = u_i(\mathcal{C}_+(x) + 2\,m) = u_i(\mathcal{C}_+(x) + 4\,l),$$

equation (5.17) for $\mathcal{H}_i(x,\, v^{\pm}(x))$ implies that for the corresponding partial derivative with respect to $v_i^{+,\,(2l)}$ we obtain the affine linear map

$$\begin{aligned}\partial_{v_i^{+,\,(2l)}}\mathcal{H}_i(x,\, v^{\pm}(x)) = {} & \gamma_i\, v_i^{+,(2l)}(x) + (b_{+,i}(x)+r_i)^T A^{m_+(x)}\, \left(\delta_l^{\kappa_+(x)} m_+(x) A_i^{m_+(x)} + (1-\delta_l^{\kappa_+(x)})\, A_i\right)\\ & +2\left(\alpha(\mathcal{B}_+(x)) + m_+(x)\, A^{m_+(x)}\, v_+^{(2\kappa_+(x))}(x) + \sum_{j=0}^{\kappa_+(x)-1} A\, v_+^{(2j)}(x) + v_+^{(2j+1)}(x)\right)^T\\ & \times\left(\delta_l^{\kappa_+(x)} m_+(x) A_i^{m_+(x)} + (1-\delta_l^{\kappa_+(x)})\, A_i\right).\end{aligned}$$

As before, let e_i denote the ith column of the identity matrix I. If if $m = 2\,l+1$ is odd, for

$$v_i^{+,(m)}(x) = u_i(\mathcal{C}_+(x) + 2\,m) = u_i(\mathcal{C}_+(x) + 4\,l + 2)$$

the corresponding partial derivative of $\mathcal{H}_i(x,\, v^{\pm}(x))$ is given by the affine linear map

$$\begin{aligned}\partial_{v_i^{+,\,(2l+1)}}\mathcal{H}_i(x,\, v^{\pm}(x)) = {} & \gamma_i\, v_i^{+,(2l+1)}(x) + (b_{+,i}(x)+r_i)^T\, A^{m_+(x)}\, e_i\\ & +2\left(\alpha(\mathcal{B}_+(x)) + m_+(x)\, A^{m_+(x)}\, v_+^{(2\kappa_+(x))}(x) + \sum_{j=0}^{\kappa_+(x)-1} A\, v_+^{(2j)}(x) + v_+^{(2j+1)}(x)\right)^T e_i.\end{aligned}$$

Thus the optimality conditions (6.1) imply that for all $l \in \{0, \ldots, \kappa_+(x)\}$ we have

$$\begin{aligned}v_i^{+,(2l)}(x) = {} & -\frac{1}{\gamma_i}\,(b_{+,i}(x)+r_i)^T\, A^{m_+(x)}\, \left(\delta_l^{\kappa_+(x)} m_+(x) A_i^{m_+(x)} + (1-\delta_l^{\kappa_+(x)})\, A_i\right)\\ & -\frac{2}{\gamma_i}\left(\alpha(\mathcal{B}_+(x)) + m_+(x)\, A^{m_+(x)}\, v_+^{(2\kappa_+(x))}(x) + \sum_{j=0}^{\kappa_+(x)-1} A\, v_+^{(2j)}(x) + v_+^{(2j+1)}(x)\right)^T\\ & \times\left(\delta_l^{\kappa_+(x)} m_+(x) A_i^{m_+(x)} + (1-\delta_l^{\kappa_+(x)})\, A_i\right).\end{aligned} \tag{6.6}$$

For $l \in \{0, \ldots, \kappa_+(x)-1\}$, we observe that the right-hand side in (6.6) does not depend on l. Note that Lemma 4.2 states that if $\mathcal{C}_+(x) + 4\kappa_+(x) \le T$, we have $m_+(x) = 1$. In this case , we also have $v_i^{+,(2\,\kappa_+(x))}(x) = v_i^{+,(0)}(x)$. If $m_+(x) = 0$, we have $v_i^{+,(2\,\kappa_+(x))}(x) = 0$. Thus we can write for all $l \in \{0, \ldots, \kappa_+(x)\}$,

$$v_i^{+,(2\,l)}(x) = \left[(1-\delta_l^{\kappa_+(x)}) + \delta_l^{\kappa_+(x)}\, m_+(x)\right]\, v_i^{+,(0)}(x).$$

Thus for all $x \in (0,\,1)$ such that $\mathcal{C}_+(x) + 4\,l \in (0,\,T)$, the value of $u(\mathcal{C}_+(x) + 4\,l)$ does not depend on l. Similarly, we obtain

$$\begin{aligned}v_i^{+,(2l+1)}(x) = {} & -\frac{1}{\gamma_i}\,(b_{+,i}(x)+r_i)^T\; A^{m_+(x)}\, e_i\\ & -\frac{2}{\gamma_i}\left(\alpha(\mathcal{B}_+(x)) + m_+(x)\, A^{m_+(x)}\, v_+^{(2\kappa_+(x))}(x) + \sum_{j=0}^{\kappa_+(x)-1} A\, v_+^{(2j)}(x) + v_+^{(2j+1)}(x)\right)^T e_i.\end{aligned} \tag{6.7}$$

Again, we observe that the value of $v_i^{+,(2l+1)}(x)$ is the same for all $l \in \{0, \dots, \kappa_+(x)-1\}$. Thus $u_i(\mathcal{C}_+(x)+4\,l+2)$ does not depend on l. Since $v_i^{+,(2l)}(x) = v_i^{+,(0)}(x)$ and $v_i^{+,(2l+1)}(x) = v_i^{+,(1)}(x)$, (6.7) implies

$$
\begin{aligned}
v_+^{(2l+1)}(x) = &\left(-\frac{1}{\gamma_i}\,(b_{+,i}(x)+r_i)^T\;A^{m_+(x)}\,e_i\right)_{i=1}^N - \left(\frac{2}{\gamma_i}\left(m_+(x)\,A^{m_+(x)}\,v_+^{(2\kappa_+(x))}(x)\right)^T\;e_i\right)_{i=1}^N\\
&-\left(\frac{2}{\gamma_i}\left(\alpha_i(\mathcal{B}_+(x)) + \sum_{j=0}^{\kappa_+(x)-1} v_i^{+,(2j+1)}(x)\right)\right)_{i=1}^N - \left(\frac{2}{\gamma_i}\sum_{j=0}^{\kappa_+(x)-1}\left(A\,\left(v_i^{+,(2j)}(x)\right)_{i=1}^N\right)^T\;e_i\right)_{i=1}^N\\
= &\left(-\frac{1}{\gamma_i}\,(b_{+,i}(x)+r_i)^T\;A^{m_+(x)}\,e_i\right)_{i=1}^N - \left(\frac{2}{\gamma_i}\left(m_+(x)\,A^{m_+(x)}\,v_+^{(2\kappa_+(x))}(x)\right)^T\;e_i\right)_{i=1}^N\\
&-\left(\frac{2}{\gamma_i}\left(\alpha_i(\mathcal{B}_+(x)) + \kappa_+(x)\,v_i^{+,(1)}(x)\right)\right)_{i=1}^N - \left(\frac{2}{\gamma_i}\,\kappa_+(x)\,\left(A\,\left(v_i^{+,(0)}(x)\right)_{i=1}^N\right)^T\;e_i\right)_{i=1}^N.
\end{aligned}
$$

Now the definition of D implies (6.2). Moreover, (6.6) yields (6.3). Equations (6.4) and (6.5) follow analogously. $\square$

Lemma 6.2 states that the optimal strategies are 4-periodic. Due to this important observation, it suffices to find the optimal control functions on the time-interval $(0, 4)$. In the proof of Lemma 6.2 we have shown that for all $i \in \{1, \dots, N\}$, $l \in \{0, \dots, \kappa_\pm(x)\}$ and $x \in (0, 1)$ such that $\mathcal{C}_\pm(x) + 4l \in (0, T)$ almost everywhere we have $v_i^{\pm,(2l)}(x) = v_i^{\pm,(0)}(x)$. Moreover, for all $i \in \{1, \dots, N\}$, $l \in \{0, \dots, \kappa_\pm(x)-1\}$ and $x \in (0, 1)$ such that $\mathcal{C}_\pm(x) + 4l + 2 \in (0, T)$ almost everywhere we have $v_i^{\pm,(2l+1)}(x) = v_i^{\pm,(1)}(x)$.

For the 4–periodic controls, we can express the objective functions J_i as functions of $v_i^{\pm,(0)}(x)$ and $v_i^{\pm,(1)}(x)$. Hence for all $i \in \{1, \dots, N\}$, for equilibrium strategies $v^\pm$ that satisfy (6.1), we can restate $\mathcal{H}_i$ from (5.17) as the quadratic function

$$
\begin{aligned}
\mathcal{H}_i(x,\,v^\pm) = &\frac{\gamma_i}{2}\,\left((\kappa_+(x)+m_+(x))\left(v_i^{+,(0)}(x)\right)^2 + \kappa_+(x)\left(v_i^{+,(1)}(x)\right)^2\right)\\
&+\frac{\gamma_i}{2}\,\left((\kappa_-(x)+m_-(x))\left(v_i^{-,(0)}(x)\right)^2 + \kappa_-(x)\left(v_i^{-,(1)}(x)\right)^2\right)\\
&+\frac{1}{2}\,\|r_i\|_2^2 - r_i^T h_i(x) + d_i(x)\\
&+\left\|\alpha(\mathcal{B}_+(x)) + (\kappa_+(x)+m_+(x))A\,v_+^{(0)}(x) + \kappa_+(x)\,v_+^{(1)}(x)\right\|_2^2\\
&+\left\|\alpha(\mathcal{B}_-(x)) + (\kappa_-(x)+m_-(x))\,A\,v_-^{(0)}(x) + \kappa_-(x)\,v_-^{(1)}(x)\right\|_2^2\\
&+(b_{+,i}(x)+r_i)^T\;A^{m_+(x)}\left(\alpha(\mathcal{B}_+(x)) + (\kappa_+(x)+m_+(x))A\,v_+^{(0)}(x) + \kappa_+(x)\,v_+^{(1)}(x)\right)\\
&+(b_{-,i}(x)+r_i)^T\;A^{1+m_-(x)}\left(\alpha(\mathcal{B}_-(x) + (\kappa_-(x)+m_-(x))\,A\,v_-^{(0)}(x) + \kappa_-(x)\,v_-^{(1)}(x)\right).
\end{aligned}
$$

For all $i \in \{1, \dots, N\}$, (6.3) implies

$$
D\,v_+^{(0)}(x) + \left((b_{+,i}(x)+r_i)^T\;A^{m_+(x)}\;A_i\right)_{i=1}^N = -2\left(A\,\alpha(\mathcal{B}_+(x)) + (\kappa_+(x)+m_+(x))\,v_+^{(0)}(x) + \kappa_+(x)\,A\,v_+^{(1)}(x)\right). \tag{6.8}
$$

This yields

$$(D + 2(\kappa_+(x) + m_+(x)))\ v_+^{(0)}(x) + \Big((b_{+,i}(x) + r_i)^T A^{m_+(x)} A_i\Big)_{i=1}^N = -2\, A\, \alpha(\mathcal{B}_+(x)) - 2\, \kappa_+(x)\, A\, v_+^{(1)}(x). \quad (6.9)$$

Moreover, (6.2) implies

$$(D + 2\kappa_+(x))\ v_+^{(1)}(x) + \Big((b_{+,i}(x) + r_i)^T A^{m_+(x)}\, e_i\Big)_{i=1}^N = -2\, \alpha(\mathcal{B}_+(x)) - 2\, (\kappa_+(x) + m_+(x))\, A\, v_+^{(0)}(x). \quad (6.10)$$

Analogously we compute the optimal strategy vectors $v_-^{(0)}(x)$ and $v_-^{(1)}(x)$. Equation (6.4) implies

$$(D + 2(\kappa_-(x) + m_-(x)))\ v_-^{(0)}(x) + \Big((b_{-,i}(x) + r_i)^T A^{m_-(x)} e_i\Big)_{i=1}^N = -2\, \alpha(\mathcal{B}_-(x)) - 2\, \kappa_-(x)\, A\, v_-^{(1)}(x). \quad (6.11)$$

Moreover, from (6.5) we obtain

$$(D + 2\kappa_-(x))\ v_-^{(1)}(x) + \Big((b_{-,i}(x) + r_i)^T A^{m_-(x)}\, A_i\Big)_{i=1}^N = -2\, \alpha(\mathcal{B}_-(x)) - 2\, (\kappa_-(x) + m_-(x))\, A\, v_-^{(0)}(x). \quad (6.12)$$

Now we show how we can determine the equilibrium strategies explicitly.

7. An explicit representation of the equilibrium strategies

In Section 4 we have represented the state in terms of traveling waves. The representation of the state from Theorem 4.1 allows us to determine the Nash equilibrium. Before we proceed, let us consider the following lemma where the inverse of a certain matrix is studied. As before assume that $T = 4\,K + \Delta$ where K is a natural number.

Lemma 7.1. *Define the matrix*

$$\mathcal{M}_\pm(x) = \begin{pmatrix} D + 2(\kappa_\pm(x) + m_\pm(x))\, I & 2\, \kappa_\pm(x)\, A \\ 2\, (\kappa_\pm(x) + m_\pm(x))\, A & D + 2\kappa_\pm(x)\, I \end{pmatrix} \quad (7.1)$$

and the diagonal $N \times N$-matrix

$$D = \operatorname{diag}\, (\gamma_i)_{i=1}^N. \quad (7.2)$$

If $D = \gamma\, I$ or γ_i is sufficiently large for all $i \in \{1, \dots, N\}$, the matrix $\mathcal{M}_\pm(x)$ is regular. In the case $D = \gamma I$ the inverse is given by

$$\mathcal{M}_\pm(x)^{-1} = \begin{pmatrix} \theta_\pm(x)\, I & \eta_\pm(x)\, A \\ \xi_\pm(x)\, A & \nu_\pm(x)\, I \end{pmatrix} \quad (7.3)$$

where

$$\theta_\pm(x) = \frac{\gamma + 2\, \kappa_\pm(x)}{\gamma\, (\gamma + 4\, \kappa_\pm(x) + 2\, m_\pm(x))} \quad (7.4)$$

$$\eta_\pm(x) = -\frac{2\, \kappa_\pm(x)}{\gamma\, (\gamma + 4\, \kappa_\pm(x) + 2\, m_\pm(x))} \quad (7.5)$$

$$\xi_{\pm}(x) = -\frac{2\,\kappa_{\pm}(x) + 2\,m_{\pm}(x)}{\gamma\,(\gamma + 4\,\kappa_{\pm}(x) + 2\,m_{\pm}(x))} \tag{7.6}$$

$$\nu_{\pm}(x) = \frac{\gamma + 2\,\kappa_{\pm}(x) + 2\,m_{\pm}(x)}{\gamma\,(\gamma + 4\,\kappa_{\pm}(x) + 2\,m_{\pm}(x))}. \tag{7.7}$$

Proof. To simplify the notation let us drop the dependence on x and the index $\pm$. Next, define the number

$$\digamma = (\gamma + 2\kappa)(\gamma + 2(\kappa + m)) - 4\kappa(\kappa + m) = \gamma(\gamma + 4\kappa + 2m).$$

The statement of Lemma 7.1 can be checked by computing the matrix product $\mathcal{M}_{\pm}(x)\,\mathcal{M}_{\pm}(x)^{-1}$ with $\mathcal{M}_{\pm}(x)^{-1}$ from (7.3) which yields the identity matrix I. In fact, we have

$$\begin{aligned}
\digamma\,\mathcal{M}\,\mathcal{M}^{-1} &= \begin{pmatrix} (\gamma + 2(\kappa + m))\,I & 2\,\kappa\,A \\ 2\,(\kappa + m)\,A & (\gamma + 2\kappa)\,I \end{pmatrix} \begin{pmatrix} (\gamma + 2\kappa)\,I & -2\kappa\,A \\ -2(\kappa + m)\,A & (\gamma + 2(\kappa + m))\,I \end{pmatrix} \\
&= \begin{pmatrix} \gamma(\gamma + 4\kappa + 2m)\,I & 0 \\ 0 & \gamma(\gamma + 4\kappa + 2m)\,I \end{pmatrix} = \digamma\,I.
\end{aligned}$$

$\square$

Now we define another matrix that we need for the explicit representation of the equilibrium states.

Lemma 7.2. *Let*

$$\varUpsilon_{\pm}(x) = \frac{1}{\gamma + 4\kappa_{\pm}(x) + 2m_{\pm}(x)}$$

and

$$\vartheta = \int_0^1 1 - (m_+(x) + 2\kappa_+(x))\varUpsilon_+(x) - (m_-(x) + 2\kappa_-(x))\varUpsilon_-(x)\,\mathrm{d}x \tag{7.8}$$

and define the $N \times N$ matrix

$$M_0 = \vartheta\,I \tag{7.9}$$

Then the matrix M_0 is regular.

Proof. For all $x \in (0,1)$, we have

$$\begin{aligned}
&1 - (m_+(x) + 2\kappa_+(x))\varUpsilon_+(x) - (m_-(x) + 2\kappa_-(x))\varUpsilon_-(x) \\
&= \left(\frac{1}{2\varUpsilon_+(x)} - m_+(x) - 2\kappa_+(x)\right)\varUpsilon_+(x) + \left(\frac{1}{2\varUpsilon_-(x)} - m_-(x) - 2\kappa_-(x)\right)\varUpsilon_-(x) \\
&= \frac{\gamma}{2}\,(\varUpsilon_+(x) + \varUpsilon_-(x)) \in (0,1).
\end{aligned}$$

The integration on $(0,1)$ then yields $\vartheta \in (0,1)$. Hence M_0 is regular. $\square$

For the representation of the equilibrium strategies, we need the vector $\varXi \in \mathbb{R}^N$ as defined in (7.21). Using $\varXi$, an explicit representation of the equilibrium strategies is given in the next theorem.

Theorem 7.3. *Let $T = 4\,K + \varDelta$ for $K \in \{1, 2, 3, \ldots\}$ and $\varDelta \in [0, 4)$. Assume that there exists a real number $\gamma > 0$ such that $\gamma_i = \gamma$ for all $i \in \{1, \ldots, N\}$. For $r_i^{(j)}$ as in (5.9), define the vector*

$$\mathcal{R} = \left(r_i^{(i)} \right)_{i=1}^{N}.$$

Then system (2.1)–(2.6) with the objective functions (2.7) has a unique Nash equilibrium that we denote by the superscript * where

$$\mathcal{R}^* = \left(r_i^{(i),*} \right)_{i=1}^{N} = \frac{1}{\vartheta}\, \varXi \tag{7.10}$$

with ϑ as defined in (7.8) and $\varXi$ as defined in (7.21). The values of $r_i^{(j),*}$ for $j \neq i$ are given by

$$r_i^{(j),*} = r_j^{(j),*} + \int_0^1 h_{\mathcal{D},\,i}^{(j)}(x) \,-\, h_{\mathcal{D},j}^{(j)}(x)\, \mathrm{d}x. \tag{7.11}$$

The optimal strategies are 4-periodic and given by

$$\begin{pmatrix} v_+^{(0)}(x) \\ v_+^{(1)}(x) \end{pmatrix} = \mathcal{M}_+(x)^{-1} \begin{pmatrix} -2\,\alpha(\mathcal{B}_+(x)) - \left((b_{+,i}(x) + r_i)^T\, A^{m_+(x)}\, A_i \right)_{i=1}^{N} \\ -2\,\alpha(\mathcal{B}_+(x)) - \left((b_{+,i}(x) + r_i)^T\, A^{m_+(x)}\, e_i \right)_{i=1}^{N} \end{pmatrix}, \tag{7.12}$$

$$\begin{pmatrix} v_-^{(0)}(x) \\ v_-^{(1)}(x) \end{pmatrix} = \mathcal{M}_-(x)^{-1} \begin{pmatrix} -2\,\alpha(\mathcal{B}_-(x)) - \left((b_{-,i}(x) + r_i)^T\, A^{m_-(x)}\, e_i \right)_{i=1}^{N} \\ -2\,\alpha(\mathcal{B}_-(x)) - \left((b_{-,i}(x) + r_i)^T\, A^{m_-(x)}\, A_i \right)_{i=1}^{N} \end{pmatrix}. \tag{7.13}$$

Proof. Lemma 6.2 states that the optimal strategies are 4-periodic. Using this fact, we have constructed controls such that for all $i \in \{i, \ldots, N\}$, the optimality conditions (6.1) hold. In the statement of Theorem 7.3, we assume that the assumptions of Lemma 7.2 hold.

With the matrix $\mathcal{M}_+(x)$ as defined in (7.1) we can summarize (6.9) and (6.10) in the form

$$\mathcal{M}_+(x) \begin{pmatrix} v_+^{(0)}(x) \\ v_+^{(1)}(x) \end{pmatrix} = \begin{pmatrix} -2\,\alpha(\mathcal{B}_+(x)) - \left((b_{+,i}(x) + r_i)^T\, A^{m_+(x)}\, A_i \right)_{i=1}^{N} \\ -2\,\alpha(\mathcal{B}_+(x)) - \left((b_{+,i}(x) + r_i)^T\, A^{m_+(x)}\, e_i \right)_{i=1}^{N} \end{pmatrix}$$

The matrix $\mathcal{M}_+(x)$ is regular, since $D = \gamma I$. Hence, we obtain (7.12).

With the matrix $\mathcal{M}_-(x)$ as defined in (7.1) we can summarize (6.11) and (6.12) in the form

$$\mathcal{M}_-(x) \begin{pmatrix} v_-^{(0)}(x) \\ v_-^{(1)}(x) \end{pmatrix} = \begin{pmatrix} -2\,\alpha(\mathcal{B}_-(x)) - \left((b_{-,i}(x) + r_i)^T\, A^{m_-(x)}\, e_i \right)_{i=1}^{N} \\ -2\,\alpha(\mathcal{B}_-(x)) - \left((b_{-,i}(x) + r_i)^T\, A^{m_-(x)}\, A_i \right)_{i=1}^{N} \end{pmatrix}.$$

Since $\mathcal{M}_-(x)$ is regular, this yields (7.13).

Thus, we have also obtained the optimal strategy vectors $v_+^{(0)}(x)$, $v_+^{(1)}(x)$, $v_-^{(0)}(x)$ and $v_-^{(1)}(x)$ as affine linear functions of the r_i. So due to (7.11) (that also holds without the exponent *) the optimal strategies can be expressed as an affine linear function of the vector $(r_i^{(i)})_{i=1}^N$. It remains to determine the numbers $r_i^{(i)}$ in such a way that the values of the objective functions J_i that we obtain for the strategy vectors $v_+(x)$ and $v_-(x)$ that we have determined above are minimal. This is the case if (5.10) holds.

In order to use these relations to derive a linear equation (equation (7.18)) in terms of $\left(r_i^{(i)}\right)_{i=1}^N$, we make the following observations.

For $i \in \{1, \ldots, N\}$ we have $r_i^T e_i = r_i^{(i)} = (I\,\mathcal{R})_i$ and due to (7.11)

$$
\begin{aligned}
r_i^T A_i &= \sum_{j=1}^N r_i^{(j)}\, a_{ij} \\
&= \sum_{j=1}^N \left[r_j^{(j)}\, a_{ij} + \int_0^1 h_{\mathcal{D},i}^{(j)}(x) \; - h_{\mathcal{D},j}^{(j)}(x)\, \mathrm{d}x\, a_{ij} \right] \\
&= \left(\sum_{j=1}^N r_j^{(j)}\, a_{ij} \right) + \left(\sum_{j=1}^N \int_0^1 h_{\mathcal{D},i}^{(j)}(x) \; - h_{\mathcal{D},j}^{(j)}(x)\, \mathrm{d}x\, a_{ij} \right) \\
&= (A\,\mathcal{R})_i + \left(\sum_{j=1}^N \int_0^1 h_{\mathcal{D},i}^{(j)}(x) \; - h_{\mathcal{D},j}^{(j)}(x)\, \mathrm{d}x\, a_{ij} \right).
\end{aligned}
$$

Hence

$$
r_i^T A^{m_\pm(x)} A_i = \left(A^{(1-m_\pm(x))}\, \mathcal{R} \right)_i + (1 - m_\pm(x)) \left(\sum_{j=1}^N \int_0^1 h_{\mathcal{D},i}^{(j)}(x) \; - h_{\mathcal{D},j}^{(j)}(x)\, \mathrm{d}x\, a_{ij} \right).
$$

Moreover, we have

$$
r_i^T A^{m_\pm(x)} e_i = \left(A^{m_\pm(x)}\, \mathcal{R} \right)_i + m_\pm(x) \left(\sum_{j=1}^N \int_0^1 h_{\mathcal{D},i}^{(j)}(x) \; - h_{\mathcal{D},j}^{(j)}(x)\, \mathrm{d}x\, a_{ij} \right).
$$

With the notation $\overline{I}_i$ from (5.4) we have for $m \in \{0, 1\}$

$$
r_i^T A^m A_i = \left(A^{(1-m)}\, \mathcal{R} \right)_i + (1 - m)\overline{I}_i, \qquad r_i^T A^m e_i = (A^m\, \mathcal{R})_i + m\overline{I}_i. \tag{7.14}
$$

For $D = \gamma I$, with $\Upsilon_\pm(x) = \theta_\pm(x) + \eta_\pm(x) = \xi_\pm(x) + \nu_\pm(x)$ as in Lemma 7.2, (7.12) and (7.13) yield furthermore

$$
\begin{aligned}
\begin{pmatrix} v_+^{(0)}(x) \\ v_+^{(1)}(x) \end{pmatrix} &= -2 \begin{pmatrix} \theta_+(x)\, I + \eta_+(x) A \\ \xi_+(x)\, A + \nu_+(x) I \end{pmatrix} \alpha(\mathcal{B}_+(x)) - \mathcal{M}_+(x)^{-1} \begin{pmatrix} \left((b_{+,i}(x))^T\, A^{m_+(x)}\, A_i \right)_{i=1}^N \\ \left((b_{+,i}(x))^T\, A^{m_+(x)}\, e_i \right)_{i=1}^N \end{pmatrix} \\
&\quad - \begin{pmatrix} \Upsilon_+(x) A^{1+m_+(x)} \\ \Upsilon_+(x) A^{m_+(x)} \end{pmatrix} \mathcal{R} - \begin{pmatrix} \theta_+(x)\, (1 - m_+(x))\, I + \eta_+(x)\, m_+(x)\, A \\ \xi_+(x)\, (1 - m_+(x))\, A + \nu_+(x)\, m_+(x)\, I \end{pmatrix} \left(\overline{I}_i \right)_{i=1}^N ,
\end{aligned} \tag{7.15}
$$

$$
\begin{aligned}
\begin{pmatrix} v_-^{(0)}(x) \\ v_-^{(1)}(x) \end{pmatrix} &= -2 \begin{pmatrix} \theta_-(x)\, I + \eta_-(x) A \\ \xi_-(x)\, A + \nu_-(x) I \end{pmatrix} \alpha(\mathcal{B}_-(x)) - \mathcal{M}_-(x)^{-1} \begin{pmatrix} \left((b_{-,i}(x))^T\, A^{m_-(x)}\, e_i \right)_{i=1}^N \\ \left((b_{-,i}(x))^T\, A^{m_-(x)}\, A_i \right)_{i=1}^N \end{pmatrix} \\
&\quad - \begin{pmatrix} \Upsilon_-(x) A^{m_-(x)} \\ \Upsilon_-(x) A^{1+m_-(x)} \end{pmatrix} \mathcal{R} - \begin{pmatrix} \theta_-(x)\, m_-(x)\, I + \eta_-(x)\, (1 - m_-(x))\, A \\ \xi_-(x)\, m_-(x)\, A + \nu_-(x)\, (1 - m_-(x))\, I \end{pmatrix} \left(\overline{I}_i \right)_{i=1}^N .
\end{aligned} \tag{7.16}
$$

Now, for $j = i \in \{1, \ldots, N\}$, (5.10) yields a linear equation for the numbers $r_i^{(i)}$, that we use to obtain a system of linear equations for the numbers $r_i^{(i)}$. From (4.14) we get

$$\begin{aligned}
\alpha(T+x) &= A^{m_+(x)}\alpha(\mathcal{B}_+(x)) + m_+(x)\, u(\mathcal{C}_+(x) + 4\,\kappa_+(x)) \\
&\quad + \sum_{j=0}^{\kappa_+(x)-1} A^{1-m_+(x)}\, u(\mathcal{C}_+(x) + 4\,j) + A^{m_+(x)} u(\mathcal{C}_+(x) + 4\,j + 2) \\
&= A^{m_+(x)}\alpha(\mathcal{B}_+(x)) + m_+(x)\, u(\mathcal{C}_+(x)) \\
&\quad + \kappa_+(x) A^{1-m_+(x)}\, u(\mathcal{C}_+(x)) + \kappa_+(x) A^{m_+(x)} u(\mathcal{C}_+(x) + 2) \\
&= A^{m_+(x)}\alpha(\mathcal{B}_+(x)) + m_+(x)\, v_+^{(0)}(x) \\
&\quad + \kappa_+(x) A^{1-m_+(x)}\, v_+^{(0)}(x) + \kappa_+(x) A^{m_+(x)} v_+^{(1)}(x).
\end{aligned}$$

Similarly, from (4.15) we get

$$\begin{aligned}
\alpha(T-x) &= A^{m_-(x)}\alpha(\mathcal{B}_-(x)) + m_-(x)\, v_-^{(0)}(x) \\
&\qquad + \kappa_-(x) A^{1-m_-(x)}\, v_-^{(0)}(x) + \kappa_-(x) A^{m_-(x)} v_-^{(1)}(x).
\end{aligned}$$

and thus we get

$$\begin{aligned}
\beta_i(x-T) &= -A_i^T\, \alpha(T-x) \\
&= -A_i^T\, A^{m_-(x)}\alpha(\mathcal{B}_-(x)) - m_-(x) A_i^T\, v_-^{(0)}(x) \\
&\quad - \kappa_-(x) e_i^T\, A^{m_-(x)}\, v_-^{(0)}(x) - \kappa_-(x) A_i^T\, A^{m_-(x)} v_-^{(1)}(x).
\end{aligned}$$

Hence (5.10) for $j = i$ yields

$$\begin{aligned}
r_i^{(i)} &= \int_0^1 h_{\mathcal{D},i}^{(i)}(x)\, - e_i^T A^{m_+(x)}\alpha(\mathcal{B}_+(x)) - A_i^T\, A^{m_-(x)}\alpha(\mathcal{B}_-(x))\, \mathrm{d}x \\
&\quad - \int_0^1 m_+(x) e_i^T v_+^{(0)}(x) + m_-(x) A_i^T v_-^{(0)}\, \mathrm{d}x \\
&\quad - \int_0^1 \kappa_+(x) e_i^T A^{1-m_+(x)}\, v_+^{(0)}(x) + \kappa_+(x) e_i^T A^{m_+(x)} v_+^{(1)}(x)\, \mathrm{d}x \\
&\quad - \int_0^1 \kappa_-(x) e_i^T\, A^{m_-(x)}\, v_-^{(0)}(x) + \kappa_-(x) A_i^T\, A^{m_-(x)} v_-^{(1)}(x)\, \mathrm{d}x.
\end{aligned} \tag{7.17}$$

Now we insert the expressions for $(v_i^{+,(0)}(x))_{i=1}^N$, $(v_i^{+,(1)}(x))_{i=1}^N$, $(v_i^{-,(0)}(x))_{i=1}^N$ and $(v_i^{-,(1)}(x))_{i=1}^N$ from (7.15) and (7.16) to obtain a linear system for $\mathcal{R}$.

$$\begin{aligned}
\mathcal{R} &= \left(\int_0^1 m_+(x) \Upsilon_+(x) A^{(1-m_+(x))} + m_-(x) \Upsilon_-(x) A^{(1-m_-(x))}\, \mathrm{d}x \right. \\
&\quad \left. + \int_0^1 2\kappa_+(x) \Upsilon_+(x) I + 2\kappa_-(x) \Upsilon_-(x) I\, \mathrm{d}x \right) \mathcal{R} + \Xi,
\end{aligned} \tag{7.18}$$

where the vector $\mathit{\Xi}$ contains the remaining terms that do not depend on $\mathcal{R}$. In order to obtain the explicit representation of $\mathit{\Xi}$ we introduce the auxiliary vectors

$$\begin{pmatrix}\tilde v_+^{(0)}(x)\\ \tilde v_+^{(1)}(x)\end{pmatrix} = \mathcal{M}_+(x)^{-1}\begin{pmatrix}-2\,\alpha(\mathcal{B}_+(x)) - \left(b_{+,i}(x)^T\,A^{m_+(x)}\,A_i + (1-m_+(x))\,\overline{I}_i\right)_{i=1}^N\\ -2\,\alpha(\mathcal{B}_+(x)) - \left(b_{+,i}(x)^T\,A^{m_+(x)}\,e_i + m_+(x)\,\overline{I}_i\right)_{i=1}^N\end{pmatrix}, \tag{7.19}$$

$$\begin{pmatrix}\tilde v_-^{(0)}(x)\\ \tilde v_-^{(1)}(x)\end{pmatrix} = \mathcal{M}_-(x)^{-1}\begin{pmatrix}-2\,\alpha(\mathcal{B}_-(x)) - \left(b_{-,i}(x)^T\,A^{m_-(x)}\,e_i + m_-(x)\,\overline{I}_i\right)_{i=1}^N\\ -2\,\alpha(\mathcal{B}_-(x)) - \left(b_{-,i}(x)^T\,A^{m_-(x)}\,A_i + (1-m_-(x))\,\overline{I}_i\right)_{i=1}^N\end{pmatrix}. \tag{7.20}$$

These vectors are derived using equations (7.12), (7.13) and (7.14) by setting $\mathcal{R}=0$. Inserting (7.19) and (7.20) in (7.17) yields the following equations for the N components of $\mathit{\Xi}$:

$$\begin{aligned}\mathit{\Xi} = &\int_0^1 \left(h_{\mathcal{D},i}^{(i)}(x)\right)_{i=1}^N - \left(A^{m_+(x)}\alpha(\mathcal{B}_+(x)) +\, A^{1-m_-(x)}\alpha(\mathcal{B}_-(x))\right)\,\mathrm{d}x\\ &-\int_0^1 \left(m_+(x)\,\tilde v_+^{(0)}(x) + m_-(x)\,A\,\tilde v_-^{(0)}(x)\right)\,\mathrm{d}x\\ &-\int_0^1 \kappa_+(x)\,\left(A^{1-m_+(x)}\,\tilde v_+^{(0)}(x) + A^{m_+(x)}\,\tilde v_+^{(1)}(x)\right)\,\mathrm{d}x\\ &-\int_0^1 \kappa_-(x)\,\left(A^{m_-(x)}\,\tilde v_-^{(0)}(x) +\, A^{1-m_-(x)}\,\tilde v_-^{(1)}(x)\right)\,\mathrm{d}x.\end{aligned} \tag{7.21}$$

We then have

$$M_0\,\mathcal{R} = \mathit{\Xi} \tag{7.22}$$

with the matrix M_0 defined in (7.9). By Lemma 7.2 the matrix M_0 is regular, *i.e.* the system of linear equations (7.22) has a unique solution. In fact the solution is given by $\mathcal{R} = \vartheta^{-1}\mathit{\Xi}$. This implies that the boundary control game with the wave equation has a unique Nash-equilibrium, that consists of 4-periodic strategies.

Hence $\mathcal{R}$ as defined in (7.10) is the unique solution of equation (7.22). This implies that for the corresponding weights $r_i^{(j)}$, that equation (5.9) holds. Thus we have constructed a unique Nash equilibrium. □

Now Theorem 7.3 implies Theorem 3.1.

Remark 7.4. In Theorem 7.3 we have given an explicit representation of the linear operator that maps the initial state and the desired states to the corresponding Nash equilibrium. The representation implies that boundedness of this operator as a map from the corresponding function spaces to the control space $(L^2(0,\,T))^N$ and thus the stability of the Nash equilibria with respect to perturbations of the initial and the desired states.

Now, in order to illustrate our outcomings, we briefly discuss two small examples.

Example 7.5. If all the desired velocities are zero, that is $h_{\mathcal{D},i}^{(j)} = 0$, (5.1) and (5.2) imply that we have

$$b_{+,i}^{(j)}(x) = -g_{\mathcal{D},i}^{(j)},\; b_{-,i}^{(j)}(x) = g_{\mathcal{D},i}^{(j)}.$$

Thus $b_{+,i}^{(j)}(x) + b_{-,i}^{(j)}(x) = 0$. Moreover, (5.4) implies $\overline{I_i} = 0$. If we start at the zero position (that is $y_0^{(i)} = 0$) with zero velocity, that is with the initial velocities $y_1^{(i)} = 0$, (4.2) implies $\alpha_i(x) = 0$. Note that for $D = \gamma\, I$ we have

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \mathcal{M}_\pm(x) = \mathcal{M}_\pm(x) \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

Assume that $\kappa_+(x) = \kappa_-(x)$ and $m_+(x) = m_-(x)$. (This is the case if $\Delta = 0$. Then $\kappa_+(x) = \kappa_-(x) = K$ and $m_+(x) = m_-(x) = 0$.) Then we have $\mathcal{M}_+(x) = \mathcal{M}_-(x)$. This implies

$$\begin{aligned}
\begin{pmatrix} \tilde v_+^{(0)}(x) + A\,\tilde v_-^{(0)}(x) \\ \tilde v_+^{(1)}(x) + A\,\tilde v_-^{(1)}(x) \end{pmatrix} &= \mathcal{M}_+(x)^{-1} \begin{pmatrix} \left(-b_{+,i}(x)^T A^{m_+(x)} A_i\right)_{i=1}^N \\ -\left(b_{+,i}(x)^T A^{m_+(x)} e_i\right)_{i=1}^N \end{pmatrix} + \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \mathcal{M}_-(x)^{-1} \begin{pmatrix} -\left(b_{-,i}(x)^T A^{m_-(x)} e_i\right)_{i=1}^N \\ -\left(b_{-,i}(x)^T A^{m_-(x)} A_i\right)_{i=1}^N \end{pmatrix} \\
&= \mathcal{M}_+(x)^{-1} \left[\begin{pmatrix} \left(-b_{+,i}(x)^T A^{m_+(x)} A_i\right)_{i=1}^N \\ -\left(b_{+,i}(x)^T A^{m_+(x)} e_i\right)_{i=1}^N \end{pmatrix} + \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} -\left(b_{-,i}(x)^T A^{m_-(x)} e_i\right)_{i=1}^N \\ -\left(b_{-,i}(x)^T A^{m_-(x)} A_i\right)_{i=1}^N \end{pmatrix} \right] \\
&= -\mathcal{M}_+(x)^{-1} \begin{pmatrix} \left(b_{+,i}(x)^T A^{m_+(x)} A_i\right)_{i=1}^N + \left(b_{-,i}(x)^T A^{m_-(x)} A_i\right)_{i=1}^N \\ \left(b_{+,i}(x)^T A^{m_+(x)} e_i\right)_{i=1}^N + \left(b_{-,i}(x)^T A^{m_-(x)} e_i\right)_{i=1}^N \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\end{aligned}$$

Now equation (7.21) implies that we have $\Xi = 0$ and (7.10) yields $\mathcal{R} = 0$. Thus by (7.11) it follows that $r_i^{(j)} = 0$ for all $i, j = 1, \dots, N$.

Assume that $\Delta = 0$. Then we have $\mathcal{B}_\pm(x) = \pm x$. Hence if we start with zero velocity, that is with the initial velocities $y_1^{(i)} = 0$, also for nonzero initial positions $y_0^{(i)}$, we have $\alpha(\mathcal{B}_-(x)) = \alpha(-x) = -A\beta(x) = -A\alpha(x) = -A\alpha(\mathcal{B}_+(x))$. Hence in this case we have $\alpha(\mathcal{B}_+(x)) + A\,\alpha(\mathcal{B}_-(x)) = 0$. Similar as above, this yields $\begin{pmatrix} \tilde v_+^{(0)}(x) + A\,\tilde v_-^{(0)}(x) \\ \tilde v_+^{(1)}(x) + A\,\tilde v_-^{(1)}(x) \end{pmatrix} = 0$ and thus $\Xi = 0$ also holds if $y_0^{(i)} \neq 0$.

Example 7.6. Assume that $T = 4K$ ($K \in \{1, 2, 3, \dots, \}$) and $D = \gamma\, I$ for some real number $\gamma > 0$. Moreover, assume $y_0^{(j)}(x) = 0$, $y_1^{(j)}(x) = 0$, $h_{\mathcal{D},i}^{(j)}(x) = 0$ and $N = 3$. Then using the notation $\delta^{\det} = \gamma(\gamma + 4K)$ we have

$$A = \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}, \quad \mathcal{M}_\pm = \begin{pmatrix} (\gamma+2K)I & 2K\,A \\ 2K\,A & (\gamma+2K)I \end{pmatrix}, \quad \mathcal{M}_\pm{}^{-1} = \frac{1}{\delta^{\det}} \begin{pmatrix} (\gamma+2K)I & -2K\,A \\ -2K\,A & (\gamma+2K)I \end{pmatrix}.$$

Let $g_{\mathcal{D},1}^{(j)}(x) = g_{\mathcal{D},2}^{(j)}(x) = 0$ and $g_{\mathcal{D},3}^{(j)}(x) = 1$. Then Example 7.5 implies that in this case $r_i^{(j)} = 0$ for all $i, j = 1, \dots, 3$. We have $\alpha(x) = \beta(x) = 0$. For each player the equilibrium strategy attains at most four values and is constant on intervals of the length 1. Applying equation (7.12) we obtain

$$\left(v_i^{+,(0),*}(x)\right)_{i=1}^3 = \frac{1}{3\,\delta^{\det}} \begin{pmatrix} 4K \\ 4K \\ -3\gamma - 8K \end{pmatrix}. \tag{7.23}$$

Moreover, comparing the associated equations for $v^{+,(1),*}(x)$, and (7.13) for $v^{-,(0),*}(x)$ and $v^{-,(1),*}(x)$ with the one for $v^{+,(0),*}(x)$ we directly get

$$v^{+,(1),*}(x) = -v^{+,(0),*}(x), \quad v^{-,(0),*}(x) = v^{+,(0),*}(x) \quad \text{and} \quad v^{-,(1),*}(x) = -v^{+,(0),*}(x). \tag{7.24}$$

Next, using the transformed objective given in (5.17) and assuming the opponents optimal strategies are given, we obtain the following stationarity system for $v_i = (v_i^{+,(0)}, v_i^{+,(1)}, v_i^{-,(0)}, v_i^{-,(1)})$

$$\begin{pmatrix} \gamma+2K & 2KA_{ii} & 0 & 0 \\ 2KA_{ii} & \gamma+2K & 0 & 0 \\ 0 & 0 & \gamma+2K & 2KA_{ii} \\ 0 & 0 & 2KA_{ii} & \gamma+2K \end{pmatrix} v_i = \begin{pmatrix} \frac{4K}{3}\sum_{j\neq i} v_j^{+,(1)} + A_i^T g_i \\ \frac{4K}{3}\sum_{j\neq i} v_j^{+,(0)} + e_i^T g_i \\ \frac{4K}{3}\sum_{j\neq i} v_j^{-,(1)} - e_i^T g_i \\ \frac{4K}{3}\sum_{j\neq i} v_j^{-,(0)} - A_i^T g_i \end{pmatrix}$$

whose solutions are exactly those ones given by (7.23) and (7.24), which (by the convexity properties of the objective function) proves that these represent in fact a Nash-equilibrium.

8. Conclusion

We have constructed a Nash equilibrium for a Dirichlet boundary control game with N players that act on the boundary nodes of a star of N vibrating strings. Each player has an objective function that is the sum of a quadratic control cost term and a quadratic tracking term for the final state at the time T. We have shown that if all players have the same control cost weight a unique Nash equilibrium exists. The equilibrium controls are 4-periodic. The result that the equilibrium controls are 4-periodic, also applies for the more general case where the weights γ_i are allowed to differ, provided that a Nash equilibrium exists. In fact this follows directly from the proof of Lemma 6.2. In the analysis for the general case the matrices $\mathcal{M}_\pm$ from Lemma 7.1 appear. For the proof of the existence of a Nash equilibrium it is necessary that these matrices are regular. Then similarly as in the proof that we have presented in this paper, a linear system $M_0\mathcal{R} = \varXi$ can be derived, but the corresponding matrix M_0 has a more complicated structure. If M_0 is regular, a unique Nash equlibrium exists.

Our results provide a basis for more involved future studies where in a multi-stage process a transportation system operator fixes prices and trades feasible intervals with the customers. In the applications of gas networks these would be fixed in contracts about the nominated gas consumption/delivery of the customers. These contracts have to be made in such a way that the operator can guarantee that the obligations can be satisfied within the technical operation range that is usually defined by box constraints for the admissible states. In the applications, additional challenges are caused by the complicated network graphs and the fact that the gas pressure also has to be increased at points in the network by compressors to drive the gas flow, see [14].

Acknowledgements. This work has been supported by the German Research Foundation (DFG) in the framework of the Collaborative Research Centre CRC/Transregio 154, Mathematical Modelling, Simulation and Optimization Using the Example of Gas Networks, project C03 and the SPP 1962 "Non-smooth and Complementarity-based Distributed Parameter Systems: Simulation and Hierarchical Optimization", project STE 2063/2-1. We want to thank the referees for their careful reading and helpful suggestions.

References

[1] M.K. Banda, M. Herty and A. Klar, Coupling conditions for gas networks governed by the isothermal Euler equations. *Netw. Heterog. Media* **1** (2006) 295–314.

[2] A. Bressan, Bifurcation analysis of a non-cooperative differential game with one weak player. *J. Differ. Equ.* **248** (2010) 1297–1314.

[3] A. Bressan and K. Han, Existence of optima and equilibria for traffic flow on networks. *Netw. Heterog. Media* **8** (2013) 627–648.

[4] R. Dager, E. Zuazua, Wave propagation, in Observation and Control in 1-d Flexible Multi-Structures. Vol. 50 of *Mathématiques et Applications*. Springer-Verlag, Berlin, Heidelberg (2006).

[5] E.J. Dockner, S. Jorgensen, N. Van Long and G. Sorger, Differential Games in Economics and Management Science. Cambridge University Press (2000)

[6] S. Ervedoza, E. Zuazua, The Wave Equation: Control and Numerics. Vol. 2048 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, Heidelberg (2012) 245–339.

[7] R. Dager, Observation and control of vibrations in tree-shaped networks of strings. *SIAM J. Control Optim.* **43** (2004) 590–623.

[8] T.L. Friesz, Dynamic Optimization and Differential Games. Springer (2010).

[9] P. Frihauf, M. Krstic, T. Basar, Nash equilibrium seeking in noncooperative games. *IEEE Trans. Autom. Control* **57** (2012) 1192–1207.

[10] M. Gugat, G. Leugering and G. Sklyar, L^p-optimal boundary control for the wave equation. *SIAM J. Control Optim.* **44** (2005) 49–74.

[11] M. Gugat, Penalty techniques for state constrained optimal control problems with the wave equation. *SIAM J. Control Optim.* **48** (2009) 3026–3051.

[12] M. Gugat and M. Sigalotti, Stars of vibrating strings: switching boundary feedback stabilization. *Netw. Heterog. Media* **5** (2010) 299–314.

[13] M. Gugat, Boundary feedback stabilization by time delay for one-dimensional wave equations. *IMA J. Math. Control Inf.* **27** (2010) 189–203.

[14] M. Gugat, M. Herty, V. Schleper, Flow control in gas networks: exact controllability to a given demand. *Math. Methods Appl. Sci.* **34** (2011) 745–757.

[15] M.R. Hestenes, Calculus of Variations and Optimal Control Theory. Wiley & Sons, Inc., New York (1980).

[16] M. Jungers, E. Trélat and H. Abou-Kandil, Min-max and min-min Stackelberg strategies with closed-loop information structure. *J. Dyn. Control Syst.* **17** (2011) 387–425.

[17] M. Knauer, Fast and save container cranes as bilevel optimal control problems. Special Issue: Modelling of fuel cells and chemical engineering applications. *Math. Comp. Model. Dyn. Syst.: Methods, Tools Appl. Eng. Related Sci.* **18** (2012) 465–486.

[18] K. Kunisch, P. Trautmann and B. Vexler, Optimal control of the undamped linear wave equation with measure valued controls. *SIAM J. Control Optim.* **54** (2016) 1212–1244.

[19] L.L. Lions, Optimal Control of Systems Governed by Partial Differential Equations, 1st ed. Vol. 170 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin (1971).

[20] D.G. Luenberger, Optimization by Vector Space Methods. John Wiley & Sons (1997).

[21] J. Nash, Non-cooperative games. *Ann. Math.* **54** (1951) 286–295.

[22] P. Nie, L. Chen and M. Fukushima, Dynamic programming approach to discrete time dynamic feedback Stackelberg games with independent and dependent followers. *Eur. J. Oper. Res.* **169** (2006) 310–328.

[23] P. Nie, Discrete time dynamic multi-leader-follower games with feedback perfect information. *Int. J. Syst. Sci.* **38** (2007) 247–255.

[24] D.F. Novella Rodriguez, E. Wintrant and O. Sename, Control-oriented modeling of fluid networks: a time-delay approach, in Recent Results on Nonlinear Delay Control Systems. Springer International Publishing, Switzerland (2016).

[25] E.J.P.G. Schmidt, On the modelling and exact controllability of networks of vibrating strings. *SIAM J. Control Optim.* **30** (1992) 229–245.

[26] L. Schwartz, Méthodes mathématiques pour les sciences physiques. Hermann, Paris (1998).

[27] J.L. Troutman, Variational Calculus and Optimal Control. Springer, New York (1996).