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COMBINED EFFECTS FOR FRACTIONAL SCHRÖDINGER–KIRCHHOFF SYSTEMS WITH CRITICAL NONLINEARITIES

Xiang Mingqi[1], Vicenţiu D. Rădulescu[2,3] and Binlin Zhang[4,*]

Abstract. In this paper, we investigate the existence of solutions for critical Schrödinger–Kirchhoff type systems driven by nonlocal integro–differential operators. As a particular case, we consider the following system:

$$\begin{cases} M\left([(u,v)]_{s,p}^p + \|(u,v)\|_{p,V}^p\right)\left((-\Delta)_p^s u + V(x)|u|^{p-2}u\right) = \lambda H_u(x,u,v) + \frac{\alpha}{p_s^*}|v|^{\beta}|u|^{\alpha-2}u & \text{in } \mathbb{R}^N \\ M\left([(u,v)]_{s,p}^p + \|(u,v)\|_{p,V}^p\right)\left((-\Delta)_p^s v + V(x)|u|^{p-2}u\right) = \lambda H_v(x,u,v) + \frac{\beta}{p_s^*}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \mathbb{R}^N, \end{cases}$$

where $(-\Delta)_p^s$ is the fractional $p-$Laplace operator with $0 < s < 1 < p < N/s$, $\alpha, \beta > 1$ with $\alpha+\beta = p_s^*$, $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a continuous function, $V : \mathbb{R}^N \to \mathbb{R}^+$ is a continuous function, $\lambda > 0$ is a real parameter. By applying the mountain pass theorem and Ekeland's variational principle, we obtain the existence and asymptotic behaviour of solutions for the above systems under some suitable assumptions. A distinguished feature of this paper is that the above systems are degenerate, that is, the Kirchhoff function could vanish at zero. To the best of our knowledge, this is the first time to exploit the existence of solutions for fractional Schrödinger–Kirchhoff systems involving critical nonlinearities in $\mathbb{R}^N$.

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1. Introduction

In this paper, we investigate the existence of solutions for an elliptic system of Schrödinger–Kirchhoff type involving nonlocal integro–differential operators. More precisely, we first the following consider system:

$$\begin{cases} M\left(\|(u,v)\|^p\right)\left(\mathcal{L}_p^s u + V(x)|u|^{p-2}u\right) & = \lambda H_u(x,u,v) + \frac{\alpha}{p_s^*}|v|^{\beta}|u|^{\alpha-2}u \quad \text{in} \quad \mathbb{R}^N \\ M\left(\|(u,v)\|^p\right)\left(\mathcal{L}_p^s v + V(x)|v|^{p-2}v\right) & = \lambda H_v(x,u,v) + \frac{\beta}{p_s^*}|u|^{\alpha}|v|^{\beta-2}v \quad \text{in} \quad \mathbb{R}^N, \end{cases} \qquad (\mathcal{S})$$

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[1] Tianjin Key Lab for Advanced Signal Processing and College of Science, Civil Aviation University of China, Tianjin, 300300, P.R. China.

[2] Department of Mathematics, Faculty of Sciences, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia.

[3] Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania.

[4] Department of Mathematics, Heilongjiang Institute of Technology, Harbin, 150050, P.R. China.

[*] Corresponding author: `zhangbinlin2012@163.com`

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where $\|(u,v)\|^p = \iint_{\mathbb{R}^{2N}} (|u(x)-u(y)|^p + |v(x)-v(y)|^p)\, K(x-y)\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(x)(|u|^p+|v|^p)\mathrm{d}x$, $\alpha,\beta>1$ with $\alpha+\beta = p_s^*$, the Kirchhoff function $M:\mathbb{R}_0^+\to\mathbb{R}_0^+$ is continuous, the potential function $V:\mathbb{R}^N\to\mathbb{R}^+$ is continuous, $\lambda>0$ is a real parameter, the perturbed terms H_u, H_v are two Carathéodory functions satisfying subcritical growth condition and $\mathcal{L}_p^s$ is a nonlocal fractional operator defined as

$$\mathcal{L}_p^s\varphi(x) = \lim_{\varepsilon\to 0^+} 2\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} |\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))K(x-y)\,\mathrm{d}y$$

for $x\in\mathbb{R}^N$ along all $\varphi\in C_0^\infty(\mathbb{R}^N)$ and $K:\mathbb{R}^N\setminus\{0\}\to\mathbb{R}^+$ is a measurable function with the property that

(a) $\gamma K\in L^1(\mathbb{R}^N)$, where $\gamma(x)=\min\{|x|^p,1\}$;

(b) there exists $K_0>0$ and $s\in(0,1)$ such that $K(x)\ge K_0\,|x|^{-(N+ps)}$ for any $x\in\mathbb{R}^N\setminus\{0\}$.

Here $B_\varepsilon(x)$ denotes the ball in $\mathbb{R}^N$ of radius $\varepsilon>0$ at the center $x\in\mathbb{R}^N$. Throughout the paper, without further mentioning, we always assume that $0<s<1<p<\infty$, $sp<N$, $p_s^*=Np/(N-sp)$ and K satisfies (a) and (b).

A typical example of K is given by $K(x)=|x|^{-(N+sp)}$. In this case, system $(\mathcal{S})$ becomes

$$\begin{cases} M\left([(u,v)]_{s,p}^p + \|(u,v)\|_{p,V}^p\right)((-\Delta)_p^s u + V(x)|u|^{p-2}u) = \lambda H_u(x,u,v) + \frac{\alpha}{p_s^*}|v|^\beta|u|^{\alpha-2}u & \text{in } \mathbb{R}^N\\ M\left([(u,v)]_{s,p}^p + \|(u,v)\|_{p,V}^p\right)((-\Delta)_p^s v + V(x)|u|^{p-2}u) = \lambda H_v(x,u,v) + \frac{\beta}{p_s^*}|u|^\alpha|v|^{\beta-2}v & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $(-\Delta)_p^s$ is the fractional $p-$Laplace operator which (up to normalization factors) may be defined as

$$(-\Delta)_p^s\varphi(x) = 2\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{|\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,\mathrm{d}y$$

for $x\in\mathbb{R}^N$ along all $\varphi\in C_0^\infty(\mathbb{R}^N)$, see [17, 37] and the references therein for further details on the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ and some recent results on the fractional $p-$Laplacian. When $s\to 1^-$ in (1.1), then problem (1.1) reduces to

$$\begin{cases} M(\|(\nabla u,\nabla v)\|_p^p + \|(u,v)\|_{p,V}^p)(-\Delta_p u + V(x)|u|^{p-2}u) = \lambda H_u(x,u,v) + \frac{\alpha}{p^*}|v|^\beta|u|^{\alpha-2}u & \text{in } \mathbb{R}^N\\ M(\|(\nabla u,\nabla v)\|_p^p + \|(u,v)\|_{p,V}^p)(-\Delta_p v + V(x)|v|^{p-2}v) = \lambda H_v(x,u,v) + \frac{\beta}{p^*}|u|^\alpha|v|^{\beta-2}v & \text{in } \mathbb{R}^N, \end{cases}$$

where Δ_p is the $p-$Laplace operator, $p^*=Np/(N-p)$ is the critical exponent of the classical Sobolev space $W^{1,p}(\mathbb{R}^N)$, and

$$\|(\nabla u,\nabla v)\|_p = \left(\int_{\mathbb{R}^N} |\nabla u|^p + |\nabla v|^p\mathrm{d}x\right)^{1/p}.$$

Much interest has grown on problems involving critical exponents, starting from the celebrated paper by *Brézis* and *Nirenberg* [7], where the case $p=2$ is considered. We refer the reader to [16, 19, 28] and references therein for the study of problems with critical exponents.

In recent years, a great attention has been focused on the study of problems involving fractional and non-local operators. This type of operators arises in a quite natural way in many different applications, such as optimization, finance, crystal dislocation, soft thin films, continuum mechanics, phase transition phenomena, population dynamics, minimal surfaces and game theory, as they are the typical outcome of stochastically stabilization of Lévy processes, see [8, 15, 21] and the references therein. The literature on non–local operators and their applications is interesting and large, we refer the interested reader to [3, 5, 25–27] and the references therein.

Recently, *Fiscella* and *Valdinoci* in [18] studied the following single fractional Kirchhoff problem involving critical exponent:

$$\begin{cases} M(\|u\|^2)\mathcal{L}_K u = \lambda f(x,u) + |u|^{2_s^*-2}u & \text{in} \quad \Omega \\ u = 0 & \text{in} \quad \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.2}$$

where $\|u\|^2 = \iint_{\mathbb{R}^{2N}} |u(x)-u(y)|^2 K(x-y)\mathrm{d}x\mathrm{d}y$, $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N > 2s$, $2_s^* = 2N/(N-2s)$ is the critical exponent of the fractional Sobolev space $H^s(\mathbb{R}^N)$, the function f is a subcritical term, λ is a positive parameter and nonlocal fractional operator $\mathcal{L}_K$ is defined for any $x \in \mathbb{R}^N$ by

$$\mathcal{L}_K\varphi(x) = \frac{1}{2}\int_{\mathbb{R}^N}(\varphi(x+y)+\varphi(x-y)-2\varphi(x))K(y)\mathrm{d}y,$$

along any $\varphi \in C_0^\infty(\mathbb{R}^N)$, where the kernel $K : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^+$ is a measurable function satisfying the following property: there exists $\theta > 0$ and $s \in (0,1)$ such that

$$\theta|x|^{-(N+2s)} \le K(x) \le \theta^{-1}|x|^{-(N+2s)} \quad \text{for any} \quad x \in \mathbb{R}^N \setminus \{0\},$$

and the Kirchhoff function $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is continuous, increasing and $M(0) > 0$. In this paper, the authors obtained the existence of solutions of problem (1.2) by using a truncation argument and the mountain pass theorem. In the Appendix A in [18], the authors considered problem (1.2) as a stationary Kirchhoff variational equation which models the *nonlocal* aspect of the tension arising from nonlocal measurements of the fractional length of the string. In other words, problem (1.2) is a fractional version of a model, the so–called Kirchhoff equation, introduced by *Kirchhoff*. It is worth pointing out that the Kirchhoff equation received much attention only after *Lions* [23] proposed an abstract framework to the problem, see [4,5,29] for some recent results.

In [4], *Autuori, Fiscella* and *Pucci* proceeded to study problem (1.2) in the degenerate case, that is, this paper covered a case that $M(0) = 0$. Here, we call the problem (1.2) associated with the Kirchhoff function to be *non-degenerate* if $M(0) > 0$, and *degenerate* if $M(0) = 0$. For example, the existence of solutions of non–degenerate Kirchhoff–type problems is treated in [14,36] and degenerate problems in [12,24,31,38,39]. Under some suitable assumptions, the existence of nonnegative mountain pass solutions is established in [4] in the degenerate case. However, the authors in [4] only obtained the above result under the condition $2s < N < 4s$, that is $N \in \{1,2,3\}$. A natural question is whether or not there exists nontrivial solutions of equation (1.2) in $\mathbb{R}^N$? Another question is whether or not we can extend the existence results for single equation to the corresponding system? With these questions, we start to work in the superlinear and sublinear cases. The main novelty of our paper is to cover the degenerate case of system ($\mathcal{S}$) in the setting of fractional p–Laplacian involving critical exponents.

Without further mentioning, we always assume that $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a continuous function satisfying the following conditions:

(M_1) there exists $\theta \in [1, p_s^*/p)$ such that $M(t)t \le \theta\mathscr{M}(t)$ for all $t \in \mathbb{R}_0^+$, where $\mathscr{M}(t) = \int_0^t M(\tau)d\tau$;
(M_2) for all $\delta > 0$ there exists $\kappa = \kappa(\delta) > 0$ such that $M(t) \ge \kappa$ for all $t \ge \delta$;
(M_3) there exist $m_0 > 0$ and $\theta_1 > 0$ such that $M(t) \ge m_0 t^{\theta_1}$ for all $t \in [0,1]$, where

$$\begin{cases} \theta_1 \in \left(0, \dfrac{p_s^*-p}{p}\right) & \text{if} \quad p \ge 2; \\ \theta_1 \in \left(0, \dfrac{p_s^*-2}{2}\right) & \text{if} \quad \max\{1, 2N/(N+2s)\} < p < 2. \end{cases}$$

Obviously, (M_1)–(M_3) cover the degenerate case that corresponds to $M(0) = 0$. Note that assumptions (M_1) and (M_2) are first used to study the multiplicity of solutions of a class of higher order p–Kirchhoff equations in [12]. In order to obtain the existence of solutions to fractional problems of Kirchhoff type, a natural assumption that M is a nondecreasing function on $\mathbb{R}_0^+$ was often employed, see for example [18,29]. However, under assumption (M_1), we can also deal with cases in which M is not monotone as $M(t) = (1+t)^k + (1+t)^{-1}$ for $t \ge 0$, with $0 < k < 1$.

Now we impose the following hypothesis on the potential function V:

(V_1) $V \in C(\mathbb{R}^N)$ satisfies $\inf_{x\in\mathbb{R}^N} V(x) \geq V_0 > 0$, where V_0 is a constant;

For the perturbed terms, we assume that

(H_1) $H \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R}^+)$, $H_z(x,0,0) = 0$ for all $x \in \mathbb{R}^N$ and there exist $c > 0$ and $q \in (\theta p, p_s^*)$ such that

$$|H_z(x,z)| \leq c(1 + |z|^{q-1}), \quad \text{for each} \quad (x,z) \in \mathbb{R}^N \times \mathbb{R}^2,$$

where $H_z(x,z) = (H_u(x,z), H_v(x,z)$ and H_u, H_v stand for the partial derivatives of H with respect to the second and third variable;

(H_2) $H_z(x,z) = o(|z|^{\theta p-1})$ as $|z| \to 0$, uniformly for $x \in \mathbb{R}^N$;

(H_3) there exists $\mu \in (\theta p, p_s^*)$ such that

$$0 \leq \mu H(x,z) \leq H_z(x,z)\cdot z, \quad \text{for all} \quad (x,z) \in \mathbb{R}^N \times \mathbb{R}^2.$$

Before giving our main result, we introduce some notations. Set

$$L^p(\mathbb{R}^N, V) = \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ measurable} : \int_{\mathbb{R}^N} V|u(x)|^p \mathrm{d}x < \infty \right\},$$

endowed with the norm

$$|u|_{p,V} = \left(\int_{\mathbb{R}^N} V|u(x)|^p \mathrm{d}x \right)^{1/p}.$$

Denote

$$W^{s,p}_{K,V}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N, V) : \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y < \infty \right\},$$

endowed with the norm $\|\varphi\|_{W^{s,p}_{K,V}(\mathbb{R}^N)} := ([\varphi]^p_{s,p} + |\varphi|^p_{p,V})^{1/p}$, where

$$[\varphi]_{s,p} = \left(\iint_{\mathbb{R}^{2N}} |\varphi(x)-\varphi(y)|^p K(x-y)\mathrm{d}x\mathrm{d}y \right)^{1/p}.$$

Note that (a) assures that $C_0^\infty(\mathbb{R}^N) \subset W^{s,p}_{K,V}(\mathbb{R}^N)$. Actually, when $K(x) = \frac{1}{|x|^{N+ps}}$, Lemma 2.4 of [31] gives that $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{s,p}_{K,V}(\mathbb{R}^N)$.

Let $\mathbf{W} = W^{s,p}_{K,V}(\mathbb{R}^N) \times W^{s,p}_{K,V}(\mathbb{R}^N)$, endowed with the norm

$$\|(u,v)\| = \left([(u,v)]^p_{s,p} + |(u,v)|^p_{p,V} \right)^{1/p},$$

for $(u,v) \in \mathbf{W}$, where

$$[(u,v)]_{s,p} = \left([u]^p_{s,p} + [v]^p_{s,p} \right)^{1/p}.$$

Then $(\mathbf{W}, \|\cdot\|_{\mathbf{W}})$ is a uniformly convex Banach space, see [30] for more details.

Now we give the definition of weak solutions for problem $(\mathcal{S})$.

Definition 1.1. *We say that* $(u,v) \in \mathbf{W}$ *is a (weak) solution of problem* $(\mathcal{S})$*, if*

$$\begin{aligned} &M(\|(u,v)\|^p) \Big[\iint_{\mathbb{R}^{2N}} |u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y) \\ &\quad + |v(x)-v(y)|^{p-2}(v(x)-u(y))(\psi(x)-\psi(y))K(x-y)\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(|u|^{p-2}u\varphi + |v|^{p-2}v\psi)\mathrm{d}x \Big] \\ &\quad = \lambda \int_{\mathbb{R}^N} H_u(x,u,v)\varphi(x) + H_v(x,u,v)\psi(x)\mathrm{d}x + \int_{\mathbb{R}^N} \frac{\alpha}{p_s^*}|v|^\beta |u|^{\alpha-2}u\varphi + \frac{\beta}{p_s^*}|u|^\alpha |v|^{\beta-2}v\psi \mathrm{d}x, \end{aligned}$$

for all $(\varphi,\psi) \in \mathbf{W}$.

For the superlinear case, that is, $q \in (\theta p, p_s^*)$, we state the first result of our paper as follows.

Theorem 1.2. *Suppose that V satisfies (V_1), M satisfies (M_1)–(M_3) and H satisfies (H_1)–(H_3). Then there exists $\lambda^* > 0$ such that system $(\mathcal{S})$ admits at least one nontrivial solution in $\mathbf{W}$ for all $\lambda \geq \lambda^*$.*

For the sublinear case, that is, $q \in (1, \theta p)$, we consider the separate case that $H(x, u, v) = h(x)f(u, v)$. More precisely, we study the following system

$$\begin{cases} M\left(\|(u,v)\|^p\right)\left(\mathcal{L}_p^s u + V(x)|u|^{p-2}u\right) = h(x)f_u(u,v) + \lambda\frac{\alpha}{p_s^*}|v|^\beta|u|^{\alpha-2}u & \text{in} \quad \mathbb{R}^N \\ M\left(\|(u,v)\|^p\right)\left(\mathcal{L}_p^s v + V(x)|v|^{p-2}v\right) = h(x)f_v(u,v) + \lambda\frac{\beta}{p_s^*}|u|^\alpha|v|^{\beta-2}v & \text{in} \quad \mathbb{R}^N, \end{cases} \tag{$\mathcal{S}'$}$$

where V satisfies (V_1), $\lambda > 0$ and f satisfies

(f_1) $f \in C^1(\mathbb{R}^2, \mathbb{R}^+)$ and there exist $C > 0$ and $q \in (1, \theta p)$ such that

$$|f_z(z)| \leq C|z|^{q-1} \quad \text{for all} \quad z \in \mathbb{R}^2,$$

where $f_z(z) = (f_u(z), f_v(z))$ and f_u, f_v stand for the partial derivatives of f with respect to the first and second variable;

(f_2) there exist $a_0 > 0$, $\delta > 0$ and $q_1 \in (1, p)$ such that

$$f(z) \geq a_0|z|^{q_1} \quad \text{for all} \quad z \in \mathbb{R}^2 \quad \text{with } |z| \leq \delta.$$

Let $\eta(t) := \frac{1}{2p}\mathscr{M}(1)t^{\theta p} - \frac{C_{p_s^*}^{p_s^*}}{p_s^*}t^{p_s^*}$ for all $t \geq 0$. Clearly, by $p_s^* > \theta p$, there exists $\mu_0 = \left(\frac{\theta \mathscr{M}(1)}{2C_{p_s^*}^{p_s^*}}\right)^{\frac{1}{p_s^*-\theta p}}$ such that $\eta(\mu_0) = \max_{t\geq 0}\eta(t) = \mathscr{M}(1)^{\frac{p_s^*}{p_s^*-\theta p}}\left(\frac{\theta}{2C_{p_s^*}^{p_s^*}}\right)^{\frac{\theta p}{p_s^*-\theta p}}\left(\frac{1}{2p} - \frac{\theta}{2p_s^*}\right) > 0$.

Throughout this paper, we assume that

(h_1) $0 \leq h \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)\bigcap L^{\frac{p_s^*}{p_s^*-q}}(\mathbb{R}^N)$ and

$$\eta(\mu_0) > \left(\frac{1}{2p}\mathscr{M}(1)\right)^{-\frac{q}{\theta p-q}}\left(CC_{p_s^*}^q\|h\|_{\frac{p_s^*}{p_s^*-q}}\right)^{\frac{\theta p}{\theta p-q}},$$

where C and q are defined in (f_1) and $C_{p_s^*} > 0$ is the embedding constant of fractional Sobolev space ($cf.$ (2.1));

(h_2) there exists an open bounded nonempty set $\Omega \subset \mathbb{R}^N$ such that $\inf_{x\in\Omega} h(x) > 0$.

Our second result reads as follows.

Theorem 1.3. *Suppose that V satisfies (V_1), M satisfies $(M_1) - (M_3)$, f satisfies $(f_1) - (f_2)$ and h satisfies $(h_1) - (h_2)$. Then there exists $\lambda^{**} > 0$ such that system $(\mathcal{S}')$ admits at least one nontrivial solution in $\mathbf{W}$ for all $\lambda \in (0, \lambda^{**}]$.*

Finally, let us simply describe the main approaches to obtain Theorems 1.2 and 1.3. To show the existence of at least one critical point of the energy functional, we shall use the mountain pass theorem of *Ambrosetti* and *Rabinowitz* [2]. However, since system $(\mathcal{S})$ contains a critical nonlinearity, it is difficult to get the global Palais-Smale condition. To overcome the lack of compactness due to the presence of a critical nonlinearity, we employ some tricks borrowed from the paper [4], where a critical Kirchhoff problem involving the fractional Laplacian has been studied. We first show that the energy functional associated with system $(\mathcal{S})$ satisfies the Palais-Smale condition at suitable levels c_λ. In this process, the key point is to study the asymptotical behaviour of c_λ as $\lambda \to \infty$, see Lemma 3.4 for more details. For λ small, we show that $(\mathcal{S}')$ has at least one nontrivial solutions by using Ekeland's variational principle. To the best of our knowledge, Theorems 1.2 and 1.3 are new even in the study of fractional Laplacian.

This paper is organized as follows. In Section 2, we give some necessary definitions and properties of space $\mathbf{W}$. In Section 3, using the mountain pass theorem, we establish the existence of nontrivial solutions for system ($\mathcal{S}$). In Section 4, by using Ekeland's variational principle, we obtain the existence of nontrivial solutions of system ($\mathcal{S}'$).

2. Variational Framework

In this section we introduce the variational framework for problem ($\mathcal{S}$), in which most of results can be found in [35], see also [32–34]. It is worth pointing out that the functional setting was first introduced by *Autuori* and *Pucci* in [5] as $p = 2$.

Let p_s^* be the fractional Sobolev critical exponent defined by $p_s^* = Np/(N - sp)$. Let $D_K^{s,p}(\mathbb{R}^N)$ be the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $[\varphi]_{s,p}$ for $\varphi \in C_0^\infty(\mathbb{R}^N)$. In the fractional Sobolev space $D_K^{s,p}(\mathbb{R}^N)$, the following fractional Sobolev inequality is well known: there is a constant $C_{p_s^*} > 0$ such that

$$\|u\|_{L^{p_s^*}(\mathbb{R}^N)} \le C_{p_s^*}[u]_{s,p} \quad \text{for all} \quad u \in D_K^{s,p}(\mathbb{R}^N), \tag{2.1}$$

by Theorem 6.5 of [17] and (a). Furthermore, it follows by the Hölder inequality that

$$\begin{aligned}\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \mathrm{d}x &\le \left(\int_{\mathbb{R}^N} |u|^{p_s^*}\mathrm{d}x\right)^{\alpha/p_s^*}\left(\int_{\mathbb{R}^N} |v|^{p_s^*}\mathrm{d}x\right)^{\beta/p_s^*}\\ &\le C_{p_s^*}^{p_s^*}[u]_{s,p}^\alpha[v]_{s,p}^\beta \le C_{p_s^*}^{p_s^*}[(u,v)]_{s,p}^{p_s^*},\end{aligned} \tag{2.2}$$

for all $\alpha, \beta > 1$ with $\alpha + \beta = p_s^*$.

A similar discussion as Lemma 2.4 of [35] gives that $\left(D_K^{s,p}(\mathbb{R}^N), [\cdot]_{s,p}\right)$ is a uniformly convex Banach space, so that a reflexive Banach space. Moreover, $(W_{K,V}^{s,p}(\mathbb{R}^N), \|\cdot\|_{W_{K,V}^{s,p}(\mathbb{R}^N)})$ is a reflexive Banach space. Let $\mathbf{W} = W_{K,V}^{s,p}(\mathbb{R}^N) \times W_{K,V}^{s,p}(\mathbb{R}^N)$, endowed with the norm

$$\|(u,v)\| = \left([(u,v)]_{s,p}^p + |(u,v)|_{p,V}^p\right)^{1/p},$$

for $(u,v) \in \mathbf{W}$, where

$$[(u,v)]_{s,p} = ([u]_{s,p}^p + [v]_{s,p}^p)^{1/p} \quad \text{and} \quad |(u,v)|_{p,V} = (|u|_{p,V}^p + |v|_{p,V}^p)^{1/p}.$$

Consequently, by Theorem 1.23 of [1], we know that $(\mathbf{W}, \|\cdot\|)$ is a reflexive Banach space.

For $(u,v) \in \mathbf{W}$, we define

$$I(u,v) = \frac{1}{p}\mathscr{M}\left(\|(u,v)\|^p\right) - \lambda\int_{\mathbb{R}^N} H(x,u,v)\mathrm{d}x - \frac{1}{p_s^*}\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\mathrm{d}x.$$

Obviously, under the assumption (H_1), the energy functional $I : \mathbf{W} \to \mathbb{R}$ associated with problem ($\mathcal{S}$) is well defined and $I \in C^1(\mathbf{W}, \mathbb{R})$ and

$$\begin{aligned}\langle I'(u,v),(\varphi,\psi)\rangle = M\left(\|(u,v)\|^p\right)\Bigg[\iint_{\mathbb{R}^N} &|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)\\ &+ |v(x)-v(y)|^{p-2}(v(x)-v(y))(\psi(x)-\psi(y))K(x-y)\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(|u|^{p-2}u\varphi + |v|^{p-2}v\psi)\mathrm{d}x\Bigg]\\ &- \lambda\int_\Omega H_u(x,u,v)\varphi + H_v(x,u,v)\psi\mathrm{d}x - \frac{\alpha}{p_s^*}\int_{\mathbb{R}^N}|u|^{\alpha-2}u|v|^\beta\mathrm{d}x - \frac{\beta}{p_s^*}\int_{\mathbb{R}^N}|u|^\alpha|v|^{\beta-2}v\mathrm{d}x,\end{aligned}$$

for all $(u, v) \in \mathbf{W}$ and $(\varphi, \psi) \in \mathbf{W}$, see for example [30], (Lems. 2 3) and [35], (Lems. 3.1 and 3.2). It follows that the critical points of functional I are weak solutions of system $(\mathcal{S})$.

Lemma 2.1. (see [30], Lem. 1) *Assume* (V_1). *If* $\nu \in [p, p_s^*)$, *then the embedding* $W^{s,p}_{K,V}(\mathbb{R}^N) \hookrightarrow L^\nu(\mathbb{R}^N)$ *is continuous, and there exists a constant* $C_\nu > 0$ *such that*

$$|u|_\nu \leq C_\nu \|u\|_{W^{s,p}_{K,V}(\mathbb{R}^N)} \quad \text{for all} \quad u \in W^{s,p}_{K,V}(\mathbb{R}^N).$$

Moreover, by Lemma 2.1, we have the following property.

Lemma 2.2. *For each* $(u, v) \in \mathbf{W}$ *and* $\nu \in [p, p_s^*)$, *we have*

$$\left\| \sqrt{u^2 + v^2} \right\|_{L^\mu(\mathbb{R}^N)} \leq C \|(u, v)\|,$$

where $C = 2^{\frac{p-1}{p}} C_\nu$.

Proof. By Lemma 2.1, for each $(u, v) \in \mathbf{W}$ and $\nu \in [p, p_s^*)$, we get

$$\begin{aligned} \left\| \sqrt{u^2 + v^2} \right\|_{L^\nu(\mathbb{R}^N)} &\leq \|u + v\|_{L^\nu(\mathbb{R}^N)} \\ &\leq \|u\|_{L^\nu(\mathbb{R}^N)} + \|v\|_{L^\nu(\mathbb{R}^N)} \\ &\leq C_\nu (\|u\|_{W^{s,p}_{K,V}(\mathbb{R}^N)} + \|v\|_{W^{s,p}_{K,V}(\mathbb{R}^N)}) \\ &\leq 2^{\frac{p-1}{p}} C_\nu \|(u, v)\|. \end{aligned}$$

Hence the desired conclusion follows. □

Theorem 2.3. *Assume* (V_1). *Let* $\nu \in (p, p_s^*)$ *be a fixed exponent and let* $\{(u_n, v_n)\}_n$ *be a bounded sequence in* $\mathbf{W}$. *Then there exists* $(u, v) \in \mathbf{W}$ *such that up to a subsequence,* $(u_n, v_n) \to (u, v)$ *strongly in* $L^\nu(\mathbb{R}^N) \times L^\nu(\mathbb{R}^N)$ *as* $n \to \infty$.

Proof. The proof can be proved similarly as that of Lemma 2.2 in [11]. For completeness, here we give a short proof. Since $\{(u_n, v_n)\}_n$ is bounded in $\mathbf{W}$, there exist a constant $C > 0$ and $(u, v) \in \mathbf{W}$, and a subsequence of $\{(u_n, v_n)\}_n$, still denoted by $\{(u_n, v_n)\}_n$, such that $\|(u_n, v_n)\| \leq C$, $\|(u, v)\| \leq C$ and

$$\begin{cases} (u_n, v_n) \rightharpoonup (u, v) & \text{weakly in} \quad \mathbf{W}, \\ (u_n, v_n) \to (u, v) & \text{a.e. in} \quad \mathbb{R}^N, \\ (u_n, v_n) \to (u, v) & \text{in} \quad L^\nu_{\text{loc}}(\mathbb{R}^N) \times L^\nu_{\text{loc}}(\mathbb{R}^N). \end{cases} \tag{2.3}$$

Thus, we obtain $u_n \to u$ and $v_n \to v$ in $L^\nu(B_R)$ for all $R > 0$. In the following, we prove that there exists $R_0 > 0$ such that for $R \geq R_0$,

$$\lim_{n \to} \|u_n\|^\nu_{L^\nu(B_R^c)} = \|u\|^\nu_{L^\nu(B_R^c)}, \tag{2.4}$$

$$\lim_{n \to} \|v_n\|^\nu_{L^\nu(B_R^c)} = \|v\|^\nu_{L^\nu(B_R^c)}. \tag{2.5}$$

We first prove (2.4). For this, it is enough to show that for any $\varepsilon > 0$ there exists $R_0 > 0$ such that for $R \geq R_0$,

$$\limsup_{n \to \infty} \|u_n\|^\nu_{L^\nu(B_R^c)} \leq \varepsilon. \tag{2.6}$$

Note that for any $\varepsilon > 0$ there exist $0 < a_0 < b_0$ such that $|a|^\nu \le \varepsilon |a|^p$ if $|a| \le a_0$ and $|a|^\nu \le \varepsilon |a|^{p_s^*}$ if $|a| \ge b_0$, thanks to $p < \nu < p_s^*$. Hence,

$$|a|^\nu \le \varepsilon(|a|^p + |a|^{p_s^*}) + \chi_{[a_0,b_0]}(|a|)|a|^\nu,$$

where $\chi_{[a_0,b_0]}$ denotes the characteristic function associated with the interval $[a_0, b_0]$. Then,

$$\int_{B_R^c} |u_n|^\nu \mathrm{d}x \le \varepsilon \int_{B_R^c} |u_n|^p + |u_n|^{p_s^*} \mathrm{d}x + b_0^\nu \int_{A_n \bigcap B_R^c} \chi_{[a_0,b_0]}(|u_n|)\mathrm{d}x, \quad \forall n \in \mathbb{N}, \tag{2.7}$$

where $A_n = \{x \in \mathbb{R}^N : a_0 \le |u_n| \le b_0\}$. Furthermore, it follows from (2.1) that there exists $C > 0$ such that

$$\int_{\mathbb{R}^N} |u_n|^p + |u_n|^{p_s^*} \mathrm{d}x \le C, \quad \forall n \in \mathbb{N}. \tag{2.8}$$

Therefore

$$a_0^p |A_n| \le \int_{\mathbb{R}^N} |u_n|^p + |u_n|^{p_s^*} \mathrm{d}x \le C, \quad \forall n \in \mathbb{N},$$

where $|A_n| = \mathrm{meas}(A_n)$, this implies that $\sup_{n\in\mathbb{N}} |A_n| \le C a_0^{-p}$. We claim that $\lim_{R\to\infty} |A_n \bigcap B_R^c| = 0$ uniformly in $n \in \mathbb{N}$. To begin with, we show that

$$\lim_{R\to\infty} |A_n \bigcap B_R^c| = 0, \quad \forall n \in \mathbb{N}. \tag{2.9}$$

Indeed, if the assertion is not true, then there exist $n_0 \ge 1$, $\delta > 0$ and $R_j \uparrow \infty$ such that

$$|A_{n_0} \bigcap B_{R_j}^c| \ge \delta, \quad \forall n \in \mathbb{N}. \tag{2.10}$$

Clearly, $|A_{n_0} \bigcap B_{R_j}^c| \le |A_{n_0}| \le C a_0^{-p}$ $\forall n \in \mathbb{N}$. Set $\Omega_j = B_{R_j}^c \setminus \overline{B_{j+1}^c}$, $\forall j \in \mathbb{N}$. Then $\Omega_m \bigcap \Omega_k = \emptyset$, if $m \ne k$, and

$$B_{R_j}^c = \bigcup_{k=1}^\infty \Omega_k, \ |A_{n_0} \bigcap B_{R_j}^c| = \sum_{k=j}^\infty |A_{n_0} \bigcap \Omega_k| \ge \delta, \quad \forall j \in \mathbb{N} \tag{2.11}$$

and $\sum_{k=1}^\infty |A_{n_0} \bigcap \Omega_k| = \infty$. This is a contradiction. Hence the limit (2.9) is proved. On the other hand, it follows from (2.3) that $u \in W_{V,K}^{s,p}(\mathbb{R}^N)$ and $u_n \to u$ a.e. in $\mathbb{R}^N$. Hence, for any $\varepsilon > 0$ there exists $R_0 > 1$ such that $R \ge R_0$,

$$\int_{B_R^c} |u(x)|^p \mathrm{d}x \le \varepsilon.$$

For fixed $\varepsilon > 0$, we choose $t_1 = R_0$, $t_k \uparrow \infty$ such that $D_k = B_{t_k}^c \setminus \overline{B_{t_{k+1}}^c}$, $B_{R_0}^c = \bigcup_{k-1}^\infty D_k$ and

$$\int_{D_k} |u(x)|^p \mathrm{d}x \le 2^{-k}\varepsilon, \quad \forall k \in \mathbb{N}.$$

Obviously, for each fixed $k \in \mathbb{N}$, D_k is a bounded domain and $D_k \bigcap D_m = \emptyset$ $(k \ne m)$. Moreover, $a_0 \le |u_n| \le b_0$ in $D_k \bigcap A_n$. By the Fatou lemma, we have for each $k \in \mathbb{N}$,

$$\begin{aligned}\limsup_{n\to\infty} \int_{D_k \bigcap A_n} |u_n|^p \mathrm{d}x &\le \int_{D_k \bigcap A_n} \limsup_{n\to\infty} |u_n|^p \mathrm{d}x \\ &= \int_{D_k \bigcap A_n} |u|^p \mathrm{d}x \le \int_{D_k} |u|^p \mathrm{d}x \le 2^{-k}\varepsilon.\end{aligned}$$

Then for any $n \in \mathbb{N}$, we obtain

$$a_0^p |A_n \bigcap B_{R_0}^c| \le \int_{A_n \bigcap B_{R_0}^c} |u_n|^p \mathrm{d}x$$
$$\le \sum_{k=1}^{\infty} \limsup_{n\to\infty} \int_{D_k \bigcap A_n} |u_n|^p \mathrm{d}x \le \varepsilon. \tag{2.12}$$

Observe that for all $R \ge R_0$ and $n \in \mathbb{N}$, we have $(A_n \bigcap B_R^c) \subset (A_n \bigcap B_{R_0}^c)$. Hence, using (2.9) and (2.12), we deduce that $\lim_{R\to\infty} |A_n \bigcap B_R^c| = 0$ uniformly in $n \in \mathbb{N}$.

Then, for any $\varepsilon > 0$ there exist $R_0 \ge 1$ and $\delta_0 \in \left(0, \varepsilon/(Ca_0^{-p})\right)$ such that $|A_n \bigcap R_R^c| < \delta_0$ for all $n \in \mathbb{N}$ and $R \ge R_0$, and

$$\int_{A_n \bigcap B_R^c} \chi_{[a_0,b_0]}(|u_n|)\mathrm{d}x \le |A_n \bigcap B_R^c| \le \delta_0 \le \frac{\varepsilon a_0^p}{C}, \quad \forall n \in \mathbb{N}. \tag{2.13}$$

Hence, from (2.7) and (2.8), we deduce that

$$\|u\|_{L^\nu(B_R^c)}^\nu \le \liminf_{n\to\infty} \|u_n\|_{L^\nu(B_R^c)}^\nu \le \varepsilon C, \quad \text{for} \quad R \ge R_0, \tag{2.14}$$

which implies (2.6). Therefore, we get $\lim_{n\to\infty} \|u_n\|_{L^\nu(\mathbb{R}^N)}^\nu = \|u\|_{L^\nu(\mathbb{R}^N)}^\nu$. Moreover, by the Brézis-Lieb lemma we obtain $u_n \to u$ in $L^\nu(\mathbb{R}^N)$. Arguing as above, we conclude that $v_n \to v$ in $L^\nu(\mathbb{R}^N)$. Thus the proof is complete. □

Lemma 2.4. *Let $(u_n, v_n) \rightharpoonup (u, v)$ weakly in $\mathbf{W}$ and $(u_n, v_n) \to (u, v)$ a.e. in $\mathbb{R}^N$. Then for fixed $\alpha, \beta > 1$ with $\alpha + \beta = p_s^*$, up to a subsequence,*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n - u|^\alpha |v_n - v|^\beta \mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta \mathrm{d}x - \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \mathrm{d}x.$$

Proof. The proof is similar to [20], (Lem. 2.2), we give it here just for completeness. We first observe that

$$\int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta \mathrm{d}x - \int_{\mathbb{R}^N} |u_n - u|^\alpha |v_n - v|^\beta \mathrm{d}x = \alpha \int_{\mathbb{R}^N} \int_0^1 |u_n - tu|^{\alpha-2}(u_n - tu)u|v_n|^\beta dt\mathrm{d}x$$
$$+ \beta \int_{\mathbb{R}^N} \int_0^1 |v_n - tv|^{\beta-2}(v_n - tv)v|u_n - u|^\alpha dt\mathrm{d}x$$
$$= \alpha \int_{\mathbb{R}^N} \int_0^1 f_n u \mathrm{d}x\mathrm{d}t + \beta \int_{\mathbb{R}^N} \int_0^1 g_n v \mathrm{d}x\mathrm{d}t, \tag{2.15}$$

where

$$f_n = |u_n - tu|^{\alpha-2}(u_n - tu)|v_n|^\beta, g_n = |v_n - tv|^{\beta-2}(v_n - tv)|u_n - u|^\alpha, \quad t \in [0,1].$$

Since $u_n \to u$ and $v_n \to v$ a.e. in $\mathbb{R}^N$, we have

$$f_n \to (1-t)^{\alpha-1}|u|^{\alpha-2}u|v|^\beta \quad \text{and} \quad g_n \to 0 \text{ a.e. in } \mathbb{R}^N \times (0,1).$$

Moreover, by the Hölder inequality, we get

$$\int_{\mathbb{R}^N} \int_0^1 |f_n|^{\frac{\alpha+\beta}{\alpha+\beta-1}} \mathrm{d}x\mathrm{d}t \le \left(\int_{\mathbb{R}^N} \int_0^1 |u_n - tu|^{\alpha+\beta} \mathrm{d}x\mathrm{d}t\right)^{\frac{\alpha-1}{\alpha+\beta-1}} \left(\int_{\mathbb{R}^N} \int_0^1 |v_n|^{\alpha+\beta} \mathrm{d}x\mathrm{d}t\right)^{\frac{\beta}{\alpha+\beta-1}} \le C,$$

and

$$\int_{\mathbb{R}^N}\int_0^1 |g_n|^{\frac{\alpha+\beta}{\alpha+\beta-1}}\mathrm{d}x\mathrm{d}t \le \left(\int_{\mathbb{R}^N}\int_0^1 |u_n-u|^{\alpha+\beta}\mathrm{d}x\mathrm{d}t\right)^{\frac{\alpha}{\alpha+\beta-1}}\left(\int_{\mathbb{R}^N}\int_0^1 |v_n-tv|^{\alpha+\beta}\right)^{\frac{\beta-1}{\alpha+\beta-1}} \le C.$$

Thus, up to a subsequence, we deduce that

$$f_n \rightharpoonup (1-t)^{\alpha-1}|u|^{\alpha-2}u|v|^{\beta} \quad \text{and} \quad g_n \rightharpoonup 0 \;\text{ weakly in } L^{\frac{\alpha+\beta}{\alpha+\beta-1}}(\mathbb{R}^N\times(0,1)),$$

as $n\to\infty$. Therefore

$$\alpha\int_{\mathbb{R}^N}\int_0^1 f_n u\mathrm{d}x\mathrm{d}t \to \alpha\int_{\mathbb{R}^N}\int_0^1 (1-t)^{\alpha-1}|u|^{\alpha}|v|^{\beta}\mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}^N}|u|^{\alpha}|v|^{\beta}\mathrm{d}x \tag{2.16}$$

and

$$\beta\int_{\mathbb{R}^N}\int_0^1 g_n v\mathrm{d}x\mathrm{d}t \to 0, \tag{2.17}$$

as $n\to\infty$. Inserting (2.16) and (2.17) into (2.15), we get the desired result. □

In the sequel, we will make use of the mountain pass theorem of *Ambrosetti–Rabinowitzin* [2] which will be used later.

Theorem 2.5. *Let E be a real Banach space and $J\in C^1(E,\mathbb{R})$ with $J(0)=0$. Suppose that J satisfies* (PS) *condition and*:

(i) *there exist $\rho,\alpha>0$ such that $J(u)\ge\alpha$ for all $u\in E$, with $\|u\|_E=\rho$;*
(ii) *there exists $e\in E$ satisfying $\|e\|_E>\rho$ such that $J(e)<0$.*

Define

$$\Gamma=\{\gamma\in C^1([0,1];E):\gamma(0)=1,\gamma(1)=e\}.$$

Then

$$c=\inf_{\gamma\in\Gamma}\max_{0\le t\le 1}J(\gamma(t))\ge\alpha$$

is a critical value of J.

3. Proof of Theorem 1.2

In this section, we prove the first main result of this paper. To apply Theorem 2.5, we first verify the validness of the conditions of Theorem 2.5. In what follows, we shortly denote by $\|\cdot\|_q$ the norm of the Lebesgue space $L^q(\mathbb{R}^N)$.

Lemma 3.1. *For any $\lambda\in\mathbb{R}^+$, there exist $\alpha_0,\rho_0>0$ such that $I(u,v)\ge\alpha_0>0$ for any $(u,v)\in\mathbf{W}$, with $\|(u,v)\|=\rho_0$.*

Proof. By (M_1), we have

$$\mathscr{M}(t)\ge\mathscr{M}(1)t^{\theta} \quad \text{for all} \quad t\in[0,1]. \tag{3.1}$$

By (H_2), for any $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ such that

$$|H_z(x,z)|\le\varepsilon|z|^{\theta p-1} \tag{3.2}$$

for all $x \in \mathbb{R}^N$ and $|z| \leq \delta$. Moreover, by (H_1), we obtain

$$|H_z(x,z)| \leq \left(c + \frac{c}{\delta}\right)|z|^{q-1} \tag{3.3}$$

for any $x \in \mathbb{R}^N$ and all $|z| \geq \delta$. From (3.2) and (3.3), we have

$$|H_z(x,z)| \leq \varepsilon|z|^{\theta p-1} + C_\varepsilon|z|^{q-1} \quad \text{for all} \quad (x,z) \in \mathbb{R}^N \times \mathbb{R}^2, \tag{3.4}$$

where $C_\varepsilon = \left(c + \frac{c}{\delta}\right)$. Note that $H(x,z) = \int_0^1 \frac{d}{dt}H(x,tz)dt = \int_0^1 H_z(x,tz)\cdot zdt$. It follows from (3.4) that

$$|H(x,z)| \leq \varepsilon|z|^{\theta p} + C_\varepsilon|z|^q \quad \text{for all} \quad (x,z) \in \mathbb{R}^N \times \mathbb{R}^2. \tag{3.5}$$

Thus, by using (3.1), (3.4), Lemma 2.2 and (2.2), we obtain for all $(u,v) \in \mathbf{W}$ with $\|(u,v)\| \leq 1$

$$\begin{aligned} I(u,v) &\geq \frac{1}{p}\mathscr{M}(1)\|(u,v)\|^{\theta p} - \lambda\int_{\mathbb{R}^N} \varepsilon(u^2+v^2)^{\theta p/2} + C_\varepsilon(u^2+v^2)^{q/2}\mathrm{d}x - \frac{1}{p_s^*}\|u\|_{p_s^*}^\alpha\|v\|_{p_s^*}^\beta \\ &\geq \frac{1}{p}\mathscr{M}(1)\|(u,v)\|^{\theta p} - \lambda\varepsilon 2^{\theta(p-1)}C_{\theta p}^{\theta p}\|(u,v)\|^{\theta p} \\ &\quad - \lambda C_\varepsilon 2^{\frac{q(p-1)}{p}}C_q^q\|(u,v)\|^q - \frac{1}{p_s^*}\|u\|_{p_s^*}^\alpha\|v\|_{p_s^*}^\beta \\ &\geq \left[\frac{1}{p}\mathscr{M}(1) - \lambda 2^{\theta(p-1)}C_{\theta p}^{\theta p}\varepsilon\right]\|(u,v)\|^{\theta p} \\ &\quad - \lambda C_\varepsilon 2^{\frac{q(p-1)}{p}}C_q^q\|(u,v)\|^q - \frac{C_{p_s^*}^{p_s^*}}{p_s^*}\|(u,v)\|^{p_s^*}. \end{aligned}$$

Now we choose $\varepsilon > 0$ small enough such that $\frac{1}{p}\mathscr{M}(1) - \lambda 2^{\theta(p-1)}C_{\theta p}^{\theta p}\varepsilon > 0$. Taking $\rho := \|(u,v)\| \leq 1$ small enough, we get the desired result because of the fact that $\theta p < q < p_s^*$. □

Lemma 3.2. *For any $\lambda \in \mathbb{R}^+$, there exists $(e_1,e_2) \in C_0^\infty(\mathbb{R}^N)\times C_0^\infty(\mathbb{R}^N)$ independent of λ such that $I(e_1,e_2) < 0$ and $\|(e_1,e_2)\| > \rho_0$, where ρ_0 is the number given in Lemma* 3.1.

Proof. The assumption (M_1) implies that

$$\mathscr{M}(t) \leq \mathscr{M}(1)t^\theta \quad \text{for all} \quad t \geq 1. \tag{3.6}$$

By (H_4), we have $H(x,z) = \int_0^1 H_z(x,tz)\cdot zdt = \int_0^1 g(x,t|z|)|z|^2dt \geq 0$. Thus by $\theta p < p_s^*$, (2.2) and (3.6), we have

$$\begin{aligned} I(tu,tv) &= \frac{1}{p}\mathscr{M}(\|t(u,v)\|^p) - \lambda\int_{\mathbb{R}^N} H(x,tu,tv)\mathrm{d}x - \frac{t^{p_s^*}}{p_s^*}\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\mathrm{d}x \\ &\leq \mathscr{M}(1)t^{\theta p}\|(u,v)\|^{\theta p} - \frac{t^{p_s^*}}{p_s^*}\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\mathrm{d}x \\ &= \mathscr{M}(1)t^{\theta p} - \frac{t^{p_s^*}}{p_s^*}\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\mathrm{d}x \\ &\to -\infty \quad \text{as } t \to \infty, \end{aligned}$$

for $(u,v) \in C_0^\infty(\mathbb{R}^N)\times C_0^\infty(\mathbb{R}^N)$ satisfying $\|(u,v)\| = 1$ and $\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\mathrm{d}x > 0$. The lemma is proved by taking $(e_1,e_2) = T(u,v)$ with $T > 0$ large enough. □

Definition 3.3. *Let X be a real Banach space and $I : X \to \mathbb{R}$ be a functional of class $C^1(X)$. A sequence $\{u_n\}_n \subset X$ is said to be a Palais-Smale sequence of I, (PS) sequence for shortness, if $\{I(u_n)\}_n$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$. The functional I satisfies (PS) condition, if any (PS) sequence admits a convergent subsequence.*

Now we discuss the compactness property for the functional I, given by the (PS) condition at a suitable level. To this aim, we fix $\lambda > 0$ and set

$$c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)), \tag{3.7}$$

where

$$\Gamma = \{\gamma \in C([0,1]; \mathbf{W}) : \gamma(0) = 0,\ \gamma(1) = e\}.$$

Obviously, $c_\lambda > 0$ by Lemma 3.1. Moreover, we have the following result.

Lemma 3.4. *Suppose that M satisfies (M_1) and (M_2) and H satisfies (H_1) and (H_4). Then*

$$\lim_{\lambda \to \infty} c_\lambda = 0,$$

where c_λ is given by (3.7).

Proof. For (e_1, e_2) given by Lemma 3.2, we have $\lim_{t \to \infty} I(te_1, te_2) = -\infty$, then there exists $t_\lambda > 0$ such that $I(t_\lambda e_1, t_\lambda e_2) = \max_{t \geq 0} I(te_1, te_2)$. Hence, by $I'(t_\lambda e_1, t_\lambda e_2) = 0$, we have

$$t_\lambda^{p-1} M(\|t_\lambda(e_1,e_2)\|^p)\|(e_1,e_2)\|^p = \lambda \int_{\mathbb{R}^N} H_u(x, t_\lambda e_1, t_\lambda e_2) e_1 \mathrm{d}x + \lambda \int_{\mathbb{R}^N} H_v(x, t_\lambda e_1, t_\lambda e_2) e_2 \mathrm{d}x + t_\lambda^{p_s^*-1} \int_{\mathbb{R}^N} |e_1|^\alpha |e_2|^\beta \mathrm{d}x.$$

Furthermore,

$$M(\|t_\lambda(e_1,e_2)\|^p)\|t_\lambda(e_1,e_2)\|^p = \lambda \int_{\mathbb{R}^N} H_u(x, t_\lambda e_1, t_\lambda e_2) t_\lambda e_1 \mathrm{d}x + \lambda \int_{\mathbb{R}^N} H_v(x, t_\lambda e_1, t_\lambda e_2) t_\lambda e_2 \mathrm{d}x + t_\lambda^{p_s^*} \int_{\mathbb{R}^N} |e_1|^\alpha |e_2|^\beta \mathrm{d}x. \tag{3.8}$$

In the following we prove that $\{t_\lambda\}_\lambda$ is bounded. Without loss of generality, we assume that $t_\lambda \geq 1$ for all $\lambda > 0$. By (M_1), we obtain that for all $t \geq 1$

$$\mathscr{M}(t) \leq \mathscr{M}(1) t^\theta. \tag{3.9}$$

Combining (3.8) with (3.9) and (M_1), we obtain

$$\begin{aligned}\theta \mathscr{M}(1)\|t_\lambda(e_1,e_2)\|^{\theta p} &\geq \theta \mathscr{M}(\|t_\lambda(e_1,e_2)\|^p) \geq M(\|t_\lambda(e_1,e_2)\|^p)\|t_\lambda(e_1,e_2)\|^p \\ &\geq \lambda t_\lambda \int_{\mathbb{R}^N} H_u(x, t_\lambda e_1, e_2) e_1 \mathrm{d}x + \lambda t_\lambda \int_{\mathbb{R}^N} H_v(x, e_2, t_\lambda e_2) e_2 \mathrm{d}x + t_\lambda^{p_s^*} \int_{\mathbb{R}^N} |e_1|^\alpha |e_2|^\beta \mathrm{d}x \\ &\geq t_\lambda^{p_s^*} \int_{\mathbb{R}^N} |e_1|^\alpha |e_2|^\beta \mathrm{d}x,\end{aligned} \tag{3.10}$$

thanks to $\lambda > 0$ and assumption (H_4). Therefore, we arrive at

$$\theta \mathscr{M}(1)\|(e_1, e_2)\|^{\theta p} \geq t_\lambda^{p_s^* - \theta p} \int_{\mathbb{R}^N} |e_1|^\alpha |e_2|^\beta \mathrm{d}x.$$

This, together with $\theta p < p_s^*$, yields that $\{t_\lambda\}_\lambda$ is bounded.

Fix arbitrarily a sequence $\{\lambda_n\}_n \subset \mathbb{R}^+$ such that $\lambda_n \to \infty$ as $n \to \infty$. Obviously $\{t_{\lambda_n}\}_n$ is bounded. Thus, there exist a subsequence, still denoted by $\{\lambda_n\}_n$, with $\lambda_n \to \infty$, and $t_0 \geq 0$ such that $t_{\lambda_n} \to t_0$. Thus, there exists $C > 0$ such that

$$M(\|t_{\lambda_n}(e_1, e_2)\|^p)\|t_{\lambda_n}(e_1, e_2)\|^p \leq C \quad \text{for all} \quad n.$$

We claim that $t_0 = 0$. If $t_0 > 0$, then the above inequality combined with Lebesgue's dominated convergence theorem and relation (3.8) imply that

$$\lambda_n t_{\lambda_n} \int_{\mathbb{R}^N} H_u(x, t_{\lambda_n} e_1, t_{\lambda_n} e_2) e_1 + H_v(x, t_{\lambda_n} e_1, t_{\lambda_n} e_2) e_2 \mathrm{d}x$$

$$+ t_{\lambda_n}^{p_s^*} \int_{\mathbb{R}^N} |e_1|^\alpha |e_2|^\beta \mathrm{d}x \to \infty \leq C, \quad \text{as } n \to \infty,$$

which is impossible, consequently $t_0 = 0$. Thus, we obtain $t_\lambda \to 0$ as $\lambda \to \infty$.

Let $\overline{\gamma}(t) = t(e_1, e_2)$. Clearly $\overline{\gamma} \in \Gamma$, thus

$$0 < c_\lambda \leq \max_{t \geq 0} I(\overline{\gamma}(t)) = I(t_\lambda e_1, t_\lambda e_2) \leq \frac{1}{p} \mathscr{M}(\|t_\lambda(e_1, e_2)\|^p).$$

Then the desired assertion follows immediately from the fact that $\mathscr{M}(\|t_\lambda(e_1, e_2)\|^p) \to 0$ as $\lambda \to \infty$, by the continuity of $\mathscr{M}$. ☐

Lemma 3.5. *Let* (H_1)*–*(H_4) *hold and suppose that* M *satisfies* (M_1) *and* (M_2)*. Then there exists* $\lambda^* > 0$ *such that for all* $\lambda > \lambda^*$*,* I *satisfies the* $(\mathrm{PS})_{c_\lambda}$ *condition on* $\mathbf{W}$*.*

Proof. For any sequence $\{(u_n, v_n)\}_n \subset \mathbf{W}$ such that $I(u_n, v_n)$ is bounded and $I'(u_n, v_n) \to 0$ as $n \to \infty$, there exists $C > 0$ such that $|\langle I'(u_n, v_n), (u_n, v_n)\rangle| \leq C\|(u_n, v_n)\|$ and $|I(u_n, v_n)| \leq C$. Two possible cases can arise: either $\inf_n \|(u_n, v_n)\|_W = d > 0$ or $\inf_n \|(u_n, v_n)\|_W = 0$, so that we distinguish the following two situations.

Case 1. $\inf\limits_n \|(u_n, v_n)\| = d > 0$. We begin by proving that $\{(u_n, v_n)\}_n$ is bounded. Denote by $\kappa = \kappa(d)$ the number corresponding to $\sigma = d^p$ in (M_1), so that

$$M(\|(u_n, v_n)\|^p) \geq \kappa \quad \text{for all} \quad n. \tag{3.11}$$

By (M_2), (3.11) and (H_3), we get

$$\begin{aligned} C + C\|(u_n, v_n)\| &\geq I(u_n, v_n) - \frac{1}{\mu}\langle I'(u_n, v_n), (u_n, v_n)\rangle \\ &= \frac{1}{p}\mathscr{M}(\|(u_n, v_n)\|^p) - \frac{1}{\mu} M(\|(u_n, v_n)\|^p)\|(u_n, v_n)\|^p \\ &\quad - \lambda \int_{\mathbb{R}^N} \left(H(x, u_n, v_n) - \frac{1}{\mu} H_u(x, u_n, v_n) u_n - \frac{1}{\mu} H_v(x, u_n, v_n) v_n \right) \mathrm{d}x \\ &\quad + \left(\frac{1}{\mu} - \frac{1}{p_s^*} \right) \int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta \mathrm{d}x \\ &\geq \left(\frac{1}{\theta p} - \frac{1}{\mu} \right) \kappa \|(u_n, v_n)\|^p. \end{aligned} \tag{3.12}$$

Hence we conclude from $1 < p$ that $\{(u_n, v_n)\}_n$ is bounded in $\mathbf{W}$.

Furthermore, by the Hölder inequality, we obtain

$$\int_{\mathbb{R}^N} \left||u_n|^{\alpha-1}|v_n|^{\beta}\right|^{\frac{p_s^*}{p_s^*-1}} \mathrm{d}x \le \|u_n\|_{p_s^*}^{\frac{p_s^*(\alpha-1)}{p_s^*-1}} \|v_n\|_{p_s^*}^{\frac{p_s^*(p_s^*-\alpha)}{p_s^*-1}} \le C_{p_s^*}^{p_s^*}[u_n]_{s,p}^{\frac{p_s^*(\alpha-1)}{p_s^*-1}}[v_n]^{\frac{p_s^*(p_s^*-\alpha)}{p_s^*-1}} \le C.$$

Similarly,

$$\int_{\mathbb{R}^N} \left||v_n|^{\beta-1}|u_n|^{\alpha}\right|^{\frac{p_s^*}{p_s^*-1}} \mathrm{d}x \le C.$$

By applying the boundedness of $\{(u_n,v_n)\}_n$ in $\mathbf{W}$ and [6], (Thm. 4.9), there exist $(u_\lambda,v_\lambda)\in\mathbf{W}$ and $\alpha_\lambda\ge 0$ such that up to a subsequence, still denoted by $\{(u_n,v_n)\}_n$, we have

$$\begin{aligned}
&(u_n,v_n)\rightharpoonup(u_\lambda,v_\lambda)\quad \text{weakly in}\quad \mathbf{W},\\
&(u_n,v_n)\to(u_\lambda,v_\lambda)\quad \text{a.e. in}\quad \mathbb{R}^N,\\
&\|(u_n,v_n)\|\to\alpha_\lambda,\ \int_{\mathbb{R}^N}|u_n-u_\lambda|^{\alpha}|v_n-v_\lambda|^{\beta}\mathrm{d}x\to\delta_\lambda,\\
&|u_n|^{\alpha-2}u_n|v_n|^{\beta}\rightharpoonup|u_\lambda|^{\alpha-2}u_\lambda|v_\lambda|^{\eta}\quad \text{weakly in}\quad L^{(p_s^*)'}(\mathbb{R}^N)\\
&|v_n|^{\beta-2}v_n|u_n|^{\alpha}\rightharpoonup|v_\lambda|^{\beta-2}v_\lambda||u_\lambda|^{\alpha}|^{\eta}\quad \text{weakly in}\quad L^{(p_s^*)'}(\mathbb{R}^N),
\end{aligned}\tag{3.13}$$

where $(p_s^*)'=p_s^*/(p_s^*-1)$. By using (3.4), for any $\varepsilon>0$, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} H_u(x,u_n,v_n)(u_n-u)+H_v(x,u_n,v_n)(v_n-u)\mathrm{d}x\right|\\
&\le\varepsilon\int_{\mathbb{R}^N}|(u_n,v_n)|^{\theta p-1}|(u_n,v_n)-(u_\lambda,v_\lambda)|+C_\varepsilon|(u_n,v_n)|^{q-1}|(u_n,v_n)-(u_\lambda,v_\lambda))|\mathrm{d}x\\
&\le C\varepsilon+CC_\varepsilon\|(u_n,v_n)-(u_\lambda,v_\lambda))\|_{L^q(\mathbb{R}^N)}.
\end{aligned}$$

Since $p\le\theta p<q<p_s^*$, we deduce by Theorem 2.3 that

$$\limsup_{n\to\infty}\left|\int_{\mathbb{R}^N} H_u(x,u_n,v_n)(u_n-u)+H_v(x,u_n,v_n)(v_n-u)\mathrm{d}x\right|\le C\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, we deduce that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} H_u(x,u_n,v_n)(u_n-u)+H_v(x,u_n,v_n)(v_n-u)\mathrm{d}x=0.\tag{3.14}$$

Next we show $\lim_{\lambda\to\infty}\alpha_\lambda=0$. It follows from $\inf_{n\ge1}\|(u_n,v_n)\|_{s,p}=d>0$ that $\alpha_\lambda>0$. Hence $M(\|(u_n,v_n)\|^p)\to M(\alpha_\lambda^p)>0$ as $n\to\infty$, by the continuity of M. We claim that $\lim_{\lambda\to\infty}\alpha_\lambda=0$. Otherwise, there exists sequence λ_k, with $\lambda_k\to\infty$ as $k\to\infty$, such that $\alpha_{\lambda_k}\to\alpha_0>0$ as $k\to\infty$. Note that

$$c_{\lambda_k}=\lim_{n\to\infty}\left(I(u_n,v_n)-\frac{1}{\mu}\langle I'(u_n,v_n),(u_n,v_n)\rangle\right).$$

A similar discussion as in (3.12) gives that

$$c_{\lambda_k}\ge\left(\frac{1}{\theta p}-\frac{1}{\mu}\right)M(\alpha_{\lambda_k}^p)\alpha_{\lambda_k}^p.$$

Letting $k \to \infty$ in above inequality and using Lemma 3.3, we get

$$0 \geq \left(\frac{1}{\theta p} - \frac{1}{\mu}\right) M(\alpha_0^p)\alpha_0^p > 0,$$

which is impossible. Thus, we obtain that

$$\lim_{\lambda\to\infty} \alpha_\lambda = 0. \tag{3.15}$$

By $(u_n, v_n) \rightharpoonup (u_\lambda, v_\lambda)$ weakly in $\mathbf{W}$, we get $\|(u_\lambda, v_\lambda)\| \leq \lim_{n\to\infty} \|(u_n, v_n)\| = \alpha_\lambda$, this together with (3.15) and (2.1) gives that

$$\lim_{\lambda\to\infty} \|(u_\lambda, v_\lambda)\|_{p_s^*} = \lim_{\lambda\to\infty} \|(u_\lambda, v_\lambda)\| = 0. \tag{3.16}$$

Now we introduce a simple notation. Let $(\varphi, \psi) \in \mathbf{W}$ be fixed and denote by $L(\varphi, \psi)$ the linear functional on $\mathbf{W}$ defined by

$$\begin{aligned}
&\langle L(\varphi, \psi), (\omega_1, \omega_2)\rangle \\
&= \iint_{\mathbb{R}^{2N}} |\varphi(x) - \varphi(y)|^{p-2}(\varphi(x) - \varphi(y))(\omega_1(x) - \omega_2(y))K(x-y)\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(x)|u|^{p-2}u\omega_1\mathrm{d}x \\
&\quad + \iint_{\mathbb{R}^{2N}} |\psi(x) - \psi(y)|^{p-2}(\psi(x) - \psi(y))(\omega_2(x) - \omega_2(y))K(x-y)\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(x)|v|^{p-2}v\omega_2\mathrm{d}x
\end{aligned}$$

for all $(\omega_1, \omega_2) \in \mathbf{W}$. Evidently, by the Hölder inequality, $L(\varphi, \psi)$ is also continuous, being

$$\begin{aligned}
|\langle L(\varphi, \psi), (\omega_1, \omega_2)\rangle| &\leq [\varphi]_{s,p}^{p-1}[\omega_1]_{s,p} + \|\varphi\|_{p,V}^{p-1}\|\omega_1\|_{p,V} + [\psi]_{s,p}^{p-1}[\omega_2]_{s,p} + \|\psi\|_{p,V}^{p-1}\|\omega_2\|_{p,V} \\
&\leq \left([\varphi]_{s,p}^{p-1} + \|\varphi\|_{p,V}^{p-1} + [\psi]_{s,p}^{p-1} + \|\psi\|_{p,V}^{p-1}\right)\|(\omega_1, \omega_2)\|.
\end{aligned}$$

Hence, the weak convergence of $\{(u_n, v_n)\}_n$ in $\mathbf{W}$ gives that

$$\lim_{n\to\infty} \langle L(u_\lambda, v_\lambda), (u_n - u_\lambda, v_n - v_\lambda)\rangle = 0. \tag{3.17}$$

Moreover, $\{L(u_n, v_n)\}_n$ is bounded in $\mathbf{W}^*$, where $\mathbf{W}^*$ denotes the dual space of $\mathbf{W}$. Furthermore, there exists a subsequence of $\{(u_n, v_n)\}_n$ still denoted by $\{(u_n, v_n)\}_n$ and some functional ξ such that

$$L(u_n, v_n) \rightharpoonup \xi \quad \text{weakly in} \quad \mathbf{W}^*,$$

that is,

$$\lim_{n\to\infty} \langle L(u_n, v_n), (\omega_1, \omega_2)\rangle = \langle \xi, (\omega_1, \omega_2)\rangle,$$

for all $(\omega_1, \omega_2) \in \mathbf{W}$. In particular, we have

$$\lim_{n\to\infty} \langle L(u_n, v_n), (u_\lambda, v_\lambda)\rangle = \langle \xi, (u_\lambda, v_\lambda)\rangle.$$

Furthermore, by (3.13) and $I'(u_n, v_n) \to 0$, we get

$$\begin{aligned}
M(\alpha_\lambda^p)\langle \xi, (\omega_1, \omega_2)\rangle &= \lambda \int_{\mathbb{R}^N} H_u(x, u_\lambda, v_\lambda)\omega_1 + H_v(x, u_\lambda, v_\lambda)\omega_2\mathrm{d}x + \frac{\alpha}{p_s^*}\int_{\mathbb{R}^N} |u_\lambda|^{\alpha-2}u_\lambda|v_\lambda|^\beta\omega_1\mathrm{d}x \\
&\quad + \frac{\beta}{p_s^*}\int_{\mathbb{R}^N} |v_\lambda|^{\beta-2}v_\lambda|u_\lambda|^\alpha\omega_2\mathrm{d}x,
\end{aligned} \tag{3.18}$$

for all $(\omega_1, \omega_2) \in \mathbf{W}$. Taking $(\omega_1, \omega_2) = (u_\lambda, v_\lambda)$ in (3.18), we obtain

$$M(\alpha_\lambda^p)\langle \xi, (u_\lambda, v_\lambda)\rangle = \lambda \int_{\mathbb{R}^N} H_u(x, u_\lambda, v_\lambda) u_\lambda \mathrm{d}x + \lambda \int_{\mathbb{R}^N} H_v(x, u_\lambda, v_\lambda) v_\lambda \mathrm{d}x + \int_{\mathbb{R}^N} |u_\lambda|^\alpha |v_\lambda|^\beta \mathrm{d}x,$$

which together with (H_2) and (M_2) implies that

$$\langle \xi, (u_\lambda, v_\lambda)\rangle \geq 0. \tag{3.19}$$

Since $\{(u_n, v_n)\}_n$ is a (PS) sequence, we deduce from Lemma 2.4 and (3.14) that

$$\begin{aligned}
o(1) &= \langle I'(u_n, v_n) - I'_{\alpha_\lambda}(u_\lambda, v_\lambda), (u_n, v_n) - (u_\lambda, v_\lambda)\rangle \\
&= M(\|(u_n, v_n)\|^p)\|(u_n, v_n)\|^p + M(\alpha_\lambda^p)\|(u_\lambda, v_\lambda)\|^p \\
&\quad - \langle L(u_n, v_n), (u_\lambda, v_\lambda)\rangle M(\|(u_n, v_n)\|^p) - \langle L(u_\lambda, v_\lambda), (u_n, v_n)\rangle M(\alpha_\lambda^p) \\
&\quad - \lambda \int_{\mathbb{R}^N} [H_u(x, u_n, v_n) - H_u(x, u_\lambda, v_\lambda)](u_n - u_\lambda)\mathrm{d}x \\
&\quad - \lambda \int_{\mathbb{R}^N} [H_v(x, u_n, v_n) - H_v(x, u_\lambda, v_\lambda)](v_n - v_\lambda)\mathrm{d}x \\
&\quad - \frac{\alpha}{p_s^*} \int_{\mathbb{R}^N} |v_n|^\beta (|u_n|^{\alpha-2}u_n - |u_\lambda|^{\alpha-2}u_\lambda)(u_n - u_\lambda)\mathrm{d}x \\
&\quad - \frac{\beta}{p_s^*} \int_{\mathbb{R}^N} |u_n|^\alpha (|v_n|^{\beta-2}v_n - |v_\lambda|^{\beta-2}v_\lambda)(v_n - v_\lambda)\mathrm{d}x \\
&= M(\alpha_\lambda^p)[\alpha_\lambda^p - \langle \xi, (u_\lambda, v_\lambda)\rangle] - \int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta \mathrm{d}x + \int_{\mathbb{R}^N} |u_\lambda|^\alpha |v_\lambda|^\beta \mathrm{d}x + o(1) \\
&= M(\alpha_\lambda^p)[\alpha_\lambda^p - \langle \xi, (u_\lambda, v_\lambda)\rangle] - \int_{\mathbb{R}^N} |u_n - u_\lambda|^\alpha |v_n - v_\lambda|^\beta \mathrm{d}x + o(1) \\
&= M(\alpha_\lambda^p)\langle L(u_n, v_n) - L(u_\lambda, v_\lambda), (u_n, v_n) - (u_\lambda, v_\lambda)\rangle - \int_{\mathbb{R}^N} |u_n - u_\lambda|^\alpha |v_n - v_\lambda|^\beta \mathrm{d}x + o(1),
\end{aligned} \tag{3.20}$$

where I_{α_λ} is defined as follows:

$$I_{\alpha_\lambda}(u_\lambda, v_\lambda) = \frac{1}{p} M(\alpha_\lambda^p)\|(u_\lambda, v_\lambda)\|^p - \lambda \int_{\mathbb{R}^N} H(x, u_\lambda, v_\lambda)\mathrm{d}x - \frac{1}{p_s^*} \int_{\mathbb{R}^N} |u_\lambda|^\alpha |v_\lambda|^\beta \mathrm{d}x.$$

Here we use the following facts:

$$\begin{aligned}
|u_\lambda|^{\alpha-2}u_\lambda(u_n - u_\lambda) \to 0 \quad &\text{strongly in} \quad L^{\frac{p_s^*}{\alpha}}(\mathbb{R}^N), \\
|v_\lambda|^{\alpha-2}v_\lambda(v_n - v_\lambda) \to 0 \quad &\text{strongly in} \quad L^{\frac{p_s^*}{\beta}}(\mathbb{R}^N),
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
&\lim_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^\beta |u_\lambda|^{\alpha-2}u_\lambda(u_n - u_\lambda)\mathrm{d}x = 0, \\
&\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^\alpha |v_\lambda|^{\beta-2}v_\lambda(v_n - v_\lambda)\mathrm{d}x = 0.
\end{aligned} \tag{3.22}$$

Now we give a proof of (3.21). Notice that for any measurable subset $U \subset \mathbb{R}^N$, we have

$$\begin{aligned}\int_U ||u_\lambda|^{\alpha-2}u_\lambda(u_n-u_\lambda)|^{p_s^*/\alpha}\mathrm{d}x &\le \left(\int_U |u_\lambda|^{p_s^*}\mathrm{d}x\right)^{(\alpha-1)/(\alpha p_s^*)} \|u_n-u_\lambda\|_{p_s^*}^{(\alpha-1)/\alpha}\\ &\le C\left(\int_U |u_\lambda|^{p_s^*}\mathrm{d}x\right)^{(\alpha-1)/(\alpha p_s^*)},\end{aligned}$$

and

$$\begin{aligned}\int_U |v_\lambda|^{\beta-2}v_\lambda(v_n-v_\lambda)|^{p_s^*/\beta}\mathrm{d}x &\le \left(\int_U |v_\lambda|^{p_s^*}\mathrm{d}x\right)^{(\beta-1)/(\beta p_s^*)} \|v_n-v_\lambda\|_{p_s^*}^{(\beta-1)/\beta}\\ &\le C\left(\int_U |v_\lambda|^{p_s^*}\mathrm{d}x\right)^{(\beta-1)/(\beta p_s^*)}.\end{aligned}$$

It follows from $(u_\lambda, v_\lambda) \in L^{p_s^*}(\mathbb{R}^N) \times L^{p_s^*}(\mathbb{R}^N)$ that $|u_\lambda|^{\alpha-2}u_\lambda(u_n-u_\lambda)|^{\frac{p_s^*}{\alpha}}$ and $|v_\lambda|^{\beta-2}v_\lambda(v_n-v_\lambda)|^{\frac{p_s^*}{\beta}}$ are equi-integrable in $\mathbb{R}^N$. Clearly, $|u_\lambda|^{\alpha-2}u_\lambda(u_n-u_\lambda) \to 0$ and $|v_\lambda|^{\beta-2}v_\lambda(v_n-v_\lambda) \to 0$ a.e. in $\mathbb{R}^N$. Hence (3.21) follows from the Vitali convergence theorem.

Since $\{|v_n|^\beta\}_n$ is bounded in $L^{p_s^*/\beta}(\mathbb{R}^N)$, we obtain by (3.21), $\alpha+\beta = p_s^*$ and the Hölder inequality that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |v_n|^\beta |u_\lambda|^{\alpha-2}u_\lambda(u_n-u_\lambda)\mathrm{d}x = 0.$$

Similarly,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |u_n|^\alpha |v_\lambda|^{\beta-2}v_\lambda(v_n-v_\lambda)\mathrm{d}x = 0.$$

Therefore, (3.22) holds.

It follows from (3.20) that

$$\begin{aligned}&M(\alpha_\lambda^p)\lim_{n\to\infty}\langle L(u_n,v_n)-L(u_\lambda,v_\lambda),(u_n,v_n)-(u_\lambda,v_\lambda)\rangle\\ &=\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n-u_\lambda|^\alpha|v_n-v_\alpha|^\beta\mathrm{d}x.\end{aligned}\tag{3.23}$$

Applying (3.20) and (3.23), we deduce that

$$\delta_\lambda = \lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n-u_\alpha|^\alpha|v_n-v_\alpha|^\beta\mathrm{d}x = M(\alpha_\lambda^p)[\alpha_\lambda^p - \langle\xi,(u_\lambda,v_\lambda)\rangle] \le M(\alpha_\lambda^p)\alpha_\lambda^p,$$

this together with (3.15) implies that$\lim_{\lambda\to\infty}\delta_\lambda = 0$.

Let us now recall the well–known Simon inequalities:

$$|\xi-\eta|^p \le \begin{cases} C_p\left(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta\right)\cdot(\xi-\eta) & \text{for}\quad p\ge 2\\ \widetilde{C}_p\left[\left(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta\right)\cdot(\xi-\eta)\right]^{p/2}\left(|\xi|^p+|\eta|^p\right)^{(2-p)/2} & \text{for}\quad 1<p<2,\end{cases}\tag{3.24}$$

for all $\xi, \eta \in \mathbb{R}^N$, where C_p and $\widetilde{C}_p$ are positive constants depending only on p.

According to the Simon inequalities (3.24), we divide the discussion into two cases. We first consider the case $p \geq 2$. By (3.24) and (2.2), we have

$$\begin{aligned}\langle L(u_n, v_n) - L(u_\lambda, v_\lambda), (u_n, v_n) - (u_\lambda - v_\lambda)\rangle &\geq \frac{1}{C_p}\|(u_n, v_n) - (u_\lambda - v_\lambda)\|^p \\ &\geq \frac{C_{p_s^*}^{-p}}{C_p}\left(\int_{\mathbb{R}^N} |u_n - u_\lambda|^\alpha |v_n - v_\lambda|^\beta \mathrm{d}x\right)^{p/p_s^*}. \end{aligned} \tag{3.25}$$

Combining (3.25) with (3.13) and (3.23), we get

$$\delta_\lambda \geq \frac{C_{p_s^*}^{-1}}{C_p} M(\alpha_\lambda^p)\delta_\lambda^{\frac{p}{p_s^*}}. \tag{3.26}$$

Define

$$\lambda^* = \sup\{\lambda > 0 : \delta_\lambda > 0\}.$$

Now we claim that $\lambda^* < \infty$. Otherwise there exists a sequence $\{\lambda_k\}_k$, with $\lambda_k \to \infty$ as $k \to \infty$, such that $\delta_{\lambda_k} > 0$ and $\alpha_{\lambda_k} \in (0, 1]$, thanks to $\alpha_\lambda \to 0$ as $\lambda \to \infty$. Using (3.20) again, we obtain

$$M(\alpha_{\lambda_k}^p)[\alpha_{\lambda_k}^p - \langle \xi, (u_{\lambda_k}, v_{\lambda_k})\rangle] = \delta_{\lambda_k},$$

which together with (3.26) implies that

$$\left[M(\alpha_{\lambda_k}^p)(\alpha_{\lambda_k}^p - \langle \xi, (u_{\lambda_k}, v_{\lambda_k}\rangle)\right]^{\frac{p_s^*-p}{p_s^*}} = (\delta_{\lambda_k})^{\frac{p_s^*-p}{p_s^*}} \geq \frac{C_{p_s^*}^{-1}}{C_p} M(\alpha_{\lambda_k}^p). \tag{3.27}$$

By (M_3), we have $M(t) \geq m_0 t^{\theta_1}$ for all $t \in [0, 1]$. Therefore, we conclude from (3.27), $\langle \xi, u_\lambda\rangle \geq 0$ and $\{\alpha_{\lambda_k}\}_k \subset (0, 1]$ that

$$\alpha_{\lambda_k}^{\frac{p(p_s^*-p)}{p_s^*}} \geq (\alpha_{\lambda_k}^p - \langle \xi, (u_{\lambda_k}, v_{\lambda_k})\rangle)^{\frac{(p_s^*-p)}{p_s^*}} \geq \frac{C_{p_s^*}^{-1}}{C_p}(M(\alpha_{\lambda_k}^p))^{\frac{p}{p_s^*}} \geq \frac{C_{p_s^*}^{-1}}{C_p}(m_0)^{\frac{p}{p_s^*}}\alpha_{\lambda_k}^{\frac{\theta_1 p^2}{p_s^*}}.$$

Hence, we obtain

$$\alpha_{\lambda_k}^{\frac{p}{p_s^*}(p_s^*-(\theta_1+1)p)} \geq \frac{C_{p_s^*}^{-1}}{C_p}(m_0)^{\frac{p}{p_s^*}},$$

thanks to $\alpha_k > 0$ for all k. This contradicts with (3.15), since $0 < \theta_1 < (p_s^* - p)/p$ by assumption (M_3). In conclusion, the assertions is proved.

Thus, for all $\lambda \geq \lambda^*$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |u_n - u_\lambda|^\alpha |v_n - v_\lambda|^\beta \mathrm{d}x = 0,$$

this together with (3.20) gives that $(u_n, v_n) \to (u_\lambda, v_\lambda)$ strongly in $\mathbf{W}$ as $n \to \infty$.

Finally, it remains to consider the case $1 < p < 2$. Now by (3.24) and the Hölder inequality

$$\begin{aligned}&\|(u_n, v_n) - (u_{\lambda,\mu}, v_\lambda)\|^p \\ &\leq \widetilde{C}_p\left[\langle L(u_n, v_n) - L(u_\lambda, v_\lambda), (u_n, v_n) - (u_\lambda, v_\lambda)\rangle\right]^{p/2}\left(\|(u_n, v_n)\|^p + \|(u_\lambda, v_\lambda)\|^p\right)^{(2-p)/2} \\ &\leq \widetilde{C}_p\left[\langle L(u_n, v_n) - L(u_\lambda, v_\lambda), (u_n, v_n) - (u_\lambda, v_\lambda)\rangle\right]^{p/2}\left(\|(u_n, v_n)\|^{p(2-p)/2} + \|(u_\lambda, v_\lambda)\|^{p(2-p)/2}\right) \\ &\leq C[\langle L(u_n, v_n) - L(u_\lambda, v_\lambda), (u_n, v_n) - (u_\lambda, v_\lambda)\rangle]^{p/2}, \end{aligned} \tag{3.28}$$

where C is a positive constant. Combining (4.9) with (3.13) and (3.23), we have

$$\delta_\lambda^{p_s^*} \geq C_{p_s^*}^{-2p_s^*} C^{-2p_s^*/p} M^{p_s^*}(\alpha_\lambda^p)\delta_\lambda^2.$$

Now we claim that there exists $\lambda^* > 0$, which defined similarly as above, such that $\delta_\lambda = 0$ for all $\lambda \geq \lambda^*$. Otherwise there exist sequences $\{\lambda_k\}_k$, with $\lambda_k \to \infty$ as $k \to \infty$, such that $\delta_{\lambda_k} > 0$ and $\alpha_{\lambda_k} \in (0,1]$, thanks to $\alpha_\lambda \to 0$ as $\lambda \to \infty$. Using (3.20), we obtain

$$M(\alpha_{\lambda_k}^p)(\alpha_{\lambda_k}^p - \langle \xi, (u_{\lambda_k}, v_{\lambda_k})\rangle) = \delta_{\lambda_k}.$$

Similar to the case $p \geq 2$, we have

$$\alpha_{\lambda_k}^{p(p_s^*-2\theta_1-2)} \geq C_{p_s^*}^{2p_s^*} C^{2p_s^*/p}(m_0)^2,$$

which contradicts with the fact $\alpha_\lambda \to 0$ as $\lambda \to \infty$, since $\theta_1 < (p_s^*-2)/2$ by assumption. A similar discussion as the case $p \geq 2$ gives that $(u_n, v_n) \to (u_\lambda, v_\lambda)$ strongly in $\mathbf{W}$. In conclusion, we get $(u_n, v_n) \to (u_\lambda, v_\lambda)$ in $\mathbf{W}$ as $n \to \infty$.

Case 2. $\inf\limits_n \|(u_n, v_n)\| = 0$. Either 0 is an accumulation point of the sequence $\{(u_n, v_n)\}_n$ and so there exists a subsequence of $\{(u_n, v_n)\}_n$ strongly converging to $(u_\lambda, v_\lambda) = (0,0)$, or (0,0) is an isolated point of the sequence $\{(u_n, v_n)\}_n$ and so there exists a subsequence, still denoted by $\{(u_n, v_n)\}_n$, such that $\inf_n \|(u_n, v_n)\| > 0$. In the first case we are done, while in the latter case we can proceed as Case 1. □

Proof of Theorem 1.2. By Lemma 3.1, Lemma 3.2 and Lemma 3.5 we know that I satisfies all assumptions in Theorem 2.5. Then for all $\lambda \geq \lambda_* > 0$, there exists $(u_\lambda, v_\lambda) \in \mathbf{W}$ such that (u_λ, v_λ) is a solution of system ($\mathcal{S}$) by Theorem 2.5. Furthermore, $\lim_{\lambda\to\infty} \|(u_\lambda, v_\lambda)\| = 0$. □

4. Proof of Theorem 1.3

Let

$$\mathcal{I}(u,v) = \frac{1}{p}\mathscr{M}(\|(u,v)\|^2) - \int_{\mathbb{R}^N} h(x)f(u,v)\mathrm{d}x - \frac{\lambda}{p_s^*}\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \mathrm{d}x,$$

for all $(u,v) \in \mathbf{W}$.

By (M_1) and (M_3), (V_1) and (f_1), we deduce that $\mathcal{I}$ is of class C^1 and a critical point of $\mathcal{I}$ is a weak solution of system ($\mathcal{S}'$).

Lemma 4.1. *For any $\lambda \in (0,1)$, there exist $\alpha_1, \rho_1 > 0$ such that $I(u,v) \geq \alpha_1 > 0$ for any $(u,v) \in \mathbf{W}$, with $\|(u,v)\| = \rho_1$.*

Proof. By using (3.1), Lemma 2.2 and (2.2), we obtain for all $(u,v) \in \mathbf{W}$ with $\|(u,v)\| \leq 1$

$$\begin{aligned}\mathcal{I}(u,v) &\geq \frac{1}{p}\mathscr{M}(1)\|(u,v)\|^{\theta p} - C\int_{\mathbb{R}^N} h(x)|(u,v)|^q \mathrm{d}x - \frac{\lambda}{p_s^*}\|u\|_{p_s^*}^\alpha \|v\|_{p_s^*}^\beta \\ &\geq \frac{1}{p}\mathscr{M}(1)\|(u,v)\|^{\theta p} - C\|h\|_{\frac{p_s^*}{p_s^*-q}} \|(u,v)\|_{p_s^*}^q - \frac{\lambda}{p_s^*} C_{p_s^*}^{p_s^*}\|(u,v)\|^{p_s^*}.\end{aligned}$$

By the Young inequality, we have for any $\varepsilon > 0$

$$CC_{p_s^*}^q \|h\|_{\frac{p_s^*}{p_s^*-q}} \|(u,v)\|^q \leq \varepsilon\|(u,v)\|^{\theta p} + \varepsilon^{-\frac{q}{\theta p - q}}\left(CC_{p_s^*}^q\|h\|_{\frac{p_s^*}{p_s^*-q}}\right)^{\frac{\theta p}{\theta p - q}},$$

thanks to $q < \theta p$. It follows that

$$\mathcal{I}(u,v) \geq \left(\frac{1}{p}\mathscr{M}(1) - \varepsilon\right)\|(u,v)\|^{\theta p} - \varepsilon^{-\frac{q}{\theta p - q}}\left(CC_{p_s^*}^q \|h\|_{\frac{p_s^*}{p_s^* - q}}\right)^{\frac{\theta p}{\theta p - q}} - \lambda \frac{C_{p_s^*}^{p_s^*}}{p_s^*}\|(u,v)\|^{p_s^*}.$$

Now we choose $\varepsilon = \frac{\mathscr{M}(1)}{2p}$. Then,

$$\mathcal{I}(u,v) \geq \frac{1}{2p}\mathscr{M}(1)\|(u,v)\|^{\theta p} - \left(\frac{\mathscr{M}(1)}{2p}\right)^{-\frac{q}{\theta p - q}}\left(CC_{p_s^*}^q \|h\|_{\frac{p_s^*}{p_s^* - q}}\right)^{\frac{\theta p}{\theta p - q}} - \frac{C_{p_s^*}^{p_s^*}}{p_s^*}\|(u,v)\|^{p_s^*},$$

since $\lambda \in (0,1)$. Let $\eta(t) := \frac{1}{2p}\mathscr{M}(1)t^{\theta p} - \frac{C_{p_s^*}^{p_s^*}}{p_s^*}t^{p_s^*}$ for all $t \geq 0$. Clearly, by $p_s^* > \theta p$, there exists $\mu_0 = (\frac{\theta \mathscr{M}(1)}{2C_{p_s^*}^{p_s^*}})^{\frac{1}{p_s^* - \theta p}}$ such that

$$\eta(\mu_0) = \max_{t \geq 0}\eta(t) = \mathscr{M}(1)^{\frac{p_s^*}{p_s^* - \theta p}}\left(\frac{\theta}{2C_{p_s^*}^{p_s^*}}\right)^{\frac{\theta p}{p_s^* - \theta p}}\left(\frac{1}{2p} - \frac{\theta}{2p_s^*}\right) > 0.$$

Taking $\rho_1 := \|(u,v)\| = \mu_0 > 0$, we obtain from condition (h_1)

$$\mathcal{I}(u,v) \geq \eta(\mu_0) - \left(\frac{\mathscr{M}(1)}{2p}\right)^{-\frac{q}{\theta p - q}}\left(CC_{p_s^*}^q \|h\|_{\frac{p_s^*}{p_s^* - q}}\right)^{\frac{\theta p}{\theta p - q}} > 0,$$

Therefore, the proof is finished. □

Lemma 4.2. *Set*

$$c_\lambda := \inf\left\{\mathcal{I}(u,v) : (u,v) \in \overline{B_{\rho_1}}\right\},$$

where $B_{\rho_1} = \{(u,v) \in \mathbf{W} : \|(u,v)\| < \rho_1\}$ and $\rho_1 > 0$ defined in Lemma 4.1. Then $c_\lambda < 0$ for all $\lambda \in (0,1)$.

Proof. Let $x_0 \in \Omega$, $0 < R < 1$ such that $B_{2R}(x_0) \subset \Omega$. Choose $\varphi, \psi \in C_0^\infty(B_{2R}(x_0))$ satisfying $0 \leq \varphi, \psi \leq 1$, $\|(\varphi,\psi)\| \leq C(R)$, and $\int_{B_R(x_0)} |(\varphi,\psi)|^{q_1}\mathrm{d}x > 0$, where $C(R) > 0$ only depending on R. Then for all $t \in (0,1)$ small enough, we have by (f_2) and (H_2)

$$\begin{aligned}
I(t\varphi, t\psi) &= \frac{1}{p}\mathscr{M}(\|t(\varphi,\psi)\|^p) - \int_\Omega h(x)f(t\varphi,t\psi)\mathrm{d}x - \frac{t^{p_s^*}\lambda}{p_s^*}\int_\Omega |\varphi|^\alpha|\psi|^\beta \mathrm{d}x \\
&\leq \frac{1}{p}\left(\max_{\xi\in[0,C(R)]} M(\xi)\right)t^p\|(\varphi,\psi)\|^p - \int_\Omega h(x)f(t\varphi,t\psi)\mathrm{d}x \\
&\leq \frac{1}{p}\left(\max_{\xi\in[0,C(R)]} M(\xi)\right)t^p\|(\varphi,\psi)\|^p - t^q a_0 \inf_{x\in\Omega} h(x)\int_\Omega |(\varphi,\psi)|^{q_1}\mathrm{d}x \\
&= C(R)\left(\frac{1}{p}\max_{\xi\in[0,C(R)]} M(\xi)\right)t^p - a_0\int_\Omega |(\varphi,\psi)|^q\mathrm{d}x\left(\inf_{x\in\Omega} h(x)\right)t^{q_1} \\
&< 0,
\end{aligned}$$

thanks to $1 < q_1 < p$. Hence the lemma is proved. □

By Lemmas 4.1–4.2 and the Ekeland variational principle, there exists a sequence $(u_n, v_n) \subset B_{\rho_1}$ such that $c_\lambda \leq \mathcal{I}(u_n,v_n) \leq c_\lambda + \frac{1}{n}$ and $\mathcal{I}(u,v) \geq \mathcal{I}(u_n,v_n) - \frac{1}{n}\|(u,v) - (u_n,v_n)\|$for all $(u,v) \in \overline{B_{\rho_1}}$. Then a standard procedure gives that (u_n,v_n) is a bounded $(PS)_{c_\lambda}$ sequence of $\mathcal{I}$.

Lemma 4.3. *There exists* $\lambda^{**} > 0$ *such that, up to a subsequence,* $\{(u_n, v_n)\}_n$ *strongly converges to some* (u_λ, v_λ) *in* $\mathbf{W}$ *for all* $\lambda \in (0, \lambda^{**})$.

Proof. Since $\{(u_n, v_n)\}_n \subset B_{\rho_1}$, there exists a subsequence of $\{(u_n, v_n)\}_n$ still denoted by $\{(u_n, v_n)\}_n$, such that $(u_n, v_n) \rightharpoonup (u_\lambda, v_\lambda)$ in $\mathbf{W}$ and $(u_n, v_n) \to (u_\lambda, v_\lambda)$ a.e. in $\mathbb{R}^N$.

Case 1. $\inf_n \|(u_n, v_n)\| = d > 0$. Reasoning as in Section 2, we obtain

$$\int_{\mathbb{R}^N} \left||u_n|^{\alpha-1}|v_n|^\beta\right|^{p_s^*/(p_s^*-1)} \mathrm{d}x \le C$$

and

$$\int_{\mathbb{R}^N} \left||v_n|^{\beta-1}|u_n|^\alpha\right|^{p_s^*/(p_s^*-1)} \mathrm{d}x \le C.$$

By applying the boundedness of $\{(u_n, v_n)\}_n$ in $\mathbf{W}$ and [6], (Thm. 4.9), there exist $(u_\lambda, v_\lambda) \in \mathbf{W}$, $\alpha_\lambda \ge 0$ and $\delta_\lambda \ge 0$ such that up to a subsequence, still denoted by $\{(u_n, v_n)\}_n$,

$$\begin{aligned}
&(u_n, v_n) \rightharpoonup (u_\lambda, v_\lambda) \quad \text{weakly in} \quad \mathbf{W},\\
&(u_n, v_n) \to (u_\lambda, v_\lambda) \quad \text{a.e. in} \quad \mathbb{R}^N,\\
&\|(u_n, v_n)\| \to \alpha_\lambda, \ \int_{\mathbb{R}^N} |u_n - u_\lambda|^\alpha |v_n - v_\lambda|^\beta \mathrm{d}x \to \delta_\lambda,\\
&|u_n|^{\alpha-2}u_n|v_n|^\beta \rightharpoonup |u_\lambda|^{\alpha-2}u_\lambda|v_\lambda|^\eta \quad \text{weakly in} \quad L^{(p_s^*)'}(\mathbb{R}^N)\\
&|v_n|^{\beta-2}v_n|u_n|^\alpha \rightharpoonup |v_\lambda|^{\beta-2}v_\lambda||u_\lambda|^\alpha|^\eta \quad \text{weakly in} \quad L^{(p_s^*)'}(\mathbb{R}^N),
\end{aligned} \tag{4.1}$$

where $(p_s^*)' = p_s^*/(p_s^* - 1)$.

Now we show that $\int_{\mathbb{R}^N} h|(u_n, v_n) - (u_\lambda, v_\lambda)|^q \mathrm{d}x \to 0$. Since $h \in L^{\frac{p_s^*}{p_s^*-q}}(\mathbb{R}^N)$ and $\{(u_n, v_n)\}_n$ is bounded in $\mathbf{W}$, there exists $R > 0$ such that

$$\int_{\mathbb{R}^N \setminus B_R} h|(u_n, v_n) - (u_\lambda, v_\lambda)|^q \mathrm{d}x \le \left(\int_{\mathbb{R}^N \setminus B_R} |h|^{p_s^*/(p_s^*-q)} \mathrm{d}x\right)^{(p_s^*-q)/p_s^*} \|(u_n, v_n) - (u_\lambda, v_\lambda)\|_{p_s^*}^p \le \frac{\varepsilon}{2},$$

for any $\varepsilon > 0$, where B_R is the ball in $\mathbb{R}^N$ with radius $R > 0$ centered at point 0. By the boundedness of $\{(u_n, v_n)\}_n$ in $\mathbf{W}$, the compact embedding (see Cor. 7.2 of [17]), and $h \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$, for above $\varepsilon > 0$, there exists $n_0 > 0$ such that $\int_{B_R} h|(u_n, v_n) - (u_\lambda, v_\lambda)|^q \mathrm{d}x \le \frac{\varepsilon}{2}$ as $n \ge n_0$. Therefore, we arrive at

$$\int_{\mathbb{R}^N} h|(u_n, v_n) - (u_\lambda, v_\lambda)|^q \mathrm{d}x \le \int_{\mathbb{R}^N \setminus B_R} h|(u_n, v_n) - (u_\lambda, v_\lambda)|^q \mathrm{d}x + \int_{B_R} h|(u_n, v_n) - (u_\lambda, v_\lambda)|^q \mathrm{d}x \le \varepsilon,$$

as $n \ge n_0$. This means that $\lim_{n\to\infty} \int_{n\to\infty} h|(u_n, v_n) - (u_\lambda, v_\lambda)|^q \mathrm{d}x = 0$. By (f_1), we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} hf_u(u_n, v_n)(u_n - u_\lambda) + hf_v(u_n, v_n)(v_n - u_\lambda)\mathrm{d}x\right|\\
&\le C \int_{\mathbb{R}^N} h|(u_n, v_n)|^{q-1}|(u_n, v_n) - (u_\lambda, v_\lambda)|\mathrm{d}x\\
&\le C \left(\int_{\mathbb{R}^N} h|(u_n, v_n) - (u_\lambda, v_\lambda)|^q \mathrm{d}x\right)^{1/q},
\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} hf_u(u_n,v_n)(u_n-u_\lambda)+hf_v(u_n,v_n)(v_n-v_\lambda)\mathrm{d}x=0. \tag{4.2}$$

Similar to Section 3, in accordance to the weak convergence of $\{(u_n,v_n)\}_n$ in $\mathbf{W}$, we have

$$\lim_{n\to\infty}\langle L(u_\lambda,v_\lambda),(u_n-u_\lambda,v_n-v_\lambda)\rangle=0, \tag{4.3}$$

where the operator L is defined in Section 3. Moreover, $\{L(u_n,v_n)\}_n$ is bounded in $\mathbf{W}^*$, where $\mathbf{W}^*$ denotes the dual space of $\mathbf{W}$. Furthermore, there exist a subsequence of $\{(u_n,v_n)\}_n$ still denoted by $\{(u_n,v_n)\}_n$ and some functional ξ such that

$$L(u_n,v_n)\rightharpoonup\xi \quad \text{weakly in} \quad \mathbf{W}^*,$$

that is,

$$\lim_{n\to\infty}\langle L(u_n,v_n),(\omega_1,\omega_2)\rangle=\langle\xi,(\omega_1,\omega_2)\rangle,$$

for all $(\omega_1,\omega_2)\in\mathbf{W}$. In particular, we have

$$\lim_{n\to\infty}\langle L(u_n,v_n),(u_\lambda,v_\lambda)\rangle=\langle\xi,(u_\lambda,v_\lambda)\rangle.$$

Since $\{(u_n,v_n)\}_n$ is a (PS) sequence, by using the same discussion as Section 3, we deduce

$$\begin{aligned}
o(1)&=\langle\mathcal{I}'(u_n,v_n)-\mathcal{I}'_{\alpha_\lambda}(u_\lambda,v_\lambda),(u_n,v_n)-(u_\lambda,v_\lambda)\rangle\\
&=M(\alpha_\lambda^p)[\alpha_\lambda^p-\langle\xi,(u_\lambda,v_\lambda)\rangle]-\lambda\int_{\mathbb{R}^N}|u_n|^\alpha|v_n|^\beta\mathrm{d}x+\lambda\int_{\mathbb{R}^N}|u_\lambda|^\alpha|v_\lambda|^\beta\mathrm{d}x+o(1)\\
&=M(\alpha_\lambda^p)[\alpha_\lambda^p-\langle\xi,(u_\lambda,v_\lambda)\rangle]-\lambda\int_{\mathbb{R}^N}|u_n-u_\lambda|^\alpha|v_n-v_\lambda|^\beta\mathrm{d}x+o(1)\\
&=M(\alpha_\lambda^p)\langle L(u_n,v_n)-L(u_\lambda,v_\lambda),(u_n,v_n)-(u_\lambda,v_\lambda)\rangle-\lambda\int_{\mathbb{R}^N}|u_n-u_\lambda|^\alpha|v_n-v_\lambda|^\beta\mathrm{d}x+o(1),
\end{aligned} \tag{4.4}$$

where $\mathcal{I}_{\alpha_\lambda}$ is defined as follows:

$$\mathcal{I}_{\alpha_\lambda}(u_\lambda,v_\lambda)=\frac{1}{p}M(\alpha_\lambda^p)\|(u_\lambda,v_\lambda)\|^p-\lambda\int_{\mathbb{R}^N}hf(u_\lambda,v_\lambda)\mathrm{d}x-\frac{\lambda}{p_s^*}\int_{\mathbb{R}^N}|u_\lambda|^\alpha|v_\lambda|^\beta\mathrm{d}x.$$

Here we apply (4.2) and the following facts (which can be proved by using the same discussion as Sect. 3):

$$|u_\lambda|^{\alpha-2}u_\lambda(u_n-u_\lambda)\to0 \quad \text{strongly in} \quad L^{\frac{p_s^*}{\alpha}}(\mathbb{R}^N),$$

$$|v_\lambda|^{\alpha-2}v_\lambda(v_n-v_\lambda)\to0 \quad \text{strongly in} \quad L^{\frac{p_s^*}{\beta}}(\mathbb{R}^N),$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|v_n|^\beta|u_\lambda|^{\alpha-2}u_\lambda(u_n-u_\lambda)\mathrm{d}x=0,$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^\alpha|v_\lambda|^{\beta-2}v_\lambda(v_n-v_\lambda)\mathrm{d}x=0.$$

It follows from (4.4) that

$$\begin{aligned} &M(\alpha_\lambda^p)\lim_{n\to\infty}\langle L(u_n,v_n)-L(u_\lambda,v_\lambda),(u_n,v_n)-(u_\lambda,v_\lambda)\rangle \\ &=\lambda\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n-u_\lambda|^\alpha|v_n-v_\alpha|^\beta\mathrm{d}x. \end{aligned} \tag{4.5}$$

According to the Simon inequalities (3.24), we divide the discussion into two cases. We first consider the case $p\geq 2$. By (3.24) and (2.2), we have

$$\begin{aligned} \langle L(u_n,v_n)-L(u_\lambda,v_\lambda),(u_n,v_n)-(u_\lambda-v_\lambda)\rangle &\geq C_p^{-1}\|(u_n,v_n)-(u_\lambda-v_\lambda)\|^p \\ &\geq C_{p_s^*}^{-p}C_p^{-1}\left(\int_{\mathbb{R}^N}|u_n-u_\lambda|^\alpha|v_n-v_\lambda|^\beta\mathrm{d}x\right)^{p/p_s^*}. \end{aligned} \tag{4.6}$$

Combining (4.6) with (4.5), we get

$$\lambda\delta_\lambda\geq C_{p_s^*}^{-1}C_p^{-1}M(\alpha_\lambda^p)\delta_\lambda^{\frac{p}{p_s^*}}. \tag{4.7}$$

Define

$$\lambda^{**}=\begin{cases}\inf\{\lambda>0:\delta_\lambda>0\}, & \text{if}\quad \delta_\lambda\not\equiv 0,\\ 1, \quad \text{if } \delta_\lambda\equiv 0.\end{cases}$$

If $\delta_\lambda\not\equiv 0$, then $\lambda^{**}=\inf\{\lambda>0:\delta_\lambda>0\}>0$. Otherwise, there exists a sequence $\{\lambda_k\}$, with $\delta_{\lambda_k}>0$, such that $\lambda_k\to 0$ as $k\to\infty$. Thus, (4.7) implies that

$$\lambda_k\delta_{\lambda_k}^{1-p/p_s^*}>C_{p_s^*}^{-1}C_p^{-1}M(\alpha_{\lambda_k}^p). \tag{4.8}$$

In view of Lemma 4.1, we know that $\{\alpha_\lambda\}_\lambda$ is uniformly bounded for λ, since ρ_1 is independent of λ. Clearly, $\{\delta_\lambda\}_\lambda$ is also uniformly bounded for λ. Hence, up to a subsequence, by (4.8) we can assume that $\alpha_{\lambda_k}\to 0$ as $k\to\infty$. Without loss of generality, we assume that $\{\alpha_{\lambda_k}\}\subset(0,1]$. Then, using a similar discussion as Section 3, we get

$$\alpha_{\lambda_k}^{p-(\theta_1+1)p^2/p_s^*}\geq C_{p_s^*}^{-1}C_p^{-1}(m_0)^{p/p_s^*},$$

which is a contradiction, and hence $\lambda^{**}>0$. Thus $\delta_\lambda=0$ for all $\lambda\in(0,\lambda^{**})$, that is, for all $\lambda\in(0,\lambda^{**}]$, we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n-u_\lambda|^\alpha|v_n-v_\lambda|^\beta\mathrm{d}x=0,$$

this together with (4.4) gives that $(u_n,v_n)\to(u_\lambda,v_\lambda)$ strongly in $\mathbf{W}$ as $n\to\infty$.

Finally, it remains to consider the case: $1<p<2$. Now by the Simon inequality and the Hölder inequality

$$\begin{aligned} &\|(u_n,v_n)-(u_\lambda,v_\lambda)\|^p \\ &\leq\widetilde{C}_p\left[\langle L(u_n,v_n)-L(u_\lambda,v_\lambda),(u_n,v_n)-(u_\lambda,v_\lambda)\rangle\right]^{p/2}\left(\|(u_n,v_n)\|^p+\|(u_\lambda,v_\lambda)\|^p\right)^{(2-p)/2} \\ &\leq C\left[\langle L(u_n,v_n)-L(u_\lambda,v_\lambda),(u_n,v_n)-(u_\lambda,v_\lambda)\rangle\right]^{p/2}, \end{aligned} \tag{4.9}$$

where C is positive constant. Combining (4.9) with (4.5) and (3.23), we have

$$(\lambda\delta_\lambda)^{p_s^*}\geq C_{p_s^*}^{-2p_s^*}C^{-2p_s^*/p}M^{p_s^*}(\alpha_\lambda^p)\delta_\lambda^2.$$

Now we claim that there exists $\lambda^{**}>0$, which defined similarly as above, such that $\delta_\lambda=0$ for all $\lambda\in(0,\lambda^{**})$. A similar discussion as the case $p\geq 2$ gives that $(u_n,v_n)\to(u_\lambda,v_\lambda)$ strongly in $\mathbf{W}$.

In conclusion, we get $(u_n,v_n)\to(u_\lambda,v_\lambda)$ in $\mathbf{W}$ as $n\to\infty$.

Case 2. $\inf\limits_n\|(u_n,v_n)\|=0$. Either 0 is an accumulation point of the sequence $\{(u_n,v_n)\}_n$ and so there exists a subsequence of $\{(u_n,v_n)\}_n$ strongly converging to $(u_\lambda,v_\lambda)=(0,0)$, or (0,0) is an isolated point of the sequence $\{(u_n,v_n)\}_n$ and so there exists a subsequence, still denoted by $\{(u_n,v_n)\}_n$, such that $\inf_n\|(u_n,v_n)\|>0$. In the first case we are done, while in the latter case we can proceed as in Case 1. □

Proof of Theorem 1.3. By Lemmas 4.1–4.2, there exists a $(PS)_{c_\lambda}$ sequence $\{(u_n,v_n)\}_n$, where $c_\lambda<0$ defined as in Lemma 4.2. Moreover, by Lemma 4.3, there exists $\lambda^{**}>0$ such that, up to a subsequence, $\{(u_n,v_n)\}_n$ strongly converges to (u_λ,v_λ), and $c_\lambda=\mathcal{I}(u_\lambda,v_\lambda)<0$ and $\mathcal{I}'(u_\lambda,v_\lambda)=0$, for all $0<\lambda<\lambda^{**}$, which imply that (u_λ,v_λ) is a nontrivial solution for system ($\mathcal{S}'$). □

In conclusion, we give the following example to illustrate a simple application of our results.

Example 4.4. *We consider the following system*

$$\begin{cases}(a+b\|(u,v)\|^p)[(-\Delta)^s_p u+V(x)|u|^{p-2}u] & =\lambda h(x)|(u,v)|^{q-2}u+\gamma\frac{\alpha}{p_s^*}|u|^{\alpha-2}u|v|^{\beta}\quad\text{in}\quad\mathbb{R}^N\\ (a+b\|(u,v)\|^p)[(-\Delta)^s_p v+V(x)|v|^{p-2}v] & =\lambda h(x)|(u,v)|^{q-2}v+\gamma\frac{\beta}{p_s^*}|v|^{\beta-2}v|u|^{\alpha}\quad\text{in}\quad\mathbb{R}^N,\end{cases}$$

where $a\geq 0, b\geq 0, a+b>0$, $1<q<p_s^*$, $1<\alpha,\beta$ *and* $\alpha+\beta=p_s^*$, $0\leq h\in L^{\frac{p_s^*}{p_s^*-q}}(\mathbb{R}^N)\bigcap L^\infty(\mathbb{R}^N)$ *with* $\inf_{x\in\mathbb{R}^N}h(x)>0$, *and* $\gamma\in(0,1)$. *For this case,* $M(t)=a+bt$, $H(x,u,v)=\frac{1}{q}h(x)|(u,v)|^q$ *and*

$$I(u,v)=\frac{a}{p}\|(u,v)\|^p+\frac{b}{2p}\|(u,v)\|^{2p}-\frac{\lambda}{q}\int_{\mathbb{R}^N}h(x)|(u,v)|^q\mathrm{d}x-\frac{\gamma}{p_s^*}\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\mathrm{d}x.$$

Obviously, M *satisfies* (M_1) *with* $\theta=2$, (M_2), *and* (M_3) *with* $m_0=b$ *and* $\theta_1=1$, H *satisfies* $(H_1),(H_2)$ *and* (M_3) *with* $\mu=q$. *If* $q\in(\theta p,p_s^*)$, *Then by Theorem 1.2, for each* $\gamma>0$ *there exists* $\lambda^*>0$ *such that for all* $\lambda\geq\lambda^*$ *system (3.16) admits at least one nontrivial solution* (u_0,v_0) *in* $\mathbf{W}$ *with positive energy. If* $1<q<p$, *then by Theorem 1.3, for each* $\lambda>0$ *there exists* $\gamma^*>0$ *such that for all* $\gamma\in(0,\gamma^*]$ *system (3.16) admits at least one nontrivial solution* (u_0,v_0) *in* $\mathbf{W}$ *with negative energy.*

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