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ON A DECOMPOSITION OF REGULAR DOMAINS INTO JOHN DOMAINS WITH UNIFORM CONSTANTS

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Abstract. We derive a decomposition result for regular, two-dimensional domains into John domains with uniform constants. We prove that for every simply connected domain $\Omega \subset \mathbb{R}^2$ with C^1-boundary there is a corresponding partition $\Omega = \Omega_1 \cup \ldots \cup \Omega_N$ with $\sum_{j=1}^N \mathcal{H}^1(\partial\Omega_j \setminus \partial\Omega) \leq \theta$ such that each component is a John domain with a John constant only depending on θ. The result implies that many inequalities in Sobolev spaces such as Poincaré's or Korn's inequality hold on the partition of Ω for uniform constants, which are independent of Ω.

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1. Introduction

It is a fundamental question to identify classes of domains for which the existence of solutions for partial differential equations or the validity of inequalities in Sobolev spaces can be guaranteed. The last decades have witnessed a tremendous process in establishing results for different assumptions on the domains.

For instance, one of the first proofs of Korn's inequality, being a widely studied inequality due to its importance in the analysis of elasticity equations, was given by Friedrichs [21] for domains allowing for a finite number of corners or edges on the boundary. Subsequently, generalizations appeared including versions for star-shaped sets [27], general Lipschitz domains [35], and more recently results [17] were obtained for the broader class of *uniform domains* using a modification of the extension operator by Jones [26].

On the other hand, it has been known for a long time that many inequalities are false on domains with external cusps. Several arguments have been provided for this fact (see [22, 40]), but the oldest is due to Friedrichs [20], who studied an inequality for analytic complex functions (*cf.* also [1]).

Recently Acosta, Durán, and Muschietti [1] investigated the existence of solutions of the divergence operator on *John domains* (see [25, 31, 33]). Apart from its application to the study of the Stokes equation the result is of interest due to its connection to Poincaré's and Korn's inequality, which may be deduced herefrom. Roughly speaking, a domain is a John domain if it satisfies a *twisted cone condition* such that each two points can be connected by a curve not getting too close to the boundary of the domain in terms of a corresponding *John constant* (we refer to Sect. 2.1 below for an exact definition).

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John domains represent a very general class allowing for sets with fractal boundary (*e.g.* Koch's *snowflake*), but at the same time excluding the formation of external cusps. They may be regarded as a very natural and in some sense most general notion of sets for the investigation of problems alluded to above since in [1,6] it has been shown that for domains satisfying the separation property (*e.g.* for simply connected planar domains) the validity of Poincaré's or Korn's inequality implies that the set is a John domain. Moreover, as already observed by Bojarski [5], the constant involved in the estimates essentially only depends on the John constant.

Difficulties concerning the properties and regularity of domains become even more challenging in models dealing with varying domains, *e.g. free boundary* or *shape optimization problems*, where the best shape of a set in dependence of a cost functional is identified as the solution of a variational problem (we refer to [7] for an introduction). Another important class is given by *free discontinuity problems* in the language of Ambrosio and De Giorgi [14] with various applications in fields of fracture mechanics or digital image segmentation, where the set of discontinuities of the function of interest is not preassigned, but determined from an energy minimization principle (*cf.* [3]).

Obviously without additional conditions there is no hope to derive uniform estimates being independent of the set shape as can be seen, *e.g.*, by considering a sequence of smooth sets converging to a domain with external cusp. Moreover, one may think of Neumann sieve type phenomena (see [32]) where the set is only connected by a small periodically distributed contact zone.

Therefore, many works appeared analyzing the behavior of constants in terms of the domain (*cf.* [24] and the references therein) or investigating special structures as convex, star-shaped or thin domains (see *e.g.* [16,23,30]). Another approach particularly used in the study of free discontinuity problems is based on the idea to establish results for a certain class of admissible (discontinuity) sets for which uniform estimates can be shown (we refer *e.g.* to [29,34,37]).

Also the present article is devoted to the derivation of uniform estimates being independent of the particular set shape. However, we will not restrict ourselves to a specific class of sets with certain properties, but rather show that for a generic domain one may construct a partition of the set such that the shape of each component can be controlled. The main result of this contribution is the following.

Theorem 1.1. *Let $\theta > 0$. Then there is $\varrho = \varrho(\theta) > 0$ such that the following holds: for all open, bounded and simply connected sets $\Omega \subset \mathbb{R}^2$ with C^1-boundary there is a partition $\Omega = \Omega_1 \cup \ldots \cup \Omega_N$ (up to a set of negligible measure) such that the sets $\Omega_1, \ldots, \Omega_N$ are ϱ-John domains with Lipschitz boundary and*

$$\sum_{j=1}^{N} \mathcal{H}^1(\partial\Omega_j) \leq (1+\theta)\mathcal{H}^1(\partial\Omega). \tag{1.1}$$

Loosely speaking, the result states that in spite of the fact that there is no uniform control of the John constant for generic domains, it is at least possible to establish uniform estimates locally in certain regions of the set. Here it is essential that the fineness of the partition can be bounded in terms of the length of the boundary of Ω. The original motivation for the derivation of Theorem 1.1 is a piecewise Korn inequality [18] for special functions of bounded deformation (see [2,4]). We hope, however, that the result may be also applied in various other situations due to the fact that John domains are a very general class and indeed many estimates only depend on the John constant (*cf.* [15]).

It is a natural question if it is possible to derive a partition of the form (1.1) into sets satisfying more specific properties, *e.g.* convexity. By constructing an example related to Koch's snowflake we see, however, that in general this is not the case and similarly as in the results for the validity of Poincaré's and Korn's inequality (again see [1,6]) also in the present context John domains appear to be an appropriate notion.

Let us remark the the regularity assumption in Theorem 1.1 is no real restriction as in many applications domains can be approximated by smooth sets (see [3], Thm. 3.42) or discontinuities can be regularized by density arguments (see [10,12]). Moreover, the result may be generalized to sets with Lipschitz boundary whose complements have a uniformly bounded number of connected components (see Thm. 6.4), which is a frequently used condition for various models in fracture mechanics or shape optimization (*cf.* [8, 9, 13, 38]). However, the limitation to sets with a specific topology is crucial as without a requirement of this type the problem is essentially, again up to a density argument, equivalent to the derivation of a version of Theorem 1.1 in the space of functions of bounded variation. This is an even more challenging issue and we refer to [18] for a deeper analysis.

The essential step in the proof of Theorem 1.1 is the derivation of a version for polygons and the general case then follows by approximation of regular sets. Although the methods we apply are rather elementary, the proof is comparably long and technical. Therefore, we restrict our decomposition scheme and analysis to a planar setting as in higher dimensions an analogous treatment of polyhedra leads to further technical difficulties. Let us remark, however, that based on Theorem 1.1 in [18] various estimates of Korn and Korn−Poincaré type are derived, which hold in arbitrary space dimension.

Our strategy is twofold. We introduce two special subclasses of polygons, which we call *semiconvex polygons* and *rotund polygons*. We then show that (1) each polygon can be partitioned into semiconvex and rotund polygons and (2) the specific characteristics of these subclasses of polygons are essentially equivalent to the property of John domains.

Loosely speaking, in semiconvex polygons concave vertices are not 'too close to opposite segments of the boundary' (see Def. 3.2) and rotund polygons contain a ball whose diameter is comparable to the diameter of the polygon (see Def. 4.1). The decomposition scheme presented below is based on the idea to separate the domain by segments and in this context it is crucial that (1) by an iterative partition we do not violate properties which have already been established in a previous step and (2) the overall length of added segments is controllable in terms of $\mathcal{H}^1(\partial\Omega)$.

The proof that semiconvex, rotund polygons are John domains for a John constant only depending on θ is constructive by defining appropriate piecewise affine curves between generic points of the domain. Hereby we crucially exploit the fact that concave vertices are not 'too close to opposite parts of the boundary' and that polygons are not "too thin". Despite the specific properties of the subclasses of polygons we still have to face additional difficulties concerning the geometry of the curves, which may, *e.g.*, partially have the form of a helix.

The paper is organized as follows. In Section 2.1 we first recall the definition of John domains and state fundamental properties. In Section 2.2 we present a version of Theorem 1.1 for polygons and give a more thorough overview of the proof. Here we also discuss an example giving some intuition why John domains appear to be the appropriate notion for the formulation of the problem. In Section 2.3 we introduce basic notation.

The subsequent sections are then devoted to the derivation of the result for polygons. In Section 3 we introduce the notion of semiconvex polygons, prove basic properties and present a decomposition scheme. Afterwards, in Section 4 we provide a fine analysis on the position of concave vertices and see that semiconvex polygons essentially coincide with convex polygons up to at most two small regions. In spite of their special structure, convex polygons are not necessarily rotund and we therefore discuss a further method to partition convex polygons. Finally, in Section 5 we prove that semiconvex and rotund polygons are John domains with controllable John constant.

In Section 6.1 we extend our findings to sets with C^1-boundary and in Section 6.2 we discuss a variant of Theorem 1.1 for sets with Lipschitz boundary allowing for a bounded number of components of the complement. Here we also present a piecewise Korn inequality as an application of our main result.

2. Preliminaries

2.1. John domains

We first introduce the notion of *John domains* and state some basic properties. Consider rectifiable curves $\gamma : [0, l(\gamma)] \to \mathbb{R}^d$ with length $l(\gamma)$ and assume that they are parameterized by arc length. For $0 < \eta < 1$ we define the η-*cigar* by

$$\mathrm{cig}(\gamma,\eta) := \bigcup_{t\in[0,l(\gamma)]} B(\gamma(t),\eta\min\{t,l(\gamma)-t\}), \tag{2.1}$$

where $B(x,r) \subset \mathbb{R}^d$ denotes the open ball with radius $r \geq 0$ and midpoint $x \in \mathbb{R}^d$. Likewise, we define the η-*carrot* by

$$\mathrm{car}(\gamma,\eta) := \bigcup_{t\in[0,l(\gamma)]} B(\gamma(t),\eta t). \tag{2.2}$$

Definition 2.1. Let $\varrho > 0$. We say a bounded domain $\Omega \subset \mathbb{R}^d$ is a ϱ-John domain if there is a point $p \in \Omega$ such that for all $x \in \Omega \setminus \{p\}$ there is a rectifiable curve $\gamma : [0,l(\gamma)] \to \Omega$ with $\gamma(0) = x$ and $\gamma(l(\gamma)) = p$ such that $\mathrm{car}(\gamma,\varrho) \subset \Omega$.

The point p will be called the *John center* and ϱ is the *John constant*. Domains of this form were introduced by John [25] to study problems in elasticity theory. The term was first used by Martio and Sarvas [31]. Roughly speaking, a domain is a John domain if it is possible to connect two arbitrary points without getting too close to the boundary of the domain.

Remark 2.2. A lot of different equivalent definitions can be found in [33]. We will also use the following characterization: a bounded domain Ω is a ϱ-John domain if for each pair of distinct points $x_1, x_2 \in \Omega$ there is a curve $\gamma : [0,l(\gamma)] \to \Omega$ with $\gamma(0) = x_1$ and $\gamma(l(\gamma)) = x_2$ such that $\mathrm{cig}(\gamma,\varrho) \subset \Omega$. Such a curve will be called *John curve between* x_1 *and* x_2.

The class of John domains is much larger than Lipschitz domains and contains sets with fractal boundaries or internal cusps, while the formation of external cusps is excluded. For instance the interior of Koch's *snowflake* is a John domain. We state a simple property (see *e.g.* [39]).

Lemma 2.3. *Let Ω be a ϱ-John domain. Then for each $x \in \Omega$ and $r > 0$ with $\Omega \setminus B(x,r) \neq \emptyset$, there is $z \in \overline{B(x,r)}$ with $B(z,\frac{1}{2}\varrho r) \subset \Omega$.*

Our main result will be first established for polygons. To prove Theorem 1.1, we then need to combine different John domains so that the unions are still John domains. In [39] we find the following lemma.

Lemma 2.4. *Let $\varrho, c_0 > 0$. There is $\varrho' = \varrho'(\varrho, c_0)$ such that the following holds*:

(i) *If $D_1, D_2 \subset \mathbb{R}^d$ are ϱ-John domains with $\min\{|D_1|,|D_2|\} \leq c_0|D_1 \cap D_2|$, then $D_1 \cup D_2$ is a ϱ'-John domain.*
(ii) *If $D_0, D_1, \ldots,$ is a sequence of ϱ-John domains in $\mathbb{R}^d$ with $|D_j| \leq c_0|D_0 \cap D_j|$ for all $j \geq 1$, then $\bigcup_{j\geq 0} D_j$ is a ϱ'-John domain.*

2.2. Formulation of the main result for polygons

The general strategy in this article is to derive the partition result first for polygons, which is easier due to the specific geometry of the boundary. In this section we present the main result for polygons and give an overview of the proof.

Our partition technique for polygons will differ from widely used algorithms as triangulation, trapezoidalization or the Hertel and Mehlhorn Algorithm (see [36]) in the sense that we do not provide an optimal partition (concerning number of pieces or runtime), but one where the length of the boundary of all polygons is comparable to the length of the boundary of the original polygon.

We consider sets $P \subset \mathbb{R}^2$ being the region enclosed by a simple polygon. For convenience sets of this form will be called *polygons* in the following although the notion typically refers only to the boundary of such sets. We always assume that polygons are closed. We notice that, according to our definition, every polygon P is simply connected and coincides with the closure of its interior, which is nonempty. In particular the Lebesgue measure $|P|$ of P is strictly positive.

We intent to prove the following theorem.

Theorem 2.5. *Let $\varepsilon, \theta > 0$. Then it exists $\varrho = \varrho(\theta) > 0$ such that for all polygons P there is a partition $P = P_0 \cup \ldots \cup P_N$ with $\mathcal{H}^1(\partial P_0) \leq \varepsilon$ and the polygons $P_1, \ldots, P_N$ are ϱ-John domains satisfying*

$$\sum_{j=1}^{N} \mathcal{H}^1(\partial P_j) \leq (1+\theta)\mathcal{H}^1(\partial P). \tag{2.3}$$

We start with a short outline of the proof. In particular, we indicate how an arbitrary polygon may be partitioned to satisfy the condition in Definition 2.1 for a John constant ϱ.

First of all, the property of ϱ-John domains may be violated if the polygon has a 'star shape', *i.e.* there are concave vertices for which the distance to other concave vertices or opposite segments of the boundary is small. We see that if this distance is too small, we can partition the polygon by introducing a short segment between a concave vertex and another point of the boundary. By this procedure we construct what we call *semiconvex polygons* (see Sect. 3, in particular Def. 3.2). Intuitively, such sets have the property that, separating the set by a short segment between a concave vertex and another point of the boundary, the "bulk part" of the polygon lies on one side.

Clearly, for convex sets it is much easier to satisfy the condition in Definition 2.1. It turns out, however, that even a convex polygon is possibly not a ϱ-John domains if the set is long and thin or has small interior angles. The presence of the latter phenomenon cannot be avoided and therefore the introduction of the set P_0 in Theorem 2.5 is possibly necessary. To tackle the first problem, we introduce so called *rotund polygons* (see Sect. 4) which are sets containing a ball whose size is comparable to the diameter of the set. We then show that convex polygons can be partitioned into rotund polygons up to a small exceptional set (see Lem. 4.6). Finally, this kind of partition can also be performed for semiconvex polygons, which is related to the fact that a semiconvex polygon, which is not already rotund, coincides with a convex polygon up to at most two small regions (see Thm. 4.5).

After combining the above described partitions we show in Section 5 that semiconvex and rotund polygons are indeed ϱ-John domains for a constant $\varrho = \varrho(\theta)$, which essentially only depends on the length of the additional boundary induced by the partition (*cf.* (2.3)). The basic idea is to take a shortest path between two points (which will 'touch' the boundary of the polygon in concave vertices) and to modify this path in such a way that the condition in Definition 2.1 is satisfied. To do this, it is essential that (1) the polygons contain a ball whose size is comparable to the diameter of the set and (2) concave vertices are 'not too close to opposite parts of the boundary'.

We remark that the definitions and terms of the subclasses of polygons introduced in the following sections (see Sects. 3, and 4) are not taken from the literature but tailored for the present exposition in order to avoid the ongoing repetition of technical assumptions. Let us also remark that, once the basic definition of semiconvex and rotund polygons have been internalized, Sections 3 and 5 can be read rather independently from each other.

Before we start to prove Theorem 2.5, let us note that it does not appear to be possible to provide a partition for which the sets satisfy a stronger property than the one given in Definition 2.1. To give some intuition, we consider the following example being a modification of *Koch's snowflake*.

Example 2.6. Let $0 < \eta < 1$. Let S_0 be an equilateral triangle. As in the construction of Koch's snowflake we replace the middle third of each segment by two segments of equal length which enclose an angle $\frac{\pi}{3}$ with the original segment. Hereby, we obtain S_1. Then S_2 is obtained by replacing the middle third of each segment of S_1 by two segments which enclose an angle $\frac{\pi}{3}\eta$ with the original one. We continue with this construction where in the definition of S_i the new segments enclose an angle $\frac{\pi}{3}\eta^{i-1}$ with the original ones.

Although the construction is very similar to the one of Koch's snowflake, we find $\mathcal{H}^1(\partial S_i) \leq C$ for all $i \in \mathbb{N}$ for some $C = C(\eta)$. Moreover, one can show that all S_i are ϱ-John domains for some $\varrho > 0$. Let us assume that the polygon S_i for i large could be partitioned into sets with 'better properties' (*e.g.* convexity). Due to the geometry of S_i we note that after separating S_i into two sets by a segment there is one set which essentially has the same shape as S_i. Consequently, to derive a partition into sets with more specific properties, it appears to be necessary to introduce all boundaries $\bigcup_{j \leq i-1} \partial S_j$. This, however, violates (2.3).

2.3. Notation

Let us fix the main notations for polygons which will be used in the following proof of Theorem 2.5. Recall that polygons P are always assumed to be closed subsets of $\mathbb{R}^2$. We denote the *vertices* of P by $\mathcal{V}_P$ and for $v \in \mathcal{V}_P$ we let $\sphericalangle(v,P)$ be the corresponding interior angle. A vertex $v \in \mathcal{V}_P$ with $\sphericalangle(v,P) > \pi$ is called *concave*, otherwise *convex*. Denote the subset of concave vertices by $\mathcal{V}'_P$.

Sometimes we will understand vertices v as complex numbers and let $\arg(v) \in [0,2\pi)$ be the phase of the complex number so that $v = |v|\mathrm{e}^{i\arg(v)}$. For $\varphi \in [0,2\pi)$ we denote by $v + \mathbb{R}_+\mathrm{e}^{i\varphi}$ open half lines with initial point v, where $\mathbb{R}_+ = (0,\infty)$. The line segment between two given points $p_1, p_2 \in \mathbb{R}^2$ is denoted by $[p_1;p_2]$ and $|[p_1;p_2]|$ is its length. For a segment $|[p_1;p_2]|$ we also introduce the notation (recall (2.1))

$$\mathrm{cig}([p_1;p_2],\eta) := \mathrm{cig}(\gamma,\eta),$$

where $\gamma : [0,l(\gamma)] \to [p_1;p_2]$ is the (affine) curve, parametrized by arc length, with $\gamma(0) = p_1$, $\gamma(l(\gamma)) = p_2$ and length $l(\gamma) = |[p_1;p_2]|$. Moreover, we define the *visible region* of $[v;w]$ by

$$\mathrm{cig}_P([v;w],\eta) = \Big\{ x \in \overline{\mathrm{cig}([v;w],\eta)} : \ \exists\, p \in [v;w] \text{ s.t. } [p;x] \subset P \Big\}$$

(see Fig. 1 below). We define an *intrinsic metric* on P by

$$d_P(p,p') = \min\{l(\gamma) : \gamma : [0,l(\gamma)] \to P \text{ Lipschitz curve with } \gamma(0) = p,\ \gamma(l(\gamma)) = p'\}$$

for $p,p' \in P$, where the curves are always assumed to be parameterized by arc length. We notice that the minimum exists as P is closed and that it is attained by a piecewise affine curve, where the endpoints of each segment lie in $\mathcal{V}'_P \cup \{p,p'\}$. Likewise, for $p \in P$ and $S \subset P$ we let $\mathrm{dist}_P(p,S) = \inf_{p' \in S} d_P(p,p')$. Let the *intrinsic diameter* of a polygon be given by

$$d(P) = \max_{p,p' \in P} d_P(p,p').$$

We find $d(P) \le \frac{1}{2}\mathcal{H}^1(\partial P)$ by considering a pair p,p' maximizing $d_P(p,p')$ and the corresponding piecewise affine curve. The following definition will be used frequently.

Definition 2.7. Let P be a polygon. We say a segment $[p;q] \subset P$ with $p,q \in \partial P$ induces a partition of P if there are two polygons Q_1, Q_2 with $P = Q_1 \cup Q_2$ and $[p;q] = Q_1 \cap Q_2$.

Note that, according to our definition of polygon, we have $|Q_1|, |Q_2| > 0$. Moreover, $[p;q] = \partial Q_1 \cap \partial Q_2$ and every continuous path connecting a point of Q_1 with a point of Q_2 must meet the segment $[p;q]$.

3. Semiconvex polygons

We first refine Definition 2.7.

Definition 3.1. Let $\eta > 0$ and P be a polygon. We say a segment $[v;w]$ between a concave vertex $v \in \mathcal{V}'_P$ and some $w \in \partial P$ which induces a partition of $P = Q_1 \cup Q_2$ according to Definition 2.7 satisfies the *segmentation property* (SP) if

$$\mathcal{V}'_P \cap \mathrm{cig}_P([v;w],\eta) \subset \{v,w\} \tag{3.1}$$

and for $i = 1,2$

$$Q_i \text{ is a triangle} \quad \Rightarrow \quad \sphericalangle(v,Q_i) > \frac{1}{2}\arcsin\eta. \tag{3.2}$$

These technical conditions are necessary to avoid the formation of geometrical artefacts in the partition process in Section 3.2 such as degenerated triangles and polygons where a concave vertex is very close to an opposite side. We now introduce the notion of *semiconvexity*.

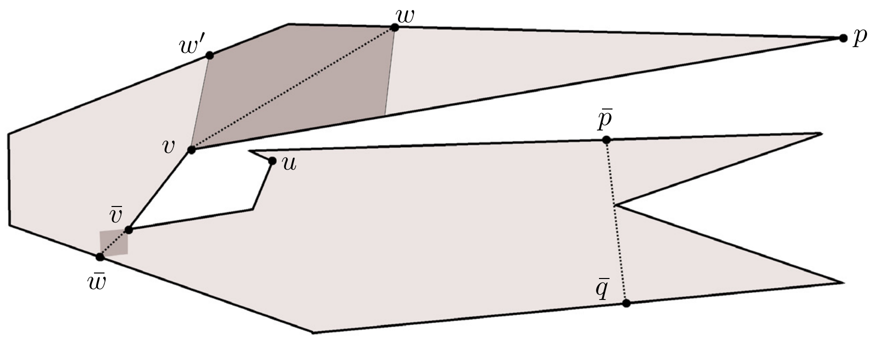

FIGURE 1. In dark gray we depicted $\mathrm{cig}_P([v;w],\eta)$ and $\mathrm{cig}_P([\bar{v};\bar{w}],\eta)$. Observe that $u \notin \mathrm{cig}_P([v;w],\eta)$ although $u \in \mathrm{cig}([v;w],\eta)$. Consequently, $[v;w]$ satisfies (3.1) but not (SP) due to the small interior angle at v in the triangle formed by v, w, p. The segment $[v;w']$ satisfies (SP). The segment $[\bar{v},\bar{w}]$ is a typical example of a segment which satisfies (SP) such that condition (3.3) is violated. Note that $[\bar{p};\bar{q}]$ does not induce a partition into two simple polygons.

Definition 3.2. Let $0 < \vartheta, \eta < 1$.

(i) We say a polygon P is ϑ-semiconvex if for each segment $[v;w]$ between a concave vertex $v \in \mathcal{V}'_P$ and some $w \in \partial P$ which induces a partition $P = Q_1 \cup Q_2$ one has

$$|[v;w]| \geq \vartheta \min_{k=1,2} d(Q_k). \tag{3.3}$$

(ii) We say a polygon P is (SP)-ϑ-semiconvex if for each segment $[v;w]$ between a concave vertex $v \in \mathcal{V}'_P$ and some $w \in \partial P$ which induces a partition $P = Q_1 \cup Q_2$ and satisfies (SP) one has (3.3).

For simplicity we will often drop the parameters and will call a polygon semiconvex and (SP)-semiconvex if no confusion arises.

Remark 3.3. Intuitively, the definition states that, separating the set by a short segment between a concave vertex and another point of the boundary, the “bulk part” of the polygon lies on one side. The semiconvexity of a polygon together with rotundness considered in Section 4 is the essential property to control the John constant of polygons. We note that in (3.3) the intrinsic diameter is the suitable notion and cannot be replaced by the length of the boundary although it seems to be another natural choice. To see this, consider Koch's *snowflake* which is a John domain with finite intrinsic diameter but whose boundary is of infinite $\mathcal{H}^1$-measure.

In Section 3.1 we study the relation between semiconvex and (SP)-semiconvex polygons deriving that the notions are very similar. In Sections 4–6 we will only need the concept of semiconvex polygons. However, for the partition of polygons into semiconvex polygons performed in Section 3.2 it is convenient to consider also the more technical notion in Definition 3.2(ii).

3.1. Properties of semiconvex polygons

By definition we clearly have that each semiconvex polygon is also (SP)-semiconvex. We now investigate the reverse direction.

Theorem 3.4. *Let $0 < \vartheta, \eta < 1$ with $\vartheta \leq \frac{1}{2}\eta$. Then for $\eta > 0$ small enough there is some $\bar{\vartheta} = \bar{\vartheta}(\vartheta) \leq \vartheta$ such that each* (SP)-*ϑ-semiconvex polygon is $\bar{\vartheta}$-semiconvex.*

Proof. Let P be a (SP)-ϑ-semiconvex polygon. Let $v \in \mathcal{V}'_P$ and some $w \in \partial P$ be given inducing a partition of P. The goal is to confirm (3.3) for $[v;w]$. To this end, we will construct a chain of segments consisting of concave vertices and combining v with w such that each segment satisfies (SP) and therefore (3.3) is applicable by assumption.

Step 1. *Cigar condition.*

We first assume that $[v;w]$ induces a partition $P = Q_1 \cup Q_2$ and that

$$\mathcal{V}'_P \cap \mathrm{cig}_P([v;w], 2\eta) \subset \{v, w\}. \tag{3.4}$$

(Compare with (3.1) and note that in contrast to Definition 3.1 we do not require (3.2)). We show that

$$|[v;w]| \geq \frac{\vartheta}{2} \min_{k=1,2} d(Q_k). \tag{3.5}$$

We distinguish the following cases:

(a) If each Q_k is either not a triangle or a triangle where the interior angle at v exceeds $\alpha_\eta := \frac{1}{2}\arcsin \eta$, we find that (3.1)–(3.2) hold and thus $|[v;w]| \geq \vartheta \min_{k=1,2} d(Q_k)$ by Definition 3.2(ii).

(b) Otherwise, we can suppose without restriction that Q_1 is a triangle consisting of the vertices v, w, p with $\sphericalangle(v, Q_1) \leq \alpha_\eta$ (*cf.* Fig. 1). Thus, if η small enough, we get $d(Q_1) = \max\{|[v;p]|, |[v;w]|\}$ and may assume $|[v;p]| \geq |[v;w]|$ as otherwise (3.5) follows directly. If $\sphericalangle(w, Q_1) \leq \pi - 2\alpha_\eta$, we apply the sine rule $\frac{|[v;p]|}{\sin \sphericalangle(w,Q_1)} = \frac{|[v;w]|}{\sin \sphericalangle(p,Q_1)}$ and the fact that $\sphericalangle(p, Q_1) \geq \alpha_\eta$ to see

$$d(Q_1) = |[v;p]| = \frac{\sin \sphericalangle(w, Q_1)}{\sin \sphericalangle(p, Q_1)} |[v;w]| \leq \frac{|[v,w]|}{\sin \alpha_\eta} \leq \frac{4}{\eta} |[v,w]| \leq \frac{2}{\vartheta} |[v,w]|$$

for η small, where we used $\sin \alpha_\eta \geq \frac{1}{4}\eta$ by a Taylor expansion and $\vartheta \leq \frac{1}{2}\eta$.

(c) Otherwise, we have $\sphericalangle(w, Q_1) > \pi - 2\alpha_\eta$. First suppose $w \in \mathcal{V}'_P$, which means that we can change the roles of v and w. We see that (3.1) holds by assumption. Moreover, we have $\sphericalangle(w, Q_1) > \pi - 2\alpha_\eta > \alpha_\eta$ for η small and that Q_2 is not a triangle since P has at least five vertices due to $\{v, w\} \subset \mathcal{V}'_P$. Consequently, also (3.2) holds and we can proceed as in (a) to find $|[v;w]| \geq \vartheta \min_{k=1,2} d(Q_k)$.

Observe that in (b) we used a purely geometrical argument and in (a), (c) we only showed that (3.2) holds, whereby Definition 3.2(ii) was applicable. In the following last case, however, we will explicitly use (3.4).

(d) Finally, we suppose that $\sphericalangle(w, Q_1) > \pi - 2\alpha_\eta$ and that w is not a concave vertex. Understanding the vertices as complex numbers we define the phase $\varphi_0 = \arg(w - v)$. Let $f : D \to \mathbb{R}^2$ so that $f(\varphi)$ denotes the closest point to v on $(v + \mathbb{R}_+ \mathrm{e}^{i(\varphi_0+\varphi)}) \cap \partial P$, where $D \subset [-\pi, \pi)$ contains a neighborhood of 0 and satisfies $|D| = \sphericalangle(v, P)$. (Recall $\mathbb{R}_+ = (0, \infty)$). For $\varphi > 0$ small let $\triangle_\varphi$ the triangle formed by v, p, $f(\varphi)$ and up to changing the sign of φ we may assume that $\sphericalangle(v, \triangle_\varphi) > \sphericalangle(v, Q_1)$ for $\varphi > 0$ small. Observe that due to the fact that w is not a concave vertex and $\sphericalangle(w, Q_1) > \pi - 2\alpha_\eta = \pi - \arcsin \eta$ we have

$$f(\varphi) \in \mathrm{cig}_P([v;w], 2\eta)$$

for φ small. This then implies $f(\varphi) \notin \mathcal{V}'_P$ for $\varphi \in [0, 2\alpha_\eta]$ since otherwise (3.4) would be violated. Consequently, letting $w' = f(2\alpha_\eta)$ we find that $[v;w']$ induces a partition $P = Q'_1 \cup Q'_2$, where the sets are labeled such that $p \in Q'_1$. Moreover, $[v;w']$ satisfies (SP). In fact, the angle condition (3.2) follows directly by construction. Moreover, we get $\mathrm{cig}_P([v;w'], \eta) \subset \mathrm{cig}_P([v;w], 2\eta)$ and thus (3.1) follows from (3.4) and the fact that $w \notin \mathcal{V}'_P$. Consequently, as P is (SP)-ϑ-semiconvex, we obtain by (3.3)

$$|[v;w']| \geq \vartheta \min_{k=1,2} d(Q'_k).$$

Consider the convex polygon $\hat{P} := Q'_1 \cap Q_2$ and note that for η small $d(\hat{P}) = |[v;w]|$ (see Fig. 1) as well as $|[v;w']| \leq |[v;w]|$. Moreover, $\min_{k=1,2} d(Q_k) \leq \min_{k=1,2} d(Q'_k) + d(\hat{P})$ and therefore we obtain since $\vartheta < 1$

$$|[v;w]| \geq \frac{1}{2}|[v;w']| + \frac{1}{2}|[v;w]| \geq \frac{\vartheta}{2} \min_{k=1,2} d(Q_k) - \frac{\vartheta}{2}|[v;w]| + \frac{1}{2}|[v;w]| \geq \frac{\vartheta}{2} \min_{k=1,2} d(Q_k).$$

Step 2. *Chains of vertices.*

Now we only assume that $[v;w]$, $v \in \mathcal{V}'_P$, $w \in \partial P$, induces a partition of P. We construct a chain $(y_1, \ldots, y_n)$ between v and w with $y_1 = v$, $y_n = w$ and $y_i \in \mathcal{V}'_P$ for $i = 2, \ldots, n-1$ such that

$$|[y_i;y_{i+1}]| \leq 3|[v;w]|, \qquad d_P(v,y_i) \leq \frac{3}{2}|[v;w]|, \quad i = 1, \ldots, n-1 \tag{3.6}$$

and the segments $[y_i;y_{i+1}] \subset P$ induce a partition satisfying (3.4) with y_i, y_{i+1} in place of v, w (*cf.* Fig. 3). (See Sect. 2.3 for the definition of $d_P(v,y_i)$).

The strategy is to define the chain between v and w inductively. Let $\mathcal{C}_0 = (y_1^0, y_2^0) = (v,w)$ and assume $\mathcal{C}_k = (y_1^k, \ldots, y_{2+k}^k)$ with $y_1^k = v$, $y_{2+k}^k = w$ and $[y_j^k; y_{j+1}^k] \subset P$ for $j = 1, \ldots, k+1$ has been constructed. If

$$\mathcal{V}'_P \cap \mathrm{cig}_P([y_j^k; y_{j+1}^k], 2\eta) \subset \{y_j^k, y_{j+1}^k\} \qquad \text{for all } j = 1, \ldots, k+1, \tag{3.7}$$

we stop. Otherwise, we find some $J \in \{1, \ldots, k+1\}$ and $\hat{v}_k \in \mathcal{V}'_P \setminus \{y_J^k, y_{J+1}^k\}$ such that $\hat{v}_k \in \mathrm{cig}_P([y_J^k; y_{J+1}^k], 2\eta)$ and $[y_J^k; \hat{v}_k] \cup [\hat{v}_k; y_{J+1}^k] \subset P$. (Choose $\hat{v}_k$ as the concave vertex in $\mathrm{cig}_P([y_J^k; y_{J+1}^k], 2\eta)$ with minimal distance to $[y_J^k; y_{J+1}^k]$). We define

$$\mathcal{C}_{k+1} = (y_1^k, \ldots, y_J^k, \hat{v}_k, y_{J+1}^k, \ldots, y_{k+2}^k).$$

Note that the triangle formed by $[y_J^k; y_{J+1}^k]$ and $\hat{v}_k$ is contained in P since P is simply connected. As in each step we choose a different $\hat{v}_k$ and $\#\mathcal{V}'_P < \infty$, after a finite number of steps we find a chain $(y_1, \ldots, y_n)$ such that (3.7) is satisfied.

We now show that (3.6) holds. To this end, we fix y_i, $i = 2, \ldots, n-1$, and identify the iteration steps that 'led to the definition of y_i'. Let k_0 be the index such that $\hat{v}_{k_0} := y_i \in \mathcal{C}_{k_0+1}$. Choose J_0 such that $y_i \in \mathrm{cig}_P(S_0, 2\eta)$ with $S_0 = [y_{J_0}^{k_0}; y_{J_0+1}^{k_0}]$.

Assume steps $k_0 > k_1 > \ldots > k_n$ and $(J_i)_{i=0}^n$ have been found with corresponding $\hat{v}_{k_i}$ such that $\hat{v}_{k_i} \in \mathrm{cig}_P(S_i, 2\eta)$ with $S_i := [y_{J_i}^{k_i}; y_{J_i+1}^{k_i}]$.

We then choose the largest value $k_{n+1} < k_n$ such that one of the points $y_{J_n}^{k_n}, y_{J_n+1}^{k_n}$ is not contained in $\mathcal{C}_{k_{n+1}}$, *e.g.* $y_{J_n}^{k_n} =: \hat{v}_{k_{n+1}}$. We then find J_{n+1} such that $\hat{v}_{k_{n+1}} \in \mathrm{cig}_P(S_{n+1}, 2\eta)$ with $S_{n+1} = [y_{J_{n+1}}^{k_{n+1}}; y_{J_{n+1}+1}^{k_{n+1}}]$, where one of the endpoints of S_{n+1} coincides with $y_{J_n+1}^{k_n}$. For later purpose we note that S_n, S_{n+1} have a common endpoint and $\hat{v}_{k_{n+1}}$ is an endpoint of S_n. Finally, after a finite number of steps, denoted by N, we arrive at $S_N = [v;w]$.

Recalling the geometry of $\mathrm{cig}(S_n, 2\eta)$ an elementary computation yields that the angles at the endpoints of S_n in the triangle formed by S_n and $\hat{v}_{k_n}$ are larger than (*cf.* Fig. 2)

$$\varphi_n := \arctan(g_n) \quad \text{ with } \quad g_n := \frac{\mathrm{dist}(S_n, \hat{v}_{k_n})}{\mathcal{H}^1(S_n) - (\tan(4\alpha_\eta))^{-1}\,\mathrm{dist}(S_n, \hat{v}_{k_n})}.$$

We note $\mathrm{dist}(S_n, \hat{v}_{k_n}) \leq \frac{1}{2}\tan(4\alpha_\eta)\mathcal{H}^1(S_n)$ and thus $g_n \leq \tan(4\alpha_\eta)$. Recalling that the segments S_{n-1}, S_n have one common endpoint (either $y_{J_n}^{k_n}$ or $y_{J_n+1}^{k_n}$) and $\hat{v}_{k_n}$ is an endpoint of S_{n-1}, we find

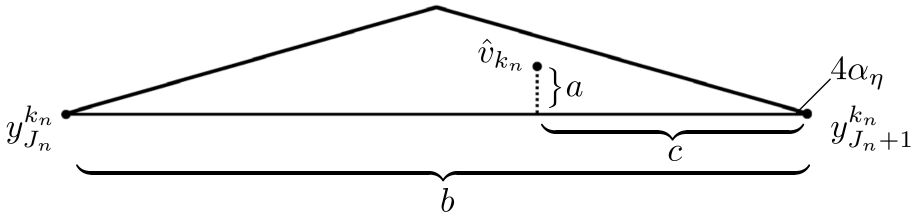

FIGURE 2. We set $a = \mathrm{dist}(S_n, \hat{v}_{k_n})$ and $b = \mathcal{H}^1(S_n)$. Elementary trigonometry yields $c \geq (\tan(4\alpha_\eta))^{-1}a$. Note that S_{n-1} is the segment between $\hat{v}_{k_n}$ and the left or right endpoint of S_n.

$\mathcal{H}^1(S_{n-1}) \leq (\sin\varphi_n)^{-1}\,\mathrm{dist}(S_n, \hat{v}_{k_n})$ for all $n = 1, \ldots, N$. Then we obtain by a Taylor expansion for η small and some large $C > 0$ independent of η (observe that for small x one has $\arctan(x), \sin(x) \approx x$)

$$\begin{aligned}\mathcal{H}^1(S_{n-1}) &\leq \frac{1}{g_n - Cg_n^2}\,\mathrm{dist}(S_n, \hat{v}_{k_n}) \leq (1 + Cg_n)\left(\mathcal{H}^1(S_n) - \frac{\mathrm{dist}(S_n, \hat{v}_{k_n})}{\tan(4\alpha_\eta)}\right)\\ &= \mathcal{H}^1(S_n) - (\tan(4\alpha_\eta))^{-1}\,\mathrm{dist}(S_n, \hat{v}_{k_n}) + C\,\mathrm{dist}(S_n, \hat{v}_{k_n}).\end{aligned}$$

Note that $\mathrm{dist}(x, S_n) \leq \mathrm{dist}(\hat{v}_{k_n}, S_n)$ for all $x \in S_{n-1}$. Then using the previous estimate and summing over all n we find for η small (such that $0 < \tan(4\alpha_\eta)(1 - C\tan(4\alpha_\eta))^{-1} \leq \frac{1}{2}$)

$$\begin{aligned}d_P(y_i, [v;w]) = d_P(y_i, S_N) &\leq \mathrm{dist}(\hat{v}_{k_0}, S_0) + \sum_{n=1}^{N} \max_{x \in S_{n-1}} \mathrm{dist}(x, S_n) \leq \sum_{n=0}^{N} \mathrm{dist}(\hat{v}_{k_n}, S_n)\\ &\leq \frac{\tan(4\alpha_\eta)}{1 - C\tan(4\alpha_\eta)} \sum_{n=1}^{N} (\mathcal{H}^1(S_n) - \mathcal{H}^1(S_{n-1})) + \mathrm{dist}(\hat{v}_{k_0}, S_0)\\ &\leq \frac{\tan(4\alpha_\eta)}{1 - C\tan(4\alpha_\eta)}(|[v;w]| - \mathcal{H}^1(S_0)) + \frac{1}{2}\tan(4\alpha_\eta)\mathcal{H}^1(S_0) \leq \frac{1}{2}|[v;w]|,\end{aligned}$$

where we used $\mathrm{dist}(S_0, \hat{v}_{k_0}) \leq \frac{1}{2}\tan(4\alpha_\eta)\mathcal{H}^1(S_0)$. This together with the triangle inequality yields (3.6).

Step 3. *Semiconvexity.*

We now show that P is semiconvex by confirming (3.3) for the segment $[v;w]$ with $\bar{\vartheta} = (3 + 12\vartheta^{-1})^{-1}$. As each of the segments $[y_i; y_{i+1}]$ satisfies (3.4) (with y_i, y_{i+1} in place of v, w), we obtain by (3.5)

$$|[y_i; y_{i+1}]| \geq \frac{\vartheta}{2} \min_{k=1,2} d(Q_k^{(i)}) \tag{3.8}$$

for $i = 1, \ldots, n-1$, where $P = Q_1^{(i)} \cup Q_2^{(i)}$ is the corresponding partition. Let $P = Q_1 \cup Q_2$ be the partition induced by $[v;w]$. It suffices to consider the case $|[v;w]| \leq \frac{1}{8}\min_{j=1,2} d(Q_j)$ as otherwise the assertion is clear provided we choose $\bar{\vartheta} \leq \frac{1}{8}$. We choose $p_j^1, p_j^2 \in Q_j$ with $\mathrm{dist}_P(p_j^1, p_j^2) = d(Q_j)$ and as $\mathrm{dist}_P(p_j^1, p_j^2) \leq \mathrm{dist}_P(p_j^1, v) + \mathrm{dist}_P(p_j^2, v)$, we obtain possibly after relabeling $\mathrm{dist}_P(p_j^1, v) \geq \frac{1}{2}d(Q_j)$ for $j = 1, 2$.

Let $B = \{x \in P : \mathrm{dist}_P(x, v) < 4|[v;w]|\}$. We now show that two arbitrary points $q_1 \in Q_1 \setminus B$, $q_2 \in Q_2 \setminus B$ do not lie in the same connected component of $P \setminus \bigcup_{i=1}^{n-1}[y_i; y_{i+1}]$.

Indeed, let T be a connected component of $P \setminus \bigcup_{i=1}^{n-1}[y_i; y_{i+1}]$. It suffices to show that $(T \setminus B) \cap Q_j = \emptyset$ for some $j = 1, 2$. If $T \subset Q_j$ for some $j = 1, 2$, this is clear. Otherwise, we find some $j = 1, 2$ such that $T' := T \cap Q_j$ satisfies $\partial T' \subset [v;w] \cup \bigcup_{i=1}^{n-1}[y_i; y_{i+1}]$ (see also Fig. 3). Now combining the two inequalities

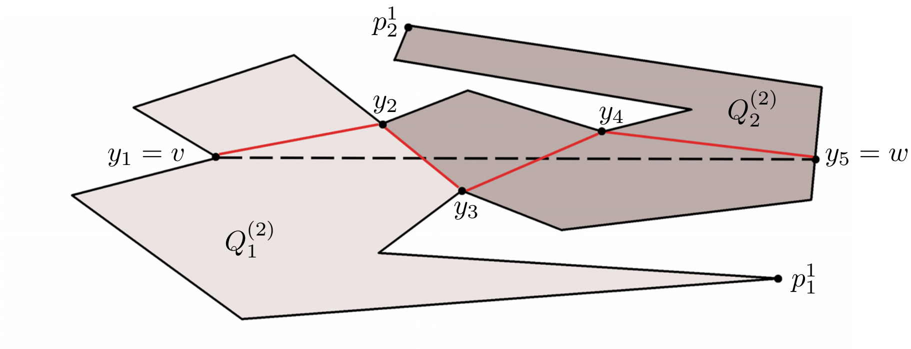

FIGURE 3. The segments $[y_i; y_{i+1}]$ inducing partitions of P are depicted in red. In light and dark gray the partition $Q_1^{(2)} \cup Q_2^{(2)}$ is sketched, where $p_j^1 \in Q_j^{(2)}$ for $j = 1, 2$. (Color online)

in (3.6), we get $d_P(v, x) \le 3|[v; w]|$ for all $x \in \partial T'$. Then also $d_P(v, x) \le 3|[v; w]|$ for all $x \in T'$ and this shows $(T \setminus B) \cap Q_j = T' \setminus B = \emptyset$.

Therefore, recalling $|[v; w]| \le \frac{1}{8} \min_j d(Q_j) \le \frac{1}{4} \min_j \operatorname{dist}_P(p_j^1, v)$, we find that $p_1^1 \in Q_1 \setminus B$ and $p_2^1 \in Q_2 \setminus B$ lie in different connected components of $P \setminus \bigcup_{i=1}^{n-1} [y_i; y_{i+1}]$. Thus, there is at least one $i = 1, \dots, i-1$ such that possibly after relabeling we have $p_1^1 \in Q_1^{(i)}$ and $p_2^1 \in Q_2^{(i)}$. Using (3.6) we find

$$\begin{aligned} \min_j d(Q_j^{(i)}) &\ge \min_j \operatorname{dist}_P(y_i, p_j^1) \\ &\ge \min_j \big(\operatorname{dist}_P(v, p_j^1) - \operatorname{dist}_P(y_i, v)\big) \ge \frac{1}{2} \min_j d(Q_j) - \frac{3}{2} |[v; w]|. \end{aligned}$$

By (3.6) and (3.8) we conclude with $\bar{\vartheta} = (3 + 12\vartheta^{-1})^{-1}$

$$\min_j d(Q_j) \le 3|[v; w]| + 4\vartheta^{-1} |[y_i; y_{i+1}]| \le (3 + 12\vartheta^{-1}) |[v; w]| = \bar{\vartheta}^{-1} |[v; w]|.$$

This shows (3.3) and concludes the proof. □

We now show that a similar property may derived if the condition in Definition 3.2(ii) only holds on a part of ∂P. To this end, we need to introduce a further notion. Suppose $[v; w]$ induces a partition of $P = Q_1 \cup Q_2$ according to Definition 2.7. We define $N'(Q_j) = \#\{u \in \mathcal{V}'_P \setminus \{v, w\} : u \in \partial Q_j\}$ for $j = 1, 2$ and the auxiliary set

$$Q_{v,w} = \begin{cases} Q_1 & \text{if} \quad N'(Q_1) < N'(Q_2) \text{ or } N'(Q_1) = N'(Q_2),\ |Q_1| < |Q_2|, \\ Q_1 \cup Q_2 & \text{if} \quad N'(Q_1) = N'(Q_2), |Q_1| = |Q_2|, \\ Q_2 & \text{else.} \end{cases} \tag{3.9}$$

Definition 3.5. We say a segment $[v; w]$ satisfies the *weak segmentation property* (WSP) if in Definition 3.1 condition (3.1) is replaced by

$$\mathcal{V}'_P \cap \operatorname{cig}_P([v; w], \eta) \cap Q_{v,w} \subset \{v, w\}. \tag{3.10}$$

We note that for (WSP) we still require (3.2). Loosely speaking, condition (3.10) only concerns the part of the polygon containing less concave vertices and is thus in general weaker than (3.1).

Corollary 3.6. *Let $0 < \vartheta, \eta < 1$ with $\vartheta \le \frac{1}{2}\eta$. Consider a polygon P and suppose $[v;w]$ induces a partition $P = Q_1 \cup Q_2$ satisfying* (WSP) *and*

$$\textit{either} \quad N'(Q_1) < N'(Q_2) \quad \textit{or} \quad N'(Q_1) = N'(Q_2),\ |Q_1| < |Q_2|. \tag{3.11}$$

Assume that for each pair $v', w' \in \mathcal{V}'_{Q_1} \cup \{v, w\}$ such that $[v';w'] \neq [v;w]$ and $[v';w']$ induces a partition $P = Q'_1 \cup Q'_2$ satisfying (WSP) *one has*

$$|[v';w']| \ge \vartheta \min_{k=1,2} d(Q'_k). \tag{3.12}$$

Then for $\eta > 0$ small enough there is $\bar\vartheta = \bar\vartheta(\vartheta) \le \vartheta$ independent of P such that each pair $\bar v, \bar w \in \mathcal{V}'_{Q_1} \cup \{v, w\}$ inducing a partition of $P = R_1 \cup R_2$ with

$$\begin{aligned} &(i) \quad [\bar v; \bar w] \neq [v;w], \\ &(ii) \quad w \in \{\bar v, \bar w\} \quad \Rightarrow \quad [\bar v;\bar w] \cap \mathrm{cig}([v;w], \eta) = \emptyset \end{aligned} \tag{3.13}$$

fulfills

$$|[\bar v;\bar w]| \ge \bar\vartheta \min_{k=1,2} d(R_k). \tag{3.14}$$

For partitions of polygons into semiconvex polygons described in Section 3.2 below we will use this corollary to show that Q_1 is semiconvex. The essential point is that for a segment $[\bar v;\bar w]$ as in (3.13) we do not assume the validity of (SP) and that (3.12) is only required for the vertices contained in Q_1. For an illustration of (3.13)(ii) we refer to Figure 4.

Proof. We follow the proof of Theorem 3.4 and only indicate the necessary changes. Fix $\bar v, \bar w \in \mathcal{V}'_{Q_1} \cup \{v, w\}$ such that $[\bar v;\bar w]$ induces a partition $P = R_1 \cup R_2$ fulfilling (3.13). Note that one of the sets, say R_1, satisfied $R_1 \subset Q_1$ and thus $N'(R_1) \le N'(Q_1)$, $|R_1| \le |Q_1|$. This yields $Q_{\bar v, \bar w} = R_1 \subset Q_1$ (see (3.9) and (3.11)). We first suppose

$$\mathcal{V}'_P \cap \mathrm{cig}_P([\bar v;\bar w], 2\eta) \cap R_1 \subset \{\bar v, \bar w\} \tag{3.15}$$

(compare to (3.4)) and show that under this assumption we have

$$|[\bar v;\bar w]| \ge \frac{\vartheta}{2} \min_{k=1,2} d(R_k). \tag{3.16}$$

The idea is to proceed as in Step 1 of the previous proof using (3.12) in place of (3.3). To this end, we notice that conditions (3.12) and (3.15) are sufficient to treat the cases (a)$-$(c). Indeed, as remarked below case (c), case (b) was a purely geometrical argument and in (a), (c) we have only shown (3.2). As by (3.16) and $Q_{\bar v,\bar w} = R_1$ also condition (3.10) holds (with $\bar v, \bar w$ in place of v, w), we derive that in case (a), (c) $[\bar v;\bar w]$ satisfies (WSP). This then implies (3.16) by (3.12). In cases (a)$-$(c) we therefore obtain (3.16). We now show that case (d) never occurs, which concludes the proof of (3.16).

Suppose case (d) occurs. Then we have that, *e.g.*, R_1 is a triangle with vertices $\bar v, \bar w, p$ such that $\sphericalangle(\bar v, R_1) \le \alpha_\eta = \frac{1}{2}\arcsin \eta$, $\sphericalangle(\bar w, R_1) > \pi - 2\alpha_\eta$ and $\bar w \notin \mathcal{V}'_P$. The latter immediately implies $\bar w = w$ since $\bar w \in \mathcal{V}'_{Q_1} \cup \{v, w\} \subset \mathcal{V}'_P \cup \{w\}$. Then $\bar v \neq w$ and $\bar v \neq v$ by (3.13)(i) and (3.13)(ii) yields that the angle enclosed by the segments $[v;w]$ and $[w;\bar v]$ is at least $2\alpha_\eta$ (*cf.* Fig. 4). This, however, contradicts the assumptions $\sphericalangle(\bar w, R_1) > \pi - 2\alpha_\eta$ and $\bar w \notin \mathcal{V}'_P$. Consequently, case (d) never occurs.

Now we consider an arbitrary segment $[\bar v;\bar w]$ with $\bar v, \bar w \in \mathcal{V}'_{Q_1} \cup \{v, w\}$ which satisfies (3.13) and induces a partition $P = R_1 \cup R_2$ with $R_1 \subset Q_1$. Note that (3.13) implies

$$w \in \partial R_1 \quad \Rightarrow \quad v \notin \partial R_1 \ \text{ and } \ R_1 \cap \mathrm{cig}([v;w], \eta) = \emptyset. \tag{3.17}$$

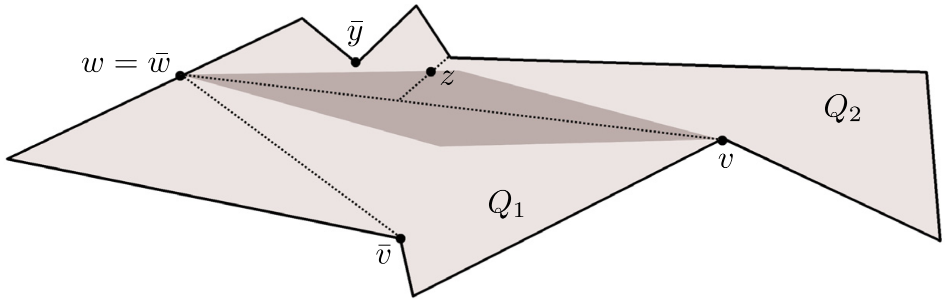

FIGURE 4. A situation with $N'(Q_1) < N'(Q_2)$ is depicted, where $[\bar{v};\bar{w}]$ does not intersect the (open) set $\mathrm{cig}([v;w],\eta)$. For the proof of Lemma 3.8 below we note that $|[w;z]| < |[v;z]|$. Therefore, $\bar{y}$ is 'nearer to w than to v' and thus $\bar{y} \notin \mathrm{cig}([v;w],\eta)$ implies $[w;\bar{y}] \cap \mathrm{cig}([v;w],\eta) = \emptyset$.

As in Step 2 of the proof of Theorem 3.4 we find a chain $(y_1,\dots,y_n)$ between $\bar{v}$ and $\bar{w}$ with $y_1 = \bar{v}$, $y_n = \bar{w}$ and $y_i \in \mathcal{V}'_{R_1} \setminus \{\bar{v},\bar{w}\}$ for $i = 2,\dots,n-1$ such that

$$|[y_i;y_{i+1}]| \le 3|[\bar{v};\bar{w}]|, \qquad d_P(\bar{v},y_i) \le \frac{3}{2}|[\bar{v};\bar{w}]|, \quad i = 1,\dots,n-1 \tag{3.18}$$

and the segments $[y_i;y_{i+1}] \subset P$ induce a partition satisfying (3.15) (with y_i, y_{i+1} in place of $\bar{v},\bar{w}$). Note that in repeating the argument in (3.7) we only select concave vertices contained in R_1, *i.e.* the essential difference to the previous proof is given by the fact that due to the replacement of (3.4) by (3.15) we can ensure that each y_i, $i = 2,\dots,n-1$, is contained in $\partial R_1 \cap \partial P$, more precisely in $\mathcal{V}'_{R_1} \setminus \{\bar{v},\bar{w}\}$.

Note that $y_i \in \mathcal{V}'_{Q_1} \cup \{v,w\}$ for all $i = 1,\dots,n$. Each segment $[y_i;y_{i+1}]$, $i = 1,\dots,n-1$, induces a partition $P = R_1^{(i)} \cup R_2^{(i)}$ such that after relabeling $R_1^{(i)} \subset R_1$. By (3.17) we have that each $[y_i;y_{i+1}]$ satisfies (3.13) (with y_i, y_{i+1} in place of $\bar{v},\bar{w}$). Consequently, as also

$$\mathcal{V}'_P \cap \mathrm{cig}_P([y_i;y_{i+1}],2\eta) \cap R_1^{(i)} \subset \{y_i,y_{i+1}\}$$

holds by (3.15) and $R_1^{(i)} \subset R_1$, for each $[y_i;y_{i+1}]$ we may proceed as above and get $|[y_i;y_{i+1}]| \ge \frac{\vartheta}{2}\min_{k=1,2} d(R_k^{(i)})$ by (3.16). This together with (3.18) allows us to proceed exactly as in Step 3 in the proof of Theorem 3.4 and we get (3.14) for $\bar{\vartheta} = (3 + 12\vartheta^{-1})^{-1}$. □

3.2. Partition of semiconvex polygons

We now show that each polygon can be partitioned into semiconvex polygons.

Theorem 3.7. *Let $0 < \theta < 1$. Then for $\eta > 0$ small there exists $\vartheta = \vartheta(\theta,\eta)$ such that for every polygon P there is a partition $P = P_1 \cup \ldots \cup P_N$ into* (SP)*-ϑ-semiconvex polygons $P_1,\dots,P_N$ such that*

$$\sum_{j=1}^{N} \mathcal{H}^1(\partial P_j) \le \left(1 + \frac{2\theta}{1-\theta}\right)\mathcal{H}^1(\partial P). \tag{3.19}$$

Clearly, by Theorem 3.4 the sets $P_1,\dots,P_N$ are then also $\bar{\vartheta}$-semiconvex for some $\bar{\vartheta}$ small enough. As a preparation we derive a partition $P = Q_1 \cup Q_2$ into two polygons such that Q_1 is (SP)-semiconvex. Then Theorem 3.7 follows by iterative application. For the proof of Theorem 3.7 it is essential that (1) the added boundary is small compared to $\mathcal{H}^1(\partial Q_1)$ (see (3.20)) and (2) Q_1 does not need to be further modified in subsequent iteration steps since hereby the overall added boundary can be controlled (see (3.31) below).

Lemma 3.8. *Let $0 < \theta < 1$. Then for $\eta > 0$ sufficiently small there is $\tilde{\vartheta} = \tilde{\vartheta}(\theta, \eta)$ such that for every polygon P, which is not an* (SP)*-$\tilde{\vartheta}$-semiconvex polygon, the following holds*: *We find a segment $[v; w]$ between a concave vertex $v \in \mathcal{V}'_P$ and some $w \in \partial P$ which satisfies* (WSP) *and induces a partition $P = Q_1 \cup Q_2$ such that Q_1 is* (SP)*-$\tilde{\vartheta}$-semiconvex and*

$$|[v; w]| \le \theta \mathcal{H}^1(\partial Q_1 \setminus [v; w]). \tag{3.20}$$

Moreover, if Q_1 is a triangle we have $\sphericalangle(v, Q_2) < \sphericalangle(v, P) - \frac{1}{2}\arcsin \eta$.

Proof. Let $0 < \theta < 1$ be given and define $\vartheta = \frac{\theta}{2}$. Let $\bar{\vartheta} \le \vartheta$ and $\eta > 0$ small as in Corollary 3.6. Define $\tilde{\vartheta} = \bar{\vartheta}\eta(4\eta + 2)^{-1}$. Let P be a non (SP)-$\tilde{\vartheta}$-semiconvex polygon.

Step 1. *Choice of $[v; w]$.*

As P is not (SP)-$\tilde{\vartheta}$-semiconvex, there is at least one segment $[v; w]$, between a concave vertex $v \in \mathcal{V}'_P$ and some $w \in \partial P$ which satisfies (SP) (and thus also (WSP)) and induces a partition $P = Q_1 \cup Q_2$ with $|[v; w]| < \tilde{\vartheta} d(Q_k) \le \vartheta d(Q_k)$ for $k = 1, 2$. In the following we label the sets such that we always have $N'(Q_1) \le N'(Q_2)$ (recall (3.9)). Choose (possibly not uniquely) a pair v, w satisfying (WSP) and

$$|[v; w]| < \vartheta \min_{k=1,2} d(Q_k) = \frac{\theta}{2} \min_{k=1,2} d(Q_k) \tag{3.21}$$

in such a way that $N'(Q_1)$ is minimized among all pairs satisfying (WSP) and (3.21). If $N'(Q_1) = N'(Q_2)$, we may suppose $|Q_1| \le |Q_2|$ after possible relabeling. After a small perturbation of the point w we may assume that $|Q_1| < |Q_2|$ and (WSP), (3.21) are still satisfied. (Recall here that $\mathrm{cig}_P([v; w], \eta)$ is closed). Moreover, we note that v, w can be selected such that

$$[w_*; w] \subset \partial P \quad \text{and} \quad |[w_*; w]| \le \frac{1}{2}|[v; w]| \tag{3.22}$$

for all $w_* \in \partial Q_1$ with the property that $[v; w_*]$ induces $P = Q_1^* \cup Q_2^*$ satisfying

$$\text{(WSP)}, \quad N'(Q_1^*) = N'(Q_1), \quad \text{and} \quad |[v; w_*]| < \vartheta \min_{k=1,2} d(Q_k^*). \tag{3.23}$$

In fact, if (3.22) is violated for some w_* which satisfies (3.23), we can replace the pair v, w by the pair v, w_* in the above choice (accordingly, we replace Q_1 by the smaller set Q_1^*). Possibly repeating this procedure at most $\mathcal{V}_P + \lfloor \frac{2\mathcal{H}^1(\partial P)}{d_v} \rfloor$ times, where $d_v := \inf\{|[v; w']| : [v; w'] \text{ induces a partition of } P\} > 0$, we obtain a (not relabeled) pair v, w such that (3.22) holds for all w_* satisfying (3.23).

Choose $p, p' \in \partial Q_1$ with $d(Q_1) = \mathrm{dist}_{Q_1}(p, p')$. Since $d(Q_1) \le \mathrm{dist}_{Q_1}(p, v) + \mathrm{dist}_{Q_1}(v, p')$, we can without restriction assume that $\mathrm{dist}_{Q_1}(p, v) \ge \frac{1}{2}d(Q_1)$ and thus by (3.21) we get

$$\mathrm{dist}_{Q_1}(p, v) \ge \theta^{-1}|[v; w]| \quad \text{and} \quad \mathrm{dist}_{Q_1}(p, w) \ge \big(\theta^{-1} - 1\big)|[v; w]|.$$

Consequently, $\mathcal{H}^1(\partial Q_1 \setminus [v; w]) \ge \mathrm{dist}_{Q_1}(p, v) + \mathrm{dist}_{Q_1}(p, w)$ and in view of $\theta < 1$ a short calculation yields (3.20). The additional assertion after (3.20) follows directly from the fact that $[v; w]$ satisfies (WSP), particularly (3.2), where we use $\sphericalangle(v, Q_2) = \sphericalangle(v, P) - \sphericalangle(v, Q_1)$. It remains to show that Q_1 is (SP)-$\tilde{\vartheta}$-semiconvex.

Step 2. *Semiconvexity of Q_1.*

As a preparation we show that the assumptions of Corollary 3.6 are satisfied. Consider a pair $v', w' \in \mathcal{V}'_{Q_1} \cup \{v, w\}$ such that the segment $[v'; w'] \ne [v; w]$ induces a partition $P = Q_1' \cup Q_2'$ satisfying (WSP) with $Q_1' \subset Q_1$. As either (i) $v' \ne v$ and $w' \ne v$ or (ii) up to relabeling $v' = v$, $w' \ne w$ with $w' \in \mathcal{V}'_P$, we get $\#\{u \in \mathcal{V}'_P \setminus \{v', w'\} : u \in \partial Q_1'\} < \#\{u \in \mathcal{V}'_P \setminus \{v, w\} : u \in \partial Q_1\}$. Thus, $N'(Q_1') < N'(Q_1)$ (see before (3.9)). As (again up to relabeling

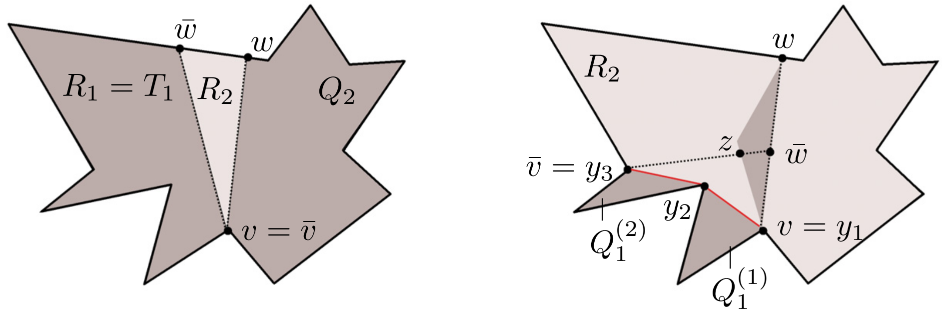

FIGURE 5. On the left case (a) is depicted, where $R_2 = Q_1 \cap T_2$ and $Q_1 = T_1 \cup R_2$. On the right case (b) with $|[v;z]| < |[w;z]|$ is illustrated.

of the points) $v' \in \mathcal{V}'_P$ and $w' \in \partial P$, we observe that $N'(Q'_1) < N'(Q_1)$ together with the choice of $N'(Q_1)$ and (3.21) implies

$$|[v';w']| \geq \vartheta \min_{k=1,2} d(Q'_k) \tag{3.24}$$

and thus (3.12) holds. Moreover, we recall that (3.11) is satisfied by the choice of Q_1 (see before (3.22)).

We now show that Q_1 is (SP)-$\tilde{\vartheta}$-semiconvex. To this end, consider a pair $\bar{v} \in \mathcal{V}'_{Q_1}$ and $\bar{w} \in \partial Q_1$ such that $[\bar{v};\bar{w}]$ induces a partition $Q_1 = R_1 \cup R_2$ satisfying (SP). In particular, $\mathcal{V}'_{Q_1} \cap \mathrm{cig}_{Q_1}([\bar{v};\bar{w}],\eta) \subset \{\bar{v},\bar{w}\}$ by (3.1). We distinguish the cases (a) $\bar{w} \notin [v;w] \setminus \{v,w\}$ and (b) $\bar{w} \in [v;w] \setminus \{v,w\}$.

(a) Assume $\bar{w} \notin [v;w] \setminus \{v,w\}$. Clearly, $[\bar{v};\bar{w}]$ induces also a partition $P = T_1 \cup T_2$, where we label the sets such that $R_1 = T_1 \subset Q_1$. Since (3.11) holds, we have $Q_{\bar{v},\bar{w}} = T_1$ and therefore

$$\mathcal{V}'_P \cap \mathrm{cig}_P([\bar{v};\bar{w}],\eta) \cap Q_{\bar{v},\bar{w}} \subset \mathcal{V}'_{Q_1} \cap \mathrm{cig}_{Q_1}([\bar{v};\bar{w}],\eta) \subset \{\bar{v},\bar{w}\}.$$

Consequently, $[\bar{v};\bar{w}]$ satisfies (WSP) with respect to the partition $P = T_1 \cup T_2$.

(a1) Assume $\bar{w} \notin [v;w] \setminus \{v,w\}$ and $N'(T_1) < N'(Q_1)$. Since $\bar{w} \in \partial P$ and $\bar{v} \in \mathcal{V}'_P$, we may proceed as in (3.24), particularly using (3.21), to find

$$|[\bar{v};\bar{w}]| \geq \vartheta \min_{k=1,2} d(T_k) \geq \vartheta \min_{k=1,2} d(R_k).$$

(a2) Now suppose $\bar{w} \notin [v;w] \setminus \{v,w\}$ and $N'(T_1) = N'(Q_1)$. Since $T_1 \subset Q_1$, this is only possible if $\bar{v} = v$. If $|[\bar{v};\bar{w}]| \geq \vartheta \min_{k=1,2} d(T_k)$, we proceed as in (a). Otherwise, by (3.22) we obtain $[w;\bar{w}] \subset \partial P$ and $|[w;\bar{w}]| \leq \frac{1}{2}|[v;w]|$. Consequently, we get $|[\bar{v};\bar{w}]| \geq \frac{1}{2}|[v;w]|$. As $R_2 = T_2 \cap Q_1$ is a triangle with vertices $v, w, \bar{w}$ (*cf.* (3.22) and Fig. 5), we deduce

$$\min_{k=1,2} d(R_k) \leq d(R_2) = d(T_2 \cap Q_1) \leq |[v;w]| + |[w;\bar{w}]| \leq \frac{3}{2}|[v;w]| \leq 3|[\bar{v};\bar{w}]|.$$

Thus, in both cases (a1), (a2) condition (3.3) holds since $\tilde{\vartheta} \leq \min\{\vartheta, \frac{1}{3}\}$ for η small.

(b) It now remains to treat the case $\bar{w} \in [v;w] \setminus \{v,w\}$. We label the sets such that $v \in R_1$ and $w \in R_2$. As $[v;w]$ satisfies (WSP) and $Q_{v,w} = Q_1$ by (3.11), $\bar{v} \notin \mathrm{cig}_P([v;w],\eta)$ holds. Thus, $\bar{v} \notin \mathrm{cig}([v;w],\eta)$ since $[\bar{v};\bar{w}] \subset P$. Let z be the intersection point of $\partial\mathrm{cig}([v;w],\eta)$ with $[\bar{v};\bar{w}]$ (see Fig. 5). We first treat the case $|[v;z]| \leq |[w;z]|$ and present the necessary adaptions for the other case at the end of the proof. Recall that the goal is to show $\tilde{\vartheta} \min_{k=1,2} d(R_k) \leq |[\bar{v};\bar{w}]|$. Now $\bar{v} \notin \mathrm{cig}([v;w],\eta)$ and $|[v;z]| \leq |[w;z]|$ imply (*cf.* Fig. 5)

$$|[\bar{v};\bar{w}]| \geq |[\bar{w};z]| \geq \eta|[v;z]|. \tag{3.25}$$

Choose the (unique) chain $(y_1 = v, y_2, \ldots, y_n = \bar{v})$ with $y_i \in \mathcal{V}'_P \cap \mathcal{V}_{R_1}$ for $i = 2, \ldots, n-1$ such that $d_P(v, \bar{v}) = \sum_{i=1}^{n-1} |[y_i; y_{i+1}]|$ and $[y_i; y_{i+1}]$ induces a partition $P = Q_1^{(i)} \cup Q_2^{(i)}$, where the sets are labeled such that $w \in Q_2^{(i)}$. Observe that $Q_1^{(i)} \subset R_1$ as $[\bar{v}; \bar{w}]$ induces a partition of $Q_1 = R_1 \cup R_2$. Then by (3.25) and $[v; \bar{w}] \cup [\bar{v}; \bar{w}] \subset P$

$$d_P(v, \bar{v}) \leq |[v; \bar{w}]| + |[\bar{v}; \bar{w}]| \leq |[v; z]| + |[z; \bar{w}]| + |[\bar{v}; \bar{w}]| \leq (2 + \eta^{-1})|[\bar{v}; \bar{w}]|. \tag{3.26}$$

Note that each $[y_i; y_{i+1}]$ satisfies (3.13) as $w \notin \partial R_1$. Then using Corollary 3.6, in particular (3.14), we find for all $i = 1, \ldots, n-1$

$$|[y_i; y_{i+1}]| \geq \bar{\vartheta} \min_{k=1,2} d(Q_k^{(i)}). \tag{3.27}$$

First assume there was some i such that $d(Q_2^{(i)}) \leq d(Q_1^{(i)})$. Then we calculate using $Q_2^{(i)} \supset R_2$ and (3.26)

$$d(R_2) \leq d(Q_2^{(i)}) \leq \bar{\vartheta}^{-1}|[y_i; y_{i+1}]| \leq \bar{\vartheta}^{-1} d_P(v, \bar{v}) \leq \bar{\vartheta}^{-1}(2 + \eta^{-1})|[\bar{v}; \bar{w}]|.$$

Otherwise, we find by (3.26), (3.27) and $d_P(v, \bar{v}) = \sum_{i=1}^{n-1} |[y_i; y_{i+1}]|$ (*cf.* Fig. 5)

$$\begin{aligned} d(R_1) &\leq 2 \max_{p \in R_1} d_P(v, p) \leq 2 \max\{|[v; \bar{w}]| + |[\bar{w}; \bar{v}]|, \max_{i=1,\ldots,n-1} (d_P(v, y_i) + d(Q_1^{(i)}))\} \\ &\leq 2 \max\{|[v; \bar{w}]| + |[\bar{w}; \bar{v}]|, \bar{\vartheta}^{-1} d_P(v, \bar{v})\} \leq 2\bar{\vartheta}^{-1}(2 + \eta^{-1})|[\bar{v}; \bar{w}]|. \end{aligned}$$

Collecting the last two estimates and recalling $\tilde{\vartheta} = \bar{\vartheta}\eta(4\eta + 2)^{-1}$ we get $\tilde{\vartheta} \min_{k=1,2} d(R_k) \leq |[\bar{v}; \bar{w}]|$, as desired.

It remains to treat the case $|[v; z]| > |[w; z]|$. We may proceed as before with w in place of v with the only difference that, due to the fact that the chain $(y_1 = w, \ldots, y_n = \bar{v}) \subset \mathcal{V}_{R_2}$ contains w, for the application of (3.14) we have to check that (3.13)(ii) holds for $[y_1; y_2]$. Indeed, since $[v; w]$ satisfies (WSP), the fact that $Q_1 = Q_{v,w}$ implies $y_2 \notin \mathrm{cig}_P([v; w], \eta)$ and then $y_2 \notin \mathrm{cig}([v; w], \eta)$ since $[y_1; y_2] \subset P$. Then $|[v; z]| > |[w; z]|$ together with $y_2 \in R_2$ yield $[y_1; y_2] \cap \mathrm{cig}([v; w], \eta) = \emptyset$ (*cf.* illustration of $\bar{y}$ in Fig. 4). ☐

Now we can give the proof of Theorem 3.7.

Proof of Theorem 3.7. We construct the partition inductively. Assume $P_1, \ldots, P_n$ have been constructed and set $R_n = \overline{P \setminus \bigcup_{j=1}^n P_j}$. (For $n = 0$ we set $R_0 = P$). Moreover, suppose that

$$\mathcal{H}^1 \left(\partial P_j \setminus \bigcup\nolimits_{i=0}^{j-1} \partial P_i \right) \leq \theta \mathcal{H}^1 \left(\partial P_j \cap \bigcup\nolimits_{i=0}^{j-1} \partial P_i \right) \tag{3.28}$$

for $j = 1, \ldots, n$, where $P_0 := P$. If R_n is (SP)-semiconvex, we set $P_{n+1} = R_n$ and stop. Otherwise, by Lemma 3.8 we find a partition $R_n = P_{n+1} \cup R_{n+1}$ such that P_{n+1} is (SP)-semiconvex and $R_{n+1} = \overline{R_n \setminus P_{n+1}} = \overline{P \setminus \bigcup_{j=1}^{n+1} P_j}$. Furthermore, we obtain by (3.20)

$$\begin{aligned} \mathcal{H}^1 \left(\partial P_{n+1} \setminus \bigcup\nolimits_{i=0}^{n} \partial P_i \right) &= \mathcal{H}^1(P_{n+1} \cap R_{n+1}) \leq \theta \mathcal{H}^1(\partial P_{n+1} \setminus (P_{n+1} \cap R_{n+1})) \\ &= \theta \mathcal{H}^1 \left(\partial P_{n+1} \cap \bigcup\nolimits_{i=0}^{n} \partial P_i \right), \end{aligned}$$

which gives (3.28) for $j = n+1$.

Recall that in each step the number of vertices of the remaining polygon decreases (namely if P_{n+1} is not a triangle) or the angle of a concave vertex in the remaining polygon decreases by at least $\frac{1}{2} \arcsin \eta$ (if P_{n+1} is a triangle). Thus, there is some $N \in \mathbb{N}$ such that the polygon $P_N := R_{N-1}$ is (SP)-semiconvex since for large $n \in \mathbb{N}$ the polygon R_{n-1} is eventually convex and thus also (SP)-semiconvex. It remains to show (3.19). First, we note

$$\mathcal{H}^1(\partial P_n \cap \partial P_i) = \mathcal{H}^1(\partial R_{n-1} \cap \partial P_i) - \mathcal{H}^1(\partial R_n \cap \partial P_i) \tag{3.29}$$

for $1 \le n \le N$ and $0 \le i \le n-1$, where we set $R_N := \emptyset$ and $R_0 = P$. Moreover, by (3.28) we get for $2 \le n \le N$

$$\mathcal{H}^1(\partial R_{n-1} \cap \partial P_{n-1}) = \mathcal{H}^1\left(\partial P_{n-1} \setminus \bigcup\nolimits_{i=0}^{n-2} \partial P_i\right) \le \theta \mathcal{H}^1\left(\partial P_{n-1} \cap \bigcup\nolimits_{i=0}^{n-2} \partial P_i\right). \tag{3.30}$$

Then by (3.29)–(3.30) we obtain $\mathcal{H}^1(\partial P_1 \cap \partial P) = \mathcal{H}^1(\partial P) - \mathcal{H}^1(\partial R_1 \cap \partial P)$ and for $2 \le n \le N$

$$\begin{aligned}\mathcal{H}^1\left(\partial P_n \cap \bigcup\nolimits_{i=0}^{n-1} \partial P_i\right) &= \sum_{i=0}^{n-1} \mathcal{H}^1\left(\partial P_n \cap \partial P_i\right) \le \theta \mathcal{H}^1\left(\partial P_{n-1} \cap \bigcup\nolimits_{i=0}^{n-2} \partial P_i\right) \\ &\quad + \sum_{i=0}^{n-2} \mathcal{H}^1(\partial R_{n-1} \cap \partial P_i) - \sum_{i=0}^{n-1} \mathcal{H}^1(\partial R_n \cap \partial P_i).\end{aligned}$$

By summation and an index shift we derive

$$\begin{aligned}\sum_{n=1}^{N} \mathcal{H}^1\left(\partial P_n \cap \bigcup\nolimits_{i=0}^{n-1} \partial P_i\right) &\le \mathcal{H}^1(\partial P_1 \cap \partial P) + \theta \sum_{n=1}^{N-1} \mathcal{H}^1\left(\partial P_n \cap \bigcup\nolimits_{i=0}^{n-1} \partial P_i\right) \\ &\quad + \sum_{n=1}^{N-1} \sum_{i=0}^{n-1} \mathcal{H}^1(\partial R_n \cap \partial P_i) - \sum_{n=2}^{N} \sum_{i=0}^{n-1} \mathcal{H}^1(\partial R_n \cap \partial P_i) \\ &\le \mathcal{H}^1(\partial P) + \theta \sum_{n=1}^{N} \mathcal{H}^1\left(\partial P_n \cap \bigcup\nolimits_{i=0}^{n-1} \partial P_i\right),\end{aligned}$$

where we used that $\partial R_N = \emptyset$ and $\mathcal{H}^1(\partial R_1 \cap \partial P) + \mathcal{H}^1(\partial P_1 \cap \partial P) = \mathcal{H}^1(\partial P)$. This yields $\sum_{n=1}^{N} \mathcal{H}^1(\partial P_n \cap \bigcup_{i=0}^{n-1} \partial P_i) \le (1-\theta)^{-1} \mathcal{H}^1(\partial P)$. Together with (3.28) and the fact that every $x \in \bigcup_{n=1}^{N} \partial P_n \setminus \partial P$ is contained in the boundary of exactly two sets, we conclude

$$\begin{aligned}\sum_{n=1}^{N} \mathcal{H}^1(\partial P_n) &= \mathcal{H}^1(\partial P) + 2\mathcal{H}^1\left(\bigcup\nolimits_{n=1}^{N} \partial P_n \setminus \partial P\right) \\ &= \mathcal{H}^1(\partial P) + 2 \sum_{n=1}^{N} \mathcal{H}^1\left(\partial P_n \setminus \bigcup\nolimits_{i=0}^{n-1} \partial P_i\right) \\ &\le \mathcal{H}^1(\partial P) + 2\theta \sum_{n=1}^{N} \mathcal{H}^1\left(\partial P_n \cap \bigcup\nolimits_{i=0}^{n-1} \partial P_i\right) \le \left(1 + \frac{2\theta}{1-\theta}\right) \mathcal{H}^1(\partial P).\end{aligned} \tag{3.31}$$

□

Later in Section 6.1 for the proof of Theorem 1.1 we will need the following observations.

Remark 3.9.

(i) Recall that by construction the partition in Theorem 3.7 arises from P by introducing a finite number of segments. As by this procedure no additional concave vertices are introduced, we find $v \in \partial P$ for all $v \in \bigcup_{j=1}^{N} \mathcal{V}'_{P_j}$.

(ii) By a slight modification of the segments $[v; w]$ introduced in Lemma 3.8 (*cf.* Remark below (3.21)) we can always ensure that the segments $[v_i; w_i] = P_i \cap P_{i+1}$ have the property that the points w_i are not vertices of P and are pairwise distinct.

(iii) The partition can be chosen with the following additional property: if two convex polygons $P^1, P^2 \subset (P_j)_{j=1}^N$ share some $v \in \mathcal{V}'_P$ with $\sphericalangle(v, P^i) \le \frac{\pi}{4}$ for $i = 1, 2$, then $\mathcal{H}^1(\partial P^1 \cap \partial P^2) = 0$. Indeed, otherwise we find $w \in \mathcal{V}_{P^1} \cap \mathcal{V}_{P^2}$ such that $\partial P^1 \cap \partial P^2 = [v; w]$. Then with $P_* = P^1 \cup P^2$ we have $\sphericalangle(v, P_*) \le \frac{\pi}{2}$ and $\sphericalangle(w, P_*) = \pi$ by (ii). Consequently, P_* is a convex polygon and we can replace in the partition P^1, P^2 by P_*.

We close this section with a further criterion for the partition of a semiconvex polygon.

Lemma 3.10. *Let $0 < \alpha, \vartheta < 1$. Then there is $\bar\vartheta = \bar\vartheta(\alpha, \vartheta) > 0$ such that for all ϑ-semiconvex polygons P the following holds: If there is a segment $[u_1; u_2]$ inducing a partition $P = P_1 \cup P_2$ such that for each concave vertex $v \in \mathcal{V}'_{P_1}$ one has that*

$$\max_{k=1,2} \sphericalangle(u_k, \triangle_v) \ge \alpha,$$

where $\triangle_v$ is the triangle with vertices v, u_1, u_2, then P_1 is $\bar\vartheta$-semiconvex.

Proof. Let P and the partition $P = P_1 \cup P_2$ with the above properties be given. To see that P_1 is $\bar\vartheta$-semiconvex for some $\bar\vartheta \le \vartheta$ to be specified below, it suffices to show that for each segment $[v; w]$ between a concave vertex $v \in \mathcal{V}'_{P_1}$ and some $w \in [u_1; u_2]$, which induces a partition $P_1 = Q_1 \cup Q_2$, one has

$$|[v; w]| \ge \bar\vartheta \min_{k=1,2} d(Q_k). \tag{3.32}$$

Indeed, for $w \in \partial P_1 \setminus [u_1; u_2]$ the property follows directly from the fact that P is ϑ-semiconvex. Without restriction we assume $\sphericalangle(u_1, \triangle_v) \ge \sphericalangle(u_2, \triangle_v)$ and label the sets such that $u_1 \in Q_1$. Similarly as in the proof of Lemma 3.8 we choose the unique chain $(y_1 = v, y_2, \dots, y_n = u_1)$ with $y_i \in \mathcal{V}'_P$ for $i = 2, \dots, n-1$ such that $d_P(v, u_1) = \sum_{i=1}^{n-1} |[y_i; y_{i+1}]|$ and $[y_i; y_{i+1}]$ induces a partition $P = Q_1^{(i)} \cup Q_2^{(i)}$, where the sets are labeled such that $u_2 \in Q_2^{(i)}$. Since P is ϑ-semiconvex, we get by (3.3)

$$|[y_i; y_{i+1}]| \ge \vartheta \min_{k=1,2} d(Q_k^{(i)}) \tag{3.33}$$

for $i = 1, \dots, n-1$. Observe that $Q_1^{(i)} \subset Q_1$ as $[v; w]$ induces a partition of P_1. Using $\sphericalangle(u_1, \triangle_v) \ge \alpha$ and the cosine formula we find by an elementary computation

$$|[v; w]| \ge \sqrt{|[v; u_1]|^2 + |[u_1; w]|^2 - 2|[v; u_1]||[u_1, w]| \cos\alpha} \ge C_\alpha(|[v; u_1]| + |[u_1; w]|)$$

for $C_\alpha > 0$ small depending only on α. Using that $[v; w] \cup [w; u_1] \subset P$ we then derive

$$d_P(v, u_1) \le |[v; w]| + |[w; u_1]| \le |[v; w]| + C_\alpha^{-1}|[v; w]| = (1 + C_\alpha^{-1})|[v; w]|. \tag{3.34}$$

We now proceed as in the proof of Lemma 3.8. First assume there is some i such that $d(Q_2^{(i)}) \le d(Q_1^{(i)})$. Then we calculate using $Q_2^{(i)} \supset Q_2$, (3.33) and (3.34)

$$d(Q_2) \le d(Q_2^{(i)}) \le \vartheta^{-1}|[y_i; y_{i+1}]| \le \vartheta^{-1} d_P(v, u_1) \le (1 + C_\alpha^{-1})\vartheta^{-1}|[v; w]|.$$

Otherwise, we find again by (3.33) and (3.34)

$$\begin{aligned} d(Q_1) &\le 2 \max_{p \in Q_1} d_P(v, p) \le 2 \max\{|[v; w]| + |[w; u_1]|, \max_{i=1,\dots,n-1} (d_P(v, y_i) + d(Q_1^{(i)}))\} \\ &\le 2 \max\{|[v; w]| + |[w; u_1]|, \vartheta^{-1} d_P(v, u_1)\} \le 2\vartheta^{-1}(1 + C_\alpha^{-1})|[v; w]|. \end{aligned}$$

Consequently, (3.32) holds for $\bar\vartheta = \vartheta C_\alpha (2 + 2C_\alpha)^{-1}$ and thus P_1 is $\bar\vartheta$-semiconvex. □

4. Semiconvex and rotund polygons

In the section we introduce a further subclass of polygons.

Definition 4.1. Let $\omega > 0$. We say a polygon P is ω-rotund if there is a ball $B(x, r) \subset P$ with $x \in P$ and $r \geq \omega d(P)$.

Similarly as before, we drop the parameter ω if no confusion arises. This property together with the semiconvexity will be the main ingredient to show that polygons may be partitioned into John domains with controllable John constant. In Section 4.1 we study the relation between the notions introduced in Definitions 3.2 and 4.1. In Section 4.2 we then show that semiconvex polygons can be partitioned into semiconvex and rotund polygons.

4.1. Properties of semiconvex and rotund polygons

To avoid confusion with further subscripts we will from now on denote by $x\mathbf{e}_j$ the jth component of points $x \in \mathbb{R}^2$. For sets $A \subset \mathbb{R}^2$ and $R \in SO(2)$ we let $|A|_{\Pi,R} = \sup_{x,y\in A} |(x-y)R\mathbf{e}_1|$. We will also use the notation $|A|_{\Pi,j} = \sup_{x,y\in A} |(x-y)\mathbf{e}_j|$ for $j = 1, 2$. By $\mathrm{int}(A)$ we denote the interior of a set. Recall also the notions introduced in Section 2.3. We begin with a simple property of convex polygons.

Lemma 4.2. *Every convex polygon P contains a ball with radius*

$$\frac{1}{4} \min_{R\in SO(2)} |P|_{\Pi,R}.$$

Proof. By [28] we find that for each convex polygon P there is a rectangle S and a homothetic copy S' of S such that $S \subset P \subset S'$ and the positive homothety ratio is at most 2. As $P \subset S'$, both rectangle sides of S' are larger than $\min_{R\in SO(2)} |P|_{\Pi,R}$ and thus each rectangle side of S is larger than $\frac{1}{2} \min_{R\in SO(2)} |P|_{\Pi,R}$. □

We now show that the intrinsic diameter of semiconvex polygons P can be controlled in terms of $|P|_{\Pi,R}$.

Lemma 4.3. *Let $0 < \vartheta < 1$ and let P be a ϑ-semiconvex polygon. Then*

$$\vartheta d(P) \leq 2 \max_{R\in SO(2)} |P|_{\Pi,R}.$$

Proof. If P is convex, the assertion is clear. Otherwise, choose $p_1, p_2 \in P$ with $d_P(p_1, p_2) = d(P)$ and let $\gamma : [0, l(\gamma)] \to P$ be a piecewise affine curve between p_1, p_2, parametrized by arc length, with $l(\gamma) = d(P)$.

Since P is not convex, there is some $v \in \mathcal{V}'_P$ such that $[v; \gamma(\frac{l(\gamma)}{2})] \subset P$. (Possibly we have to take $v = \gamma(\frac{l(\gamma)}{2})$). Then we can choose $w \in \partial P$ such that $\gamma(\frac{l(\gamma)}{2}) \in [v; w]$ and $[v; w]$ induces a partition $P = Q_1 \cup Q_2$ according to Definition 2.7. The choice of γ implies $\min_{k=1,2} d(Q_k) \geq \frac{1}{2} d(P)$ and thus we conclude, using that P is ϑ-semiconvex

$$\frac{\vartheta}{2} d(P) \leq \vartheta \min_{k=1,2} d(Q_k) \leq |[v; w]| \leq \max_{R\in SO(2)} |P|_{\Pi,R}.$$ □

We now formulate the first main result of this section stating that semiconvex polygons are rotund if the lengths of shortest and longest extend are comparable.

Theorem 4.4. *Let $0 < \vartheta, \lambda < 1$. Then there is an $\omega = \omega(\vartheta, \lambda) > 0$ such that all ϑ-semiconvex polygons P with*

$$\min_{R\in SO(2)} |P|_{\Pi,R} \geq \lambda \max_{R\in SO(2)} |P|_{\Pi,R}$$

are ω-rotund.

Whereas the statement is straightforward for convex polygons by Lemma 4.2, the argument for nonconvex polygons relies on the observation that concave vertices are ‘not too close to opposite parts of the boundary’ due to condition (3.3).

Proof. Choose $p_1, p_2 \in P$ with $d_P(p_1,p_2) = d(P)$ and let $\gamma : [0, l(\gamma)] \to P$ be a piecewise affine curve between p_1, p_2, parametrized by arc length, with $l(\gamma) = d(P)$. As noticed in Section 2.3, recall that the endpoints of each segment of γ are contained in $\mathcal{V}'_P \cup \{p_1, p_2\}$. Define $\delta = \frac{1}{14}\vartheta\lambda$ and set for shorthand $q_1 = \gamma(\delta)$ and $q_2 = \gamma(1-\delta)$. We distinguish two cases:

(a) First assume $\gamma([\delta, 1-\delta]) = [q_1; q_2]$ is a segment with $q_1, q_2 \notin \partial P$ and suppose that after translation and rotation we have $q_1 = (t_1, 0)$, $q_2 = (t_2, 0)$ with $t_1 < t_2$. For $k = 1,2$ denote by S_k the connected component of $(\{t_k\} \times \mathbb{R}) \cap \mathrm{int}(P)$ containing q_k. The segments $\overline{S_1}, \overline{S_2}$ induce a partition $P = P_1 \cup P' \cup P_2$ of P with $P_k \cap P' = \overline{S_k}$ for $k = 1,2$ (*cf.* Fig. 6). First, by the fact that $l(\gamma) = d(P)$ and $2\delta \le \frac{1}{2}$ we get

$$d(P') \ge |P'|_{\Pi,1} = |[q_1; q_2]| \ge (1-2\delta)d(P) \ge \frac{1}{2}d(P). \tag{4.1}$$

Moreover, we obtain for $k = 1,2$ by the choice of γ and q_1, q_2

$$\mathrm{dist}_P(x, S_k) \le 3\delta d(P) \text{ for all } x \in P_k. \tag{4.2}$$

Indeed, *e.g.* for $k = 1$, we observe $\mathrm{dist}_P(y, q_2) \ge (1-2\delta)d(P)$ for all $y \in S_1$ and $\mathrm{dist}_P(q_2, p_2) = \delta d(P)$. This implies $\mathrm{dist}_P(y, p_2) \ge (1-3\delta)d(P)$ for all $y \in S_1$, from which (4.2) follows. Consequently, by (4.2), Lemma 4.3 and $\delta = \frac{1}{14}\vartheta\lambda$ we obtain

$$\min_{R \in SO(2)} |P'|_{\Pi,R} \ge \min_{R \in SO(2)} |P|_{\Pi,R} - 6\delta d(P) \ge \lambda \max_{R \in SO(2)} |P|_{\Pi,R} - 6\delta d(P) \ge \delta d(P). \tag{4.3}$$

(a1) If P' is a convex polygon, we find by (4.3) and Lemma 4.2 that P' contains a ball $B(x,r)$ with $r = \frac{\delta}{4}d(P)$ and thus, since $P \supset P'$, P is $\frac{\delta}{4}$-rotund.

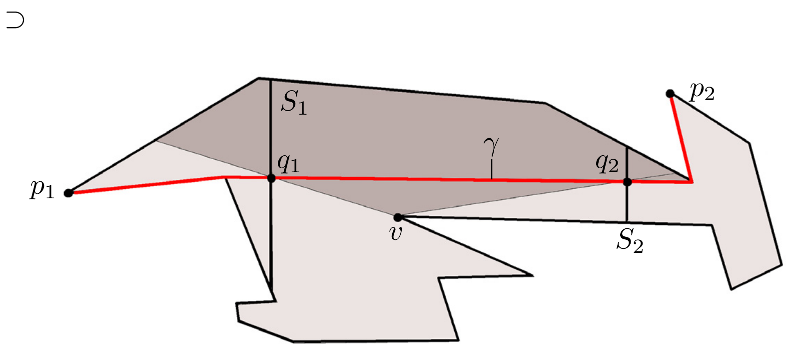

FIGURE 6. We sketched case (a2), where T is contained in the dark gray set.

(a2) Otherwise, we choose a concave vertex $v \in \mathcal{V}'_{P'}$ which minimizes the distance to $[q_1; q_2]$. This implies that the triangle with vertices v, q_1, q_2 is contained in P' (see Fig. 6). Understanding the vertices as complex numbers we define the phases $\varphi_1 = \arg(q_1 - v) \in [0, 2\pi)$ and $\varphi_2 = \arg(q_2 - v) \in [0, 2\pi)$, where possible after reflection of P' along $\mathbb{R} \times \{0\}$ and a rotation we can suppose that $0 \le \varphi_2 < \varphi_1 < 2\pi$ with $\varphi_1 - \varphi_2 < \pi$ (*cf.* Fig. 6).

We define the function $f : [\varphi_2, \varphi_1] \to P$ so that $f(\varphi)$ denotes the closest point to v on $(v + \mathbb{R}_+ e^{i\varphi}) \cap \partial P$. Observe that for $\varphi \in [\varphi_2, \varphi_1]$ each $[v; f(\varphi)]$ induces a partition $P = Q_1^\varphi \cup Q_2^\varphi$ according to Definition 2.7 with $p_k, q_k \in Q_k^\varphi$ for $k = 1, 2$. Consequently, by definition of γ and q_k, $k = 1, 2$, we get $\min_{k=1,2} d(Q_k^\varphi) \geq \delta d(P)$ for all $\varphi \in [\varphi_1, \varphi_2]$ and then we obtain

$$|[v; f(\varphi)]| \geq \vartheta \min_{k=1,2} d(Q_k^\varphi) \geq \vartheta\delta d(P) \tag{4.4}$$

since P is ϑ-semiconvex. Consequently, we derive that the circular sector

$$T := \{x \in \mathbb{R}^2 : \arg(x - v) \in [\varphi_2, \varphi_1],\ |[x; v]| \leq \vartheta\delta d(P)\}$$

is contained in P. Clearly, we have $\operatorname{dist}(v, [q_1; q_2]) \leq d(P)$, which in view of $|[q_1; q_2]| \geq \frac{1}{2} d(P)$ (see (4.1)) implies $\varphi_1 - \varphi_2 \geq \arctan(1/2)$ by elementary trigonometry. Then it is not hard to see that there is a ball $B(x, r) \subset T \subset P$ with $r \geq c\vartheta\delta d(P)$ for a universal $c > 0$ small enough. This yields that P is ω-rotund for some ω only depending on ϑ, λ.

(b) We now suppose that $\gamma([\delta, 1 - \delta])$ is not a segment or $q_k \in \partial P$ for some $k = 1, 2$, *i.e.* we find $v \in \mathcal{V}'_P$ with $v \in \gamma([\delta, 1 - \delta])$. Choose $v_-, v_+ \in \gamma$ such that $[v_-; v]$ and $[v; v_+]$ are contained in γ. Let $\varphi_- = \arg(v_- - v)$, $\varphi_+ = \arg(v_+ - v)$ and without restriction, possibly after a rotation and reflection, we can assume that $0 \leq \varphi_+ < \varphi_- < 2\pi$ with $\varphi_- - \varphi_+ > \pi$. We now proceed as in (a):

We see that $[v; f(\varphi)]$ induces a partition $P = Q_1^\varphi \cup Q_2^\varphi$ for all $\varphi \in (\varphi_+, \varphi_-)$ with $p_k \in Q_k^\varphi$, $k = 1, 2$, and $|[v; f(\varphi)]| \geq \vartheta \min_{k=1,2} d(Q_k^\varphi) \geq \vartheta \min_{k=1,2} d_P(v, p_k) \geq \vartheta\delta d(P)$ (*cf.* (4.4)). Then as before the set $\{x \in \mathbb{R}^2 : \arg(x - v) \in (\varphi_+, \varphi_-),\ |[x; v]| \leq \vartheta\delta d(P)\}$ is contained in P. Since $\varphi_- - \varphi_+ > \pi$, we conclude that P contains a ball with radius larger than $c\vartheta\delta d(P)$. ☐

The result shows that if $\max_{R \in SO(2)} |P|_{\Pi,R}$ and $\min_{R \in SO(2)} |P|_{\Pi,R}$ are comparable, the polygon P already has the desired properties. Otherwise, we will perform a partition of semiconvex polygons into semiconvex and rotund polygons as described in Section 4.2 below. To this end, it is crucial to characterize the position of concave vertices in a semiconvex polygon. The following result shows that for a semiconvex polygon, which is not already rotund, one can identify (at most) two regions which contain the concave vertices.

Theorem 4.5. *Let $0 < \vartheta < 1$. Then there is a constant $C = C(\vartheta) > 0$ such that the following holds for all ϑ-semiconvex polygons P: There are two segments S_1, S_2 inducing a partition of $P = P_1 \cup P' \cup P_2$ with $P_i \cap P' = S_i$ for $i = 1, 2$ such that P' is a convex polygon and the polygons P_i satisfy*

$$\begin{aligned} &(i) && \mathcal{H}^1(S_i) \leq \vartheta \mathcal{H}^1(\partial P),\\ &(ii) && \max_{R \in SO(2)} |P_i|_{\Pi,R} \leq C \min_{R \in SO(2)} |P_i|_{\Pi,R},\\ &(iii) && \max_{R \in SO(2)} |P_i|_{\Pi,R} \leq C \operatorname{dist}(v, S_i) \quad \textit{for all} \ \ v \in \mathcal{V}'_{P_i}. \end{aligned} \tag{4.5}$$

We remark that the choice $P_i = \emptyset$, $i = 1, 2$, is admissible. (In this case also the corresponding segment is empty). Moreover, also the choice $P_1 = P$, $P' = P_2 = \emptyset$ is possible, where Theorem 4.4 and (4.5)(ii) then imply that P is rotund. Later, condition (4.5)(iii) will be crucial to show that P_i are semiconvex using Lemma 3.10. Theorem 4.4 together with (4.5)(ii) will then yield that the polygons P_i are rotund.

Proof. Possibly after rotation we have $\min_{R \in SO(2)} |P|_{\Pi,R} = |P|_{\Pi,2}$. Without restriction we can assume that $\vartheta^2 |P|_{\Pi,1} > 12|P|_{\Pi,2}$ as otherwise the claim holds for $P_1 = P$, $P' = P_2 = \emptyset$ and $S_1 = S_2 = \emptyset$ with $C = 12/\vartheta^2 + 1$, where (4.5)(ii) for P_1 follows from $\max_{R \in SO(2)} |P|_{\Pi,R} \leq |P|_{\Pi,1} + |P|_{\Pi,2}$ and (4.5)(i), (iii) are trivial. Moreover, possibly after another infinitesimal rotation we can suppose $\vartheta^2 |P|_{\Pi,1} > 12|P|_{\Pi,2}$ and

$$v_1 \mathbf{e}_1 \neq v_2 \mathbf{e}_1 \quad \text{for all} \ \ v_1, v_2 \in \mathcal{V}_P,\ v_1 \neq v_2. \tag{4.6}$$

Choose $p_1, p_2 \in \partial P$ with $(p_2 - p_1)\mathbf{e}_1 = |P|_{\Pi,1}$ and let $\gamma : [0, l(\gamma)] \to P$ be the piecewise affine curve between p_1, p_2, parametrized by arc length, with $d_P(p_1, p_2) = l(\gamma)$. Define $\mathcal{U}_1 = \{v \in \mathcal{V}'_P : d_P(v, p_1) \leq d_P(v, p_2)\}$ and $\mathcal{U}_2 = \mathcal{V}'_P \setminus \mathcal{U}_1$. We first cut off two small pieces near p_1, p_2 to obtain an auxiliary convex polygon. Afterwards, we define P' and show (4.5).

Step 1. *Definition of an auxiliary polygon.*

Let $\mathcal{V}_* \subset \mathcal{V}'_P$ be the vertices v for which there is some $w \in \partial P$ such that $[v; w]$ is parallel to the $\mathbf{e}_2$-axis, $\gamma \cap [v; w] \neq \emptyset$ and $[v; w]$ induces a partition of P according to Definition 2.7 (see Fig. 7). Note particularly that $v \in \mathcal{V}_*$ for each $v \in \mathcal{V}'_P$ with $v \in \gamma$. Let $I = \{i = 1, 2 : \mathcal{U}_i \cap \mathcal{V}_* \neq \emptyset\}$. For $i \in I$ we choose $v_i \in \mathcal{U}_i \cap \mathcal{V}_*$ with

$$|(p_i - v_i)\mathbf{e}_1| = \max_{v \in \mathcal{U}_i \cap \mathcal{V}_*} |(p_i - v)\mathbf{e}_1|.$$

For v_i we find a corresponding w_i^1 such that $[v_i; w_i^1]$ is parallel to the $\mathbf{e}_2$-axis, intersects γ and induces a partition $P = Q_1^{(i,1)} \cup Q_2^{(i,1)}$, where the sets are labeled such that $p_k \in Q_k^{(i,1)}$ for $k = 1, 2$. Note that there may exist a second segment $[v_i; w_i^2]$ parallel to the the $\mathbf{e}_2$-axis inducing a partition $P = Q_1^{(i,2)} \cup Q_2^{(i,2)}$ with $p_1, p_2 \notin Q_i^{(i,2)}$, *cf.* Figure 7. (If such a segment does not exist, we set $Q_i^{(i,2)} = \emptyset$ and $[v_i; w_i^2] = \emptyset$). Note that $w_i^1, w_i^2 \notin \mathcal{V}_P$ by (4.6). Since P is ϑ-semiconvex, we derive using $\vartheta^2 |P|_{\Pi,1} \geq 12|P|_{\Pi,2}$

$$\min_{k=1,2} d(Q_k^{(i,j)}) \leq \vartheta^{-1}|[v_i; w_i^j]| \leq \vartheta^{-1}|P|_{\Pi,2} \leq \frac{1}{12}\vartheta|P|_{\Pi,1}.$$

As for $l = 1, 2$, $l \neq i$, we have $\mathrm{dist}(v_i, p_l) \geq \frac{1}{2} d_P(p_1, p_2) \geq \frac{1}{2}|P|_{\Pi,1}$ by definition of $\mathcal{U}_i$, we get $d(Q_l^{(i,j)}) \geq \frac{1}{2}|P|_{\Pi,1}$ and thus obtain

$$r_{i,j} := d(Q_i^{(i,j)}) \leq \vartheta^{-1}|[v_i; w_i^j]| \leq \frac{1}{12}\vartheta|P|_{\Pi,1}. \tag{4.7}$$

If $i \notin I$, we set $r_{i,j} = 0$ for $j = 1, 2$, $v_i = p_i$ and introduce the trivial partitions $P = Q_1^{(i,j)} \cup Q_2^{(i,j)}$ with $Q_i^{(i,j)} = \{p_i\}$ and $Q_k^{(i,j)} = P$ for $k \neq i$. For shorthand we define $\bar{r}_i = \max_{j=1,2} r_{i,j}$ for $i = 1, 2$.

By the fact that $d_P(p_i, v_i) \leq d(Q_i^{(i,1)}) \leq \bar{r}_i$ and (4.7) we have that the sets

$$T := [v_1\mathbf{e}_1, v_2\mathbf{e}_1] \times \mathbb{R}, \qquad T' := [p_1\mathbf{e}_1 + 2\bar{r}_1, p_2\mathbf{e}_1 - 2\bar{r}_2] \times \mathbb{R} \tag{4.8}$$

satisfy $\emptyset \subsetneq T' \subset T$. Consider the polygon $\hat{P} := Q_2^{(1,1)} \cap Q_2^{(1,2)} \cap Q_1^{(2,1)} \cap Q_1^{(2,2)}$, which is confined, if existent, by the segments $[v_i^j; w_i^j]$, $i, j = 1, 2$. Moreover, let P_1^* be the connected component of $P \cap T$ contained in $\hat{P}$. (See Fig. 7. Below we will see that $P_1^* = \hat{P}$).

As P_1^* is connected, it is a polygon. We now show that P_1^* is convex. Note that $v \in \mathrm{int}(T)$ for all $v \in \mathcal{V}'_{P_1^*}$ since $P_1^* \subset T$. Moreover, $v \notin \mathrm{int}(T)$ for all $\mathcal{V}'_P \cap \mathcal{V}_*$ by definition of v_1, v_2. Consequently, $\gamma \cap P_1^*$ does not contain a concave vertex of P and is thus a segment. Assume $\mathcal{V}'_{P_1^*} \neq \emptyset$ and choose a vertex $v \in \mathcal{V}'_{P_1^*} \subset \mathcal{V}'_P$ minimizing $\mathrm{dist}(v, \gamma \cap P_1^*)$. Then there is $p \in \gamma$ such that $[v; p] \subset P$ and $[v; p]$ parallel to the $\mathbf{e}_2$-axis. This then implies $v \in \mathcal{V}_*$, which gives a contradiction and shows that P_1^* is convex.

The convexity of P_1^* together with the fact that $w_i^j \notin \mathcal{V}_P$ (see (4.6)) also implies $P_1^* \cap \partial T = \bigcup_{i,j=1,2} [v_i^j; w_i^j]$ and this yields $\hat{P} = P_1^*$. Moreover, we derive

$$P_2^* := P \cap T' = P_1^* \cap T'.$$

Indeed, as $P_1^* = \hat{P}$, we obtain $P \setminus P_1^* \subset \bigcup_{j=1}^2 (Q_1^{(1,j)} \cup Q_2^{(2,j)})$. Then the definition of $r_{i,j}$ (see (4.7)) together with $r_{i,j} \leq \bar{r}_i$, (4.8) and $Q_i^{(i,j)} \cap \partial T \neq \emptyset$, $i, j = 1, 2$, yields $Q_i^{(i,j)} \cap T' = \emptyset$ for $i, j = 1, 2$. This implies the claim. Since P_1^* is convex, also P_2^* is convex.

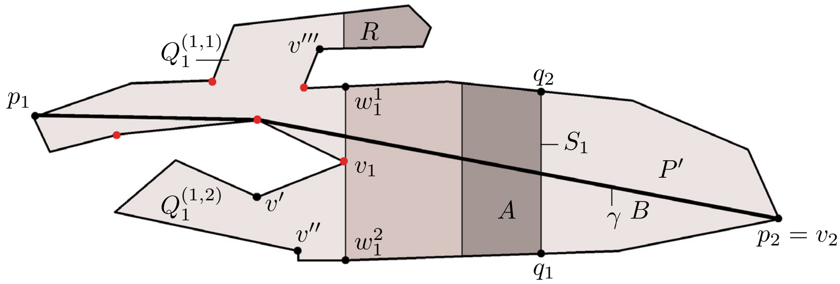

FIGURE 7. We illustrate a case with $I = \{1\}$. In red the vertices $\mathcal{V}_*$ are depicted, where $v', v'', v''' \notin \mathcal{V}_*$. Moreover, we have $P_2^* = P' \cup A = P \cap T'$ and $P_1^* = \hat{P} = P_2^* \cup B$. Note that $R \subset T \cap P$, but $R \cap P_1^* = \emptyset$.

Step 2. *Definition of P'.*
We are now in a position to define P'. As $3\bar{r}_i \leq \frac{1}{4}|P|_{\Pi,1}$ by (4.7), we can choose $t_1 < t_2$ with

$$p_1\mathbf{e}_1 + 3\bar{r}_1 \leq t_1 \leq p_1\mathbf{e}_1 + \frac{1}{4}|P|_{\Pi,1}, \quad p_2\mathbf{e}_1 - \frac{1}{4}|P|_{\Pi,1} \leq t_2 \leq p_2\mathbf{e}_1 - 3\bar{r}_2 \tag{4.9}$$

such that $S_i := P \cap (\{t_i\} \times \mathbb{R})$ satisfy

$$\mathcal{H}^1(S_i) \leq |t_i - p_i\mathbf{e}_1| \leq \max\{\mathcal{H}^1(S_i), 3\bar{r}_i\}. \tag{4.10}$$

This follows from a continuity argument taking $|P|_{\Pi,1} \geq 12|P|_{\Pi,2}$ into account. Clearly, $P' := P \cap ([t_1, t_2] \times \mathbb{R})$ is a convex polygon (*cf.* again Fig. 7). Denote the closures of the at most two connected components of $P \setminus P'$ by P_1, P_2, where $P_i = \emptyset$ if and only if $i \notin I$ and note that indeed $S_i = P' \cap P_i$ for $i \in I$. It remains to confirm (4.5). As a preparation, we show that there is a universal $C > 0$ such that for $i \in I$

$$d(P_i) \leq C\vartheta^{-1}(|t_i - p_i\mathbf{e}_1| + \mathcal{H}^1(S_i)). \tag{4.11}$$

We confirm the claim *e.g.* for $i = 1$. Let q_1, q_2 be the endpoints of the segment S_1. As P_1^* is convex, we have that the closed triangle $\triangle$ with vertices q_1, q_2 and v_1 is contained in $P_1^* \subset P$. If $[q_j; v_1]$ is not completely contained in ∂P_1^* for $j = 1, 2$, it induces a partition $P = R_1^{(j)} \cup R_2^{(j)}$, where the sets are labeled such that $R_1^{(j)} \subset P_1$. If $[q_j; v_1] \subset \partial P_1^*$, we set $R_1^{(j)} = \emptyset$. We obtain $P_1 = R_1^{(1)} \cup R_1^{(2)} \cup \triangle$. Note that $d(\triangle) \leq |t_1 - p_1\mathbf{e}_1| + \mathcal{H}^1(S_1)$ and due to the fact that P is ϑ-semiconvex, we get

$$\min_{k=1,2} d(R_k^{(j)}) \leq \vartheta^{-1}|[q_j; v_1]| \leq \vartheta^{-1}(|t_1 - p_1\mathbf{e}_1| + \mathcal{H}^1(S_1)) \leq \frac{1}{3}|P|_{\Pi,1},$$

where in the last step we used (4.7), (4.10) and that by assumption $\mathcal{H}^1(S_1) \leq |P|_{\Pi,2} \leq \frac{1}{12}\vartheta^2|P|_{\Pi,1}$. Since $d(R_2^{(j)}) \geq \frac{3}{4}|P|_{\Pi,1}$ by (4.9), we derive $d(R_1^{(j)}) \leq d(R_2^{(j)})$ and then (4.11) follows.

We now show (4.5). First, (i) follows from $\mathcal{H}^1(S_i) \leq |P|_{\Pi,2}$ and $12|P|_{\Pi,2} \leq \vartheta^2|P|_{\Pi,1} \leq \vartheta^2\mathcal{H}^1(\partial P)$. If $R\mathbf{e}_1$, $R \in SO(2)$, encloses an angle smaller than $\frac{\pi}{4}$ with the $\mathbf{e}_2$-axis, we find $|P_i|_{\Pi,R} \geq \frac{1}{\sqrt{2}}\max\{|[v_i; w_i^1]| + |[v_i; w_i^2]|, \mathcal{H}^1(S_i)\}$. Likewise, if $R\mathbf{e}_1$ encloses an angle smaller than $\frac{\pi}{4}$ with the $\mathbf{e}_1$-axis, we get in view of (4.9)

$$|P_i|_{\Pi,R} \geq |\gamma \cap (P_i \cap T)|_{\Pi,R} \geq c|t_i - v_i\mathbf{e}_1| \geq c(|t_i - p_i\mathbf{e}_1| - \bar{r}_i) \geq c|t_i - p_i\mathbf{e}_1|$$

for a universal c small enough, where we used that $\gamma \cap (P_i \cap T)$ is a segment enclosing a small angle with $\mathbf{e}_1$ since $\vartheta^2|P|_{\Pi,1} \geq 12|P|_{\Pi,2}$ (*cf.* Fig. 7). By (4.7) and (4.10) this implies

$$\min_{R\in SO(2)} |P_i|_{\Pi,R} \geq c\min\{\max\{\vartheta\bar{r}_i, \mathcal{H}^1(S_i)\}, |t_i - p_i\mathbf{e}_1|\} \geq c\vartheta|t_i - p_i\mathbf{e}_1|.$$

Consequently, (4.10) and (4.11) yield

$$\max_{R\in SO(2)}|P_i|_{\Pi,R}\le d(P_i)\le C\vartheta^{-1}2|t_i-p_i\mathbf{e}_1|\le C\vartheta^{-2}\min_{R\in SO(2)}|P_i|_{P,R}.$$

This gives (ii). Finally, to see (iii), we recall that $P_2^*=P\cap T'$ is a convex polygon and thus in view of (4.8), (4.9), we get $\mathrm{dist}(v,S_i)\ge|t_i-p_i\mathbf{e}_1|-2\bar r_i\ge\frac13|t_i-p_i\mathbf{e}_1|=\frac13|P_i|_{\Pi,1}$ for all $v\in\mathcal{V}'_{P_i}$. The claim now follows from (4.5)(ii). □

4.2. Partitions into semiconvex and rotund polygons

We now show that semiconvex polygons can be partitioned into semiconvex and rotund polygons. We start with the partition of convex polygons into rotund polygons by introducing segments parallel to the direction of shortest extend.

Lemma 4.6. *Let $\theta>0$. Then there is $\omega=\omega(\theta)>0$ such that for all convex polygons P, satisfying $\sphericalangle(v,P)\ge\frac{\pi}{4}$ for all vertices $v\in\mathcal{V}_P$, there is a partition $P=P_1\cup\ldots\cup P_N$ with*

$$\mathcal{H}^1\left(\bigcup\nolimits_{j=1}^N\partial P_j\setminus\partial P\right)\le\theta\mathcal{H}^1(\partial P)\tag{4.12}$$

and the polygons $(P_j)_{j=1}^N$ are ω-rotund.

Proof. After rotation we may assume that $\min_{R\in SO(2)}|P|_{\Pi,R}=|P|_{\Pi,2}$. Clearly, it is not restrictive to suppose that $\theta\le\theta_0$ for some $\theta_0\le1$ to be specified below. If $|P|_{\Pi,1}<7\theta^{-1}|P|_{\Pi,2}$, we obtain $\max_{R\in SO(2)}|P|_{\Pi,R}\le|P|_{\Pi,1}+|P|_{\Pi,2}\le(1+7\theta^{-1})|P|_{\Pi,2}$ and P is ω-rotund by Theorem 4.4 for ω only depending on θ.

Now assume $|P|_{\Pi,1}\ge7\theta^{-1}|P|_{\Pi,2}$. For $t\in\mathbb{R}$ we denote by S_t the segments $S_t=(\{t\}\times\mathbb{R})\cap P$ which induce partitions $P=Q_1^t\cup Q_2^t$, where $Q_2^t\subset\{x_1\ge t\}$. For shorthand we write $\varphi_\theta=\arctan\theta$. Choose the smallest s_1 and the largest s_2 such that the polygon $P':=\overline{P\setminus(Q_1^{s_1}\cup Q_2^{s_2})}$ with $\mathcal{V}_{P'}=(u_1,\ldots,u_n)$ satisfies

$$[u_i;u_{i+1}]\ne S_{s_1},S_{s_2}\ \Rightarrow\ \arg(u_{i+1}-u_i)\in\big(\{0,\pi\}+[-\varphi_\theta,\varphi_\theta]\big)\mathrm{mod}2\pi,\tag{4.13}$$

i.e. $u_{i+1}-u_i$ and $\mathbf{e}_1$ enclose an angle smaller than φ_θ. We show that for $j=1,2$ one has

$$\begin{aligned}&(i)\ \ |Q_1^{s_1}|_{\Pi,1}+|Q_2^{s_2}|_{\Pi,1}\le\frac12|P|_{\Pi,1},\\&(ii)\ \ |Q_j^{s_j}|_{\Pi,1}\le3\theta^{-1}\mathcal{H}^1(S_{s_j}),\\&(iii)\ \ 0<|Q_j^{s_j}|_{\Pi,2}\le4\mathcal{H}^1(S_{s_j}).\end{aligned}\tag{4.14}$$

By convexity of P and the choice in (4.13) we find curves γ_j in $\partial Q_j^{s_j}$ with $|\gamma_j|_{\Pi,1}=|Q_j^{s_j}|_{\Pi,1}$ such that the angle enclosed by $\mathbf{e}_1$ and the tangent vector γ_j' of γ_j is larger than φ_θ (see Fig. 8). By $|P|_{\Pi,1}\ge7\theta^{-1}|P|_{\Pi,2}$ a short calculation then yields

$$\sum\nolimits_j\theta|Q_j^{s_j}|_{\Pi,1}=\sum\nolimits_j\tan(\varphi_\theta)|Q_j^{s_j}\cap\gamma_j|_{\Pi,1}\le\sum\nolimits_j|Q_j^{s_j}|_{\Pi,2}\le2|P|_{\Pi,2}\le\frac{2\theta}{7}|P|_{\Pi,1},$$

which gives (i). The first inequality in (iii) follows from the fact that $Q_j^{s_j}$ cannot be degenerated to a single vertex as for θ_0 small in view of (4.13) this would contradict the lower bound $\frac{\pi}{4}$ on the interior angles of P. Note, however, that $|Q_j^{s_j}|_{\Pi,1}=0$ is possible, where in this case we have $Q_j^{s_j}=S_{s_j}$.

We now show (ii)−(iii), *e.g.* for S_{s_1}. Recalling the previous observation we see that the claim is clear if $|Q_1^{s_1}|_{\Pi,1}=0$ and we therefore assume $|Q_1^{s_1}|_{\Pi,1}>0$. For convenience define $[u;v]:=S_{s_1}$. We now get

$$(a)\ \min\{\sphericalangle(u,Q_1^{s_1}),\sphericalangle(v,Q_1^{s_1})\}\le\frac{\pi}{2}-\varphi_\theta,\qquad(b)\ \max\{\sphericalangle(u,Q_1^{s_1}),\sphericalangle(v,Q_1^{s_1})\}\le\frac{\pi}{2}+\frac{\varphi_\theta}{2}\cdot$$

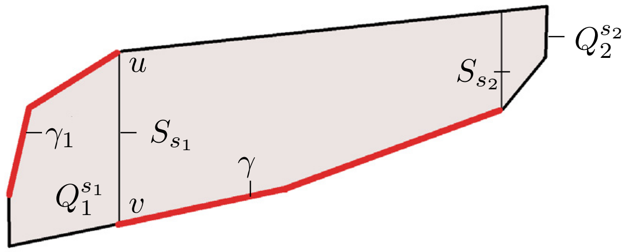

FIGURE 8. We depicted the curves γ_1 and γ considered above in the proof of (4.14). Note that similar arguments involving the angle between tangent vectors and $\mathbf{e}_1$ are also used in (4.15) and (4.19).

If (b) was false, as above using the convexity of P we would find a curve γ in $\partial P'$ with $|\gamma|_{\Pi,1} = |P'|_{\Pi,1}$ such that the angle enclosed by $\mathbf{e}_1$ and γ' is larger than $\frac{\varphi_\theta}{2}$ (see Fig. 8). But then similarly as in the proof of (4.14)(i) we would find, by a Taylor expansion for θ_0 small, and $|P'|_{\Pi,1} \geq \frac{1}{2}|P|_{\Pi,1}$ (see (4.14)(i))

$$|P|_{\Pi,2} \geq |P'|_{\Pi,2} \geq \tan\Big(\frac{\varphi_\theta}{2}\Big)|P' \cap \gamma|_{\Pi,1} \geq \frac{2\theta}{5}|P'|_{\Pi,1} \geq \frac{\theta}{5}|P|_{\Pi,1},$$

which contradicts the assumption. To see (a), we observe that the construction of P' implies that, up to changing the roles of u and v, $|\sphericalangle(u, Q_1^{s_1}) - \frac{\pi}{2}| \geq \varphi_\theta$. As in the proof of (b), we derive that it is not possible that the angle is obtuse. This gives (a).

Combining (a) and (b) and recalling the convexity of P we derive

$$\mathcal{H}^1(S_{s_1}) + \tan(\varphi_\theta/2)|Q_1^{s_1}|_{\Pi,1} - \tan(\varphi_\theta)|Q_1^{s_1}|_{\Pi,1} \geq 0$$

and then for θ_0 small by a Taylor expansion $\mathcal{H}^1(S_{s_1}) \geq \frac{\theta}{3}|Q_1^{s_1}|_{\Pi,1}$, *i.e.* (ii) holds.

To see the second inequality in (iii), we again use (a), (b) and (ii) to obtain for θ small

$$|Q_1^{s_1}|_{\Pi,2} \leq \mathcal{H}^1(S_{s_1}) + \tan(\varphi_\theta/2)|Q_1^{s_1}|_{\Pi,1} \leq \mathcal{H}^1(S_{s_1}) + \theta|Q_1^{s_1}|_{\Pi,1} \leq 4\mathcal{H}^1(S_{s_1}). \tag{4.15}$$

For later purpose note that (4.14)(ii) and the assumption $|P|_{\Pi,1} \geq 7\theta^{-1}|P|_{\Pi,2}$ also imply

$$|P'|_{\Pi,1} \geq |P|_{\Pi,1} - 3\theta^{-1}(\mathcal{H}^1(S_{s_1}) + \mathcal{H}^1(S_{s_2})) \geq |P|_{\Pi,1} - 6\theta^{-1}|P|_{\Pi,2} \geq \theta^{-1}|P|_{\Pi,2}. \tag{4.16}$$

We are now in a position to partition P' with vertical segments: we assume segments $S_{t_1}, S_{t_2}, \ldots, S_{t_n}$ with $s_1 = t_1 < t_2 < \ldots < t_n$ and $P'_j = \overline{Q_2^{t_j} \setminus Q_2^{t_{j+1}}}$ for $j = 1, \ldots, n-1$, $n \in \mathbb{N}$, have been constructed with

$$0 < |P'_j|_{\Pi,1} = \theta^{-1}\mathcal{H}^1(S_{t_{j+1}}),\ j = 1, \ldots, n-1, \qquad |Q_2^{t_n} \cap P'|_{\Pi,1} \geq \theta^{-1}\mathcal{H}^1(S_{s_2}). \tag{4.17}$$

We observe that the latter condition in (4.17) holds in the case $n = 1$, where no set has been constructed yet. In fact, we have $Q_2^{t_1} \cap P' = P'$ and then $|Q_2^{t_1} \cap P'|_{\Pi,1} \geq \theta^{-1}|P|_{\Pi,2} \geq \theta^{-1}\mathcal{H}^1(S_{s_2})$ by (4.16). Moreover, recall $\mathcal{H}^1(S_{t_1}) > 0$ by (4.14)(iii).

If $|Q_2^{t_n} \cap P'|_{\Pi,1} \leq 4\theta^{-1}\mathcal{H}^1(S_{s_2})$, we set $P'_n = Q_2^{t_n} \cap P'$, $t_{n+1} = s_2$, $S_{t_{n+1}} = S_{s_2}$ and stop. For later reference we note that in this case

$$\theta^{-1}\mathcal{H}^1(S_{s_2}) \leq |P'_n|_{\Pi,1} \leq 4\theta^{-1}\mathcal{H}^1(S_{s_2}). \tag{4.18}$$

Otherwise, we have $|Q_2^{t_n} \cap P'|_{\Pi,1} > 4\theta^{-1}\mathcal{H}^1(S_{s_2})$. As $\mathcal{H}^1(S_t)$ is continuous in t and $\mathcal{H}^1(S_{t_n}) > 0$ by (4.17), we apply the intermediate value theorem and find some $t_{n+1} \in [t_n, s_2]$ such that $P'_n := \overline{Q_2^{t_n} \setminus Q_2^{t_{n+1}}}$ satisfies $|P'_n|_{\Pi,1} = \theta^{-1}\mathcal{H}^1(S_{t_{n+1}})$. This gives

$$|Q_2^{t_{n+1}} \cap P'|_{\Pi,1} = |Q_2^{t_n} \cap P'|_{\Pi,1} - |P'_n|_{\Pi,1} \geq \theta^{-1}(4\mathcal{H}^1(S_{s_2}) - \mathcal{H}^1(S_{t_{n+1}})).$$

Then using $\mathcal{H}^1(S_{t_{n+1}}) \leq \mathcal{H}^1(S_{s_2}) + 2\theta|Q_2^{t_{n+1}} \cap P'|_{\Pi,1}$ by (4.13) (see (4.15) for a similar argument), we get

$$3|Q_2^{t_{n+1}} \cap P'|_{\Pi,1} \geq \theta^{-1}(4\mathcal{H}^1(S_{s_2}) - \mathcal{H}^1(S_{t_{n+1}})) + 2|Q_2^{t_{n+1}} \cap P'|_{\Pi,1} \geq 3\theta^{-1}\mathcal{H}^1(S_{s_2}),$$

which gives the second part of (4.17). We now proceed with the next iteration step and observe that the construction stops after a finite number of steps with a partition $P' = P'_1 \cup \ldots P'_N$ since by convexity of P we have $\mathcal{H}^1(S_t) \geq \min_{i=1,2}\mathcal{H}^1(S_{s_i}) > 0$ for all $t \in [s_1, s_2]$.

Note that by (4.17), (4.18) each P'_n contains a triangle with a base of length $\mathcal{H}^1(S_{t_{n+1}})$ and a height with length in the intervall $[\theta^{-1}\mathcal{H}^1(S_{t_{n+1}}), 4\theta^{-1}\mathcal{H}^1(S_{t_{n+1}})]$. By (4.13) it is not hard to see that each of these triangles contains a ball with radius larger than $C\mathcal{H}^1(S_{t_{n+1}})$ for a universal $C > 0$ small enough. Likewise, again arguing as in (4.15), by (4.13), (4.17), (4.18) we also find

$$|P'_n|_{\Pi,2} \leq \mathcal{H}^1(S_{t_{n+1}}) + 2\theta|P'_n|_{\Pi,1} \leq 9\mathcal{H}^1(S_{t_{n+1}}). \tag{4.19}$$

Consequently, again by (4.17), (4.18) we derive $d(P'_n) \leq (4\theta^{-1}+9)\mathcal{H}^1(S_{t_{n+1}})$ and thus we conclude that $(P'_n)_{n=1}^N$ are ω-rotund for some $\omega = \omega(\theta)$ small enough.

Define $P_1 = P'_1 \cup Q_1^{s_1}$, $P_N = P'_N \cup Q_2^{s_2}$ and $P_n = P'_n$ for $n = 2, \ldots, N-1$. Clearly, $P_2, \ldots, P_{N-1}$ are ω-rotund. Applying (4.14)(ii), (iii) and (4.17) we get

$$d(P_1) \leq d(P'_1) + C\theta^{-1}\mathcal{H}^1(S_{s_1}) \leq d(P'_1) + C\theta^{-1}(\mathcal{H}^1(S_{t_2}) + 2\theta|P'_1|_{\Pi,1}) \leq Cd(P'_1),$$

where in the penultimate step we once again exploited (4.13). A similar expression holds for P_N. Consequently, possibly passing to a smaller ω also P_1, P_N are ω-rotund. Finally, to see (4.12), we compute by (4.17)

$$\mathcal{H}^1\left(\bigcup\nolimits_{j=1}^N \partial P_j \setminus \partial P\right) = \sum_{j=2}^N \mathcal{H}^1(S_{t_j}) \leq \theta\sum_{j=1}^N |P'_j|_{\Pi,1} \leq \theta|P|_{\Pi,1} \leq \theta\mathcal{H}^1(\partial P). \qquad \square$$

We now finally show that semiconvex polygons can be partitioned into semiconvex and rotund polygons up to an arbitrary small set.

Theorem 4.7. *Let $\theta, \vartheta, \epsilon > 0$. Then there are $\omega = \omega(\vartheta, \theta)$, $\bar\vartheta = \bar\vartheta(\vartheta, \theta)$ and a universal constant $C > 0$ such that the following holds: For all ϑ-semiconvex polygons P there is a partition $P = P_0 \cup \ldots \cup P_N$ with $\mathcal{H}^1(\partial P_0) \leq \epsilon$ and*

$$\sum_{i=1}^N \mathcal{H}^1(\partial P_i) \leq (1 + C\theta)\mathcal{H}^1(\partial P) \tag{4.20}$$

such that the polygons $(P_i)_{i=1}^N$ are $\bar\vartheta$-semiconvex and ω-rotund.

Proof. Possibly by passing to a smaller ϑ, we can assume that $\vartheta \leq \theta$ in the following since (3.3) still holds for a smaller value of ϑ. We apply Theorem 4.5 to obtain a partition $P = P_1 \cup P' \cup P_2$ such that P' is a convex polygon and P_i satisfy (4.5) with $S_i := P_i \cap P'$ for $i = 1, 2$. (Recall that some of the polygons may be empty).

We now first concern ourselves with P'. By $\mathcal{V}_\sphericalangle$ we denote the vertices $v \in \mathcal{V}_{P'}$ with $\sphericalangle(v, P') < \frac{\pi}{4}$. For each $v \in \mathcal{V}_\sphericalangle$ we choose a (closed) isosceles triangle $\triangle_v \subset P'$ with $v \in \triangle_v$ such that $\sphericalangle(v, \triangle_v) = \sphericalangle(v, P')$ is the only angle smaller than $\frac{\pi}{4}$ and we obtain $\mathcal{H}^1(\partial P_0) \le \epsilon$ as well as $\sphericalangle(u, P'') \ge \frac{\pi}{4}$ for all $u \in \mathcal{V}_{P''}$, where $P_0 = \bigcup_{v \in \mathcal{V}_\sphericalangle} \triangle_v$ and $P'' = \overline{P' \setminus P_0}$. We notice that by the triangle inequality

$$\mathcal{H}^1(\partial P'') \le \mathcal{H}^1(\partial P') = \mathcal{H}^1(\partial P) + \sum_{i=1}^{2} (\mathcal{H}^1(S_i) - \mathcal{H}^1(\partial P_i \setminus S_i)) \le \mathcal{H}^1(\partial P). \tag{4.21}$$

We apply Lemma 4.6 on P'' to obtain a partition $P'' = P_3 \cup \ldots \cup P_N$ with

$$\mathcal{H}^1\Big(\bigcup\nolimits_{j=3}^N \partial P_j \setminus \partial P''\Big) \le \theta \mathcal{H}^1(\partial P'') \le \theta \mathcal{H}^1(\partial P)$$

such that the polygons $(P_j)_{j=3}^N$ are convex and ω-rotund for some ω only depending on θ. Since each $x \in \bigcup_{j=3}^N \partial P_j \setminus \partial P''$ is contained in exactly two components, we compute by (4.5)(i), (4.21) and $\vartheta \le \theta$

$$\begin{aligned}\sum\nolimits_{j=1}^N \mathcal{H}^1(\partial P_j) &\le \sum\nolimits_{i=1}^2 \mathcal{H}^1(\partial P_i) + \mathcal{H}^1(\partial P'') + 2\theta\mathcal{H}^1(\partial P)\\ &\le \mathcal{H}^1(\partial P) + \sum\nolimits_{i=1}^2 (\mathcal{H}^1(\partial P_i) + \mathcal{H}^1(S_i) - \mathcal{H}^1(\partial P_i \setminus S_i)) + 2\theta\mathcal{H}^1(\partial P)\\ &= \mathcal{H}^1(\partial P) + 2\theta\mathcal{H}^1(\partial P) + 2\sum\nolimits_{i=1}^2 \mathcal{H}^1(S_i) \le \mathcal{H}^1(\partial P) + 6\theta\mathcal{H}^1(\partial P).\end{aligned}$$

This gives (4.20). It remains to show that P_1 and P_2, if existent, are semiconvex and rotund. We denote the endpoints of the segments S_i by u_1^i, u_2^i, $i = 1, 2$. First, by (4.5)(iii) each $v \in \mathcal{V}'_{P_i}$ satisfies $\mathcal{H}^1(S_i) \le C \operatorname{dist}(v, S_i)$ for $C = C(\vartheta)$ and therefore an elementary geometric argument implies that there is an angle $\alpha = \alpha(\vartheta) > 0$ such that $\max_{k=1,2} \sphericalangle(\triangle_v, u_k^i) \ge \alpha$ for all $v \in \mathcal{V}'_{P_i}$, where $\triangle_v$ denotes the triangle formed by u_1^i, u_2^i and v. Thus, recalling that P is ϑ-semiconvex, we get that P_i are $\bar\vartheta$-semiconvex by Lemma 3.10 for $\bar\vartheta$ only depending on ϑ.

Finally, the fact that P_i is $\bar\vartheta$-semiconvex together with (4.5)(ii) yields that P_i is ω-rotund by Theorem 4.4 for ω only depending on ϑ. □

Remark 4.8. As in Remark 3.9 we note that by the partition no additional concave vertices are introduced.

5. Equivalence of John domains and semiconvex, rotund polygons

In this section we study the relation of semiconvex, rotund polygons and John domains. This together with the partitions introduced in the last sections will allow us to give the proof of Theorem 2.5. In the following for convenience we will say that a polygon P is a ϱ-John domain if $\operatorname{int}(P)$ is a ϱ-John domain. We first observe that polygons, which are ϱ-John domains, are semiconvex and rotund.

Lemma 5.1. *Let $0 < \varrho \le 1$. Each polygon P which is a ϱ-John domain is ϑ-semiconvex and ω-rotund for ϑ, ω only depending on ϱ.*

Proof. Since there is $x \in P$ with $P \setminus B(x, d(P)/2) \neq \emptyset$, Lemma 2.3 implies that P is $\frac{1}{4}\varrho$-rotund. If P was not ϑ-semiconvex for $\vartheta = \frac{\varrho}{4}$, there would be $u_1, u_2 \in \partial P$, $u_1 \in \mathcal{V}'_P$, inducing a partition $P = Q_1 \cup Q_2$ such that

$$|[u_1; u_2]| < \vartheta \min_{k=1,2} d(Q_k) \le \frac{1}{4} \min_{k=1,2} d(Q_k).$$

We can choose $v_k \in Q_k$ such that $d_P(v_k, w) \ge \frac{1}{4} d(Q_k) \ge \frac{1}{4}\vartheta^{-1}|[u_1; u_2]|$ for all $w \in [u_1; u_2]$. Let γ be a John curve between v_1, v_2 (see Rem. 2.2) and let w_* be an intersection point of γ with $[u_1; u_2]$. As $\operatorname{cig}(\gamma, \varrho) \subset P$, we derive $B(w_*, \frac{\varrho}{4\vartheta}|[u_1; u_2]|) \subset P$. In view of $\vartheta = \frac{\varrho}{4}$, this gives a a contradiction. □

We now show that semiconvex and rotund polygons are John domains with controllable John constant. Recall the notation $x\mathbf{e}_j$ for the jth component of points $x \in \mathbb{R}^2$ and that sometimes points are understood as complex numbers (see Sect. 2.3).

Theorem 5.2. *Let $\vartheta, \omega > 0$. Then there is $\varrho = \varrho(\vartheta, \omega)$ such that each ϑ-semiconvex and ω-rotund polygon P is a ϱ-John domain.*

Proof. By $0 < c < 1, C \geq 1$ we denote generic constants which are always independent of ϑ, ω. Possibly by passing to smaller ϑ, ω we can assume that ϑ, ω are sufficiently small with respect to C and ϑ is small with respect to ω in the following proof since the properties in Definition 3.2(i) and Definition 4.1 still hold for smaller values of ϑ, ω. As P is ω-rotund, we find some $p \in P$ and $r \geq \omega d(P)$ such that $B(p, r) \subset P$. Let $x \in \mathrm{int}(P)$ arbitrary. The goal is to construct a curve γ between x and p such that for ϑ small enough

$$\mathrm{car}(\gamma, \vartheta^3) \subset P, \tag{5.1}$$

where $\mathrm{car}(\gamma, \vartheta^3)$ as in (2.2). This then shows that $\mathrm{int}(P)$ is a ϑ^3-John domain. The construction will involve several steps.

Step 1. *Preparations.*

Choose the (unique) curve $\gamma_0 : [0, l(\gamma_0)] \to P$ with $\gamma_0(0) = x$ and $\gamma_0(l(\gamma_0)) = p$ such that $d_P(x, p) = l(\gamma_0)$ (see Fig. 9). As observed in Section 2.3 there are $0 = t_0 < t_1 < \ldots < t_n = l(\gamma_0)$ such that γ_0 is piecewise affine on $[t_i, t_{i+1}]$ and $v_i := \gamma_0(t_i) \in \mathcal{V}_P'$ are concave vertices for $i = 1, \ldots, n-1$. Moreover, define $v_0 = x$ and $v_n = p$. We consider a concave vertex $v \in \mathcal{V}_P'$ and $q \in \partial P$ such that $[v; q]$ induces a partition $P = Q_1^{(v,q)} \cup Q_2^{(v,q)}$ according to Definition 2.7 with $x \in Q_1^{(v,q)}$ and $p \in Q_2^{(v,q)}$. For convenience we will call such a segment $[v; q]$ in the following a segment which *separates* x *and* p (*cf.* Fig. 9). Since P is semiconvex, we have

$$|[v; q]| \geq \vartheta \min_{k=1,2} d(Q_k^{(v,q)}) \geq \vartheta \min\{\max\{d_P(v, x), d_P(q, x)\},\ \omega d(P)\}, \tag{5.2}$$

where we used that $d(Q_2^{(v,q)}) \geq r \geq \omega d(P)$. In particular, if $v = v_i$ we note that $d_P(v, x) = t_i$ and thus for ϑ small with respect to ω

$$|[v_i; q]| \geq \omega\vartheta t_i \geq 4\vartheta^2 t_i. \tag{5.3}$$

Likewise, if $v = v_i$, $q = v_{i+1}$ and $[v_i; v_{i+1}]$ separates x and p, we find

$$|[v_i; v_{i+1}]| \geq 4\vartheta^2 t_{i+1}. \tag{5.4}$$

Consider the subset

$$\{0 = i_0, 1 = i_1, i_2, \ldots, i_m = n-1\} \subset \{0, \ldots, n-1\}$$

with corresponding vertices $\hat{v}_j := \gamma_0(t_{i_j})$, $\hat{v}_j' := \gamma_0(t_{i_j+1})$, $j = 0, \ldots, m$ such that for $j = 1, \ldots, m-1$ the segments $[\hat{v}_j; \hat{v}_j']$ separate x and p or satisfy

$$|[\hat{v}_j; \hat{v}_j']| = t_{i_j+1} - t_{i_j} \geq 4\vartheta^2 t_{i_j+1}. \tag{5.5}$$

(Observe that the first and the last segment $[\hat{v}_0; \hat{v}_0']$ and $[\hat{v}_m; \hat{v}_m']$ do not induce a partition). Note that $i_j + 1 = i_{j+1}$ is possible, namely if a pair of directly consecutive segments separate x and p or satisfy (5.5). We then obtain

$$|[\hat{v}_j; \hat{v}_j']| = t_{i_j+1} - t_{i_j} \geq 4\vartheta^2 t_{i_j+1} \tag{5.6}$$

for all $j = 0, \ldots, m$. If (5.5) holds, this follows directly. For $j \in \{0, m\}$ we observe $t_0 = 0$ and $t_n - t_{n-1} \geq r \geq \omega d(P) \geq 4\vartheta^2 d(P)$ for ϑ small with respect to ω. Otherwise, $[\hat{v}_j; \hat{v}_j']$ separates x and p and the assertion follows from (5.4) with $i = i_j$, *i.e.* $v_i = \hat{v}_j$ and $v_{i+1} = \hat{v}_j'$. This property will essentially be important to estimate the length of the curve γ defined in Step 6 (*cf.* (5.34) below).

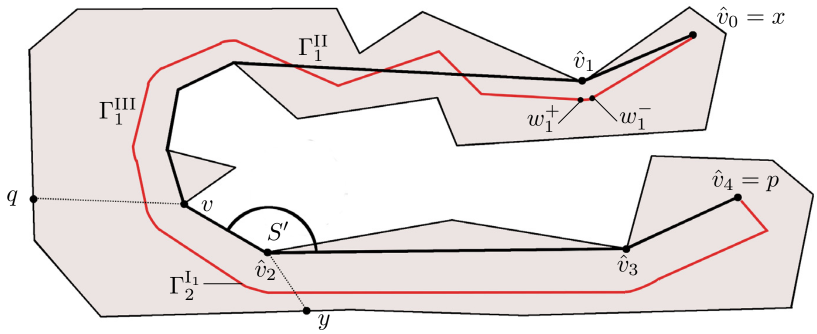

FIGURE 9. The path γ_0 is depicted in black and the curve γ in red, which 'changes the side of the boundary' on Γ_1^{II} and has a signed curvature on Γ_1^{III}. The segments $[v;q]$, $[\hat v_2;y]$ separate x and p. The segment $[\hat v_2;\hat v_3]$ does not separate x and p, but satisfies (5.5). We have also illustrated a circular sector S' contained in the cone S and the part of the circle $\Gamma_2^{\mathrm{I}_1}$ defined in Step 2.

For each $1 \le i \le n-1$ choose the unique $\nu_i^-, \nu_i^+ \in S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$ such that $\nu_i^- \perp v_i - v_{i-1}$, $\nu_i^+ \perp v_{i+1} - v_i$ and $v_i + \varepsilon\nu_i^\pm \in P$ for $\varepsilon > 0$ small. Define

$$w_i^- = v_i + 2\vartheta^2 t_i \nu_i^-, \qquad w_i^+ = v_i + 2\vartheta^2 t_i \nu_i^+. \tag{5.7}$$

Moreover, we set $w_0^- = w_0^+ = x$ and $w_n^- = p + 2\vartheta^2 t_n \nu_{n-1}^+$. The goal is to construct a curve $\gamma : [0, l(\gamma)] \to P$ with $\gamma(0) = x$, $\gamma(l(\gamma)) = p$ with $\mathrm{car}(\gamma, \vartheta^3) \subset P$, where we essentially connect the points $w_i^\pm$ defined above. We have to construct curves between

$$\text{(I) } w_{i_j}^- \text{ and } w_{i_j}^+,\ w_{i_j+1}^- \text{ and } w_{i_j+1}^+,\quad \text{(II) } w_{i_j}^+ \text{ and } w_{i_j+1}^-,\quad \text{(III) } w_{i_j+1}^+ \text{ and } w_{i_{j+1}}^- \tag{5.8}$$

for $j = 0, \dots, m-1$ and at the end of the curve a path between w_{n-1}^- and p.

The most delicate cases are (II) and (III), where in (II) γ_0 typically 'changes the side of the boundary' and in (III) the part of γ_0 has signed curvature, possibly the form of a 'helix' (*cf.* Fig. 9, Fig. 11). In Step 2–Step 5 we construct the various parts of the curve, where one has to ensure that (1) the length of γ is comparable to the length of γ_0 (see (5.9)(i), (5.17), (5.30), (5.33)(i)) and (2) the distance of γ from the boundary is sufficiently large (see (5.9)(ii), (5.19), (5.31), (5.33)(ii)). In Step 6 we finally show that the constructed curve satisfies the property stated in Definition 2.1.

Step 2. *Construction of curves* (I).
Let $j \in \{0, \dots, m-1\}$ and recall (5.7)–(5.8). Let $\Gamma_j^{\mathrm{I}_1}$ and $\Gamma_j^{\mathrm{I}_2}$ be the parts of the two circles with midpoints $v_{i_j} = \hat v_j$, $v_{i_j+1} = \hat v_j'$ and radii $2\vartheta^2 t_{i_j}$, $2\vartheta^2 t_{i_j+1}$ connecting $w_{i_j}^-, w_{i_j}^+$ and $w_{i_j+1}^-, w_{i_j+1}^+$ respectively. (Note that $\Gamma_0^{\mathrm{I}_1} = \emptyset$). We have

$$\begin{aligned} &(i)\quad l(\Gamma_j^{\mathrm{I}_1}) \le 4\pi\vartheta^2 t_{i_j} \le \pi(t_{i_j+1} - t_{i_j}), \quad l(\Gamma_j^{\mathrm{I}_2}) \le 4\pi\vartheta^2 t_{i_j+1} \le \pi(t_{i_j+1} - t_{i_j}),\\ &(ii)\quad \mathrm{dist}(\partial P, \Gamma_j^{\mathrm{I}_1}) \ge \vartheta^2 t_{i_j}, \qquad \mathrm{dist}(\partial P, \Gamma_j^{\mathrm{I}_2}) \ge \vartheta^2 t_{i_j+1}. \end{aligned} \tag{5.9}$$

Indeed, the first inequality in (i) is clear and the second follows from (5.6). We show (ii) for $i = i_j$, the proof for $i_j + 1$ is similar. Let $\varphi_- = \arg(v_{i-1} - v_i)$, $\varphi_+ = \arg(v_{i+1} - v_i)$ and suppose that possibly after rotation and reflection we have $0 \le \varphi_+ < \varphi_- < 2\pi$ and $\varphi_- - \varphi_+ < \pi$. We define the (infinite) cone $S = \{x \in \mathbb{R}^2 : \arg(x - v) \in [\varphi_+, \varphi_-]\}$ and note that $\mathrm{dist}(\Gamma_j^{\mathrm{I}_1}, S) \ge 2\vartheta^2 t_{i_j}$ by (5.7). If (ii) was wrong, we would find some $y \in \partial P \setminus S$ such that $[y; \hat{v}_j] \subset P$ and $|[y;\hat{v}_j]| < 3\vartheta^2 t_{i_j}$. As $y \notin S$ and γ_0 is the shortest path between x and p, we get that $[y;\hat{v}_j]$ separates x and p (*cf.* Fig. 9). This contradicts (5.3).

Step 3. *Construction of curves* (II).

Let $j \in \{0,\ldots,m\}$ and recall (5.7)–(5.8). (Because of Step 5 below we also consider $j = m$). To simplify the notation we assume $v_{i_j} = \hat{v}_j = 0$, $v_{i_j+1} = \hat{v}'_j = (0,d)$, where $d := t_{i_j+1} - t_{i_j}$, and define the rectangle $R = [-2\vartheta^2 t_{i_j+1}, 2\vartheta^2 t_{i_j+1}] \times [0,d]$. Observe that $w^+_{i_j}, w^-_{i_j+1} \in \partial R$. For notational convenience we will write $w = w^+_{i_j}$ and $w' = w^-_{i_j+1}$ in the following. Recall that $\hat{v}_j, \hat{v}'_j \in \mathcal{V}'_P$ for $j \in \{1,\ldots,m-1\}$. In the other cases we have (recall $t_0 = 0$, $v_n = p$)

$$\begin{aligned} &B(v_0,t_0) = B(\hat{v}_0,t_0) = \emptyset \subset P, \quad v_1 = \hat{v}'_0 \in \mathcal{V}'_P,\\ &v_{n-1} = \hat{v}_m \in \mathcal{V}'_P, \quad B(v_n,\omega d(P)) = B(\hat{v}'_m,\omega d(P)) \subset P. \end{aligned} \tag{5.10}$$

Define the set of vertices $\mathcal{U}_R := \{v \in \mathcal{V}'_P : v \in R\} \cup \{\hat{v}_j,\hat{v}'_j\}$. For convenience we now first treat the case $j \in \{1,\ldots,m-1\}$ and indicate the minor adaptions for $j \in \{0,m\}$, necessary due to $\hat{v}_0, \hat{v}'_m \notin \mathcal{V}'_P$, at the end of Step 3.

Note that $x\mathbf{e}_1 \neq 0$ for all $x \in (\partial P \cap R) \setminus \{\hat{v}_j,\hat{v}'_j\}$ as $[\hat{v}_j;\hat{v}'_j]$ induces a partition of P. Let $\mathrm{sgn}(y) = 1$ for $y > 0$ and $\mathrm{sgn}(y) = -1$ for $y < 0$. By convention we set $\mathrm{sgn}(\hat{v}_j\mathbf{e}_1) = -\mathrm{sgn}(w\mathbf{e}_1)$ and $\mathrm{sgn}(\hat{v}'_j\mathbf{e}_1) = -\mathrm{sgn}(w'\mathbf{e}_1)$. We let

$$\mathcal{V}_\pm = \{v \in \mathcal{U}_R : \pm\mathrm{sgn}(v\mathbf{e}_1) > 0,\ [v;(0,v\mathbf{e}_2)] \subset P\} \tag{5.11}$$

and show that

$$|[v;u]| \geq 8\vartheta^2(t_{i_j} + v\mathbf{e}_2) \quad \text{for all } v \in \mathcal{V}_- \cup \mathcal{V}_+, u \in \partial P \text{ with } \mathrm{sgn}(v\mathbf{e}_1) \neq \mathrm{sgn}(u\mathbf{e}_1). \tag{5.12}$$

To see this, assume *e.g.* that $v \in \mathcal{V}_+$ and suppose first $u\mathbf{e}_2 < v\mathbf{e}_2$. Clearly, $[v;u]$ does not necessarily induce a partition of P as possibly $[v;u] \not\subset P$. However, due to the fact that $\{0\} \times [0,d], [v;(0,v\mathbf{e}_2)] \subset P$, we see that there have to exist $v' \in \mathcal{V}_+$ and $u' \in \partial P$ with

$$0 \leq v'\mathbf{e}_1 \leq v\mathbf{e}_1, \quad \frac{v'\mathbf{e}_2 - u\mathbf{e}_2}{v\mathbf{e}_2 - u\mathbf{e}_2} \geq \frac{|[v';u]|}{|[v;u]|}, \quad v'\mathbf{e}_2 \leq v\mathbf{e}_2, \quad u'\mathbf{e}_1 < 0, \quad |[v';u']| \leq |[v';u]| \tag{5.13}$$

such that $[v';u']$ induces a partition of P (see Fig. 10). In fact, choose v' as a concave vertex in $[0,v\mathbf{e}_1] \times [u\mathbf{e}_2, v\mathbf{e}_2]$ lying on or above the segment $[v;u]$ with minimal distance to $\{0\} \times [0,d]$ (note that possibly $v' = v$) and let u' be the point on $\partial P \cap [v';u]$ closest to v'. The second property in (5.13) follows from the fact that v' lies on or above the segment $[v;u]$. Since $\mathrm{sgn}(v'\mathbf{e}_1) \neq \mathrm{sgn}(u'\mathbf{e}_1)$, $[v';u']$ separates x and p. Now suppose the statement was wrong. We then obtain using $v\mathbf{e}_2 > u\mathbf{e}_2 \geq 0$, $v'\mathbf{e}_2 \leq v\mathbf{e}_2$ as well as (5.13)

$$|[v';u']| \leq |[v';u]| \leq |[v;u]| \frac{v'\mathbf{e}_2 - u\mathbf{e}_2}{v\mathbf{e}_2 - u\mathbf{e}_2} < 8\vartheta^2(t_{i_j} + v\mathbf{e}_2)\frac{v'\mathbf{e}_2}{v\mathbf{e}_2} \leq 8\vartheta^2(t_{i_j} + v'\mathbf{e}_2).$$

Consequently, we have $|[v';u']| < \vartheta\omega d(P)$ for ϑ small with respect to ω. Moreover, for ϑ small we get $2\vartheta^{-1} \leq (8\vartheta^2)^{-1}$ and thus by $|v'\mathbf{e}_1| \leq |[v';u']|$

$$\max\{d_P(v',x), d_P(u',x)\} \geq t_{i_j} + v'\mathbf{e}_2 - |v'\mathbf{e}_1| > \big(2\vartheta^{-1} - 1\big)|[v';u']| \geq \vartheta^{-1}|[v';u']|.$$

The last two estimates contradict (5.2). This shows (5.12) in the case $u\mathbf{e}_2 < v\mathbf{e}_1$. For $u\mathbf{e}_2 \geq v\mathbf{e}_1$ we proceed similarly, where the second and third property in (5.13) are replaced by $v'\mathbf{e}_2 \geq v\mathbf{e}_2$ and $|[v';u]| \leq |[v;u]|$.

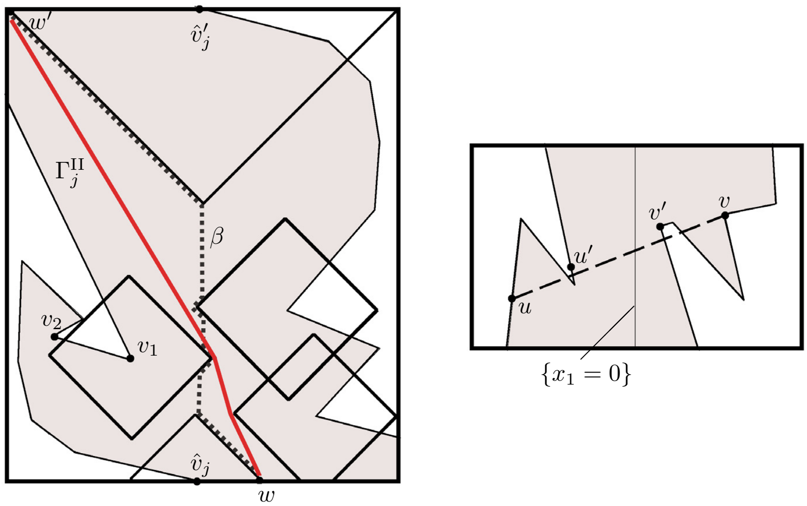

FIGURE 10. On the left we have depicted β in dotted lines and Γ_j^{II} in red. Note $\operatorname{sgn}(\hat{v}_j\mathbf{e}_1) = -1$, $\operatorname{sgn}(\hat{v}'_j\mathbf{e}_1) = 1$, $C_- = C(\hat{v}_j) \cup C(v_1)$ and $v_2 \notin \mathcal{V}_-$. The main idea in the construction is that Γ_j^{II} is 'not too close to concave vertices'. On the right the situation of (5.13) is illustrated.

Recalling (5.11) we let $C(v)$ be the closed square with midpoint $v \in \mathcal{V}_- \cup \mathcal{V}_+$ and diagonal $4\vartheta^2 l(v)$ with faces parallel to $\mathbf{e}_1 + \mathbf{e}_2$ and $\mathbf{e}_1 - \mathbf{e}_2$, where

$$l(v) = t_{i_j} + v\mathbf{e}_2 \in [t_{i_j}, t_{i_j+1}]. \tag{5.14}$$

Moreover, define $C_\pm = \bigcup_{v\in\mathcal{V}_\pm} C(v)$ and let H_+, H_- be the closed half space right and left of $\{0\}\times\mathbb{R}$, respectively. We show

$$\begin{aligned} &(i)\ \ C_+ \cap C_- = \emptyset, \qquad (ii)\ \ (C_+ \cap H_-) \cup (C_- \cap H_+) \subset P,\\ &(iii)\ \ w, w' \in (\partial C_+ \cap H_-) \cup (\partial C_- \cap H_+). \end{aligned} \tag{5.15}$$

To see (i), note that (5.12) implies $|[v_1; v_2]| \geq 8\vartheta^2 \max\{l(v_1), l(v_2)\}$ for $v_1 \in \mathcal{V}_+, v_2 \in \mathcal{V}_-$ and thus $C(v_1) \cap C(v_2) = \emptyset$. Likewise, if (ii) was wrong, there would be, *e.g.*, $v \in \mathcal{V}_+$ and $u \in \partial P$ with $\operatorname{sgn}(u\mathbf{e}_1) < 0$ such that $|[v; u]| \leq 2\vartheta^2 l(v)$, which contradicts (5.12). Finally, we always have $C(\hat{v}_j) \cap C(\hat{v}'_j) = \emptyset$ by (5.6). Consequently, (5.7), (5.14), and the convention $\operatorname{sgn}(\hat{v}_j\mathbf{e}_1) = -\operatorname{sgn}(w\mathbf{e}_1)$, $\operatorname{sgn}(\hat{v}'_j\mathbf{e}_1) = -\operatorname{sgn}(w'\mathbf{e}_1)$ show that each of the points w, w' is contained in $\partial C_+ \cap H_-$ or $\partial C_- \cap H_+$.

We define $P_R = \overline{(P \cap R) \setminus (C_+ \cup C_-)}$. Then $w, w' \in P_R$ by (5.15)(iii). Moreover, by (5.15) we find a continuous, piecewise affine path β between w, w' in the set

$$\big(\{0\} \times [0, d] \cup (\partial C_+ \cap H_-) \cup (\partial C_- \cap H_+)\big) \cap P_R$$

such that the tangent vector of β is a.e. contained in $\{\frac{1}{\sqrt{2}}(-1, 1), (0, 1), \frac{1}{\sqrt{2}}(1, 1)\}$ (see Fig. 10). In particular, w, w' are in the same connected component of P_R, which will be denoted by P_R^{con} in the following. Let $\Gamma_j^{\mathrm{II}} : [0, l(\Gamma_j^{\mathrm{II}})] \to P_R^{\mathrm{con}}$ be the shortest curve between w and w' parametrized by arc length. The goal will be to establish (5.17) and (5.19) below. Since P_R^{con} is a polygon, we find that Γ_j^{II} is piecewise affine and changes its direction only in concave vertices of P_R^{con}. We show that for all $0 \leq s < s' \leq l(\Gamma_j^{\mathrm{II}})$ one has

$$(i)\ \ [\Gamma_j^{\mathrm{II}}(s); (0, \Gamma_j^{\mathrm{II}}(s)\mathbf{e}_2)] \subset P, \qquad (ii)\ \ \arg(\Gamma_j^{\mathrm{II}}(s') - \Gamma_j^{\mathrm{II}}(s)) \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]. \tag{5.16}$$

First, (5.16)(i) holds for β in place of Γ_j^{II} by (5.15). Since P_R^{con} is simply connected, the bounded connected components of $\mathbb{R}^2 \setminus (\beta \cup \Gamma_j^{\mathrm{II}})$ are contained in P_R^{con}. Then the fact that Γ_j^{II} is the shortest path between w, w' in P_R^{con} together with (5.16)(i) for β implies (5.16)(i) for Γ_j^{II}.

From (5.16)(i) we deduce that $\arg(\Gamma_j^{\mathrm{II}}(s') - \Gamma_j^{\mathrm{II}}(s)) \in [0, \pi]$ for all $s < s'$ since Γ_j^{II} is the shortest path between w, w'. Select $s_1 < s_2$ and $t_1 < t_2$ such that $\beta(t_k) = \Gamma_j^{\mathrm{II}}(s_k)$ for $k = 1, 2$ and $\Gamma_j^{\mathrm{II}}((s_1, s_2)) \cap \beta((t_1, t_2)) = \emptyset$. Let P_* be the polygon with boundary $\Gamma_j^{\mathrm{II}}([s_1, s_2]) \cup \beta([t_1, t_2])$. Since $P_* \subset P_R^{\mathrm{con}}$ and Γ_j^{II} is the shortest path between w, w' in P_R^{con}, P_* only has concave vertices on $\Gamma_j^{\mathrm{II}}((s_1, s_2))$. Recalling that the tangent vector of β is a.e. contained in $\{\frac{1}{\sqrt{2}}(-1, 1), (0, 1), \frac{1}{\sqrt{2}}(1, 1)\}$ and that the paths $\Gamma_j^{\mathrm{II}}([s_1, s_2])$, $\beta([t_1, t_2])$ have a common start and endpoint, we derive (5.16)(ii) for $s_1 \le s < s' \le s_2$. Herefrom we also deduce

$$l(\Gamma_j^{\mathrm{II}}) \le \sqrt{2}(t_{i_j+1} - t_{i_j}). \tag{5.17}$$

Additionally, we obtain that if Γ_j^{II} changes its direction in s, then

$$\Gamma_j^{\mathrm{II}}(s) = v + 2\vartheta^2 l(v)\mathbf{e}_1 \ \text{ for } v \in \mathcal{V}_- \quad \text{ or } \quad \Gamma_j^{\mathrm{II}}(s) = v - 2\vartheta^2 l(v)\mathbf{e}_1 \ \text{ for } v \in \mathcal{V}_+. \tag{5.18}$$

In fact, Γ_j^{II} changes its direction only in concave vertices of P_R^{con}. First, if $\Gamma_j^{\mathrm{II}}(s) \in \mathcal{U}_R$ (recall definition below (5.10)), then $\Gamma_j^{\mathrm{II}}(s) \in \mathcal{V}_- \cup \mathcal{V}_+$ by (5.16)(i) and thus $\Gamma_j^{\mathrm{II}}(s) \notin P_R$ since $C(\Gamma_j^{\mathrm{II}}(s)) \cap P_R = \emptyset$. This gives a contradiction. Consequently, $\Gamma_j^{\mathrm{II}}(s)$ is a corner of $C(v)$ for some $v \in \mathcal{V}_- \cup \mathcal{V}_+$. Then (5.16) together with the geometry of $C(v)$ and the fact that Γ_j^{II} is a shortest path implies that $\Gamma_j^{\mathrm{II}}(s)$ has to be the left or right corner of $C(v)$, respectively. This yields (5.18). We now finally show

$$\mathrm{dist}(\partial P, \Gamma_j^{\mathrm{II}}(s)) \ge c\vartheta^2(t_{i_j} + s) \quad \text{for } s \in [0, l(\Gamma_j^{\mathrm{II}})] \tag{5.19}$$

for some universal $c > 0$ small. First, in view of (5.9) we observe that (5.19) holds for $s = 0, l(\Gamma_j^{\mathrm{II}})$ since $\hat{v}_j = \Gamma_j^{\mathrm{II}}(0) \in \Gamma_j^{\mathrm{I}_1}$ and $\hat{v}'_j = \Gamma_j^{\mathrm{II}}(l(\Gamma_j^{\mathrm{II}})) \in \Gamma_j^{\mathrm{I}_2}$. For each $s \in [0, l(\Gamma_j^{\mathrm{II}})]$ we denote by $q^{\pm}(s)$ the nearest point to $\Gamma_j^{\mathrm{II}}(s)$ on $\partial P \cap \big(\Gamma_j^{\mathrm{II}}(s) \pm \mathbb{R}_+\mathbf{e}_1\big)$. Moreover, we set $f^{\pm}(s) = |q^{\pm}(s) - \Gamma_j^{\mathrm{II}}(s)|$. For later note that $f^{\pm}$ is a lower semicontinuous function and it is possibly discontinuous at s only if $q^{\pm}(s)$ is a concave vertex. The fact that (5.19) holds for $s = 0, l(\Gamma_j^{\mathrm{II}})$ and (5.16)(ii) show that it suffices to prove

$$f^{\pm}(s) > \vartheta^2(t_{i_j} + s) \tag{5.20}$$

for $s \in [0, l(\Gamma_j^{\mathrm{II}})]$ as herefrom (5.19) follows for $c > 0$ sufficiently small. Consider, *e.g.*, f^+. First, we show that (5.20) holds for s with

$$(a) \ \ q^+(s) \in \mathcal{U}_R \qquad \text{or} \qquad (b) \ \ \Gamma_j^{\mathrm{II}} \text{ changes its direction in } s.$$

In fact, in case (a), (5.11), (5.16)(i) imply $q^+(s) \in \mathcal{V}_+$ and thus by (5.14) and $\Gamma_j^{\mathrm{II}} \subset P_R$ we have $f^+(s) \ge 2\vartheta^2(t_{i_j} + \Gamma_j^{\mathrm{II}}(s)\mathbf{e}_2)$. If (a) does not hold, we consider case (b) and recall (5.18). If we had $\Gamma_j^{\mathrm{II}}(s) = v - 2\vartheta^2 l(v)\mathbf{e}_1$ for some $v \in \mathcal{V}_+$, (a) would be satisfied since then $q_+(s) = v \in \mathcal{U}_R$. Consequently, we have $\Gamma_j^{\mathrm{II}}(s) = v + 2\vartheta^2 l(v)\mathbf{e}_1$ for some $v \in \mathcal{V}_-$ and then $f^+(s) \ge 6\vartheta^2(t_{i_j} + \Gamma_j^{\mathrm{II}}(s)\mathbf{e}_2)$ by (5.12) as $\mathrm{sgn}(v\mathbf{e}_1) \neq \mathrm{sgn}(q^+(s)\mathbf{e}_1)$. In all cases (5.20) follows from the fact that $\Gamma_j^{\mathrm{II}}(s)\mathbf{e}_2 \ge \frac{s}{\sqrt{2}}$ by (5.16)(ii).

We now show (5.20) by contradiction. Choose the largest value $0 < s < l(\Gamma_j^{\mathrm{II}})$ such that (5.20) is violated. Then neither (a) nor (b) hold. Since (a) does not hold, f^+ is continuous in a neighborhood of s and thus

$$f^+(s) = \vartheta^2(t_{i_j} + s) \tag{5.21}$$

by the choice of s. Choose the largest value $s' < s$ such that for s' one of the conditions (a), (b) holds. (If this is not possible, set $s' = 0$). We now show that (5.21) implies $f^+(s') \le \vartheta^2(t_{i_j} + s')$ which contradicts the fact that (5.20) holds for s'. This will conclude the proof of (5.20) and then (5.19) is proved.

Let $t' = \Gamma_j^{\mathrm{II}}(s')\mathbf{e}_2$, $t = \Gamma_j^{\mathrm{II}}(s)\mathbf{e}_2$ and $T : [0,d] \to [0, l(\Gamma_j^{\mathrm{II}})]$ be the (increasing) function with $\tau = \Gamma_j^{\mathrm{II}}(T(\tau))\mathbf{e}_2$ for $\tau \in [0,d]$. Due to the fact that Γ_j^{II} does not change its direction on $(s', s]$ we observe that $\tau \mapsto T(\tau)$ and $\tau \mapsto \Gamma_j^{\mathrm{II}}(T(\tau))\mathbf{e}_1$ are affine on $(t', t+\varepsilon)$ for ε small enough. Moreover, as (a) does not hold, we get that $\tau \mapsto q^+(T(\tau))\mathbf{e}_1$ is concave in $(t', t+\varepsilon)$ (*cf.* upper part in Fig. 10). Define $g : [0,d] \to [0,\infty)$ by

$$g(\tau) = q^+(T(\tau))\mathbf{e}_1 - \Gamma_j^{\mathrm{II}}(T(\tau))\mathbf{e}_1$$

and observe that g is concave in $(t', t+\varepsilon)$. More precisely, g is differentiable up to a finite number of points. To avoid further notation involving the superdifferential of concave functions, we will for simplicity assume that g is smooth. In fact, this can be always obtained by a slight modification of g on (t', t) without affecting the following arguments.

Since g is concave and T is affine on $(t', t+\varepsilon)$, we get $\bar{T} > 0$ such that

$$g(\tau) \le g(t) + g'(t)(\tau - t), \qquad T(\tau) = T(t) + \bar{T}(\tau - t) \tag{5.22}$$

for $\tau \in (t', t]$. The function $h : [0,d] \to [0,\infty)$ defined by $h(\tau) = g(\tau)(t_{i_j} + T(\tau))^{-1}$ satisfies $h(t) = \vartheta^2$ by (5.21) and $T(t) = s$. Note also that $h'(t) \ge 0$ due to the maximal choice of s. Consequently, $(t_{i_j} + T(t))g'(t) - \bar{T}g(t) \ge 0$ and this together with (5.22) and $\bar{T} > 0$ yields for $\tau \in (t', t]$

$$g(\tau) \le g(t) + g'(t)(\tau - t) \le g'(t)\bar{T}^{-1}(t_{i_j} + T(t)) + g'(t)(\tau - t) = g'(t)\bar{T}^{-1}(t_{i_j} + T(\tau)).$$

Using that g' is non-increasing and $\bar{T} > 0$ we then find for $\tau \in (t', t]$

$$\begin{aligned} h'(\tau) &= (t_{i_j} + T(\tau))^{-2}\big((t_{i_j} + T(\tau))g'(\tau) - \bar{T}g(\tau)\big) \\ &\ge (t_{i_j} + T(\tau))^{-1}(g'(\tau) - g'(t)) \ge 0 \end{aligned}$$

and thus $h(\tau) \le \vartheta^2$ on (t', t). This yields $f^+(\sigma) \le \vartheta^2(t_{i_j} + \sigma)$ for all $\sigma \in (s', s]$. As f^+ is lower semicontinuous, we get the desired contradiction $f^+(s') \le \vartheta^2(t_{i_j} + s')$.

To conclude Step 3, it remains to treat the cases announced in (5.10). First, for $j = 0$, (5.12) trivially holds for $v = v_0 = 0$ and $t_0 = 0$ and (5.19) is true for $s = 0$ since $t_0 = 0$. For $j = m$, (5.12) follows from $v = v_n = p$ and (5.10) with ϑ small with respect to ω. Finally, (5.19) is satisfied for $s = l(\Gamma_m^{\mathrm{II}})$ again by $B(p, \omega d(P)) \subset P$. The rest remains unchanged.

Step 4. *Construction of curves* (III).
Let $j \in \{0, \dots, m-1\}$ and recall (5.7)–(5.8). Let us first observe that if $i_j + 1 = i_{j+1}$, then $w^-_{i_j+1}$ and $w^-_{i_{j+1}}$ coincide. Therefore, in this particular case we set $\Gamma_j^{\mathrm{I}_2} = \Gamma_j^{\mathrm{III}} = \emptyset$. Now suppose $i_j + 1 < i_{j+1}$. First, as $[v_i; v_{i+1}]$ do not separate x and p for $i_j + 1 \le i \le i_{j+1} - 1$ (recall definition before (5.5)), we see that $\gamma_0([t_{i_j+1}, t_{i_{j+1}}])$ has the form of a helix, *i.e.* γ_0 has in $[t_{i_j+1}, t_{i_{j+1}}]$ a signed curvature. (Clearly, a 'degenerated helix' with less than a full winding is possible). More precisely, γ_0 may consist of an *outward helix* and an *inward helix* in the following sense: define

$$\varphi_k = \arg(v_{k+1} - v_k) \quad \text{ for } \; i_j + 1 \le k \le i_{j+1} - 1$$

and let $S_k = v_k + \mathbb{R}_+ \mathrm{e}^{i\varphi_k}$ with $\mathbb{R}_+ = (0,\infty)$. Let k^* be the smallest index such that

$$S_{k^*} \cap \gamma_0([t_{i_j+1}, t_{k^*}]) \ne \emptyset$$

and let $\gamma_0([t_{i_j+1}, t_{k^*}])$, $\gamma_0([t_{k^*}, t_{i_{j+1}}))$ be the outward and inward part of the helix, respectively. Indeed, beyond v_{k^*} the helix can not further growth outwardly as this would inavoidably imply self-intersection of the polygon (see Fig. 11).

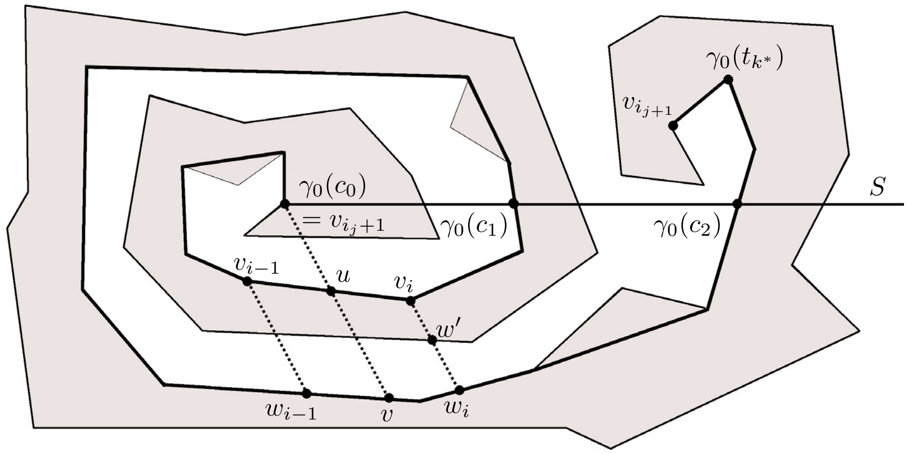

FIGURE 11. The part of γ_0 between t_{i_j+1} and $t_{i_{j+1}}$ has been depicted in black (for illustration purposes with $m = 2$) and we sketched a segment $[u; v]$ as considered below (5.23). Note that the outward helix ends in $\gamma_0(t_{k^*})$.

Recalling (5.7) we let $\Gamma_j^{\rm III} : [0, l(\Gamma_j^{\rm III})] \to \mathbb{R}^2$ be the arc length parametrized curve with $\Gamma_j^{\rm III}(s_i^\pm) = w^\pm_{i_j+i}$ for suitable $0 = s_1^+ < s_2^- < s_2^+ < \ldots < s_{N-1}^+ < s_N^- = l(\Gamma_j^{\rm III})$, $N = i_{j+1} - i_j$, which is affine on $[s_{i-1}^+, s_i^-]$ and on $[s_i^-, s_i^+]$ a part of a circle with midpoint v_{i_j+i} and radius $2\vartheta^2 t_{i_j+i}$ (see also Step 2 and Fig. 9). The crucial point is to show that the length of $\Gamma_j^{\rm III}$ is comparable to $t_{i_{j+1}} - t_{i_j}$ (*cf.* (5.30) below). To this end, we have to ensure that up to a finite number of 'windings' of the helix, the 'radius of a winding' can be suitably bounded from below.

We first concentrate on the part $\gamma_0([t_{i_j+1}, t_{k^*}])$. Possibly after a translation, rotation and reflection we can assume $S_{i_j+1} = \{0\} \times (0, \infty)$ (*i.e.* $v_{i_j+1} = 0$ and $v_{i_j+2} \in \{0\} \times (0, \infty)$) and $\arg(v_{i_j+3} - v_{i_j+2}) \in (\frac{\pi}{2}, \frac{3\pi}{2})$. Let $S = [0, \infty) \times \{0\}$. Let $c_k \in [t_{i_j+1}, t_{k^*}]$, $t_{i_j+1} = c_0 < c_1 < c_2 < \ldots < c_m$ be the points for which $\gamma_0(c_k) \in S$. Note that the number of points $\#(c_k)_k$ can be interpreted as the winding number of the outward helix. We now show for $3 \le k \le m - 1$

$$\gamma_0([c_{k-1}, c_k]) \cap B(0, r_k) = \emptyset, \tag{5.23}$$

where $B(0, r_k)$ denotes the open ball with radius $r_k = \vartheta^2 c_{k-3}$. If $m \le 3$, there is nothing to show. Therefore, we suppose $m \ge 4$. Choose an arbitrary $v \in \gamma_0([c_{k-1}, c_k])$ for $3 \le k \le m-1$. Let $u \in \gamma_0([c_{k-2}, c_{k-1}])$ be the (unique) point on $[0; v]$. In particular, we have $|[v; u]| \le |v|$. Select the index i such that $u \in [v_{i-1}; v_i] \subset \gamma_0([c_{k-3}, c_k])$.

Denote the intersection points on $(v_{i-1} + \mathbb{R}_+(v - u)) \cap \gamma_0$ and $(v_i + \mathbb{R}_+(v - u)) \cap \gamma_0$ nearest to v_{i-1} and v_i, respectively, by w_{i-1} and w_i. Due to the geometry of $\gamma_0([t_{i_j+1}, t_{k^*}])$ we have $w_{i-1}, w_i \in \gamma_0([c_{k-3}, c_{k+1}])$ and $\min_{l=i-1,i} |[v_l; w_l]| \le |[u; v]|$. Suppose, *e.g.*, $|[w_i; v_i]| \le |[u; v]| \le |v|$. Then we find some $w' \in \partial P$ with $w' \in [v_i; w_i]$ such that $[v_i; w'] \subset P$ and $[v_i; w']$ separates x and p. Now in view of $|[w'; v_i]| \le |v|$ and $t_i \ge c_{k-3}$ (recall $v_i \in \gamma_0([c_{k-3}, c_k])$), (5.3) implies

$$|v| \ge |[v_i; w']| \ge 4\vartheta^2 t_i \ge \vartheta^2 c_{k-3} = r_k, \tag{5.24}$$

which gives (5.23). For the part $\gamma_0([t_{k^*}, t_{i_{j+1}}])$ we proceed analogously. Let $\hat{S} = v_{i_{j+1}} + [0, \infty)\mathrm{e}^{i\varphi}$ for $\varphi \in [0, 2\pi)$ such that $\gamma_0(c_m) \in \hat{S}$. Let $\hat{c}_k \in [c_m, t_{i_{j+1}}]$, $c_m = \hat{c}_0 < \hat{c}_1 < \hat{c}_2 < \ldots < \hat{c}_{\hat{m}} = t_{i_{j+1}}$ be the points for which $\gamma_0(\hat{c}_k) \in \hat{S}$. Similarly as before we can show that for $1 \le k \le \hat{m} - 2$ one has

$$\gamma_0([\hat{c}_{k-1}, \hat{c}_k]) \cap B(v_{i_{j+1}}, \hat{r}_k) = \emptyset, \tag{5.25}$$

where $\hat{r}_k = \vartheta^2 \hat{c}_{k-1}$. In fact, select some $v \in \gamma_0([\hat{c}_{k-1}, \hat{c}_k])$ with $\hat{m} - k \geq 2$. Let $u \in \gamma_0([\hat{c}_k, \hat{c}_{k+1}])$ be the (unique) point on $[v_{i_{j+1}}; v]$. In particular, we have $|[v;u]| \leq |[v; v_{i_{j+1}}]|$. Assume $u \in [v_{i-1}; v_i]$, where $v_{i-1}, v_i \in \gamma_0([\hat{c}_{k-1}, \hat{c}_{k+2}])$. Arguing exactly as before we find some $w' \in \partial P$ such that $[v_i; w'] \subset P$, $|[w'; v_i]| \leq |[v; v_{i_{j+1}}]|$ and $[v_i; w']$ separates x and p. (Note that as before we possibly have to replace v_i by v_{i-1}). Then as in (5.24) using $v_i \in \gamma_0([\hat{c}_{k-1}, \hat{c}_{k+2}])$ we have $|[v_i; w']| \geq 4\vartheta^2 t_i \geq \vartheta^2 \hat{c}_{k-1}$. Consequently, we obtain $|[v; v_{i_{j+1}}]| \geq \hat{r}_k$.

By (5.23), (5.25) and the fact that γ_0 is parametrized by arc length we deduce

$$\begin{aligned} &c_k - c_{k-1} \geq 2\pi\vartheta^2 c_{k-3}, \quad 3 \leq k \leq m-1, \\ &\hat{c}_l - \hat{c}_{l-1} \geq 2\pi\vartheta^2 \hat{c}_{l-1}, \quad 1 \leq l \leq \hat{m} - 2. \end{aligned} \tag{5.26}$$

For $1 \leq k \leq m$ let $\mathcal{N}_k = \{n \in \mathbb{N} : v_{i_j+n} \in \gamma_0([c_{k-1}, c_k])\}$. By construction of $\mathit{\Gamma}_j^{\mathrm{III}}$ and (5.7) we obtain for ϑ small

$$\begin{aligned} s_n^- - s_{n-1}^+ &= \sqrt{(t_{i_j+n} - t_{i_j+n-1})^2 + (2\vartheta^2(t_{i_j+n} - t_{i_j+n-1}))^2} \leq 2(t_{i_j+n} - t_{i_j+n-1}), \\ s_n^+ - s_n^- &= 2\vartheta^2 t_{i_j+n}\tilde{\varphi}_n, \end{aligned} \tag{5.27}$$

where $\tilde{\varphi}_n$ denotes the angle enclosed by $\nu^-_{i_j+n}$, $\nu^+_{i_j+n}$ smaller than π (recall (5.7)). Let $1 \leq k \leq m$ and consider $n \in \mathcal{N}_k$. If $5 \leq k \leq m$, let n_k be the largest index in $\mathcal{N}_{k-4}$. Otherwise, set $n_k = 0$. Using (5.26) we first observe

$$\begin{aligned} \sum_{l=1}^{n_k} t_{i_j+l}\,\tilde{\varphi}_{i_j+l} &= \sum_{t=1}^{k-4}\sum_{l\in\mathcal{N}_t} t_{i_j+l}\,\tilde{\varphi}_{i_j+l} \leq \sum_{t=1}^{k-4}\sum_{l\in\mathcal{N}_t} c_t\,\tilde{\varphi}_{i_j+l} \leq 3\pi\sum_{t=1}^{k-4} c_t \\ &\leq \frac{3}{2\vartheta^2}\sum_{t=1}^{k-4}(c_{t+3} - c_{t+2}) \leq \frac{3}{2\vartheta^2}(c_{k-1} - c_3) \leq \frac{3}{2\vartheta^2}(c_{k-1} - c_0), \end{aligned}$$

where in the third step a calculation yields $\sum_{l\in\mathcal{N}_t}\tilde{\varphi}_{i_j+l} \leq 2\pi + 2\frac{\pi}{2} = 3\pi$ for all t. Similarly, we one can show $\sum_{l=n_k+1}^{n}\tilde{\varphi}_{i_j+l} \leq 8\pi + 2\frac{\pi}{2}$ and thus by (5.27) we derive

$$\begin{aligned} s_n^+ = s_n^+ - s_1^+ &= \sum_{l=2}^{n} s_l^+ - s_{l-1}^+ \leq 2(t_{i_j+n} - t_{i_j+1}) + 2\vartheta^2\sum_{l=2}^{n} t_{i_j+l}\,\tilde{\varphi}_{i_j+l} \\ &\leq 2(t_{i_j+n} - t_{i_j+1}) + 3(c_{k-1} - c_0) + 18\pi\vartheta^2 t_{i_j+n}. \end{aligned}$$

Recalling $c_0 = t_{i_j+1}$ and $c_{k-1} \leq t_{i_j+n}$ since $n \in \mathcal{N}_k$, we then find

$$s_n^+ \leq 5(t_{i_j+n} - t_{i_j+1}) + 18\pi\vartheta^2 t_{i_j+n} \leq C(t_{i_j+n} - t_{i_j+1}) + C\vartheta^2 t_{i_j+1}. \tag{5.28}$$

Now letting $\hat{\mathcal{N}}_k = \{n \in \mathbb{N} : v_{i_j+n} \in \gamma_0([\hat{c}_{k-m-1}, \hat{c}_{k-m}])\}$ for $m+1 \leq k \leq \hat{m}+m$ and repeating the above arguments we find for $n \in \hat{\mathcal{N}}_k$

$$s_n^+ \leq C(t_{i_j+n} - t_{i_j+1}) + C\vartheta^2 t_{i_j+1}, \tag{5.29}$$

where we set $s_N^+ := s_N^- = l(\mathit{\Gamma}_j^{\mathrm{III}})$. Thus, in particular for $n = N = i_{j+1} - i_j$ we have by (5.6)

$$l(\mathit{\Gamma}_j^{\mathrm{III}}) \leq C(t_{i_{j+1}} - t_{i_j+1}) + C\vartheta^2 t_{i_j+1} \leq C(t_{i_{j+1}} - t_{i_j+1}) + C(t_{i_j+1} - t_{i_j}). \tag{5.30}$$

We finally show that for $s \in [0, l(\mathit{\Gamma}_j^{\mathrm{III}})]$ one has

$$\mathrm{dist}(\mathit{\Gamma}_j^{\mathrm{III}}(s), \partial P) \geq c\vartheta^2(t_{i_j+1} + s) \tag{5.31}$$

for some $c > 0$ sufficiently small. Let $s \in [s^+_{n-1}, s^+_n]$ and $P = Q_1 \cup Q_2$ be the partition induced by $[v_{i_j+n-1}; v_{i_j+n}]$ with $x, p \in Q_1$ since the segment does not separate x and p. (Observe that $Q_2 = \emptyset$ is possible). As (5.5) does not hold, we have $t_{i_j+n} - t_{i_j+n-1} < 4\vartheta^2 t_{i_j+n} = 4\vartheta^2 t_{i_j+n-1} + 4\vartheta^2(t_{i_j+n} - t_{i_j+n-1})$ and then with ϑ small we get

$$t_{i_j+n} - t_{i_j+n-1} \leq 5\vartheta^2 t_{i_j+n-1} = 5\vartheta^2(t_{i_j+n-1} - t_{i_j+1}) + 5\vartheta^2 t_{i_j+1}.$$

Consequently, we find by (5.28), (5.29) for $C \geq 2$ and ϑ small (such that $C\vartheta^2 \leq 1$)

$$\begin{aligned} s + t_{i_j+1} &\leq s^+_n + t_{i_j+1} \leq C(t_{i_j+n} - t_{i_j+1}) + C\vartheta^2 t_{i_j+1} + t_{i_j+1} \\ &\leq C(t_{i_j+n-1} - t_{i_j+1}) + C\vartheta^2 t_{i_j+1} + t_{i_j+1} \leq C(t_{i_j+n-1} - t_{i_j+1}) + 2t_{i_j+1} \\ &\leq C t_{i_j+n-1}. \end{aligned} \tag{5.32}$$

Fix $u \in \partial P$. If $u \in Q_2$, we get by construction of $\varGamma^{\mathrm{III}}_j$ that $\mathrm{dist}(u, \varGamma^{\mathrm{III}}_j) \geq 2\vartheta^2 t_{i_j+n-1}$ (*cf.* Step 2 for a similar argument) and therefore by (5.32)

$$\mathrm{dist}(u, \varGamma^{\mathrm{III}}_j) \geq c\vartheta^2(s + t_{i_j+1})$$

for $c > 0$ small enough. In this case (5.31) holds. On the other hand, if $u \in Q_1$, we find $u' \in |[u; v_{i_j+n}]| \cap \partial P$ such that $|[u'; v_{i_j+n}]|$ separates x and p. If we had $\mathrm{dist}(\varGamma^{\mathrm{III}}_j(s), u) < \vartheta^2(t_{i_j+1} + s)$, we would get by (5.27)

$$\begin{aligned} |[u'; v_{i_j+n}]| &\leq \mathrm{dist}(u, \varGamma^{\mathrm{III}}_j(s)) + s^+_n - s^+_{n-1} < \vartheta^2(t_{i_j+1} + s) + s^+_n - s^-_n + s^-_n - s^+_{n-1} \\ &\leq \vartheta^2(t_{i_j+1} + s^+_n) + 2\pi\vartheta^2 t_{i_j+n} + 2(t_{i_j+n} - t_{i_j+n-1}). \end{aligned}$$

Then the fact that (5.5) does not hold and (5.32) yield for ϑ small with respect to ω and C

$$|[u; v_{i_j+n}]| \leq \vartheta^2(t_{i_j+1} + s^+_n) + (2\pi + 8)\vartheta^2 t_{i_j+n} \leq C\vartheta^2 t_{i_j+n} < \min\{\vartheta t_{i_j+n}, \vartheta\omega d(P)\}.$$

This contradicts (5.2) and concludes the proof of (5.31).

Step 5. *A curve between w^-_{n-1} and p.*
It remains to define a path between w^-_{n-1} and p (*cf.* below (5.8)). Define a path $\varGamma^{\mathrm{I}_1}_m$ between w^-_{n-1}, w^+_{n-1} as in Step 2 satisfying (5.9). Moreover, take a path $\varGamma^{\mathrm{II}}_m$ between w^+_{n-1}, w^-_n as in Step 3 such that (5.17) and (5.19) hold. Let $\varGamma^{\mathrm{I}_2}_m = \varGamma^{\mathrm{III}}_m = \emptyset$ and let $\varGamma^{\mathrm{IV}}$ be the segment between $w^-_n = p + 2\vartheta^2 t_n \nu^+_{n-1}$ and p. Clearly, since $B(p, \omega d(P)) \subset P$, we have for ϑ small with respect to ω

$$(i)\ \ l(\varGamma^{\mathrm{IV}}) = 2\vartheta^2 t_n, \quad (ii)\ \ \mathrm{dist}(\varGamma^{\mathrm{IV}}(s), \partial P) \geq \vartheta^2 d(P) \geq \vartheta^2 t_n = \vartheta^2 l(\gamma_0). \tag{5.33}$$

Step 6. *The curve γ and the carrot condition.*
Now define $\gamma : [0, l(\gamma)] \to P$ such that γ is parametrized by arc length and

$$\gamma([0, l(\gamma)]) = \bigcup_{j=0}^{m} \left(\varGamma^{\mathrm{I}_1}_j \cup \varGamma^{\mathrm{II}}_j \cup \varGamma^{\mathrm{I}_2}_j \cup \varGamma^{\mathrm{III}}_j\right) \cup \varGamma^{\mathrm{IV}}$$

with $\gamma(0) = x$ and $\gamma(l(\gamma)) = p$. We now show that (5.1) holds for ϑ sufficiently small. We have to derive that $B(\gamma(\tau), \vartheta^3\tau) \subset P$ for all $\tau \in [0, l(\gamma)]$. Let $\gamma(\tau) \in \hat{\varGamma}$, where $\hat{\varGamma} \in \{\varGamma^{\mathrm{IV}}\} \cup \{\varGamma^{\mathrm{X}}_j, \mathrm{X} = \mathrm{I}_1, \mathrm{I}_2, \mathrm{II}, \mathrm{III}, j = 0, \dots, m\}$. Choose $\tau_0 \leq \tau$ such that $\gamma(\tau_0) = \hat{\varGamma}(0)$ and $i \in \{0, \dots, n\}$ such that $\gamma(\tau_0) \in \{w^-_i, w^+_i\}$. (Note that $i = i_j$ or $i = i_j + 1$ for some $j = 0, \dots, m$, *cf.* (5.8)). Combining (5.9)(i), (5.17), (5.30) and (5.33)(i) we derive by a telescope sum argument

$$\tau_0 \leq \hat{C} t_i \tag{5.34}$$

for some universal $\hat{C} \geq 1$ large enough, *i.e.* γ is at most $\hat{C}$ times longer than the original curve γ_0. Letting $s = \tau - \tau_0$ and using

$$\mathrm{dist}(\hat{\Gamma}(s), \partial P) \geq c\vartheta^2(t_i + s)$$

by (5.9)(ii), (5.19), (5.31), (5.33)(ii), respectively, we conclude by (5.34) for ϑ small

$$\begin{aligned}\mathrm{dist}(\gamma(\tau), \partial P) &= \mathrm{dist}(\hat{\Gamma}(s), \partial P) \geq c\vartheta^2(t_i + s) \geq c\vartheta^2(t_i + \hat{C}^{-1}s) \\ &= c\hat{C}^{-1}\vartheta^2\tau + c\vartheta^2\big(t_i - \hat{C}^{-1}\tau_0\big) \geq c\hat{C}^{-1}\vartheta^2\tau \geq \vartheta^3\tau. \qquad \square\end{aligned}$$

6. Proof of the main result and application

This section is devoted to the proof of Theorem 1.1 and an application. First, we prove the main partition result for polygons, which with the preparations in the last sections is now straightforward.

Proof of Theorem 2.5. Let $\theta, \varepsilon > 0$. By Theorem 3.7 and Theorem 3.4 we first partition $P = P_1' \cup \ldots P_m'$ into $\bar{\vartheta}$-semiconvex polygons with $\bar{\vartheta} = \bar{\vartheta}(\theta)$ such that $\sum_{j=1}^m \mathcal{H}^1(\partial P_j') \leq (1 + C\theta)\mathcal{H}^1(\partial P)$ for $C > 0$ universal. Applying Theorem 4.7 on each P_j' with $\epsilon = \frac{1}{m}\varepsilon$ and $\vartheta = \bar{\vartheta}$ we find $\tilde{\vartheta} = \tilde{\vartheta}(\theta)$, $\omega = \omega(\theta)$ and for each P_j' a partition $P_j' = P_0^j \cup P_{i_j+1} \cup \ldots \cup P_{i_{j+1}}$ with $i_1 = 0$, $i_{m+1} = N$ such that $\sum_{j=1}^m \mathcal{H}^1(\partial P_0^j) \leq \varepsilon$ and the polygons $(P_i)_{i=1}^N$ are $\tilde{\vartheta}$-semiconvex and ω-rotund with

$$\sum_{i=1}^N \mathcal{H}^1(\partial P_i) = \sum_{j=1}^m \sum_{k=i_j+1}^{i_{j+1}} \mathcal{H}^1(\partial P_k) \leq \sum_{j=1}^m (1 + C\theta)\mathcal{H}^1(\partial P_j') \leq (1 + C\theta)\mathcal{H}^1(\partial P).$$

Define $P_0 = \bigcup_{j=1}^m P_0^j$. Starting the proof with θC^{-1} instead of θ, we obtain (2.3). The fact that the polygons $(P_i)_{i=1}^N$ are ϱ-John domains for $\varrho = \varrho(\theta)$ follows from Theorem 5.2. $\square$

The reader more interested in applications of our main result may now skip Section 6.1 and continue with Section 6.2.

6.1. Proof of the main result

To derive the result for sets with C^1-boundary we will have to combine different John domains. We start with an adaption of Lemma 2.4. In the following $\mathrm{diam}(D)$ denotes the diameter of a set $D \subset \mathbb{R}^2$.

Lemma 6.1. *Let $0 < \varrho, c' < 1$. Then for some $\varrho' = (c', \varrho) > 0$ the following holds:*

(i) *Let $D_1, D_2 \subset \mathbb{R}^2$ be simply connected ϱ-John domains with Lipschitz boundary and $D_1 \cap D_2 = \emptyset$ such that $\partial D_1 \cap \partial D_2$ contains a segment S with*

$$\mathcal{H}^1(S) \geq c' \min\{\mathrm{diam}(D_1), \mathrm{diam}(D_2)\}.$$

Then $D = \mathrm{int}(\overline{D_1} \cup \overline{D_2})$ is a ϱ'-John domain.

(ii) *Let P be a polygon and $\triangle$ a closed triangle with $\mathrm{int}(P) \cap \mathrm{int}(\triangle) = \emptyset$ such that $\mathrm{int}(P)$ is a ϱ-John domain and $\partial P \cap \partial\triangle$ contains the longest edge of $\triangle$. Then $D = \mathrm{int}(\triangle \cup P)$ is a ϱ'-John domain with $\varrho'|D| \leq |P|$.*

Proof. (i) After rotation and translation we suppose $S = [(-2d, 0); (2d, 0)]$ with $d^2 \geq c\min\{|D_1|, |D_2|\}$ for $c = c(c')$, where the inequality follows from the assumption and the isodiametric inequality. For $\eta > 0$ let $Q_\eta = (-\eta d, \eta d)^2$, $Q_\eta^1 = (-\eta d, \eta d) \times (0, \eta d)$ and $Q_\eta^2 = (-\eta d, \eta d) \times (-\eta d, 0)$. We now show that there is $\eta > 0$ only depending on ϱ such that possibly after changing the roles of D_1 and D_2 we have

$$Q_\eta^i \subset D_i \qquad \text{for} \qquad i = 1, 2.$$

We show the claim for $i = 1$. As ∂D_1 is Lipschitz, we get that $[-d, d] \times (0, \varepsilon] \subset D_1$ for ε small enough. Let γ be a John curve in D_1 connecting $(-d, \varepsilon)$ and (d, ε) (*cf.* Rem. 2.2). Since D_1 is a ϱ-John domain, we find $0 < \eta < 1$ only depending on ϱ such that $\gamma \cap Q^1_\eta = \emptyset$. As $[-d, d] \times (0, \varepsilon] \subset D_1$ and D_1 is simply connected, we then derive $Q^1_\eta \subset D_1$ as desired.

Define $D'_i = D_i \cup Q_\eta$ and $D' = D'_1 \cup D'_2$. Each D'_i is a $\bar{\varrho}$-John domain for $\bar{\varrho} = \bar{\varrho}(\varrho)$ by Lemma 2.4(i) since $|Q_\eta| = 2|Q^i_\eta| = 2|Q^i_\eta \cap D_i|$. Moreover, we find $\varrho' = \varrho'(c', \varrho)$ such that D' is a ϱ'-John domain by Lemma 2.4(i) as

$$\min\{|D'_1|, |D'_2|\} \le 2 \min\{|D_1|, |D_2|\} \le 2c^{-1}d^2 \le C|Q_\eta|$$

for C only depending on c, η and thus only depending on ϱ, c'. Finally, possibly passing to a smaller ϱ' also D is a ϱ'-John domain since $D \setminus D' \subset \partial D_1 \cap \partial D_2$.

(ii) Note that (i) is not directly applicable as $\mathrm{int}(\triangle)$ is possibly not a ϱ-John domain. Suppose $S = [(-d, 0); (d, 0)]$ is the longest edge of $\triangle$ and $\triangle \subset [-d, d] \times [0, \infty)$. Arguing as in (i), we find η only depending on ϱ such that the closed triangle $\triangle'$ with vertices $(-d, 0)$, $(d, 0)$ and $(0, -d\eta)$ is completely contained in P. As $\mathcal{H}^1(S) = \mathrm{diam}(\triangle)$, it is not hard to see that $B := \mathrm{int}(\triangle \cup \triangle')$ is a ϱ'-John domain with ϱ' only depending on ϱ. Moreover, we find

$$|\triangle| \le C|\triangle'| \tag{6.1}$$

for $C = C(\varrho)$. Then by Lemma 2.4(i) and (6.1) also $D = \mathrm{int}(P \cup B) = \mathrm{int}(P \cup \triangle)$ is a John domain for a possibly smaller ϱ' only depending on ϱ. Finally, $\varrho'|D| \le |P|$ follows from (6.1) for ϱ' small enough. □

Before we concern ourselves with sets with C^1-boundary we state the following corollary of Theorem 2.5.

Corollary 6.2. *Let be given the situation of Theorem 2.5. If $\sphericalangle(v, P) \ge \frac{\pi}{4}$ for all $v \in \mathcal{V}_P$, one can set $P_0 = \emptyset$.*

Proof. First, we apply Theorem 2.5 to get a partition of $P = \bigcup_{j=0}^N P_j$. Recall that by the construction in the proof of Theorems 4.7 and 2.5 the component P_0 is the finite union of closed, isosceles triangles with exactly one interior angle smaller than $\frac{\pi}{4}$ (see before (4.21)). We first see that each two triangles $\triangle_1$, $\triangle_2$ do not share a segment. Indeed, otherwise the corresponding convex polygons, denoted by P'_1, P'_2, from which the triangles are cut out, share a segment and contain $v \in \mathcal{V}_{P'_1} \cap \mathcal{V}_{P'_2}$ with $v \in \triangle_1 \cap \triangle_2$ and $\sphericalangle(v, P'_i) \le \frac{\pi}{4}$ for $i = 1, 2$. As the partition can be constructed such that endpoints of introduced segments never coincide unless there are concave vertices of P (see Rem. 3.9(ii)), we derive $v \in \mathcal{V}'_P$. Then, however, Remark 3.9(iii) implies $\mathcal{H}^1(\partial P'_1 \cap \partial P'_2) = 0$, which gives a contradiction.

Moreover, it is not restrictive to assume that each edge of a triangle $\triangle$ is completely contained in ∂P or some ∂P_j since otherwise we choose an isosceles $\triangle' \subset \triangle$ with the desired property. We then note that the $\triangle \setminus \triangle'$ is a convex polygon with interior angles larger than $\frac{\pi}{4}$ and thus we can apply Lemma 4.6 to obtain a refined partition of $\triangle \setminus \triangle'$ consisting of ϱ-John domains such that (2.3) still holds.

The assumption $\sphericalangle(v, P) \ge \frac{\pi}{4}$ for all $v \in \mathcal{V}_P$ implies that for each $\triangle$ at least one of the two longer edges is contained in some ∂P_i. Then $\mathrm{int}(P_i \cup \triangle)$ is a John domain by Lemma 6.1(ii) for a John constant only depending on θ and $|\mathrm{int}(P_i \cup \triangle)| \le C|P_i|$ for $C = C(\theta)$. Hereby we can define a partition $(P'_i)_i$ of Ω satisfying (2.3) such that each component P'_i is the union of P_i with some triangles adjacent to P_i. Now Lemma 2.4(ii) (with $D_0 = P_i$, $D_j = \mathrm{int}(P_i \cup \triangle)$ for $j \ge 1$) yields that all P'_j are John domains for a John constant only depending on θ. □

We now extend the result to sets with smooth boundary, where we first derive a version without the sharp estimate (1.1).

Theorem 6.3. *Theorem 1.1 holds with $\sum_{j=1}^N \mathcal{H}^1(\partial\Omega_j) \le C\mathcal{H}^1(\partial\Omega)$ in place of (1.1) for a universal $C > 1$.*

Proof. As Ω has C^1-boundary and $\partial\Omega$ is connected due to the fact that Ω is simply connected, we can find $p_0,\dots,p_{n-1} \in \partial\Omega$ such that the closed squares Q_i with diagonal $[p_i;p_{i+1}]$ for $i = 0,1,\dots,n-1$ (set $p_n = p_0$ and $Q_n = Q_0$) satisfy

$$(i)\ \ d := \min_{i=0,\dots,n-1} |[p_i;p_{i+1}]| \geq \frac{1}{2} \max_{i=0,\dots,n-1} |[p_i;p_{i+1}]|, \tag{6.2}$$

$$(ii)\ \ Q_i \cap Q_{i+1} = \{p_{i+1}\},\ \ \mathrm{dist}(Q_i, Q_{(i+k)\,\mathrm{mod} n}) \geq \frac{d}{2} \quad \text{for } i = 0,\dots,n-1,\ |k| \geq 2$$

and $\partial\Omega \cap Q_i$ is the graph of a C^1 function, where the angle enclosed by $p_{i+1} - p_i$ and the tangent vector of $\partial\Omega$ in $\partial\Omega \cap Q_i$ is smaller than $\frac{\pi}{8}$. Moreover, this can be done in the way that all interior angles of the interior polygon $P_{\mathrm{int}} := \overline{\Omega \setminus \bigcup_{i=0}^{n-1} Q_i}$ are larger than $\frac{\pi}{4}$. Define also the sets

$$P_i^{\mathrm{out}} = \Omega \cap \mathrm{int}(Q_i) \tag{6.3}$$

for $i = 0,\dots,n-1$. The geometry of P_i^{out} implies that P_i^{out} has Lipschitz boundary and is a c-John domain for a universal constant $c > 0$. Moreover, we observe that $\mathcal{H}^1(\partial P_{\mathrm{int}}) + \sum_{i=0}^{n-1} \mathcal{H}^1(\partial P_i^{\mathrm{out}}) \leq C\mathcal{H}^1(\partial\Omega)$. The claim follows from Corollary 6.2 applied on P_{int}. □

This together with Lemma 6.1 allows to give the proof of Theorem 1.1.

Proof of Theorem 1.1. By Corollary 6.2 we find a partition $P_{\mathrm{int}} = P_1 \cup \ldots \cup P_N$ of the polygon P_{int} constructed in the proof of Theorem 6.3, where by (2.3) for $C > 0$ universal

$$\sum_{j=1}^{N} \mathcal{H}^1(\partial P_j \setminus \partial P_{\mathrm{int}}) \leq \theta\mathcal{H}^1(\partial P_{\mathrm{int}}) \leq C\theta\mathcal{H}^1(\partial\Omega). \tag{6.4}$$

The goal is now to combine each P_j with certain $(P_i^{\mathrm{out}})_{i=0}^{n-1}$ defined in (6.3) such that the resulting sets are still John domains and (1.1) holds. Let J be the set of indices such that $j \in J$ if and only if $\mathrm{diam}(P_j) < \frac{d}{4}$ with d as in (6.2). By (6.2) we see that each P_j, $j \in J$, intersects at most two sets $\overline{P_i^{\mathrm{out}}}$, $i = 0,\dots,n-1$. Recalling the geometry of $(P_i^{\mathrm{out}})_i$ and the fact that the interior angles of the polygon P_{int} are larger than $\frac{\pi}{4}$, we find $\mathcal{H}^1(\partial P_j) \leq C\mathcal{H}^1(\partial P_j \setminus \partial P_{\mathrm{int}})$ for $j \in J$ for a universal constant $C > 0$ and thus by (6.4)

$$\sum_{j\in J} \mathcal{H}^1(\partial P_j) \leq C \sum_{j\in J} \mathcal{H}^1(\partial P_j \setminus \partial P_{\mathrm{int}}) \leq \theta\mathcal{H}^1(\partial P_{\mathrm{int}}) \leq C\theta\mathcal{H}^1(\partial\Omega). \tag{6.5}$$

Recall the definition of Q_i in (6.2) and denote by Q_i' the enlarged square with the same center and orientation, but with diagonal length $\frac{5}{4}|[p_i;p_{i+1}]|$. Note that all sets $P_{\mathrm{int}} \cap Q_i'$ are Lipschitz and are all related to a square of sidelength d through Lipschitz homeomorphism with Lipschitz constants of both the homeomorhism itself and its inverse uniformly bounded independently of i. Let $\bar{c} > 0$ to be specified below in (6.8)–(6.10). We observe that there is $\bar{C} = \bar{C}(\bar{c}) > 0$ such that

$$\#I \leq \bar{C}\theta d^{-1}\mathcal{H}^1(\partial\Omega), \text{ where } I := \left\{ i : \mathcal{H}^1\left(\mathrm{int}(P_{\mathrm{int}} \cap Q_i') \cap \bigcup_{j=1}^{N} \partial P_j\right) \geq \bar{c}d \right\}. \tag{6.6}$$

Indeed, this follows from (6.4) and (6.2).

Consider $i \notin I$. For $j = 1,\dots,N$ define the components $A_{j,i} := P_j \cap Q_i'$ of $P_{\mathrm{int}} \cap Q_i'$ and denote by $(A_{j,i}^k)_k$ the connected components of $A_{j,i}$. Then the result in ([19], Lem. 4.6), which essentially relies on the relative isoperimetric inequality, shows that for $\bar{c}$ sufficiently small there is exactly one component $B_i := P_{j_i} \cap Q_i' \subset (A_{j,i})_{j=1}^N$ with $|B_i| > \frac{1}{2}|P_{\mathrm{int}} \cap Q_i'|$ and the other components $A_{j,i} \neq B_i$ satisfy

$$\mathrm{diam}(A_{j,i}^k) \leq C\mathcal{H}^1(\partial A_{j,i}^k \cap \mathrm{int}(P_{\mathrm{int}} \cap Q_i')) \leq C\bar{c}d \quad \text{for all} \quad A_{j,i}^k \tag{6.7}$$

for a universal $C > 0$, particularly independent of i and $A^k_{j,i}$. Then using the fact that $\mathcal{H}^1(\partial(P_{\rm int} \cap Q'_i) \cap \partial A^k_{j,i}) \le C{\rm diam}(A^k_{j,i})$ by the geometry of $P_{\rm int} \cap Q'_i$, we get by (6.6)–(6.7)

$$\sum_{A_{j,i} \neq B_i} \mathcal{H}^1(\partial A_{j,i}) \le C \sum_{A_{j,i} \neq B_i} \mathcal{H}^1(\partial A_{j,i} \cap {\rm int}(P_{\rm int} \cap Q'_i)) + C \sum_{A_{j,i} \neq B_i} \sum_k {\rm diam}(A^k_{j,i}) \le C\bar{c}d. \tag{6.8}$$

Moreover, as ${\rm dist}(\partial Q'_i, \partial Q_i) \ge \frac{1}{8\sqrt{2}}d$, for $\bar{c}$ small enough we derive by (6.7)

$$A_{j,i} = (P_j \cap Q'_i) \neq B_i \text{ and } \partial A_{j,i} \cap \partial P^{\rm out}_i \neq \emptyset \quad \Rightarrow \quad P_j \subset Q'_i \text{ and } j \in J. \tag{6.9}$$

Moreover, we find

$$(i) \;\; \mathcal{H}^1(\partial B_i \cap \partial P^{\rm out}_i) \ge \frac{d}{2}, \qquad (ii) \;\; \partial B_i \cap \partial P^{\rm out}_i \;\text{ connected.} \tag{6.10}$$

First, (i) follows for $\bar{c}$ small from (6.8) and the fact that $\mathcal{H}^1(\partial P^{\rm out}_i \cap \partial P_{\rm int}) \ge d$. If (ii) was false, we would find that the polygon $B_i = P_{i_j} \cap Q'_i$ has at least one concave vertex not lying on $\partial P_{\rm int}$. This, however, contradicts the construction of the partition, *cf.* Remark 3.9(i) and Remark 4.8. Note that (6.10)(i) implies $j_i \notin J$. By Lemma 6.1(i), (6.2) and (6.10) we find

$$D_i := {\rm int}(P_{j_i} \cup \overline{P^{\rm out}_i}) \tag{6.11}$$

is a ϱ'-John domain with Lipschitz boundary for $\varrho' = \varrho'(\theta)$.

We are now in the position to define the partition of Ω. For all $j \notin J$, let $I_j \subset \{0, \ldots, n-1\} \setminus I$ be the index set such that $P_j \cap Q'_i = B_i$ if and only if $i \in I_j$, where $B_i = P_{j_i} \cap Q'_i$ as above. Note that the above arguments in (6.9) show that the sets $(I_j)_j$ are pairwise disjoint and also observe that I_j may be empty. Define $P'_j = \bigcup_{i \in I_j} D_i$ for $j \notin J$ with D_i as in (6.11) and consider the partition $(\Omega_j)_j$ consisting of the sets

$$(P'_j)_{j \notin J} \;\cup\; ({\rm int}(P_j))_{j \in J} \;\cup\; (P^{\rm out}_i)_{i \in I}.$$

Note that the sets cover Ω up to a set of negligible measure since each $P^{\rm out}_i$, $i \notin I$, is contained in some P'_j, $j \notin J$. Note that by (6.9) we derive

$$\bigcup\nolimits_j (\partial\Omega_j \setminus \partial\Omega) \subset \bigcup\nolimits_{i \in I} (\partial P^{\rm out}_i \cap \partial P_{\rm int}) \cup \bigcup\nolimits_{j=1}^N (\partial P_j \setminus \partial P_{\rm int}) \cup \bigcup\nolimits_{j \in J} (\partial P_j \cap \partial P_{\rm int}).$$

This together with (6.4), (6.5) and $\sum_{i \in I} \mathcal{H}^1(\partial P^{\rm out}_i \cap \partial P_{\rm int}) \le Cd\,\#I \le C\theta\mathcal{H}^1(\partial\Omega)$ (see (6.6)) yields $\sum_j \mathcal{H}^1(\partial\Omega_j \setminus \partial\Omega) \le C\theta\mathcal{H}^1(\partial\Omega)$ and herefrom we indeed derive (1.1) since we can replace θ by $C^{-1}\theta$ in the above proof. Finally, observe that all components are John domains with Lipschitz boundary for a John constant only depending on θ, where for the sets $(P'_j)_{j \notin J}$ we use (6.11) and Lemma 2.4(ii). ☐

6.2. A generalization and application

We now present a generalized version of Theorem 1.1 for Lipschitz sets which are not necessarily simply connected. This version will be one of the main ingredients of [18]. For a bounded set $D \subset \mathbb{R}^2$ we introduce the *saturation* of D defined by ${\rm sat}(D) = {\rm int}(\mathbb{R}^2 \setminus E^0)$, where E^0 denotes the unique unbounded connected component of $\mathbb{R}^2 \setminus D$.

Theorem 6.4. *Let $\varepsilon > 0$ and $M \in \mathbb{N}$. Then there is a universal constant $\varrho > 0$ and $C = C(M) > 0$ such that for all bounded domains $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary and the property that ${\rm sat}(\Omega) \setminus \Omega$ consists of at most M components the following holds: There is a partition $\Omega = \Omega_0 \cup \ldots \cup \Omega_N$ such that $|\Omega_0| \le \varepsilon$ and the sets $\Omega_1, \ldots, \Omega_N$ are ϱ-John domains with Lipschitz boundary with*

$$\sum\nolimits_{j=0}^N \mathcal{H}^1(\partial\Omega_j) \le C\mathcal{H}^1(\partial\Omega).$$

Proof. Let $\varepsilon > 0$ be given and let $U_1, \ldots, U_m$ be the connected components of $\mathrm{sat}(\Omega) \setminus \Omega$ with $m \le M$. For each U_j we can choose a segment S_j with $\mathcal{H}^1(S_j) \le \mathrm{diam}(\Omega) \le \mathcal{H}^1(\partial\Omega)$ such that $\Theta_j := \partial U_j \cup S_j \cup \partial(\mathrm{sat}(\Omega))$ is connected. Consequently, we get that each connected component of $\Omega \setminus \bigcup_{j=1}^m \Theta_j$ is simply connected. For $s > 0$ we cover $\mathbb{R}^2$ with squares of the form $Q(p) = p + [-s, s]^2$, $p \in 2s\mathbb{Z}^2$. Let

$$\mathcal{Q}_s := \left\{ Q(p) : Q(p) \cap \Omega \neq \emptyset,\ Q(p) \cap \left(\partial\Omega \cup \bigcup\nolimits_{j=1}^m S_j \right) \neq \emptyset \right\}.$$

Since Ω has Lipschitz boundary, we find that for s sufficiently small

$$s\#\mathcal{Q}_s \le C\mathcal{H}^1 \left(\partial\Omega \cup \bigcup\nolimits_{j=1}^m S_j \right) \le C\mathcal{H}^1(\partial\Omega) + CM\mathrm{diam}(\Omega) \le C\mathcal{H}^1(\partial\Omega) \tag{6.12}$$

with $C = C(M)$. By $(P_i)_i$ we denote the connected components of $\mathbb{R}^2 \setminus \bigcup_{Q(p) \in \mathcal{Q}_s} Q(p)$ having nonempty intersection with Ω. Since each P_i is the union of squares and the connected components of $\Omega \setminus \bigcup_j \Theta_j$ are simply connected, also P_i is simply connected and thus $\overline{P_i}$ is a polygon with interior angles not smaller than $\frac{\pi}{2}$. Moreover, we find by (6.12)

$$\sum\nolimits_i \mathcal{H}^1(\partial P_i) \le 8s\#\mathcal{Q}_s \le C\mathcal{H}^1(\partial\Omega).$$

Likewise, if we choose s small enough, we get that $\Omega_0 := \Omega \setminus \bigcup_i P_i$ satisfies

$$|\Omega_0| \le 4s^2\#\mathcal{Q}_s \le Cs^2\mathcal{H}^1(\partial\Omega) \le \varepsilon, \qquad \mathcal{H}^1(\partial\Omega_0) \le C\mathcal{H}^1(\partial\Omega).$$

The result now follows from Corollary 6.2 applied on each P_i for $\theta = 1$. (Note that alternatively one may also apply Theorem 2.5 on each P_i choosing the occurring exceptional sets P_0^i small enough in terms of ε). ☐

Finally, we derive a piecewise Korn inequality for a certain subclass of SBD (we refer to [2,4] for more details on this function space). Although this problem will be thoroughly discussed in [18], we include a simplified analysis in the present exposition to give a first application of the main results of this article.

Let $1 < p < \infty$ and $M \in \mathbb{N}$. For an open, bounded set $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary we let $\mathcal{W}^p_M(\Omega)$ be the set of functions in $SBD^p(\Omega)$ whose jump set $J_y = \bigcup_{j=1}^m \Gamma_j^y$ is the finite union of closed connected pieces of Lipschitz curves with at most M components (*i.e.* $m \le M$) and $y|_{\Omega \setminus J_y} \in W^{1,p}(\Omega \setminus J_y)$. Note that similar assumptions have been used, *e.g.*, in [9,13,29,34].

Theorem 6.5. *Let $p \in (1, \infty)$ and $M \in \mathbb{N}$. Then there is $c = c(p) > 0$ and $C = C(M) > 0$ such that for all $\Omega \subset \mathbb{R}^2$ open, bounded with Lipschitz boundary with the property that $\mathrm{sat}(\Omega) \setminus \Omega$ consists of at most M components the following holds: For each $y \in \mathcal{W}^p_M(\Omega)$ there is a partition $(\Omega_j)_{j=0}^N$ of Ω with*

$$\sum\nolimits_{j=0}^N \mathcal{H}^1(\partial\Omega_j) \le C(\mathcal{H}^1(J_y) + \mathcal{H}^1(\partial\Omega)) \tag{6.13}$$

and corresponding $A_j \in \mathbb{R}^{2\times 2}_{\mathrm{skew}}$, $b_j \in \mathbb{R}^2$ such that $u := y - \sum_{j=0}^N (A_j\ \cdot + b_j)\chi_{\Omega_j}$ satisfies

$$(\mathrm{diam}(\Omega))^{-1}\|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)} \le c\|e(y)\|_{L^p(\Omega)},$$

where $e(y) = \frac{1}{2}(\nabla y^T + \nabla y)$.

Note that one essential point is that the constant c does not depend on Ω and C depends on Ω only in terms of the number of components of $\mathrm{sat}(\Omega) \setminus \Omega$. Therefore, the result is also interesting in the case of varying domains Ω and functions $y \in W^{1,p}(\Omega)$.

Proof. A classical result states that y is piecewise rigid if $\|e(y)\|_{L^p(\Omega)} = 0$ (see also [11]), so we can concentrate on the case $\|e(y)\|_{L^p(\Omega)} > 0$. Applying the following results on each connected component of Ω separately, it is not restrictive to assume that Ω is connected. Moreover, we may suppose that Ω is simply connected as otherwise we consider $\mathrm{sat}(\Omega)$ and define an extension $\bar{y}$ with $\bar{y} = 0$ on $\mathrm{sat}(\Omega) \setminus \Omega$, where we obtain $\bar{y} \in \mathcal{W}^p_{2M}(\mathrm{sat}(\Omega))$.

We now repeat the arguments in the proof of Theorem 6.4 on $(\Gamma^y_j)_j$ instead of $(U_j)_j$: we introduce segments to obtain simply connected components of $\mathrm{sat}(\Omega)$ and covering the boundary with squares we obtain an estimate of the form (6.12), where the right hand side now also depends on $\mathcal{H}^1(J_y)$. As before this yields a partition $(\Omega_j)_{j=0}^N$ of Ω such that $|\Omega_0| \le \varepsilon$ for an arbitrarily small $\varepsilon > 0$ and $\Omega_1, \ldots, \Omega_N$ are ϱ-John domains for a universal constant ϱ. Then (6.13) follows as in Theorem 6.4.

As Korn's inequality holds on John domains with a constant only depending on the John constant (see *e.g.* [1]), we get by an elementary scaling argument

$$\sum\nolimits_{j=1}^N \big((\mathrm{diam}(\Omega_j))^{-p}\|y - (A_j\ \cdot + b_j)\|^p_{L^p(\Omega_j)} + \|\nabla y - A_j\|^p_{L^p(\Omega_j)}\big) \le c\|e(y)\|^p_{L^p(\Omega)}$$

for suitable $A_j \in \mathbb{R}^{2\times 2}_{\mathrm{skew}}$, $b_j \in \mathbb{R}^2$ and $c = c(p)$. Finally, as $y \in L^p(\Omega)$, $\nabla y \in L^p(\Omega)$ and $|\Omega_0| \le \varepsilon$, we find $(\mathrm{diam}(\Omega))^{-1}\|y\|_{L^p(\Omega_0)} + \|\nabla y\|_{L^p(\Omega_0)} \le \|e(y)\|_{L^p(\Omega)}$ for ε small enough so that the assertion holds for $u = y - \sum_{j=1}^N (A_j\ \cdot + b_j)\chi_{\Omega_j}$. □

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