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ON THE HORSESHOE CONJECTURE FOR MAXIMAL DISTANCE MINIMIZERS

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Abstract. We study the properties of sets Σ having the minimal length (one-dimensional Hausdorff measure) over the class of closed connected sets $\Sigma \subset \mathbb{R}^2$ satisfying the inequality $\max_{y\in M} \operatorname{dist}(y, \Sigma) \le r$ for a given compact set $M \subset \mathbb{R}^2$ and some given $r > 0$. Such sets play the role of shortest possible pipelines arriving at a distance at most r to every point of M, where M is the set of customers of the pipeline. We describe the set of minimizers for M a circumference of radius $R > 0$ for the case when $r < R/4.98$, thus proving the conjecture of Miranda, Paolini and Stepanov for this particular case. Moreover we show that when M is the boundary of a smooth convex set with minimal radius of curvature R, then every minimizer Σ has similar structure for $r < R/5$. Additionaly, we prove a similar statement for local minimizers.

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1. Preliminaries

1.1. Introduction

For a given compact set $M \subset \mathbb{R}^2$ consider the functional

$$F_M(\Sigma) := \sup_{y\in M} \operatorname{dist}(y, \Sigma),$$

where Σ is a subset of $\mathbb{R}^2$ and $\operatorname{dist}(y, \Sigma)$ stands for the Euclidean distance between y and Σ (naturally, $F_M(\emptyset) := +\infty$). The quantity $F_M(\Sigma)$ will be called the *energy* of Σ. Consider the class of closed connected sets $\Sigma \subset \mathbb{R}^2$ satisfying $F_M(\Sigma) \le r$ for some $r > 0$. We are interested in the properties of sets of minimal length (one-dimensional Hausdorff measure) $\mathcal{H}^1(\Sigma)$ over the mentioned class. Such sets will be further called *minimizers*. They can be viewed as shortest possible pipelines arriving at a distance at most r to every point of M which in this case is considered as the set of customers of the pipeline.

It is proven (in fact, even in the general n-dimensional case $M \subset \mathbb{R}^n$; see [12] for the rigorous statement and details) that the set $OPT^*_\infty(M)$ of minimizers (for all $r > 0$) is nonempty and coincides with the set $OPT_\infty(M)$

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of solutions of the dual problem: minimize F_M over all compact connected sets $\Sigma \subset \mathbb{R}^2$ with prescribed bound on the total length $\mathcal{H}^1(\Sigma) \leq l$. The latter minimizing problem is quite similar to many other problems of minimizing other functionals over closed connected sets, for instance the average distance with respect to some finite Borel measure (see [3,5,6,10,11]) or similar urban planning problems (see [4]). If one minimizes maximum or average distance functional over discrete sets with an a priori restriction on the number of connected components (rather than over connected one-dimensional sets) one gets another class of closely related problems known as k-center problem and k-median problem (see *e.g.* [8,15,16] as well as [1,2] and references therein).

Some basic properties of minimizers for the above mentioned problem in n-dimensional case (like the absence of loops and Ahlfors regularity) have been proven in [13]. Further, in [12] the following characterization of minimizers has been studied. Let $B_\rho(x)$ be the open ball of radius ρ centered at a point x, and let $B_\rho(M)$ be the open ρ-neighborhood of M *i.e.*

$$B_\rho(M) := \bigcup_{x\in M} B_\rho(x).$$

Further, we introduce

Definition 1.1. A point $x \in \Sigma$ is called *energetic*, if for all $\rho > 0$ one has

$$F_M(\Sigma \setminus B_\rho(x)) > F_M(\Sigma).$$

Denote the set of all energetic points of Σ by G_Σ.

Let us consider a minimizer Σ with energy $F_M(\Sigma) = r$ (the subset of $OPT^*_\infty(M)$ of minimizers with energy r will be further denoted by $OPT^*_\infty(M,r)$). Then the set Σ can be split into three disjoint subsets:

$$\Sigma = E_\Sigma \sqcup \mathrm{X}_\Sigma \sqcup \mathcal{S}_\Sigma,$$

where $X_\Sigma \subset G_\Sigma$ is the set of isolated energetic points (*i.e.* every $x \in X_\Sigma$ is energetic and there is a $\rho > 0$ possibly depending on x such that $B_\rho(x) \cap G_\Sigma = \{x\}$), $E_\Sigma := G_\Sigma \setminus X_\Sigma$ is the set of non isolated energetic points and $S_\Sigma := \Sigma \backslash G_\Sigma$ is the set of non energetic points also called the Steiner part of Σ. In [12] the following assertions have been proven:

(a) For every point $x \in G_\Sigma$ there exists a point $Q_x \in M$ (possibly non unique) such that $\mathrm{dist}\,(x, Q_x) = r$ and $B_r(Q_x) \cap \Sigma = \emptyset$. If X_Σ is not finite, the limit points of X_Σ belong to E_Σ.
(b) For all $x \in S_\Sigma$ there exists an $\varepsilon > 0$ such that $S_\Sigma \cap B_\varepsilon(x)$ is either a line segment or a regular tripod, *i.e.* the union of three line segments with an endpoint in x and relative angles of $2\pi/3$. If a point $x \in S_\Sigma$ is a center of a regular tripod, then it called a *Steiner point* (or a *branching point*) of Σ.

Note that the finiteness of $\mathcal{H}^1(\Sigma)$ implies that Σ is path-connected (see, for example, [7]). By the absence of loops the path in Σ between every couple of points of Σ is unique.

This paper is organized as follows: Section 1.2 contains a very brief survey on the Steiner problem, Section 1.3 describes basic notations, Section 2 is devoted to the statement of the main result with Section 2.1 containing a sketch of the proof of the main theorem (Thm. 2.2), Section 3 contains complete proofs of lemmas stated in Section 2.

1.2. The Steiner problem

The Steiner problem which has several different but more or less equivalent formulations, is that of finding a set S with minimal length (one-dimensional Hausdorff measure $\mathcal{H}^1(S)$) such that $S \cup A$ is connected, where A is a given compact subset of a given complete metric space X.

Namely, if we define

$$\mathrm{Ntw}\,(A) := \{S \subset X : S \cup A \text{ is connected}\}$$

then the Steiner problem is to find an element of $\mathrm{Ntw}\,(A)$ with minimal $\mathcal{H}^1$-length.

If S is a solution to the Steiner problem for a given set A (in the case when X is proper and connected, a solution exists [14]), then the set $\Sigma := \overline{S}$ is called a Steiner tree for the set A (or a Steiner tree connecting the set A, or just a Steiner set). It has been proven in [14] that in the case $\mathcal{H}^1(S) < +\infty$ the following properties hold:

(1) $\overline{S}$ contains no loops (homeomorphic images of S^1);
(2) $S\setminus A$ has at most countably many connected components, and each of the latter has strictly positive length;
(3) the closure of every connected component of $S \setminus A$ is a topological tree with endpoints on A and with at most one endpoint belonging to each connected component of A.

From now on we will consider the Steiner problem in the case when the ambient space X is the Euclidean plane $\mathbb{R}^2$. Then

(4) $\Sigma\setminus A$ consists of line segments (this follows from the result of [14] stating that away from the data every Steiner tree is an embedded graph consisting of geodesic segments);
(5) the angle between two segments adjacent to the same vertex is greater or equal to $2\pi/3$ [9];
(6) let us call *a Steiner (or branching) point* such a point of Σ that does not belong to A and which is not an interior point of a segment of Σ; the degree (in the graph theoreric sense) of a Steiner point x is equal to 3. In this case the angle between any pair of segments of Σ adjacent to x is equal to $2\pi/3$ (see [9]). Such a set is called *regular tripod*;
(7) It is well-known that for $A = \{x_1, x_2, x_3\} \subset \mathbb{R}^2$ there is the unique solution of the Steiner problem. We denote it by $\mathcal{S}t(x_1, x_2, x_3)$.

We will say that a set $S \in \mathrm{Ntw}\,(A)$ is a *locally minimal network* for the given set A if for an arbitrary point $x \in S$ there exists a neighbourhood $U \ni x$ such that $S \cap \overline{U}$ is a Steiner tree for $S \cap \partial U$. If a neighbourhood of a point $x \in S$ is a regular tripod then it is still called a *Steiner point*.

A locally minimal network satisfies all the properties of a Steiner tree mentioned above except the first one (see [9, 14]).

In this paper we will use the locally minimal networks for a set A that consists of at most four points. It is well known that there are only 7 possible combinatorial types of such networks which one can find in Figures 20 and 21 (see [9, 14]).

1.3. Notation

We introduce the following notation.

- For a given set $X \subset \mathbb{R}^2$ we denote by $\overline{X}$ its closure, by $\mathrm{Int}(X)$ its interior and by ∂X its topological boundary.
- For given points B, C we use the notation $[BC]$, $[BC)$ and (BC) for the corresponding (closed) line segment, ray and line respectively. We denote by $]BC]$ and $]BC[$ the corresponding semiopen and open segments, and by $|BC|$ the length of these segments.
- By a *closed convex curve* we mean a *boundary of a convex compact set*.
- We call a *chord* of a closed convex curve Z a line segment connecting two points of Z.
- A subset of a planar curve Z is called an *arc* of Z if it is a continuous injective image of an interval (possibly degenerate). We say that an arc of Z is *closed*, if it is a relatively closed subset of Z. The images of the endpoints of the interval will be called *ends* of the arc; the images of internal points of the interval will be called *internal* points of the arc. Whenever there is no confusion the closed arc with ends B, C will be denoted by $[\breve{BC}]$ and its length by $|\breve{BC}|$ (not to be confused with the length of the segment connecting B and C which is denoted by $|BC|$).
- For a convex closed set $N \subset \mathbb{R}^2$ we define the *minimal radius of curvature* of its boundary by the formula

$$R(\partial N) := \inf_{x\in\partial N} \sup \{\rho \colon \overline{B_\rho(O)} \cap \partial \ni x \text{ for some } O \in N \text{ such that } B_R(O) \subset \mathrm{Int}(N)\}.$$

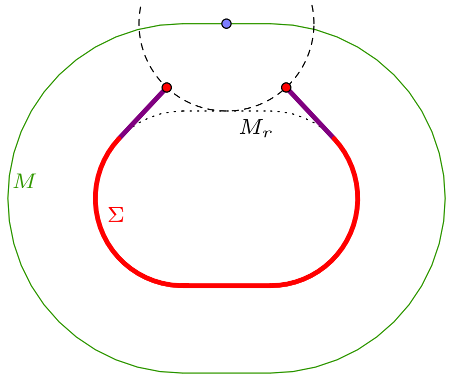

FIGURE 1. A horseshoe.

- For a convex closed set $N \subset \mathbb{R}^2$ we define the *inner set* N_r to be the set of all points of N lying at a distance of at least r from the boundary, namely, $N_r := N \setminus B_r(\partial N)$.
- From now on we define $N := \text{conv}\,(M)$, where *conv* stands for the closed convex hull, and $M_r := \partial N_r$. Note that N, N_r, M and M_r are closed sets.

From now M is a convex closed curve with minimal radius of curvature $R > r$. Clearly, M_r is convex closed curve and has minimal radius of curvature at least $R - r$. Also the condition on curvature of M_r and convexity of N imply $C^{1,1}$ smoothness of M_r.

- If a line segment $[BC]$ is an arc of M_r and a chord of M_r simultaneously (this happens if M_r is not strictly convex) we will work with $[BC]$ as with the arc in the case when $]BC[$ has an energetic point, and as with the chord otherwise. In Section 2 it will be explained in details.
- We say that an arc $[\breve{BC}] \subset \Sigma$ of M_r *is continued by a chord* in the set Σ if for some $J \in \{B, C\}$ there is a chord $[JD]$ of M_r such that $[JD] \subset \Sigma$.
- For a point $x \in \Sigma \cap M_r$ let $Q_x \in M$ be such a point that $\text{dist}\,(x, Q_x) = r$ (in this case Q_x is unique because of the condition on curvature of M). Also, in this case $[xQ_x]$ is a part of the normal to M_r at x and of the normal to M at Q_x.
- For an energetic point $x \in G_\Sigma$ let Q_x be a point mentioned in the property (a) of the set of energetic points. We may consider this choice of $Q_x \in M$ as a canonical choice of a point on M at the distance r from x.
- We say that a set $Z \subset \mathbb{R}^2$ *covers* a subset $Q \subset M$ if $Q \subset \overline{B_r(Z)}$. Usually we use the latter notion for the case when Q is an arc of M.
- For a set $Z \subset \mathbb{R}^2$ we define the *diameter* of Z as $\sup \{\text{dist}\,(x, y)\,|\, x, y \in Z\}$, and denote it by $\text{diam}\,(Z)$.
- We fix the clockwise orientation of the plane.
- For rays $[BC)$, $[CD)$ let $\angle([BC), [CD))$ stand for the *directed angle* from $[BC)$ to $[CD)$ with respect to the clockwise orientation.
- When using the asymptotic expressions $o(\cdot)$, $O(\cdot)$, we will always be silently assuming that the respective variable tends to some limit; both the variable and the limit will be usually clear from the context (if it necessary to avoid confusion, the variable name will be indicated in the lower index of the asymptotic symbols).

2. MAIN RESULTS

Definition 2.1. Let M be a closed convex curve with minimal radius of curvature $R > r$. Then the connected curve Σ is called a *horseshoe*, if $F_M(\Sigma) = r$ and Σ is a union of an arc q of M_r with two non degenerate tangent segments to M_r at the different ends of q ending with energetic points (as shown in Fig. 1).

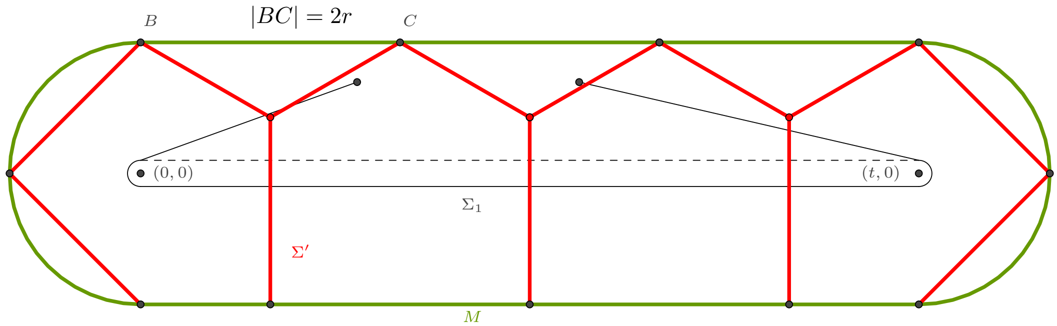

FIGURE 2. The horseshoe Σ_1 and the better competitor Σ' in Example 2.5.

The following theorem proves the particular case of the conjecture of Miranda, Paolini and Stepanov from [12] about the set $OPT^*_\infty(M, r)$ of minimizers for $M := \partial B_R(O)$ if $r < R/4.98$. It shows even more: namely, that *every* closed convex curve M has minimizers of the same structure, if minimal radius of curvature of M is at least $5r$. Note however that it does not prove the whole conjecture (which has been formulated in [12] for circumference $M = \partial B_R(O)$ and every $r < R$).

Theorem 2.2. *For every closed convex curve M with minimal radius of curvature R and for every $r < R/5$ the set of minimizers $OPT^*_\infty(M, r)$ contains only horseshoes. For the circumference $M = \partial B_R(O)$ the claim is true for $r < R/4.98$.*

Definition 2.3. Let $M \subset \mathbb{R}^2$ be a planar compact set. A connected set Σ is called *local minimizer* if it covers M and there is an $\varepsilon > 0$ such that for every connected Σ' covering M and satisfying $\operatorname{diam}(\Sigma \Delta \Sigma') \le \varepsilon$ one has $\mathcal{H}^1(\Sigma) \le \mathcal{H}^1(\Sigma')$.

Corollary 2.4. *Let $\hat{\Sigma}$ be a local minimizer for some closed convex curve M with minimal radius of curvature $R > 5r$. Then if $\hat{\Sigma}$ is not a horseshoe, one has $\mathcal{H}^1(\hat{\Sigma}) - \mathcal{H}^1(\Sigma) \ge (R - 5r)/2$, where $\Sigma \in OPT^*_\infty(M, r)$ is an arbitrary (global) minimizer.*

It is worth mentioning that the claim of Theorem 2.2 does not hold without some assumptions on the dependence of R on r, as the following example shows.

Example 2.5 (see Fig. 2)**.** Consider the stadium

$$N^t := \overline{\cup_{x \in [0,t] \times \{0\}} B_1(x)}, \;\; M^t := \partial N^t.$$

Note that $M = M^t$ is the border of the stadium and has minimal radius of curvature 1 for every t. Let us choose $1 > r > 1 - \varepsilon$ for sufficiently small ε, and a sufficiently large t such that $t/r \in 2\mathbb{N}$. Clearly, any horseshoe has length $2t - O(1)$ as $t \to \infty$. Consider the points $x_{2k} = (2kr, 1), x_{2k+1} = ((2k+1)r, -1) \in M^t$ for $k = 0, 1, \dots, t/(2r)$. Let X be the union of the sets $\mathcal{S}t(x_{2k}, x_{2k+1}, x_{2k+2})$ for $k = 0, 1, \dots, t/(2r) - 1$. Every such tree is a tripod; its length tends to $2 + \sqrt{3}$ when $\varepsilon \to 0+$. Note that

$$\Sigma' := X \cup [(0, 1), (-1, 0)] \cup [(-1, 0), (0, -1)] \cup [(t, 1), (t + 1, 0)] \cup [(t + 1, 0), (t, -1)]$$

is connected and $M^t \subset \overline{B_r(\Sigma')}$ (it is true, because $\operatorname{dist}(x_i, x_{i+2}) = 2r$, so the set $\{x_i\}$ covers horizontal lines of M^t; four additional segments cover semicircles of M^t). The length of X is $(2 + \sqrt{3} + o_\varepsilon(1))\frac{t}{2r} + 4\sqrt{2} \le (1 + \frac{\sqrt{3}}{2} + o_\varepsilon(1))\frac{t}{r} < 2t - O_t(1)$, and therefore one can choose ε sufficiently small and t sufficiently large such that Σ' is a better competitor than a horseshoe, hence a horseshoe is not a global minimizer for M^t.

2.1. The outline of the proof

Here is the sketch of the proof of Theorem 2.2. Let us recall that in [13] the following statement is proven.

Lemma 2.6. *Let $M \subset \mathbb{R}^2$ be a compact set, $\Sigma \in OPT_\infty^*(M)$. Then Σ has no loops.*

In the sequel the union of the closures of all connected components of $\Sigma \cap \mathrm{Int}(N_r)$ is denoted by Σ_r.

Lemma 2.7. *Let M be a convex closed curve with minimal radius of curvature R and $\Sigma \in OPT_\infty^*(M, r)$ be a minimizer with the energy $r < R$. Then the following assertions hold.*

(i) *The closure of every connected component of $\Sigma \cap \mathrm{Int}(N_r)$ is a solution of the Steiner problem for some set of points belonging to M_r, and in particular consists only of line segments of positive length.*
(ii) *Σ consists of arcs of M_r (possibly degenerate) and line segments of positive length with disjoint interiors.*
(iii) *The length of each line segment in Σ_r does not exceed $a_M(r)$ for some $a_M(r) \leq 2r$. For the circumference $\partial B_R(O)$ one can take $a_{\partial B_R(O)}(r) = 2r\sqrt{1 - \frac{r^2}{4R^2}}$.*

Note that we do not show in this Lemma that the number of line segments in Σ is finite.

A statement similar to Lemma 2.7 may be proven for M being a boundary of a not necessarily convex set, but we restrict the statement to the convex case to avoid excessive technicalities.

Lemma 2.8. *Let M be a closed convex curve with minimal radius of curvature $R > 2a_M(r) + r$, where a_M is such that the length of each line segment in Σ_r does not exceed $a_M(r)$ (in particular one can take a_M as in Lem. 2.7), $\Sigma \in OPT_\infty^*(M, r)$. Then Σ has no Steiner point in $\mathrm{Int}(N_r) \cup (S_\Sigma \cap M_r)$. Thus $\Sigma \cap \mathrm{Int}(N_r)$ consists of disjoint interiors of chords of M_r.*

Let us consider the set of the closures of connected components of $\Sigma \setminus N_r$. Denote it by $V_C(G)$ (further it will be associated with a subset of the vertex set of a graph). Note that Σ is connected (and does not reduce to a single point), so every $S \in V_C(G)$ has positive length. In our setting M is compact, thus every $\Sigma \in OPT_\infty^*(M, r)$ has finite length, hence the set $V_C(G)$ is at most countable.

Consider an arbitrary $S \in V_C(G)$. Note that by connectedness of S the set $\overline{B_r(S)} \cap M$ is always a closed arc. We denote it by q_S.

Consider the set of all maximal arcs of M_r in the set Σ, which are not contained in the closure of a connected component of $\Sigma \setminus N_r$. Let us denote by $V_A(G)$ the subset of such arcs having an energetic point in their interior. Note that if M is not strictly convex, then an arc $[\breve{BC}]$ of M_r can be a chord of M_r. In this situation if $]\breve{BC}[$ has no energetic point then we will consider it as a chord of M_r: note that if $\Sigma \setminus]\breve{BC}[$ does not cover $Q_x \in M$ for some $x \in]BC[$, then x is energetic; thus if $]\breve{BC}]$ has no energetic point then $[BC] = [\breve{BC}]$ has all the properties of a standard chord of M_r .

Obviously, an arc $[\breve{BC}] \in V_A(G)$ of M_r covers an arc $q_{[\breve{BC}]} := [Q_B \breve{Q}_C]$ of M, where $Q_B, Q_C \in M$ are the unique points such that $\mathrm{dist}\,(B, Q_B) = \mathrm{dist}\,(C, Q_C) = r$.

Definition 2.9. Let M be a closed convex curve with minimal radius of curvature $R > r$, $\Sigma \in OPT_\infty^*(M, r)$. Let $S \in V_C(G)$, a closure of a connected component of $\Sigma \setminus N_r$.

(i) Denote by $n(S)$ the number of energetic points in S.
(ii) A point $x \in S \cap M_r$ is called an *entering point*. Denote the number of entering points of S by $m(S)$.

The following lemma says in particular that $n(S)$, $m(S)$ are finite.

Lemma 2.10. *Let M be a closed convex curve with minimal radius of curvature $R > 2a_M(r) + r$, $\Sigma \in OPT_\infty^*(M, r)$. Let S be the closure of a connected component of $\Sigma \setminus N_r$. Then $n(S) \leq 2$, $m(S) \leq 2$. Further, S is a locally minimal network connecting the set of entering points of S and energetic points of $S \setminus M_r$.*

By the previous Lemma, S is a locally minimal network for at most $n(S)+m(S)\le 4$ points. All the possible combinatorial types of such networks are listed in Figures 20 and 21.

Lemma 2.11. *Under conditions of Theorem 2.2 if $S\in V(G):=V_C(G)\sqcup V_A(G)$ does not reduce to a point, then*

$$q_S \not\subset \bigcup_{S'\in V(G)\setminus\{S\}} q_{S'}.$$

Moreover, every set $S\in V(G)$ has an energetic point.

Lemma 2.12. *Under conditions of Theorem 2.2 the set $V(G)=V_C(G)\sqcup V_A(G)$ is finite.*

Note that a singleton of $\Sigma\cap M_r$ (a maximal arc $\xi\subset\Sigma\cap M_r$ of zero length not contained in the closure of a connected component of $\Sigma\setminus N_r$) cannot be energetic (by the previous Lemma the union of q_S over $S\in V(G)\setminus\xi$ is closed as a finite union of closed sets, hence it coincides with M because $q_\xi=\{Q_\xi\}$), so a neighbourhood of ξ is a segment or a tripod (the latter is impossible by Lem. 2.8). Summing up, every point of $\Sigma\cap M_r$ is contained in a maximal arc of M_r of positive length or in the closure of a connected component of $\Sigma\setminus N_r$. Also by Lem. 2.10 every connected component of $\Sigma\setminus N_r$ contains at most 5 segments, thus Σ consists of a finite number of segments and arcs of M_r.

Lemma 2.13. *Under conditions of Theorem 2.2 let $[BI]\subset\Sigma$ be a chord of M_r. Then $I\in S_\Sigma$ and moreover there exists such an $\varepsilon>0$ that $\overline{B_\varepsilon(I)}\cap\Sigma=[I_1I_2]$, for some $I_1,I_2\in\partial B_\varepsilon(I)$.*

Lemma 2.14. *Under conditions of Theorem 2.2 every maximal arc $[B\breve{}C]\in V_A(G)$ is continued by segments lying on tangent lines to M_r in the sense that there exists such an open $U\supset[B\breve{}C]$ that $\Sigma\cap\overline{U}=[B'B]\cup[B\breve{}C]\cup[CC']$, where $[B'B]$ and $[CC']$ are subsets of tangent lines to M_r at points B, C respectively.*

Lemma 2.15. *Under conditions of Theorem 2.2 let $C\in M_r\cap\Sigma$. Then Σ has the tangent line at C, in particular for every $\varepsilon>0$ there is a $\delta>0$ such that for every couple of points $B,D\in\Sigma\cap B_\delta(C)\setminus C$, holds $\min(|\angle BCD-\pi|,|\angle BCD|)<\varepsilon$.*

Definition 2.16. Under conditions of Theorem 2.2 consider the following abstract graph $G=(V(G),E(G))$ (recall that the set of vertices $V(G)=V_C(G)\sqcup V_A(G)$; by Lem. 2.12 it is finite), where the set of edges $E(G)$ is defined as follows:

- in the case S_1, $S_2\in V_C(G)$ there is an edge between them if they are connected in Σ by a chord of M_r or if $S_1\cap S_2\neq\emptyset$;
- in the case $S_1\in V_C(G)$, $[B\breve{}C]\in V_A(G)$ there is an edge between S_1 and $[B\breve{}C]$, if $S_1\cap[B\breve{}C]\neq\emptyset$;
- and finally in the case $[B_1\breve{}C_1]$, $[B_2\breve{}C_2]\in V_A(G)$ there is no edge between them.

Corollary 2.17. *Under conditions of Theorem 2.2 graph G has no cycles; it has exactly two vertices of degree 1 and all the other vertices have degree 2. In other words G is a path with at least one edge.*

Proof. First, by Lemma 2.12 the graph is finite. By Lemma 2.13 every chord of M_r in Σ connects exactly two vertices in $V(G)$. Thus, the inequality $m(S)\le 2$ (Lem. 2.10) implies $\deg(v)\le 2$ for $v\in V_C(G)$; for $v\in V_A(G)$ the inequality $\deg(v)\le 2$ holds by Lemma 2.14.

Note that if $(S_1,S_2)\in E(G)$ then there is a path between S_1 and S_2 in Σ not intersecting other sets $S\in V(G)$, $S\notin\{S_1,\ S_2\}$. It means that if G has a cycle C then so has Σ, contradicting Lemma 2.6. Moreover, the path between two points in Σ belonging to two different vertices of $V(G)$ naturally induces a path in G (in fact, if a path in Σ connects two different vertices $S_1,S_2\in V(G)$ wihtout touching other vertices, then $(S_1,S_2)\in E(G)$; therefore for a generic path in Σ connecting two different vertices of G it is enough to split it in a finite number of paths connecting different vertices in G and not passing throw other vertices). Therefore, connectedness of Σ gives us that G is connected. We conclude that G is a path.

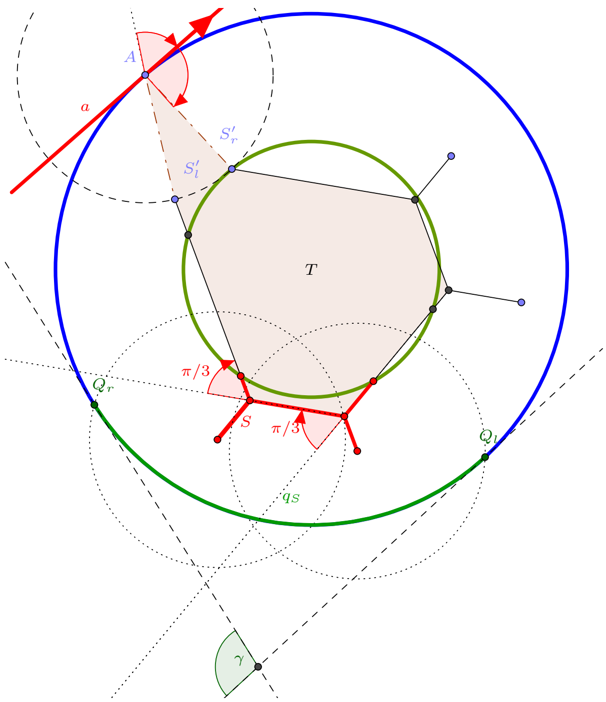

FIGURE 3. Figure to the construction of T.

Now we have to show that $\#V(G) > 1$. Suppose the contrary, *i.e.* $V(G) = \{v\}$. If $v \in V_C(G)$, then $m(v) = 0$, so v is a segment that is impossible. Otherwise v is an arc, but $q_v = M$, so $v = M_r$ contains a loop. We got again a contradiction with Lemma 2.6. $\square$

Thus under conditions of Theorem 2.2 there are two connected components of $\Sigma \setminus N_r$ with one entering point; these components correspond to the leaves of our graph. We call them *ending components* and denote by S_l and S_r (calling them *left* and *right* respectively); the other components will be called *middle components*.

By Lemma 2.11 every point of M is covered by at most two sets from $V(G)$. By Corollary 2.17 graph G is a path, so if S_1, S_2 are connected by an edge in G, then $q_{S_1} \cap q_{S_2} \neq \emptyset$. Moreover, the same reasoning gives $q_{S_l} \cap q_{S_r} \neq \emptyset$, because otherwise there would be some part of M not covered by Σ.

Lemma 2.18. *The arcs q_{S_l} and q_{S_r} have disjoint interiors.*

Denote by A an arbitrary point of the intersection of q_{S_l} and q_{S_r} (see Fig. 3); by Lemma 2.18 there are at most 2 such points. Consider the set $\hat{\Sigma} := \Sigma \cup [AS'_l] \cup [AS'_r]$, where $[AS'_l]$ and $[AS'_r]$ are segments of length r connecting A with S_l and S_r respectively. In view of Lemma 2.6 and the fact that $B_r(A) \cap \Sigma = \emptyset$, the set $\hat{\Sigma}$ bounds the unique region which we further denote by T (see Fig. 3).

Previous Lemmas give us the following corollary.

Corollary 2.19. *The boundary of T is a closed curve consisting of a finite number of arcs of M_r and a finite number of line segments.*

Consider the behavior of the tangent line to the boundary of T. Corollary 2.19 and Lemma 2.15 imply that all points where tangent direction is discontinuous (*i.e.* points where the tangent line to ∂T does not exist) except A belong to connected components of $\Sigma \setminus N_r$.

Definition 2.20. Let γ be a C^1-smooth injective planar curve. We say that the *turning* of γ is the following object:

$$\mathrm{turn}(\gamma) := \int_0^1 d\arg(\gamma'(t)),$$

where $\gamma : [0,1] \to \mathbb{R}^2$ is some injective parameterization of γ with $\gamma'(t) \neq 0$ and arg is a continuous branch of the multifunction Arg.

Let $\gamma : [0,1] \to \mathbb{R}^2$ be a piecewise C^1-smooth injective planar curve, with a finite number of discontinuity points $\{t_i\}_{i=1}^N$ of γ', $t_i < t_j$ if $i < j$ (it means that γ is C^1-smooth on every $[t_i t_{i+1}]$). We define

$$\mathrm{turn}(\gamma) := \sum_{i=1}^N \mathrm{turn}(\gamma([t_i, t_{i+1}])) + \sum_{i=1}^N \angle([\gamma(t_i - 0)\gamma(t_i)), [\gamma(t_i)\gamma(t_i + 0))).$$

Let $\gamma : [0,1] \to \mathbb{R}^2$ be a simple closed (*i.e.* $\gamma(0) = \gamma(1)$) piecewise C^1-smooth planar curve, with a finite number of discontinuity points of γ' $\{t_i\}_{i=1}^N$, $t_i < t_j$ if $i < j$. We define

$$\mathrm{turn}(\gamma) := \sum_{i=1}^N \mathrm{turn}(\gamma([t_i, t_{i+1}])) + \sum_{i=1}^N \angle([\gamma(t_i - 0)\gamma(t_i)), [\gamma(t_i)\gamma(t_i + 0))) + \angle([\gamma(1-0)\gamma(1)), [\gamma(0)\gamma(0+0))).$$

In our setting, the turning of an open curve γ will almost always coincide with the directed angle between the tangent lines to the ends of γ. Note that for a self-avoiding closed piecewise C^1-smooth planar curve γ we always have $\mathrm{turn}(\gamma) = 2\pi$ (for a parameterization with respect to the clockwise orientation).

Now we define the same quantity for the closure of a connected component of $\Sigma \setminus N_r$.

Definition 2.21. Under conditions of Theorem 2.2 let S be the closure of a connected component of $\Sigma \setminus N_r$. Then $\mathrm{turn}(S)$ stands for the turning number of the $S \cap \partial T$ parameterized in the clockwise order. In particular, if $S = S_l$ then $\mathrm{turn}(S)$ stands for the turning number of the curve $S \cap \partial T$ parameterized so that it starts at the entering point and ends at point S'_l, and if $S = S_r$ then $\mathrm{turn}(S)$ stands for the turning number of $S \cap \partial T$ parameterized so that it starts at point S'_r and ends at the entering point.

Now we are ready to state the central Lemma. Figure 3 should simplify the reading of its statement.

Lemma 2.22. *Under conditions of Theorem 2.2 let $\Sigma \in OPT^*_\infty(M,r)$ be a minimizer, $S \in V(G)$ be the closure of a connected component of $\Sigma \setminus N_r$ or an arc of M_r. Then the following assertions hold.*

- *If S is a middle component or an arc of M_r then $\mathrm{turn}(q_S) \leq \mathrm{turn}(S)$. The equality holds if and only if S is an arc of M_r.*
- *If S is an ending component then for the left and the right components we have*

$$\mathrm{turn}(q_{S_l}) \leq \mathrm{turn}(S_l) + \angle([C_l S'_l), [S'_l A)) + \angle([S'_l A), a),$$

$$\mathrm{turn}(q_{S_r}) \leq \angle(a, [AS'_r)) + \angle([AS'_r), [S'_r C_r)) + \mathrm{turn}(S_r),$$

where a stands for the tangent ray to M at the point A directed from the left to the right (see Figure 3, angles $\angle([S'_l A), a)$, $\angle(a, [AS'_r))$ are marked red) and C_i is the branching point of S_i if S_i is a tripod and the entering point of S_i in other cases, where $i \in \{l, r\}$ (the definition is correct by Lem. 2.10). The equality holds if and only if S is a segment of the tangent line to M_r.

Remark 2.23. If in Lemma 2.22 we assume that Σ has no Steiner points in N_r then it is enough to request the inequality $r < R/2.9$ (see proof of Lem. 2.22, Case 1a).

Now the proof of Theorem 2.2 is just few lines.

Proof of Theorem 2.2. By Lemma 2.7(iii) $2a_M(r) + r < 5r$ for general M, and $2a_M(r) + r < 4.98r$ when M is the circumference. Note that

$$2\pi = \mathrm{turn}(\partial T) = \sum_{S \in V(G)} \mathrm{turn}(S) + \angle([C_l S_l'], [S_l' A)) + \angle([S_l' A), a) + \angle([AS_r'), [S_r' C_r)) + \angle(a, [AS_r'))$$

by Lemmas 2.14 and 2.15, and also $\mathrm{turn}(M) = 2\pi$. Hence by Lemma 2.22

$$\begin{aligned} 2\pi &= \sum_{S \in V(G)} \mathrm{turn}(S) + \angle([C_l S_l'], [S_l' A)) + \angle([S_l' A), a) + \angle([AS_r'), [S_r' C_r)) + \angle(a, [AS_r')) \\ &\geq \sum_{S \in V(G)} \mathrm{turn}(q_S) \geq \mathrm{turn}(M) = 2\pi. \end{aligned}$$

Thus all the inequalities in Lemma 2.22 are equalities. Summing up, every global minimizer $\Sigma \in OPT_\infty^*(M, r)$ consists of arcs of M_r and segments of tangent lines to M_r, *i.e.* components of the combinatorial type (a) in Figure 20, tangent to M_r. Every vertex, corresponding to a component of the combinatorial type (a) in Figure 20 has degree 1 in G. Thus Σ has the unique arc of M_r, and because of the absence of loops it cannot coincide with M_r. By Lemma 2.14 every maximal arc $[\breve{BC}] \in V_A(G)$ is connected in the graph G with two vertices, corresponding to connected components of $\Sigma \setminus N_r$. Hence any minimizer is a horseshoe. $\square$

3. Proofs

Recall that Σ is an arbitrary minimizer for some convex closed curve M and $N = \mathrm{conv}\,(M)$. Clearly $\Sigma \subset N$ (N is a convex set, so one can project on N the part of Σ belonging to $\mathbb{R}^2 \setminus N$ on N and length of Σ will strictly decrease).

The following well-known fact will be used during the proof.

Lemma 3.1. *Let M be a convex closed curve with minimal radius of curvature R and $B_R(O)$ be a ball of radius R centered at point $O \in N$. If $\partial B_R(O)$ touches M (tangentially to M), then $B_R(O) \subset N$.*

Further on we assume by default M and Σ are as in Theorem 2.2. Sometimes we will request weaker conditions. The following assertion is valid.

Lemma 3.2. *Let M be a convex closed curve with minimal radius of curvature $R > r$ and Σ be an arbitrary minimizer for M. Then the set E_Σ of non-discrete energetic points of Σ is a subset of M_r.*

Proof. Suppose the contrary. Then there are such a point $x \in E_\Sigma \setminus M_r$ that $\mathrm{dist}\,(x, M) < r - \varepsilon$ for some positive ε and a sequence $\{x_k\}$ of energetic points from $B_{\varepsilon/2}(x)$ converging to x. Let us choose such a sequence of positive numbers $\{\varepsilon_k\}$ that $B_{\varepsilon_i}(x_i) \cap B_{\varepsilon_j}(x_j) = \emptyset$ for $i \neq j$.

Because of convexity of N and the fact that minimal radius of curvature of M exceeds r, one has that each $\gamma_k := \overline{B_{r+\varepsilon_k}(x_k)} \cap M$ is connected, thus we can say that each $\overline{B_{\varepsilon_k}(x_k)}$ covers the arc γ_k. All γ_j have a common point: in fact, for $z \in M$ such that $\mathrm{dist}\,(x, z) = \mathrm{dist}\,(x, M)$ one has

$$\mathrm{dist}\,(x_j, z) \leq \mathrm{dist}\,(x, z) + \mathrm{dist}\,(x_j, x) < r - \varepsilon + \varepsilon/2 = r - \varepsilon/2,$$

thus $z \in \gamma_j$ for all j. Therefore $\gamma_i \subset (\gamma_j \cup \gamma_l)$ for some distinct i, j, l. So one of the points x_i, x_j, x_l is not energetic because $F_M(\Sigma) = F_M(\Sigma \setminus B_{\varepsilon_i}(x_i))$ which is the desired contradiction. $\square$

Proof of Lemma 2.7. Proof of (i): No change in the set $\mathrm{Int}(\Sigma \cap N_r)$ influences the value of $F_M(\Sigma)$, so if we take the closure S of any connected component of $\Sigma \cap \mathrm{Int}(N_r)$ and substitute it by a Steiner tree connecting $S \cap M_r$ (which must be nonempty if $\Sigma \cap \mathrm{Int}(N_r) \neq \emptyset$ because of connectedness of Σ and the requirement

$F_M(\Sigma) \leq r$ which gives $\Sigma \setminus \mathrm{Int}(N_r) \neq \emptyset$), then the length of the resulting set should remain the same by optimality of Σ, and thus S is itself a Steiner tree connecting $S \cap M_r$ as claimed.

Proof of (ii): Recall that $\Sigma = E_\Sigma \sqcup X_\Sigma \sqcup S_\Sigma$, where X_Σ is a discrete set of points, S_Σ consists of Steiner trees (hence of line segments) and $E_\Sigma \subset M_r$ by Lemma 3.2.

Proof of (iii): Remove an arbitrary open line segment Δ from the set $\Sigma \cap \mathrm{Int}(N_r)$. The value of F_M does not change, *i.e.* $F_M(\Sigma\setminus\Delta) = F_M(\Sigma)$, and by Lemma 2.6 $\Sigma\setminus\Delta$ splits into two connected components Σ_1 and Σ_2, so that $\Sigma\setminus\Delta = \Sigma_1 \sqcup \Sigma_2$ (Σ is closed, so Σ_1, Σ_2 are closed too). Obviously $M \subset \overline{B_r(\Sigma_1)} \cup \overline{B_r(\Sigma_2)}$. Then by connectedness of M there is such a point $A \in M$ that $A \in \overline{B_r(\Sigma_1)} \cap \overline{B_r(\Sigma_2)}$, but then there are points $B \in \overline{\Sigma_1}$ and $C \in \overline{\Sigma_2}$ such that $|AB| \leq r$, $|AC| \leq r$. Hence the distance between Σ_1 and Σ_2 does not exceed $|BC| \leq 2r$ but the length of the deleted segment Δ does not exceed the distance between the Σ_1 and Σ_2 in view of optimality of Σ (otherwise one could connect Σ_1 with Σ_2 with a shorter segment). We let then $a_M(r)$ be the supremum of $|BC|$ over all the possible choices of Δ, so that we have proven $a_M(r) \leq 2r$.

In the case $M = \partial B_R(O)$ the length of the segment $[BC]$ reaches its maximal value when $[BC]$ is a chord and $|AB| = |AC| = r$. Then we can calculate the maximal value of length of $[BC]$ in this case:

$$\sin\frac{\angle AOC}{2} = \frac{|AC|}{2|OC|} = \frac{r}{2R},$$

so that

$$|BC| = 2|OC|\sin\angle AOC = 4|OC|\sin\frac{\angle AOC}{2}\cos\frac{\angle AOC}{2} = 2r\sqrt{1-\frac{r^2}{4R^2}}.$$

□

Proof of Lemma 2.8. Assume the contrary *i.e.* that Σ has a Steiner point $X \in \mathrm{Int}(N_r) \cup (S_\Sigma \cap M_r)$. In view of Lemma 3.1 there is a point $O \in N$ such that $X \in B_R(O)$ and $B_R(O) \subset \mathrm{Int}(N)$ (hence $B_{R-r}(O) \subset \mathrm{Int}(N_r)$, and in particular, $O \in \mathrm{Int}(N_r)$). Recall that as defined in Lemma 2.7 Σ_r is the union of the closures of all connected components of $\Sigma \cap \mathrm{Int}(N_r)$. Now denote by X_0 one of the Steiner points of $\Sigma_r \cup (S_\Sigma \cap M_r)$ nearest to O, and let $t := |OX_0|$. We claim that $X_0 \in \mathrm{Int}(N_r)$. In fact, otherwise $X_0 \in M_r$ and hence

$$t = \mathrm{dist}\,(O, M_r) = \mathrm{dist}\,(O, M) - r \geq R - r > 3.98r,$$

but X_0 is a Steiner point, hence, in view of the smoothness and convexity of M_r there are two line segments $[X_0Z_i] \subset \Sigma$, $i = 1, 2$ at angle $2\pi/3$ with respect to each other, intersecting $\mathrm{Int}(N_r)$. Suppose without loss of generality that $\angle OX_0Z_1 \leq \pi/3$. Then $Z_1 \in B_t(O) \subset \mathrm{Int}(N_r)$, since otherwise there is an $Y \in [X_0Z_1] \cap \partial B_t(O) \subset \Sigma \cap \partial B_t(O)$ such that the line segment $[X_0Y] \subset \Sigma$ is a chord of $\partial B_t(O)$, which provides the estimate

$$|X_0Y| = 2t\cos\angle OX_0Z_1 \geq t > 3.98r$$

contrary to Lemma 2.7(iii), this contradiction proving the claim.

Let Σ' stand for the closure of the connected component of $\Sigma \cap \mathrm{Int}(N_r)$ containing X_0. By the structure of a Steiner tree sivce X_0 belongs to $\mathrm{Int}(N_r)$ then there are three maximal line segments of Σ' starting from X_0. Consider such a pair of them $[X_0X_{-1}], [X_0X_1]$ that the point O belongs to the angle $\angle X_{-1}X_0X_1$ (not excluding the case it belongs to one of the sides of this angle). Recall that $\angle X_{-1}X_0X_1 = 2\pi/3$. Also note that points X_{-1}, X_1 lie outside of $B_t(O)$. Hence either $[X_0X_1]$ or $[X_0X_{-1}]$ intersects $B_t(O)$. We assume without loss of generality that it is $[X_0X_1]$. Denote the intersection of the segment $[X_0X_1]$ and the circumference $\partial B_t(O)$ by C.

We claim that $t \leq a_M(r)$. Supposing the contrary, since $|X_0C| \leq a_M(r)$ and $|OX_0| = |OC| = t > a_M(r) \geq |X_0C|$, we have $\angle OX_0C > \pi/3$, hence the segment $[X_0X_{-1}]$ also intersects $B_t(O)$. Denote the intersection of the segment $[X_0X_{-1}]$ with $\partial B_t(O)$ by D and note that also $\angle OX_0D > \pi/3$, and hence $\angle CX_0D > 2\pi/3$ which contradicts the local optimality of Σ, showing the claim.

Note that X_1, X_{-1} belong to $\mathrm{Int}(N_r)$ because $R - r > 2a_M(r) \geq t + a_M(r)$, and hence X_1, X_{-1} are Steiner points. Also by Lemma 2.7 the lengths $[X_0X_{-1}]$ and $[X_0X_1]$ do not exceed $a_M(r)$. Consider a regular hexagon

P with sidelength $a_M(r)$ such that X_0 is a vertex of P and the segments $[X_0X_1]$, $[X_0X_{-1}]$ belong to two sides of P. The following assertions hold.

- $\operatorname{diam} P = 2a_M(r)$.
- The line segment $[OX_0]$ splits the angle $\angle X_{-1}X_0X_1 = 2\pi/3$ in two angles, at least one of them is acute. Denote the latter angle by $\angle OX_0B$, where B is the corresponding vertex of P (so that $|X_0B| = a_M(r)$). Then the angle $\angle OBX_0$ is also acute because $|OX_0| = t \le a_M(r) = |X_0B|$. Therefore the perpendicular from O to the line (X_0B) intersects the latter inside $[X_0B]$, so that O is inside the square built on $[X_0B]$. But this square is a subset of P hence $O \in P$.
- The above assertions imply that $P \subset \overline{B_{2a_M(r)}(O)}$, and hence $P \subset \operatorname{Int}(N_r)$.

Now let us pick such vertices X_{-2} and X_2 that $[X_1X_2], [X_{-1}X_{-2}] \subset \Sigma_r$ and O belongs to both angles $\angle X_0X_1X_2$ and $\angle X_0X_{-1}X_{-2}$. Clearly $X_2, X_{-2} \in P \subset \operatorname{Int}(N_r)$ so they again are Steiner points. Let us define the points X_3, X_{-3} in the same way: $[X_2X_3], [X_{-2}X_{-3}] \in \Sigma_r$ and O belongs to the angles $\angle X_1X_2X_3$ and $\angle X_{-1}X_{-2}X_{-3}$. Points X_3, X_{-3} also belong to P, hence to $\operatorname{Int}(N_r)$, hence they also are Steiner points. The six constructed line segments belong to $\operatorname{Int}(N_r)$, so there is no endpoint there. Continuing inductively this construction, we arrive at two paths in $P \subset \operatorname{Int}(N_r)$: one path (starting from $X_0, X_1, X_2, X_3, \dots$) turns left every time and the other one (starting from $X_0, X_{-1}, X_{-2}, X_{-3}, \dots$) turns right every time. Thus $\Sigma \cap P \subset \Sigma \cap \operatorname{Int}(N_r)$ contains a cycle or an endpoint of Σ in $\operatorname{Int}(N_r)$, but both cases are impossible for a Steiner tree by Lemmas 2.6 and 2.7. ☐

Proof of Lemma 2.10. Let $S \in V_C(G)$ be the closure of a connected component of $\Sigma \setminus N_r$. First we prove that $n(S) \le 2$. By property (a) of the set of energetic points for every energetic point $x \in S$ of Σ there is such a point $Q_x \in M$ that $\operatorname{dist}(x, Q_x) = r$ and $B_r(Q_x) \cap \Sigma = \emptyset$. Then Q_x can be only an end of the arc q_S, otherwise $S = S \setminus B_r(Q_x)$ is not connected. If an end of q_S corresponds to two different energetic points W_1, W_2 of S then $q_{W_1} \subset q_{W_2}$ or $q_{W_2} \subset q_{W_1}$ which is impossible, and hence $n(S) \le 2$ as claimed.

Now let us prove $m(S) \le 2$. Assume the contrary *i.e.* the existence of at least three different entering points in S. Let us denote them I_1, I_2 and I_3 such that $Q_{I_2} \in [Q_{I_1}\breve{}Q_{I_3}] \subset q_S$. Note that I_2 cannot be energetic, because Q_{I_2} is not an end of q_S. So I_2 has such a neighbourhood U that $U \cap \Sigma$ is a segment or a regular tripod; by Lemma 2.8 it is a segment.

We claim that Σ contains a chord $[I_2J]$ of M_r. It is true if Σ is not tangent to M_r at I_2. Now, let Σ be tangent to M_r at I_2, so I_2 belongs to closures of two different connected components of $\Sigma \setminus N_r$; one of them is S; denote the second one by S'. Let P_1 be the region bounded by the arc $[I_1\breve{}I_2]$ of M_r (choosing in such a way that P_1 does not contain N_r) and the unique path between I_1 and I_2 in S. Define P_3 analogously (with I_3 in place of I_1). Obviously, $S' \subset P_1$ or $S' \subset P_3$. Hence $q(S') \subset q(S)$ and replacing S' in Σ by a Steiner tree for $S' \cap M_r$ we get a connected competitor to Σ still covering M. Also, any Steiner tree for $S' \cap M_r$ belongs to N_r by the convexity of M_r, so this replacement decreases the length, which is impossible. Hence, we get the claim, *i.e.* there is a chord $[I_2J] \subset \Sigma$ of M_r.

Then $|I_2J| \le |I_1J|$ (otherwise we can replace $[I_2J]$ by $[I_1J]$ in Σ producing the competitor of strictly lower length), and analogously $|I_2J| \le |I_3J|$. Note that $J \notin S$ because by Lemma 2.6 Σ has no loops. One can see that points I_1, I_2, I_3, J belong to M_r in the natural (clockwise) order otherwise the arc q_{S_J} is a subset of q_S, where S_J is the closure of the connected component of $\Sigma \setminus N_r$ containing J, which is impossible.

Hence $|JI_2|$ is at least the diameter d of the maximal ball inscribed in N_r and touching M_r at point I_2, *i.e.* the double *inradius* of M_r. Since $d \ge 2(R - r)$, we have $|JI_2| \ge 2(R - r) > 2r$ contradicting Lemma 2.7(iii), showing the claim $m(S) \le 2$.

Finally, note that S should be locally minimal in a neighbourhood of any point $x \in S$ except energetic and entering points of S. We have proved that $n(S) \le 2$, a non energetic point $x \in S$ has a neighbourhood U_x such that $\Sigma \cap U_x$ is either a segment or a regular tripod. If $x \in S$ is a non energetic endpoint of S then $\Sigma \cap U_x \ne S \cap U_x$, so x is an entering point. So by definition of a locally minimal network, S is a locally minimal network for its entering and energetic points. ☐

Remark 3.3. During the proof of Lemma 2.10 (claim $n(S) \leq 2$) we show that if $x \in G_\Sigma \cap S$ then Q_x can be only an end of the arc q_S. So in the case $n(S) = 2$ there is the unique one-to-one correspondence between energetic points of S and endpoints of q_S.

Proof of Lemma 2.11. The fact that S has an energetic point immediately implies that q_S does not belong to the union of $q_{S'}$ over $S' \in V(G)\setminus\{S\}$. Suppose the contrary, *i.e.* that S has no energetic point.

If S is the closure of a connected component of $\Sigma\setminus N_r$, then by Lemma 2.10 S is a locally minimal network for its entering points, but $m(S) \leq 2$, hence S is a segment with endpoints on M_r, which is impossible for a connected component of $\Sigma\setminus N_r$.

If S is a non degenerate arc $[\breve{BC}]$, then $]\breve{BC}[\subset S_\Sigma$, which is impossible by the definition of $V_A(G)$. □

Proof of Lemma 2.12. Suppose the contrary. Consider an arbitrary $\varepsilon > 0$ (which later will be chosen sufficiently small). First, note that Lemma 2.11 implies that every point of M belongs to at most two different arcs q_S, where $S \in V(G)$ (otherwise, there are three arcs of M containing a point $x \in M$, so one of them is contained in the union if others, which is impossible by Lem. 2.11). Thus the sum of $\mathcal{H}^1(q_S)$ over $V(G)$ is at most $2\mathcal{H}^1(M)$, and therefore there is only a finite number of connected components and arcs with $\mathcal{H}^1(q_S) \geq \varepsilon$. Denote by $V_\varepsilon(G)$ the infinite set of such $S \in V(G)$ that $\mathcal{H}^1(q_S) < \varepsilon$.

Obviously, if $V(G)$ is an infinite set, then $V_C(G)$ is an infinite set. Let us show that there are infinitely many chords of M_r in Σ intersecting $\mathrm{Int}(N_r)$ (if N, and hence N_r, is strictly convex then in fact every chord of M_r intersects $\mathrm{Int}(N_r)$). Suppose the contrary. Then $\Sigma\setminus\mathrm{Int}(N_r)$ has a finite number of connected components; but $V_C(G)$ is infinite, hence there are components containing infinitely many elements of $V_C(G)$; let K be one of these components containing at least five different elements of $V_C(G)$. Obviously, $q_K := \overline{B_r(K)} \cap \Sigma$ is connected. By Lemma 2.11 $K\setminus M_r$ contains 5 energetic points, such that they belong to different elements of $V_C(G)$. Call them W_1, W_2, W_3, W_4, W_5 such that Q_{W_1}, Q_{W_2}, Q_{W_3}, Q_{W_4}, $Q_{W_5} \in q_K$ belong to M_r in the natural (clockwise) order. Then $B_r(Q_{W_i}) \cap \Sigma = \emptyset$, $i = 1, \dots, 5$ and therefore K should contain the points I_2, I_3, $I_4 \in M_r$ such that

$$\mathrm{dist}\,(Q_{W_2}, I_2) = \mathrm{dist}\,(Q_{W_3}, I_3) = \mathrm{dist}\,(Q_{W_4}, I_4) = r$$

(because $K\setminus I_j$ must be disconnected, $j = 2, 3, 4$). Consider the path between I_2 and I_4 in K. It should coincide with $[\breve{I_2 I_4}] \subset M_r$, otherwise we reduce the length of Σ, projecting the path on M_r. So W_3 should belong to M_r which is impossible by the choice of W_i, $i = 1, \dots, 5$ and gives the desired contradiction. Thus the set Ch of chords of M_r in Σ intersecting $\mathrm{Int}(N_r)$ is infinite.

There is at most a finite number of chords of length at least ε because $\mathcal{H}^1(\Sigma)$ is finite. Let us exclude from the infinite set Ch a finite set of chords of length at least ε and a finite set of chords adjacent to a component not in $V_\varepsilon(G)$; denote the resulting set by Ch': chords in Ch' are adjacent only to the elements of $V_\varepsilon(G)$ and have length strictly less than ε. Let us show that any of the chords in Ch' connects components without Steiner points. Suppose the contrary. The following three cases have to be considered:

(i) A chord in Ch' is adjacent to a connected component $S \in V_\varepsilon(G)$ with $m(S) = 2$ containing a Steiner point. Then the angle between the entering segments of the component is at most $2\pi/3$ (in fact, it must be between $\pi/3$ and $2\pi/3$). Recall that $\mathcal{H}^1(q_S) < \varepsilon$, hence by the triangle inequality S is a subset of an ε-neighbourhood of M_r (otherwise $\mathrm{dist}\,(x, y) \leq r - \varepsilon$ for some $x \in S$, $y \in M$, so $B_\varepsilon(y) \cap M \subset q_S$ which contradicts $\mathcal{H}^1(q_S) < \varepsilon$). So, when ε is sufficiently small, recalling smoothness of M_r one has that one of the entering segments has angle with M_r at least $\pi/12$. It implies that the entering point I of this segment is not energetic, so by Lemma 2.8 its neighbourhood is a segment and it is an end of a chord $[IJ] \subset \Sigma$ of M_r. So by the constraint on the radius of curvature of M chord $[IJ]$ has length more than ε, which gives a contradiction with the assumption that our chord is in Ch'.

(ii) A chord in Ch' is adjacent to a connected component $S \in V_\varepsilon(G)$ with $m(S) = 1$ containing a Steiner point. Then it has the combinatorial type (b) in Figure 20. Let us consider the triangle ΔQCI, where Q is an end of q_S, C is the branching point of S, I is the entering point of S. Since $\angle QCI = 2\pi/3$, we have

$\angle QIC \le \pi/3$, so the angle between the entering segment $[CI]$ and M_r is at least $\pi/6$. Then again the chord $[IJ]$ has length more than ε, that contradicts the choice of the chord.

(iii) Finally, a chord in Ch' is adjacent to an arc $S \in V_\varepsilon(G)$ containing a Steiner point x. Then $x \in M_r$, and x is an end of a chord of M_r in Σ which forms angle $\pi/3$ with M_r. Again by the condition on the radius of curvature of M_r and with the choice of ε sufficiently small, this chord has length more than ε which is impossible.

Let us consider any chord $[I_1I_2] \in Ch'$, such that it connects some components from $V_\varepsilon(G)$ (which do not have Steiner points as proven). Note that the set $]I_2I_1) \cap \Sigma$ (resp. $]I_1I_2) \cap \Sigma$) contains an energetic point (it may coincide with I_1 (I_2); if I_1 (I_2) is not energetic, an energetic point in $]I_2I_1) \cap \Sigma$ (resp. $]I_1I_2) \cap \Sigma$) exists by Lemma 2.8 and the absence of Steiner points in the considered connected components and arcs); denote the nearest to I_1 (resp. I_2) energetic point of $]I_2I_1) \cap \Sigma$ (resp. $]I_1I_2) \cap \Sigma$) by W_1 (resp. W_2).

Consider the region P bounded by the segments $[W_1Q_{W_1}]$, $[W_2Q_{W_2}]$, $[W_1W_2]$ and the lesser arc $[Q_{W_1}\breve{}Q_{W_2}]$ of M. Let us show that the intersection of $\mathrm{Int}(P)$ with Σ is nonempty. There are two tangent lines to M_r parallel to $[W_1W_2]$; let l be the nearest line to $[W_1W_2]$. Note that $[I_1I_2] \in Ch' \subset Ch$, so $[I_1I_2] \cap \mathrm{Int}(N_r) \ne \emptyset$ and $l \cap [W_1W_2] = \emptyset$. Consider a point $w \in l \cap M_r$ and note that Q_w is not covered by Σ, because $\mathrm{dist}\,(Q_W, \Sigma) = \mathrm{dist}\,(Q_w, [W_1, W_2]) > \mathrm{dist}\,(Q_w, l) = r$. We got a contradiction, so $\mathrm{Int}(P) \cap \Sigma \ne \emptyset$.

Let us pick a point $x \in \mathrm{Int}(P) \cap \Sigma$ and consider the path in Σ connecting x with the segment $[W_1W_2]$. The existence of this path gives that for some $i \in \{1, 2\}$ (say, without loss of generality, $i = 1$) one has $W_i = I_i$ (in fact, $]W_1W_2[\subset S_\Sigma$, which means that this path connects x with W_1 without touching $]W_1W_2]$, but a neighbourhood in Σ of an energetic point of $\Sigma \setminus N_r$ is either a single line segment or two line segments with angle at least $2\pi/3$, see Figs. 20 and 21, and thus $W_1 \in M_r$) and $B_\delta(I_1) \cap \mathrm{Int}(P) \cap \Sigma \ne \emptyset$ for sufficiently small $\delta > 0$. Let k be the tangent line to M_r at $I_1 = W_1$. Since $|I_1I_2| \le \varepsilon$, the angle between k and $[I_1I_2]$ is $O(\varepsilon)$. Consider an arbitrary point $y \in \partial B_\delta(I_1) \cap \mathrm{Int}(P) \cap \Sigma$. Since $B_r(Q_{I_1}) \cap \Sigma = \emptyset$ and $|yI_1| = \delta$ the angle between k and $[yI_1]$ is $O(\delta)$. Let z be a projection of y on $[I_1I_2]$. Then $\angle yI_1z = O(\varepsilon + \delta)$ is the smallest angle (for sufficiently small ε, δ) in the right-angled triangle ΔyI_1z. Hence one can replace $]I_1z[$ by $[zy]$ in Σ. The new set is still connected, covers M and has strictly lower length than Σ. We got in this way a contradiction with the optimality of Σ, concluding the proof. $\square$

Proof of Lemma 2.13. Note that in Σ there are at most two chords of M_r ending at I. It is true because of the properties of a locally minimal network: the angle between two segments ending at the same point is greater or equal to $2\pi/3$.

Let us show that $I \in S_\Sigma$. Assume the contrary: let $I \in G_\Sigma$. Then $B_r(Q_I) \cap \Sigma = \emptyset$. There are two possibilities:

(1) $I \in S$, where $S \in V_C(G)$;

(2) $I \in S$, where $S \in V_A(G)$ (as mentioned after Lem. 2.12 S is non degenerate *i.e.* does not reduce to a single point I).

Recall that Σ consists of a finite number of segments and a finite number of arcs of M_r. In the case (1) the smoothness of M_r, Lemma 2.10 and the fact $B_r(Q_I) \cap \Sigma = \emptyset$ imply that the intersection of a small neighbourhood of I with $S \setminus N_r$ is a subset of the tangent line to M_r at I.

Thus the set $\Sigma \cap B_\varepsilon(I) \setminus \mathrm{Int}(N_r)$ is contained in the union of the tangent line τ to M_r at I and the arc $M_r \cap \partial B_\varepsilon(I)$. Both $\tau \cap B_\varepsilon(I)$ and $M_r \cap B_\varepsilon(I)$ are split by I into 2 segments $[IE_1']$, $[IE_2']$ and 2 arcs $[I\breve{E}_1]$, $[I\breve{E}_2]$ of M_r, respectively, where $E_1, E_2, E_1', E_2' \in \partial B_\varepsilon(I)$. We may assume E_1 in the same halfplane with E_1' bounded by the normal to M_r passing throw I. At least one arc and one segment (say, $[I\breve{E}_1]$ and $[IE_1']$) have angle at most $\pi/2$ with the chord $[IB]$. The cases (i) and (ii) below deal with the situation with nonempty set $\Sigma \cap ([I\breve{E}_1] \cup [IE_1'])$. In the remaining cases $\Sigma \cap B_\varepsilon(I) \setminus \mathrm{Int}(N_r)$ is a subset of $[I\breve{E}_2'] \cup [IE_2]$ and therefore in (iii)–(vi) we deal with all the possible cases of $B_\varepsilon(I) \cap [I\breve{E}_2']$ and $B_\varepsilon(I) \cap [IE_2]$ empty/nonempty:

(i) there is such a segment $[IE] \subset \Sigma$, that (IE) is the tangent line to M_r, $|IE| = \varepsilon$ and $\angle BIE \le \pi/2$;

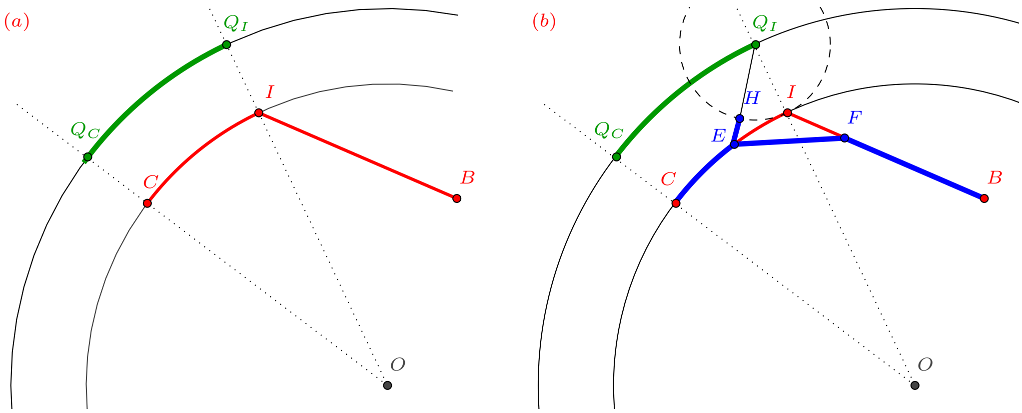

FIGURE 4. The case (iv) of Lemma 2.13: (a) the (impossible) part of the minimizer; (b) the better competitor.

(ii) there is such an $\varepsilon > 0$ and an arc $[\breve{IE}] \subset \Sigma \cap M_r$ that $|IE| = \varepsilon$ and $\angle BIE \leq \pi/2$;
(iii) there is such a small $\varepsilon > 0$ that $\overline{B_\varepsilon(I)} \cap \Sigma$ is equal to $[FI] \cup [IE]$ where $F, E \in \partial B_\varepsilon(I)$, $[FI] \subset [BI]$ and $[IE]$ is a subset of the tangent line to M_r at point I;
(iv) there is such a small $\varepsilon > 0$ that $\overline{B_\varepsilon(I)} \cap \Sigma$ is equal to $[FI] \cup [\breve{IE}]$, where $F, E \in \partial B_\varepsilon(I)$, $[FI] \subset [IB]$ and $[\breve{IE}] \subset M_r$;
(v) there is such a small $\varepsilon > 0$ that $\overline{B_\varepsilon(I)} \cap \Sigma$ contains $[FI] \cup [IC] \cup [\breve{ID}]$ where $[IC]$ is a subset of the tangent line to M_r at point I, $[FI] \subset [BI]$, $[\breve{ID}] \subset M_r$ and $\angle CID < \pi/6$;
(vi) there is such an $\varepsilon > 0$ that $\overline{B_\varepsilon(I)}$ is a subset of chord $[IB]$.

We will show that all these cases are impossible. Let ξ stand for the segment $[IE]$ in the cases (i) and (iii), and for $[\breve{IE}]$ in the cases (ii) and (iv).

CASES (i), (ii): Let $F := [BI] \cap B_\varepsilon(I)$ and l^ε be the lesser arc of $\partial B_\varepsilon(I)$ limited by intersections with $\partial B_r(Q_I)$ and M_r. It is easy to see that $\mathcal{H}^1(l^\varepsilon) = O(\varepsilon^2)$ and $|FI| + \mathcal{H}^1(\xi) - \mathcal{H}^1(\mathcal{S}t(F,I,E)) = c\varepsilon + o(\varepsilon)$ with $c > 0$, where $\mathcal{S}t(F,I,E)$ is a Steiner tree connecting points F, I, E. Then the length of $\Sigma' := \Sigma \setminus ([FI] \cup \xi) \cup l^\varepsilon \cup \mathcal{S}t(F,I,E)$ is less than $\mathcal{H}^1(\Sigma)$ for sufficiently small ε. Moreover Σ' is still connected and $F_M(\Sigma') \leq F_M(\Sigma)$. This gives us a contradiction with optimality of Σ.

CASES (iii), (iv): Note that $|FI| = |IE| = \varepsilon$ (see Fig. 4a), so $\mathcal{H}^1(\xi) = \varepsilon + o(\varepsilon)$ when $\varepsilon \to 0^+$, because M_r is smooth. Let H be the point of intersection of $[EQ_I]$ and $\partial B_r(Q_I)$ (see Fig. 4b). Note that (IQ_I) is perpendicular to the tangent line to M_r at the point I. Thus

$$\begin{aligned} |EH| &= |EQ_I| - |Q_I H| = \sqrt{|EI|^2 + r^2} - r = \sqrt{\varepsilon^2 + r^2} - r \\ &= r\sqrt{1 + o(\varepsilon)} - r = o(\varepsilon). \end{aligned}$$

Now, since the angle between ξ and the segment $[FI]$ is less than π, we get

$$|EF| = \sqrt{2\varepsilon^2 - 2\varepsilon^2 \cos \angle EIF} = \sqrt{2}\varepsilon\sqrt{1 - \cos \angle EIF} < 2\varepsilon - c\varepsilon, \quad \text{for some} \quad c > 0$$

and therefore

$$|EH| + |EF| < \mathcal{H}^1(\xi) + |IF| = 2\varepsilon + o(\varepsilon)$$

for sufficiently small $\varepsilon > 0$. So we have a contradiction with the optimality of Σ, because we show that $(\Sigma \setminus B_\varepsilon(I)) \cup [EH] \cup [EF]$ is the better competitor.

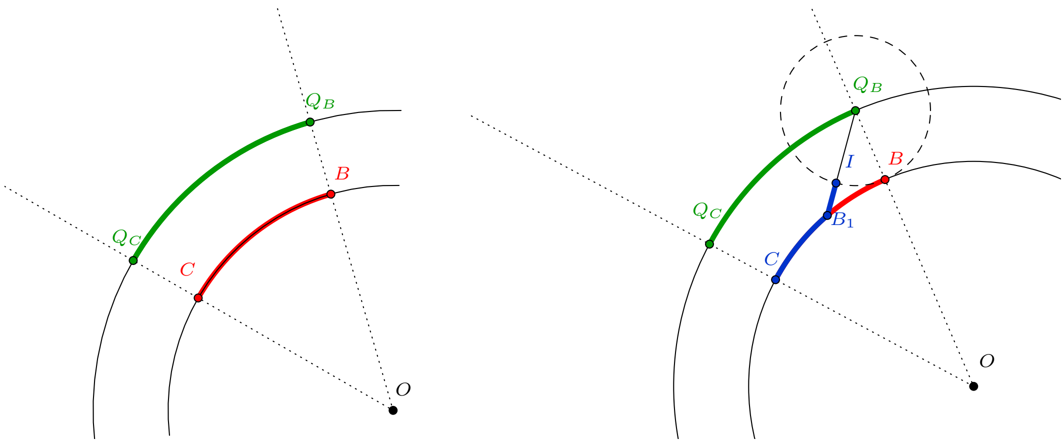

FIGURE 5. Picture to Lemma 2.14. An end of an arc of $M_r \cap \Sigma$ cannot be an endpoint of Σ.

CASE (v): Let $H \in [IC)$ be such a point that $(DH) \perp (IC)$. Then the set

$$\Sigma' = \Sigma \setminus]\breve{ID}] \cup [HD]$$

is still connected, has energy F_M not greater than Σ and strictly smaller length, since $|HD| < |ID|/2 \leq \mathcal{H}^1([\breve{ID}])/2$. It means Σ' is the better competitor than Σ, again a contradiction.

CASE (vi): In this case $S \in V_A(G)$ and $S = \{I\}$, which is impossible.

So all cases are impossible and we have a contradiction which implies $I \in S_\Sigma$. Because of Lemma 2.8 I can not be a Steiner point. Then there exists an $\varepsilon > 0$ such that $S_\Sigma \cap B_\varepsilon(I)$ is a segment. $\square$

Proof of Lemma 2.14. Let $\breve{BC}$ be as in the statement being proven.

Suppose that there is a segment $[IJ] \subset \Sigma$ such that $I =]\breve{BC}[\cap [IJ]$. We claim that $B_\varepsilon(I) \cap \Sigma \subset [\breve{BC}]$. In fact, by Lemma 2.13 $[IJ]$ cannot be a part of a chord of M_r, so $]IJ] \subset \Sigma \setminus \mathrm{Int}(N_r)$. Note that in this case I is energetic (because $B_\varepsilon(I)$ is not a segment or a tripod for every $\varepsilon > 0$). Hence $B_r(Q_I) \cap \Sigma = \emptyset$, so $[IJ]$ is a part of the tangent line to M_r at I. Let us choose an $\varepsilon > 0$ and set $\{D_1, D_2\} := [\breve{BC}] \cap \partial B_\varepsilon(I)$, $E := [IJ] \cap \partial B_\varepsilon(I)$. If $\varepsilon > 0$ is sufficiently small one of the angles $\angle D_1IE$, $\angle D_2IE$ is less than $\pi/6$ (say $\angle D_1IE$). Let $H \in [IJ)$ be such a point that $(D_1H) \perp (IJ)$. Then the set

$$\Sigma' := \Sigma \setminus]\breve{ID_1}] \cup [HD_1]$$

is still connected, has energy F_M not greater than $F_M(\Sigma)$ and strictly smaller length, since $|HD_1| < |ID_1|/2 \leq \mathcal{H}^1([\breve{ID_1}])/2$. It means that Σ' is better competitor than Σ. We got a contradiction, showing thus $B_\varepsilon(I) \cap \Sigma \subset [\breve{BC}]$ for $I \in]\breve{BC}[$.

Let us prove now that $B_\varepsilon(B) \setminus [\breve{BC}]$ is a subset of the tangent line to M_r at B (the analogous statement for the point C is completely symmetric). By Lemma 2.13 there is no chord of M_r in Σ with endpoint B. So the set $B_\varepsilon(B) \setminus [\breve{BC}]$ is a subset of $\Sigma \setminus N_r$.

We claim first that B is not an endpoint of Σ *i.e.* $B_\varepsilon(B) \setminus [\breve{BC}] \neq \emptyset$. Assume the contrary and recall that $Q_B, Q_C \in M$ are such points that $\mathrm{dist}\,(B, Q_B) = \mathrm{dist}\,(C, Q_C) = r$. Then one can set $B_1 := \partial B_\varepsilon(B) \cap [\breve{BC}]$ and replace $[\breve{B_1B}]$ by the segment $[B_1I] := [B_1Q_B] \setminus B_r(Q_B)$, producing the competitor of strictly lower length because $[\breve{BC}] \setminus [\breve{B_1B}] \cup [B_1I] = [\breve{B_1C}] \cup [B_1I]$ still covers the arc $[Q_BQ_C]$ of M (when ε is sufficiently small) (see Fig. 5).

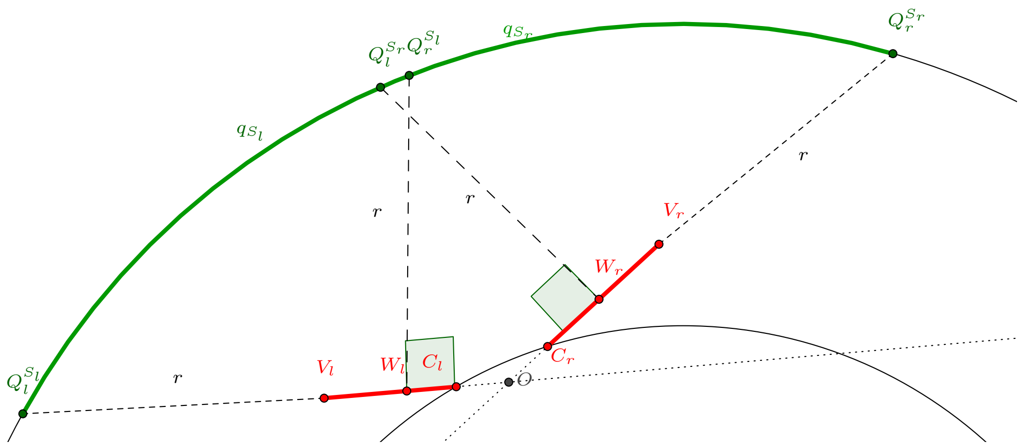

FIGURE 6. Picture to Lemma 2.18.

Therefore we have proven that for sufficiently small $\varepsilon > 0$ the set $B_\varepsilon(B)\setminus[\breve{BC}]$ is a nonempty subset of $\Sigma\setminus N_r$. If B is energetic then $B_r(Q_B) \cap \Sigma = \emptyset$, hence $B_\varepsilon(B)\setminus[\breve{BC}]$ is a subset of the tangent line to M_r at point B showing the claim. So $B \in S_\Sigma$, hence $B_\varepsilon(B)$ is a segment or a tripod for sufficiently small $\varepsilon > 0$. But the case of a tripod is impossible by Lemma 2.8, while the case of a segment is only possible recalling smoothness of M_r (and part of M_r in a neighbourhood of B is in fact flat).

Summing up, the only segments intersecting $[\breve{BC}]$ are segments tangent to M_r at points B and C. As a consequence of Lemma 2.12 Σ consists of a finitely many segments and maximal arcs of M_r, so when ε is small, $B_\varepsilon([\breve{BC}])$ contains only 2 segments which is proven to be tangent to M_r at points B and C, respectively. The statement is proven. □

Proof of Lemma 2.15. Consider a point $C \in M_r \cap \Sigma$. By Lemma 2.14 if C belongs to some non degenerate arc of $\Sigma \cap M_r$ with an energetic point in its interior (*i.e.* an element of $V_A(G)$) the statement is true. Note that if there is a chord $[IC] \subset \Sigma$ of M_r then Lemma 2.13 implies the claim. Thus $B_\varepsilon(C)\cap \mathrm{Int}(N_r) = \emptyset$. If $C \in S_\Sigma$ then by Lemma 2.8 its neighbourhood cannot be a tripod, so it is a segment and the statement of Lemma is obvious. It remains to consider the case when $B_\varepsilon(C) \cap \mathrm{Int}(N_r) = \emptyset$ and C is energetic, which implies $B_r(Q_C) \cap \Sigma = \emptyset$ so the set $B_\varepsilon(C) \cap \Sigma$ is just a segment (because Σ consists of a finite number of arcs of M_r and segments by Lem. 2.12) which must be a subset of the tangent line to M_r at C, the claim follows. □

Proof of Lemma 2.18. Recall that $m(S_l) = m(S_r) = 1$. Denote the ends of q_{S_l} and q_{S_r} in the following way: $q_{S_l} = [Q_l^{S_l}\breve{\ }Q_r^{S_l}]$, $q_{S_r} = [Q_l^{S_r}\breve{\ }Q_r^{S_r}]$. Suppose the contrary, *i.e.* that $Q_r^{S_l} \in]Q_l^{S_r}\breve{\ }Q_r^{S_r}[$, $Q_l^{S_r} \in]Q_l^{S_l}\breve{\ }Q_r^{S_l}[$. Suppose that $n(S_l) = 2$ or $n(S_r) = 2$ (let $n(S_l) = 2$, the case $n(S_r) = 2$ is completely analogous). Then by Remark 3.3 there is an energetic point of S_l corresponding to the point $Q_r^{S_l}$. But $B_r(Q_r^{S_l}) \cap \Sigma \neq \emptyset$, because $Q_r^{S_l} \in]Q_l^{S_r}\breve{\ }Q_r^{S_r}[= q_{S_r}$. So we have a contradiction with the assumption $n(S_l) = 2$, and hence S_l coincides with the segment $[C_lV_l]$. Clearly, V_l, C_l and $Q_l^{S_l}$ lie on the same line (otherwise one can replace $[V_lV']$ by the part of the segment $[V'Q_l^{S_l}]$, where $V' := \partial B_\varepsilon(V_l)\cap[V_lC_l]$ producing a competitor of strictly lower length). Hence $[C_lV_l]$ is tangent to $B_r(Q_r^{S_l})$ (see Fig. 6).

Let W_l be such a point of $[C_lV_l]$ that $\mathrm{dist}\,(W_l, Q_r^{S_l}) = r$, W_r be such a point of $[C_rV_r]$ that $\mathrm{dist}\,(W_r, Q_l^{S_r}) = r$. Note that the points C_l, V_l, $Q_l^{S_l}$ lie on the same line, so $\mathrm{dist}\,(W_lQ_l^{S_l}) \geq r = \mathrm{dist}\,(W_l, Q_r^{S_l})$, so $\angle Q_r^{S_l}Q_l^{S_l}W_l \leq \angle Q_l^{S_l}Q_r^{S_l}W_l$. The segment $[C_lV_l]$ is tangent to $B_r(Q_r^{S_l})$, hence $(Q_r^{S_l}W_l) \perp (V_lC_l)$. Calculating angles in triangle $\Delta Q_r^{S_l}Q_l^{S_l}W_l$ we have $\angle Q_r^{S_l}Q_l^{S_l}W_l \leq \pi/4$. Obviously, $\angle Q_r^{S_r}Q_l^{S_l}W_l \leq \angle Q_r^{S_l}Q_l^{S_l}W_l$, so $\angle Q_r^{S_r}Q_l^{S_l}W_l \leq \pi/4$.

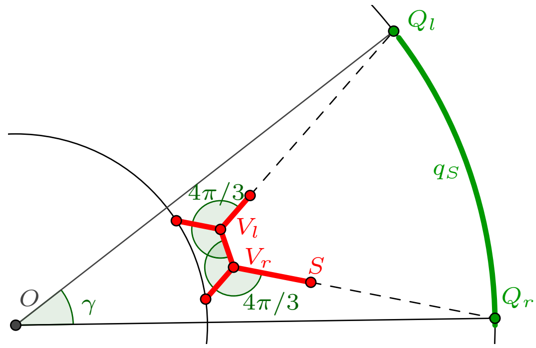

FIGURE 7. Picture to the case 3.1: middle component, $n = 2, m = 2$.

By symmetry we have inequality $\angle Q_l^{S_l} Q_r^{S_r} W_r \leq \pi/4$. Denote by O the intersection point of $(V_l C_l)$ and $(V_r C_r)$. From the triangle $\Delta Q_r^{S_r} Q_l^{S_l} O$ we have $\angle Q_r^{S_r} O Q_l^{S_l} \geq \pi/2$.

Note that $2r > |W_l W_r| \geq |C_l C_r|$ and $\angle Q_r^{S_r} O Q_l^{S_l} = \angle C_l O C_r \geq \pi/2$. It means that $|C_l O| < 2r$ and $|C_r O| < 2r$. Hence the intersection point of the rays $[V_l C_l)$ and $[V_r C_r)$ belongs to N_r, that contradicts the optimality of Σ. $\square$

3.1. Proof of the central Lemma

First, let us prove the following technical lemma.

Lemma 3.4. *Let S be the closure of a connected component of $\Sigma \setminus N_r$ such that $n(S) = 2$. Let $W \in G_\Sigma \cap S$ be an energetic point of S, such that $\overline{B_\varepsilon(W)} \cap S = [J_1 W] \cup [W J_2]$. Then (QW) is the bisector of $\angle J_1 W J_2$, where Q is the end of arc q_S corresponding to W (in the sense of Rem. 3.3).*

Proof of Lemma 3.4. Suppose the contrary *i.e.* that $\angle J_1 W Q \neq \angle Q W J_2$. Let l be the tangent line to $B_r(Q)$ at the point W. Then $\mathcal{H}^1(\Sigma \cap B_\varepsilon(W)) - \mathcal{H}^1([J_1 Y] \cup [Y J_2]) = O(\varepsilon)$, where Y is such a point in l that $\angle J_1 Y Q = \angle Q Y J_2$. On the other hand, $\operatorname{dist}(Y, \partial B_r(Q)) = O(\varepsilon^2)$; let $V \in \partial B_r(Q)$ be such a point that $\operatorname{dist}(Y, \partial B_r(Q)) = \operatorname{dist}(Y, V)$. Then the set $(\Sigma \setminus B_\varepsilon(W)) \cup [J_1 Y] \cup [J_2 Y] \cup [Y V]$ is connected, covers $q(S)$ and has strictly lower length than Σ, giving a desired contradiction. $\square$

Finally, we are ready to prove the central Lemma.

Proof of Lemma 2.22. Obviously, if S is an arc, then the compared values are equal.

It suffices thus to consider the case when S is the closure of a connected component of $\Sigma \setminus N_r$. Denote by Q_l and Q_r the ends of q_S. Let O be an intersection point of the normals to M at points Q_l and Q_r. It exists unless $\operatorname{turn}(q_S) = 0$ in which case the claim is obvious. Note that $\operatorname{turn}(q_S) = \angle Q_l O Q_r$ and denote for brevity thus value by γ. Also one has $|Q_l O| \geq R$, $|Q_r O| \geq R$. Note that Lemmas 2.8 and 2.10 as well as Corollary 2.17 hold true when $R > 2a_M(r) + r$ which is guaranteed when $R > 5r$ (or $R > 4.98r$ in the case when M is a circumference of radius R), *i.e.* under the conditions of the statement being proven.

By Lemma 2.10 S is a locally minimal network for at most $n(S) + m(S) \leq 4$ points. All the possible combinatorial types of such networks are listed in Figures 20 and 21. Note that if S is a middle component then $m(S) = 2$, otherwise $m(S) = 1$. Let us analyze all the possible types one by one, first when S is a middle component, then for S an ending component.

(1) Let S be a middle component. By Lemma 2.10 it is a locally minimal network, moreover it has two entering points (if one, then it is an ending component) and one or two energetic points.

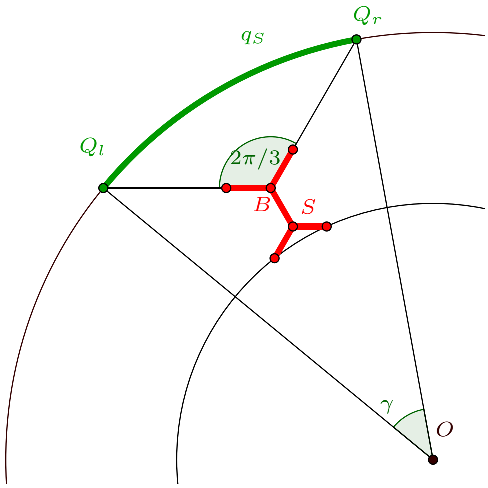

FIGURE 8. A general picture to the case 3.1: middle component, $n = 2, m = 2$.

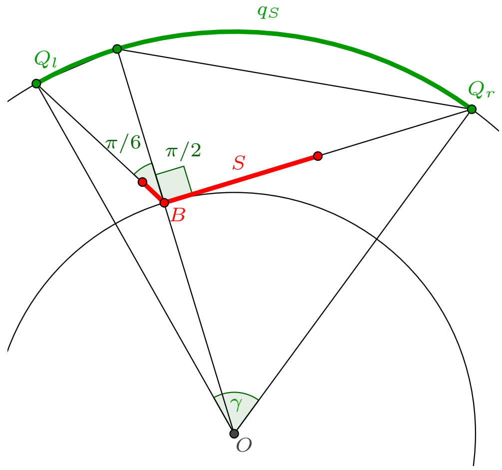

FIGURE 9. A marginal picture to the case 3.1: middle component, $n = 2, m = 2$.

(a) *The case $n = 2$, $m = 2$, the combinatorial type (a) in Figure 21 (see Fig. 7).* Denote the Steiner points of S by V_l and V_r. In this case $\text{turn}(S) = \pi/3 + \pi/3 = 2\pi/3$. Assuming the contrary (it means that $\gamma \geq 2\pi/3$) and connecting O with Q_l and Q_r, we get a (non convex) pentagon $Q_lV_lV_rQ_rO$ with two angles equal to $4\pi/3$ and one angle at least $2\pi/3$, which is impossible.

(b) *The case $n = 2, m = 2$, the combinatorial type (b) in Figure 21 (see Fig. 8).* Note that in this case there exists a Steiner point adjacent to both entering points, and also there exists a Steiner point (we call it B) adjacent to both energetic points. Clearly $\text{turn}(S) = \pi/3$. Let us prove that $\text{turn}(q_S) < \pi/3$. We evaluate the arc of M bounded by continuations of segments starting from B. Clearly this arc is maximal when B belongs to M_r (it is the marginal case). Hence it is enough to look at the angle in $N \backslash N_r$ of size $2\pi/3$ with vertex B on M_r. It is well-known that the arc is maximal when S is tangent to M_r and when M is a circumference. In this case the normal to M_r at B splits the angle $\angle Q_lBQ_r = 2\pi/3$ in two angles: one of size $\pi/2$ and another of size $\pi/6$ (see Fig. 9), so that the size of the arc is

$$\arccos\left(1 - \frac{1}{\delta}\right) + \frac{\pi}{6} - \arcsin\left(\frac{1}{2}\left(1 - \frac{1}{\delta}\right)\right),$$

where $\delta := R/r$, hence it is strictly less than $\pi/3$ for $\delta \geq 2.9$.

(c) *The case $n = 2$, $m = 2$, the combinatorial type (c) in Figure 21.*
There are two possibilities for S in this case, see Figures 10 and 11.
THE CASE IN FIGURE 11 can be reduced to the previous case 3.1. Obviously, $\text{turn}(S) = \pi/3$. Let us fix the entering points Y_l, Y_r and the left energetic point W_l and move the right energetic point W_r to the right (in the direction of the ray $[W_lW_r)$). Then at some time the combinatorial type changes to (b) in Figure 21, during this process $\text{turn}(S) = \pi/3$, and $\text{turn}(q_S)$ grows, but $\text{turn}(q_S) \leq \pi/3$. By case 3.1.
THE CASE IN FIGURE 10: denote the energetic points of S by W_l and W_r, and the entering points by Y_l, Y_r respectively, and the branching point by V_l (without loss of generality it is connected with W_l and Y_l). Let $2\beta := \angle V_lW_rY_r$, and note that $\angle Y_lV_lW_r = 2\pi/3$. Then $\text{turn}(S) = (\pi - 2\pi/3) + (\pi - 2\beta) = 4\pi/3 - 2\beta$. Assume the contrary (*i.e.* in this case $\gamma \geq 4\pi/3 - 2\beta$) and call L the point of intersection of (Q_lW_l)

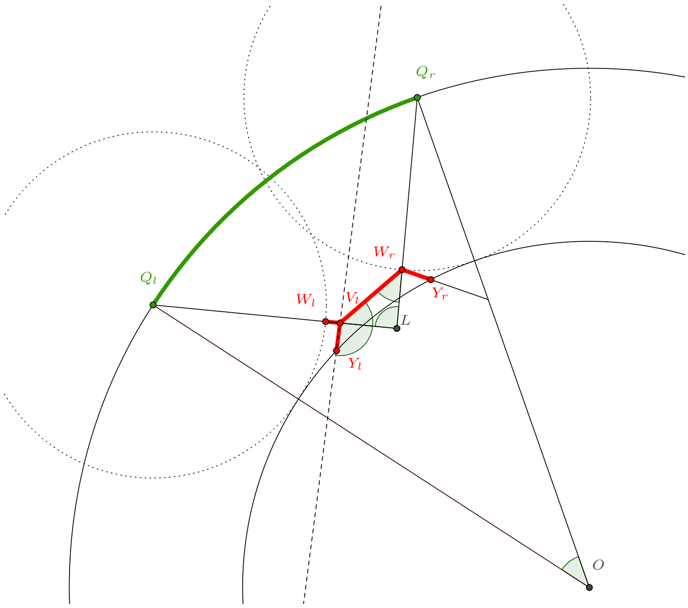

FIGURE 10. Picture to the case 3.1: middle component, $n = 2$, $m = 2$.

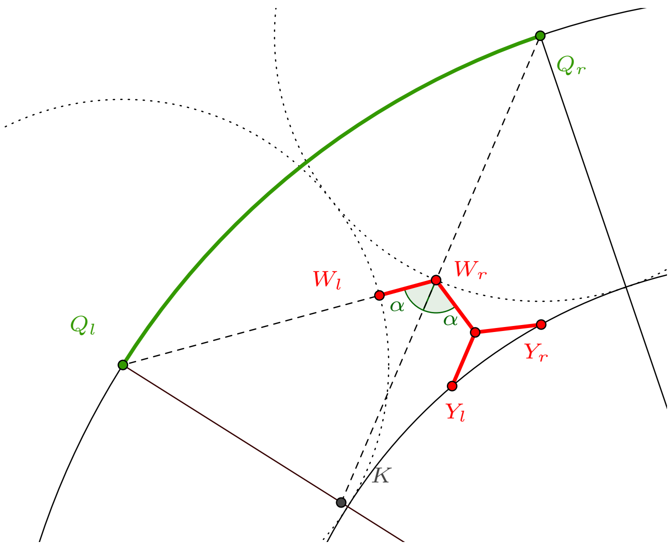

FIGURE 11. Picture to the case 3.1: middle component, $n = 2$, $m = 2$.

and (Q_rW_r). By Lemma 3.4 $\angle LW_rV_l = \angle Y_rW_rV_l/2 = \beta$. Then

$$\pi - \pi/3 - \beta = \angle Q_lLQ_r > \angle Q_lOQ_r = \gamma,$$

(the first equality coming from ΔV_lW_rL) which implies

$$\gamma \geq 4\pi/3 - 2\beta > 2\pi/3 - \beta = \angle Q_lLQ_r > \gamma,$$

a contradiction.

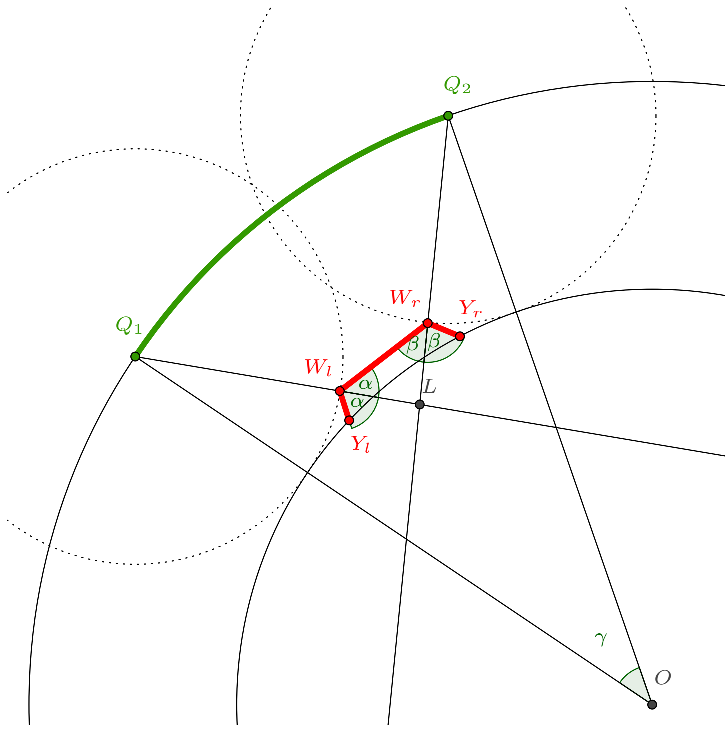

FIGURE 12. Picture to the case 3.1: middle component, $n = 2$, $m = 2$.

(d) *The case $n = 2$, $m = 2$, the combinatorial type (d) in Figure 21 (see Fig. 12).* Denote the energetic points of S by W_l and W_r, and the entering points by Y_l, Y_r respectively. Let $2\alpha := \angle Y_l W_l W_r$, $2\beta := \angle W_l W_r Y_r$. Then $\mathrm{turn}(S) = (\pi - 2\alpha) + (\pi - 2\beta)$. Assume the contrary (it means that $\gamma \geq 2\pi - 2\alpha - 2\beta$) and denote by L the point of intersection of $(Q_l W_l)$ and $(Q_r W_r)$. By Lemma 3.4 $\angle L W_l W_r = \angle Y_l W_l W_r / 2 = \alpha$, $\angle L W_r W_l = \angle Y_r W_r W_l / 2 = \beta$. Then

$$\pi - \alpha - \beta = \angle Q_l L Q_r > \angle Q_l O Q_r = \gamma,$$

(the first equality coming from $\Delta W_l W_r L$) which implies

$$\gamma \geq 2\pi - 2\alpha - 2\beta > \pi - \alpha - \beta = \angle Q_l L Q_r > \gamma,$$

a contradiction.

(e) *The case $n = 1$, $m = 2$, the combinatorial type (b) in Figure 20 (see Figs. 13–15).*
Clearly, $\mathrm{turn}(S) = \pi/3$. To prove the statement, assume the contrary (*i.e.* $\gamma \geq \pi/3$) and as in the previous case connect O with Q_l and Q_r. Denote the energetic point of S by W. Let us consider three subcases:
- the point W covers both Q_r and Q_l (see Fig. 13);
- the point W covers Q_l and Q_r is covered by an entering point (see Fig. 14);
- W covers Q_l and Q_r is covered by $H \in S \setminus (M_r \cup W)$ (see Fig. 15).

IN THE SUBCASE (i) $|WQ_r| = |WQ_l| = r$. Let us connect O with W, and note that the angle $\angle Q_l O Q_r = \gamma$ splits into two parts; let us pick the largest one (without loss of generality it is $\angle W O Q_r$). Consider the triangle $\Delta O Q_r W$ with side $|OQ_r| \geq R$ and acute angle (α in Fig. 13) at least $\pi/6$ against the side $|WQ_r| = r$. Recalling that $R > 2r$ and denoting by $\beta := \angle O W Q_r$, by the law of sines for triangle

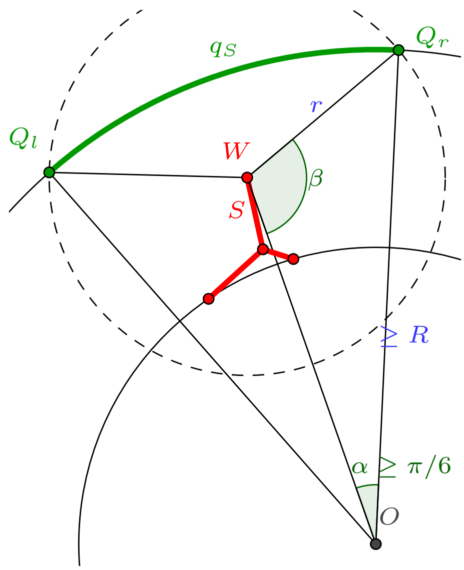

FIGURE 13. Picture to the case 3.1: middle component, $m = 2, n = 1$.

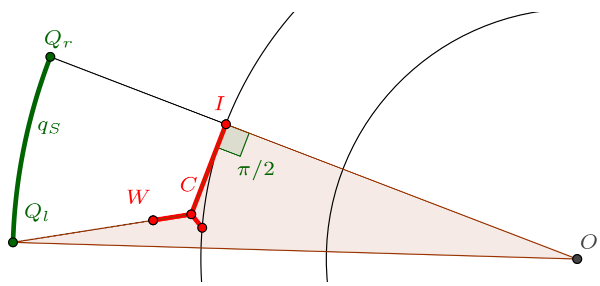

FIGURE 14. Picture to the case 3.1: middle component, $m = 2, n = 1$.

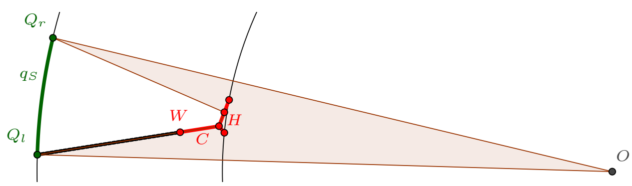

FIGURE 15. Picture to the case 3.1: middle component, $m = 2, n = 1$.

$\varDelta OQ_rW$ we get

$$\sin\beta = \frac{|OQ_r|}{r}\sin\alpha \geq \frac{R}{2r} > 1,$$

a contradiction.

IN THE SUBCASE (ii) Q_r is covered by the entering point I. Then (CI) is perpendicular to (IQ_r), where C is the branching point of S, so points Q_r, O, I lie on the same line. Consider the sum of the angles in the non convex quadrilateral Q_lCIO: it is $\angle Q_l + \angle C + \angle I + \angle O \geq \angle Q_l + 4\pi/3 + \pi/2 + \pi/3 > 2\pi$, a contradiction.

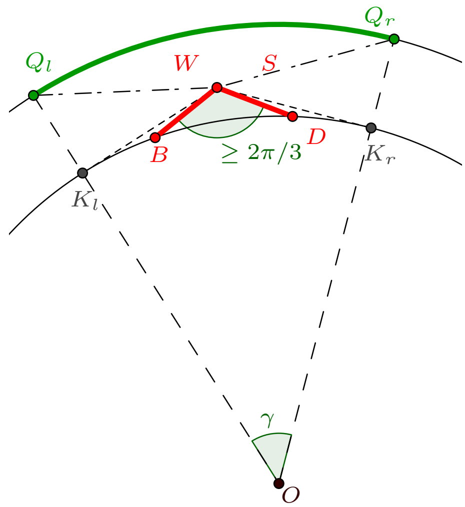

FIGURE 16. Picture to the case 3.1: middle component, $m = 2$, $n = 1$.

IN THE SUBCASE (iii) Q_r is covered by $H \in]CI[$, where C is the branching point of S, I is an entering point of S. Note that (CI) is perpendicular to (HQ_r); points Q_l, W, C lie on the same line. Consider the sum of the angles in the non convex pentagon Q_lCHQ_rO: it is $\angle Q_l + \angle C + \angle H + \angle Q_r + \angle O \geq \angle Q_l + 4\pi/3 + 3\pi/2 + \angle Q_r + \pi/3 > 3\pi$, a contradiction.

(f) *The last case $n = 1$, $m = 2$, the combinatorial type (c) in Figure 20 (see Fig. 16).* Then S consists of two segments, *i.e.* $S = [BW] \cup [WD]$, where $B, D \in M_r$ are entering points, W is energetic and $\angle BWD \geq 2\pi/3$. In this case $\operatorname{turn}(S) = \pi - \angle BWD$.
First, connect O with Q_l and Q_r then denote $K_l = [OQ_l] \cap M_r$ and $K_r = [OQ_r] \cap M_r$. Now consider the convex quadrilateral $P = K_lOK_rW$. The sum of the angles $\angle K_l + \angle K_r + \angle W$ of P is at least $\pi/2 + \angle BWD + \pi/2$, so that the remaining angle (which is equal to γ) is at most $\pi - \angle BWD = \operatorname{turn}(S)$ as claimed.
If one has the equality then both $[BW]$ and $[WD]$ are tangent to M_r, but W is not energetic point in this case, because Q_l is covered by $B = K_l$, Q_r is covered by $D = K_r$, so we got a contradiction.

(2) *Let S be an ending component (without loss of generality let it be the left one, so $Q_r = A$). Recall that C denotes the branching point if S is a tripod and the entering point if S is a seqment. Then there are two options:*

(a) *The case $n = 1$, $m = 1$, the combinatorial type (a) in Figure 20 (see Fig. 17).* In this case $S = [CS'_l]$, where $C \in M_r$, $|S'_lQ_r| = r$, and $\operatorname{turn}(S) = 0$. Denote by K such a point that $K \in [OQ_l)$ and $\angle OQ_rK = \pi/2$. Define the points $L := [S'_lC) \cap (OQ_l)$ and $P := [CS'_l) \cap (Q_rK)$, and introduce the angles $\alpha := \angle PS'_lQ_r$ and $\beta := \angle S'_lQ_rK$.
The following two situations have to be considered. Note that $|S'_lQ_l| = r$, otherwise one can replace $[CS'_l] \cap B_\varepsilon(S'_l)$ in Σ by the part $[DF]$ of the segment $[DQ_r]$ where $D = [CS'_l] \cap \partial B_\varepsilon(S'_l)$, F is the point satisfying $\operatorname{dist}(F, Q_r) = r$, producing the competitor of strictly lower length.

- Case $\angle CS'_lQ_r \leq \pi$ (see the top picture in Fig. 17).
Then $\angle([S'_lA), a) = \beta$ and $\angle([CS'_l), [S'_lA)) = \alpha$, so that

$$\operatorname{turn}(S) + \angle([CS'_l), [S'_lA)) + \angle([S'_lA), a) = \alpha + \beta.$$

Note that $\angle S'_lPK = \alpha + \beta$ and $\angle OKQ_r = \pi/2 - \gamma$. If $\alpha + \beta \leq \gamma$ (contrary to the claim being proven), then $\angle OKP + \angle KPS'_l < \pi/2$ so $\angle KLP > \pi/2$, which is impossible because then $|CQ_l| < |S'_lQ_l|$ which contradicts $|S'_lQ_l| = r$, $|CQ_l| \geq r$.

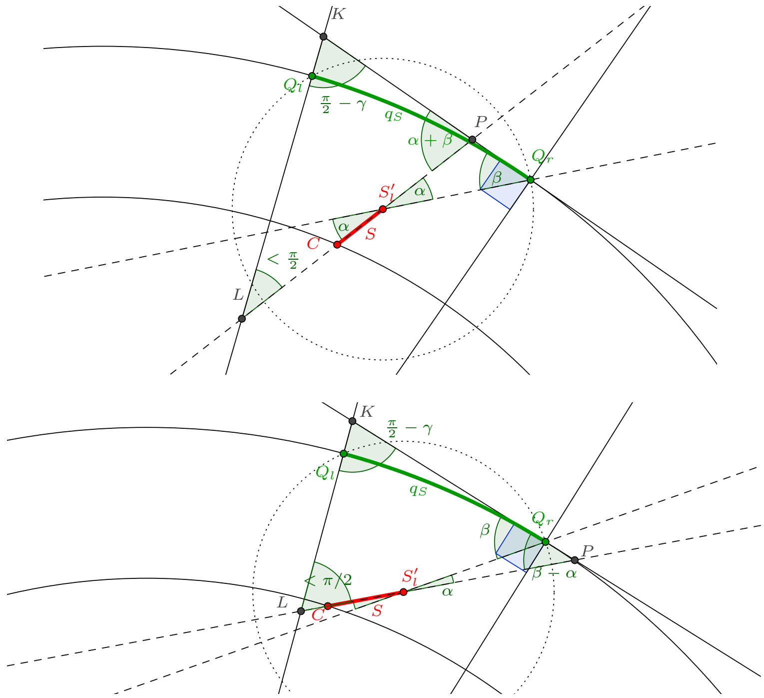

FIGURE 17. Picture to the case 3.1: ending component, $n = 1$, $m = 1$.

- Case $\angle CS_l'Q_r > \pi$ (see the bottom picture in Fig. 17). In this case $\angle([S_l'A), a) = \beta$ and $\angle([CS_l'), [S_l'A)) = -\alpha$, so

  $$\mathrm{turn}(S) + \angle([CS_l'), [S_l'A)) + \angle([S_l'A), a) = \beta - \alpha$$

  and we know that $\angle KPC = \beta - \alpha$. If $\beta - \alpha \le \gamma$ (the contrary to the claim being proven), then $\angle OKP + \angle KPC < \pi/2$, which is impossible because then $|CQ_l| < |S_l'Q_l|$ which contradicts $|S_l'Q_l| = r$, $|CQ_l| \ge r$.

(b) *The case $n = 2$, $m = 1$, the combinatorial type (b) in Figure 20 (see Fig. 18).*
Note that S is a tripod: $S = [BC] \cup [CW] \cup [CS_l'] \subset \overline{(N \setminus N_r)}$, where $B \in M_r$. Let us prove that $Q_r = [CS_l') \cap M$ and $Q_l = [CW) \cap M$. Suppose the contrary *i.e.* without loss of generality C, S_l', and Q_r do not lie on the same line. Let us pick a sufficiently small $\varepsilon > 0$ and denote by J the intersection point of $\partial B_\varepsilon(S_l')$ with $[CS_l']$. Then one may replace $[JS_l']$ by $[JI]$ in Σ, where I stands for the intersection point of $\partial B_r(Q_r)$ with $[JQ_r]$. Clearly the resulting set covers q_{S_l}, so it has the same energy F_M; by the triangle inequality it has strictly lower length, so we got a contradiction.
Note that $|S_l'Q_r| = r = |WQ_l|$; $B_r(Q_r) \cap \Sigma = B_r(Q_l) \cap \Sigma = \emptyset$. Let $K \in [OQ_l)$ be the point satisfying $(Q_rK) \perp (OQ_r)$. Then $\alpha := \mathrm{turn}(S) = \angle([BC), [CQ_r)) = \pi/3$, $\angle([CS_l'), [S_l'A)) = 0$ and $\beta := ([CQ_r), [Q_rK)) = \angle([S_l'A), a)$. We have to show $\alpha + \beta > \gamma$. Let P be the point of intersection of $(KQ_r]$ and $[BC)$. Then $\angle OKP = \pi/2 - \gamma$ and $\angle KPC = \alpha + \beta$. Assume the contrary, *i.e.* $\alpha + \beta \le \gamma$. Then $\angle OKP + \angle KPC \le \pi/2$ hence $\angle KLP \ge \pi/2$, where L is the point of intersection of (BC) and (OK), but since $\angle Q_lCL = 2\pi/3$ the sum of the angles of the triangle ΔCLQ_l exceeds π, which is impossible.

(c) *The case $n = 2$, $m = 1$, the combinatorial type (c) in Figure 20 (see Fig. 19).* In this case $A = Q_r$, $S_l' = W_r$. Denote $\angle([CW_r), [W_rQ_r))$ by α, $\angle([S_l'A), a) = \angle([W_rQ_r), a)$ by β. Clearly $\mathrm{turn}(S) = \alpha + \beta$, $\mathrm{turn}(q_S) = \gamma$. Let L be the point of intersection of (W_rC) and (Q_lO). Suppose the contrary, *i.e.*

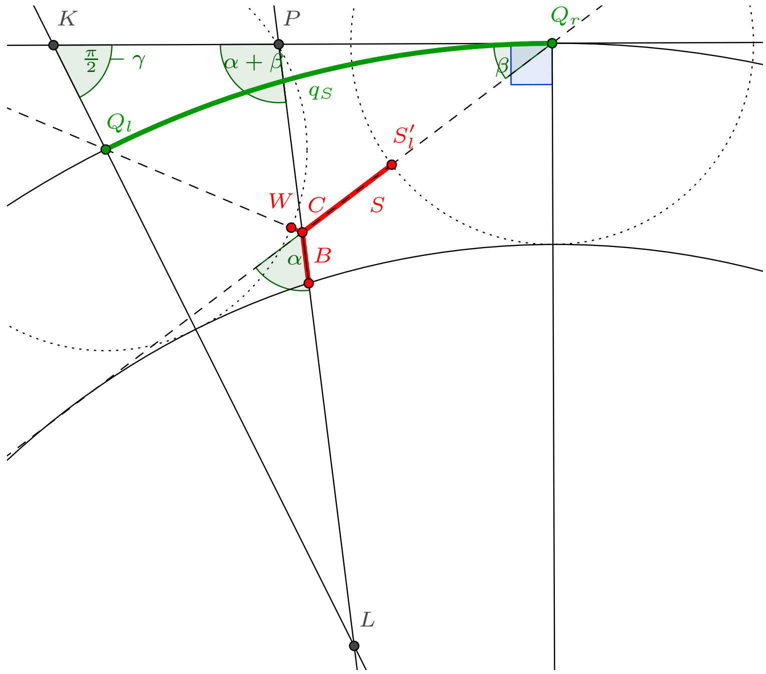

FIGURE 18. Picture to the case 3.1: ending component, $n = 2$, $m = 1$.

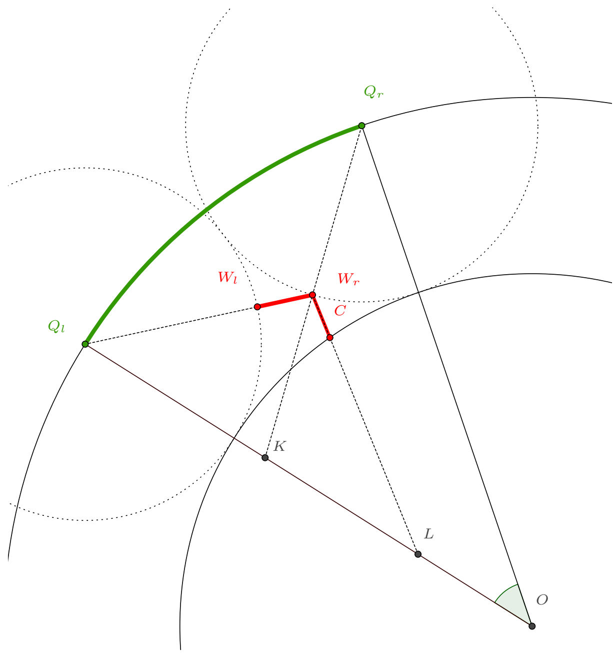

FIGURE 19. Picture to the case 3.1: ending component, $n = 2$, $m = 1$.

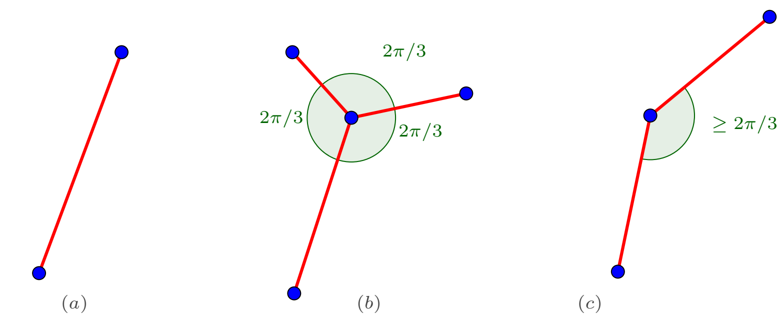

FIGURE 20. Locally miminal networks for sets of 2 and 3 points.

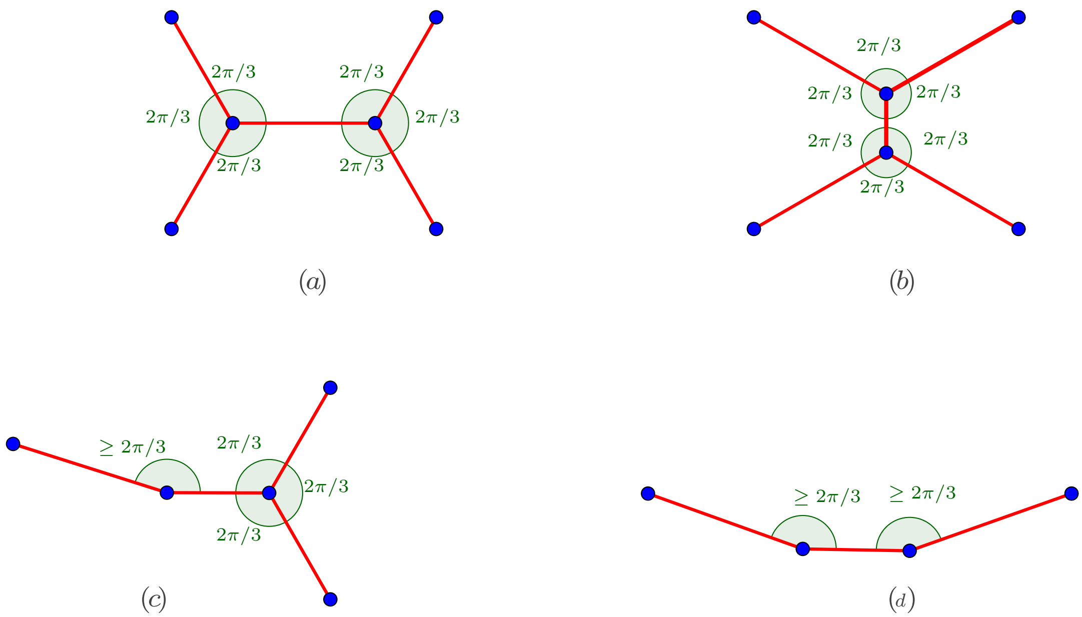

FIGURE 21. Locally miminal networks for sets of 4 points.

$\gamma \geq \alpha + \beta$. Then

$$\angle W_r L Q_l = \pi - \angle W_r L O = \pi - (2\pi - \angle L W_r Q_r - \angle W_r Q_r O - \angle Q_r O L) =$$
$$\pi - (2\pi - (\pi - \alpha) - (\pi/2 - \beta) - \gamma) = \pi/2 - \beta - \alpha + \gamma \geq \pi/2,$$

which is impossible because then $|CQ_l| < |S'_l Q_l|$, which contradicts $|S'_l Q_l| = r$, $|CQ_l| \geq r$. $\square$

Proof of Corollary 2.4. Let $\hat{\Sigma}$ be a local minimizer in the sense of Definition 2.3. Suppose the claim is false, *i.e.*

$$\mathcal{H}^1(\hat{\Sigma}) - \mathcal{H}^1(\Sigma) < (R - 5r)/2 \tag{3.1}$$

and $\hat{\Sigma}$ is not a horseshoe. Suppose first that $\hat{\Sigma}_r$ contains no line segment of length exceeding

$$a'_M(r) := 2r + \mathcal{H}^1(\hat{\Sigma}) - \mathcal{H}^1(\Sigma) < 2r + (R - 5r)/2.$$

Then Lemma 2.8 remains true for this situation with a'_M instead of a_M, because $2a'_M(r) + r < R$. Lemma 2.10 also remains true with $a'_M(r)$ instead of a_M by the same reason. We may repeat now line by line the proof of Theorem 2.2 without any change because all the arguments used in this proof as well as in Lemma 2.22 are local, except the Lemma 2.8 and part of Lemma 2.10 (the claim $m(S) \leq 2$) which hold true with a'_M instead of a_M. This proves that $\hat{\Sigma}$ is a horseshoe in the considered case.

On the other hand it is impossible to $\hat{\Sigma}_r$ to have a segment of length at least $a'_M(r)$, otherwise using the replacement from Lemma 2.7(iii) and get a contradiction with (3.1). □

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