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OPTIMAL CONTROL OF PIECEWISE DETERMINISTIC MARKOV PROCESSES: A BSDE REPRESENTATION OF THE VALUE FUNCTION

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Abstract. We consider an infinite-horizon discounted optimal control problem for piecewise deterministic Markov processes, where a piecewise open-loop control acts continuously on the jump dynamics and on the deterministic flow. For this class of control problems, the value function can in general be characterized as the unique viscosity solution to the corresponding Hamilton−Jacobi−Bellman equation. We prove that the value function can be represented by means of a backward stochastic differential equation (BSDE) on infinite horizon, driven by a random measure and with a sign constraint on its martingale part, for which we give existence and uniqueness results. This probabilistic representation is known as nonlinear Feynman−Kac formula. Finally we show that the constrained BSDE is related to an auxiliary dominated control problem, whose value function coincides with the value function of the original non-dominated control problem.

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1. Introduction

The aim of the present paper is to prove that the value function in an infinite-horizon optimal control problem for piecewise deterministic Markov processes (PDMPs) can be represented by means of an appropriate backward stochastic differential equation. Piecewise deterministic Markov processes, introduced in [21], evolve through random jumps at random times, while the behavior between jumps is described by a deterministic flow. We consider optimal control problems of PDMPs where the control acts continuously on the jump dynamics and on the deterministic flow as well.

Let us start by describing our setting in an informal way. Let $(E, \mathcal{E})$ be a general measurable space. A PDMP on $(E, \mathcal{E})$ can be described by means of three local characteristics, namely a continuous flow $\phi(t, x)$, a jump rate $\lambda(x)$, and a transition measure $Q(x, \mathrm{d}y)$, according to which the location of the process at the jump time is chosen. The PDMP dynamic can be described as follows: starting from some initial point $x \in E$, the motion of the process follows the flow $\phi(t, x)$ until a random jump T_1, verifying

$$\mathbb{P}(T_1 > s) = \exp\left(-\int_0^s \lambda(\phi(r, x))\,\mathrm{d}r\right), \quad s \geq 0.$$

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At time T_1 the process jumps to a new point X_{T_1} selected with probability $Q(x, \mathrm{d}y)$ (conditional on T_1), and the motion restarts from this new point as before.

Now let us introduce a measurable space $(A, \mathcal{A})$, which will denote the space of control actions. A controlled PDMP is obtained starting from a jump rate $\lambda(x, a)$ and a transition measure $Q(x, a, \mathrm{d}y)$, depending on an additional control parameter $a \in A$, and a continuous flow $\phi^{\beta}(t, x)$, depending on the choice of a measurable function $\beta(t)$ taking values on $(A, \mathcal{A})$. A natural way to control a PDMP is to chose a control strategy among the set of *piecewise open-loop policies*, *i.e.*, measurable functions that depend only on the last jump time and post jump position. We can mention [1, 8, 20, 21, 24], as a sample of works that use this kind of approach. Roughly speaking, at each jump time T_n, we choose an open loop control α_n depending on the initial condition $E_n \in \mathcal{E}$ to be used until the next jump time. A control α in the class of admissible control laws $\mathcal{A}_{ad}$ has the explicit form

$$\alpha_t = \alpha_0(t, x)\, \mathbb{1}_{[0,\, T_1)}(t) + \sum_{n=1}^{\infty} \alpha_n(t - T_n, E_n)\, \mathbb{1}_{[T_n,\, T_{n+1})}(t), \tag{1.1}$$

and the controlled process X is

$$X_t = \begin{cases} \phi^{\alpha_0}(t, x) & \text{if} \quad t \in [0,\, T_1), \\ \phi^{\alpha_n}(t - T_n, E_n) & \text{if} \quad t \in [T_n,\, T_{n+1}),\ \ n \in \mathbb{N} \setminus \{0\}. \end{cases}$$

We denote by $\mathbb{P}^x_\alpha$ the probability measure such that, for every $n \geq 1$, the conditional survival function of the inter-jump time $T_{n+1} - T_n$ and the distribution of the post jump position $X_{T_{n+1}}$ are

$$\mathbb{P}^x_\alpha(T_{n+1} - T_n > s \,|\, \mathcal{F}_{T_n}) = \exp\left(- \int_{T_n}^{T_n + s} \lambda(\phi^{\alpha_n}(r - T_n, X_{T_n}), \alpha_n(r - T_n, X_{T_n}))\, \mathrm{d}r \right), \quad \forall\, s \geq 0,$$
$$\mathbb{P}^x_\alpha(X_{T_{n+1}} \in B |\, \mathcal{F}_{T_n},\, T_{n+1}) = Q(\phi^{\alpha_n}(T_{n+1} - T_n, X_{T_n}), \alpha_n(T_{n+1} - T_n, X_{T_n}), B), \quad \forall B \in \mathcal{E},$$

on $\{T_n < \infty\}$.

In the classical infinite-horizon control problem one wants to minimize over all control laws α a functional cost of the form

$$J(x, \alpha) = \mathbb{E}^x_\alpha \left[\int_0^\infty \mathrm{e}^{-\delta\, s}\, f(X_s, \alpha_s)\, \mathrm{d}s \right] \tag{1.2}$$

where $\mathbb{E}^x_\alpha$ denotes the expectation under $\mathbb{P}^x_\alpha$, f is a given real function on $E \times A$ representing the running cost, and $\delta \in (0,\, \infty)$ is a discounting factor. The value function of the control problem is defined in the usual way:

$$V(x) = \inf_{\alpha \in \mathcal{A}_{ad}} J(x, \alpha), \quad x \in E. \tag{1.3}$$

Let now E be an open subset of $\mathbb{R}^d$, and $h(x, a)$ be a bounded Lipschitz continuous function such that $\phi^\alpha(t, x)$ is the unique solution of the ordinary differential equation

$$\dot{x}(t) = h(x(t), \alpha(t)), \quad x(0) = x \in E.$$

We will assume that λ and f are bounded functions, uniformly continuous, and Q is a Feller stochastic kernel. In this case, V is known to be the unique viscosity solution on $[0,\, \infty) \times E$ of the Hamilton−Jacobi−Bellman (HJB) equation

$$\delta v(x) = \inf_{a \in A} \left(h(x, a) \cdot \nabla v(x) + \lambda(x, a) \int_E (v(y) - v(x))\, Q(x, a, \mathrm{d}y) \right), \quad x \in E. \tag{1.4}$$

The characterization of the optimal value function as the viscosity solution of the corresponding integro-differential HJB equation is an important approach to tackle the optimal control problem of PDMPs, and can be found for instance in [22, 25, 42]. Alternatively, the control problem can be reformulated as a discrete-stage Markov decision model, where the stages are the jump times of the process and the decision at each stage is the control function that solves a deterministic optimal control problem. The reduction of the optimal control problem to a discrete-time Markov decision process is exploited for instance in [1, 8, 20, 21].

In the present work our aim is to represent the value function $V(x)$ by means of an appropriate Backward Stochastic Differential Equation (BSDE for short). We are interested in the general case when the probability measures $\{\mathbb{P}^x_\alpha\}_\alpha$ are not absolutely continuous with respect to the law of a given, uncontrolled process. This really increases the complexity of the problem since, roughly speaking, reflects the fully nonlinear character of the HJB equation (see Rem. 2.5 for a comparison with the case of dominated models). This basic difficulty has prevented the effective use of BSDE techniques in the context of optimal control of PDMPs until now. In fact, we believe that this is the first time that this difficulty is coped with and this connection is established. It is our hope that the great development that BSDE theory has now gained will produce new results in the optimization theory of PDMPs. In the context of diffusions, probabilistic formula for the value function for non-dominated models have been discovered only in the recent year. In this sense, a fundamental role is played by [38], where a new class of BSDEs with nonpositive jumps is introduced in order to provide a probabilistic formula, known as nonlinear Feynman−Kac formula, for fully nonlinear integro-partial differential equations, associated to the classical optimal control for diffusions. This approach was later applied to many cases within optimal switching and impulse control problems, see [28,29,39], and developed with extensions and applications, see [12, 13, 18, 19, 30]. In all the above mentioned cases the controlled processes are diffusions constructed as solutions to stochastic differential equations of Itô type driven by a Brownian motion.

We wish to adapt to the PDMPs framework the theory developed in the context of optimal control for diffusions. The fundamental idea behind the derivation of the Feynman−Kac representation, borrowed from [38], concerns the so-called *randomization of the control*, that we are going to describe below in our framework. A first step in the generalization of this method to the non-diffusive processes context was done in [4], where a probabilistic representation for the value function associated to an optimal control problem for pure jump Markov processes was provided. As in the pure jump case, also in the PDMPs framework the correct formulation of the randomization method requires some efforts, and can not be modeled on the diffusive case, since the controlled processes are not defined as solutions to stochastic differential equations. In addition, the presence of the controlled flow between jumps in the PDMP's dynamics makes the treatment more difficult and suggests to use the viscosity solution theory. Finally, we notice that we consider PDMPs with unbounded state space E. This restriction is due to the fact that the presence of the boundary would induce technical difficulties on the study of the associated BSDE, which would be driven by a non quasi-left continuous random measure, see Remark 2.3. For such general BSDEs the existence and uniqueness results were at disposal only in particular frameworks, see *e.g.* [14] for the deterministic case, and counter-examples were provided in the general case, see Section 4.3 in [17]. Only recently this problem has been faced and solved in a general context in [2] (see also [5, 6]), where a technical condition is provided in order to achieve existence and uniqueness of the BSDE. The mentioned condition turns out to be verified in the case of control problems related to PDMPs with discontinuities at the boundary of the domain, see Remark 4.5 in [2]. This fact opens to the possibility to apply the BSDEs techniques also in this context, which is left as future development of the method.

Let us now informally describe the randomization method in the PDMPs framework. The first step, for any starting point $x \in E$, consists in replacing the state trajectory and the associated control process (X_s, α_s) by an (uncontrolled) PDMP (X_s, I_s). In particular, I is a process with values in the space of control actions A, whose intensity is given by a deterministic measure $\lambda_0(\mathrm{d}b)$, which is arbitrary but finite and with full support. The PDMP (X, I) is constructed on a different probability space by means of a new triplet of local characteristics and takes values on the enlarged state space $E \times A$. Let us denote by $\mathbb{P}^{x,a}$ the corresponding law, where (x, a) is the starting point in $E \times A$. Then we formulate an auxiliary optimal control problem where we control the intensity of the process I: for any predictable, bounded and positive random field $\nu_t(b)$, by means of a theorem of Girsanov type, we construct a probability measure $\mathbb{P}_\nu^{x,a}$ under which the compensator of I is given by the random measure $\nu_t(\mathrm{d}b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}t$ (under $\mathbb{P}_\nu^{x,a}$ the law of X is also changed) and we minimize the functional

$$J(x,a,\nu) = \mathbb{E}_\nu^{x,a}\left[\int_0^\infty \mathrm{e}^{-\delta\, s} f(X_s, I_s)\,\mathrm{d}s\right] \tag{1.5}$$

over all possible choices of ν. This will be called the *dual* control problem. Notice that the family $\{\mathbb{P}_\nu^{x,a}\}_\nu$ is a dominated model. One of our main results states that the value function of the dual control problem, denoted as $V^*(x,a)$, can be represented by means of a well-posed constrained BSDE. The latter is an equation over an infinite horizon of the form

$$\begin{aligned} Y_s^{x,a} &= Y_T^{x,a} - \delta \int_s^T Y_r^{x,a}\,\mathrm{d}r + \int_s^T f(X_r, I_r)\,\mathrm{d}r - (K_T^{x,a} - K_s^{x,a}) \\ &\quad - \int_s^T \int_A Z_r^{x,a}(X_r,\, b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}r - \int_s^T \int_{E\times A} Z_r^{x,a}(y,\, b)\, q(\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}b), \quad 0 \leqslant s \leqslant T < \infty, \end{aligned} \tag{1.6}$$

with unknown triplet $(Y^{x,a}, Z^{x,a}, K^{x,a})$ where q is the compensated random measure associated to (X, I), $K^{x,a}$ is a predictable increasing càdlàg process, $Z^{x,a}$ is a predictable random field, where we additionally add the sign constraint

$$Z_s^{x,a}(X_{s-}, b) \geqslant 0. \tag{1.7}$$

The reference filtration is now the canonical one associated to the pair (X, I). We prove that this equation has a unique maximal solution, in an appropriate sense, and that the value of the process $Y^{x,a}$ at the initial time represents the dual value function:

$$Y_0^{x,a} = V^*(x,a). \tag{1.8}$$

Our main purpose is to show that the maximal solution to (1.6)−(1.7) at the initial time also provides a Feynman−Kac representation to the value function (1.3) of our original optimal control problem for PDMPs. To this end, we introduce the deterministic real function on $E \times A$

$$v(x,a) := Y_0^{x,a}, \tag{1.9}$$

and we prove that v is a viscosity solution to (1.4). By the uniqueness of the solution to the HJB equation (1.4) we conclude that the value of the process Y at the initial time represents both the original and the dual value function:

$$Y_0^{x,a} = V^*(x,a) = V(x). \tag{1.10}$$

Identity (1.10) is the desired BSDE representation of the value function for the original control problem, and a nonlinear Feynman−Kac formula for the HJB equation (1.4).

Formula (1.10) can be used to design algorithms based on the numerical approximation of the solution to the constrained BSDE (1.6)−(1.7), and therefore to get probabilistic numerical approximations for the value function of the addressed optimal control problem. In the recent years there has been much interest in this problem, and numerical schemes for constrained BSDEs have been proposed and analyzed in the diffusive framework, see [36,37]. We hope that our results may be used to get similar methods in the PDMPs context as well.

The paper is organized as follows. Section 2 is dedicated to define a setting where the optimal control (1.3) is solved by means of the corresponding HJB equation (1.4). We start by recalling the construction of a PDMP given its local characteristics. In order to apply techniques based on BSDEs driven by general random measures, we work in a canonical setting and we use a specific filtration. The construction is based on the well-posedness of the martingale problem for multivariate marked point processes studied in [32], and is the object of Section 2.1. This general procedure is then applied in Section 2.2 to formulate in a precise way the optimal control problem we are interested in. At the end of Section 2.2 we recall a classical result on existence and uniqueness of the viscosity solution to the HJB equation (1.4), and its identification with the value function V, provided by [22].

In Section 3 we start to develop the control randomization method. Given suitable local characteristics, we introduce an auxiliary process (X, I) on $E \times A$ by relying on the construction in Section 2.1, and we formulate a dual optimal control problem for it under suitable conditions. The formulation of the randomized process is very different from the diffusive framework, since our data are the local characteristics of the process rather than the coefficients of some stochastic differential equations solved by it. In particular, we need to choose a specific probability space under which the pair (X, I) remains a PDMP.

In Section 4 we introduce the constrained BSDE (1.6)−(1.7) over infinite horizon. By a penalization approach, we prove that under suitable assumptions the above mentioned equation admits a unique maximal solution (Y, Z, K) in a certain class of processes. Moreover, the component Y at the initial time coincides with the value function V^* of the dual optimal control problem. This is the first of our main results, and is the object of Theorem 4.7.

In Section 5 we prove that the initial value of the maximal solution $Y^{x,a}$ to (1.6)−(1.7) provides a viscosity solution to (1.4). This is the second main result of the paper, which is stated in Theorem 5.1. As a consequence, by means of the uniqueness result for viscosity solutions to (1.4) recalled in Section 2.2, we get the desired nonlinear Feynman−Kac formula, as well as the equality between the value functions of the primal and the dual control problems, see Corollary 5.2. The proof of Theorem 5.1 is based on arguments from the viscosity theory, and combines BSDEs techniques with control-theoretic arguments. A relevant task is to derive the key property that the function v in (1.9) does not depend on a, as consequence of the A-nonnegative constrained jumps. Recalling the identification in Theorem 4.7, we are able to give a direct proof of the non-dependence of v on a by means of control-theoretic techniques, see Proposition 5.6 and the comments below. This allows us to consider very general spaces A of control actions. Moreover, differently from the previous literature, we provide a direct proof of the viscosity solution property of v, which does not rely on a penalized HJB equation. Indeed, we obtain a dynamic programming principle in the dual control framework and we directly derive from it the HJB equation, see Propositions 5.8 and 5.9.

Finally, for a better readability, some technical proofs have been reported in Section 6.

2. Piecewise deterministic controlled Markov Processes

2.1. The construction of a PDMP given its local characteristics

Given a topological space F, in the sequel $\mathcal{B}(F)$ will denote the Borel σ-field associated with F, and $\mathbb{C}_b(F)$ the set of all bounded continuous functions on F. The Dirac measure concentrated at some point $x \in F$ will be denoted δ_x.

Let E be a Borel space (*i.e.*, a topological space homeomorphic to a Borel subset of a compact metric space, see *e.g.* Def. 16-(a) in [23]), and $\mathcal{E}$ the corresponding σ-algebra. We will often need to construct a PDMP in E with a given triplet of local characteristics (ϕ, λ, Q). We assume that $\phi : \mathbb{R}_+ \times E \to E$ is a continuous function, $\lambda : E \mapsto \mathbb{R}_+$ is a nonnegative continuous function satisfying

$$\sup_{x \in E} \lambda(x) < \infty, \tag{2.1}$$

and that Q maps E into the set of probability measures on $(E, \mathcal{E})$, and is a stochastic Feller kernel, *i.e.*, for all $v \in \mathbb{C}_b(E)$, the map $x \mapsto \int_E v(y)\, Q(x, \mathrm{d}y)$ $(x \in E)$ is continuous.

We recall the main steps of the construction of a PDMP given its local characteristics. The existence of a Markovian process associated with the triplet (ϕ, λ, Q) is a well known fact (see, *e.g.*, [20, 21]). Nevertheless, we need special care in the choice of the corresponding filtration, since this will be crucial in sequel, when we will solve the associated BSDE and we will implicitly apply a version of the martingale representation theorem. For this reason, we will use an explicit construction that we are going to describe. Many of the techniques we are going to use are borrowed from the theory of multivariate (marked) point processes. We will often follow [32], but we also refer the reader to the treatise [33] for a more systematic exposition.

We start by constructing a suitable sample space to describe the jumping mechanism of the Markov process. Let Ω' denote the set of sequences $\omega' = (t_n, e_n)_{n\geq 1}$ in $((0,\infty)\times E)\cup\{(\infty,\Delta)\}$, where $\Delta \notin E$ is adjoined to E as an isolated point, satisfying in addition

$$t_n \leq t_{n+1}; \qquad t_n < \infty \implies t_n < t_{n+1}. \tag{2.2}$$

To describe the initial condition we will use the measurable space $(E,\mathcal{E})$. Finally, the sample space for the Markov process will be $\Omega = E \times \Omega'$. We define canonical functions $T_n : \Omega \to (0,\infty]$, $E_n : \Omega \to E \cup \{\Delta\}$ as follows: writing $\omega = (x,\omega')$ in the form $\omega = (x, t_1, e_1, t_2, e_2, \ldots)$ we set for $t \geq 0$ and for $n \geq 1$

$$T_n(\omega) = t_n, \qquad E_n(\omega) = e_n, \qquad T_\infty(\omega) = \lim_{n\to\infty} t_n, \qquad T_0(\omega) = 0, \qquad E_0(\omega) = x.$$

We also introduce, for any $B \in \mathcal{E}$, the counting process $N(s,B) = \sum_{n\in\mathbb{N}} \mathbb{1}_{T_n\leq s}\mathbb{1}_{E_n\in B}$, and we define the process $X : \Omega \times [0,\infty) \to E \cup \Delta$ setting

$$X_t = \begin{cases} \phi(t - T_n, E_n) & \text{if} \quad T_n \leq t < T_{n+1}, \quad \text{for} \quad n \in \mathbb{N}, \\ \Delta & \text{if} \quad t \geq T_\infty. \end{cases} \tag{2.3}$$

In Ω we introduce for all $t \geq 0$ the σ-algebras $\mathcal{G}_t = \sigma(N(s,B) : s \in (0,t], B \in \mathcal{E})$. To take into account the initial condition we also introduce the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, where $\mathcal{F}_0 = \mathcal{E} \otimes \{\emptyset, \Omega'\}$, and, for all $t \geq 0$, $\mathcal{F}_t$ is the σ-algebra generated by $\mathcal{F}_0$ and $\mathcal{G}_t$. $\mathbb{F}$ is right-continuous and will be called the natural filtration. In the following all concepts of measurability for stochastic processes (adaptedness, predictability *etc.*) refer to $\mathbb{F}$. We denote by $\mathcal{F}_\infty$ the σ-algebra generated by all the σ-algebras $\mathcal{F}_t$. The symbol $\mathcal{P}$ denotes the σ-algebra of $\mathbb{F}$-predictable subsets of $[0,\infty)\times\Omega$.

On the filtered sample space $(\Omega,\mathbb{F})$ we have so far introduced the canonical marked point process $(T_n,E_n)_{n\geq 1}$. The corresponding random measure p is, for any $\omega \in \Omega$, a σ-finite measure on $((0,\infty)\times E, \mathcal{B}(0,\infty)\otimes\mathcal{E})$ defined as

$$p(\omega, \mathrm{d}s\,\mathrm{d}y) = \sum_{n\in\mathbb{N}} \mathbb{1}_{\{T_n(\omega)<\infty\}}\, \delta_{(T_n(\omega),E_n(\omega))}(\mathrm{d}s\,\mathrm{d}y), \tag{2.4}$$

where δ_k denotes the Dirac measure at point $k \in (0,\infty)\times E$. For notational convenience the dependence on ω will be suppressed and, instead of $p(\omega,\mathrm{d}s\,\mathrm{d}y)$, it will be written $p(\mathrm{d}s\,\mathrm{d}y)$.

Proposition 2.1. *Assume that (2.1) holds, and fix $x \in E$. Then there exists a unique probability measure on $(\Omega,\mathcal{F}_\infty)$, denoted by $\mathbb{P}^x$, such that its restriction to $\mathcal{F}_0$ is δ_x, and the $\mathbb{F}$-compensator of the measure p under $\mathbb{P}^x$ is the random measure*

$$\tilde{p}(\mathrm{d}s\,\mathrm{d}y) = \sum_{n\in\mathbb{N}} \mathbb{1}_{[T_n,\,T_{n+1})}(s)\,\lambda(\phi(s-T_n,E_n))\,Q(\phi(s-T_n,E_n),\mathrm{d}y)\,\mathrm{d}s.$$

Moreover, $\mathbb{P}^x(T_\infty = \infty) = 1$.

Proof. The result is a direct application of Theorem 3.6 in [32]. The fact that, $\mathbb{P}^x$-a.s., $T_\infty = \infty$ follows from the boundedness of λ, see Proposition 24.6 in [21]. □

For fixed $x \in E$, the sample path of the process X in (2.3) under $\mathbb{P}^x$ can be defined iteratively, by means of (ϕ,λ,Q), in the following way. Set

$$F(s,x) = \exp\left(-\int_0^s \lambda(\phi(r,x))\,\mathrm{d}r\right), \quad s \geq 0.$$

We have

$$\mathbb{P}^x(T_1 > s) = F(s,x), \quad s \geq 0, \tag{2.5}$$

$$\mathbb{P}^x(X_{T_1} \in B | \, T_1) = Q(x,B), \quad B \in \mathcal{E}, \tag{2.6}$$

on $\{T_1 < \infty\}$, and, for every $n \geq 1$,

$$\mathbb{P}^x(T_{n+1} - T_n > s \,|\, \mathcal{F}_{T_n}) = \exp\left(- \int_{T_n}^{T_n+s} \lambda(\phi(r - T_n, X_{T_n}))\, \mathrm{d}r \right), \quad s \geq 0, \tag{2.7}$$

$$\mathbb{P}^x(X_{T_{n+1}} \in B | \, \mathcal{F}_{T_n}, T_{n+1}) = Q(\phi(T_{n+1} - T_n, X_{T_n}), B), \quad B \in \mathcal{E}, \tag{2.8}$$

on $\{T_n < \infty\}$. In particular, it follows from (2.7) that

$$\mathbb{P}^x(T_{n+1} > s \,|\, \mathcal{F}_{T_n}) = \exp\left(- \int_{T_n}^{s} \lambda(\phi(r - T_n, X_{T_n}))\, \mathrm{d}r \right), \quad \forall\, s \geq T_n. \tag{2.9}$$

Proposition 2.2. *In the probability space $\{\Omega, \mathcal{F}_\infty, \mathbb{P}^x\}$ the process X has distribution δ_x at time zero, and it is a homogeneous Markov process, i.e., for any $x \in E$, nonnegative times t, s, $t \leq s$, and for every bounded measurable function f,*

$$\mathbb{E}^x[f(X_{t+s}) \,|\, \mathcal{F}_t] = P_s(f(X_t)), \tag{2.10}$$

where $P_t f(x) := \mathbb{E}^x[f(X_t)]$.

Proof. From (2.9), taking into account the semigroup property $\phi(t+s,x) = \phi(t, \phi(s,x))$, for any $s \geq 0$ we have

$$\begin{aligned}
\mathbb{P}^x(T_{n+1} > t+s \,|\, \mathcal{F}_t)\, \mathbb{1}_{\{t \in [T_n, T_{n+1})\}} &= \frac{\mathbb{P}^x(T_{n+1} > t+s \,|\, \mathcal{F}_{T_n})}{\mathbb{P}^x(T_{n+1} > t \,|\, \mathcal{F}_{T_n})}\, \mathbb{1}_{\{t \in [T_n, T_{n+1})\}} \\
&= \exp\left(- \int_t^{t+s} \lambda(\phi(r - T_n, X_{T_n}))\, \mathrm{d}r \right) \mathbb{1}_{\{t \in [T_n, T_{n+1})\}} \\
&= \exp\left(- \int_0^{s} \lambda(\phi(r + t - T_n, X_{T_n}))\, \mathrm{d}r \right) \mathbb{1}_{\{t \in [T_n, T_{n+1})\}} \\
&= \exp\left(- \int_0^{s} \lambda(\phi(r, X_t))\, \mathrm{d}r \right) \mathbb{1}_{\{t \in [T_n, T_{n+1})\}} \\
&= F(s, X_t)\, \mathbb{1}_{\{t \in [T_n, T_{n+1})\}}.
\end{aligned} \tag{2.11}$$

Hence, denoting $N_t = N(t, E)$, it follows from (2.11) that

$$\mathbb{P}^x(T_{N_t+1} > t+s \,|\, \mathcal{F}_t) = F(s, X_t);$$

in other words, conditional on $\mathcal{F}_t$, the jump time after t of a PDMP started at x has the same distribution as the first jump time of a PDMP started at X_t. Since the remaining interarrival times and postjump positions are independent on the past, we have shown that (2.10) holds for every bounded measurable function f. □

Remark 2.3. In the present paper we restrict the analysis to the case of PDMPs on an unbounded domain E. This choice is motivated by the fact that the presence of jumps at the boundary of the domain would induce discontinuities in the compensator of the random measure associated to the process. Since we have in mind to apply techniques based on BSDEs driven by the compensated random measure associated to the PDMP (see Sect. 4), this fact would considerably complicates the tractation.

More precisely, consider a PDMP on an open state space E with boundary ∂E. In this case, when the process reaches the boundary a forced jump occurs, and the process immediately goes back to the interior of the domain. According to (26.2) in [21], the compensator of the counting measure p in (2.4) admits the form

$$\tilde{p}(\mathrm{d}s\, \mathrm{d}y) = \lambda(X_{s-})\, Q(X_{s-}, \mathrm{d}y)\, \mathbb{1}_{\{X_{s-} \in E\}}\, \mathrm{d}s + R(X_{s-}, \mathrm{d}y)\, \mathbb{1}_{\{X_{s-} \in \Gamma\}}\, \mathrm{d}p^*_s,$$

where

$$p_s^* = \sum_{n=1}^{\infty} \mathbb{1}_{\{s \geq T_n\}} \, \mathbb{1}_{\{X_{T_n -} \in \Gamma\}}$$

is the process counting the number of jumps of X from the active boundary $\Gamma \in \partial E$ (for the precise definition of Γ see p. 61 in [21]), and R defined on $\partial E \times \mathcal{E}$ is the transition probability measure describing the distribution of the process after the forced jumps from the boundary. In particular, the compensator $\tilde{p}$ can be rewritten as

$$\tilde{p}(\mathrm{d}s\, \mathrm{d}y) = \Phi(X_{s-}, \mathrm{d}y)\, \mathrm{d}A_s,$$

where $\Phi(X_{s-}, \mathrm{d}y) = Q(X_{s-}, \mathrm{d}y)\, \mathbb{1}_{\{X_{s-} \in E\}} + R(X_{s-}, \mathrm{d}y)\, \mathbb{1}_{\{X_{s-} \in \Gamma\}}$, and $\mathrm{d}A_s = \lambda(X_{s-})\, \mathbb{1}_{\{X_{s-} \in E\}}\, \mathrm{d}s + \mathrm{d}p_s^*$ is a predictable and discontinuous process, with jumps $\Delta A_s = \mathbb{1}_{\{X_{s-} \in \Gamma\}}$. The presence of these discontinuities in the compensator of p induces very technical difficulties in the study of the associated BSDE, see the recent paper [2]. The above mentioned case is left as a future improvement of the theory.

2.2. Optimal control of PDMPs

In the present section we aim at formulating an optimal control problem for piecewise deterministic Markov processes, and to discuss its solvability. The PDMP state space E will be an open subset of $\mathbb{R}^d$, and $\mathcal{E}$ the corresponding σ-algebra. In addition, we introduce a Borel space A, endowed with its σ-algebra $\mathcal{A}$, called the space of control actions. The additional hypothesis that A is compact is not necessary for the majority of the results, and will be explicitly asked whenever needed. The other data of the problem consist in three functions f, h and λ on $E \times A$, and in a probability transition Q from $(E \times A, \mathcal{E} \otimes \mathcal{A})$ to $(E, \mathcal{E})$, satisfying the following conditions.

(HhλQ)

(i) $h : E \times A \mapsto E$ is a bounded, uniformly continuous, function satisfying

$$\begin{cases} \forall x,\, x' \in E, \quad \text{and } \forall a, a' \in A, \qquad |h(x,a) - h(x',a')| \leqslant L_h\, (|x - x'| + |a - a'|), \\ \forall x \in E \quad \text{and} \quad \forall a \in A, \qquad |h(x,a)| \leqslant M_h, \end{cases}$$

where L_h and M_h are constants independent of $a, a' \in A$, $x, x' \in E$.

(ii) $\lambda : E \times A \mapsto \mathbb{R}^+$ is a nonnegative, bounded, uniformly continuous function, satisfying

$$\sup_{(x,a) \in E \times A} \lambda(x,a) < \infty. \tag{2.12}$$

(iii) Q maps $E \times A$ into the set of probability measures on $(E, \mathcal{E})$, and is a stochastic Feller kernel. *i.e.*, for all $v \in \mathbb{C}_b(E)$, the map $(x,a) \mapsto \int_{\mathbb{R}^d} v(y)\, Q(x,a,\mathrm{d}y)$ is continuous (hence it belongs to $\mathbb{C}_b(E \times A)$).

(Hf) $f : E \times A \mapsto \mathbb{R}^+$ is a nonnegative, bounded, uniformly continuous function. In particular, there exists a positive constant M_f such that

$$0 \leqslant f(x,a) \leqslant M_f, \quad \forall\, x \in E,\, a \in A.$$

The requirement that $Q(x, a, \{x\}) = 0$ for all $x \in E$, $a \in A$ is natural in many applications, but here is not needed. h, λ and Q depend on the control parameter $a \in A$ and play respectively the role of and controlled drift, controlled jump rate and controlled probability transition. Roughly speaking, we may control the dynamics of the process by changing dynamically its deterministic drift, its jump intensity and its post jump distribution.

Let us give a more precise definition of the optimal control problem under study. To this end, we first construct Ω, $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, $\mathcal{F}_\infty$ as in the previous paragraph.

We will consider the class of *piecewise open-loop controls*, first introduced in [44] and often adopted in this context, see for instance [1, 20, 21]. Let X be the (uncontrolled) process constructed in a canonical way from a marked point process (T_n, E_n) as in Section 2.1. The class of admissible control law $\mathcal{A}_{ad}$ is the set of all Borel-measurable maps $\alpha : [0, \infty) \times E \to A$, and the control applied to X is of the form:

$$\alpha_t = \alpha_0(t, x)\, \mathbb{1}_{[0,\, T_1)}(t) + \sum_{n=1}^{\infty} \alpha_n(t - T_n, E_n)\, \mathbb{1}_{[T_n,\, T_{n+1})}(t). \tag{2.13}$$

In other words, at each jump time T_n, we choose an open loop control α_n depending on the initial condition E_n to be used until the next jump time.

By abuse of notation, we define the controlled process $X : \Omega \times [0, \infty) \to E \cup \{\Delta\}$ setting

$$X_t = \begin{cases} \phi^{\alpha_0}(t, x) & \text{if} \quad t \in [0,\, T_1), \\ \phi^{\alpha_n}(t - T_n, E_n) & \text{if} \quad t \in [T_n,\, T_{n+1}),\ n \in \mathbb{N} \setminus \{0\}, \end{cases} \tag{2.14}$$

where $\phi^{\beta}(t, x)$ is the unique solution to the ordinary differential equation

$$\dot{x}(t) = h(x(t), \beta(t)), \quad x(0) = x \in E,$$

with β an $\mathcal{A}$-measurable function. Then, for every starting point $x \in E$ and for each $\alpha \in \mathcal{A}_{ad}$, by Proposition 2.1 there exists a unique probability measure on $(\Omega, \mathcal{F}_\infty)$, denoted by $\mathbb{P}^x_\alpha$, such that its restriction to $\mathcal{F}_0$ is δ_x, and the $\mathbb{F}$-compensator under $\mathbb{P}^x_\alpha$ of the measure $p(\mathrm{d}s\, \mathrm{d}y)$ is

$$\tilde{p}^{\alpha}(\mathrm{d}s\, \mathrm{d}y) = \sum_{n=1}^{\infty} \mathbb{1}_{[T_n,\, T_{n+1})}(s)\, \lambda(X_s, \alpha_n(s - T_n, E_n))\, Q(X_s, \alpha_n(s - T_n, E_n), \mathrm{d}y)\, \mathrm{d}s.$$

According to Proposition 2.2, under $\mathbb{P}^x_\alpha$ the process X in (2.14) is Markovian with respect to $\mathbb{F}$.

Denoting by $\mathbb{E}^x_\alpha$ the expectation under $\mathbb{P}^x_\alpha$, we finally define, for $x \in E$ and $\alpha \in \mathcal{A}_{ad}$, the functional cost

$$J(x, \alpha) = \mathbb{E}^x_\alpha \left[\int_0^\infty \mathrm{e}^{-\delta\, s}\, f(X_s, \alpha_s)\, \mathrm{d}s \right] \tag{2.15}$$

and the value function of the control problem

$$V(x) = \inf_{\alpha \in \mathcal{A}_{ad}} J(x, \alpha), \tag{2.16}$$

where $\delta \in (0, \infty)$ is a discounting factor that will be fixed from here on. By the boundedness assumption on f, both J and V are well defined and bounded.

Let us consider the Hamilton−Jacobi−Bellman equation (for short, HJB equation) associated to the optimal control problem: this is the following elliptic nonlinear equation on $[0, \infty) \times E$:

$$H^v(x, v, Dv) = 0, \tag{2.17}$$

where

$$H^{\psi}(z, v, p) = \sup_{a \in A} \left\{ \delta\, v - h(z, a) \cdot p - \int_E (\psi(y) - \psi(z))\, \lambda(z, a)\, Q(z, a, \mathrm{d}y) - f(z, a) \right\}.$$

Remark 2.4. The HJB equation (2.17) can be rewritten as

$$\delta\, v(x) = \inf_{a \in A} \{\mathcal{L}^a v(x) + f(x, a)\} = 0, \tag{2.18}$$

where $\mathcal{L}^a$ is the operator depending on $a \in A$ defined as

$$\mathcal{L}^a v(x) := h(x, a) \cdot \nabla v(x) + \lambda(x, a) \int_E (v(y) - v(x))\, Q(x, a, \mathrm{d}y). \tag{2.19}$$

Remark 2.5. A different way to tackle optimal control problems for jump processes consists in dealing with dominated models, *i.e.*, in considering controlled processes that have laws that are absolutely continuous with respect to the law of a given, uncontrolled process. The corresponding optimal control problems are sometimes called *intensity control problems*, and are formulated by means of a change of probability measure of Girsanov type, see *e.g.* [10, 27]. This type of models have been considered for instance in [15, 16] in the pure jump case, and in [3] in the semi-Markov framework. In the PDMPs context, this would correspond to construct $\mathbb{P}^x_\alpha$ in such a way that, under $\mathbb{P}^x_\alpha$, the process X would have the same (uncontrolled) flow $\phi(t,x)$ and transition measure $Q(x,\mathrm{d}y)$, while the intensity $\lambda(x)$ would be multiplied by $r(x,\alpha_t,y)$, with r some function given in advance as another datum of the problem. Compared to the non-dominated model, the complexity of the problem decreases considerably. Indeed, the corresponding HJB equation would reduce to

$$\delta v(x) = \tilde{\mathcal{L}}v(x) + \tilde{f}(x, v(y)-v(x)), \quad x \in E,$$

where $\tilde{\mathcal{L}}$ denotes the linear operator

$$\tilde{\mathcal{L}}v(x) := h(x)\cdot\nabla v(x) + \lambda(x)\int_E (v(y)-v(x))\,Q(x,\mathrm{d}y),$$

and $\tilde{f}$ is the Hamiltonian function

$$\tilde{f}(x,z(\cdot)) := \lambda(x)\,\inf_{a\in A}\left\{\int_E z(y)\,(r(x,a,y)-1)\,Q(x,\mathrm{d}y)\right\}.$$

Let us recall the following facts. Given a locally bounded function $z : E \to \mathbb{R}$, we define its lower semicontinuous (l.s.c. for short) envelope z_*, and its upper semicontinuous (u.s.c. for short) envelope z^*, by

$$z_*(x) = \liminf_{\substack{y\to x\\ y\in E}} z(y), \qquad z^*(x) = \limsup_{\substack{y\to x\\ y\in E}} z(y), \quad \text{for all } x\in E.$$

Definition 2.6. Viscosity solution to (2.17).

(i) A locally bounded u.s.c. function w on E is called a *viscosity supersolution* (resp. *viscosity subsolution*) of (2.17) if

$$H^w(x_0, w(x_0), D\varphi(x_0)) \geqslant \ (\textit{resp.}\ \leqslant)\ 0.$$

for any $x_0 \in E$ and for any $\varphi\in C^1(E)$ such that

$$(u-\varphi)(x_0) = \min_E (u-\varphi)\ (\textit{resp.}\ \max_E (u-\varphi)).$$

(ii) A function z on E is called a *viscosity solution* of (2.17) if it is locally bounded and its u.s.c. and l.s.c. envelopes are respectively subsolution and supersolution of (2.17).

The HJB equation (2.17) admits a unique viscosity solution, which coincides with the value function V in (2.16). The following result is stated in Theorem 7.5 in [22].

Theorem 2.7. *Let* **(HhλQ)** *and* **(Hf)** *hold, and assume that A is compact. Then the value function V of the PDMPs optimal control problem is the unique viscosity solution to* (2.17). *Moreover, V is continuous.*

3. Control randomization and dual optimal control problem

In this section we start to implement the control randomization method. In the first step, for an initial time $t \geq 0$ and a starting point $x \in E$, we construct an (uncontrolled) PDMP (X, I) with values in $E \times A$ by specifying its local characteristics, see (3.1)–(3.3) below. Next we formulate an auxiliary optimal control problem where, roughly speaking, we optimize a functional cost by modifying the intensity of the process I over a suitable family of probability measures.

This dual problem is studied in Section 4 by means of a suitable class of BSDEs. In Section 5 we will show that the same class of BSDEs provides a probabilistic representation of the value function introduced in the previous section. As a byproduct, we also get that the dual value function coincides with the one associated to the original optimal control problem.

3.1. A dual control system

Let E still denote an open subset of $\mathbb{R}^d$ with σ-algebra $\mathcal{E}$, and A be a Borel space with corresponding σ-algebra $\mathcal{A}$. Let moreover h, λ and Q be respectively two real functions on $E \times A$ and a probability transition from $(E \times A, \mathcal{E} \otimes \mathcal{A})$, satisfying **(Hh$\lambda$Q)** as before. We denote by $\phi(t, x, a)$ the unique solution to the ordinary differential equation

$$\dot{x}(t) = h(x(t), a), \quad x(0) = x \in E, \;\; a \in A.$$

In particular, $\phi(t, x, a)$ corresponds to the function $\phi^{\beta}(t, x)$, introduced in Section 2.2, when $\beta(t) \equiv a$. Let us now introduce another finite measure λ_0 on $(A, \mathcal{A})$ satisfying the following assumption:
(Hλ_0) λ_0 is a finite measure on $(A, \mathcal{A})$ with full topological support.
The existence of such a measure is guaranteed by the fact that A is a separable space with metrizable topology. We define

$$\tilde{\phi}(t, x, a) := (\phi(t, x, a), \quad a), \tag{3.1}$$

$$\tilde{\lambda}(x, a) := \lambda(x, a) + \lambda_0(A), \tag{3.2}$$

$$\tilde{Q}(x, a, \mathrm{d}y\,\mathrm{d}b) := \frac{\lambda(x, a)\, Q(x, a, \mathrm{d}y)\, \delta_a(\mathrm{d}b) + \lambda_0(\mathrm{d}b)\, \delta_x(\mathrm{d}y)}{\tilde{\lambda}(x, a)}. \tag{3.3}$$

We wish to construct a PDMP (X, I) as in Section 2.1 but with enlarged state space $E \times A$ and local characteristics $(\tilde{\phi}, \tilde{\lambda}, \tilde{Q})$. Firstly, we need to introduce a suitable sample space to describe the jump mechanism of the process (X, I) on $E \times A$. Accordingly, we set Ω' as the set of sequences $\omega' = (t_n, e_n, a_n)_{n \geq 1}$ contained in $((0, \infty) \times E \times A) \cup \{(\infty, \Delta, \Delta')\}$, where $\Delta \notin E$ (resp. $\Delta' \notin A$) is adjoined to E (resp. to A) as an isolated point, satisfying (2.2). In the sample space $\Omega = E \times A \times \Omega'$ we define the random variables $T_n : \Omega \to (0, \infty]$, $E_n : \Omega \to E \cup \{\Delta\}$, $A_n : \Omega \to A \cup \{\Delta'\}$, as follows: writing $\omega = (x, a, \omega')$ in the form $\omega = (x, a, t_1, e_1, a_1, t_2, e_2, a_2, \ldots)$, we set for $t \geq 0$ and for $n \geq 1$,

$$T_n(\omega) = t_n, \qquad T_\infty(\omega) = \lim_{n \to \infty} t_n, \qquad T_0(\omega) = 0,$$

$$E_n(\omega) = e_n, \qquad A_n(\omega) = a_n, \qquad E_0(\omega) = x, \qquad A_0(\omega) = a.$$

We define the process (X, I) on $(E \times A) \cup \{\Delta, \Delta'\}$ setting

$$(X, I)_t = \begin{cases} (\phi(t - T_n, E_n, A_n), A_n) & \text{if} \quad T_n \leq t < T_{n+1}, \text{ for } n \in \mathbb{N}, \\ (\Delta, \Delta') & \text{if} \quad t \geq T_\infty. \end{cases} \tag{3.4}$$

In Ω we introduce for all $t \geq 0$ the σ-algebras $\mathcal{G}_t = \sigma(N(s, G) \,:\, s \in (0, t], G \in \mathcal{E} \otimes \mathcal{A})$ generated by the counting processes $N(s, G) = \sum_{n \in \mathbb{N}} \mathbb{1}_{T_n \leq s} \mathbb{1}_{(E_n, A_n) \in G}$, and the σ-algebra $\mathcal{F}_t$ generated by $\mathcal{F}_0$ and $\mathcal{G}_t$, where $\mathcal{F}_0 = \mathcal{E} \otimes \mathcal{A} \otimes \{\emptyset, \Omega'\}$. We still denote by $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and $\mathcal{P}$ the corresponding filtration and predictable σ-algebra. The random measure p is now defined on $(0, \infty) \times E \times A$ as

$$p(\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}b) = \sum_{n\in\mathbb{N}} \mathbb{1}_{\{T_n, E_n, A_n\}}(\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}b). \tag{3.5}$$

Given any starting point $(x, a) \in E\times A$, by Proposition 2.1, there exists a unique probability measure on $(\Omega, \mathcal{F}_\infty)$, denoted by $\mathbb{P}^{x,a}$, such that its restriction to $\mathcal{F}_0$ is $\delta_{(x,a)}$ and the $\mathbb{F}$-compensator of the measure $p(\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}b)$ under $\mathbb{P}^{x,a}$ is the random measure

$$\tilde{p}(\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}b) = \sum_{n\in\mathbb{N}} \mathbb{1}_{[T_n,\, T_{n+1})}(s)\, \Lambda(\phi(s - T_n, E_n, A_n), A_n, \mathrm{d}y\,\mathrm{d}b)\,\mathrm{d}s,$$

where

$$\Lambda(x, a, \mathrm{d}y\,\mathrm{d}b) = \lambda(x, a)\, Q(x, a, \mathrm{d}y)\, \delta_a(\mathrm{d}b) + \lambda_0(\mathrm{d}b)\, \delta_x(\mathrm{d}y), \quad \forall (x, a) \in E \times A.$$

We denote by $q = p - \tilde{p}$ the compensated martingale measure associated to p.

As in Section 2.1, the sample path of a process (X, I) with values in $E \times A$, starting from a fixed initial point $(x, a) \in E \times A$ at time zero, can be defined iteratively by means of its local characteristics $(\tilde{\phi}, \tilde{\lambda}, \tilde{Q})$ in the following way. Set

$$F(s, x, a) = \exp\left(-\int_0^s (\lambda(\phi(r, x, a), a) + \lambda_0(A))\,\mathrm{d}r\right).$$

We have

$$\mathbb{P}^{x,a}(T_1 > s) = F(s, x, a), \quad s \geq 0, \tag{3.6}$$

$$\mathbb{P}^{x,a}(X_{T_1} \in B, I_{T_1} \in C|\, T_1) = \tilde{Q}(x, B \times C), \quad B \in \mathcal{E},\, C \in \mathcal{A}, \tag{3.7}$$

on $\{T_1 < \infty\}$, and, for every $n \geq 1$,

$$\mathbb{P}^{x,a}(T_{n+1} > s \,|\, \mathcal{F}_{T_n}) = \exp\left(-\int_{T_n}^s (\lambda(\phi(r - T_n, X_{T_n}, I_{T_n}), I_{T_n}) + \lambda_0(A))\,\mathrm{d}r\right), \quad s \geq T_n, \tag{3.8}$$

$$\mathbb{P}^{x,a}(X_{T_{n+1}} \in B, I_{T_{n+1}} \in C|\, \mathcal{F}_{T_n}, T_{n+1}) = \tilde{Q}(\phi(T_{n+1} - T_n, X_{T_n}, I_{T_n}), I_{T_n}, B \times C), \quad B \in \mathcal{E},\, C \in \mathcal{A}, \tag{3.9}$$

on $\{T_n < \infty\}$.

Finally, an application of Proposition 2.2 provides that (X, I) is a Markov process on $[0, \infty)$ with respect to $\mathbb{F}$. For every real-valued function defined on $E \times A$, the infinitesimal generator is given by

$$\mathcal{L}\varphi(x, a) := h(x, a) \cdot \nabla_x \varphi(x, a) + \int_E (\varphi(y, a) - \varphi(x, a))\, \lambda(x, a)\, Q(x, a, \mathrm{d}y) + \int_A (\varphi(x, b) - \varphi(x, a))\, \lambda_0(\mathrm{d}b).$$

For our purposes, it will be not necessary to specify the domain of the previous operator (for its formal definition we refer to Thm. 26.14 in [21]); in the sequel the operator $\mathcal{L}$ will be applied to test functions with suitable regularity.

3.2. The dual optimal control problem

We now introduce a dual optimal control problem associated to the process (X, I), and formulated in a weak form. For fixed (x, a), we consider a family of probability measures $\{\mathbb{P}^{x,a}_\nu,\, \nu \in \mathcal{V}\}$ in the space $(\Omega, \mathcal{F}_\infty)$, whose effect is to change the stochastic intensity of the process (X, I).

Let us proceed with precise definitions. We still assume that **(HhλQ)**, **(Hλ_0)** and **(Hf)** hold. We recall that $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$ is the augmentation of the natural filtration generated by p in (3.5). We define

$$\mathcal{V} = \{\nu : \Omega \times [0, \infty) \times A \to (0, \infty)\ \mathcal{P} \otimes \mathcal{A}\text{-measurable and bounded}\}.$$

For every $\nu \in \mathcal{V}$, we consider the predictable random measure

$$\tilde{p}^\nu(ds\,\mathrm{d}y\,\mathrm{d}b) := \nu_s(b)\,\lambda_0(\mathrm{d}b)\,\delta_{\{X_{s-}\}}(\mathrm{d}y)\,\mathrm{d}s + \lambda(X_{s-},\, I_{s-})\,Q(X_{s-},\, I_{s-},\, \mathrm{d}y)\,\delta_{\{I_{s-}\}}(\mathrm{d}b)\,\mathrm{d}s. \tag{3.10}$$

In particular, by the Radon Nikodym theorem one can find two nonnegative functions d_1, d_2 defined on $\Omega \times [0,\,\infty) \times E \times A$, $\mathcal{P} \otimes \mathcal{E} \otimes \mathcal{A}$, such that

$$\begin{aligned} \lambda_0(\mathrm{d}b)\,\delta_{\{X_{t-}\}}(\mathrm{d}y)\,\mathrm{d}t &= d_1(t,y,b)\,\tilde{p}(\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}b) \\ \lambda(X_{t-},\, I_{t-},\, \mathrm{d}y)\,\delta_{\{I_{t-}\}}(\mathrm{d}b)\,\mathrm{d}t &= d_2(t,y,b)\,\tilde{p}(\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}b), \\ d_1(t,y,b) + d_2(t,y,b) &= 1, \qquad \tilde{p}(\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}b)\text{-a.e.} \end{aligned}$$

and we have $d\tilde{p}^\nu = (\nu\, d_1 + d_2)\, d\tilde{p}$. For any $\nu \in \mathcal{V}$, consider then the Doléans$-$Dade exponential local martingale L^ν defined

$$\begin{aligned} L_s^\nu &= \exp\left(\int_0^s \int_{E\times A} \log(\nu_r(b)\,d_1(r,y,b) + d_2(r,y,b))\,p(\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}b) - \int_0^s \int_A (\nu_r(b)-1)\lambda_0(\mathrm{d}b)\,\mathrm{d}r\right) \\ &= \mathrm{e}^{\int_0^s \int_A (1-\nu_r(b))\lambda_0(\mathrm{d}b)\,\mathrm{d}r} \prod_{n\geqslant 1: T_n \leqslant s} (\nu_{T_n}(A_n)\,d_1(T_n,E_n,A_n) + d_2(T_n,E_n,A_n)), \end{aligned} \tag{3.11}$$

for $s \geq 0$. When $(L_t^\nu)_{t\geq 0}$ is a true martingale, for every time $T > 0$ we can define a probability measure $\mathbb{P}_{\nu,T}^{x,a}$ equivalent to $\mathbb{P}^{x,a}$ on $(\Omega,\, \mathcal{F}_T)$ by

$$\mathbb{P}_{\nu,T}^{x,a}(\mathrm{d}\omega) = L_T^\nu(\omega)\,\mathbb{P}^{x,a}(\mathrm{d}\omega). \tag{3.12}$$

By the Girsanov theorem for point processes (see Thm. 4.5 in [32]), the restriction of the random measure p to $(0,T] \times E \times A$ admits $\tilde{p}^\nu = (\nu\, d_1 + d_2)\,\tilde{p}$ as compensator under $\mathbb{P}_{\nu,T}^{x,a}$. We set $q^\nu := p - \tilde{p}^\nu$, and we denote by $\mathbb{E}_{\nu,T}^{x,a}$ the expectation operator under $\mathbb{P}_{\nu,T}^{x,a}$. Previous considerations are formalized in the following lemma, which is a direct consequence of Lemma 3.2 in [4].

Lemma 3.1. *Let assumptions* **(HhλQ)** *and* **(Hλ_0)** *hold. Then, for every* $(x,a) \in E \times A$ *and* $\nu \in \mathcal{V}$, *under the probability* $\mathbb{P}^{x,a}$, *the process* $(L_t^\nu)_{t\geq 0}$ *is a martingale. Moreover, for every time* $T > 0$, L_T^ν *is square integrable, and, for every* $\mathcal{P}_T \otimes \mathcal{E} \otimes \mathcal{A}$*-measurable function* $H : \Omega \times [0,T] \times E \times A \to \mathbb{R}$ *such that* $\mathbb{E}^{x,a}\left[\int_0^T \int_{E\times A} |H_s(y,b)|^2\,\tilde{p}(ds\,\mathrm{d}y\,\mathrm{d}b)\right] < \infty$, *the process* $\int_0^\cdot \int_{E\times A} H_s(y,b)\,q^\nu(ds\,\mathrm{d}y\,\mathrm{d}b)$ *is a* $\mathbb{P}_{\nu,T}^{x,a}$*-martingale on* $[0,T]$.

We aim at extending the previous construction to the infinite horizon, in order to get a suitable probability measure on $(\Omega, \mathcal{F}_\infty)$. We have the following result, which is essentially based on the Kolmogorov extension theorem for product spaces, see *e.g.* Theorem 1.1.10 in [43].

Proposition 3.2. *Let assumptions* **(HhλQ)** *and* **(Hλ_0)** *hold. Then, for every* $(x,a) \in E \times A$ *and* $\nu \in \mathcal{V}$, *there exists a unique probability* $\mathbb{P}_\nu^{x,a}$ *on* $(\Omega, \mathcal{F}_\infty)$, *under which the random measure* $\tilde{p}^\nu$ *in* (3.10) *is the compensator of the measure* p *in* (3.5) *on* $(0,\,\infty) \times E \times A$. *Moreover, for any time* $T > 0$, *the restriction of* $\mathbb{P}_\nu^{x,a}$ *on* $(\Omega, \mathcal{F}_T)$ *is given by the probability measure* $\mathbb{P}_{\nu,T}^{x,a}$ *in* (3.12).

Proof. See Section 6.1. □

Finally, for every $x \in E$, $a \in A$ and $\nu \in \mathcal{V}$, we introduce the dual functional cost

$$J(x,a,\nu) := \mathbb{E}_\nu^{x,a}\left[\int_0^\infty \mathrm{e}^{-\delta\, t}\, f(X_t, I_t)\,\mathrm{d}t\right], \tag{3.13}$$

and the dual value function

$$V^*(x,a) := \inf_{\nu\in\mathcal{V}} J(x,a,\nu), \tag{3.14}$$

where $\delta > 0$ in (3.13) is the discount factor introduced in Section 2.2.

4. Constrained BSDEs and the dual value function representation

In this section we introduce a BSDE with a sign constrain on its martingale part, for which we prove the existence and uniqueness of a maximal solution, in an appropriate sense. This constrained BSDE is then used to give a probabilistic representation formula for the dual value function introduced in (3.14).

Throughout this section we still assume that **(HhλQ)**, **(Hλ_0)** and **(Hf)** hold. The random measures p, $\tilde p$ and q, as well as the dual control setting $\Omega, \mathbb{F}, (X,I), \mathbb{P}^{x,a}$, are the same as in Section 3.1. We recall that $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$ is the augmentation of the natural filtration generated by p, and that $\mathcal{P}_T$, $T > 0$, denotes the σ-field of $\mathbb{F}$-predictable subsets of $[0,\,T]\times\Omega$.

For any $(x,a)\in E\times A$ we introduce the following notation.

- $\mathbf{L^2_{x,a}}(\mathcal{F}_\tau)$, the set of $\mathcal{F}_\tau$-measurable random variables ξ such that $\mathbb{E}^{x,a}\left[|\xi|^2\right] < \infty$; here $\tau \geqslant 0$ is an $\mathbb{F}$-stopping time.
- $\mathbf{S}^\infty$ the set of real-valued càdlàg adapted processes $Y = (Y_t)_{t\geqslant 0}$ which are uniformly bounded.
- $\mathbf{S^2_{x,a}(0,T)}$, $T > 0$, the set of real-valued càdlàg adapted processes $Y = (Y_t)_{0\leqslant t\leqslant T}$ satisfying

$$||Y||_{\mathbf{S^2_{x,a}(0,\,T)}} := \mathbb{E}^{x,a}\left[\sup_{0\leqslant t\leqslant T}|Y_t|^2\right] < \infty.$$

- $\mathbf{L^2_{x,a}}(\mathbf{0},\mathbf{T})$, $T > 0$, the set of real-valued progressive processes $Y = (Y_t)_{0\leqslant t\leqslant T}$ such that

$$||Y||^2_{\mathbf{L^2_{x,a}(0,\,T)}} := \mathbb{E}^{x,a}\left[\int_0^T |Y_t|^2\,\mathrm{d}t\right] < \infty.$$

  We also define $\mathbf{L^2_{x,a,loc}} := \cap_{T>0}\mathbf{L^2_{x,a}}(\mathbf{0},\mathbf{T})$.
- $\mathbf{L^2_{x,a}}(\mathrm{q};\mathbf{0},\mathbf{T})$, $T > 0$, the set of $\mathcal{P}_T\otimes\mathcal{B}(E)\otimes\mathcal{A}$-measurable maps $Z : \Omega\times[0,\,T]\times E\times A\to\mathbb{R}$ such that

$$\begin{aligned}||Z||^2_{\mathbf{L^2_{x,a}}(\mathrm{q};\mathbf{0},\,\mathbf{T})} :=& \,\mathbb{E}^{x,a}\left[\int_0^T\int_{E\times A}|Z_t(y,b)|^2\,\tilde p(\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}b)\right]\\ =& \,\mathbb{E}^{x,a}\left[\int_0^T\int_E |Z_t(y,I_t)|^2\,\lambda(X_t,I_t)\,Q(X_t,I_t,\mathrm{d}y)\,\mathrm{d}t\right]\\ &+\mathbb{E}^{x,a}\left[\int_0^T\int_A |Z_t(X_t,b)|^2\,\lambda_0(\mathrm{d}b)\,\mathrm{d}t\right] < \infty.\end{aligned}$$

  We also define $\mathbf{L^2_{x,a,loc}}(\mathrm{q}) := \cap_{T>0}\mathbf{L^2_{x,a}}(\mathrm{q};\mathbf{0},\mathbf{T})$.
- $\mathbf{L^2}(\lambda_0)$, the set of $\mathcal{A}$-measurable maps $\psi : A\to\mathbb{R}$ such that

$$|\psi|^2_{\mathbf{L^2}(\lambda_0)} := \int_A |\psi(b)|^2\,\lambda_0(\mathrm{d}b) < \infty.$$

- $\mathbf{L^2_{x,a}}(\lambda_0; \mathbf{0, T})$, $T > 0$, the set of $\mathcal{P}_T \otimes \mathcal{A}$-measurable maps $W : \Omega \times [0, T] \times A \to \mathbb{R}$ such that

$$|W|^2_{\mathbf{L^2_{x,a}}(\lambda_0; \mathbf{0, T})} := \mathbb{E}^{x,a} \left[\int_0^T \int_A |W_t(b)|^2 \, \lambda_0(\mathrm{d}b) \, \mathrm{d}t \right] < \infty.$$

  We also define $\mathbf{L^2_{x,a,loc}}(\lambda_0) := \cap_{T>0} \mathbf{L^2_{x,a}}(\lambda_0; \mathbf{0, T})$.
- $\mathbf{K^2_{x,a}}(\mathbf{0, T})$, $T > 0$, the set of nondecreasing càdlàg predictable processes $K = (K_t)_{0 \leqslant t \leqslant T}$ such that $K_0 = 0$ and $\mathbb{E}^{x,a}\left[|K_T|^2\right] < \infty$. We also define $\mathbf{K^2_{x,a,loc}} := \cap_{T>0} \mathbf{K^2_{x,a}}(\mathbf{0, T})$.

We are interested in studying the following family of BSDEs with partially nonnegative jumps over an infinite horizon, parametrized by (x, a): $\mathbb{P}^{x,a}$-a.s.,

$$\begin{aligned} Y_s^{x,a} &= Y_T^{x,a} - \delta \int_s^T Y_r^{x,a} \, \mathrm{d}r + \int_s^T f(X_r, I_r) \, \mathrm{d}r - (K_T^{x,a} - K_s^{x,a}) \\ &\quad - \int_s^T \int_A Z_r^{x,a}(X_r, b) \, \lambda_0(\mathrm{d}b) \, \mathrm{d}r - \int_s^T \int_{E \times A} Z_r^{x,a}(y, b) \, q(\mathrm{d}r \, \mathrm{d}y \, \mathrm{d}b), \quad 0 \leqslant s \leqslant T < \infty, \end{aligned} \tag{4.1}$$

with

$$Z_s^{x,a}(X_{s-}, b) \geqslant 0, \qquad \mathrm{d}s \otimes \mathrm{d}\mathbb{P}^{x,a} \otimes \lambda_0(\mathrm{d}b) \text{ -a.e. on } [0, \infty) \times \Omega \times A, \tag{4.2}$$

where δ is the positive parameter introduced in Section 2.2.

We look for a *maximal solution* $(Y^{x,a}, Z^{x,a}, K^{x,a}) \in \mathbf{S}^\infty \times \mathbf{L^2_{x,a,loc}}(\mathrm{q}) \times \mathbf{K^2_{x,a,loc}}$ to (4.1)–(4.2), in the sense that for any other solution $(\tilde{Y}, \tilde{Z}, \tilde{K}) \in \mathbf{S}^\infty \times \mathbf{L^2_{x,a,loc}}(\mathrm{q}) \times \mathbf{K^2_{x,a,loc}}$ to (4.1)–(4.2), we have $Y_t^{x,a} \geqslant \tilde{Y}_t$, $\mathbb{P}^{x,a}$-a.s., for all $t \geqslant 0$.

Proposition 4.1. *Let Hypotheses* **(HhλQ)**, **(Hλ_0)** *and* **(Hf)** *hold. Then, for any $(x, a) \in E \times A$, there exists at most one maximal solution $(Y^{x,a}, Z^{x,a}, K^{x,a}) \in \mathbf{S}^\infty \times \mathbf{L}^2_{x,a,\mathbf{loc}}(\mathrm{q}) \times \mathbf{K}^2_{x,a,\mathbf{loc}}$ to the BSDE with partially nonnegative jumps* (4.1)–(4.2).

Proof. Let (Y, Z, K) and (Y', Z', K') be two maximal solutions of (4.1)–(4.2). By definition, we clearly have the uniqueness of the component Y. Regarding the other components, taking the difference between the two backward equations we obtain: $\mathbb{P}^{x,a}$-a.s.

$$\begin{aligned} 0 = &- (K_t - K_t') - \int_0^t \int_A (Z_s(X_s, b) - Z_s'(X_s, b)) \, \lambda_0(\mathrm{d}b) \, \mathrm{d}s \\ &- \int_0^t \int_{E \times A} (Z_s(y, b) - Z_s'(y, b)) \, q(\mathrm{d}s \, \mathrm{d}y \, \mathrm{d}b), \quad 0 \leqslant t \leqslant T < \infty, \end{aligned}$$

that can be rewritten as

$$\begin{aligned} &\int_0^t \int_{E \times A} (Z_s(y, b) - Z_s'(y, b)) \, p(\mathrm{d}s \, \mathrm{d}y \, \mathrm{d}b) = -(K_t - K_t') \\ &+ \int_0^t \int_E (Z_s(y, I_s) - Z_s'(y, I_s)) \, \lambda(X_s, I_s) \, Q(X_s, I_s, \mathrm{d}y) \, \mathrm{d}s, \quad 0 \leqslant t \leqslant T < \infty. \end{aligned} \tag{4.3}$$

The right-hand side of (4.3) is a predictable process, therefore it has no totally inaccessible jumps (see, *e.g.*, Prop. 2.24, Chap. I, in [35]); on the other hand, the left side is a pure jump process with totally inaccessible jumps. This implies that $Z = Z'$, and as a consequence the component K is unique as well. □

In the sequel we prove by a penalization approach the existence of the maximal solution to (4.1)–(4.2), see Theorem 4.7. In particular, this will provide a probabilistic representation of the dual value function V^* introduced in Section 3.2.

4.1. Penalized BSDE and associated dual control problem

Let us introduce the family of penalized BSDEs on $[0,\infty)$ associated to (4.1)−(4.2), parametrized by the integer $n \geqslant 1$: $\mathbb{P}^{x,a}$-a.s.,

$$\begin{aligned} Y_s^{n,x,a} &= Y_T^{n,x,a} - \delta \int_s^T Y_r^{n,x,a}\,\mathrm{d}r + \int_s^T f(X_r, I_r)\,\mathrm{d}r \\ &\quad -n \int_s^T \int_A [Z_r^{n,x,a}(X_r, b)]^-\,\lambda_0(\mathrm{d}b)\,\mathrm{d}r - \int_s^T \int_A Z_r^{n,x,a}(X_r, b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}r \\ &\quad - \int_s^T \int_{E\times A} Z_r^{n,x,a}(y, b)\,q(\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}b), \quad 0 \leqslant s \leqslant T < \infty, \end{aligned} \tag{4.4}$$

where $[z]^- = \max(-z, 0)$ denotes the negative part of z.

We shall prove that there exists a unique solution to equation (4.4), and provide an explicit representation to (4.4) in terms of a family of dual control problems. To this end, we start by considering, for fixed $T > 0$, the family of BSDEs on $[0, T]$: $\mathbb{P}^{x,a}$-a.s.,

$$\begin{aligned} Y_s^{T,n,x,a} &= -\delta \int_s^T Y_r^{T,n,x,a}\,\mathrm{d}r + \int_s^T f(X_r, I_r)\,\mathrm{d}r \\ &\quad -n \int_s^T \int_A [Z_r^{T,n,x,a}(X_r, b)]^-\,\lambda_0(\mathrm{d}b)\,\mathrm{d}r - \int_s^T \int_A Z_r^{T,n,x,a}(X_r, b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}r \\ &\quad - \int_s^T \int_{E\times A} Z_r^{T,n,x,a}(y, b)\,q(\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}b), \quad 0 \leqslant s \leqslant T, \end{aligned} \tag{4.5}$$

with zero final cost at time $T > 0$.

Remark 4.2. The penalized BSDE (4.5) can be rewritten in the equivalent form: $\mathbb{P}^{x,a}$-a.s.,

$$Y_s^{T,n,x,a} = \int_s^T f^n(X_r,\, I_r,\, Y_r^{T,n,x,a},\, Z_r^{T,n,x,a})\,\mathrm{d}s - \int_s^T \int_{E\times A} Z_r^{T,n,x,a}(y, b)\,q(\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}b),$$

$s \in [0, T]$, where the generator f^n is defined by

$$f^n(x, a, u, \psi) := f(x, a) - \delta u - \int_A \big\{n\,[\psi(a)]^- + \psi(b)\big\}\,\lambda_0(\mathrm{d}b), \tag{4.6}$$

for all $(x, a, u, \psi) \in E \times A \times \mathbb{R} \times \mathbf{L^2}(\lambda_0)$.

We notice that, under Hypotheses **(HhλQ)**, **(Hλ_0)** and **(Hf)**, f^n is Lipschitz continuous in ψ with respect to the norm of $\mathbf{L^2}(\lambda_0)$, uniformly in (x, a, u), *i.e.*, for every $n \in \mathbb{N}$, there exists a constant L_n, depending only on n, such that for every $(x, a, u) \in E \times A \times \mathbb{R}$ and $\psi,\ \psi' \in \mathbf{L^2}(\lambda_0)$,

$$|f^n(x, a, u, \psi') - f^n(x, a, u, \psi)| \leqslant L_n |\psi - \psi'|_{\mathbf{L^2}(\lambda_0)}.$$

For every integer $n \geqslant 1$, let $\mathcal{V}^n$ denote the subset of elements $\nu \in \mathcal{V}$ valued in $(0, n]$. We have the following result, which is based on a fixed point argument and an application of the Itô formula.

Proposition 4.3. *Let Hypotheses* **(HhλQ)**, **(Hλ_0)** *and* **(Hf)** *hold. For every* $(x, a, n, T) \in E \times A \times \mathbb{N} \times (0, \infty)$, *there exists a unique solution* $(Y^{T,n,x,a}, Z^{T,n,x,a}) \in \mathbf{S}^\infty \times \mathbf{L}^{\boldsymbol{2}}_{\boldsymbol{x,a}}(\mathrm{q}; \mathbf{0}, \mathbf{T})$ *to* (4.5). *Moreover, the following uniform estimate holds:* $\mathbb{P}^{x,a}$*-a.s.,*

$$Y_s^{T,n,x,a} \leqslant \frac{M_f}{\delta}, \quad \forall\, s \in [0, T]. \tag{4.7}$$

Proof. See Section 6.2. □

With the help of Proposition 4.3, one can prove the existence and uniqueness of a solution to equation (4.4), as well as an explicit representation formula in terms of the dual controls $\nu \in \mathcal{V}^n$.

Proposition 4.4. *Let Hypotheses* **(HhλQ)**, **(Hλ_0)** *and* **(Hf)** *hold. Then, for every* $(x, a, n) \in E \times A \times \mathbb{N}$*, there exists a unique solution* $(Y^{n,x,a}, Z^{n,x,a}) \in \mathbf{S}^\infty \times \mathbf{L}^2_{x,a,\mathbf{loc}}(\mathrm{q})$ *to* (4.4).

Moreover, $(Y^{n,x,a}, Z^{n,x,a})$ *admits the following explicit representation:* $\mathbb{P}^{x,a}$*-a.s.,*

$$Y_s^{n,x,a} = \operatorname*{ess\,inf}_{\nu \in \mathcal{V}^n} \mathbb{E}_\nu^{x,a} \left[\int_s^\infty \mathrm{e}^{-\delta\,(r-s)}\, f(X_r, I_r)\, \mathrm{d}r \Big| \mathcal{F}_s \right], \quad s \geqslant 0. \tag{4.8}$$

Proof. See Section 6.3. □

Finally, let us define

$$K_t^{n,x,a} := n \int_0^t \int_A [Z_s^{n,x,a}(X_s, b)]^- \, \lambda_0(\mathrm{d}b)\, \mathrm{d}s, \quad t \geqslant 0.$$

Using the fact that $\Delta Y_r^{n,x,a} = \int_{E\times A} Z_r^{n,x,a}(y,b)\, p(\{r\}\, \mathrm{d}y\, \mathrm{d}b)$ and the uniform estimate (4.7), we are able to provide the following a priori uniform estimate on the sequence $(Z^{n,x,a}, K^{n,x,a})_{n\geqslant 0}$.

Lemma 4.5. *Assume that hypotheses* **(HhλQ)**, **(Hλ_0)** *and* **(Hf)** *hold. For every* $(x, a, n) \in E \times A \times \mathbb{N}$*, and for every* $T > 0$*, there exists a constant* C *depending only on* M_f*,* δ *and* T *such that*

$$||Z^{n,x,a}||^2_{\mathbf{L}^2_{x,a}(\mathrm{q};\mathbf{0},\,\mathbf{T})} + ||K^{n,x,a}||^2_{\mathbf{K}^2_{x,a}(\mathbf{0},\,\mathbf{T})} \leqslant C. \tag{4.9}$$

Proof. See Section 6.4. □

4.2. BSDE representation of the dual value function

In order to prove the main result of this section we first give a preliminary result, which is a consequence of the definition of a solution to the constrained BSDE (4.1)–(4.2) and of Lemma 3.1.

Lemma 4.6. *Assume that Hypotheses* **(HhλQ)**, **(Hλ_0)** *and* **(Hf)** *hold. For every* $(x, a) \in E \times A$*, let* $(Y^{x,a}, Z^{x,a}, K^{x,a}) \in \mathbf{S}^\infty \times \mathbf{L}^2_{x,a,\mathbf{loc}}(\mathrm{q}) \times \mathbf{K}^2_{x,a,\mathbf{loc}}$ *be a solution to the BSDE with partially nonnegative jumps* (4.1)–(4.2)*. Then,*

$$Y_s^{x,a} \leqslant \operatorname*{ess\,inf}_{\nu \in \mathcal{V}} \mathbb{E}_\nu^{x,a} \left[\int_s^\infty \mathrm{e}^{-\delta(r-s)}\, f(X_r, I_r)\, \mathrm{d}r \Big| \mathcal{F}_s \right], \ \forall\, s \geqslant 0. \tag{4.10}$$

Proof. See Section 6.5. □

Now we are ready to state the main result of the section.

Theorem 4.7. *Under Hypotheses* **(HhλQ)**, **(Hλ_0)** *and* **(Hf)**, *for every* $(x, a) \in E \times A$*, there exists a unique maximal solution* $(Y^{x,a}, Z^{x,a}, K^{x,a}) \in \mathbf{S}^\infty \times \mathbf{L}^2_{x,a,\mathbf{loc}}(\mathrm{q}) \times \mathbf{K}^2_{x,a,\mathbf{loc}}$ *to the BSDE with partially nonnegative jumps* (4.1)–(4.2)*. In particular,*

(i) $Y^{x,a}$ *is the nonincreasing limit of* $(Y^{n,x,a})_n$*;*
(ii) $Z^{x,a}$ *is the weak limit of* $(Z^{n,x,a})_n$ *in* $\mathbf{L}^2_{x,a,\mathbf{loc}}(\mathrm{q})$*;*
(iii) $K_s^{x,a}$ *is the weak limit of* $(K_s^{n,x,a})_n$ *in* $\mathbf{L}^2(\mathcal{F}_s)$*, for any* $s \geqslant 0$*;*

Moreover, $Y^{x,a}$ has the explicit representation:

$$Y_s^{x,a} = \operatorname*{ess\,inf}_{\nu \in \mathcal{V}} \mathbb{E}_\nu^{x,a} \left[\int_s^\infty \mathrm{e}^{-\delta(r-s)} f(X_r, I_r)\, \mathrm{d}r \Big| \mathcal{F}_s \right], \ \forall\, s \geqslant 0. \tag{4.11}$$

In particular, setting $s = 0$, we have the following representation formula for the value function of the dual control problem:

$$V^*(x,a) = Y_0^{x,a}, \quad (x,a) \in E \times A. \tag{4.12}$$

Proof. Let $(x,a) \in E \times A$ be fixed. From the representation formula (4.8) it follows that $Y_s^n \geqslant Y_s^{n+1}$ for all $s \geqslant 0$ and all $n \in \mathbb{N}$, since by definition $\mathcal{V}^n \subset \mathcal{V}^{n+1}$ and $(Y^n)_n$ are càdlàg processes. Moreover, recalling the boundedness of f, from (4.8) we see that $(Y^n)_n$ is positive. Then $(Y^{n,x,a})_n \in \mathbf{S}^\infty$ converges decreasingly to some adapted process $Y^{x,a}$, which is moreover uniformly bounded by Fatou's lemma. Furthermore, for every $T > 0$, the Lebesgue's dominated convergence theorem insures that the convergence of $(Y^{n,x,a})_n$ to $Y^{x,a}$ also holds in $\mathbf{L^2_{x,a}(0,T)}$.

Let us fix $T \geqslant 0$. By the uniform estimates in Lemma 4.5, the sequence $(Z^{n,x,a}_{|[0,\,T]})_n$ is bounded in the Hilbert space $\mathbf{L^2_{x,a}}(\mathrm{q}; \mathbf{0}, \mathbf{T})$. Then, we can extract a subsequence which weakly converges to some $\tilde{Z}^T$ in $\mathbf{L^2_{x,a}}(\mathrm{q}; \mathbf{0}, \mathbf{T})$. Let us then define the following mappings

$$\begin{aligned} I^1_\tau := \qquad & Z \longmapsto \int_0^\tau \int_{E\times A} Z_s(y,b)\, q(\mathrm{d}s\, \mathrm{d}y\, \mathrm{d}b) \\ & \mathbf{L^2_{x,a}}(\mathrm{q}; \mathbf{0}, \mathbf{T}) \longrightarrow \mathbf{L^2_{x,a}}(\mathcal{F}_\tau), \\ I^2_\tau := \qquad & Z(X_s, \cdot) \longmapsto \int_0^\tau \int_A Z_s(X_s, b)\, \lambda_0(\mathrm{d}b)\, \mathrm{d}s \\ & \mathbf{L^2_{x,a}}(\lambda_0; \mathbf{0}, \mathbf{T}) \longrightarrow \mathbf{L^2_{x,a}}(\mathcal{F}_\tau), \end{aligned}$$

for every stopping time $0 \leqslant \tau \leqslant T$. We notice that I^1_τ (resp. I^2_τ) defines a linear continuous operator (hence weakly continuous) from $\mathbf{L^2_{x,a}}(\mathrm{q}; \mathbf{0}, \mathbf{T})$ (resp. $\mathbf{L^2_{x,a}}(\lambda_0; \mathbf{0}, \mathbf{T})$) to $\mathbf{L^2_{x,a}}(\mathcal{F}_\tau)$. Therefore $I^1_\tau Z^{n,x,a}_{|[0,\,T]}$ (resp., $I^2_\tau Z^{n,x,a}_{|[0,\,T]}(X, \cdot)$) weakly converges to $I^1_\tau \tilde{Z}^T$ (resp., $I^2_\tau \tilde{Z}^T(X, \cdot)$) in $\mathbf{L^2_{x,a}}(\mathcal{F}_\tau)$. Since

$$\begin{aligned} K_\tau^{n,x,a} = Y_\tau^{n,x,a} - Y_0^{n,x,a} - \delta \int_0^\tau Y_r^{n,x,a}\, \mathrm{d}r + \int_0^\tau f(X_r, I_r)\, \mathrm{d}r - \int_0^\tau \int_A Z_r^{n,x,a}(X_r, b)\, \lambda_0(\mathrm{d}b)\, \mathrm{d}r \\ - \int_0^\tau \int_{E\times A} Z_r^{n,x,a}(y,b)\, q(\mathrm{d}r\, \mathrm{d}y\, \mathrm{d}b), \quad \forall\, \tau \in [0,\,T], \end{aligned}$$

we also have the following weak convergence in the space $\mathbf{L^2_{x,a}}(\mathcal{F}_\tau)$:

$$K_\tau^{n,x,a} \rightharpoonup \tilde{K}_\tau^T = K_\tau^T, \tag{4.13}$$

where

$$\begin{aligned} K_s^T := Y_s^{x,a} - Y_0^{x,a} - \delta \int_0^s Y_r^{x,a}\, \mathrm{d}r + \int_0^s f(X_r, I_r)\, \mathrm{d}r \\ - \int_0^s \int_A \tilde{Z}_r^T(X_r, b)\, \lambda_0(\mathrm{d}b)\, \mathrm{d}r \\ - \int_0^s \int_{E\times A} \tilde{Z}_r^T(y,b)\, q(\mathrm{d}r\, \mathrm{d}y\, \mathrm{d}b), \quad \forall\, s \in [0,\,T]. \end{aligned}$$

We have that $\mathbb{E}^{x,a}\left[|\tilde{K}^T|^2\right] < \infty$ and $\tilde{K}_0^T = 0$. We prove in the following that, since the process $(K_s^{n,x,a})_{s\in[0,\,T]}$ is nondecreasing and predictable, the limit process $\tilde{K}^T$ on $[0,\,T]$ remains nondecreasing and predictable.

Let us show that $\tilde K^T$ is a predictable process. To this end, we notice that $K_\tau^{n,x,a}$ also converges weakly in the Hilbert space $\mathbf{L^2_{x,a}(0,T)}$. Indeed, let $\xi \in \mathbf{L^2_{x,a}(0,T)}$; then, by Fubini's theorem,

$$\mathbb{E}\left[\int_0^T \xi_r\,(K_r^{n,x,a} - \tilde K_r^T)\,\mathrm{d}r\right] = \int_0^T \mathbb{E}\left[\xi_r\,(K_r^{n,x,a} - \tilde K_r^T)\right]\,\mathrm{d}r.$$

Since $\xi_r \in \mathbf{L^2_{x,a}}(\mathcal{F}_s)$ for a.e. $s \in [0,\,T]$, we conclude by Lebesgue's dominated convergence theorem that

$$\int_0^T \mathbb{E}\left[\xi_r\,(K_r^{n,x,a} - \tilde K_r^T)\right]\,\mathrm{d}r \overset{n\to\infty}{\longrightarrow} 0.$$

This implies that $\tilde K^T$ is a predictable process. Indeed, the space of predictable processes is a vectorial space and is strongly closed for the strong topology in the Hilbert space $\mathbf{L^2_{x,a}(0,T)}$. On the other hand, any convex subspace of a Banach space is closed for the strong topology if and only if it is closed for the weak topology, see Theorem III.7 in [11], and the conclusion follows. Similarly, since $(Y^{n,x,a})_n$ are càdlàg adapted processes, the limit $Y^{x,a} \in \mathbf{L^2_{x,a}(0,T)}$ remains an optional process. Moreover, since the process in the right-hand side of (4.13) is an optional process and is equal to $\tilde K^T$ for all stopping times valued in $[0,\,T]$, by the optional section theorem (see *e.g.* Cor. 4.11, Chap. IV, in [31]) it is indistinguishable to $\tilde K^T$. Then, it follows from Lemma 2.2 in [40] that $\tilde K^T$ and $Y^{x,a}$ are càdlàg processes.

Let us now prove that $\tilde K^T$ is a nondecreasing process. For any pair $u,\ s$ with $t \le u \le s \le T$, we have $\tilde K_u^T \le \tilde K_s^T$, $\mathbb{P}^{x,a}$-a.s. Indeed, let $\xi \in \mathbf{L^2_{x,a}}(\mathcal{F}_s)$ be nonnegative, then, from the martingale representation theorem, we see that there exists a random variable $\zeta \in \mathbf{L^2_{x,a}}(\mathcal{F}_u)$ and a random field η in $\mathbf{L^2_{x,a}}(\mathrm{q};\mathbf{0,T})$ such that

$$\xi = \zeta + \int_u^s \eta_r(y,b)\,q(\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}b).$$

Therefore

$$\begin{aligned}
0 &\le \mathbb{E}^{x,a}\left[\xi\,(K_s^{n,x,a} - K_u^{n,x,a})\right]\\
&= \mathbb{E}^{x,a}\left[\xi\,K_s^{n,x,a}\right] - \mathbb{E}^{x,a}\left[\zeta\,K_u^{n,x,a}\right] - \mathbb{E}^{x,a}\left[\mathbb{E}^{x,a}\left[K_u^{n,x,a}\int_u^s \eta_r(y,b)\,q(\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}b)\Big|\mathcal{F}_u\right]\right]\\
&= \mathbb{E}^{x,a}\left[\xi\,K_s^{n,x,a}\right] - \mathbb{E}^{x,a}\left[\zeta\,K_u^{n,x,a}\right]\\
&\overset{n\to\infty}{\longrightarrow} \mathbb{E}^{x,a}\left[\xi\,\tilde K_s^T\right] - \mathbb{E}^{x,a}\left[\zeta\,\tilde K_u^T\right] = \mathbb{E}^{x,a}\left[\xi\,(\tilde K_s^T - \tilde K_u^T)\right],
\end{aligned}$$

that shows that $\tilde K_u^T \le \tilde K_s^T$, $\mathbb{P}^{x,a}$-a.s. As a consequence, there exists a null measurable set $N \subset \Omega$ such that $\tilde K_u^T(\omega) \le \tilde K_s^T(\omega)$ for all $\omega \in \Omega \setminus N$, with $u, s \in \mathbb{Q} \cap [0,\,T]$, $u < s$. Since $\tilde K^T$ is càdlàg, this is enough to conclude that $\tilde K^T$ is a nondecreasing process. Therefore $\tilde K^T \in \mathbf{K^2_{x,a}(0,T)}$ and $Y^{x,a} \in \mathbf{S}^\infty$.

Then we notice that $\tilde Z^{T'}_{|[0,\,T]} = \tilde Z^T$, $\tilde K^{T'}_{|[0,\,T]} = \tilde K^T$, for any $0 \leqslant T \leqslant T' < \infty$. Indeed, for $i = 1,2$, $I^i\,\tilde Z^{T'}_{|[0,\,T]}$, as $I^i\,\tilde Z^T$, is the weak limit in $\mathbf{L^2_{x,a}}(\mathcal{F}_s)$ of $(I^i\,Z^{n,x,a}_{|[0,\,T]})_{n\geqslant 0}$, while $\tilde K^{T'}_{|[0,\,T]}$, as $\tilde K^T$, is the weak limit in $\mathbf{L^2_{x,a}}(\mathcal{F}_s)$ of $(K^{n,x,a}_{|[0,\,T]})_{n\geqslant 0}$, for every $s \in [0,\,T]$. Hence, we define $Z_s^{x,a} = \tilde Z_s^T$, $K_s^{x,a} = \tilde K_s^T$ for all $s \in [0,\,T]$ and for any $T > 0$. Observe that $Z^{x,a} \in \mathbf{L^2_{x,a,loc}}(\mathrm{q})$ and $K^{x,a} \in \mathbf{K^2_{x,a,loc}}$. Moreover, for any $T > 0$, for $i = 1,2$, $(I^i\,Z^{n,x,a}_{|[0,\,T]})_{n\geqslant 0}$ weakly converges to $I^i\,Z^{x,a}_{|[0,\,T]}$ in $\mathbf{L^2_{x,a}}(\mathcal{F}_s)$, and $(K^{n,x,a}_{|[0,\,T]})_{n\geqslant 0}$ weakly converges to $K^{x,a}_{|[0,\,T]}$ in $\mathbf{L^2_{x,a}}(\mathcal{F}_s)$, for $s \in [0,\,T]$. In conclusion, we have: $\mathbb{P}^{x,a}$-a.s.,

$$\begin{aligned}
Y_s^{x,a} = &\,Y_T^{x,a} - \delta\int_s^T Y_r^{x,a}\,\mathrm{d}r + \int_s^T f(X_r, I_r)\,\mathrm{d}r - (K_T^{x,a} - K_s^{x,a}) - \int_s^T\int_A Z_r^{x,a}(X_r,b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}r\\
&- \int_s^T\int_{E\times A} Z_r^{x,a}(y,b)\,q(\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}b), \quad 0 \leqslant s \leqslant T.
\end{aligned}$$

Since T is arbitrary, it follows that $(Y^{x,a}, Z^{x,a}, K^{x,a})$ solves equation (4.1) on $[0,\,\infty)$.

To show that the jump constraint (4.2) is satisfied, we consider the functional

$$G : \mathbf{L^2_{x,a}}(\lambda_0; \mathbf{0}, \mathbf{T}) \to \mathbb{R}$$

given by

$$G(V(\cdot)) \;:=\; \mathbb{E}\left[\int_0^T \int_A [V_s(b)]^- \, \lambda_0(\mathrm{d}b)\, \mathrm{d}s\right], \quad \forall\, V \in \mathbf{L^2_{x,a}}(\lambda_0; \mathbf{0}, \mathbf{T}).$$

Notice that $G(Z^{n,x,a}(X,\cdot)) = \mathbb{E}^{x,a}[K_T^{n,x,a}/n]$, for any $n \in \mathbb{N}$. From uniform estimate (4.9), we see that $G(Z^{n,x,a}(X,\cdot)) \to 0$ as $n \to \infty$. Since G is convex and continuous in the strong topology of $\mathbf{L^2_{x,a}}(\lambda_0; \mathbf{0}, \mathbf{T})$, then G is lower semicontinuous in the weak topology of $\mathbf{L^2_{x,a}}(\lambda_0; \mathbf{0}, \mathbf{T})$, see, *e.g.*, Corollary III.8 in [11]. Therefore, we find

$$G(Z^{x,a}(X,\cdot)) \leqslant \liminf_{n\to\infty} G(Z^{n,x,a}(X,\cdot)) = 0,$$

which implies the validity of jump constraint (4.2) on $[0, T]$, and the conclusion follows from the arbitrariness of T. Hence, $(Y^{x,a}, Z^{x,a}, K^{x,a})$ is a solution to the constrained BSDE (4.1)–(4.2) on $[0, \infty)$.

It remains to prove the representation formula (4.11) and the maximality property for $Y^{x,a}$. Firstly, since by definition $\mathcal{V}^n \subset \mathcal{V}$ for all $n \in \mathbb{N}$, it is clear from representation formula (4.8) that

$$Y_s^{n,x,a} = \operatorname*{ess\,inf}_{\nu\in\mathcal{V}^n} \mathbb{E}_\nu^{x,a}\left[\int_s^\infty \mathrm{e}^{-\delta\,(r-s)}\, f(X_r, I_r)\,\mathrm{d}r \Big| \mathcal{F}_s\right] \geqslant \operatorname*{ess\,inf}_{\nu\in\mathcal{V}} \mathbb{E}_\nu^{x,a}\left[\int_s^\infty \mathrm{e}^{-\delta\,(r-s)}\, f(X_r, I_r)\,\mathrm{d}r \Big| \mathcal{F}_s\right],$$

for all $n \in \mathbb{N}$, for all $s \geqslant 0$. Moreover, being $Y^{x,a}$ the pointwise limit of $Y^{n,x,a}$, we deduce that

$$Y_s^{x,a} \geqslant \operatorname*{ess\,inf}_{\nu\in\mathcal{V}} \mathbb{E}_\nu^{x,a}\left[\int_s^\infty \mathrm{e}^{-\delta\,(r-s)}\, f(X_r, I_r)\,\mathrm{d}r | \mathcal{F}_s\right], \quad s \geqslant 0. \tag{4.14}$$

On the other hand, $Y^{x,a}$ satisfies the opposite inequality (4.10) from Lemma 4.6, and thus we achieve the representation formula (4.11).

Finally, to show that $Y^{x,a}$ is the maximal solution, let us consider a triplet $(\bar{Y}^{x,a}, \bar{Z}^{x,a}, \bar{K}^{x,a}) \in \mathbf{S}^\infty \times \mathbf{L^2_{x,a,loc}}(\mathrm{q}) \times \mathbf{K^2_{x,a,loc}}$ solution to (4.1)–(4.2). By Lemma 4.6, $(\bar{Y}^{x,a}, \bar{Z}^{x,a}, \bar{K}^{x,a})$ satisfies inequality (4.10). Then, from the representation formula (4.11) it follows that $\bar{Y}_s^{x,a} \leqslant Y_s^{x,a}$, $\forall\, s \geqslant 0$, $\mathbb{P}^{x,a}$-a.s., *i.e.*, the maximality property holds. The uniqueness of the maximal solution directly follows from Proposition 4.1. □

5. A BSDE representation for the value function

Our main purpose is to show how maximal solutions to BSDEs with nonnegative jumps of the form (4.1)–(4.2) provide actually a Feynman–Kac representation to the value function V associated to our optimal control problem for PDMPs. We know from Theorem 4.7 that, under Hypotheses **(HhλQ)**, **(Hλ_0)** and **(Hf)**, there exists a unique maximal solution $(Y^{x,a}, Z^{x,a}, K^{x,a})$ on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^{x,a})$ to (4.1)–(4.2). Let us introduce a deterministic function $v : E \times A \to \mathbb{R}$ as

$$v(x,a) := Y_0^{x,a}, \quad (x,a) \in E \times A. \tag{5.1}$$

Our main result is as follows:

Theorem 5.1. *Assume that Hypotheses* **(HhλQ)**, **(Hλ_0)**, *and* **(Hf)** *hold. Then the function v in (5.1) does not depend on the variable a:*

$$v(x,a) = v(x,a'), \quad \forall a, a' \in A,$$

for all $x \in E$. Let us define by abuse of notation the function v on E by

$$v(x) = v(x,a), \quad \forall x \in E,$$

for any $a \in A$. Then v is a viscosity solution to (2.17).

In particular, by Theorem 2.7, v is the unique viscosity solution to (2.17), is continuous and coincides with the value function V of the PDMPs optimal control problem, which admits therefore the probabilistic representation (5.1). Finally, Theorem 4.7 implies that the dual value function V^* coincides with the value function V of the original control problem. We have therefore the following result.

Corollary 5.2. *Let Hypotheses* **(HhλQ)**, **(Hλ_0)** *and* **(Hf)** *hold, and assume that A is compact. Then the value function V of the optimal control problem defined in* (2.16) *admits the Feynman–Kac representation formula:*

$$V(x) = Y_0^{x,a}, \quad (x,a) \in E \times A.$$

Moreover, the value function V coincides with the dual value function V^* defined in (3.14)*, namely*

$$V(x) = V^*(x,a) = Y_0^{x,a}, \quad (x,a) \in E \times A. \tag{5.2}$$

The rest of the paper is devoted to prove Theorem 5.1.

5.1. The identification property of the penalized BSDE

For every $n \in \mathbb{N}$, let us introduce the deterministic function v^n defined on $E \times A$ by

$$v^n(x,a) = Y_0^{n,x,a}, \quad (x,a) \in E \times A. \tag{5.3}$$

We investigate the properties of the function v^n. Firstly, it straightly follows from (5.3) and inequality (4.7) that

$$|v^n(x,a)| \leqslant \frac{M_f}{\delta}, \quad \forall\, (x,a) \in E \times A.$$

Moreover, we have the following identification result.

Lemma 5.3 (Identification property)**.** *Under Hypotheses* **(HhλQ)**, **(Hλ_0)** *and* **(Hf)**, *for any $n \in \mathbb{N}$, the function v^n is such that, for any $(x,a) \in E \times A$, we have*

$$Y_s^{n,x,a} = v^n(X_s, I_s), \quad s \geqslant 0 \quad \mathrm{d}\mathbb{P}^{x,a} \otimes \mathrm{d}s \text{ -a.e.} \tag{5.4}$$

Proof. See Section 6.6. □

Remark 5.4. When the pair of Markov processes (X,I) is the unique strong solution to some system of stochastic differential equations, (X,I) often satisfies a stochastic flow property, and the fact that $Y_s^{n,x,a}$ is a deterministic function of (X_s, I_s) straightly follows from the uniqueness of the BSDE (see, *e.g.*, Rem. 2.4 in [7]). In our framework, we deal with the local characteristics of the state process (X,I) rather than with the stochastic differential equation solved by it. As a consequence, a stochastic flow property for (X,I) is no more directly available. The idea is then to prove the identification (5.4) using an iterative construction of the solution of standard BSDEs. This alternative approach is based on the fact that, when f does not depend on y, z, the desired identification follows from the Markov property of the state process (X,I), and it is inspired by the proof of the Theorem 4.1 in [26].

Remark 5.5. By Proposition 4.1, the maximal solution to the constrained BSDE (4.1)–(4.2) is the pointwise limit of the solution to the penalized BSDE (4.4). Then, as a byproduct of Lemma 5.3 we have the following identification property for v: $\mathbb{P}^{x,a}$-a.s.,

$$v(X_s, I_s) = Y_s^{x,a}, \quad (x,a) \in E \times A,\ s \geqslant 0. \tag{5.5}$$

5.2. The non-dependence of the function v on the variable a

We claim that the function v in (5.1) does not depend on its last argument:

$$v(x,a) = v(x,a'), \quad a, a' \in A, \quad \text{for any } x \in E. \tag{5.6}$$

We recall that, by (4.12) and (5.1), v coincides with the value function V^* of the dual control problem introduced in Section 3.2. Therefore, (5.6) holds if we prove that $V^*(x,a)$ does not depend on a.

This is guaranteed by the following result.

Proposition 5.6. *Assume that Hypotheses* **(HhλQ)**, **(Hλ_0)** *and* **(Hf)** *hold. Fix $x \in E$, $a, a' \in A$, and $\nu \in \mathcal{V}$. Then, there exists a sequence $(\nu^\varepsilon)_\varepsilon \in \mathcal{V}$ such that*

$$\lim_{\varepsilon \to 0^+} J(x, a', \nu^\varepsilon) = J(x, a, \nu). \tag{5.7}$$

Proof. See Section 6.7. □

Identity (5.7) implies that

$$V^*(x,a') \leq J(x,a,\nu) \quad x \in E,\ a, a' \in A,$$

and by the arbitrariness of ν one can conclude that

$$V^*(x,a') \leq V^*(x,a) \quad x \in E,\ a, a' \in A.$$

In other words $V^*(x,a) = v(x,a)$ does not depend on a, and (5.6) holds.

5.3. Viscosity properties of the function v

Taking into account (5.6), by abuse of notation, we define the function v on E by

$$v(x) := v(x,a), \quad \forall x \in E, \quad \text{for any } a \in A. \tag{5.8}$$

We shall prove that the function v in (5.8) provides a viscosity solution to (2.17). We separate the proof of viscosity subsolution and supersolution properties, which are different. In particular the supersolution property is more delicate and should take into account the maximality property of $Y^{x,a}$.

Remark 5.7. Identity (5.5) in Remark 5.5 gives

$$v(X_s) = Y_s^{x,a}, \quad \forall x \in E,\ s \geqslant 0, \quad \text{for any } a \in A. \tag{5.9}$$

Proof of the viscosity subsolution property to (2.17).

Proposition 5.8. *Let assumptions* **(HhλQ)**, **(Hλ_0)** *and* **(Hf)** *hold. Then, the function v in (5.8) is a viscosity subsolution to (2.17).*

Proof. Let $\bar{x} \in E$, and let $\varphi \in C^1(E)$ be a test function such that

$$0 = (v^* - \varphi)(\bar{x}) = \max_{x \in E} (v^* - \varphi)(x). \tag{5.10}$$

By the definition of $v^*(\bar{x})$, there exists a sequence $(x_m)_m$ in E such that

$$x_m \to \bar{x} \text{ and } v(x_m) \to v^*(\bar{x})$$

when m goes to infinity. By the continuity of φ and by (5.10) it follows that

$$\gamma_m := \varphi(x_m) - v(x_m) \to 0,$$

when m goes to infinity. Let η be a fixed positive constant and $\tau_m := \inf\{t \geqslant 0 : |\phi(t, x_m) - x_m| \geqslant \eta\}$. Let moreover $(h_m)_m$ be a strictly positive sequence such that

$$h_m \to 0 \text{ and } \frac{\gamma_m}{h_m} \to 0,$$

when m goes to infinity.

We notice that there exists $M \in \mathbb{N}$ such that, for every $m > M$, $h_m \wedge \tau_m = h_m$. Let us introduce $\bar{\tau} := \inf\{t \geqslant 0 : |\phi(t, \bar{x}) - \bar{x}| \geqslant \eta\}$. Clearly $\bar{\tau} > 0$. We show that it does not exists a subsequence τ_{n_k} of τ_n such that $\tau_{n_k} \to \tau_0 \in [0, \bar{\tau})$. Indeed, let $\tau_{n_k} \to \tau_0 \in [0, \bar{\tau})$. In particular $|\phi(\tau_{n_k}, \bar{x}) - \bar{x}| \geqslant \eta$. Then, by the continuity of ϕ it follows that $|\phi(\tau_0, \bar{x}) - \bar{x}| \geqslant \eta$, and this is in contradiction with the definition of $\bar{\tau}$.

Let us now fix $a \in A$, and let $Y^{x_m,a}$ be the unique maximal solution to (4.1)–(4.2) under $\mathbb{P}^{x_m,a}$. We apply the Itô formula to $\mathrm{e}^{-\delta t}\, Y_t^{x_m,a}$ between 0 and $\theta_m := \tau_m \wedge h_m \wedge T_1$, where T_1 denotes the first jump time of (X, I). Using the identification property (5.9), from the constraint (4.2) and the fact that K is a nondecreasing process it follows that $\mathbb{P}^{x_m,a}$-a.s.,

$$v(x_m) \leqslant \mathrm{e}^{-\delta\theta_m}\, v(X_{\theta_m}) + \int_0^{\theta_m} \mathrm{e}^{-\delta r}\, f(X_r, I_r)\,\mathrm{d}r - \int_0^{\theta_m} \mathrm{e}^{-\delta r} \int_E (v(y) - v(X_r))\,\tilde{q}(\mathrm{d}r\,\mathrm{d}y),$$

where $\tilde{q}(\mathrm{d}r\,\mathrm{d}y) = p(\mathrm{d}r\,\mathrm{d}y) - \lambda(X_r, I_r)\, Q(X_r, I_r, \mathrm{d}y)\,\mathrm{d}r$. In particular

$$v(x_m) \leqslant \mathbb{E}^{x_m,a}\left[\mathrm{e}^{-\delta\theta_m}\, v(X_{\theta_m}) + \int_0^{\theta_m} \mathrm{e}^{-\delta r}\, f(X_r, I_r)\,\mathrm{d}r\right].$$

Equation (5.10) implies that $v \leqslant v^* \leqslant \varphi$, and therefore

$$\varphi(x_m) - \gamma_m \leqslant \mathbb{E}^{x_m,a}\left[\mathrm{e}^{-\delta\theta_m}\, \varphi(X_{\theta_m}) + \int_0^{\theta_m} \mathrm{e}^{-\delta r}\, f(X_r, I_r)\,\mathrm{d}r\right].$$

At this point, applying Itô's formula to $\mathrm{e}^{-\delta r}\,\varphi(X_r)$ between 0 and θ_m, we get

$$-\frac{\gamma_m}{h_m} + \mathbb{E}^{x_m,a}\left[\int_0^{\theta_m} \frac{1}{h_m}\,\mathrm{e}^{-\delta r}\,[\delta\,\varphi(X_r) - \mathcal{L}^{I_r}\varphi(X_r) - f(X_r, I_r)]\,\mathrm{d}r\right] \leqslant 0, \tag{5.11}$$

where $\mathcal{L}^{I_r}\varphi(X_r) = \int_E (\varphi(y) - \varphi(X_r))\,\lambda(X_r, I_r)\,Q(X_r, I_r, \mathrm{d}y)$. Now we notice that, $\mathbb{P}^{x_m,a}$-a.s., $(X_r, I_r) = (\phi(r, x_m), a)$ for $r \in [0, \theta_m]$. Taking into account the continuity of the map $(y, b) \mapsto \delta\,\varphi(y) - \mathcal{L}^b\varphi(y) - f(y, b)$, we see that for any $\varepsilon > 0$,

$$-\frac{\gamma_m}{h_m} + (\varepsilon + \delta\,\varphi(x_m) - \mathcal{L}^a\varphi(x_m) - f(x_m, a))\,\mathbb{E}^{x_m,a}\left[\frac{\theta_m\,\mathrm{e}^{-\delta\,\theta_m}}{h_m}\right] \leqslant 0, \tag{5.12}$$

Let $f_{T_1}(s)$ denote the distribution density of T_1 under $\mathbb{P}^{x_m,a}$, see (3.6). Taking $m > M$, we have

$$\begin{aligned}\mathbb{E}^{x_m,a}\left[\frac{g(\theta_m)}{h_m}\right] &= \frac{1}{h_m}\int_0^{h_m} s\,\mathrm{e}^{-\delta\,s}\, f_{T_1}(s)\,\mathrm{d}s + \frac{h_m\,\mathrm{e}^{-\delta\,h_m}}{h_m}\,\mathbb{P}^{x_m,a}[T_1 > h_m]\\ &= \frac{1}{h_m}\int_0^{h_m} s\,\mathrm{e}^{-\delta\,s}\,(\lambda(\phi(r, x_m), a) + \lambda_0(A))\,\mathrm{e}^{-\int_0^s (\lambda(\phi(r,x_m),a)+\lambda_0(A))\,\mathrm{d}r}\,\mathrm{d}s\\ &\quad + \mathrm{e}^{-\delta\,h_m}\,\mathrm{e}^{-\int_0^{h_m}(\lambda(\phi(r,x_m),a)+\lambda_0(A))\,\mathrm{d}r}.\end{aligned} \tag{5.13}$$

By the boundedness of λ and λ_0, it is easy to see that the two terms in the right-hand side of (5.13) converge respectively to zero and one when m goes to infinity. Thus, passing into the limit in (5.12) as m goes to infinity, we obtain

$$\delta\,\varphi(\bar{x}) - \mathcal{L}^a\varphi(\bar{x}) - f(\bar{x}, a) \leqslant 0.$$

From the arbitrariness of $a \in A$ we conclude that v is a viscosity subsolution to (2.17) in the sense of Definition 2.6. □

Proof of the viscosity supersolution property to (2.17).

Proposition 5.9. *Let assumptions* **(HhλQ)**, **(Hλ_0)**, *and* **(Hf)** *hold. Then, the function v in* (5.8) *is a viscosity supersolution to* (2.17).

Proof. Let $\bar{x} \in E$, and let $\varphi \in C^1(E)$ be a test function such that

$$0 = (v_* - \varphi)(\bar{x}) = \min_{x\in E}(v_* - \varphi)(x). \tag{5.14}$$

Notice that we can assume w.l.o.g. that $\bar{x}$ is a strict minimum of $v_* - \varphi$. As a matter of fact, one can subtract to φ a positive cut-off function which behaves as $|x-\bar{x}|^2$ when $|x-\bar{x}|^2$ is small, and that regularly converges to 1 as $|x-\bar{x}|^2$ increases to 1.

Then, for every $\eta > 0$, we can define

$$0 < \beta(\eta) := \inf_{x\notin B(\bar{x},\eta)}(v_* - \varphi)(x). \tag{5.15}$$

We will show the result by contradiction. Assume thus that

$$H^{\varphi}(\bar{x}, \varphi, \nabla\varphi) < 0.$$

Then by the continuity of H, there exists $\eta > 0$, $\beta(\eta) > 0$ and $\varepsilon \in (0,\, \beta(\eta)\delta]$ such that

$$H^{\varphi}(y, \varphi, \nabla\varphi) \leqslant -\varepsilon,$$

for all $y \in B(\bar{x},\eta) = \{y \in E : |\bar{x} - y| < \eta\}$. By definition of $v_*(\bar{x})$, there exists a sequence $(x_m)_m$ taking values in $B(\bar{x},\eta)$ such that

$$x_m \to \bar{x} \text{ and } v(x_m) \to v_*(\bar{x})$$

when m goes to infinity. By the continuity of φ and by (5.14) it follows that

$$\gamma_m := v(x_m) - \varphi(x_m) \to 0,$$

when m goes to infinity. Let us fix $T > 0$ and define $\theta := \tau \wedge T$, where $\tau = \inf\{t \geqslant 0 : X_t \notin B(\bar{x},\eta)\}$.

At this point, let us fix $a \in A$, and consider the solution $Y^{n,x_m,a}$ to the penalized (4.4), under the probability $\mathbb{P}^{x_m,a}$. Notice that

$$\mathbb{P}^{x_m,a}\{\tau = 0\} = \mathbb{P}^{x_m,a}\{X_0 \notin B(\bar{x},\eta)\} = 0.$$

We apply Itô's formula to $\mathrm{e}^{-\delta t}\, Y_t^{n,x_m,a}$ between 0 and θ. Then, proceeding as in the proof of formula (4.8) in Proposition 4.4, we get the following inequality:

$$Y_0^{n,x_m,a} \geqslant \inf_{\nu\in\mathcal{V}^n} \mathbb{E}_{\nu}^{x_m,a}\left[\mathrm{e}^{-\delta\theta}\, Y_\theta^{n,x_m,a} + \int_0^\theta \mathrm{e}^{-\delta r}\, f(X_r, I_r)\,\mathrm{d}r\right]. \tag{5.16}$$

Since $Y^{n,x_m,a}$ converges decreasingly to the maximal solution $Y^{x_m,a}$ to the constrained BSDE (4.1)–(4.2), and recalling the identification property (5.9), inequality (5.16) leads to the corresponding inequality for $v(x_m)$:

$$v(x_m) \geqslant \inf_{\nu\in\mathcal{V}} \mathbb{E}_{\nu}^{x_m,a}\left[\mathrm{e}^{-\delta\theta}\, v(X_\theta) + \int_0^\theta \mathrm{e}^{-\delta r}\, f(X_r, I_r)\,\mathrm{d}r\right].$$

In particular, there exists a strictly positive, predictable and bounded function ν_m such that

$$v(x_m) \geqslant \mathbb{E}_{\nu_m}^{x_m,a}\left[\mathrm{e}^{-\delta\theta}\, v(X_\theta) + \int_0^\theta \mathrm{e}^{-\delta r}\, f(X_r, I_r)\,\mathrm{d}r\right] - \frac{\varepsilon}{2\delta}\cdot \tag{5.17}$$

Now, from equation (5.14) and (5.15) we get

$$\varphi(x_m) + \gamma_m \geqslant \mathbb{E}_{\nu_m}^{x_m,a}\left[\mathrm{e}^{-\delta\theta}\, \varphi(X_\theta) + \beta\,\mathrm{e}^{-\delta\theta}\,\mathbb{1}_{\{\tau\leqslant T\}} + \int_0^\theta \mathrm{e}^{-\delta r}\, f(X_r, I_r)\,\mathrm{d}r\right] - \frac{\varepsilon}{2\,\delta}\cdot$$

At this point, applying Itô's formula to $\mathrm{e}^{-\delta r}\,\varphi(X_r)$ between 0 and θ, we get

$$\gamma_m + \mathbb{E}_{\nu_m}^{x_m,a}\left[\int_0^\theta \mathrm{e}^{-\delta r}\,[\delta\,\varphi(X_r) - \mathcal{L}^{I_r}\varphi(X_r) - f(X_r, I_r)]\,\mathrm{d}r - \beta\,\mathrm{e}^{-\delta\theta}\,\mathbb{1}_{\{\tau\leqslant T\}}\right] + \frac{\varepsilon}{2} \geqslant 0, \tag{5.18}$$

where $\mathcal{L}^{I_r}\varphi(X_r) = \int_E (\varphi(y) - \varphi(X_r))\,\lambda(X_r, I_r)\,Q(X_r, I_r, \mathrm{d}y)$. Noticing that, for $r \in [0,\,\theta]$,

$$\begin{aligned}\delta\,\varphi(X_r) - \mathcal{L}^{I_r}\varphi(X_r) - f(X_r, I_r) &\leqslant \delta\,\varphi(X_r) - \inf_{b\in A}\{\mathcal{L}^b\varphi(X_r) - f(X_r, b)\}\\ &= H^\varphi(X_r, \varphi, \nabla\varphi)\\ &\leqslant -\varepsilon,\end{aligned}$$

from (5.18) we obtain

$$\begin{aligned}0 &\leqslant \gamma_m + \frac{\varepsilon}{2\,\delta} + \mathbb{E}_{\nu_m}^{x_m,a}\left[-\varepsilon\int_0^\theta \mathrm{e}^{-\delta r}\,\mathrm{d}r - \beta\,\mathrm{e}^{-\delta\theta}\,\mathbb{1}_{\{\tau\leqslant T\}}\right]\\ &= \gamma_m - \frac{\varepsilon}{2\,\delta} + \mathbb{E}_{\nu_m}^{x_m,a}\left[\left(\frac{\varepsilon}{\delta} - \beta\right)\mathrm{e}^{-\delta\theta}\mathbb{1}_{\{\tau\leqslant T\}} + \frac{\varepsilon}{\delta}\mathrm{e}^{-\delta\theta}\,\mathbb{1}_{\{\tau> T\}}\right]\\ &\leqslant \gamma_m - \frac{\varepsilon}{2\,\delta} + \frac{\varepsilon}{\delta}\,\mathbb{E}_{\nu_m}^{x_m,a}\left[\mathrm{e}^{-\delta\theta}\,\mathbb{1}_{\{\tau> T\}}\right]\\ &= \gamma_m - \frac{\varepsilon}{2\,\delta} + \frac{\varepsilon}{\delta}\,\mathbb{E}_{\nu_m}^{x_m,a}\left[\mathrm{e}^{-\delta T}\,\mathbb{1}_{\{\tau> T\}}\right]\\ &\leqslant \gamma_m - \frac{\varepsilon}{2\,\delta} + \mathrm{e}^{-\delta T}.\end{aligned}$$

Letting T and m go to infinity we achieve the contradiction: $0 \leqslant -\frac{\varepsilon}{2\,\delta}$. ☐

6. Other technical proofs

6.1. Proof of Proposition 3.2

For simplicity, in the sequel we will drop the dependence of $\mathbb{P}^{x,a}$ and $\mathbb{P}_\nu^{x,a}$ on (x, a), which will be denoted respectively by $\mathbb{P}$ and $\mathbb{P}^\nu$.

We notice that $\mathcal{F}_{T_n} = \sigma(T_1, E_1, A_1, \ldots, T_n, E_n, A_n)$ defines an increasing family of sub σ-fields of $\mathcal{F}_\infty$ such that $\mathcal{F}_\infty$ is generated by $\bigcup_n \mathcal{F}_{T_n}$. The idea is then to provide a consistent family $\{\mathbb{P}_n^\nu\}_n$ of probability measures on $(\Omega, \mathcal{F}_{T_n})$ (*i.e.*, $\mathbb{P}_{n+1}^\nu\big|_{\mathcal{F}_{T_n}} = \mathbb{P}_n^\nu$), under which $\tilde{p}^\nu$ is the compensator of the measure p on $(0,\,T_n] \times E \times A$. Indeed, if we have at disposal such a family of probabilities, we can naturally define on $\bigcup_n \mathcal{F}_{T_n}$ a set function $\mathbb{P}^\nu$ verifying the desired property, by setting $\mathbb{P}^\nu(B) := \mathbb{P}_n^\nu(B)$ for every $B \in \mathcal{F}_{T_n}$, $n \geq 1$. Finally, to conclude we would need to show that $\mathbb{P}^\nu$ is countably additive on $\bigcup_n \mathcal{F}_{T_n}$, and therefore can be extended uniquely to $\mathcal{F}_\infty$.

Let us proceed by steps. For every $n \in \mathbb{N}$, we set

$$\mathrm{d}\mathbb{P}_n^\nu := L_{T_n}^\nu\,\mathrm{d}\mathbb{P} \qquad \text{on} \qquad (\Omega, \mathcal{F}_{T_n}), \tag{6.1}$$

where L^ν is given by (3.11). Notice that, for every $n \in \mathbb{N}$, the probability $\mathbb{P}^\nu_n$ is well defined. Indeed, recalling the boundedness properties of ν and λ_0, we have

$$\begin{aligned} L^\nu_{T_n} &= \mathrm{e}^{\int_0^{T_n} \int_A (1-\nu_r(b))\lambda_0(\mathrm{d}b)\,\mathrm{d}r} \prod_{k=1}^{n} (\nu_{T_k}(A_k)\, d_1(T_k, E_k, A_k) + d_2(T_k, E_k, A_k)) \\ &\leq (\max(||\nu||_\infty, 1))^n\, \mathrm{e}^{\lambda_0(A)\, T_n}, \end{aligned} \tag{6.2}$$

and since T_n is exponentially distributed (see (2.7)), we get

$$\mathbb{E}\left[L^\nu_{T_n}\right] \leq (||\nu||_\infty)^n\, \mathbb{E}\left[\mathrm{e}^{\lambda_0(A)\, T_n}\right] < \infty.$$

Then, arguing as in the proof of the Girsanov theorem for point process (see, *e.g.*, the comments after Thm. 4.5 in [32]), it can be proved that the restriction of the random measure p to $(0, T_n] \times E \times A$ admits $\tilde{p}^\nu = (\nu\, d_1 + d_2)\,\tilde{p}$ as compensator under $\mathbb{P}^\nu_n$. Moreover, $\{\mathbb{P}^\nu_n\}_n$ is a consistent family of probability measures on $(\Omega, \mathcal{F}_{T_n})$, namely

$$\mathbb{P}^\nu_{n+1}\big|_{\mathcal{F}_{T_n}} = \mathbb{P}^\nu_n, \quad n \in \mathbb{N}. \tag{6.3}$$

Indeed, taking into account definition (6.1), it is easy to see that identity (6.3) is equivalent to

$$\mathbb{E}\left[L^\nu_{T_n} | \mathcal{F}_{T_{n-1}}\right] = L^\nu_{T_{n-1}}, \quad n \in \mathbb{N}. \tag{6.4}$$

By Corollary 3.6, Chapter II, in [41], and taking into account the estimate (6.2), it follows that the process $(L^\nu_{t\wedge T_n})_{t\geq 0}$ is a uniformly integrable martingale. Then, identity (6.4) follows from the optional stopping theorem for uniformly integrable martingales (see, *e.g.*, Thm. 3.2, Chap. II, in [41]).

At this point, we define the following probability measure on $\bigcup_n \mathcal{F}_{T_n}$:

$$\mathbb{P}^\nu(B) := \mathbb{P}^\nu_n(B), \quad B \in \mathcal{F}_{T_n},\ n \in \mathbb{N}. \tag{6.5}$$

In order to get the desired probability measure on $(\Omega, \mathcal{F}_\infty)$, we need to show that $\mathbb{P}^\nu$ in (6.5) is σ-additive on $\bigcup_n \mathcal{F}_{T_n}$: in this case, $\mathbb{P}^\nu$ can indeed be extended uniquely to $\mathcal{F}_\infty$, see Theorem 6.1 in [34].

Let us then prove that $\mathbb{P}^\nu$ in (6.5) is countably additive on $\bigcup_n \mathcal{F}_{T_n}$. To this end, let us introduce the product space $\tilde{E}^{\mathbb{N}}_\Delta := (E \times A \times [0, \infty) \cup \{(\Delta, \Delta', \infty)\})^{\mathbb{N}}$, with associated Borel σ-algebra $\tilde{\mathcal{E}}^{\mathbb{N}\otimes}_\Delta$. For every $n \in \mathbb{N}$, we define the following probability measure on $(\tilde{E}^n_\Delta, \tilde{\mathcal{E}}^{n\otimes}_\Delta)$:

$$\mathbb{Q}^\nu_n(A) := \mathbb{P}^\nu_n(\omega : \pi_n(\omega) \in A), \quad A \in \tilde{E}^n_\Delta, \tag{6.6}$$

where $\pi_n = (T_1, E_1, A_1, \ldots, T_n, E_n, A_n)$. The consistency property (6.3) of the family $(\mathbb{P}^\nu_n)_n$ implies that

$$\mathbb{Q}^\nu_{n+1}(A \times \tilde{E}_\Delta) = \mathbb{Q}^\nu_{n+1}(A), \quad A \in \tilde{E}^n_\Delta. \tag{6.7}$$

Let us now define

$$\begin{aligned} &\mathcal{A} := \{A \times \tilde{E}_\Delta \times \tilde{E}_\Delta \times \ldots : A \in \tilde{E}^n_\Delta, \qquad n \geq 0\}, \\ &\mathbb{Q}^\nu(A \times \tilde{E}_\Delta \times \tilde{E}_\Delta \times \ldots) := \mathbb{Q}^\nu_n(A), \quad A \in \tilde{E}^n_\Delta, \qquad n \geq 0. \end{aligned} \tag{6.8}$$

By the Kolmogorov extension theorem for product spaces (see Thm. 1.1.10 in [43]), it follows that $\mathbb{Q}^\nu$ is σ-additive on $\mathcal{A}$. Then, collecting (6.5), (6.6) and (6.8), it is easy to see that the σ-additivity of $\mathbb{Q}^\nu$ on $\mathcal{A}$ implies the σ-additivity of $\mathbb{P}^\nu$ on $\bigcup_n \mathcal{F}_{T_n}$.

Finally, we need to show that

$$\mathbb{P}^\nu\big|_{\mathcal{F}_T} = L^\nu_T\, \mathbb{P} \quad \forall\, T > 0,$$

or, equivalently, that

$$\mathbb{E}[L_T^\nu\, \psi] = \mathbb{E}^\nu[\psi] \quad \forall \psi\ \mathcal{F}_T\text{-measurable function.}$$

To this end, fix $T>0$, and let ψ be a $\mathcal{F}_{T\wedge T_n}$-measurable bounded function. In particular, ψ is $\mathcal{F}_{T\wedge T_m}$-measurable, for every $m \geq n$. Since by definition $\mathbb{P}^\nu\big|_{\mathcal{F}_{T_n}} = L_{T_n}^\nu\, \mathbb{P}, n \in \mathbb{N}$, we have

$$\begin{aligned}\mathbb{E}^\nu[\psi] &= \mathbb{E}[L_{T_m}^\nu\, \psi] \\ &= \mathbb{E}[\mathbb{E}[L_{T_m}^\nu\, \psi | \mathcal{F}_{T\wedge T_m}]] \\ &= \mathbb{E}[\psi\, \mathbb{E}[L_{T_m}^\nu | \mathcal{F}_{T\wedge T_m}]] \\ &= \mathbb{E}[\psi\, L_{T\wedge T_m}^\nu] \quad \forall m \geq n.\end{aligned}$$

Since $L_{T\wedge T_m}^\nu \underset{m\to\infty}{\longrightarrow} L_T^\nu$ a.s., and $(L_s^\nu)_{s\in[0,\,T]}$ is a uniformly integrable martingale, by Theorem 3.1, Chapter II, in [41], we get

$$\mathbb{E}^\nu[\psi] = \lim_{m\to\infty} \mathbb{E}[L_{T\wedge T_m}^\nu\, \psi] = \mathbb{E}[L_T^\nu\, \psi], \quad \forall \psi \in \bigcup_n \mathcal{F}_{T\wedge T_n}.$$

Then, by the monotone class theorem, recalling that $\bigvee_n \mathcal{F}_{T\wedge T_n} = \mathcal{F}_{\bigvee_n \mathcal{F}_{T\wedge T_n}}$ (see, *e.g.*, Cor. 3.5, point 6, in [31]), we get

$$\mathbb{E}^\nu[\psi] = \mathbb{E}[L_T^\nu\, \psi], \quad \forall \psi \in \bigvee_n \mathcal{F}_{T\wedge T_n} = \mathcal{F}_{\bigvee_n \mathcal{F}_{T\wedge T_n}} = \mathcal{F}_T.$$

This concludes the proof.

6.2. Proof of Proposition 4.3

The existence and uniqueness of a solution $(Y^{T,n,x,a}, Z^{T,n,x,a}) \in \mathbf{S^2_{x,a}(0,T)} \times \mathbf{L^2_{x,a}}(\mathrm{q}; \mathbf{0,T})$ to (4.5) is based on a fixed point argument, and uses integral representation results for $\mathbb{F}$-martingales, with $\mathbb{F}$ the natural filtration (see, *e.g.*, Thm. 5.4 in [32]). This procedure is standard and we omit it (similar proofs can be found in the proofs of Thm. 3.2 in [45], Prop. 3.2 in [9], Thm. 3.4 in [16]). It remains to prove the uniform estimate (4.7). To this end, let us apply Itô's formula to $\mathrm{e}^{-\delta\, r}\, Y_r^{T,n,x,a}$ between s and T. We get: $\mathbb{P}^{x,a}$-a.s.

$$\begin{aligned}Y_s^{T,n,x,a} = &\int_s^T \mathrm{e}^{-\delta\,(r-s)}\, f(X_r, I_r)\, \mathrm{d}r - \int_s^T \int_{E\times A} \mathrm{e}^{-\delta\,(r-s)}\, Z_r^{T,n,x,a}(y,b)\, q(\mathrm{d}r\, \mathrm{d}y\, \mathrm{d}b) \\ &- \int_s^T \int_A \mathrm{e}^{-\delta\,(r-s)}\, \{n[Z_r^{T,n,x,a}(X_r,b)]^- + Z_r^{T,n,x,a}(X_r,b)\}\, \lambda_0(\mathrm{d}b)\, \mathrm{d}r, \quad s \in [0,\,T].\end{aligned} \tag{6.9}$$

Now for any $\nu \in \mathcal{V}^n$, let us introduce the compensated martingale measure $q^\nu(\mathrm{d}s\, \mathrm{d}y\, \mathrm{d}b) = q(\mathrm{d}s\, \mathrm{d}y\, \mathrm{d}b) - (\nu_s(b)-1)\, d_1(s,y,b)\, \tilde{p}(\mathrm{d}s\, \mathrm{d}y\, \mathrm{d}b)$ under $\mathbb{P}_\nu^{x,a}$. Taking the expectation in (6.9) under $\mathbb{P}_\nu^{x,a}$, conditional on $\mathcal{F}_s$, and since $Z^{T,n,x,a}$ is in $\mathbf{L^2_{x,a}}(\mathrm{q}; \mathbf{0,T})$, from Lemma 3.1 we get that, $\mathbb{P}^{x,a}$-a.s.,

$$\begin{aligned}Y_s^{T,n,x,a} = &- \mathbb{E}_\nu^{x,a}\left[\int_s^T \int_A \mathrm{e}^{-\delta\,(r-s)}\, \{n[Z_r^{T,n,x,a}(X_r,b)]^- + \nu_r(b)\, Z_r^{T,n,x,a}(X_r,b)\}\, \lambda_0(\mathrm{d}b)\, \mathrm{d}r \Big| \mathcal{F}_s\right] \\ &+ \mathbb{E}_\nu^{x,a}\left[\int_s^T \mathrm{e}^{-\delta\,(r-s)}\, f(X_r, I_r)\, \mathrm{d}r \Big| \mathcal{F}_s\right], \quad s \in [0,\,T].\end{aligned} \tag{6.10}$$

From the elementary numerical inequality: $n[z]^- + \nu z \geqslant 0$ for all $z \in \mathbb{R}$, $\nu \in (0, n]$, we deduce by (6.10) that, for all $\nu \in \mathcal{V}^n$,

$$Y_s^{T,n,x,a} \leqslant \mathbb{E}_\nu^{x,a}\left[\int_s^T \mathrm{e}^{-\delta\,(r-s)}\, f(X_r, I_r)\,\mathrm{d}r \Big| \mathcal{F}_s\right], \quad s \in [0,\,T].$$

Therefore, $\mathbb{P}^{x,a}$-a.s.,

$$Y_s^{T,n,x,a} \leqslant \mathbb{E}_\nu^{x,a}\left[\int_s^\infty \mathrm{e}^{-\delta\,(r-s)}\, f(X_r, I_r)\,\mathrm{d}r \Big| \mathcal{F}_s\right] \leqslant \frac{M_f}{\delta}, \quad s \in [0,\,T].$$

6.3. Proof of Proposition 4.4

Uniqueness. Fix $n \in \mathbb{N}$, $(x, a) \in E \times A$, and consider two solutions $(Y^1, Z^1) = (Y^{1,n,x,a}, Z^{1,n,x,a})$, $(Y^2, Z^2) = (Y^{2,n,x,a}, Z^{2,n,x,a}) \in \mathbf{S}^\infty \times \mathbf{L}^2_{\mathbf{x,a,loc}}(\mathrm{q})$ of (4.4). Set $\bar{Y} = Y^2 - Y^1$, $\bar{Z} = Z^2 - Z^1$. Let $0 \leqslant s \leqslant T < \infty$. Then, an application of Itô's formula to $\mathrm{e}^{-2\,\delta\,r}|\bar{Y}_r|^2$ between s and T yields: $\mathbb{P}^{x,a}$-a.s.,

$$\begin{aligned}
\mathrm{e}^{-2\,\delta\,s}|\bar{Y}_s|^2 =& \,\mathrm{e}^{-2\,\delta\,T}|\bar{Y}_T|^2 - 2n\int_s^T\int_A \mathrm{e}^{-2\,\delta\,r}\,\bar{Y}_r\,\{[Z_r^2(X_s, b)]^- - [Z_r^1(X_s, b)]^-\}\,\lambda_0(\mathrm{d}b)\,\mathrm{d}r\\
&- 2\int_s^T\int_A \mathrm{e}^{-2\,\delta\,r}\,\bar{Y}_r\,\bar{Z}_r(X_s, b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}r - 2\int_s^T\int_{E\times A} \mathrm{e}^{-2\,\delta\,r}\,\bar{Y}_r\,\bar{Z}_r(y, b)\,q(\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}b)\\
&- \int_s^T\int_{E\times A} \mathrm{e}^{-2\,\delta\,r}\,|\bar{Z}_r(y, b)|^2\,p(\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}b).
\end{aligned} \tag{6.11}$$

Notice that

$$\begin{aligned}
&-n\int_s^T\int_A \mathrm{e}^{-\delta\,(r-s)}\,\bar{Y}_r\,\{[Z_r^2(X_r, b)]^- - [Z_r^1(X_r, b)]^-\}\,\lambda_0(\mathrm{d}b)\,\mathrm{d}r\\
=& \int_s^T\int_A \mathrm{e}^{-\delta\,(r-s)}\,\bar{Y}_r\,\bar{Z}_r(X_r, b)\,\nu_r^\varepsilon(b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}r\\
&-\varepsilon\int_s^T\int_A \mathrm{e}^{-\delta\,(r-s)}\,\bar{Y}_r\,\bar{Z}_r(X_r, b)\,\mathbb{1}_{\{|\bar{Y}_r|\leqslant 1\}}\,\mathbb{1}_{\{[Z_r^2(X_r,b)]^- = [Z_r^1(X_r,b)]^-,\,|\bar{Z}_r(X_r,b)|\leqslant 1\}}\,\lambda_0(\mathrm{d}b)\,\mathrm{d}r\\
&-\varepsilon\int_s^T\int_A \mathrm{e}^{-\delta\,(r-s)}\,\bar{Y}_r\,\mathbb{1}_{\{|\bar{Y}_r|\leqslant 1\}}\,\mathbb{1}_{\{[Z_r^2(X_r,b)]^- = [Z_r^1(X_r,b)]^-,\,|\bar{Z}_r(X_r,b)|> 1\}}\,\lambda_0(\mathrm{d}b)\,\mathrm{d}r\\
&-\varepsilon\int_s^T\int_A \mathrm{e}^{-\delta\,(r-s)}\,\bar{Z}_r(X_r, b)\,\mathbb{1}_{\{|\bar{Y}_r|> 1\}}\,\mathbb{1}_{\{[Z_r^2(X_r,b)]^- = [Z_r^1(X_r,b)]^-,\,|\bar{Z}_r(X_r,b)|\leqslant 1\}}\,\lambda_0(\mathrm{d}b)\,\mathrm{d}r\\
&-\varepsilon\int_s^T\int_A \mathrm{e}^{-\delta\,(r-s)}\,\mathbb{1}_{\{|\bar{Y}_r|> 1\}}\,\mathbb{1}_{\{[Z_r^2(X_r,b)]^- = [Z_r^1(X_r,b)]^-,\,|\bar{Z}_r(X_r,b)|> 1\}}\,\lambda_0(\mathrm{d}b)\,\mathrm{d}r,
\end{aligned}$$

where $\nu^\varepsilon : \mathbb{R}_+ \times \Omega \times A$ is given by

$$\begin{aligned}
\nu_r^\varepsilon(b) =& - n\,\frac{[Z_r^2(X_r, b)]^- - [Z_r^1(X_r, b)]^-}{\bar{Z}_r(X_r, b)}\mathbb{1}_{\{Z_r^2(X_r,b)]^- - [Z_r^1(X_r,b)]^- \neq 0\}}\\
&+ \varepsilon\,(\bar{Y}_r)^{-1}\,(\bar{Z}_r(X_r^{x,a}, b))^{-1}\,\mathbb{1}_{\{|\bar{Y}_r|>1\}}\,\mathbb{1}_{\{[Z_r^2(X_r^{x,a},b)]^- = [Z_r^1(X_s^{x,a},b)]^-,\,|\bar{Z}_r(X_r^{x,a},b)|>1\}}\\
&+ \varepsilon\,(\bar{Z}_r(X_r^{x,a}, b))^{-1}\,\mathbb{1}_{\{|\bar{Y}_r|\leq 1\}}\,\mathbb{1}_{\{[Z_r^2(X_r^{x,a},b)]^- = [Z_r^1(X_s^{x,a},b)]^-,\,|\bar{Z}_r(X_r^{x,a},b)|>1\}}\\
&+ \varepsilon\,(\bar{Y}_r)^{-1}\,\mathbb{1}_{\{|\bar{Y}_r|>1\}}\,\mathbb{1}_{\{[Z_r^2(X_r^{x,a},b)]^- = [Z_r^1(X_s^{x,a},b)]^-,\,|\bar{Z}_r(X_r^{x,a},b)|\leq 1\}}\\
&+ \varepsilon\,\mathbb{1}_{\{|\bar{Y}_r|\leqslant 1\}}\,\mathbb{1}_{\{[Z_r^2(X_r,b)]^- = [Z_r^1(X_r,b)]^-,\,|\bar{Z}_r(X_r,b)|\leqslant 1\}},
\end{aligned} \tag{6.12}$$

for arbitrary $\varepsilon \in (0, 1)$. In particular, ν^ε is a $\mathcal{P} \otimes \mathcal{A}$-measurable map satisfying $\nu^\varepsilon_r(b) \in [\varepsilon, n]$, $\mathrm{d}r \otimes \mathrm{d}\mathbb{P}^{x,a} \otimes \lambda_0(\mathrm{d}b)$-almost everywhere. Consider the probability measure $\mathbb{P}^{x,a}_{\nu^\varepsilon}$ on $(\Omega, \mathcal{F}_\infty)$, whose restriction to $(\Omega, \mathcal{F}_T)$ has Radon−Nikodym density:

$$L^{\nu^\varepsilon}_s := \mathcal{E}\left(\int_0^\cdot \int_{E\times A} (\nu^\varepsilon_t(b)\, d_1(t, y, b) + d_2(t, y, b) - 1)\, q(\mathrm{d}t\, \mathrm{d}y\, \mathrm{d}b)\right)_s \tag{6.13}$$

for all $0 \leqslant s \leqslant T$, where $\mathcal{E}(\cdot)_s$ is the Doléans−Dade exponential. The existence of such a probability is guaranteed by Proposition 3.2. From Lemma 3.1 it follows that $(L^{\nu^\varepsilon}_s)_{s\in[0,\,T]}$ is a uniformly integrable martingale. Moreover, $L^{\nu^\varepsilon}_T \in \mathbf{L^p}(\mathcal{F}_T)$, for any $p \geqslant 1$. Under the probability measure $\mathbb{P}^{x,a}_{\nu^\varepsilon}$, by Girsanov's theorem, the compensator of p on $[0, T] \times E \times A$ is $(\nu^\varepsilon_s(b)\, d_1(s, y, b) + d_2(s, y, b))\, \tilde{p}(\mathrm{d}s\, \mathrm{d}y\, \mathrm{d}b)$. We denote by $q^{\nu^\varepsilon}(\mathrm{d}s\, \mathrm{d}y\, \mathrm{d}b) := p(\mathrm{d}s\, \mathrm{d}y\, \mathrm{d}b) - (\nu^\varepsilon_s(b)\, d_1(s, y, b) + d_2(s, y, b))\, \tilde{p}(\mathrm{d}s\, \mathrm{d}y\, \mathrm{d}b)$ the compensated martingale measure of p under $\mathbb{P}^{x,a}_{\nu^\varepsilon}$. Therefore equation (6.11) becomes: $\mathbb{P}^{x,a}$-a.s.,

$$\mathrm{e}^{-2\,\delta\, s}|\bar{Y}_s|^2 \leqslant \mathrm{e}^{-2\,\delta\, T}|\bar{Y}_T|^2 - 2\int_s^T \int_{E\times A} \mathrm{e}^{-2\,\delta\, r}\, \bar{Y}_r\, \bar{Z}_r(X_s, b)\, q^{\nu^\varepsilon}(\mathrm{d}s\, \mathrm{d}y\, \mathrm{d}b) + 2\,\frac{\varepsilon}{\delta}\,\lambda_0(A),$$

for all $\varepsilon \in (0, 1)$. Moreover, from the arbitrariness of ε, we obtain

$$\mathrm{e}^{-2\,\delta\, s}|\bar{Y}_s|^2 \leqslant \mathrm{e}^{-2\,\delta\, T}|\bar{Y}_T|^2 - 2\int_s^T \int_{E\times A} \mathrm{e}^{-2\,\delta\, r}\, \bar{Y}_r\, \bar{Z}_r(X_s, b)\, q^{\nu^\varepsilon}(\mathrm{d}s\, \mathrm{d}y\, \mathrm{d}b). \tag{6.14}$$

From Lemma 3.1, we see that the stochastic integral in (6.14) is a martingale, so that, taking the expectation $\mathbb{E}^{x,a}_{\nu^\varepsilon}$, conditional on $\mathcal{F}_s$, with respect to $\mathbb{P}^{x,a}_{\nu^\varepsilon}$, we achieve

$$\mathrm{e}^{-2\,\delta\, s}|\bar{Y}_s|^2 \leqslant \mathrm{e}^{-2\,\delta\, T}\,\mathbb{E}^{x,a}_{\nu^\varepsilon}[|\bar{Y}_T|^2|\mathcal{F}_s]. \tag{6.15}$$

In particular, $(\mathrm{e}^{-2\,\delta\, s}|\bar{Y}_s|^2)_{t\geqslant 0}$ is a submartingale. Since $\bar{Y}$ is uniformly bounded, we see that $(\mathrm{e}^{-2\,\delta\, s}|\bar{Y}_s|^2)_{t\geqslant 0}$ is an uniformly integrable submartingale, therefore $\mathrm{e}^{-2\,\delta\, s}|\bar{Y}_s|^2 \to \xi_\infty \in \mathbf{L^1}(\Omega, \mathcal{F}, \mathbb{P}^{x,a}_{\nu^\varepsilon})$, as $s \to \infty$. Using again the boundedness of $\bar{Y}$, we obtain that $\xi_\infty = 0$, which implies $\bar{Y} = 0$. Finally, plugging $\bar{Y} = 0$ into (6.11) we conclude that $\bar{Z} = 0$.

Existence. Fix $(x, a, n) \in E \times A \times \mathbb{N}$. For $T > 0$, let $(Y^{T,n,x,a}, Z^{T,n,x,a}) = (Y^T, Z^T)$ denote the unique solution to the penalized BSDE (4.5) on $[0, T]$, whose existence is guaranteed by Proposition 4.3.

Step 1. *Convergence of* $(Y^T)_T$. Let $T, T' > 0$, with $T < T'$, and $s \in [0, T]$. We have

$$|Y^{T'}_s - Y^T_s|^2 \leqslant \mathrm{e}^{-2\,\delta\,(T-s)}\,\mathbb{E}^{x,a}_{\nu^\varepsilon}\left[|Y^{T'}_T - Y^T_T|^2|\mathcal{F}_s\right] \overset{T,\,T'\to\infty}{\longrightarrow} 0, \tag{6.16}$$

where the convergence result follows from (4.7). Let us now consider the sequence of real-valued càdlàg adapted processes $(Y^T)_T$. It follows from (6.16) that, for any $t \geqslant 0$, the sequence $(Y^T_t(\omega))_T$ is Cauchy for almost every ω, so that it converges $\mathbb{P}^{x,a}$-a.s. to some $(\mathcal{F}_t)$-measurable random variable Y_t, which is bounded from the right-hand side of (4.7). Moreover, using again (6.16) and (4.7), we see that, for any $0 \leqslant S < T \wedge T'$, with $T, T' > 0$, we have

$$\sup_{0\leqslant t\leqslant S} |Y^{T'}_t - Y^T_t| \leqslant \mathrm{e}^{-\delta\,(T\wedge T' - S)}\,\frac{M_f}{\delta} \overset{T,T'\to\infty}{\longrightarrow} 0. \tag{6.17}$$

In other words, the sequence $(Y^T)_{T>0}$ converges $\mathbb{P}^{x,a}$-a.s. to Y uniformly on compact subsets of $\mathbb{R}_+$. Since each Y^T is a càdlàg process, it follows that Y is càdlàg, as well. Finally, from estimate (4.7) we see that Y is uniformly bounded and therefore belongs to $\mathbf{S}^\infty$.

Step 2. *Convergence of* $(Z^T)_T$. Let $S,\, T,\, T' > 0$, with $S < T < T'$. Then, applying Itó's formula to $\mathrm{e}^{-2\,\delta\,s}|Y_t^{T'} - Y_t^T|^2$ between 0 and S, and taking the expectation, we find

$$\begin{aligned}
&\mathbb{E}^{x,a}\left[\int_0^S \int_{E\times A} \mathrm{e}^{-2\,\delta\,r}\,|Z_r^{T'}(y,b) - Z_r^T(y,b)|^2\,\tilde{p}(\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}b)\right]\\
&= \mathrm{e}^{-2\,\delta\,S}\mathbb{E}^{x,a}\left[|Y_S^{T'} - Y_S^T|^2\right] - |Y_0^{T'} - Y_0^T|^2\\
&\quad - 2n\,\mathbb{E}^{x,a}\left[\int_0^S \int_A \mathrm{e}^{-2\,\delta\,r}\,(Y_r^{T'} - Y_r^T)\,\{[Z_r^2(X_r,b)]^- - [Z_r^1(X_r,b)]^-\}\,\lambda_0(\mathrm{d}b)\,\mathrm{d}r\right]\\
&\quad - 2\mathbb{E}^{x,a}\left[\int_0^S \int_A \mathrm{e}^{-2\,\delta\,r}\,(Y_r^{T'} - Y_r^T)\,(Z_r^{T'}(X_r,b) - Z_r^T(X_r,b))\,\lambda_0(\mathrm{d}b)\,\mathrm{d}r\right].
\end{aligned}$$

Recalling the elementary inequality $bc \leqslant b^2 + c^2/4$, for any $b,\, c \in \mathbb{R}$, we get

$$\begin{aligned}
&\mathbb{E}^{x,a}\left[\int_0^S \int_{E\times A} \mathrm{e}^{-2\,\delta\,r}\,|Z_r^{T'}(y,b) - Z_r^T(y,b)|^2\,\tilde{p}(\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}b)\right]\\
&\leqslant \mathrm{e}^{-2\,\delta\,S}\mathbb{E}^{x,a}\left[|Y_S^{T'} - Y_S^T|^2\right] + 4(n^2+1)\,\lambda_0(A)\,\mathbb{E}^{x,a}\left[\int_0^S \mathrm{e}^{-2\,\delta\,r}\,|Y_r^{T'} - Y_r^T|^2\,\mathrm{d}r\right]\\
&\quad + \frac{1}{4}\mathbb{E}^{x,a}\left[\int_0^S \int_A \mathrm{e}^{-2\,\delta\,r}\,|[Z_r^2(X_r,b)]^- - [Z_r^1(X_r,b)]^-|^2\,\lambda_0(\mathrm{d}b)\,\mathrm{d}r\right]\\
&\quad + \frac{1}{4}\mathbb{E}^{x,a}\left[\int_0^S \int_A \mathrm{e}^{-2\,\delta\,r}\,|Z_r^{T'}(X_r,b) - Z_r^T(X_r,b)|^2\,\lambda_0(\mathrm{d}b)\,\mathrm{d}r\right].
\end{aligned}$$

Recalling the form of the compensator $\tilde{p}$, we get

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}^{x,a}\left[\int_0^S \int_{E\times A} \mathrm{e}^{-2\,\delta\,r}\,|Z_r^{T'}(y,b) - Z_r^T(y,b)|^2\,\tilde{p}(\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}b)\right]\\
&\leqslant \mathrm{e}^{-2\,\delta\,S}\mathbb{E}^{x,a}\left[|Y_S^{T'} - Y_S^T|^2\right] + 4(n^2+1)\,\lambda_0(A)\,\mathbb{E}^{x,a}\left[\int_0^S \mathrm{e}^{-2\,\delta\,r}\,|Y_r^{T'} - Y_r^T|^2\,\mathrm{d}r\right] \overset{T,T'\to\infty}{\longrightarrow} 0,
\end{aligned}$$

where the convergence to zero follows from estimate (6.17). Then, for any $S > 0$, we see that $(Z^T_{|[0,\,S]})_{T>S}$ is a Cauchy sequence in the Hilbert space $\mathbf{L^2_{x,a}}(\mathrm{q};\mathbf{0},\,\mathbf{S})$. Therefore, we deduce that there exists $\tilde{Z}^S \in \mathbf{L^2_{x,a}}(\mathrm{q};\mathbf{0},\,\mathbf{S})$ such that $(Z^T_{|[0,\,S]})_{T>S}$ converges to $\tilde{Z}^S$ in $\mathbf{L^2_{x,a}}(\mathrm{q};\mathbf{0},\,\mathbf{S})$, *i.e.*,

$$\mathbb{E}^{x,a}\left[\int_0^S \int_{E\times A} \mathrm{e}^{-2\,\delta\,r}\,|Z_r^T(y,b) - \tilde{Z}_r^S(y,b)|^2\,\tilde{p}(\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}b)\right] \overset{T\to\infty}{\longrightarrow} 0.$$

Notice that $\tilde{Z}^{S'}_{|[0,\,S]} = \tilde{Z}^S$, for any $0 \leqslant S \leqslant S' < \infty$. Indeed, $\tilde{Z}^{S'}_{|[0,\,S]}$, as $\tilde{Z}^S$, is the limit in $\mathbf{L^2_{x,a}}(\mathrm{q};\mathbf{0},\,\mathbf{S})$ of $(Z^T_{|[0,\,S]})_{T>S}$. Hence, we define $Z_s = \tilde{Z}^S_s$ for all $s \in [0,\,S]$ and for any $S > 0$. Observe that $Z \in \mathbf{L^2_{x,a,loc}}(\mathrm{q})$. Moreover, for any $S > 0$, $(Z^T_{|[0,\,S]})_{T>S}$ converges to $Z_{|[0,\,S]}$ in $\mathbf{L^2_{x,a}}(\mathrm{q};\mathbf{0},\,\mathbf{S})$, *i.e.*,

$$\mathbb{E}^{x,a}\left[\int_0^S \int_{E\times A} \mathrm{e}^{-2\,\delta\,r}\,|Z_r^T(y,b) - Z_r(y,b)|^2\,\tilde{p}(\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}b)\right] \overset{T\to\infty}{\longrightarrow} 0. \tag{6.18}$$

Now, fix $S \in [0,\, T]$ and consider the BSDE satisfied by (Y^T, Z^T) on $[0,\, S]$: $\mathbb{P}^{x,a}$-a.s.,

$$
\begin{aligned}
Y_t^T = {} & Y_S^T - \delta \int_t^S Y_r^T \,\mathrm{d}r + \int_t^S f(X_r, I_r)\,\mathrm{d}r - \int_t^S \int_{E\times A} Z_r^T(y,b)\, q(\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}b) \\
& - n \int_t^S \int_A [Z_r^T(X_r,b)]^- \,\lambda_0(\mathrm{d}b)\,\mathrm{d}r - \int_t^S \int_A Z_r^T(X_r,b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}r, \quad 0 \leqslant t \leqslant S.
\end{aligned}
$$

From (6.18) and (6.17), we can pass to the limit in the above BSDE by letting $T \to \infty$ keeping S fixed. Then we deduce that (Y, Z) solves the penalized BSDE (4.4) on $[0,\, S]$. Since S is arbitrary, it follows that (Y, Z) solves equation (4.4) on $[0,\, \infty)$.

Representation formula (4.8). Fix $n \in \mathbb{N}$, and for any $\nu \in \mathcal{V}^n$, let us introduce the compensated martingale measure $q^\nu(\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}b) = q(\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}b) - (\nu_s(b) - 1)\, d_1(s,y,b)\, \tilde{p}(\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}b)$ under $\mathbb{P}_\nu^{x,a}$. Fix $T \geqslant s$ and apply Itô's formula to $\mathrm{e}^{-\delta\, r}\, Y_r^{n,x,a}$ between s and T. Then we obtain:

$$
\begin{aligned}
Y_s^{n,x,a} = {} & \mathrm{e}^{-\delta\,(T-s)}\, Y_T^{n,x,a} + \int_s^T \mathrm{e}^{-\delta\,(r-s)}\, f(X_r, I_r)\,\mathrm{d}r \\
& - \int_s^T \int_A \mathrm{e}^{-\delta\,(r-s)} \left\{ n[Z_r^{n,x,a}(X_r,b)]^- + \nu_r(a)\, Z_r^{n,x,a}(X_r,b) \right\} \lambda_0(\mathrm{d}b)\,\mathrm{d}r \\
& - \int_s^T \int_{E\times A} \mathrm{e}^{-\delta\,(r-s)}\, Z_r^{n,x,a}(y,b)\, q^\nu(\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}b), \quad s \in [t,\, T].
\end{aligned}
\tag{6.19}
$$

Taking the expectation in (6.19) under $\mathbb{P}_\nu^{x,a}$, conditional on $\mathcal{F}_s$, and since we know that $Z^{n,x,a}$ is in $\mathbf{L^2_{loc,x,a}}(\mathrm{q})$, we get from Lemma 3.1 that, $\mathbb{P}^{x,a}$-a.s.,

$$
\begin{aligned}
Y_s^{n,x,a} = {} & \mathbb{E}_\nu^{x,a} \left[\mathrm{e}^{-\delta\,(T-s)}\, Y_T^{n,x,a} + \int_s^T \mathrm{e}^{-\delta\,(r-s)}\, f(X_r, I_r)\,\mathrm{d}r \middle| \mathcal{F}_s \right] \\
& - \mathbb{E}_\nu^{x,a} \left[\int_s^T \int_A \mathrm{e}^{-\delta\,(r-s)} \left\{ n[Z_r^{n,x,a}(X_r,b)]^- + \nu_r(a)\, Z_r^{n,x,a}(X_r,b) \right\} \lambda_0(\mathrm{d}b)\,\mathrm{d}r \middle| \mathcal{F}_s \right].
\end{aligned}
\tag{6.20}
$$

From the elementary numerical inequality: $n[z]^- + \nu z \geqslant 0$ for all $z \in \mathbb{R}$, we deduce by (6.20) that, for all $\nu \in \mathcal{V}^n$,

$$
\begin{aligned}
Y_s^{n,x,a} & \leqslant \mathbb{E}_\nu^{x,a} \left[\mathrm{e}^{-\delta\,(T-s)}\, Y_T^{n,x,a} + \int_s^T \mathrm{e}^{-\delta\,(r-s)}\, f(X_r, I_r)\,\mathrm{d}r \middle| \mathcal{F}_s \right] \\
& \leqslant \mathbb{E}_\nu^{x,a} \left[\mathrm{e}^{-\delta\,(T-s)}\, Y_T^{n,x,a} + \int_s^\infty \mathrm{e}^{-\delta\,(r-s)}\, f(X_r, I_r)\,\mathrm{d}r \middle| \mathcal{F}_s \right].
\end{aligned}
$$

Since $Y^{n,x,a}$ is in $\mathbf{S}^\infty$, sending $T \to \infty$, we obtain from the conditional version of Lebesgue dominated convergence theorem that

$$
Y_s^{n,x,a} \leqslant \mathbb{E}_\nu^{x,a} \left[\int_s^\infty \mathrm{e}^{-\delta\,(r-s)}\, f(X_r, I_r)\,\mathrm{d}r \middle| \mathcal{F}_s \right],
$$

for all $\nu \in \mathcal{V}^n$. Therefore,

$$
Y_s^{n,x,a} \leqslant \operatorname*{ess\,inf}_{\nu \in \mathcal{V}^n} \mathbb{E}_\nu^{x,a} \left[\int_s^\infty \mathrm{e}^{-\delta\,(r-s)}\, f(X_r, I_r)\,\mathrm{d}r \middle| \mathcal{F}_s \right].
\tag{6.21}
$$

On the other hand, for $\varepsilon \in (0,\, 1)$, let us consider the process $\nu^\varepsilon \in \mathcal{V}^n$ defined by:

$$
\nu_s^\varepsilon(b) = n\, \mathbb{1}_{\{Z_s^{n,x,a}(X_{s-},b) \leqslant 0\}} + \varepsilon\, \mathbb{1}_{\{0 < Z_s^{n,x,a}(X_{s-},b) < 1\}} + \varepsilon\, Z_s^{n,x,a}(X_{s-},b)^{-1}\, \mathbb{1}_{\{Z_s^{n,x,a}(X_{s-},b) \geqslant 1\}}
$$

(notice that we can not take $\nu_s(b) = n\mathbb{1}_{\{Z_s^n(X_{s-},b)\leqslant 0\}}$, since this process does not belong to $\mathcal{V}^n$ because of the requirement of strict positivity). By construction, we have

$$n[Z_s^n(X_{s-},b)]^- + \nu_s^\varepsilon(b)\, Z_s^n(X_{s-},b) \leqslant \varepsilon, \quad s \geqslant 0,\, b \in A,$$

and thus for this choice of $\nu = \nu^\varepsilon$ in (6.20):

$$Y_s^{n,x,a} \geqslant \mathbb{E}_{\nu^\varepsilon}^{x,a}\left[\mathrm{e}^{-\delta\,(T-s)}\, Y_T^{n,x,a} + \int_s^T \mathrm{e}^{-\delta\,(r-s)}\, f(X_r, I_r)\,\mathrm{d}r \Big| \mathcal{F}_s\right] - \varepsilon\, \frac{1-\mathrm{e}^{-\delta(T-s)}}{\delta}\, \lambda_0(A).$$

Letting $T \to \infty$, since f is bounded by M_f and $Y^{n,x,a}$ is in $\mathbf{S}^\infty$, it follows from the conditional version of Lebesgue dominated convergence theorem that

$$\begin{aligned} Y_s^{n,x,a} &\geqslant \mathbb{E}_{\nu^\varepsilon}^{x,a}\left[\int_s^\infty \mathrm{e}^{-\delta\,(r-s)}\, f(X_r, I_r)\,\mathrm{d}r \Big| \mathcal{F}_s\right] - \frac{\varepsilon}{\delta}\, \lambda_0(A), \\ &\geqslant \operatorname*{ess\,inf}_{\nu\in\mathcal{V}^n} \mathbb{E}_{\nu}^{x,a}\left[\int_s^\infty \mathrm{e}^{-\delta\,(r-s)}\, f(X_r, I_r)\,\mathrm{d}r \Big| \mathcal{F}_s\right] - \frac{\varepsilon}{\delta}\, \lambda_0(A). \end{aligned}$$

From the arbitrariness of ε, together with (6.21), this is enough to prove the required representation of $Y^{n,x,a}$.

6.4. Proof of Lemma 4.5

Fix $T > 0$. In what follows we shall denote by $C > 0$ a generic positive constant depending on M_f, δ and T, which may vary from line to line. Let us apply Itô's formula to $|Y_r^{n,x,a}|^2$ between 0 and T. Noticing that $K^{n,x,a}$ is continuous and $\Delta Y_r^{n,x,a} = \int_{E\times A} Z_r^{n,x,a}(y,b)\, p(\{r\}\,\mathrm{d}y\,\mathrm{d}b)$, we get: $\mathbb{P}^{x,a}$-a.s.,

$$\begin{aligned} \mathbb{E}^{x,a}\left[|Y_0^{n,x,a}|^2\right] = {}& \mathbb{E}^{x,a}\left[|Y_T^{n,x,a}|^2\right] - 2\mathbb{E}^{x,a}\left[\int_0^T |Y_r^{n,x,a}|^2\,\mathrm{d}r\right] \\ & - 2\mathbb{E}^{x,a}\left[\int_s^T Y_r^{n,x,a}\, dK_r^{n,x,a}\right] + 2\mathbb{E}^{x,a}\left[\int_0^T Y_r^{n,x,a}\, f(X_r, I_r)\,\mathrm{d}r\right] \\ & - 2\mathbb{E}^{x,a}\left[\int_0^T \int_A Y_r^{n,x,a}\, Z_r^{n,x,a}(X_r, b)\, \lambda_0(\mathrm{d}b)\,\mathrm{d}r\right] \\ & - \mathbb{E}^{x,a}\left[\int_0^T \int_{E\times A} |Z_r^{n,x,a}(y,b)|^2\, \tilde{p}(\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}b)\right]. \end{aligned}$$

Set now $C_Y := \frac{M_f}{\delta}$. Recalling the uniform estimate (4.7) on Y^n, and using elementary inequalities, we get

$$\begin{aligned} \mathbb{E}^{x,a}\left[\int_0^T \int_{E\times A} |Z_s^{n,x,a}(y,\, b)|^2\, \tilde{p}(\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}b)\right] \leqslant {}& C_Y^2 + 2\,T\,C_Y^2 + 2\,T\,C_Y\,M_f + 2\,C_Y\,T\,\mathbb{E}^{x,a}\left[|K_T^{n,x,a}|\right] \\ & + \frac{C_Y}{\alpha}\,T\,\lambda_0(A) + \alpha\, C_Y \mathbb{E}^{x,a}\left[\int_0^T \int_A |Z_r^{n,x,a}(X_s, b)|^2\, \lambda_0(\mathrm{d}b)\,\mathrm{d}r\right], \end{aligned} \tag{6.22}$$

for any $\alpha > 0$. At this point, from relation (4.4), we obtain:

$$K_T^{n,x,a} = Y_0^{n,x,a} - Y_T^{n,x,a} - \delta \int_0^T \int_A Y_s^{n,x,a}\,\mathrm{d}s + \int_0^T \int_{E\times A} Z_s^{n,x,a}(y,b)\,q(\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}b) + \int_0^T f(X_s, I_s)\mathrm{d}s + \int_0^T \int_A Z_s^{n,x,a}(X_s, b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}s. \tag{6.23}$$

Then, using the inequality $2bc \leqslant \frac{1}{\beta}b^2 + \beta c^2$, for any $\beta > 0$, and taking the expected value we have

$$2\,\mathbb{E}^{x,a}\left[|K_T^{n,x,a}|\right] \leqslant 2\,\delta\,C_Y\,T + 2\,M_f\,T + \frac{T}{\beta}\,\lambda_0(A) + \beta\,\mathbb{E}^{x,a}\left[\int_0^T \int_A |Z_s^{n,x,a}(X_s,b)|^2\,\lambda_0(\mathrm{d}b)\,\mathrm{d}s\right]. \tag{6.24}$$

Plugging (6.24) into (6.22), we get

$$\mathbb{E}^{x,a}\left[\int_0^T \int_{E\times A} |Z_s^{n,x,a}(y,\,b)|^2\,\tilde{p}(\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}b)\right] \leqslant C + C_Y\,\left(2\,T\,\beta + \alpha\right)\int_0^T \int_A |Z_s^{n,x,a}(X_s,b)|^2\,\lambda_0(\mathrm{d}b)\,\mathrm{d}s.$$

Hence, choosing $\alpha + 2\,T\,\beta = \frac{1}{2\,C_Y}$, we get

$$\frac{1}{2}\,\mathbb{E}^{x,a}\left[\int_0^T \int_{E\times A} |Z_s^{n,x,a}(y,\,b)|^2\,\tilde{p}(\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}b)\right] \leqslant C,$$

which gives the required uniform estimate for $(Z^{n,x,a})$, and also $(K^{n,x,a})$ by (6.23).

6.5. Proof of the Lemma 4.6

Let $(x,a) \in E\times A$, and consider a triplet $(Y^{x,a}, Z^{x,a}, K^{x,a}) \in \mathbf{S}^\infty \times \mathbf{L^2_{x,a,loc}}(\mathrm{q}) \times \mathbf{K^2_{x,a,loc}}$ satisfying (4.1)–(4.2). Applying Itô's formula to $\mathrm{e}^{-\delta\,r}\,Y_r^{x,a}$ between s and $T > s$ (see *e.g.* Thm. 3.89 in [33]), and recalling that $K^{x,a}$ is nondecreasing, we have

$$Y_s^{x,a} \leqslant \mathrm{e}^{-\delta\,(T-s)}\,Y_T^{x,a} + \int_s^T \mathrm{e}^{-\delta\,(r-s)}\,f(X_r, I_r)\,\mathrm{d}r - \int_s^T \int_A \mathrm{e}^{-\delta\,(r-s)}\,Z_r^{x,a}(X_r,b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}r - \int_s^T \int_{E\times A} \mathrm{e}^{-\delta\,(r-s)}\,Z_r^{x,a}(y,b)\,\tilde{q}(\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}b), \quad 0 \leqslant s \leqslant T < \infty. \tag{6.25}$$

Then for any $\nu \in \mathcal{V}$, let us introduce the compensated martingale measure $q^\nu(\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}b) = q(\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}b) - (\nu_s(b) - 1)\,d_1(s,y,b)\,\tilde{p}(\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}b)$ under $\mathbb{P}_\nu^{x,a}$. Taking expectation in (6.25) under $\mathbb{P}_\nu^{x,a}$, conditional on $\mathcal{F}_s$, and recalling that $Z^{x,a}$ is in $\mathbf{L^2_{x,a,loc}}(\mathrm{q})$, we get from Lemma 3.1 that, $\mathbb{P}^{x,a}$-a.s.,

$$Y_s^{x,a} \leqslant \mathbb{E}_\nu^{x,a}\left[\mathrm{e}^{-\delta\,(T-s)}\,Y_T^{x,a} + \int_s^T \mathrm{e}^{-\delta\,(r-s)}\,f(X_r, I_r)\,\mathrm{d}r \middle| \mathcal{F}_s\right] - \mathbb{E}_\nu^{x,a}\left[\int_s^T \int_A \mathrm{e}^{-\delta\,(r-s)}\,\nu_r(a)\,\bar{Z}_r^{x,a}(X_r,b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}r \middle| \mathcal{F}_s\right]. \tag{6.26}$$

Furthermore, since ν is strictly positive and $Z^{x,a}$ satisfies the nonnegative constraint (4.2), from inequality (6.26) we get

$$
\begin{aligned}
Y_s^{x,a} &\leqslant \mathbb{E}_\nu^{x,a}\left[\mathrm{e}^{-\delta\,(T-s)}\,Y_T^{x,a} + \int_s^T \mathrm{e}^{-\delta\,(r-s)}\,f(X_r,I_r)\,\mathrm{d}r\bigg|\mathcal{F}_s\right]\\
&\leqslant \mathbb{E}_\nu^{x,a}\left[\mathrm{e}^{-\delta\,(T-s)}\,Y_T^{x,a} + \int_s^\infty \mathrm{e}^{-\delta\,(r-s)}\,f(X_r,I_r)\,\mathrm{d}r\bigg|\mathcal{F}_s\right].
\end{aligned}
$$

Finally, sending $T\to\infty$ and recalling that $Y^{x,a}$ is in $\mathbf{S}^\infty$, the conditional version of Lebesgue dominated convergence theorem yields

$$
Y_s^{x,a} \leqslant \mathbb{E}_\nu^{x,a}\left[\int_s^\infty \mathrm{e}^{-\delta\,(r-s)}\,f(X_r,I_r)\,\mathrm{d}r\bigg|\mathcal{F}_s\right]
$$

for all $\nu\in\mathcal{V}$, and the conclusion follows from the arbitrariness of $\nu\in\mathcal{V}$.

6.6. Proof of Lemma 5.3. (Identification property)

Fix $(x,a,n)\in E\times A\times\mathbb{N}$. Let $(Y^n,Z^n)=(Y^{n,x,a},Z^{n,x,a})$ be the solution to the penalized BSDE (4.4). From Proposition 4.4 we know that there exists a sequence $(Y^{n,T},Z^{n,T})_T=(Y^{n,T,x,a},Z^{n,T,x,a})_T$ in $\mathbf{S}^\infty\times\mathbf{L^2_{x,a,loc}}(\mathrm{q})$ such that, when T goes to infinity, $(Y^{n,T})_T$ converges $\mathbb{P}^{x,a}$-a.s. to (Y^n) and $(Z^{n,T})_T$ converges to (Z^n) in $\mathbf{L^2_{x,a,loc}}(\mathrm{q})$. Let us now fix $T,S>0$, $S<T$, and consider the BSDE solved by $(Y^{n,T},Z^{n,T})$ on $[0,\,S]$:

$$
\begin{aligned}
Y_t^{n,T} \;=\; Y_S^{n,T} &- \delta\int_t^S Y_r^{n,T}\,\mathrm{d}r + \int_t^S f(X_r,I_r)\,\mathrm{d}r - \int_t^S\int_{E\times A} Z_r^{n,T}(y,b)\,q(\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}b)\\
&-n\int_t^S\int_A [Z_r^{n,T}(X_r,b)]^-\,\lambda_0(\mathrm{d}b)\,\mathrm{d}r - \int_t^S\int_A Z_r^{n,T}(X_r,b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}r,\quad 0\leqslant t\leqslant S.
\end{aligned}
$$

Then, it follows from the fixed point argument used in the proof of Proposition 4.3, that there exists a sequence $(Y^{n,T,k},Z^{n,T,k})_k=(Y^{n,T,k,x,a},Z^{n,T,k,x,a})_k$ in $\mathbf{L^2_{x,a}(0,S)}\times\mathbf{L^2_{x,a}}(\mathrm{q},\mathbf{0},\mathbf{S})$ converging to $(Y^{n,T},Z^{n,T})$ in $\mathbf{L^2_{x,a}(0,S)}\times\mathbf{L^2_{x,a}}(\mathrm{q},\mathbf{0},\mathbf{S})$, such that $(Y^{n,T,0},Z^{n,T,0})=(0,0)$ and

$$
\begin{aligned}
Y_t^{n,T,k+1} = Y_S^{n,T,k} &- \delta\int_t^S Y_r^{n,T,k}\,\mathrm{d}r + \int_t^S f(X_r,I_r)\,\mathrm{d}r\\
&-n\int_t^S\int_A [Z_r^{n,T,k}(X_r,b)]^-\,\lambda_0(\mathrm{d}b)\,\mathrm{d}r - \int_t^S\int_A Z_r^{n,T,k}(X_r,b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}r,\\
&-\int_t^S\int_{E\times A} Z_r^{n,T,k+1}(y,b)\,q(\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}b),\qquad 0\leqslant t\leqslant S.
\end{aligned}
$$

Let us define

$$
v^{n,T}(x,a) := Y_0^{n,T},\quad v^{n,T,k}(x,a) := Y_0^{n,T,k}.
$$

We start by noticing that, for $k=0$, we have, $\mathbb{P}^{x,a}$-a.s.,

$$
Y_t^{n,T,1} = \mathbb{E}^{x,a}\left[\int_t^S f(X_r,I_r)\,\mathrm{d}r\bigg|\mathcal{F}_t\right],\quad t\in[0,\,S].
$$

Then, from the Markov property of (X,I) we get

$$
Y_t^{n,T,1} = v^{n,T,1}(X_t,I_t),\quad \mathrm{d}\mathbb{P}^{x,a}\otimes\mathrm{d}t\text{ -a.e.} \tag{6.27}
$$

Furthermore, identification (6.27) implies

$$Z_t^{n,T,1}(y,b) = v^{n,T,1}(y,b) - v^{n,T,1}(X_{t-}, I_{t-}), \tag{6.28}$$

where (6.28) has to be understood as an equality (almost everywhere) between elements of the space $\mathbf{L^2_{x,a}}(\mathrm{q}; \mathbf{0}, \mathbf{S})$. At this point we consider the inductive step: $1 \leqslant k \in \mathbb{N}$, and assume that, $\mathbb{P}^{x,a}$-a.s.,

$$\begin{aligned} Y_t^{n,T,k} &= v^{n,T,k}(X_t, I_t) \\ Z_t^{n,T,k}(y,b) &= v^{n,T,k}(y,b) - v^{n,T,k}(X_{t-}, I_{t-}). \end{aligned}$$

Then

$$\begin{aligned} Y_t^{n,T,k+1} = \mathbb{E}^{x,a}\Bigg[& v^{n,T,k}(X_S, I_S) - \delta \int_t^S v^{n,T,k}(X_r, I_r)\,\mathrm{d}r + \int_t^S f(X_r, I_r)\,\mathrm{d}r \\ & -n \int_t^S \int_A [v^{n,T,k}(X_r, b) - v^{n,T,k}(X_r, I_r)]^- \,\lambda_0(\mathrm{d}b)\,\mathrm{d}r \\ & - \int_t^S \int_A v^{n,T,k}(X_r, b) - v^{n,T,k}(X_r, I_r)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}r \Bigg| \mathcal{F}_t \Bigg], \qquad 0 \leqslant t \leqslant S. \end{aligned}$$

Using again the Markov property of (X, I), we achieve that

$$Y_t^{n,T,k+1} = v^{n,T,k+1}(X_t, I_t), \quad \mathrm{d}\mathbb{P}^{x,a} \otimes \mathrm{d}t \text{ -a.e.} \tag{6.29}$$

Then, applying the Itô formula to $|Y_t^{n,T,k} - Y_t^{n,T}|^2$ and taking the supremum of t between 0 and S, one can show that

$$\mathbb{E}^{x,a}\left[\sup_{0 \leqslant t \leqslant S} \left| Y_t^{n,T,k} - Y_t^{\delta,n,T} \right|^2 \right] \to 0 \quad \text{as } k \text{ goes to infinity.}$$

Therefore, $v^{n,T,k}(x,a) \to v^{n,T}(x,a)$ as k goes to infinity, for all $(x,a) \in E \times A$, from which it follows that

$$Y_t^{n,T,x,a} = v^{n,T}(X_t, I_t), \quad \mathrm{d}\mathbb{P}^{x,a} \otimes \mathrm{d}t \text{ -a.e.} \tag{6.30}$$

Finally, from (6.17) we have that $(Y^{n,T,x,a})_T$ converges $\mathbb{P}^{x,a}$-a.s. to $(Y^{n,x,a})$, uniformly on compact sets of $\mathbb{R}$. Thus, $v^{n,T}(x,a) \to v^n(x,a)$ as T goes to infinity, for all $(x,a) \in E \times A$, and this gives the requested identification $Y_t^{n,x,a} = v^n(X_t, I_t)$, $\mathrm{d}\mathbb{P}^{x,a} \otimes \mathrm{d}t$ -a.e.

6.7. Proof of Proposition 5.6

We start by giving a technical result. In the sequel, Π^{n_1,n_2} and Γ^{n_1,n_2} will denote respectively the random sequences $(T_{n_1}, E_{n_1}, A_{n_1}, T_{n_1+1}, E_{n_1+1}, A_{n_1+1}, \ldots, T_{n_2}, E_{n_2}, A_{n_2})$ and

$(T_{n_1}, A_{n_1}, T_{n_1+1}, A_{n_1+1}, \ldots, T_{n_2}, A_{n_2})$, n_1, $n_2 \in \mathbb{N} \setminus \{0\}$, $n_1 \leq n_2$, where $(T_k,\, E_k,\, A_k)_{k \geq 1}$ is the sequence of random variables introduced in Section 3.1.

Lemma 6.1. *Assume that Hypotheses* **(HhλQ)**, **(Hλ_0)** *and* **(Hf)** *hold. Let $\nu^n : \Omega \times \mathbb{R}_+ \times (\mathbb{R}_+ \times A)^n \times A \to (0, \infty)$, $n > 1$ (resp. $\nu^0 : \Omega \times \mathbb{R}_+ \times A \to (0, \infty)$), be some $\mathcal{P} \otimes \mathcal{B}((\mathbb{R}_+ \times A)^n) \otimes \mathcal{A}$-measurable maps, uniformly bounded with respect to n (resp. a bounded $\mathcal{P} \otimes \mathcal{A}$-measurable map). Let moreover $g : \Omega \times A \to (0, \infty)$ be a bounded $\mathcal{A}$-measurable map, and set*

$$\nu_t(b) = \nu_t^0(b)\, \mathbb{1}_{\{t \leqslant T_1\}} + \sum_{n=1}^{\infty} \nu_t^n(\Gamma^{1,n}, b)\, \mathbb{1}_{\{T_n < t \leqslant T_{n+1}\}}, \tag{6.31}$$

$$\nu'_t(b) = g(b)\, \mathbb{1}_{\{t \leqslant T_1\}} + \nu_t^0(b)\, \mathbb{1}_{\{T_1 < t \leqslant T_2\}} + \sum_{n=2}^{\infty} \nu_t^{n-1}(\Gamma^{2,n}, b)\, \mathbb{1}_{\{T_n < t \leqslant T_{n+1}\}}. \tag{6.32}$$

Fix $x \in E$, $a, a' \in A$. Then, for every $n > 1$, for every $\mathcal{B}((\mathbb{R}_+ \times E \times A)^n)$-measurable function $F : (\mathbb{R}_+ \times E \times A)^n \to \mathbb{R}$,

$$\mathbb{E}_{\nu'}^{x,a'}\left[F(\Pi^{1,n})|\mathcal{F}_{T_1}\right] = \frac{\mathbb{E}_{\nu}^{x,a}\left[\mathbb{1}_{\{T_1>\tau\}}\,F(\tau,\chi,\xi,\Pi^{1,n-1})\right]}{\mathbb{P}_{\nu}^{x,a}(T_1>\tau)}\Bigg|_{\tau=T_1,\chi=X_1,\,\xi=A_1}. \tag{6.33}$$

Proof of the Lemma.

Taking into account (3.8), (3.9), and (6.32), we have: for all $r \geqslant T_1$,

$$\begin{aligned}
&\mathbb{P}_{\nu'}^{x,a'}\left[T_2 > r, E_2 \in F, A_2 \in C|\mathcal{F}_{T_1}\right]\\
&= \int_r^\infty \int_F \exp\left(-\int_{T_1}^s \lambda(\phi(t-T_1,E_1,A_1),A_1)\,\mathrm{d}t - \int_{T_1}^s\int_A \nu_t^0(b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}t\right)\\
&\quad\times \lambda(\phi(s-T_1,E_1,A_1),A_1)\,Q(\phi(s-T_1,E_1,A_1),A_1,\mathrm{d}y)\,\mathrm{d}s\\
&\quad+\int_r^\infty \int_C \exp\left(-\int_{T_1}^s \lambda(\phi(t-T_1,E_1,A_1),A_1)\,\mathrm{d}t - \int_{T_1}^s\int_A \nu_t^0(b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}t\right)\,\nu_s^0(b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}s,
\end{aligned} \tag{6.34}$$

and, for all $r \geqslant T_n$, $n > 2$,

$$\begin{aligned}
&\mathbb{P}_{\nu'}^{x,a}\left[T_{n+1} > r, E_{n+1} \in F, A_{n+1} \in C|\mathcal{F}_{T_n}\right]\\
&= \int_r^\infty \int_F \exp\left(-\int_{T_n}^s \lambda(\phi(t-T_n,E_n,A_n),A_n)\,\mathrm{d}t\right)\exp\left(-\int_{T_n}^s\int_A \nu_t^{n-1}(\Gamma^{2,n},b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}t\right)\\
&\quad\times \lambda(\phi(s-T_n,E_n,A_n),A_n)\,Q(\phi(s-T_n,E_n,A_n),A_n,\mathrm{d}y)\,\mathrm{d}s\\
&\quad+\int_r^\infty \int_C \exp\left(-\int_{T_n}^s \lambda(\phi(t-T_n,E_n,A_n),A_n)\,\mathrm{d}t\right)\\
&\quad\times \exp\left(-\int_{T_n}^s\int_A \nu_t^{n-1}(\Gamma^{2,n},b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}t\right)\nu_s^{n-1}(\Gamma^{2,n},b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}s.
\end{aligned} \tag{6.35}$$

We will prove identity (6.33) by induction. Let us start by showing that (6.33) holds in the case $n = 2$, namely that, for every $\mathcal{B}((\mathbb{R}_+ \times E \times A)^2)$-measurable function $F : (\mathbb{R}_+ \times E \times A)^2 \to \mathbb{R}$,

$$\mathbb{E}_{\nu'}^{x,a'}\left[F(\Pi^{1,2})|\mathcal{F}_{T_1}\right] = \frac{\mathbb{E}_{\nu}^{x,a}\left[\mathbb{1}_{\{T_1>\tau\}}\,F(\tau,\chi,\xi,\Pi^{1,1})\right]}{\mathbb{P}_{\nu}^{x,a}(T_1>\tau)}\Bigg|_{\tau=T_1,\chi=X_1,\,\xi=A_1}. \tag{6.36}$$

From (6.34) we get

$$\begin{aligned}
&\mathbb{E}_{\nu'}^{x,a'}\left[F(\Pi^{1,2})|\mathcal{F}_{T_1}\right] = \mathbb{E}_{\nu'}^{x,a'}\left[F(T_1,E_1,A_1,T_2,E_2,A_2)|\mathcal{F}_{T_1}\right]\\
&= \int_{T_1}^\infty \int_E F(T_1,E_1,A_1,s,y,A_1)\,\exp\left(-\int_{T_1}^s \lambda(\phi(t-T_1,E_1,A_1),A_1)\,\mathrm{d}t - \int_{T_1}^s\int_A \nu_t^0(b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}t\right)\\
&\quad\times \lambda(\phi(s-T_1,E_1,A_1),A_1)\,Q(\phi(s-T_1,E_1,A_1),A_1,\mathrm{d}y)\,\mathrm{d}s\\
&\quad+\int_{T_1}^\infty \int_A F(T_1,E_1,A_1,s,\phi(s-T_1,E_1,A_1),b)\\
&\quad\times \exp\left(-\int_{T_1}^s \lambda(\phi(t-T_1,E_1,A_1),A_1)\,\mathrm{d}t - \int_{T_1}^s\int_A \nu_t^0(b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}t\right)\,\nu_s^0(b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}s.
\end{aligned}$$

On the other hand,

$$\mathbb{P}_{\nu}^{x,a}(T_1>\tau) = \exp\left(-\int_0^\tau \lambda(\phi(t-\tau,\chi,\xi),\xi)\,\mathrm{d}t - \int_0^\tau\int_A \nu_t^0(b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}t\right),$$

and

$$
\begin{aligned}
&\mathbb{E}_{\nu}^{x,a}\left[\mathbb{1}_{\{T_1>\tau\}}\, F(\tau,\chi,\xi,\Pi^{1,1})\right] = \mathbb{E}_{\nu}^{x,a}\left[\mathbb{1}_{\{T_1>\tau\}}\, F(\tau,\chi,\xi,T_1,E_1,A_1)\right]\\
&= \int_\tau^\infty \int_E \mathbb{1}_{\{s>\tau\}}\, F(\tau,\chi,\xi,s,y,\xi)\, \exp\left(-\int_0^s \lambda(\phi(t-\tau,\chi,\xi),\xi)\,\mathrm{d}t - \int_0^s\int_A \nu_t^0(b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}t\right)\\
&\quad\times \lambda(\phi(s-\tau,\chi,\xi),\xi)\, Q(\phi(s-\tau,\chi,\xi),\xi,\mathrm{d}y)\,\mathrm{d}s\\
&\quad+ \int_\tau^\infty \int_A \mathbb{1}_{\{s>\tau\}}\, F(\tau,\chi,\xi,s,\phi(s-\tau,\chi,\xi),b)\\
&\quad\times \exp\left(-\int_0^s \lambda(\phi(t-\tau,\chi,\xi),\xi)\,\mathrm{d}t - \int_0^s\int_A \nu_t^0(b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}t\right)\nu_s^0(b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}s.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{\mathbb{E}_{\nu}^{x,a}\left[\mathbb{1}_{\{T_1>\tau\}}\, F(\tau,\chi,\xi,\Pi^{1,1})\right]}{\mathbb{P}_{\nu}^{x,a}(T_1>\tau)}\\
&= \exp\left(\int_0^\tau \lambda(\phi(t-\tau,\chi,\xi),\xi)\,\mathrm{d}t + \int_0^\tau\int_A \nu_t^0(b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}t\right)\int_\tau^\infty\int_E \mathbb{1}_{\{s>\tau\}}\, F(\tau,\chi,\xi,s,y,\xi)\\
&\quad\times \exp\left(-\int_0^s \lambda(\phi(t-\tau,\chi,\xi),\xi)\,\mathrm{d}t - \int_0^s\int_A \nu_t^0(b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}t\right)\\
&\quad\times \lambda(\phi(s-\tau,\chi,\xi),\xi)\, Q(\phi(s-\tau,\chi,\xi),\xi,\mathrm{d}y)\,\mathrm{d}s\\
&\quad+ \exp\left(\int_0^\tau \lambda(\phi(t-\tau,\chi,\xi),\xi)\,\mathrm{d}t + \int_0^\tau\int_A \nu_t^0(b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}t\right)\\
&\quad\times \int_\tau^\infty \int_A \mathbb{1}_{\{s>\tau\}}\, F(\tau,\chi,\xi,s,\phi(s-\tau,\chi,\xi),b)\\
&\quad\times \exp\left(-\int_0^s \lambda(\phi(t-\tau,\chi,\xi),\xi)\,\mathrm{d}t - \int_0^s\int_A \nu_t^0(b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}t\right)\nu_s^0(b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}s\\
&= \int_\tau^\infty\int_E \mathbb{1}_{\{s>\tau\}}\, F(\tau,\chi,\xi,s,y,\xi)\\
&\quad\times \exp\left(-\int_\tau^s \lambda(\phi(t-\tau,\chi,\xi),\xi)\,\mathrm{d}t - \int_\tau^s\int_A \nu_t^0(b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}t\right)\\
&\quad\times \lambda(\phi(s-\tau,\chi,\xi),\xi)\, Q(\phi(s-\tau,\chi,\xi),\xi,\mathrm{d}y)\,\mathrm{d}s\\
&\quad+ \int_\tau^\infty \int_A \mathbb{1}_{\{s>\tau\}}\, F(\tau,\chi,\xi,s,\phi(s-\tau,\chi,\xi),b)\\
&\quad\times \exp\left(-\int_\tau^s \lambda(\phi(t-\tau,\chi,\xi),\xi)\,\mathrm{d}t - \int_\tau^s\int_A \nu_t^0(b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}t\right)\nu_s^0(b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}s,
\end{aligned}
$$

and (6.36) follows.

Assume now that (6.33) holds for $n-1$, namely that, for every $\mathcal{B}((\mathbb{R}_+\times E\times A)^{n-1})$-measurable function $F:(\mathbb{R}_+\times E\times A)^{n-1}\to\mathbb{R}$,

$$
\mathbb{E}_{\nu'}^{x,a'}\left[F(\Pi^{1,n-1})|\mathcal{F}_{T_1}\right] = \left.\frac{\mathbb{E}_{\nu}^{x,a}\left[\mathbb{1}_{\{T_1>\tau\}}\, F(\tau,\chi,\xi,\Pi^{1,n-2})\right]}{\mathbb{P}_{\nu}^{x,a}(T_1>\tau)}\right|_{\tau=T_1,\chi=X_1,\,\xi=A_1}. \tag{6.37}
$$

We have to prove that (6.37) implies that, for every $\mathcal{B}((\mathbb{R}_+\times E\times A)^{n})$-measurable function $F:(\mathbb{R}_+\times E\times A)^{n}\to\mathbb{R}$,

$$
\mathbb{E}_{\nu'}^{x,a'}\left[F(\Pi^{1,n})|\mathcal{F}_{T_1}\right] = \left.\frac{\mathbb{E}_{\nu}^{x,a}\left[\mathbb{1}_{\{T_1>\tau\}}\, F(\tau,\chi,\xi,\Pi^{1,n-1})\right]}{\mathbb{P}_{\nu}^{x,a}(T_1>\tau)}\right|_{\tau=T_1,\chi=X_1,\,\xi=A_1}. \tag{6.38}
$$

Using (6.35), we get

$$\begin{aligned}
&\mathbb{E}^{x,a'}_{\nu'}\left[F(\Pi^{1,n})|\mathcal{F}_{T_1}\right]\\
=&\mathbb{E}^{x,a'}_{\nu'}\left[\mathbb{E}^{x,a'}_{\nu'_\varepsilon}\left[F(\Pi^{1,n})|\mathcal{F}_{T_{n-1}}\right]\Big|\mathcal{F}_{T_1}\right]\\
=&\mathbb{E}^{x,a'}_{\nu'}\Bigg[\int_{T_{n-1}}^{\infty}\int_E F(\Pi^{1,n-1},s,y,A_{n-1})\\
&\times\exp\left(-\int_{T_{n-1}}^{s}\lambda(\phi(t-T_{n-1},E_{n-1},A_{n-1}),A_{n-1})\,\mathrm{d}t-\int_{T_{n-1}}^{s}\int_A \nu_t^{n-2}(\Gamma^{1,n-1},b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}t\right)\\
&\times\lambda(\phi(s-T_{n-1},E_{n-1},A_{n-1}),A_{n-1})\,Q(\phi(s-T_{n-1},E_{n-1},A_{n-1}),A_{n-1},\mathrm{d}y)\,\mathrm{d}s\\
&+\int_{T_{n-1}}^{\infty}\int_A F(\Pi^{1,n-1},s,\phi(s-T_{n-1},E_{n-1},A_{n-1}),b)\\
&\times\exp\left(-\int_{T_{n-1}}^{s}\lambda(\phi(t-T_{n-1},E_{n-1},A_{n-1}),A_{n-1})\,\mathrm{d}t-\int_{T_{n-1}}^{s}\int_A \nu_t^{n-2}(\Gamma^{1,n-1},b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}t\right)\\
&\times\nu_s^{n-2}(\Gamma^{1,n-1},b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}s\Bigg|\mathcal{F}_{T_1}\Bigg]. \qquad (6.39)
\end{aligned}$$

At this point we observe that the term in the conditional expectation in the right-hand side of (6.39) only depends on the random sequence $\Pi^{1,n-1}$. For any sequence of random variables $(S_i,W_i,V_i)_{i\in[1,n-1]}$ with values in $([0,\,\infty)\times E\times A)^{n-1}$, $S_{i-1}\le S_i$ for every $i\in[1,\,n-1]$, we set

$$\begin{aligned}
&\psi(S_1,W_1,V_1,\dots,S_{n-1},W_{n-1},V_{n-1}):=\\
&\int_{S_{n-1}}^{\infty}\int_E F(S_1,W_1,\dots,V_{n-1},S_{n-1},W_{n-1},s,y,V_{n-1})\\
&\times\exp\left(-\int_{S_{n-1}}^{s}\lambda(\phi(t-S_{n-1},W_{n-1},V_{n-1}),V_{n-1})\,\mathrm{d}t\right.\\
&\left.-\int_{S_{n-1}}^{s}\int_A\nu_t^{n-2}(S_1,V_1,\dots,S_{n-1},V_{n-1},b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}t\right)\\
&\times\lambda(\phi(s-S_{n-1},W_{n-1},V_{n-1}),V_{n-1})\,Q(\phi(s-S_{n-1},W_{n-1},V_{n-1}),V_{n-1},\mathrm{d}y)\,\mathrm{d}s\\
&+\int_{S_{n-1}}^{\infty}\int_A F(S_1,W_1,V_1,\dots,S_{n-1},W_{n-1},V_{n-1},,s,\phi(s-S_{n-1},W_{n-1},V_{n-1}),b)\\
&\times\exp\left(-\int_{S_{n-1}}^{s}\lambda(\phi(t-S_{n-1},W_{n-1},V_{n-1}),V_{n-1})\,\mathrm{d}t\right.\\
&\left.-\int_{S_{n-1}}^{s}\int_A\nu_t^{n-2}(S_1,V_1,\dots,S_{n-1},V_{n-1},b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}t\right)\\
&\times\nu_s^{n-2}(S_1,V_1,\dots,S_{n-1},V_{n-1},b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}s.
\end{aligned}$$

Identity (6.39) can be rewritten as

$$\mathbb{E}^{x,a'}_{\nu'}\left[F(\Pi^{1,n})|\mathcal{F}_{T_1}\right]=\mathbb{E}^{x,a'}_{\nu'}\left[\psi(\Pi^{1,n-1})\Big|\mathcal{F}_{T_1}\right]. \qquad (6.40)$$

Then, by applying the inductive step (6.37), we get

$$\begin{aligned}\mathbb{E}_{\nu'}^{x,a'}\left[F(\Pi^{1,n})|\mathcal{F}_{T_1}\right] &= \mathbb{E}_{\nu'}^{x,a'}\left[\psi(\Pi^{1,n-1})\middle|\mathcal{F}_{T_1}\right]\\ &= (\mathbb{P}_{\nu}^{x,a}[T_1>\tau])^{-1}\,\mathbb{E}_{\nu}^{x,a}\left[\mathbb{1}_{\{T_1>\tau\}}\,\psi(\tau,\chi,\xi,\Pi^{1,n-2})\right]\Big|_{\tau=T_1,\chi=X_1,\,\xi=A_1}.\end{aligned}\tag{6.41}$$

Since

$$\begin{aligned}\psi(\tau,\chi,\xi,\Pi^{1,n-2}) = &\int_{T_{n-2}}^{\infty}\int_E F(\tau,\chi,\xi,\Pi^{1,n-2},s,y,A_{n-2})\\ &\times\exp\left(-\int_{T_{n-2}}^{s}\lambda(\phi(t-T_{n-2},E_{n-2},A_{n-2}),A_{n-2})\,\mathrm{d}t-\int_{T_{n-2}}^{s}\int_A\nu_t^{n-2}(\Gamma^{1,n-2},b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}t\right)\\ &\times\lambda(\phi(s-T_{n-2},E_{n-2},A_{n-2}),A_{n-2})\,Q(\phi(s-T_{n-2},E_{n-2},A_{n-2}),A_{n-2},\mathrm{d}y)\,\mathrm{d}s\\ &+\int_{T_{n-2}}^{\infty}\int_A F(\tau,\chi,\xi,\Pi^{1,n-2},s,\phi(s-T_{n-2},E_{n-2},A_{n-2}),b)\\ &\times\exp\left(-\int_{T_{n-2}}^{s}\lambda(\phi(t-T_{n-2},E_{n-2},A_{n-2}),A_1)\,\mathrm{d}t-\int_{T_{n-2}}^{s}\int_A\nu_t^{n-2}(\Gamma^{1,n-2},b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}t\right)\\ &\times\nu_s^{n-2}(\Gamma^{1,n-2},b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}s\\ =&\,\mathbb{E}_{\nu}^{x,a}[F(\tau,\chi,\xi,\Pi^{1,n-1})|\mathcal{F}_{T_{n-2}}],\end{aligned}$$

identity (6.41) can be rewritten as

$$\begin{aligned}\mathbb{E}_{\nu'}^{x,a'}\left[F(\Pi^{1,n})|\mathcal{F}_{T_1}\right] &= (\mathbb{P}_{\nu}^{x,a}[T_1>\tau])^{-1}\,\mathbb{E}_{\nu}^{x,a}\left[\mathbb{1}_{\{T_1>\tau\}}\,\mathbb{E}_{\nu}^{x,a}[F(\tau,\chi,\xi,\Pi^{1,n-1})|\mathcal{F}_{T_{n-2}}]\right]\Big|_{\tau=T_1,\chi=X_1,\,\xi=A_1}\\ &= \frac{\mathbb{E}_{\nu}^{x,a}\left[\mathbb{1}_{\{T_1>\tau\}}\,F(\tau,\chi,\xi,\Pi^{1,n-1})\right]}{\mathbb{P}_{\nu}^{x,a}(T_1>\tau)}\Bigg|_{\tau=T_1,\,\chi=E_1,\,\xi=A_1}.\end{aligned}\tag{6.42}$$

This concludes the proof of the Lemma.

Proof of Proposition 5.6.
We start by noticing that,

$$J(x,a,\nu) = \mathbb{E}_{\nu}^{x,a}\left[F(T_1,E_1,A_1,T_2,E_2,A_2,\ldots)\right],$$

where

$$\begin{aligned}F(T_1,E_1,A_1,T_2,E_2,A_2,\ldots) &= \int_0^{\infty}\mathrm{e}^{-\delta t}f(X_t,I_t)\,\mathrm{d}t\\ &= \int_0^{T_1}\mathrm{e}^{-\delta t}f(\phi(t,X_0,I_0),I_0)\,\mathrm{d}t+\sum_{n=2}^{\infty}\int_{T_{n-1}}^{T_n}\mathrm{e}^{-\delta t}f(\phi(t-T_{n-1},E_{n-1},A_{n-1}),A_{n-1})\,\mathrm{d}t.\end{aligned}\tag{6.43}$$

We aim at constructing a sequence of controls $(\nu^{\varepsilon})_{\varepsilon}\in\mathcal{V}$ such that

$$\begin{aligned}J(x,a',\nu^{\varepsilon}) &= \mathbb{E}_{\nu^{\varepsilon}}^{x,a'}\left[F(T_1,E_1,A_1,T_2,E_2,A_2,\ldots)\right]\\ &\underset{\varepsilon\to 0}{\longrightarrow}\mathbb{E}_{\nu}^{x,a}\left[F(T_1,E_1,A_1,T_2,E_2,A_2,\ldots)\right] = J(x,a,\nu).\end{aligned}\tag{6.44}$$

Since $\nu \in \mathcal{V}$, there exists a $\mathbb{P}^{x,a}$-null set $\mathcal{N}$ such that ν admits the representation

$$\nu_t(b) = \nu_t^0(b)\,\mathbb{1}_{\{t\leqslant T_1\}} + \sum_{n=1}^{\infty} \nu_t^n(T_1, A_1, T_2, A_2, \ldots, T_n, A_n, b)\,\mathbb{1}_{\{T_n<t\leqslant T_{n+1}\}}. \tag{6.45}$$

Here $(\omega, t) \in \Omega \times \mathbb{R}_+$, $\omega \notin \mathcal{N}$, and $(\nu^n)_{n>1}$ (resp. ν^0) are $\mathcal{P} \otimes \mathcal{B}((\mathbb{R}_+ \times A)^n) \otimes \mathcal{A}$-measurable (resp. is $\mathcal{P} \otimes \mathcal{A}$-measurable), uniformly bounded with respect to n (resp. bounded), see for instance Definition 26.3 in [21].

Let $\bar{B}(a, \varepsilon)$ be the closed ball centered in a with radius ε. We notice that $\varepsilon \mapsto \lambda_0(\bar{B}(a, \varepsilon))$ defines a nonnegative, right-continuous, nondecreasing function, satisfying

$$\lambda_0(\bar{B}(a, 0)) = \lambda_0(\{a\}) \geq 0, \quad \lambda_0(\bar{B}(a, \varepsilon)) > 0 \quad \forall \varepsilon > 0.$$

If $\lambda_0(\{a\}) > 0$, we set $h(\varepsilon) = \varepsilon$ for every $\varepsilon > 0$. Otherwise, if $\lambda_0(\{a\}) = 0$, we define h as the right inverse function of $\varepsilon \mapsto \lambda_0(\bar{B}(a, \varepsilon))$, namely

$$h(p) = \inf\{\varepsilon > 0 : \lambda_0(\bar{B}(a, \varepsilon)) \geq p\},\ p \geq 0.$$

From Lemma 1.37 in [31] the following property holds:

$$\forall p \geq 0, \quad \lambda_0(\bar{B}(a, h(p))) \geq p. \tag{6.46}$$

At this point, we introduce the following family of processes, parametrized by ε:

$$\begin{aligned}\nu_t^\varepsilon(b) = \frac{1}{\varepsilon}\,\frac{1}{\lambda_0(\bar{B}(a, h(\varepsilon)))}\mathbb{1}_{\{b\in\bar{B}(a,h(\varepsilon))\}}\mathbb{1}_{\{t\leqslant T_1\}} + \nu_t^0(b)\,\mathbb{1}_{\{T_1<t\leqslant T_2\}}\\ + \sum_{n=2}^{\infty} \nu_t^{n-1}(T_2, A_2, \ldots, T_n, A_n, b)\,\mathbb{1}_{\{T_n<t\leqslant T_{n+1}\}}.\end{aligned} \tag{6.47}$$

With this choice, for all $r > 0$,

$$\begin{aligned}&\mathbb{P}_{\nu^\varepsilon}^{x,a'}(T_1 > r, E_1 \in F, A_1 \in C)\\ &= \int_r^\infty \int_F \exp\left(-\int_0^s \lambda(\phi(t, x, a'), a')\,\mathrm{d}t - \frac{s}{\varepsilon}\right) \lambda(\phi(s, x, a'), a')\, Q(\phi(s, x, a'), a', \mathrm{d}y)\,\mathrm{d}s\\ &\quad + \int_r^\infty \int_C \exp\left(-\int_0^s \lambda(\phi(t, x, a'), a')\,\mathrm{d}t - \frac{s}{\varepsilon}\right) \frac{1}{\varepsilon}\,\frac{1}{\lambda_0(\bar{B}(a, h(\varepsilon)))}\mathbb{1}_{\{b\in\bar{B}(a,h(\varepsilon))\}}\,\lambda_0(\mathrm{d}b)\,\mathrm{d}s.\end{aligned} \tag{6.48}$$

To prove (6.44), it is enough to show that, for every $k > 1$,

$$\mathbb{E}_{\nu^\varepsilon}^{x,a'}[\bar{F}(\Pi^{1,k})] \underset{\varepsilon\to 0}{\longrightarrow} \mathbb{E}_{\nu}^{x,a}[\bar{F}(\Pi^{1,k})], \tag{6.49}$$

where

$$\begin{aligned}\bar{F}(S_1, W_1, V_1, \ldots, S_k, W_k, V_k) = \int_0^{S_1} \mathrm{e}^{-\delta t} f(\phi(t, X_0, I_0), I_0)\,\mathrm{d}t\\ + \sum_{n=2}^{k} \int_{S_{n-1}}^{S_n} \mathrm{e}^{-\delta t} f(\phi(t - S_{n-1}, W_{n-1}, V_{n-1}), V_{n-1})\,\mathrm{d}t,\end{aligned} \tag{6.50}$$

for any sequence of random variables $(S_n, W_n, V_n)_{n\in[1,k]}$ with values in $([0, \infty) \times E \times A)^n$, with $S_{n-1} \leq S_n$ for every n.

As a matter of fact, the remaining term

$$R(\varepsilon, k) := \mathbb{E}^{x,a'}_{\nu^\varepsilon} \left[\int_{T_k}^{\infty} \mathrm{e}^{-\delta t} f(\phi(t - T_{n-1}, E_{n-1}, A_{n-1}), A_{n-1}) \, \mathrm{d}t \right]$$

converges to zero, uniformly in ε, as k goes to infinity. To see it, we notice that

$$|R(\varepsilon, k)| \leq \frac{M_f}{\delta} \, \mathbb{E}^{x,a'}_{\nu^\varepsilon} \left[\mathrm{e}^{-\delta T_k} \right] = \frac{M_f}{\delta} \, \mathbb{E}^{x,a'} \left[L^{\nu^\varepsilon}_{T_k} \, \mathrm{e}^{-\delta T_k} \right], \tag{6.51}$$

where, L^ν is the Doléans–Dade exponential local martingale defined in (3.11). Taking into account (6.47) and (6.46), we get

$$\mathbb{E}^{x,a'} \left[L^{\nu^\varepsilon}_{T_k} \, \mathrm{e}^{-\delta T_k} \right] \leq \mathbb{E}^{x,a'} \left[\frac{\mathrm{e}^{T_1 \, \lambda_0(A)} \, \mathrm{e}^{-T_1 \, \frac{1}{\varepsilon}}}{\varepsilon^2} \, L^{\bar\nu}_{T_k} \, \mathrm{e}^{-\delta T_k} \right] \leq \frac{4}{\mathrm{e}^2} \, \mathbb{E}^{x,a'} \left[\frac{\mathrm{e}^{T_1 \, \lambda_0(A)}}{T_1^2} \, L^{\bar\nu}_{T_k} \, \mathrm{e}^{-\delta T_k} \right]$$

where

$$\bar\nu(t, b) := \mathbb{1}_{\{t \leqslant T_1\}} + \nu^0_t(b) \, \mathbb{1}_{\{T_1 < t \leqslant T_2\}} + \sum_{n=2}^{\infty} \nu^{n-1}_t(T_2, A_2, \ldots, T_n, A_n, b) \, \mathbb{1}_{\{T_n < t \leqslant T_{n+1}\}}.$$

Since $\bar\nu \in \mathcal{V}$, by Proposition 3.2 there exists a unique probability $\mathbb{P}^{x,a'}_{\bar\nu}$ on $(\Omega, \mathcal{F}_\infty)$ such that its restriction on $(\Omega, \mathcal{F}_{T_k})$ is $L^{\bar\nu}_{T_k} \, \mathbb{P}^{x,a'}$. Then (6.51) reads

$$|R(\varepsilon, k)| \leq \frac{4 \, M_f}{\delta \, \mathrm{e}^2} \, \mathbb{E}^{x,a'}_{\bar\nu} \left[\frac{\mathrm{e}^{T_1 \, \lambda_0(A)}}{T_1^2} \, \mathrm{e}^{-\delta T_k} \right], \tag{6.52}$$

and the conclusion follows by the Lebesgue dominated convergence theorem.

Let us now prove (6.49). By Lemma 6.1, taking into account (6.48), we achieve

$$\begin{aligned}
\mathbb{E}^{x,a'}_{\nu^\varepsilon}[\bar F(\Pi^{1,k})] &= \mathbb{E}^{x,a'}_{\nu^\varepsilon} \left[\mathbb{E}^{x,a'}_{\nu^\varepsilon} \left[\bar F(\Pi^{1,k}) \middle| \mathcal{F}_{T_1} \right] \right] \\
&= \mathbb{E}^{x,a'}_{\nu^\varepsilon} \left[\frac{\mathbb{E}^{x,a}_{\nu} \left[\mathbb{1}_{\{T_1 > \tau\}} \, \bar F(s, y, b, \Pi^{1,k-1}) \right]}{\mathbb{P}^{x,a}_{\nu}(T_1 > \tau)} \bigg|_{s = T_1, \, y = X_1, \, b = A_1} \right] \\
&= \int_0^\infty \int_E \frac{\mathbb{E}^{x,a}_{\nu} \left[\mathbb{1}_{\{T_1 > s\}} \, \bar F(s, y, a', \Pi^{1,k-1}) \right]}{\mathbb{P}^{x,a}_{\nu}(T_1 > s)} \\
&\quad \times \exp\left(- \int_0^s \lambda(\phi(t, x, a'), a') \, \mathrm{d}t - \frac{s}{\varepsilon} \right) \lambda(\phi(s, x, a'), a') \, Q(\phi(s, x, a'), a', \mathrm{d}y) \, \mathrm{d}s \\
&\quad + \int_0^\infty \int_A \frac{\mathbb{E}^{x,a}_{\nu} \left[\mathbb{1}_{\{T_1 > s\}} \, \bar F(s, \phi(s, x, a'), b, \Pi^{1,k-1}) \right]}{\mathbb{P}^{x,a}_{\nu}(T_1 > s)} \\
&\quad \times \exp\left(- \int_0^s \lambda(\phi(t, x, a'), a') \, \mathrm{d}t - \frac{s}{\varepsilon} \right) \frac{1}{\varepsilon} \, \frac{1}{\lambda_0(\bar B(a, h(\varepsilon)))} \mathbb{1}_{\{b \in \bar B(a, h(\varepsilon))\}} \, \lambda_0(\mathrm{d}b) \, \mathrm{d}s.
\end{aligned} \tag{6.53}$$

At this point, we set

$$\varphi(s, y, b) := \frac{\mathbb{E}^{x,a}_{\nu} \left[\mathbb{1}_{\{T_1 > s\}} \, \bar F(s, y, b, \Pi^{1,k-1}) \right]}{\mathbb{P}^{x,a}_{\nu}(T_1 > s)}, \quad s \in [0, \infty), \, y \in E, \, b \in A. \tag{6.54}$$

Notice that, for every $(y,b) \in E \times A$,

$$\bar F(s,y,b,\Pi^{1,k-1}) = \int_0^s \mathrm{e}^{-\delta t} f(\phi(t,X_0,I_0),I_0)\,\mathrm{d}t + \int_s^{T_1} \mathrm{e}^{-\delta t} f(\phi(t-s,y,b),b)\,\mathrm{d}t \\ + \sum_{n=2}^{k-1} \int_{T_{n-1}}^{T_n} \mathrm{e}^{-\delta t} f(\phi(t-T_{n-1},E_{n-1},A_{n-1}),A_{n-1})\,\mathrm{d}t,$$

so that

$$|\varphi(s,y,b)| \leq \frac{M_f}{\delta}\cdot \tag{6.55}$$

Identity (6.53) becomes

$$\begin{aligned}\mathbb{E}^{x,a'}_{\nu^\varepsilon}[\bar F(\Pi^{1,k})] &= \int_0^\infty \int_E \varphi(s,y,a')\,\exp\left(-\int_0^s \lambda(\phi(t,x,a'),a')\,\mathrm{d}t - \frac{s}{\varepsilon}\right)\\ &\quad\times \lambda(\phi(s,x,a'),a')\,Q(\phi(s,x,a'),a',\mathrm{d}y)\,\mathrm{d}s\\ &\quad+ \int_0^\infty \int_A \varphi(s,\phi(s,x,a'),b)\,\exp\left(-\int_0^s \lambda(\phi(t,x,a'),a')\,\mathrm{d}t - \frac{s}{\varepsilon}\right)\\ &\quad\times \frac{1}{\varepsilon}\,\frac{1}{\lambda_0(\bar B(a,h(\varepsilon)))}\mathbb{1}_{\{b\in \bar B(a,h(\varepsilon))\}}\,\lambda_0(\mathrm{d}b)\,\mathrm{d}s\\ &=: I_1(\varepsilon) + I_2(\varepsilon).\end{aligned}$$

Using the change of variable $s = \varepsilon\, z$, we have

$$\begin{aligned}I_1(\varepsilon) &= \int_0^\infty \int_E f_\varepsilon(z,y)\,\lambda(\phi(\varepsilon\, z,x,a'),a')\,Q(\phi(\varepsilon\, z,x,a'),a',\mathrm{d}y)\;\mathrm{d}z,\\ I_2(\varepsilon) &= \int_0^\infty \int_A g_\varepsilon(z,b)\,\lambda_0(\mathrm{d}b)\,\mathrm{d}z,\end{aligned}$$

where

$$\begin{aligned}f_\varepsilon(z,y) &:= \varepsilon\,\varphi(\varepsilon\, z,y,a')\,\exp\left(-\int_0^{\varepsilon\, z} \lambda(\phi(t,x,a'),a')\,\mathrm{d}t - z\right),\\ g_\varepsilon(z,b) &:= \varphi(\varepsilon\, z,\phi(\varepsilon\, z,x,a'),b)\,\exp\left(-\int_0^{\varepsilon\, z} \lambda(\phi(t,x,a'),a')\,\mathrm{d}t - z\right)\\ &\quad\times \frac{1}{\lambda_0(\bar B(a,h(\varepsilon)))}\,\mathbb{1}_{\{b\in \bar B(a,h(\varepsilon))\}}.\end{aligned}$$

Exploiting the continuity properties of λ, Q, ϕ and f, we get

$$I_2(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} \varphi(0,x,a), \tag{6.56}$$

where we have used that $\phi(0,x,b) = x$ for every $b \in A$. On the other hand, from the estimate (6.55), it follows that $|f_\varepsilon(z,y)| \leq \frac{M_f}{\delta}\,\mathrm{e}^{-z}\,\varepsilon$. Therefore

$$|I_1(\varepsilon)| \leq \frac{M_f}{\delta}\,\varepsilon\,||\lambda||_\infty \int_0^\infty \mathrm{e}^{-z}\,\mathrm{d}z = \frac{M_f}{\delta}\,\varepsilon\,||\lambda||_\infty \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{6.57}$$

Collecting (6.57) and (6.56), we conclude that

$$\mathbb{E}^{x,a'}_{\nu^\varepsilon}[\bar F(\Pi^{1,k})] \underset{\varepsilon\to 0}{\longrightarrow} \varphi(0,x,a). \tag{6.58}$$

Recalling the definitions of φ and $\bar{F}$ given respectively in (6.54) and (6.50), we see that

$$
\begin{aligned}
\varphi(0,x,a) &= (\mathbb{P}_\nu^{x,a}(T_1>0))^{-1}\,\mathbb{E}_\nu^{x,a}\left[\mathbb{1}_{\{T_1>0\}}\,\bar{F}(0,x,a,\Pi^{1,k-1})\right]\\
&= \mathbb{E}_\nu^{x,a}\left[\bar{F}(0,x,a,\Pi^{1,k-1})\right]\\
&= \mathbb{E}_\nu^{x,a}\left[\int_0^{T_1}\mathrm{e}^{-\delta t}f(\phi(t,x,a),a)\,\mathrm{d}t+\sum_{n=2}^{k}\int_{T_{n-2}}^{T_{n-1}}\mathrm{e}^{-\delta t}f(\phi(t-T_{n-1},E_{n-1},A_{n-1}),A_{n-1})\,\mathrm{d}t\right]\\
&= \mathbb{E}_\nu^{x,a}\left[\bar{F}(\Pi^{1,k})\right],
\end{aligned}
$$

and this concludes the proof.

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