ESAIM: COCV 24 (2018) 105–127
DOI: 10.1051/cocv/2016077

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

PRESCRIBED CONDITIONS AT INFINITY FOR FRACTIONAL PARABOLIC AND ELLIPTIC EQUATIONS WITH UNBOUNDED COEFFICIENTS

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Abstract. We investigate existence and uniqueness of solutions to a class of fractional parabolic equations satisfying prescribed point-wise conditions at infinity (in space), which can be time-dependent. Moreover, we study the asymptotic behavior of such solutions. We also consider solutions of elliptic equations satisfying appropriate conditions at infinity.

Mathematics Subject Classification. 35R11, 35K67, 35J75.

Received July 14, 2016. Accepted December 14, 2016.

1. Introduction

We are concerned with existence and uniqueness of solutions to the following linear *nonlocal* parabolic Cauchy problem:

$$\begin{cases} \partial_t u = -a\,(-\Delta)^s u + cu + f & \text{in} \quad \mathbb{R}^N \times (0,T] =: S_T \\ u = u_0 & \text{in} \quad \mathbb{R}^N \times \{0\}, \end{cases} \tag{1.1}$$

where the coefficient a is a positive function only depending on the space variable x, which becomes unbounded as $|x| \to \infty$; $(-\Delta)^s$ denotes the fractional Laplace operator of order $s \in (0,1)$, $N > 2s$, while $c, f, u_0 \in L^\infty(\mathbb{R}^N)$. Moreover, we investigate existence and uniqueness of solutions to the linear *nonlocal* elliptic equation

$$a(-\Delta)^s u - cu = f \qquad \text{in} \qquad \mathbb{R}^N; \tag{1.2}$$

in this case we also suppose that $c < 0$ (see the comments after Rem. 2.5).

(a) Parabolic problems. The well-posedness of problem (1.1) has been largely studied in the literature in the local case $s = 1$ (see, *e.g.*, [2, 8, 11–15, 18, 24]). As a matter of fact, if $N = 1, 2$ and $s = 1$, then there exists a

Keywords and phrases. Nonlocal operators, evolution equations, sub- supersolutions.

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unique bounded solution of problem (1.1). If $N \geq 3$, a special role is played by the behaviour at infinity of the coefficient a. In particular, if

$$a(x) \leq C(1+|x|^2)^{\frac{\alpha}{2}} \quad \text{for all } x \in I\!R^N, \text{ for some } C>0, \alpha \leq 2,$$

then problem (1.1) admits only one bounded solution (see [2,12]). Instead, if

$$a(x) \geq C(1+|x|^2)^{\frac{\alpha}{2}} \qquad \text{for all} \qquad x \in I\!R^N, \text{ for some } C>0, \alpha > 2,$$

then problem (1.1) admits infinitely many bounded solutions. More precisely, for any given $g \in C([0,T])$, if

$$\lim_{|x|\to\infty} u_0(x) \;=\; g(0), \tag{1.3}$$

then there exists a unique bounded solution of problem (1.1) such that

$$\lim_{|x|\to\infty} u(x,t) = \; g(t) \qquad \text{uniformly with respect to} \qquad t \in [0,T] \tag{1.4}$$

(see [11,15]). Observe that condition (1.4) can be regarded as a Dirichlet condition at infinity.

More recently, existence and uniqueness results concerning *nonlocal* Cauchy parabolic problems have been established. In this respect, in [1,16], [17] quite general integro-differential equations have been treated, requiring that there exist two constants $C_1 > 0, C_2 > 0$ such that

$$C_1 \leq a(x) \leq C_2 \qquad \text{for all} \qquad x \in I\!R^N. \tag{1.5}$$

Moreover, in [20] the uniqueness of solutions of problem (1.1) with $c \equiv 0$ in suitable weighted Lebesgue spaces is stated, under suitable assumptions on a.

In the present paper we always assume that

(H_0) there exist $C_0 > 0$, $\alpha > 2s$ such that $a(x) \geq C_0(1+|x|^2)^{\frac{\alpha}{2}}$ for all $x \in I\!R^N$.

Clearly, this case is not covered by [1, 16, 17], since (1.5) is not satisfied. Moreover, hypothesis (H_0) excludes that the assumptions on a made in [20] hold.

It is worth mentioning that the unbounded diffusion coefficient $a(x)$ is very important for the applications, see for instance, for the local case, [2,8,10,18,19]. Clearly, the same models with the unbounded diffusion coefficient $a(x)$ occurs when considering nonlocal diffusion, for instance, in association with non-Gaussian stochastic processes, that, starting from any point in $I\!R^N$, can reach *infinity* (see, *e.g.*, [5]).

We prove (see Thm. 2.7) that there exists a unique solution of problem (1.1) such that (1.4) is satisfied, provided (1.3) holds; furthermore,

$$|u| \leq Ce^{\beta T} \qquad \text{in} \qquad I\!R^N \times [0,T], \tag{1.6}$$

for some $C > 0$ and $\beta > 0$. This result generalizes to the case of nonlocal operator the results in [11] and in [15].

In proving this result, at first for any $j \in I\!N$, we consider *viscosity* solutions of approximating problems in a large cylinder $B_j \times (0,T]$; here and hereafter for each $R > 0$, $B_R := \{x \in I\!R^N \,:\, |x| < R\}$. For such problems existence, uniqueness and regularity results have been given in [3, 4]. Then using suitable super- and subsolutions and standard compactness arguments we obtain the existence of a solution of problem (1.1), satifying the estimate (1.6), which depends on T. Then, in order to show that condition (1.4) holds, proper sub- and supersolutions are introduced (see (4.24) and (4.37) below). In doing this, a special role is played by a supersolution $V = V(x)$ of equation

$$-a(-\Delta)^s V \;=\; -1 \qquad \text{in} \qquad I\!R^N \setminus \overline{B}_{R_0}, \tag{1.7}$$

for some $R_0 > 0$, such that

$$V(x) > 0 \qquad \text{for all} \qquad x \in I\!R^N, \quad \lim_{|x|\to\infty} V(x) = 0, \tag{1.8}$$

which has been appropriately constructed (see Prop. 3.1). Moreover, we show that similar results hold for problem

$$\begin{cases} \partial_t u = -a\,(-\Delta)^s u + cu + f & \text{in} \quad I\!R^N \times (0,\infty) \\ u = u_0 & \text{in} \quad I\!R^N \times \{0\}, \end{cases} \tag{1.9}$$

provided $c \le 0$ (see Thm. 2.8). Clearly, in this case, condition (1.4) is replaced by

$$\lim_{|x|\to\infty} u(x,t) = g(t) \qquad \text{uniformly with respect to} \qquad t \in [0,\infty). \tag{1.10}$$

In order to impose condition (1.10), we need to show preliminarily that the solution satisfies the bound

$$|u| \le C \qquad \text{in} \qquad I\!R^N \times (0,\infty), \tag{1.11}$$

which is global in time. In order to obtain this estimate, we use a positive supersolution $h = h(x)$ of equation

$$-a(-\Delta)^s h = -1 \qquad \text{in} \qquad I\!R^N. \tag{1.12}$$

Note that the proof of the existence of such a supersolution h is rather technical (see Prop. 3.2); indeed, we also show that

$$h(x) > 0 \qquad \text{for all} \qquad x \in I\!R^N, \quad \lim_{|x|\to\infty} h(x) = 0. \tag{1.13}$$

Let us describe in general terms the deep relation between our results and stochastic calculus for jump processes. In fact, equation (1.12) completed with condition (1.13) can be regarded as the counterpart on $I\!R^N$ for the operator $a(-\Delta)^s$ of the *first exit-time* problem in a bounded domain. Note that the first exit-time problem in B_R, in the case $a \equiv 1$, has been studied in [6,9]. In fact, in [6] and in [9] it is outlined the connection between the so-called first exit time problem

$$\begin{cases} -(-\Delta)^s u = -1 & \text{in} \quad B_R \\ u = 0 & \text{in} \quad \big(I\!R^N \setminus B_R\big), \end{cases} \tag{1.14}$$

and the first exit-time from B_R of the jump process associated to $(-\Delta)^s$, starting from any point in B_R. Now, since equation (1.12), completed with condition (1.13), corresponds to problem (1.14) in the limit case $R = \infty$, it is somehow related to reachability of *infinity* by the jump process associated to the operator $a(-\Delta)^s$ (see [5,10]). In particular, from the existence of the supersolution h it follows that *infinity* can actually be attained by the jump process starting from any point $x_0 \in I\!R^N$. This property is usually expressed saying that the process is *transient*.

Moreover, it is well-known that if any point of the boundary of a bounded domain of $I\!R^N$ can be reached by the jump process associated to a nonlocal diffusion operator starting from points inside the domain, then the Dirichlet problem admits a unique solution. Now, since in our case the jump process is transient in whole of $I\!R^N$, one can expect that there exists a unique solution of problem (1.1) which satisfies conditions of Dirichlet type at infinity. Indeed, we prove this.

We should mention that, to the best of our knowledge, in the literature no results concerning the prescription of general Dirichlet conditions at infinity for solutions of nonlocal parabolic (or elliptic) equations have been obtained before the present paper.

Finally, we prove that the solution $u(x,t)$ of problem (1.9) satisfying (1.10) admits a limit as $t\to\infty$. In fact, the function

$$W(x) := \lim_{t\to\infty} u(x,t) \quad (x\in I\!R^N)$$

is the unique solution of equation (1.2) such that

$$\lim_{|x|\to\infty} W(x) \,=\, \gamma,$$

provided

$$\gamma = \lim_{t\to\infty} g(t) \tag{1.15}$$

(see Thm. 2.11). Such result is shown by adapting to the present situation the method of sub- and supersolutions used in [21] in the case of bounded domains of $I\!R^N$ for "local" parabolic equations. Indeed, some important changes are in order, in view of the nonlocal character of the problem and since we prescribe conditions as $|x|\to\infty$.

(b) Ellipitc equations. In the local case, some existence and uniqueness results for equations (1.2) with $s=1$ can be deduced from general results in [19]. Moreover, the case $0<s<1$ has been treated in [20]; in particular, it is shown that uniqueness results hold in $L^p_\psi(I\!R^N)$, for $\psi\in C(\bar S_T), \psi>0, p\geq 1$, under suitable assumptions on a.

From the result concerning the asymptotic behaviour of solutions of problem (1.1) recalled in (**a**) above, we can infer that there exists a unique solution of equation (1.2), which satisfies (1.15). However, we also prove this existence and uniqueness result also independently, without using results for parabolic problems. In fact, we solve approximating problems in a large ball B_j for any $j\in I\!N$. In order to obtain a uniform bound for the solutions of such problems we use in crucial way the supersolution h of equation (1.12). Then, by standard compactness tools, we get a solution of equation (1.2). Using again the supersolution h, and in particular the fact that (1.13) holds, we impose that

$$\lim_{|x|\to\infty} u(x) \,=\, \gamma \quad (\gamma\in I\!R). \tag{1.16}$$

We devote the forthcoming Section 2 to the precise statement of the main results obtained in this paper (see in particular Sect. 2.1).

2. Mathematical framework and results

The fractional Laplacian $(-\Delta)^s$ can be defined by the Fourier transform $\mathfrak{F}$ for any function in the Schwartz class $\mathcal{S}$ (see *e.g.* [22]). Moreover,

$$(-\Delta)^s u = \mathfrak{F}^{-1}\big(|\xi|^{2s}\mathfrak{F}u\big), \quad \xi\in I\!R^N, u\in\mathcal{S}. \tag{2.1}$$

Suppose that for some $\gamma>0$, $u\in\mathcal{L}^s(I\!R^N)\cap C^{2s+\gamma}(I\!R^N)$ if $s<\frac12$, or $u\in\mathcal{L}^s(I\!R^N)\cap C^{1,2s+\gamma-1}_{\rm loc}(I\!R^N)$ if $s\geq\frac12$. Then we have

$$(-\Delta)^s u(x) = C_{N,s}\ \text{P.V.}\int_{I\!R^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\mathrm{d}y \quad (x\in I\!R^N), \tag{2.2}$$

where (see [7])

$$C_{N,s} = \frac{4^s s\Gamma((N+2s)/2)}{\pi^{N/2}\Gamma(1-s)} = \left(\int_{I\!R^N}\frac{1-\cos(\xi_1)}{|\xi|^{N+2s}}\mathrm{d}\xi\right)^{-1},$$

Γ being the Gamma function. In the sequel, for simplicity, we shall write

$$\int_{I\!R^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\mathrm{d}y \equiv \text{P.V.}\int_{I\!R^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\mathrm{d}y \quad (x\in I\!R^N).$$

Moreover, $(-\Delta)^s u\in C(I\!R^N)$.

Concerning the coefficients a and c, and the function f we always make the following assumption:

$$(H_1) \qquad \begin{cases} \text{(i)} \quad a \in C^{0,\sigma}_{\text{loc}}(I\!\!R^N) \ \big(\sigma \in (0,1)\big), \ \ a(x) > 0 \quad \text{for all} \ \ x \in I\!\!R^N; \\ \text{(ii)} \quad c, f \in C^{0,\sigma}_{\text{loc}}(I\!\!R^N) \cap L^\infty(I\!\!R^N). \end{cases}$$

Now we can give the definition of solution. Let $\Omega \subseteq I\!\!R^N$ be an open subset.

Definition 2.1. We say that a function u is a *subsolution* to equation

$$\partial_t u = -a\,(-\Delta)^s u + cu + f \quad \text{in} \ \ Q_T := \Omega \times (0,T], \tag{2.3}$$

if

(i) u is upper semicontinuous in S_T;
(ii) for any open bounded subset $U \subset Q_T$, for any $(x_0,t_0) \in U$, for any test function $\varphi \in C^2(S_T)$ such that $u(x_0,t_0) - \varphi(x_0,t_0) \geq u(x,t) - \varphi(x,t)$ for all $(x,t) \in U$, one has

$$\partial_t \psi(x_0,t_0) \leq -a(x_0)(-\Delta)^s \psi(x_0,t_0) \ + c(x_0)u(x_0,t_0) + f(x_0),$$

where

$$\psi := \begin{cases} \varphi \text{ in } U \\ u \text{ in } \ S_T \setminus U. \end{cases} \tag{2.4}$$

Furthermore, we say that a function u is a *supersolution* to equation (2.3) if

(i) u is lower semicontinuous in S_T;
(ii) for any open bounded subset $U \subset Q_T$, for any $(x_0,t_0) \in U$, for any test function $\varphi \in C^2(S_T)$ such that $u(x_0,t_0) - \varphi(x_0,t_0) \leq u(x,t) - \varphi(x,t)$ for all $(x,t) \in U$, one has

$$\partial_t \psi(x_0,t_0) \geq -a(x_0)(-\Delta)^s \psi(x_0,t_0) \ + c(x_0)u(x_0,t_0) + f(x_0),$$

where ψ is defined by (2.4). Finally, we say that u is a *solution* to equation (1.2) if it is both a subsolution and a supersolution to equation (1.2).

Let $g \in C([0,T])$, $u_0 \in C(I\!\!R^N)$ with

$$u_0(x,0) = g(0) \qquad \text{for all} \qquad x \in I\!\!R^N \setminus \Omega. \tag{2.5}$$

Consider the problem

$$\begin{cases} \partial_t u = -a\,(-\Delta)^s u + cu + f & \text{in} \quad Q_T \\ u = g & \text{in} \quad \big(I\!\!R^N \setminus \Omega\big) \times (0,T] \\ u = u_0 & \text{in} \quad I\!\!R^N \times \{0\}. \end{cases} \tag{2.6}$$

Definition 2.2. We say that a function u is a *subsolution* to problem (2.6) if

(i) u is upper semicontinuous in $\overline{S_T}$;
(ii) u is a subsolution to equation (2.3);
(iii) $u(x,t) \leq g(t)$ for all $x \in I\!\!R^N \setminus \Omega, t \in (0,T]$ and $u(x,0) \leq u_0(x)$ for all $x \in I\!\!R^N$.

Similarly, supersolutions are defined. Finally, we say that u is a *solution* to problem (2.6) if it is both a subsolution and a supersolution to problem (2.6).

Observe that according to our definition, any solution of problem (2.6) takes continuously the initial datum u_0 and the boundary datum g.

Definition 2.3. We say that a function u is a *subsolution* to equation

$$a\,(-\Delta)^s u \,-\, cu = f \qquad \text{in} \qquad \Omega, \tag{2.7}$$

if

(i) u is upper semicontinuous in $I\!R^N$;
(ii) for any open bounded subset $U \subset \Omega$, for any $x_0 \in U$, for any test function $\varphi \in C^2(I\!R^N)$ such that $u(x_0) - \varphi(x_0) \geq u(x) - \varphi(x)$ for all $x \in U$, one has

$$a(x_0)(-\Delta)^s\psi(x_0) \,-\, c(x_0)u(x_0) \,\leq\, f(x_0),$$

where ψ is defined by

$$\psi := \begin{cases} \varphi & \text{in} \qquad U \\ u & \text{in} \qquad I\!R^N \setminus U. \end{cases} \tag{2.8}$$

Furthermore, we say that a function u is a *supersolution* to equation (2.7) if

(i) u is lower semicontinuous in $I\!R^N$;
(ii) for any open subset $U \Subset \Omega$, for any $x_0 \in U$, for any test function $\varphi \in C^2(\Omega)$ such that $u(x_0) - \varphi(x_0) \leq u(x) - \varphi(x)$ for all $x \in U$, one has one has

$$a(x_0)(-\Delta)^s\psi(x_0) \,-\, c(x_0)u(x_0) \,\geq\, f(x_0).$$

Finally, we say that u is a *solution* to equation (2.7) if it is both a subsolution and a supersolution to equation (2.7).

$$\begin{cases} a\,(-\Delta)^s u - cu \,=\, f \text{ in } \Omega \\ u \,=\gamma \qquad\qquad\qquad \text{in } \left(I\!R^N \setminus \Omega\right), \end{cases} \tag{2.9}$$

where $\gamma \in I\!R$.

Definition 2.4. We say that a function u is a *subsolution* to problem (2.9) if

(i) u is upper semicontinuous in $I\!R^N$;
(ii) u is a subsolution to equation (2.7);
(iii) $u(x) \leq \gamma$ for all $x \in I\!R^N \setminus \Omega$.

Similarly, supersolutions and solutions are defined.

In the next two Remarks we summarize existence, uniqueness and regularity results shown in [3, 4], for problems (2.6) and (2.9), that will be used in the sequel.

Remark 2.5. Let $\Omega \subset I\!R^N$ an open bounded subset with $\partial\Omega$ of class C^1; let $\gamma \in I\!R$. Let assumption (H_1) be satisfied. Assume that $\sup_\Omega c < 0$. We have that

(i) there exists a unique solution to problem (2.9);
(ii) if u is a subsolution of problem (2.9) and v is a supersolution of problem (2.9), then $u \leq v$ in $I\!R^N$;

(iii) if u is a solution of equation (2.7), then, for some $\mu \in (0,1)$, for any open subset $\Omega' \subset\subset \Omega$,

$$\|u\|_{C^{0,\mu}(\Omega')} \leq C,$$

for some constant $C > 0$, which only depends on $\|u\|_\infty, N, a, c, f$.

Note that (i), (ii) follow from [3], (Thm. 2 and [4], Thm. 1), whereas from Theorem 2 and the comments at the end of page 2 in [4], it follows (iii).

Note that in order to apply the results from [3, 4] we need to observe that equation (2.7) is equivalent to equation

$$(-\Delta)^s u = \frac{c}{a}u + \frac{f}{a} \qquad \text{in} \qquad \Omega,$$

for which those results can be used.

Furthermore, observe that when dealing with elliptic equations, we assume that $c < 0$ in $\mathbb{R}^N$. This guarantees, together the continuity of c, that $\sup_\Omega c < 0$, which is required by Remark 2.5. If one can prove the results in Remark 2.5 under the more general hypothesis that $c \leq 0$ in Ω, all the results in the following remain true, only supposing $c \leq 0$ in $\mathbb{R}^N$.

Analogously, noting that equation (2.3) is equivalent to equation

$$\frac{1}{a}\partial_t u + (-\Delta)^s u = \frac{c}{a}u + \frac{f}{a}u \qquad \text{in} \qquad Q_T,$$

from the results in ([3], Sect. 4.3) and in [4] (see also the comments at the end of page 2 in [4]) we get the next result.

Remark 2.6. Let $\Omega \subset \mathbb{R}^N$ be an open bounded subset with $\partial\Omega$ of class C^1. Let assumption (H_1) be satisfied. Let $g \in C([0,T]), u_0 \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$; suppose that condition (2.5) is satisfied. We have that:

(i) there exists a unique solution to problem (2.6);
(ii) if u is a subsolution of problem (2.6) and v is a supersolution of problem (2.6), then $u \leq v$ in Q_T;
(iii) if u is a solution of equation (2.3), then, for some $0 < \mu < 1$, for any open subset $\Omega' \subset\subset \Omega, \tau \in (0,T]$ we have

$$|u(x,t_1) - u(y,t_2)| \leq C\big(|x-y|^\mu + |t_1 - t_2|^{\frac{\mu}{2s}}\big) \qquad \text{for all} \qquad x, y \in \Omega', t_1, t_2 \in [\tau, T],$$

for some constant $C > 0$, which only depends on $\|u\|_\infty, N, a, c, f$.

Note that the estimates in Remark 2.5(iii) and in Remark 2.6(iii), which require assumption (H_1), will have a crucial role in proving existence of solutions. In fact, they permit to use compactnees arguments for solutions of problems in suitable approximating domains.

2.1. Main results: Existence, uniqueness and asymptotic behaviour of solutions

In the following, we always assume that

$$0 < s < 1, \quad N > 2s.$$

Concerning existence and uniqueness of solutions of problem (1.1) we have the next result.

Theorem 2.7. *Let assumptions* $(H_0), (H_1)$ *be satisfied. Let* $T > 0$. *Let* $g \in C([0,T]), u_0 \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$; *suppose that condition* (1.3) *is satisfied. Then there exists a unique solution* u *to problem* (1.1) *such that condition* (1.4) *is satisfied. Furthermore,* (1.6) *holds.*

Under the extra hypothesis that $c < 0$, we have the next existence and uniqueness result for problem (1.9).

Theorem 2.8. *Let assumptions* $(H_0), (H_1)$ *be satisfied. Let* $g \in C([0,\infty)) \cap L^\infty((0,\infty)), u_0 \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), c < 0$*; suppose that condition* (1.3) *is satisfied. Then there exists a unique solution to problem* (1.9) *such that condition* (1.10) *is satisfied. Furthermore, for some* $C > 0$, (1.11) *holds.*

Remark 2.9. Observe that the estimate in (1.6) depends on $T > 0$, while that in (1.11) is independent of T. In order to get (1.11) we use the further hypothesis $c < 0$.

Concerning the elliptic equation (1.2) we show the next result.

Theorem 2.10. *Let assumptions* $(H_0), (H_1)$ *be satisfied. Let* $\gamma \in \mathbb{R}$*; suppose that* $c < 0$ *in* $\mathbb{R}^N$*. Then there exists a unique solution to equation* (1.2) *such that condition* (1.16) *is satisfied.*

The next theorem is concerned with the asymptotic behaviour as $t \to \infty$ of solutions of problem (1.9).

Theorem 2.11. *Let assumptions of Theorem* 2.8 *be satisfied. Let* $\gamma := \lim_{t\to\infty} g(t)$*. Let* u *be the unique solution to problem* (1.1) *such that* (1.4) *is satisfied. Suppose that condition* (1.15) *holds. Then*

$$\lim_{t\to\infty} u(x,t) \;=\; W(x) \quad \textit{for all } \; x \in \mathbb{R}^N,$$

where W *is the unique solution of equation* (1.2) *satisfying condition* (1.16)*.*

Remark 2.12. Note that the existence result in Theorem 2.10 can be regarded as a consequence of Theorem 2.11. In fact, from Theorem 2.11 in particular we obtain the existence of a solution $W(x) := \lim_{t\to\infty} u(x,t)$ of problem (1.2), where $u(x,t)$ is the solution of problem (1.1) with $g(t) \equiv \gamma$ and u_0 satisfying (1.3). However, in Section 4 we give an independent proof of Theorem 2.11, without using results concerning the parabolic problem. Finally, observe that the supersolution $h(x)$ of equation (1.12) plays a crucial role both in the Proof of Theorem 2.10 and in that of Theorem 2.11.

3. Construction of stationary supersolutions

For any $C > 0, \beta > 0$ define the function

$$V(x) := C|x|^{-\beta} \quad \big(x \in \mathbb{R}^N \setminus \{0\}\big). \tag{3.10}$$

Concerning the function V, we show the next result.

Proposition 3.1. *Let assumptions* $(H_0), (H_1) - (i)$ *be satisfied; let* $R_0 > 0$*. Then there exist* $C > 0, \beta > 0$ *such that the function* V *satisfies*

$$-a(x)(-\Delta)^s V(x) \;\leq\; -1 \quad \textit{for all } x \in \mathbb{R}^N \setminus \overline{B}_{R_0}. \tag{3.11}$$

In particular, V *is a supersolution of equation* (1.7) *in the sense of Definition* 2.3*. Moreover,* (1.8) *holds.*

Proof. Note that (see Sect. 2)

$$(-\Delta)^s V(x) = C_{N,s} \int_{\mathbb{R}^N} \frac{V(x) - V(y)}{|x-y|^{N+2s}} \mathrm{d}y \quad \text{for all } \; x \in \mathbb{R}^N \setminus \{0\}.$$

Moreover (see [23], Exp. 5, p. 257), we have

$$\hat{V}(\xi) = C_\beta C |\xi|^{-N+\beta}$$

for some $C_\beta > 0$. Hence, by (2.1),

$$(-\Delta)^s V(x) = C C_\beta (\mathfrak{F}^{-1} |\xi|^{2s-N+\beta})(x) = C C_\beta C_{\beta+2s} |x|^{-\beta+2s} \qquad \text{for all} \qquad x \in \mathbb{R}^N \setminus \{0\}.$$

Thus, in view of (H_0), we have

$$-a(x)(-\Delta)^s V(x) \leq -CC_0 C_\beta C_{\beta+2s}|x|^{-(\beta+2s)+\alpha} \qquad \text{for all} \qquad x \in I\!R^N \setminus \{0\}. \tag{3.12}$$

Now we choose $0 < \beta < \alpha - 2s$; so, from (3.12) it follows that (3.11) is satisfied, provided that

$$C \geq \frac{1}{R_0^{\beta+2s-\alpha} C_0 C_\beta C_{\beta+2s}}.$$

This completes the proof. □

Proposition 3.2. *Let assumptions $(H_0), (H_1)-(i)$ be satisfied. There exists a supersolution h of equation (1.12) in the sense of Definition 2.3, which satisfies (1.13).*

Proof. Let V be given by Proposition 3.1. Take $\hat{R} > 0$. From the results in [9] it follows that, for a certain $C_1 = C_1(N, s) > 0$, the function

$$\bar{W}(x) \equiv \bar{W}(|x|) := C_1\big(\hat{R}^2 - |x|^2\big)_+^{s/2} \quad (x \in I\!R^N).$$

solves

$$\begin{cases} (-\Delta)^s u \, = \, 1 & \text{in} \quad B_{\hat{R}} \\ u \, = 0 & \text{in} \quad I\!R^N \setminus B_{\hat{R}}. \end{cases} \tag{3.13}$$

Hence, it easily follows that for each $\mu_0 > 0, \mu_1 \geq \mu_0 \max_{\overline{B}_{\hat{R}}} \frac{1}{a}$ and $\mu_2 > 0$, the function

$$W(x) \equiv W(|x|) := \mu_1 \bar{W}(x) + \mu_2 \quad \big(x \in I\!R^N\big)$$

is a supersolution of problem

$$\begin{cases} -a(x)(-\Delta)^s u \, = \, -\mu_0 & \text{in} \quad B_{\hat{R}} \\ u \, = \mu_2 & \text{in} \quad I\!R^N \setminus B_{\hat{R}}. \end{cases} \tag{3.14}$$

For any $\tilde{C} > 0$ set

$$\tilde{V}(x) \equiv \tilde{V}(|x|) := \tilde{C} V(|x|) \quad \big(x \in I\!R^N\big).$$

It is easily checked that if

$$\mu_2 > C\, 2^\beta\, \hat{R}^{-\beta}\, \tilde{C}, \tag{3.15}$$

then

$$\tilde{V} < W \qquad \text{in} \qquad \left[\frac{\hat{R}}{2}, \hat{R}\right]. \tag{3.16}$$

Furthermore, since $\tilde{V}(x) \to +\infty$ as $x \to 0$, in view of (3.16), we can deduce that there exists $\bar{R} \in (0, \hat{R}/2)$ such that $W(\bar{R}) = \tilde{V}(\bar{R})$. Indeed, such $\bar{R}$ is unique. To see this, take any $\bar{R} > 0$ such that $W(\bar{R}) = \tilde{V}(\bar{R})$. In view of (3.16) and the very definition of W and $\tilde{V}$ we have that $\bar{R} \in \left(0, \frac{\hat{R}}{2}\right)$. Furthermore,

$$\hat{R}^2 - \bar{R}^2 \geq 1, \tag{3.17}$$

provided $\hat{R} > 2$. Moreover, it is direct to check that if we show that

$$W'(\bar{R}) > \tilde{V}'(\bar{R}), \tag{3.18}$$

then such $\bar{R}$ is unique. In order to show (3.18), note that (3.18) is equivalent to

$$s\mu_1 C_1 \bar{R}^2 < \beta(\hat{R}^2 - \bar{R}^2)^{1-\frac{s}{2}}\tilde{V}(\bar{R}). \tag{3.19}$$

Now, in view of (3.17) and the definition of W, (3.19) follows if we take

$$\tilde{C} > \frac{s\mu_1 C_1 \hat{R}^{2+\beta}}{\beta C}\cdot \tag{3.20}$$

Therefore,

$$\tilde{V} \geq W \quad \text{in } \ B_{\bar{R}}, \quad \tilde{V}(\bar{R}) = W(\bar{R}), \quad \tilde{V} \leq W \qquad \text{in} \qquad I\!R^N \setminus B_{\bar{R}}. \tag{3.21}$$

Take $0 < R_0 < \bar{R}$. Define

$$h := \min\{\tilde{V}, W\} \qquad \text{in} \qquad I\!R^N.$$

We claim that h is a supersolution of equation

$$-a(-\Delta)^s h = -\min\{\mu_0, \tilde{C}\} \quad \text{in } \ I\!R^N.$$

In fact, since V is a supersolution of equation (1.7), by Definition 2.3 and (2.2), for any open bounded subset $\Omega' \subset I\!R^N \setminus \overline{B}_{R_0}$, for any $x_0 \in \Omega'$, for any test function $\varphi \in C^2(I\!R^N)$ such that $\tilde{V}(x_0) - \varphi(x_0) \leq \tilde{V}(x) - \varphi(x)$ for all $x \in \Omega'$, one has

$$a(x_0) C_{N,s} \int_{I\!R^N} \frac{\psi(x_0) - \psi(y)}{|x_0 - y|^{N+2s}} \mathrm{d}y \geq \tilde{C},$$

where ψ is defined by (2.8) with u replaced by $\tilde{V}$ and U by Ω'. Hence

$$a(x_0) C_{N,s} \left\{ \int_{\Omega'} \frac{\varphi(x_0) - \varphi(y)}{|x_0 - y|^{N+2s}} \mathrm{d}y + \int_{I\!R^N \setminus \Omega'} \frac{\tilde{V}(x_0) - \tilde{V}(y)}{|x_0 - y|^{N+2s}} \mathrm{d}y \right\} \geq \tilde{C}. \tag{3.22}$$

Similarly, since W is a supersolution of problem (3.14), we have that for any open bounded subset $U \subset B_{R_0}$, for any $x_0 \in U$, for any test function $\varphi \in C^2(I\!R^N)$ such that $W(x_0) - \varphi(x_0) \leq W(x) - \varphi(x)$ for all $x \in U$, one has

$$a(x_0) C_{N,s} \left\{ \int_{U} \frac{\varphi(x_0) - \varphi(y)}{|x_0 - y|^{N+2s}} \mathrm{d}y + \int_{I\!R^N \setminus U} \frac{W(x_0) - W(y)}{|x_0 - y|^{N+2s}} \mathrm{d}y \right\} \geq \mu_0. \tag{3.23}$$

Now, take any $x_0 \in I\!R^N$ with $|x_0| \geq \bar{R}$, any open bounded subset $U \subset I\!R^N$ with $x_0 \in U$, and any test function $\varphi \in C^2(I\!R^N)$ such that $h(x_0) - \varphi(x_0) \leq h(x) - \varphi(x)$ for all $x \in U$. Set

$$\psi := \begin{cases} \varphi & \text{in} \quad U \\ h & \text{in} \quad I\!R^N \setminus U. \end{cases} \tag{3.24}$$

Note that, due to (3.21), we have

$$h(x_0) = \tilde{V}(x_0). \tag{3.25}$$

For any $0 < \epsilon < \bar{R} - R_0$, we have $\mathcal{U}_1 := U \cap \left(I\!R^N \setminus B_{R_0+\epsilon}\right) \subset I\!R^N \setminus \overline{B}_{R_0}, x_0 \in \mathcal{U}_1$. Moreover,

$$\varphi(x) \leq \tilde{V}(x) \quad \text{for all } \ x \in \mathcal{U}_1, \quad \varphi(x_0) = \tilde{V}(x_0). \tag{3.26}$$

So, from (3.22) with $\Omega' = \mathcal{U}_1$ we get

$$a(x_0) C_{N,s} \left\{ \int_{\mathcal{U}_1} \frac{\varphi(x_0) - \varphi(y)}{|x_0 - y|^{N+2s}} \mathrm{d}y + \int_{I\!R^N \setminus \mathcal{U}_1} \frac{\tilde{V}(x_0) - \tilde{V}(y)}{|x_0 - y|^{N+2s}} \mathrm{d}y \right\} \geq \tilde{C}. \tag{3.27}$$

Due to (3.21) and (3.25), since $h \le \tilde{V}$ in $I\!R^N$, we have

$$a(x_0)C_{N,s}\left\{\int_{\mathcal{U}_1}\frac{\varphi(x_0)-\varphi(y)}{|x_0-y|^{N+2s}}\mathrm{d}y+\int_{I\!R^N\setminus\mathcal{U}_1}\frac{h(x_0)-h(y)}{|x_0-y|^{N+2s}}\mathrm{d}y\right\}\ge\tilde{C}. \tag{3.28}$$

Set $\mathcal{U}_2 := U \cap B_{R_0+\epsilon}$. In view of (3.28), since $\varphi(x_0)-\varphi(y) \ge h(x_0)-h(y)$ for all $y \in \mathcal{U}_2$ we have

$$\begin{aligned} a(x_0)C_{N,s}\int_{I\!R^N}\frac{\psi(x_0)-\psi(y)}{|x_0-y|^{N+2s}}\mathrm{d}y &= a(x_0)C_{N,s}\left\{\int_U\frac{\varphi(x_0)-\varphi(y)}{|x_0-y|^{N+2s}}\mathrm{d}y+\int_{I\!R^N\setminus U}\frac{h(x_0)-h(y)}{|x_0-y|^{N+2s}}\mathrm{d}y\right\}\\ &= a(x_0)C_{N,s}\left\{\int_{\mathcal{U}_1}\frac{\varphi(x_0)-\varphi(y)}{|x_0-y|^{N+2s}}\mathrm{d}y+\int_{I\!R^N\setminus\mathcal{U}_1}\frac{h(x_0)-h(y)}{|x_0-y|^{N+2s}}\mathrm{d}y\right.\\ &\quad\left.-\int_{\mathcal{U}_2}\frac{h(x_0)-h(y)}{|x_0-y|^{N+2s}}\mathrm{d}y+\int_{\mathcal{U}_2}\frac{\varphi(x_0)-\varphi(y)}{|x_0-y|^{N+2s}}\mathrm{d}y\right\}\ge\tilde{C}. \end{aligned}\tag{3.29}$$

Now, take any $x_0 \in I\!R^N$ with $|x_0| < \bar{R}$, any open bounded subset $U \subset I\!R^N$ with $x_0 \in U$, and any test function $\varphi \in C^2(I\!R^N)$ such that $h(x_0)-\varphi(x_0) \le h(x)-\varphi(x)$ for all $x \in U$. Let ψ be defined by (3.24). Note that (3.21) gives

$$h(x_0) = W(x_0). \tag{3.30}$$

For any $0<\epsilon<\bar{R}-R_0$ we have $\mathcal{U}_1 := U \cap B_{\bar{R}-\epsilon} \subset \overline{B}_{\bar{R}}, x_0 \in \mathcal{U}_1$. Moreover,

$$\varphi(x) \le W(x) \quad \text{for all } \ x \in \mathcal{U}_1, \quad \varphi(x_0) = W(x_0). \tag{3.31}$$

So, from (3.23) with $\Omega' = \mathcal{U}_1$ we get

$$a(x_0)C_{N,s}\left\{\int_{\mathcal{U}_1}\frac{\varphi(x_0)-\varphi(y)}{|x_0-y|^{N+2s}}\mathrm{d}y+\int_{I\!R^N\setminus\mathcal{U}_1}\frac{W(x_0)-W(y)}{|x_0-y|^{N+2s}}\mathrm{d}y\right\}\ge\mu_0. \tag{3.32}$$

Due to (3.30) and (3.32), since $h \le W$ in $I\!R^N$, we have

$$a(x_0)C_{N,s}\left\{\int_{\mathcal{U}_1}\frac{\varphi(x_0)-\varphi(y)}{|x_0-y|^{N+2s}}\mathrm{d}y+\int_{I\!R^N\setminus\mathcal{U}_1}\frac{h(x_0)-h(y)}{|x_0-y|^{N+2s}}\mathrm{d}y\right\}\ge\mu_0. \tag{3.33}$$

Set $\mathcal{U}_2 := U \cap \left(I\!R^N \setminus B_{\bar{R}-\epsilon}\right)$. In view of (3.33), since $\varphi(x_0)-\varphi(y) \ge h(x_0)-h(y)$ for all $y \in \mathcal{U}_2$ we have

$$\begin{aligned} a(x_0)C_{N,s}\int_{I\!R^N}\frac{\psi(x_0)-\psi(y)}{|x_0-y|^{N+2s}}\mathrm{d}y &= a(x_0)C_{N,s}\left\{\int_U\frac{\varphi(x_0)-\varphi(y)}{|x_0-y|^{N+2s}}\mathrm{d}y+\int_{I\!R^N\setminus U}\frac{h(x_0)-h(y)}{|x_0-y|^{N+2s}}\mathrm{d}y\right\}\\ &= a(x_0)C_{N,s}\left\{\int_{\mathcal{U}_1}\frac{\varphi(x_0)-\varphi(y)}{|x_0-y|^{N+2s}}\mathrm{d}y+\int_{I\!R^N\setminus\mathcal{U}_1}\frac{h(x_0)-h(y)}{|x_0-y|^{N+2s}}\mathrm{d}y\right.\\ &\quad\left.-\int_{\mathcal{U}_2}\frac{h(x_0)-h(y)}{|x_0-y|^{N+2s}}\mathrm{d}y+\int_{\mathcal{U}_2}\frac{\varphi(x_0)-\varphi(y)}{|x_0-y|^{N+2s}}\mathrm{d}y\right\}\ge\mu_0. \end{aligned}\tag{3.34}$$

From (3.29) and (3.34) the claim follows. Therefore,

$$h := \bar{C}h \qquad \text{in} \qquad I\!R^N,$$

with $\bar{C} \ge \max\left\{\frac{1}{\mu_0}, \frac{1}{\tilde{C}}\right\}$, is a supersolution of equation (1.12); moreover, it is immediately seen that it satisfies (1.13). □

4. Proofs of existence and uniqueness results

To begin with, let us show the next quite standard comparison principle.

Proposition 4.1. *Let assumptions* $(H_0), (H_1)$ *be satisfied. Let* u *be a subsolution of problem* (1.1)*, let* v *be a supersolution of problem* (1.1)*. Suppose that both*

$$\limsup_{|x|\to\infty}(u-v) \le 0 \qquad \text{uniformly for} \quad t\in[0,T].$$

Then

$$u \le v \qquad \text{in} \qquad S_T.$$

Proof. Set $w := u - v$. Let $\epsilon > 0$. Then there exists $R_\epsilon > 0$ such that

$$|w(x,t)| \le \epsilon \quad \text{for all } \ x \in I\!R^N \setminus B_{R_\epsilon}, t \in [0,T].$$

Hence, it is easily seen that w is a subsolution of problem (in the sense of Def. 2.1)

$$\begin{cases} \partial_t w = -a\,(-\Delta)^s w + cw & \text{in} \quad B_{R_\epsilon}\times(0,T] \\ w\ = \epsilon & \text{in} \quad \big(I\!R^N\setminus B_{R_\epsilon}\big)\times(0,T] \\ w\ = 0 & \text{in} \quad I\!R^N\times\{0\}. \end{cases} \tag{4.1}$$

Moreover, it is easily seen that the function

$$z(x,t) := \epsilon\, e^{\|c\|_\infty t} \quad \big(x\in I\!R^N, t\in[0,T]\big)$$

is a supersolution of problem (4.1). By the comparison principle (see Rem. 2.6),

$$w \le z \quad \text{in } \ I\!R^N\times[0,T]. \tag{4.2}$$

Letting $\epsilon\to 0^+$, we get $w\le 0$ in $I\!R^N\times[0,T]$. Hence the proof is complete. □

Let us prove Theorem 2.7. Hereafter, $\{\zeta_j\}\subset C_c^\infty(B_j)$ will be a sequence of functions such that

$$0\le\zeta_j\le 1,\ \ \zeta_j\equiv 1 \ \text{ in } \ B_{j/2} \ \text{ for each } j\in I\!N. \tag{4.3}$$

Proof of Theorem 2.7. For any $j\in I\!N$ let u_j be the unique solution (see Rem. 2.6) of the problem

$$\begin{cases} \partial_t u = -a\,(-\Delta)^s u + cu + f & \text{in} \quad B_j\times(0,T] \\ u\ = g & \text{in} \quad \big(I\!R^N\setminus B_j\big)\times(0,T] \\ u\ = u_{0,j} & \text{in} \quad I\!R^N\times\{0\}, \end{cases} \tag{4.4}$$

where

$$u_{0,j}(x) := \zeta_j(x)u_0(x) + [1-\zeta_j(x)]g(0) \qquad \text{for all} \qquad x\in\overline{B}_j.$$

It is easily seen that the function

$$\overline{v}(x,t) := Ce^{\beta t} \quad \big((x,t)\in I\!R^N\times[0,T]\big)$$

is a supersolution of problem (4.4) for any $j\in I\!N$, provided that

$$\beta\ge 1+\|c\|_\infty, \ \ C\ge\max\{\|f\|_\infty,\|g\|_\infty,\|u_0\|_\infty\}.$$

Thus, by the comparison principle (see Rem. 2.6),

$$u_j(x,t) \le \bar{v}(x,t) \quad \text{for all } \ (x,t) \in I\!R^N \times [0,T]. \tag{4.5}$$

Furthermore, the function

$$\underline{v}(x,t) := -C\mathrm{e}^{\beta t} \quad \big((x,t) \in I\!R^N \times [0,T]\big)$$

is a subsolution of problem (4.4) for any $j \in I\!N$. Thus, by the comparison principle,

$$u_j(x,t) \ge \underline{v}(x,t) \quad \text{for all } \ (x,t) \in I\!R^N \times [0,T]. \tag{4.6}$$

From (4.5)–(4.6) we obtain

$$\big|u_j(x,t)\big| \le C\mathrm{e}^{\beta T} =: K_T \quad \text{for all } \ (x,t) \in I\!R^N \times [0,T]. \tag{4.7}$$

By the *a priori* estimates recalled in Remark 2.6(iii) and usual compactness arguments, there exists a subsequence $\{u_{j_k}\} \subset \{u_j\}$ and a function $u \in C(S_T)$ such that

$$u := \lim_{k\to\infty} u_{j_k} \qquad \text{uniformly in} \qquad D \times [\tau, T],$$

for any compact subset $D \subset I\!R^N$ and for any $\tau \in (0,T)$. For simplicity we still denote $\{u_{j_k}\}$ by $\{u_j\}$. In view of stability properties of viscosity solutions under local uniform convergence, the function u is a solution of equation

$$\partial_t u = -a\,(-\varDelta)^s u + cu + f \qquad \text{in} \qquad I\!R^N \times (0,T].$$

Claim 1. We have that

$$\lim_{t\to 0^+} u(x,t) \;=\; u_0(x) \qquad \text{for any} \qquad x \in I\!R^N.$$

In fact, let $x_0 \in I\!R^N$. Take $j_0 \in I\!N$ so large that $x_0 \in B_{j_0/2}$. In view of the definition of $\{\zeta_j\}$ (see (4.3)) there exists $\delta_0 \in (0,1)$ such that for any $j \ge j_0$

$$u_j(x,0) = u_{0,j}(x) = u_0(x) \quad \text{for all } \ x \in B_{\delta_0}(x_0). \tag{4.8}$$

Since $u_0 \in C(I\!R^N)$, for any $0 < \epsilon < 1$ there exists $\delta \in (0,\delta_0)$ such that

$$-\epsilon < u_0(x) - u_0(x_0) < \epsilon \quad \text{for all } \ x \in B_\delta(x_0). \tag{4.9}$$

From (4.8), (4.9) it follows that for any $0 < \epsilon < 1$ and any $j \ge j_0$ there holds

$$-\epsilon < u_j(x,0) - u_0(x_0) < \epsilon \quad \text{for all } \ x \in B_\delta(x_0). \tag{4.10}$$

Consider a function $\chi \in C^\infty(I\!R^N)$ such that

$$\chi(x) = |x - x_0|^2 \quad \big(x \in B_1(x_0)\big),$$

$$0 \le \chi \le 2 \quad \text{in } \ I\!R^N, \ \chi \equiv 2 \quad \big(x \in I\!R^N \setminus B_2(x_0)\big).$$

Observe that since $\chi \in C^\infty(I\!R^N) \cap L^\infty(I\!R^N)$, we have that $(-\varDelta)^s\chi \in C(I\!R^N)$ (see Sect. 2). Therefore

$$\sup_{x\in B_\delta(x_0)} \big|(-\varDelta)^s\chi(x)\big| < \infty.$$

Define

$$h(x,t) := \big[\chi(x) + At\big]\mathrm{e}^{\eta t} \quad (x \in I\!R^N, t \in [0,\delta]),$$

$$\overline{v}(x,t) := Mh(x,t) + u_0(x_0) + \epsilon \quad (x \in I\!R^N, t \in [0,\delta]),$$

where $A > 0, \eta > 0, M$ are constants to be determined. We have that

$$\partial_t \overline{v}(x,t) = M[A\mathrm{e}^{\eta t} + \eta h(x,t)] \quad (x \in I\!R^N, t \in [0,\delta]),$$

whereas

$$-a(x)(-\Delta)^s \overline{v}(x,t) + c(x)\overline{v}(x,t) + f(x)$$
$$\leq M\mathrm{e}^{\eta\delta} \max_{\overline{B}_\delta(x_0)} \left| a(-\Delta)^s \chi \right| + Mc(x)h(x,t) + \|c\|_\infty(\|u_0\|_\infty + 1) + \|f\|_\infty \quad \big(x \in B_\delta(x_0), t \in [0,\delta]\big).$$

Therefore,

$$\partial_t \overline{v}(x,t) \geq -a(x)(-\Delta)^s \overline{v}(x,t) + c(x)\overline{v}(x,t) + f(x) \quad \big(x \in B_\delta(x_0), t \in [0,\delta]\big), \tag{4.11}$$

if

$$\eta \geq \|c\|_\infty, \quad A \geq \max_{\overline{B}_\delta(x_0)} \left| a(-\Delta)^s \chi \right| + \|c\|_\infty(\|u_0\|_\infty + 1) + \|f\|_\infty. \tag{4.12}$$

Furthermore, since

$$h(x,t) \geq \delta^2 \quad \text{for all } \; x \in I\!R^N \setminus B_\delta(x_0), t \in [0,\delta],$$

it easily follows that

$$\overline{v}(x,t) \geq u_j(x,t) \quad \text{for all } \; x \in I\!R^N \setminus B_\delta(x_0), t \in [0,\delta], \tag{4.13}$$

if

$$M \geq \frac{2K_T}{\delta^2}\cdot \tag{4.14}$$

From (4.10) we get

$$\overline{v}(x,0) \geq u_j(x,0) \qquad \text{for all} \qquad x \in B_\delta(x_0), \tag{4.15}$$

while

$$\overline{v}(x,0) \geq M\delta^2 + u_0(x_0) \geq u_j(x,0) \qquad \text{for all} \qquad x \in I\!R^N \setminus B_\delta(x_0), \tag{4.16}$$

due to (4.14).

Suppose that (4.12), (4.14) hold. Then, by (4.11), (4.13), (4.15), (4.16), for any $j \in I\!N$ the function $\overline{v}$ is a supersolution (in the sense of Def. 2.2) of problem

$$\begin{cases} \partial_t v = -a\,(-\Delta)^s v + cv + f & \text{in} \quad B_\delta(x_0) \times (0,\delta] \\ v \;= u_j & \text{in} \quad \big(I\!R^N \setminus B_\delta(x_0)\big) \times (0,\delta] \\ v \;= u_j & \text{in} \quad I\!R^N \times \{0\}, \end{cases} \tag{4.17}$$

while u_j is a solution of the same problem. By the comparison principle (see Rem. 2.6) we obtain

$$u_j \leq \overline{v} \quad \text{in } \; B_\delta(x_0) \times (0,\delta]. \tag{4.18}$$

Define

$$\underline{v}(x,t) := -Mh(x,t) + u_0(x_0) - \epsilon \quad (x \in I\!R^N, t \in [0,\delta]);$$

suppose that (4.12) and (4.14) hold. By the same arguments as above, we can show that there holds

$$u_j \geq \underline{v} \qquad \text{in} \qquad B_\delta(x_0) \times (0,\delta]. \tag{4.19}$$

Inequalities (4.18)–(4.19) yield

$$-Mh(x,t) - \epsilon \leq u_j(x,t) - u_0(x_0) \;\leq\; Mh(x,t) + \epsilon \tag{4.20}$$

for all $x \in B_\delta(x_0), t \in [0,\delta]$. Letting $j \to \infty$, thus we obtain

$$-Mh(x,t) - \epsilon \le u(x,t) - u_0(x_0) \le Mh(x,t) + \epsilon \tag{4.21}$$

for all $x \in B_\delta(x_0), t \in (0,\delta]$. Letting $x \to x_0, t \to 0^+$, and then $\epsilon \to 0^+$, we get that $\lim_{x\to x_0} u(x,t) = u_0(x_0)$. Hence the Claim 1 has been shown.

Claim 2. We have that

$$\lim_{|x|\to\infty} u(x,t) = g(t) \qquad \text{uniformly with respect to} \qquad t \in [0,T].$$

In fact, fix any $t_0 \in [0,T], 0 < \epsilon < 1$. Since $g \in C([0,T])$, there exists $\delta \in (0,1)$ such that

$$g(t_0) - \frac{\epsilon}{2} \le g(t) \le g(t_0) + \frac{\epsilon}{2} \qquad \text{for any} \qquad t \in [\underline{t}_\delta, \overline{t}_\delta], \tag{4.22}$$

where

$$\underline{t}_\delta := \max\{t_0 - \delta, 0\}, \quad \overline{t}_\delta := \min\{t_0 + \delta, T\}.$$

Clearly, $\delta = \delta(\epsilon)$ does not depend on t_0. Furthermore, due to (1.3), there exists $R_\epsilon > 0$ such that

$$g(0) - \frac{\epsilon}{2} \le u_0(x) \le g(0) + \frac{\epsilon}{2} \qquad \text{for all} \qquad x \in I\!R^N \setminus B_{R_\epsilon}. \tag{4.23}$$

Let $R \ge \max\{R_0, R_\epsilon\}$ with R_0 given by Proposition 3.1; set

$$N_j^R := B_j \setminus B_R \qquad \text{for any} \qquad j > R.$$

Define

$$\underline{w}(x,t) := -MV(x)\mathrm{e}^{\eta t} - \lambda(t-t_0)^2 + g(t_0) - \epsilon \qquad \text{for all} \qquad (x,t) \in I\!R^N \times [0,T], \tag{4.24}$$

where $M > 0, \eta > 0, \lambda > 0$ are constants to be chosen in the sequel, while $V(x) \equiv V(|x|)$ is the supersolution given by Proposition 3.1.

In view of Proposition 3.1, we have

$$-a(x)(-\Delta)^s\underline{w} + c(x)\underline{w} \ge M\mathrm{e}^{\eta t} - M\,c(x)V(x)\,\mathrm{e}^{\eta t} - \|c\|_\infty(\|g\|_\infty + \lambda + 1) \qquad \text{for all} \qquad x \in N_j^R, t \in (\underline{t}_\delta, \overline{t}_\delta].$$

Therefore,

$$\begin{aligned}\partial_t\underline{w} + a(-\Delta)^s\underline{w} - c\underline{w} - f \le& -\eta MV\mathrm{e}^{\eta t} - 2\lambda(t-t_0) - M\mathrm{e}^{\eta t} + cMV\mathrm{e}^{\eta t}\\ &+ \|c\|_\infty\big(\|g\|_\infty + \lambda + 1\big) + \|f\|_\infty \le 0 \quad \text{in } N_j^R \times (\underline{t}_\delta, \overline{t}_\delta],\end{aligned} \tag{4.25}$$

if we take

$$\eta \ge \|c\|_\infty, \tag{4.26}$$

$$M \ge 2\lambda + \|f\|_\infty + \|c\|_\infty(\|g\|_\infty + \lambda + 1) + \|f\|_\infty. \tag{4.27}$$

In view of (4.7), we obtain

$$\underline{w}(x,t) \le -MV(R) + \|g\|_\infty \le -K_T \le u_j(x,t) \quad \text{for all } x \in \overline{B}_R, t \in (\underline{t}_\delta, \overline{t}_\delta), \tag{4.28}$$

if

$$M \ge \frac{\|g\|_\infty + K_T}{V(R)}. \tag{4.29}$$

From (4.22) we have

$$\underline{w}(x,t) \leq g(t) \quad \text{for all } \; x \in I\!\!R^N \setminus B_j, t \in (\underline{t}_\delta, \overline{t}_\delta). \tag{4.30}$$

Suppose that $\underline{t}_\delta = 0$ (note that this is always the case when $t_0 = 0$). From (4.22) and (4.23) we have

$$\underline{w}(x,0) \leq g(t_0) - \epsilon \leq g(0) - \frac{\epsilon}{2} \leq u_j(x,0) = u_{0,j}(x) \quad \text{for all } \; x \in I\!\!R^N \setminus B_R; \tag{4.31}$$

while

$$\underline{w}(x,0) \leq -MV(R_\epsilon) + \|g\|_\infty \leq -K_T \leq u_j(x,0) \quad \text{for all } \; x \in B_R, \tag{4.32}$$

provided that (4.29) holds.

Suppose that $\underline{t}_\delta > 0$. It follows from (4.7) that

$$\underline{w}(x,\underline{t}_\delta) \leq -\lambda\delta^2 + \|g\|_\infty \leq -K_T \leq u_j(x,\underline{t}_\delta) \quad \text{for all } \; x \in I\!\!R^N, t \in (\underline{t}_\delta, \overline{t}_\delta), \tag{4.33}$$

if

$$\lambda \geq \frac{\|g\|_\infty + K_T}{\delta^2}. \tag{4.34}$$

Now, suppose that (4.26), (4.27), (4.29), (4.34) hold. By (4.25), (4.28), (4.30), (4.31), (4.32), (4.33), for any $j \in I\!\!N, j > R$, the function $\underline{w}$ is a subsolution (in the sense of Def. 2.2) of problem

$$\begin{cases} \partial_t v = -a\,(-\Delta)^s v + cv + f & \text{in} \quad N_j^R \times (\underline{t}_\delta, \overline{t}_\delta] \\ v = u_j & \text{in} \quad \left(I\!\!R^N \setminus N_j^R\right) \times (0,T] \\ v = u_j & \text{in} \quad I\!\!R^N \times \{0\}, \end{cases} \tag{4.35}$$

while u_j is a solution of the same problem. By the comparison principle (see Rem. 2.6) we obtain

$$\underline{w} \leq u_j \qquad \text{in} \qquad N_j^R \times (\underline{t}_\delta, \overline{t}_\delta]. \tag{4.36}$$

Define

$$\overline{w}(x,t) := MV(x)\mathrm{e}^{\eta t} + \lambda(t-t_0)^2 + g(t_0) + \epsilon \qquad \text{for all} \qquad x \in I\!\!R^N, t \in [0,T]; \tag{4.37}$$

suppose that (4.26), (4.27), (4.29), (4.34). By the same arguments as above, we can show that there holds

$$\overline{w} \geq u_j \qquad \text{in} \qquad N_j^R \times (\underline{t}_\delta, \overline{t}_\delta]. \tag{4.38}$$

From (4.36) and (4.38) we get

$$-MV(x)\mathrm{e}^{\eta t} - \underline{\lambda}(t-t_0)^2 - \epsilon \leq u_j(x,t) - g(t_0) \;\leq\; MV(x)\mathrm{e}^{\eta t} + \lambda(t-t_0)^2 + \epsilon \tag{4.39}$$

for all $x \in N_j^R, t \in (\underline{t}_\delta, \overline{t}_\delta]$. Choosing $t = t_0$ in (4.39) and letting $j \to \infty$, we obtain

$$-MV(x)\mathrm{e}^{\eta T} - \epsilon \leq u(x,t_0) - g(t_0) \;\leq\; MV(x)\mathrm{e}^{\eta T} + \epsilon \qquad \text{for all} \qquad x \in I\!\!R^N \setminus B_R, t \in (\underline{t}_\delta, \overline{t}_\delta]. \tag{4.40}$$

From (4.40) it follows that

$$\sup_{t_0 \in [0,T]} \left|u(x,t_0) - g(t_0)\right| \leq \overline{C}V(x) + \epsilon \qquad \text{for all} \qquad x \in I\!\!R^N \setminus B_R, \tag{4.41}$$

where $\overline{C} := M\mathrm{e}^{\eta T}$. Due to (4.41) and (1.8), letting $|x| \to \infty, \epsilon \to 0^+$, we obtain (1.4). Hence the Claim 2 has been shown.

Finally, this solution is unique, due to Proposition 4.1. □

Now we prove Theorem 2.8. We follow the same line of arguments of the proof of Theorem 2.7, but there is an important difference. In fact, we need to substitute the estimate (4.7), which is dependent on T, by another one independent of T. In order to obtain such better estimate we use the supersolution V constructed in Proposition 3.2.

Proof of Theorem 2.8. Arguing as in the proof of Theorem 2.7 we construct the sequence $u_j(x,t)$ of solutions of problem (4.4) with $T=\infty$. Let $h(x)$ be the supersolution provided by Proposition 3.2. Then obviously

$$V_0(x) := h(x) - \inf_{\mathbb{R}^N} h + 1 \tag{4.42}$$

is also a supersolution of (1.12) and $V_0(x) \geq 1$. Let $B := \max\{\|f\|_\infty, \|u_0\|_\infty, \|g\|_\infty\}$ and V_0 be defined in (4.42). Since $c \leq 0$, we have that BV_0 is a supersolution of problem (4.4), while $-BV_0$ is a subsolution of (4.4). Thus, by the comparison principle,

$$|u_j| \leq BV_0 \qquad \text{in} \qquad B_j \times (0,\infty). \tag{4.43}$$

Passing to the limit as $j \to \infty$ we obtain that

$$|u| \leq AV_0 \leq \check{C} := B\|V_0\|_\infty \qquad \text{in} \qquad \mathbb{R}^N \times (0,\infty). \tag{4.44}$$

Note that estimate (4.44) substitutes estimate (4.7) which is depending on T. Now, consider the functions $\underline{w}$ and $\overline{w}$ defined in (4.24) and in (4.37), respectively. Assume that

$$\eta = 0, \quad M \geq \frac{\|g\|_\infty + \check{C}}{V(R)}, \quad \lambda \geq \frac{\|g\|_\infty + \check{C}}{\delta^2}.$$

Note that M and λ do not depend on T. We observe that (4.26), (4.27) and (4.34) can be replaced by the present requirement on η and M, since now we are assuming that $c \leq 0$, and (4.44) holds. By the same arguments as in the proof of Theorem 2.7 we can infer that for any $\epsilon > 0$

$$\sup_{t_0 \in [0,\infty)} \big| u(x,t_0) - g(t_0) \big| \leq MV(x) + \epsilon \qquad \text{for all} \qquad x \in \mathbb{R}^N \setminus B_R. \tag{4.45}$$

Thanks to (4.45) and (1.8), letting $|x| \to \infty, \epsilon \to 0^+$, we obtain (1.10). Finally, this solution is unique, due to Proposition 4.1, applied for each fixed $T > 0$. This completes the proof. □

We have the next quite standard comparison principle.

Proposition 4.2. *Let assumptions* $(H_0), (H_1)$ *be satisfied. Suppose that* $c \leq 0$ *in* $\mathbb{R}^N$. *Let* u *be a subsolution and* v *a supersolution to equation* (1.2) *such that*

$$\limsup_{|x| \to \infty} (u - v) \leq 0.$$

Then

$$u \leq v \qquad in \qquad \mathbb{R}^N.$$

Proof. Set $w := u - v$. Let $\epsilon > 0$. Then there exists $R_\epsilon > 0$ such that

$$|w(x)| \leq \epsilon \qquad \text{for all} \qquad x \in \mathbb{R}^N \setminus B_{R_\epsilon}.$$

Hence, it is easily checked that w is a subsolution of problem (in the sense of Def. 2.3)

$$\begin{cases} -a\,(-\Delta)^s w + cw \,=\, 0 & \text{in} \quad B_{R_\epsilon} \\ w \,=\, \epsilon & \text{in} \quad \mathbb{R}^N \setminus B_{R_\epsilon}. \end{cases} \tag{4.46}$$

Moreover, it is easily seen that the function $z \equiv \epsilon$ is a supersolution of problem (4.46). So, by the comparison principle (see Rem. 2.5),

$$w \leq \epsilon \qquad \text{in} \qquad I\!\!R^N. \tag{4.47}$$

Letting $\epsilon \to 0^+$, we get $w \leq 0$ in $I\!\!R^N$. Hence the proof is complete. □

Now, we prove Theorem 2.10.

Proof of Theorem 2.10. Let $\gamma \in I\!\!R$. For any $j \in I\!\!N$ let u_j be the unique solution (see Remark 2.5) of the problem

$$\begin{cases} a\,(-\Delta)^s u - cu \,=\, f & \text{in} \quad B_j \times (0,T] \\ u \,=\gamma & \text{in} \quad I\!\!R^N \setminus B_j. \end{cases} \tag{4.48}$$

We claim that there exists $K > 0$ such that for any $j \in I\!\!N$

$$\left|u_j(x)\right| \leq K \quad \text{for all} \qquad x \in I\!\!R^N. \tag{4.49}$$

In fact, let $h = h(x) \equiv h(|x|)$ be the supersolution given by Proposition 3.2. Define

$$\tilde{h} := C(h+1) \qquad \text{in} \qquad I\!\!R^N,$$

where $C \geq \max\{\gamma, \|f\|_\infty\}$. It is easily seen that, for any $j \in I\!\!N$, h is a supersolution of problem (4.48). Therefore, by the comparison principle (see Rem. 2.5), we get (4.49), with $K = \|\tilde{h}\|_\infty$.

By the *a priori* estimates recalled in Remark 2.5(iii) and usual compactness arguments, there exists a subsequence $\{u_{j_k}\} \subset \{u_j\}$ and a function $u \in C(I\!\!R^N)$ such that

$$u := \lim_{k\to\infty} u_{j_k} \quad \text{uniformly in} \;\; D,$$

for any compact subset $D \subset I\!\!R^N$. For simplicity, we still denote $\{u_{j_k}\}$ by $\{u_j\}$. In view of stability properties of viscosity solutions under local uniform convergence, the function u is a solution of equation

$$a\,(-\Delta)^s u - cu \,=\, f \qquad \text{in} \qquad I\!\!R^N.$$

Claim. The solution u satisfies condition (1.16).

In fact, define

$$\underline{w}(x) := -\underline{M}h(x) + \gamma \quad \text{for all} \;\; x \in I\!\!R^N, \tag{4.50}$$

where $\underline{M} > 0$ is a constant to be chosen in the sequel.

In view of Proposition 3.2, it is easily seen that, if we take

$$\underline{M} \geq \|c\|_\infty \gamma + \|f\|_\infty,$$

then $\underline{w}$ is a subsolution of problem (4.48), for any $j \in I\!\!N$. By the comparison principle (see Rem. 2.5),

$$\underline{w} \leq u_j \quad \text{in} \;\; I\!\!R^N. \tag{4.51}$$

On the other hand, by the same methods as above, we can show that

$$u_j \leq \overline{w} \quad \text{in} \;\; I\!\!R^N, \tag{4.52}$$

where

$$\overline{w}(x) := \overline{M}h(x) + \gamma \quad \text{for all} \;\; x \in I\!\!R^N,$$

for suitable $\overline{M} > 0$.

From (4.51), (4.52) it follows that

$$-\underline{M}h + \gamma \leq u_j \leq \overline{M}h + \gamma \quad \text{in} \;\; I\!\!R^N.$$

Letting $j \to \infty$, in view of (1.13) we have that (2.4) holds. So, the Claim has been shown.

Finally, the uniqueness of the solution u follows from Proposition 4.2. □

5. Asymptotic behaviour of solutions: proofs

To begin with, we show the next auxiliary result.

Proposition 5.1. *Let assumptions of Theorem 2.8 be satisfied with $g \equiv g_1$. Assume that*

$$g_1(t_1) \le g_1(t_2) \qquad \text{for any} \quad 0 \le t_1 < t_2. \tag{5.1}$$

Let $\underline{V} := -AV_0$, with

$$A \ge \|g_1\|_\infty + \|f\|_\infty \tag{5.2}$$

and V_0 defined in (4.42). Let $\underline{w}$ be the unique solution, provided by Theorem 2.8, of the problem

$$\begin{cases} \partial_t u = -a(-\Delta)^s u + cu + f & \text{in} \quad I\!R^N \times (0,\infty) \\ u = \underline{V} & \text{in} \quad I\!R^N \times \{0\} \end{cases} \tag{5.3}$$

such that

$$\lim_{|x|\to\infty} \underline{w}(x,t) = g_1(t) \qquad \text{uniformly for} \quad t \in [0,\infty). \tag{5.4}$$

Then $t \mapsto \underline{w}(x,t)$ is nondecreasing, i.e.,

$$\underline{w}(x,t_1) \le \underline{w}(x,t_2) \qquad \text{for all} \quad x \in I\!R^N, 0 \le t_1 < t_2. \tag{5.5}$$

Proof of Proposition 5.1. It is easily seen that $\underline{V}$ is a subsolution of problem (5.3). In fact, since $c \le 0$ and $\underline{V} < 0$, due to (5.2) we have (in the viscosity sense)

$$-a(-\Delta)^s \underline{V} + c\underline{V} + f \ge A - \|f\|_\infty \ge 0 = \partial_t \underline{V} \qquad \text{in} \quad I\!R^N \times (0,\infty).$$

Moreover,

$$\underline{V} - \underline{w} = 0 \quad \text{in } I\!R^N \times \{0\},$$

and by (5.2) and (5.4),

$$\limsup_{|x|\to\infty} [\underline{V}(x) - \underline{w}(x,t)] \le 0 \qquad \text{uniformly for} \quad t \in [0,\infty).$$

Since $\underline{w}$ is a solution of problem (5.3), by Proposition 4.1,

$$\underline{V}(x) = \underline{w}(x,0) \le \underline{w}(x,t) \qquad \text{for all} \quad x \in I\!R^N, t > 0. \tag{5.6}$$

In order to show (5.5), take any $t_0 > 0$ and define

$$\tilde{w}(x,t) := \underline{w}(x, t+t_0) \quad \text{for all } \; x \in I\!R^N, t > 0.$$

Note that both $\underline{w}$ and $\tilde{w}$ satisfy the equation

$$\partial_t v - a(-\Delta)^s v - cv = f \quad \text{in } \; I\!R^N \times (0,\infty).$$

Moreover, from (5.6) we obtain that

$$\tilde{w}(x,0) \ge \underline{w}(x,0) \qquad \text{for all} \quad x \in I\!R^N. \tag{5.7}$$

In addition, due to (5.1),

$$\lim_{|x|\to\infty} [\tilde{w}(x,t) - \underline{w}(x,t)] = \tilde{g}_1(t+t_0) - g_1(t) \ge 0 \quad \text{uniformly for } t \in [0,\infty).$$

Thus, by Proposition 4.1 applied for each fixed $T > 0$,

$$\tilde{w}(x,t) \geq \underline{w}(x,t) \quad \text{for all } \; x \in I\!R^N, t > 0.$$

Hence the conclusion follows. □

Similarly, we can show the next result.

Proposition 5.2. *Let assumptions of Theorem 2.8 be satisfied with $g \equiv g_2$. Assume that*

$$g_2(t_1) \;\geq\; g_2(t_2) \qquad \text{for any} \qquad 0 \leq t_1 < t_2. \tag{5.8}$$

Let $\overline{V} := AV_0$, where V_0 is defined in (4.42) and A in (5.2).

Let $\overline{w}$ be the unique solution, provided by Theorem 2.8, of the problem

$$\begin{cases} \partial_t u = -a(-\Delta)^s u \,+\, cu \,+ f \;\text{in}\; I\!R^N \times (0,\infty) \\ u \,= \overline{V} \qquad\qquad\qquad\qquad\quad \text{in} \qquad I\!R^N \times \{0\} \end{cases} \tag{5.9}$$

such that

$$\lim_{|x|\to\infty} \overline{w}(x,t) \;= g_2(t) \qquad \text{uniformly for} \qquad t \in [0,\infty). \tag{5.10}$$

Then $t \mapsto \overline{w}(x,t)$ is nonincreasing, i.e.,

$$\overline{w}(x,t_1) \geq \overline{w}(x,t_2) \qquad \text{for all} \qquad x \in I\!R^N, 0 \leq t_1 < t_2. \tag{5.11}$$

Now we prove the next result.

Proposition 5.3. *Let assumptions of Theorem 2.8 be satisfied. Let $g_1 \in C([0,\infty)) \cap L^\infty((0,\infty))$ with*

$$g_1(t) \leq g(t) \quad \text{for all } t \in [0,\infty), \tag{5.12}$$

$$\lim_{t\to\infty} g_1(t) = \lim_{t\to\infty} g(t); \tag{5.13}$$

suppose that (5.1) is satisfied. Let u be the unique solution to problem (1.9) such that condition (1.10) is satisfied, given by Theorem 2.8. Let $\underline{w}$ be given by Proposition 5.1, also supposing that

$$A \geq \|u_0\|_\infty. \tag{5.14}$$

Then

$$\underline{w}(x,t) \leq u(x,t) \quad \text{for all } x \in I\!R^N, t > 0. \tag{5.15}$$

Proof. Let $z := \underline{w} - u$. Note that z solves equation

$$\partial_t z = -a(-\Delta)^s z + cz \quad \text{in } \; I\!R^N \times (0,\infty).$$

In view of (5.14) we have

$$z(x,0) = \underline{V}(x) - u_0(x) \leq 0 \quad \text{for all } \; x \in I\!R^N.$$

Moreover, from (5.12) we obtain

$$\lim_{|x|\to\infty} z(x,t) = g_1(t) - g(t) \leq 0 \quad \text{uniformly for } \; t \in [0,\infty).$$

Hence, by Proposition 4.1 applied for each fixed $T > 0$,

$$z \leq 0 \qquad \text{for all} \qquad x \in I\!R^N, t > 0.$$

This completes the proof. □

Analogously to Proposition 5.3, the next result can be shown.

Proposition 5.4. *Let assumptions of Theorem 2.8 be satisfied. Let* $g_2 \in C([0,\infty)) \cap L^\infty((0,\infty))$ *with*

$$g_2(t) \geq g(t) \quad \text{for all } t \in [0,\infty), \tag{5.16}$$

$$\lim_{t\to\infty} g_2(t) = \lim_{t\to\infty} g(t); \tag{5.17}$$

suppose that (5.8) is satisfied. Let u *be the unique solution to problem (1.9) such that condition (1.10) is satisfied, given by Theorem 2.8. Let* $\overline{w}$ *be given by Proposition 5.2, also supposing that (5.14) holds. Then*

$$\overline{w}(x,t) \geq u(x,t) \quad \text{for all } x \in I\!R^N, t > 0. \tag{5.18}$$

Now we are in position to prove Theorem 2.11.

Proof of Theorem 2.11. Keep the same notation as in Propositions 5.1–5.4. In view of (5.5) and (5.11), we can define

$$\underline{W}(x) := \lim_{t\to\infty} \underline{w}(x,t), \quad \overline{W}(x) := \lim_{t\to\infty} \overline{w}(x,t) \quad \text{for any } x \in I\!R^N. \tag{5.19}$$

Observe that the constant C in Remark 2.6 do not depend on T, since a, c, f does not depend on t. Consequently we have that $\underline{w} \to \underline{W}$, $\overline{w} \to \overline{W}$ as $t \to \infty$ uniformly in each compact subset of $I\!R^N$; thus, $\underline{W}, \overline{W} \in C(I\!R^N)$. We claim that both $\underline{W}$ and $\overline{W}$ solve

$$a(-\Delta)^s u - cu = f \quad \text{in } I\!R^N. \tag{5.20}$$

In fact, we limit ourselves to show that $\underline{W}$ is a subsolution of equation (5.20), since the remaining part of the claim follows analogously.

Now, let $\{t_n\} \subset (0,\infty)$ be a sequence with $t_n \to \infty$ as $n \to \infty$. Set

$$\underline{w}_n(x) := \underline{w}(x,t_n) \quad (x \in I\!R^N).$$

Thus, $\underline{w}_n \to \underline{W}$ locally uniformly in $I\!R^N$ as $n \to \infty$.

Take any bounded subset $U \subset I\!R^N, x_0 \in U$, and take any test function $\varphi \in C^2(I\!R^N)$ such that

$$\underline{W}(x_0) - \varphi(x_0) \geq \underline{W}(x) - \varphi(x) \quad \text{for all } x \in U.$$

Choose $\xi \in C^2(I\!R^N)$ with

$$0 \leq \xi < 1 \quad \text{if } x \in I\!R^N \setminus \{x_0\},\ \xi(x_0) = 1. \tag{5.21}$$

Fix any $\epsilon > 0$. So,

$$\underline{W}(x_0) - [\varphi(x_0) - \epsilon\xi(x_0)] > \underline{W}(x) - [\varphi(x) - \epsilon\xi(x)] \qquad \text{for all} \qquad x \in U \setminus \{x_0\}.$$

It is easily seen that there exists $\bar{n} = \bar{n}(\epsilon) \in I\!N$ such that for any $n > \bar{n}$, for some $x_n^\epsilon \in U$,

$$\underline{w}_n(x_n^\epsilon) - [\varphi(x_n^\epsilon) - \epsilon\xi(x_n^\epsilon)] \geq \underline{w}_n(x) - [\varphi(x) - \epsilon\xi(x)] \qquad \text{for all} \qquad x \in U;$$

moreover, for each $\epsilon > 0$, $x_n^\epsilon \to x_0$ as $n \to \infty$.

Since $\underline{w}$ is a solution of (5.3), due to Definition 2.1, we have that

$$0 = \partial_t \chi(x_n^\epsilon) \leq -a(x_n^\epsilon)(-\Delta)^s \chi(x_n^\epsilon) + c(x_n^\epsilon)\underline{w}_n(x_n^\epsilon) + f(x_n^\epsilon), \tag{5.22}$$

with

$$\chi \equiv \chi_{\epsilon,n} := \begin{cases} \varphi - \epsilon\xi & \text{in} \quad U \\ \underline{w}_n & \text{in} \quad I\!R^N \setminus U. \end{cases}$$

Note that

$$(-\Delta)^s\chi(x_n) = C_{N,s}\left\{\int_U \frac{\varphi(x_n^\epsilon)-\epsilon\xi(x_n^\epsilon)-[\varphi(y)-\epsilon\xi(y)]}{|x_n^\epsilon-y|^{N+2s}}\mathrm{d}y+\int_{I\!R^N\setminus U}\frac{\underline{w}_n(x_n^\epsilon)-\underline{w}(y)}{|x_n^\epsilon-y|^{N+2s}}\mathrm{d}y\right\}. \tag{5.23}$$

Since $\varphi,\xi\in C^2(U)$, for any $\epsilon>0$ we have

$$\lim_{n\to\infty}\int_U \frac{\varphi(x_n^\epsilon)-\epsilon\xi(x_n^\epsilon)-[\varphi(y)-\epsilon\xi(y)]}{|x_n^\epsilon-y|^{N+2s}}\mathrm{d}y = \int_U\frac{\varphi(x_0)-\varphi(y)}{|x_0-y|^{N+2s}}\mathrm{d}y+\epsilon\int_U\frac{\xi(y)-\xi(x_0)}{|x_0-y|^{N+2s}}\mathrm{d}y; \tag{5.24}$$

furthermore,

$$\lim_{n\to\infty}\int_{I\!R^N\setminus U}\frac{\underline{w}_n(x_n^\epsilon)-\underline{w}_n(y)}{|x_n^\epsilon-y|^{N+2s}}\mathrm{d}y = \int_{I\!R^N\setminus U}\frac{\underline{W}(x_0)-\underline{W}(y)}{|x_0-y|^{N+2s}}\mathrm{d}y. \tag{5.25}$$

From (5.23), (5.24), (5.25), letting $n\to\infty$ in (5.22), we have, for any $\epsilon>0$,

$$0\le -a(x_0)(-\Delta)^s\psi(x_0)-a(x_0)\epsilon C_{N,s}\int_U\frac{\xi(y)-\xi(x_0)}{|x_0-y|^{N+2s}}\mathrm{d}y+c(x_0)\underline{w}(x_0)+f(x_0),$$

with

$$\psi := \begin{cases}\varphi & \text{in } U\\ \underline{W} & \text{in } I\!R^N\setminus U.\end{cases}$$

Letting $\epsilon\to 0$, the claim follows.

Note that, in view of (5.4), (5.10), (5.13), (5.17), we can infer that

$$\lim_{|x|\to\infty}\underline{W}(x)=\overline{W}(x)=\gamma,$$

where $\gamma=\lim_{t\to\infty}g(t)$. By Proposition 4.2,

$$\underline{W}(x)=\overline{W}(x)\quad\text{for all } x\in I\!R^N. \tag{5.26}$$

By (5.15) and (5.18),

$$\underline{w}(x,t)\le u(x,t)\le\overline{w}(x,t)\quad\text{for all } x\in I\!R^N, t>0.$$

Letting $t\to\infty$, due to (5.19) and (5.26), we get the thesis, with $W:=\underline{W}\equiv\overline{W}$.

□

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