ESAIM: COCV 24 (2018) 89–104
DOI: 10.1051/cocv/2016075

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

ON STABILITY OF NONLINEAR NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS *

Pham Huu Anh Ngoc[1], Thai Bao Tran[2] and Cao Thanh Tinh[3]

Abstract. We address the challenging problem of the exponential stability of nonlinear time-varying functional differential equations of neutral type. By a novel approach, we present explicit sufficient conditions for the exponential stability of nonlinear time-varying neutral functional differential equations. A discussion of the obtained results and illustrative examples are given.

Mathematics Subject Classification. 34K20.

Received May 4, 2016. Revised November 7, 2016. Accepted December 6, 2016.

1. Introduction

Delay differential equations of neutral type have many applications including population models, lossless transmission lines, chemical reactors, partial element equivalent circuits, control systems, electrodynamics mixing liquids, neutron transportation, see [2,11,13,17]. In qualitative analysis of such systems, problems of stability of solutions play an important role. That is why problems of stability of delay differential equations of neutral type have attracted a great interest of researchers during the past decades, see *e.g.* [1,3,8,24] and the references therein.

The traditional approaches to analyze stability of delay differential equations of neutral type are Lyapunov's method and its variants, see *e.g.* [8,11,13,15,23,24]. However, it is difficult to construct a Lyapunov function for such equations. Most of the existing stability conditions for time-varying delay differential equations of neutral type in the literature are given in terms of matrix inequalities or differential inequalities. In particular, stability analysis of nonlinear time-varying functional differential equations of neutral type is very difficult and complicated. Therefore, there have not been many explicit stability conditions available for such equations. Using Lyapunov functions and the comparison principle, several sufficient conditions for asymptotic stability of some specific classes of nonlinear neutral differential equations have been reported in some papers published a long

Keywords and phrases. Functional differential equation, neutral equation, exponential stability.

* *This research is supported by Vietnam National University Ho Chi Minh City (VNU-HCM) under the Grant Number C2016-26-03.*

[1] Department of Mathematics, Vietnam National University-HCMC, International University, Sai Gon, Vietnam. phangoc@hcmiu.edu.vn

[2] Department of Information Systems, Vietnam national university-HCMC, University of Information Technology, Thu Duc district, Saigon, Vietnam. trantb@uit.edu.vn.

[3] Department of Mathematics, Vietnam national university-HCMC, University of Information Technology, Thu Duc district, Saigon, Vietnam. tinhct@uit.edu.vn.

time ago, see *e.g.* [6, 11, 13, 21, 22]. Recently, some abstract sufficient conditions for the exponential stability of semi-linear neutral functional differential equations were given in [9].

In this paper, we present a novel approach to the exponential stability of nonlinear time-varying functional differential equations of neutral type. Our approach is based on a system transformation and the comparison principle. More precisely, using a simple system transformation, it is to show that solutions of the nonlinear neutral functional differential equation satisfy a coupled differential-difference equation (see *e.g.* [16]). Then, a positive linear coupled differential-difference equation (see *e.g.* [19]) is given, which stands for an "upper bound" of the coupled difference-differential system. Next, explicit stability conditions for the positive linear coupled differential-difference equation are provided and it is to prove that these conditions ensure stability of the coupled differential-difference equation and that of the nonlinear neutral functional differential equation (using the comparison principle).

Consequently, new explicit sufficient conditions for the exponential stability of nonlinear time-varying functional differential equations of neutral type are derived. Both delay-dependent and delay-independent stability conditions are presented. To the best of our knowledge, Theorem 3.3, Corollary 3.5, Theorem 4.1 of this paper are original. The stability conditions obtained are quite simple, easy to use. Furthermore, they have some potential applications, for example, they can be applied to study behavior of solutions of neutral delay logistic equations [23] and exponential stability of equilibria of various classes of neural networks of neutral type [5].

The organization of this paper is as follows. In the next section, we give some notations and preliminary results which will be used in what follows. The main results are presented in Section 3. Some new explicit sufficient conditions for the exponential stability of time-varying functional differential equations of neutral type are given. A brief discussion of the obtained results is given.

2. Preliminaries

Let $\mathbb{N}$ be the set of all natural numbers. For given $m \in \mathbb{N}$, let $\underline{m} := \{1, 2, \ldots, m\}$. For given integers $l, q \geq 1, \mathbb{R}^l$ denotes the l-dimensional vector space over $\mathbb{R}$ and $\mathbb{R}^{l\times q}$ stands for the set of all $l \times q$-matrices with entries in $\mathbb{R}$. For $A = (a_{ij}) \in \mathbb{R}^{l\times q}$ and $B = (b_{ij}) \in \mathbb{R}^{l\times q}$, $A \geq B$ means that $a_{ij} \geq b_{ij}$ for $i = 1, \ldots, l$, $j = 1, \ldots, q$. In particular, if $a_{ij} > b_{ij}$ for $i = 1, \ldots, l$, $j = 1, \ldots, q$, then we write $A \gg B$ instead of $A \geq B$. Denote by $\mathbb{R}^{l\times q}_+$ the set of all nonnegative matrices. Similar notations are adopted for vectors.

For $x \in \mathbb{R}^n$ and $P \in \mathbb{R}^{l\times q}$ we define $|x| = (|x_i|)$ and $|P| = (|p_{ij}|)$. Then one has

$$|PQ| \leq |P||Q|, \quad \forall P \in \mathbb{R}^{l\times q}, \quad \forall Q \in \mathbb{R}^{q\times r}.$$

Let I_n be the identity matrix in $\mathbb{R}^{n\times n}$. For any matrix $M \in \mathbb{R}^{n\times n}$ the *spectral abscissa* (resp. the *spectral radius*) of M is defined by $s(M) := \max\{\Re\lambda \;\; \lambda \in \sigma(M)\}$ (resp. $\rho(M) := \max\{|\lambda| \;\; \lambda \in \sigma(M)\}$) where $\sigma(M) := \{z \in \mathbb{C} : \det(zI_n - M) = 0\}$ is the spectrum of M. A matrix $M \in \mathbb{R}^{n\times n}$ is said to be Hurwitz stable (resp. Schur stable) if, $s(M) < 0$ (resp. $\rho(M) < 1$). A norm $\|\cdot\|$ on $\mathbb{R}^n$ is said to be *monotonic* if $\|x\| \leq \|y\|$ whenever $x, y \in \mathbb{R}^n, |x| \leq |y|$. Every p-norm on $\mathbb{R}^n$

$$\|x\|_p = (|x_1|^p + |x_2|^p + \ldots + |x_n|^p)^{\frac{1}{p}}, \quad 1 \leq p < \infty; \qquad \|x\|_\infty = \max_{i=1,2,\ldots,n} |x_i|,$$

is monotonic. Throughout the paper, if otherwise not stated, the norm of vectors on $\mathbb{R}^n$ is monotonic and the norm of a matrix $P \in \mathbb{R}^{l\times q}$ is understood as its operator norm associated with a given pair of monotonic vector norms on $\mathbb{R}^l$ and $\mathbb{R}^q$, that is

$$\|P\| = \max\{\|Py\| : \quad \|y\| = 1\}.$$

A matrix $M \in \mathbb{R}^{n\times n}$ is called a *Metzler matrix* if all off-diagonal elements of M are nonnegative. For given $A := (a_{ij}) \in \mathbb{R}^{n\times n}$, we associate the Metzler matrix $M(A) := (\hat{a}_{ij})$ where $\hat{a}_{ij} = |a_{ij}|$ if $i \neq j$, for $i, j \in \underline{n}$ and $\hat{a}_{ii} = a_{ii}$, for $i \in \underline{n}$. The following results are used in what follows.

Theorem 2.1 [4]**.**

(i) *Let* $A \in \mathbb{R}_+^{n\times n}$. *Then* A *is Schur stable if and only if* $Aq \ll q$, *for some* $q \in \mathbb{R}_+^n, q \gg 0$.
(ii) *Let* $A \in \mathbb{R}^{n\times n}$ *be a Metzler matrix. Then* A *is Hurwitz stable if and only if* $Ap \ll 0$, *for some* $p \in \mathbb{R}_+^n, p \gg 0$.

Lemma 2.2 [18]**.** *Let* $A \in \mathbb{R}^{n\times n}$ *be a Metzler matrix and* $B, C, D \in \mathbb{R}_+^{n\times n}$. *Then the following statements are equivalent*

(i) $\rho(D) < 1$ *and* $s(A + B(I_n - D)^{-1}C) < 0$;
(ii) *there exist* $p, q \in \mathbb{R}_+^n, p \gg 0, q \gg 0$ *such that*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \ll \begin{pmatrix} 0 \\ q \end{pmatrix}; \tag{2.1}$$

(iii) $s(A) < 0$ *and* $\rho(C(-A)^{-1}B + D) < 1$.

To make a representation self contained and dynamic, we present here some basic facts on vector-valued functions of bounded variation and related topics.

A function $\eta(\cdot) : [\alpha, \beta] \to \mathbb{R}^{m\times n}$ is said to be increasing on $[\alpha, \beta]$ if

$$\eta(\theta_2) \geq \eta(\theta_1) \quad \text{for } \alpha \leq \theta_1 \leq \theta_2 \leq \beta.$$

A matrix-valued function $\eta(\cdot) : [\alpha, \beta] \to \mathbb{R}^{m\times n}$ is said to be of bounded variation if

$$\mathrm{Var}_{[\alpha,\beta]}\eta(\cdot) := \sup_{P[\alpha,\beta]} \sum_k \|\eta(\theta_k) - \eta(\theta_{k-1})\| < +\infty,$$

where the supremum is taken over the set of all finite partitions of the interval $[\alpha, \beta]$. The set $BV([\alpha, \beta], \mathbb{R}^{m\times n})$ of all matrix functions $\eta(\cdot)$ of bounded variation on $[\alpha, \beta]$ satisfying $\eta(\alpha) = 0$ is a Banach space endowed with the norm $\|\eta\| = Var_{[\alpha,\beta]}\eta(\cdot)$. Since all matrix norms on $\mathbb{R}^{m\times n}$ are equivalent, it follows that the matrix function $\eta(\cdot) = (\eta_{ij}(\cdot)) \in \mathbb{R}^{m\times n}$ is of bounded variation if and only if each $\eta_{ij}(\cdot)$ is of bounded variation.

Let

$$NBV([\alpha, \beta], \mathbb{R}^{m\times n}) := \{\eta \in BV([\alpha, \beta], \mathbb{R}^{m\times n}); \eta \text{ is continuous from left on } (\alpha, \beta)\}.$$

Clearly, $NBV([\alpha, \beta], \mathbb{R}^{m\times n})$ is closed in $BV([\alpha, \beta], \mathbb{R}^{m\times n})$ and thus it is a Banach space with the norm $\|\eta\| = \mathrm{Var}_{[\alpha,\beta]}\eta(\cdot)$.

Given $\eta(\cdot) \in NBV([\alpha, \beta], \mathbb{R}^{m\times n})$ then for any continuous functions $\gamma \in C([\alpha, \beta], \mathbb{R})$ and $\varphi \in C([\alpha, \beta], \mathbb{R}^n)$, the integrals

$$\int_\alpha^\beta \gamma(\theta)\mathrm{d}[\eta(\theta)] \quad \text{and} \quad \int_\alpha^\beta \mathrm{d}[\eta(\theta)]\varphi(\theta)$$

exist and are defined respectively as the limits of $S_1(P) := \sum_{k=1}^p \gamma(\zeta_k)(\eta(\theta_k) - \eta(\theta_{k-1}))$ and $S_2(P) := \sum_{k=1}^p (\eta(\theta_k) - \eta(\theta_{k-1}))\varphi(\zeta_k)$ as $d(P) := \max_k |\theta_k - \theta_{k-1}| \to 0$, where $P = \{\theta_1 = \alpha \leq \theta_2 \leq \ldots \leq \theta_p = \beta\}$ is any finite partition of the interval $[\alpha, \beta]$ and $\zeta_k \in [\theta_{k-1}, \theta_k]$. It is immediate from the definition that

$$\begin{aligned} \left\| \int_\alpha^\beta \gamma(\theta)\mathrm{d}[\eta(\theta)] \right\| &\leq \max\nolimits_{\theta\in[\alpha,\beta]} |\gamma(\theta)| \, \|\eta\|, \\ \left\| \int_\alpha^\beta \mathrm{d}[\eta(\theta)]\varphi(\theta) \right\| &\leq \max\nolimits_{\theta\in[\alpha,\beta]} \|\varphi(\theta)\| \, \|\eta\|. \end{aligned} \tag{2.2}$$

Let $\mathbb{R}^{m\times n}$ be endowed with the norm $\|\cdot\|$. Denote by $C(J, \mathbb{R}^{m\times n})$, the vector space of all continuous functions on J with values in $\mathbb{R}^{m\times n}$. In particular, $C([\alpha, \beta], \mathbb{R}^{m\times n})$ is a Banach space endowed with the norm $\|\varphi\| := \max_{\theta\in[\alpha,\beta]} \| \varphi(\theta) \|$. In what follows, the Banach space $C([-h, 0], \mathbb{R}^n)$ is used frequently. For simplicity, we write $\mathcal{C}$ instead of $C([-h, 0], \mathbb{R}^n)$. For a given $r > 0$, let $\mathcal{C}_r := \{\varphi \in \mathcal{C} : \|\varphi\| \leq r\}$ and let $B_r := \{x \in \mathbb{R}^n : \|x\| \leq r\}$.

3. Exponential stability of neutral functional differential equations

Consider the nonlinear functional differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}(t;x_t) = f\big(t;x_t\big), \quad t \geq \sigma, \tag{3.1}$$

where, for each $t \in \mathbb{R}, x_t(\cdot) \in \mathcal{C}$ is defined by $x_t(\theta) := x(t+\theta), \theta \in [-h,0]$ with given $h > 0$ and $\mathcal{D}(\cdot;\cdot), f(\cdot;\cdot) : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ are given continuous functions.

For given $\varphi \in \mathcal{C}$, consider for (3.1) the initial condition

$$x_\sigma(\theta) = \varphi(\theta), \quad \theta \in [-h,0]. \tag{3.2}$$

Throughout, it is assumed that $\mathcal{D}(\cdot;\cdot) : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ is defined by

$$\mathcal{D}(t;\varphi) := \varphi(0) - \sum_{i=1}^{m} D_i(t)\varphi(-h_i) - \int_{-h}^{0} E(t,s)\varphi(s)\mathrm{d}s, \quad (t,\varphi) \in \mathbb{R} \times \mathcal{C}, \tag{3.3}$$

where $0 \leq h_i \leq h, i \in \underline{m}$ and $D_i(\cdot) : \mathbb{R} \to \mathbb{R}^{n\times n}, i \in \underline{m},\ E(\cdot,\cdot) : \mathbb{R} \times [-h,0] \to \mathbb{R}^{n\times n}$, are given continuous functions. Furthermore, suppose there exist $D_i \in \mathbb{R}_+^{n\times n}, i \in \underline{m}$ and a continuous function $E_0(\cdot) : [-h,0] \to \mathbb{R}_+^{n\times n}$ such that

$$|D_i(t)| \leq D_i, \forall t \in \mathbb{R}, \forall i \in \underline{m}; \quad |E(t,s)| \leq E_0(s),\ \forall t \in \mathbb{R}, \forall s \in [-h,0]. \tag{3.4}$$

Note that (3.3), (3.4) ensure that $\mathcal{D}$ is atomic at zero on $\mathbb{R}\times\mathcal{C}$ (see [11] for the definition and detailed information). Then (3.1) is a functional differential equation of neutral type (see [11], Def. 7.1, p. 59).

Definition 3.1. Let $\sigma \in \mathbb{R}$ and $\varphi \in \mathcal{C}$ be given. A continuous function $x(\cdot) : [-h+\sigma, \gamma) \to \mathbb{R}^n$, is said to be a solution of (3.1) through (σ,φ) if $\mathcal{D}(t;x_t)$ is continuously differentiable on $[\sigma,\gamma)$ and $x(\cdot)$ satisfies (3.1) on $[\sigma,\gamma)$ and fulfils the initial condition (3.2).

Under the above hypotheses on $\mathcal{D}$ and f, there is a solution of (3.1) through (σ,φ). In addition, if f is Lipschitz continuous in the second argument on compact subsets of $\mathbb{R}\times\mathcal{C}$, then there exists a unique solution of (3.1) through (σ,φ), ([11], Thm. 8.3, p. 65).

In what follows, assume that

$$f(t;\varphi) := g(t;\varphi(0),\varphi), \quad (t,\varphi) \in \mathbb{R} \times \mathcal{C}, \tag{3.5}$$

where $g : \mathbb{R} \times \mathbb{R}^n \times \mathcal{C} \to \mathbb{R}^n$, is continuous in all its arguments and is Lipschitz continuous in the last two arguments on compact subsets of $\mathbb{R} \times \mathbb{R}^n \times \mathcal{C}$. Then (3.1) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}(t;x_t) = g\big(t;x(t),x_t\big), \quad t \geq \sigma, \tag{3.6}$$

and there always exists a unique solution of (3.6) through (σ,φ). This solution is denoted by $x(\cdot;\sigma,\varphi)$. Furthermore, if $[\sigma-h,\gamma)$ is the maximum interval of existence of $x(\cdot;\sigma,\varphi)$ then $x(\cdot;\sigma,\varphi)$ is said to be *noncontinuable*. The existence of a noncontinuable solution follows from Zorn's lemma and the maximum interval of existence must be open. In what follows, $x(\cdot;\sigma,\varphi)$ denotes a noncontinuable solution.

Suppose $g(t;0,0) = 0, t \in \mathbb{R}$ and then $x = 0$ is a solution of (3.6).

Definition 3.2. The zero solution of (3.6) is said to be exponentially stable (shortly, ES) if there exist positive numbers r, K, β such that for each $\sigma \in \mathbb{R}$ and each $\varphi \in \mathcal{C}_r$, the solution $x(\cdot;\sigma,\varphi)$ of (3.6) through (σ,φ) exists on $[\sigma-h,+\infty)$ and furthermore satisfies

$$\|x(t;\sigma,\varphi)\| \leq K\mathrm{e}^{-\beta(t-\sigma)}, \quad \forall t \geq \sigma.$$

To analyse the exponential stability of the nonlinear neutral functional differential equation (3.6), it is assumed that:

(H_1) $g(t;\cdot,\varphi)$ is continuously differentiable on the ball B_δ for any $t\in\mathbb{R}$ and any $\varphi\in\mathcal{C}_\delta$, for some $\delta>0$ and there exist continuous functions $a_{ij}(\cdot):\mathbb{R}\to\mathbb{R}, i,j\in\underline{n}$ and $b_i(\cdot):\mathbb{R}\to\mathbb{R}, i\in\underline{n}$ such that

$$b_i(t)\le\frac{\partial g_i}{\partial x_i}(t;x,\varphi)\le a_{ii}(t)<0;\quad \left|\frac{\partial g_i}{\partial x_j}(t;x,\varphi)\right|\le a_{ij}(t),\quad i\ne j,\ i,j\in\underline{n},\tag{3.7}$$

for any $t\in\mathbb{R}$, any $x\in B_\delta$ and any $\varphi\in\mathcal{C}_\delta$ and (H_2) there is a continuous functional

$$\mathcal{L}(t;\varphi):\mathbb{R}\times\mathcal{C}\to\mathbb{R}^n;\quad (t,\varphi)\mapsto\mathcal{L}(t,\varphi):=\int_{-h}^0\mathrm{d}[\eta(t,\theta)]\varphi(\theta),\tag{3.8}$$

where $\eta(t;\cdot)\in NBV([-h,0],\mathbb{R}^{n\times n})$ for each $t\in\mathbb{R}$ such that

$$|g(t;0,\varphi)|\le|\mathcal{L}(t;\varphi)|,\quad \forall t\in\mathbb{R},\ \forall\varphi\in\mathcal{C}_\delta.\tag{3.9}$$

Let us define

$$A_0(t):=(a_{ij}(t))\in\mathbb{R}^{n\times n},\ t\in\mathbb{R};\qquad A_1(t):=(a_{ij}^{(1)}(t))\in\mathbb{R}_+^{n\times n},\ t\in\mathbb{R},\tag{3.10}$$

with

$$a_{ij}^{(1)}(t):=a_{ij}(t),\quad t\in\mathbb{R},\quad i\ne j,i,j\in\underline{n},$$

and

$$a_{ii}^{(1)}(t):=|b_i(t)|,\quad t\in\mathbb{R},i\in\underline{n}.$$

Set

$$V(t):=\big(\mathrm{Var}_{[-h,0]}\eta_{ij}(t;\cdot)\big)\in\mathbb{R}^{n\times n},\quad t\in\mathbb{R},\tag{3.11}$$

and

$$V_0(t):=\sum_{i=1}^m A_1(t)|D_i(t)|+\int_{-h}^0 A_1(t)|E(t,s)|\mathrm{d}s+V(t),\quad t\in\mathbb{R}.\tag{3.12}$$

We are now in the position to state the main result of this paper.

Theorem 3.3. *Suppose (H_1)–(H_2) hold. If there exist $\beta>0$ and $p,q\in\mathbb{R}^n_+, p,q\gg0$ such that*

$$\big(A_0(t)\big)p+V_0(t)\mathrm{e}^{\beta h}q\ll-\beta p,\quad \forall t\in\mathbb{R},\tag{3.13}$$

$$p+\left(\sum_{i=1}^m|D_i(t)|+\int_{-h}^0|E(t,s)|\mathrm{d}s\right)\mathrm{e}^{\beta h}q\ll q,\quad \forall t\in\mathbb{R},\tag{3.14}$$

then the zero solution of (3.6) is ES.

Remark 3.4. (i) Roughly speaking, (H_1) means that the "linearized part" of (3.6) is bounded above by $A_0(t), t\in\mathbb{R}$ and (H_2) says that the "nonlinear part" of (3.6) is bounded above by $\mathcal{L}(t;\varphi)$. Thus, (3.6) is "bounded above" (in some sense) by the linear neutral differential system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}(t;x_t)=A_0(t)x(t)+\int_{-h}^0\mathrm{d}[\eta(t,\theta)]x(t+\theta),\tag{3.15}$$

in a neighbourhood of 0. These are characterizations of a system which satisfies Theorem 3.3. Then (3.13), (3.14) ensures that (3.15) is exponentially stable. This implies that (3.6) is ES as well.

Theorem 3.3 now can be interpreted as follows:

"Suppose (3.6) is "bounded above" by the linear system (3.15) and (3.15) is exponentially stable. Then the zero solution of (3.6) is exponentially stable".

This is a nice surprise because it is similar to the well-known Weierstrass M-test in the theory of infinite series of functions, see *e.g.* [20].

(ii) The proof of Theorem 3.3 given below also shows that the solutions of (3.6) exponentially decay with the rate β. That is,

$$\|x(t;\sigma,\varphi)\| \le K\mathrm{e}^{-\beta(t-\sigma)}, \quad \forall t \ge \sigma, \forall \varphi \in \mathcal{C}_\delta,$$

for some $K \ge 1$. Here $\beta > 0$ satisfies (3.13)–(3.14).

Corollary 3.5. *Suppose that*

$$b_i \le \frac{\partial g_i}{\partial x_i}(t;x,\varphi) \le a_{ii} < 0; \quad \left|\frac{\partial g_i}{\partial x_j}(t;x,\varphi)\right| \le a_{ij}, \quad i \ne j,\ i,j \in \underline{n}, \tag{3.16}$$

for any $t \in \mathbb{R}$, any $x \in B_\delta$ and any $\varphi \in \mathcal{C}_\delta$ (for some $\delta > 0$) and

$$|g(t;0,\varphi)| \le |\mathcal{L}_0(\varphi)|, \quad \forall t \in \mathbb{R},\ \forall \varphi \in \mathcal{C}_\delta. \tag{3.17}$$

where $\mathcal{L}_0(\varphi) : \mathcal{C} \to \mathbb{R}^n;\ \varphi \mapsto \mathcal{L}_0\varphi := \int_{-h}^0 \mathrm{d}[\eta_0(\theta)]\varphi(\theta)$ with $\eta_0(\cdot) \in NBV([-h,0],\mathbb{R}^{n\times n})$. Let

$$A_0 := (a_{ij}) \in \mathbb{R}^{n\times n}; \qquad A_1 := (a_{ij}^{(1)}) \in \mathbb{R}_+^{n\times n}, \tag{3.18}$$

with $a_{ij}^{(1)} := a_{ij}, i \ne j, i,j \in \underline{n}$ and $a_{ii}^{(1)} := |b_i|, i \in \underline{n}$ and let

$$V_0 := \sum_{i=1}^m A_1 D_i + \int_{-h}^0 A_1 E_0(s)\mathrm{d}s + (\mathrm{Var}_{[-h,0]}\eta_{0ij}), \tag{3.19}$$

where $D_i, i \in \underline{n}$ and $E_0(\cdot)$ satisfy (3.4) and $\eta_0(\cdot) := (\eta_{0_{ij}}(\cdot))$. Then the zero solution of (3.6) is ES if one of the following conditions holds:

(i) *there exist $\beta > 0$ and $p,q \in \mathbb{R}_+^n, p \gg 0, q \gg 0$ such that*

$$A_0 p + V_0 \mathrm{e}^{\beta h} q \ll -\beta p; \qquad p + (\sum_{i=1}^m D_i + \int_{-h}^0 E_0(s)\mathrm{d}s)\mathrm{e}^{\beta h} q \ll q; \tag{3.20}$$

(ii) $\rho(\sum_{i=1}^m D_i + \int_{-h}^0 E_0(s)\mathrm{d}s) < 1$ *and* $s(A_0 + V_0(I_n - \sum_{i=1}^m D_i - \int_{-h}^0 E_0(s)\mathrm{d}s)^{-1}) < 0$*;*

(iii) $s(A_0) < 0$ *and* $\rho((-A_0)^{-1}V_0 + \sum_{i=1}^m D_i + \int_{-h}^0 E_0(s)\mathrm{d}s) < 1$;

(iv) *there exist $p,q \in \mathbb{R}_+^n, p \gg 0, q \gg 0$ such that*

$$\begin{pmatrix} A_0 & V_0 \\ I_n & \sum_{i=1}^m D_i + \int_{-h}^0 E_0(s)\mathrm{d}s \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \ll \begin{pmatrix} 0 \\ q \end{pmatrix}. \tag{3.21}$$

Proof. Note that (ii), (iii) and (iv) are equivalent, by Lemma 2.2. By continuity, (iv) implies that (i) holds for sufficiently small $\beta > 0$.

On the other hand, (3.6) is ES provided (i) holds, by Theorem 3.3. Thus, the conclusion of Corollary 3.5 follows from Theorem 3.3. This completes the proof. □

Remark 3.6. Note that if (3.20) holds for $h > 0$ then by continuity, it still holds for any $h_* \in [\underline{h}, \bar{h}]$ for some $\underline{h}, \bar{h}$ with $0 < \underline{h} < h < \bar{h}$. Thus, the zero solution of (3.6) is ES for any $h \in [\underline{h}, \bar{h}]$. This gives a delay-dependent stability condition of (3.6).

Consider the linear neutral differential system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}(t;x_t) = A(t)x(t) + \sum_{k=1}^{r} B_k(t)x(t-\tau_k) + \int_{-h}^{0} C(t,s)x(t+s)\mathrm{d}s, \quad t \geq \sigma, \tag{3.22}$$

where $\mathcal{D}(\cdot;\cdot)$ satisfies (3.3) and (3.4) and $A(\cdot):\mathbb{R}\to\mathbb{R}^{n\times n}, B_k(\cdot):\mathbb{R}\to\mathbb{R}^{n\times n}, k\in\underline{r}$ and $C(\cdot,\cdot):\mathbb{R}\times[-h,0]\to\mathbb{R}^{n\times n}$, are given continuous functions and $0\leq\tau_k\leq h, k\in\underline{r}$.

Let

$$V_1(t) := \sum_{i=1}^{m}|A(t)||D_i(t)| + \int_{-h}^{0}|A(t)||E(t,s)|\mathrm{d}s + \sum_{k=1}^{r}|B_k(t)| + \int_{-h}^{0}|C(t,s)|\mathrm{d}s, \quad t\in\mathbb{R}. \tag{3.23}$$

The following is immediate from Theorem 3.3.

Corollary 3.7. *Suppose there exist $\beta>0$ and $p,q\in\mathbb{R}^n_+, p,q\gg 0$ such that*

$$\big(M(A(t))\big)p + V_1(t)\mathrm{e}^{\beta h}q \ll -\beta p, \quad \forall t\in\mathbb{R}, \tag{3.24}$$

and (3.14) holds. Then (3.22) is exponentially stable.

Proof of Theorem 3.3. Let $x(t) := x(t;\sigma,\varphi), t\in[\sigma-h,\gamma)$ be the unique solution of (3.1) through (σ,φ) and let $y(t) := x(t) - \sum_{i=1}^{m} D_i(t)x(t-h_i) - \int_{-h}^{0} E(t,s)x(t+s)\mathrm{d}s, t\in[\sigma,\gamma)$. Then $x(\cdot)$ and $y(\cdot)$ satisfy the following system

$$\frac{\mathrm{d}y}{\mathrm{d}t} = g(t,x(t),x_t), \quad t\in[\sigma,\gamma), \tag{3.25}$$

and

$$x(t) = y(t) + \sum_{i=1}^{m} D_i(t)x(t-h_i) + \int_{-h}^{0} E(t,s)x(t+s)\mathrm{d}s, \quad t\in[\sigma,\gamma). \tag{3.26}$$

We divide the proof into three steps.

Step I. We show that there exists $r>0$ such that for any $\sigma\in\mathbb{R}$ and any $\varphi\in\mathcal{C}_r$, the solution $x(t) := x(t;\sigma,\varphi), t\in[\sigma-h,\gamma)$ satisfies

$$\|x(t)\| \leq \frac{\delta}{2}, \;\; t\in[\sigma,\gamma), \tag{3.27}$$

where $\delta>0$ is the positive number so that (3.7) and (3.9) hold.

Without loss of generality, assume that $\mathbb{R}^n$ is endowed with the maximum norm $\|\cdot\|_\infty$. Note that (3.13) and (3.14) also hold for any vectors $kp, kq\in\mathbb{R}^n, k>0$. Therefore, we can assume further that

$$\max\{\|p\|,\|q\|\} \leq \frac{\delta}{2}. \tag{3.28}$$

Let $p := (p_1,p_2,\ldots,p_n)^T; q := (q_1,q_2,\ldots,q_n)^T, p_i,q_i > 0, \forall i\in\underline{n}$. Choose $r>0$ so that $0<r<\min\{\min_{i\in\underline{n}} p_i, \min_{i\in\underline{n}} q_i\}$ and

$$|\varphi(0)| + \sum_{i=1}^{m} D_i|\varphi(-h_i)| + \int_{-h}^{0} E_0(s)|\varphi(s)|\mathrm{d}s \ll p, \quad \forall\varphi\in\mathcal{C}_r, \tag{3.29}$$

where $D_i\in\mathbb{R}^{n\times n}, i\in\underline{m}$ and $E_0(\cdot)$ satisfy (3.4).

Note that $|\varphi(t)|\ll q$, for any $t\in[-h,0]$ and for any $\varphi\in\mathcal{C}_r$. From $x(\sigma+s)=\varphi(s), s\in[-h,0]$, it follows that $|x(\sigma)| = |\varphi(0)| \ll q$. Furthermore, (3.4) and (3.29) imply

$$\begin{aligned}|y(\sigma)| &\leq |x(\sigma)| + \sum_{i=1}^{m}|D_i(\sigma)||x(\sigma-h_i)| + \int_{-h}^{0}|E(\sigma,s)||x(\sigma+s)|\mathrm{d}s \\ &\leq |\varphi(0)| + \sum_{i=1}^{m} D_i|\varphi(-h_i)| + \int_{-h}^{0} E_0(s)|\varphi(s)|\mathrm{d}s \ll p.\end{aligned}$$

We claim that

$$|x(t)| \le q, \quad t \in [\sigma, \gamma); \quad |y(t)| \le p, \quad t \in [\sigma, \gamma).$$

Assume on the contrary that there exists $t_0 \in (\sigma, \gamma)$ such that either $|x(t_0)| \not\le q$ or $|y(t_0)| \not\le p$. Set $t_1 := \inf\{t \in [\sigma, \gamma) : (|x(t)|, |y(t)|) \not\le (q, p)\}$. By continuity, $t_1 > \sigma$ and one of the following statements holds:

(C1) $|x(t)| \le q, t \in [\sigma, t_1]$ and there is $i_0 \in \underline{n}$ such that

$$|y(t)| \le p, \forall t \in [\sigma, t_1); |y_{i_0}(t_1)| = p_{i_0}, |y_{i_0}(\tau_k)| > p_{i_0}, \tag{3.30}$$

for some $\tau_k \in (t_1, t_1 + \frac{1}{k}), k \in \mathbb{N}$.

(C2) $|y(t)| \le p, t \in [\sigma, t_1]$ and there is $k_0 \in \underline{n}$ such that

$$|x(t)| \le q, \forall t \in [\sigma, t_1); |x_{k_0}(t_1)| = q_{k_0}, |x_{k_0}(\xi_k)| > q_{k_0}, \tag{3.31}$$

for some $\xi_k \in (t_1, t_1 + \frac{1}{k}), k \in \mathbb{N}$.

Assume that (C1) holds. By the monotonicity of vector norms,

$$\|x(t)\| = \||x(t)|\| \le \|q\| \le \frac{\delta}{2}, \quad t \in [\sigma, t_1].$$

By continuity, $\|x(t)\| < \delta, t \in [\sigma, t_1 + \epsilon_0)$, for some $\epsilon_0 > 0$. This implies $\|x_t\| < \delta, t \in [\sigma, t_1 + \epsilon_0)$. Set

$$z(t) := \sum_{i=1}^{m} D_i(t)x(t - h_i) + \int_{-h}^{0} E(t, s)x(t + s)\mathrm{d}s, t \in [\sigma, \gamma). \tag{3.32}$$

From (3.26), it follows that

$$x(t) = y(t) + z(t), t \in [\sigma, \gamma). \tag{3.33}$$

Since (3.25), (3.33) and the mean value Theorem [7], it follows that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|y_i(t)| &= \mathrm{sgn}(y_i(t))\frac{\mathrm{d}y_i}{\mathrm{d}t} = \mathrm{sgn}(y_i(t))g_i(t, x(t), x_t) \\
&= \mathrm{sgn}(y_i(t))\big[\big(g_i(t, x(t), x_t) - g_i(t, 0, x_t)\big) + g_i(t, 0, x_t)\big] \\
&= \mathrm{sgn}(y_i(t))\sum_{j=1}^{n}\left(\int_0^1 \frac{\partial g_i}{\partial x_j}\big(t, \xi x(t), x_t\big)\mathrm{d}\xi\right) x_j(t) + \mathrm{sgn}(y_i(t))g_i(t, 0, x_t) \\
&= \mathrm{sgn}(y_i(t))\sum_{j=1}^{n}\left(\int_0^1 \frac{\partial g_i}{\partial x_j}\big(t, \xi x(t), x_t\big)\mathrm{d}\xi\right) (y_j(t) + z_j(t)) + \mathrm{sgn}(y_i(t))g_i(t, 0, x_t) \\
&= \left(\int_0^1 \frac{\partial g_i}{\partial x_i}\big(t, \xi x(t), x_t\big)\mathrm{d}\xi\right)|y_i(t)| + \mathrm{sgn}(y_i(t))\sum_{j=1, j\ne i}^{n}\left(\int_0^1 \frac{\partial g_i}{\partial x_j}\big(t, \xi x(t), x_t\big)\mathrm{d}\xi\right) y_j(t) \\
&\quad + \mathrm{sgn}(y_i(t))\sum_{j=1}^{n}\left(\int_0^1 \frac{\partial g_i}{\partial x_j}\big(t, \xi x(t), x_t\big)\mathrm{d}\xi\right) z_j(t) + \mathrm{sgn}(y_i(t))g_i(t, 0, x_t),
\end{aligned}$$

for almost any $t \in [\sigma, t_1 + \epsilon_0)$ and any $i \in \underline{n}$. Invoking (3.7), we get the following estimates

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|y_i(t)| &\le \sum_{j=1}^{n} a_{ij}(t)|y_j(t)| + \sum_{j=1}^{n} \left|\left(\int_0^1 \frac{\partial g_i}{\partial x_j}(t, \xi x(t), x_t)\mathrm{d}\xi\right)\right| |z_j(t)| + |g_i(t,0,x_t)| \\ &\le \sum_{j=1}^{n} a_{ij}(t)|y_j(t)| + \sum_{j=1}^{n} a_{ij}^{(1)}(t)|z_j(t)| + |g_i(t,0,x_t)|,\end{aligned}$$

for almost any $t \in [\sigma, t_1 + \epsilon_0)$ and any $i \in \underline{n}$. It follows that for any $t \in [\sigma, t_1 + \epsilon_0)$ and for any $i \in \underline{n}$

$$\begin{aligned}D^+|y_i(t)| := \limsup_{h\to 0^+} \frac{|y_i(t+h)| - |y_i(t)|}{h} &= \limsup_{h\to 0^+} \frac{1}{h}\int_t^{t+h} \frac{\mathrm{d}}{\mathrm{d}\zeta}|y_i(\zeta)|\mathrm{d}\zeta \\ &\le \sum_{j=1}^{n} a_{ij}(t)|y_j(t)| + \sum_{j=1}^{n} a_{ij}^{(1)}(t)|z_j(t)| + |g_i(t,0,x_t)|, \qquad (3.34)\end{aligned}$$

where D^+ denotes the Dini upper-right derivative. Note that (3.32) yields

$$|z(t_1)| \le \sum_{i=1}^{m} |D_i(t_1)||x(t_1 - h_i)| + \int_{-h}^{0} |E(t_1,s)||x(t_1+s)|\mathrm{d}s \le \sum_{i=1}^{m} |D_i(t_1)|q + \int_{-h}^{0} |E(t_1,s)|q\mathrm{d}s,$$

and thus,

$$A_1(t_1)|z(t_1)| \le \left(\sum_{i=1}^{m} A_1(t_1)|D_i(t_1)| + \int_{-h}^{0} A_1(t_1)|E(t_1,s)|\mathrm{d}s\right) q.$$

Furthermore, (3.9) and (2.2) imply

$$|g(t_1, 0, x_{t_1})| \le \left|\int_{-h}^{0} \mathrm{d}[\eta(t_1,\theta)]x(t_1+\theta)\right| \le (\mathrm{Var}_{[-h,0]}\eta_{ij}(t_1,\cdot))q = V(t_1)q.$$

Therefore,

$$A_1(t_1)|z(t_1)| + |g(t_1,0,x_{t_1})| \le \left(\sum_{i=1}^{m} A_1(t_1)|D_i(t_1)| + \int_{-h}^{0} A_1(t_1)|E(t_1,s)|\mathrm{d}s + V(t_1)\right) q = V_0(t_1)q, \qquad (3.35)$$

where $V_0(t) := (v_{ij}^{(0)}(t))$ is defined by (3.12). Let i_0 be the index so that (3.30) holds. It follows from (3.34), (3.35) and (3.13) that

$$\begin{aligned}D^+|y_{i_0}(t_1)| &\le \sum_{j=1}^{n} a_{i_0 j}(t_1)|y_j(t_1)| + \sum_{j=1}^{n} a_{i_0 j}^{(1)}(t_1)|z_j(t_1)| + |g_{i_0}(t_1,0,x_{t_1})| \\ &\overset{(3.34)-(3.35)}{\le} \sum_{j=1}^{n} a_{i_0 j}(t_1)p_j + \sum_{j=1}^{n} v_{i_0 j}^{(0)}(t_1)q_j \overset{(3.13)}{<} 0.\end{aligned}$$

On the other hand, (3.30) implies that

$$D^+|y_{i_0}(t_1)| = \limsup_{t\to t_1^+} \frac{|y_{i_0}(t)| - |y_{i_0}(t_1)|}{t - t_1} \ge \overline{\lim}_{k\to\infty} \frac{|y_{i_0}(\tau_k)| - |y_{i_0}(t_1)|}{\tau_k - t_1} \ge 0.$$

This is a contradition.

Assume that (C2) holds. It follows from (3.14) and (3.26) that

$$|x(t_1)| \overset{(3.26)}{=} \Big|y(t_1) + \sum_{i=1}^{m} D_i(t)x(t_1-h_i) + \int_{-h}^{0} E(t_1,s)x(t_1+s)\mathrm{d}s\Big|$$

$$\leq |y(t_1)| + \sum_{i=1}^{m} |D_i(t_1)||x(t_1-h_i)| + \int_{-h}^{0} |E(t_1,s)||x(t_1+s)|\mathrm{d}s$$

$$\leq p + \left(\sum_{i=1}^{m} |D_i(t_1)| + \int_{-h}^{0} |E(t_1,s)|\mathrm{d}s\right) q \overset{(3.14)}{\ll} q.$$

This conflicts with the last inequality in (3.31). Thus,

$$|x(t)| \leq q, \quad t \in [\sigma,\gamma); \qquad |y(t)| \leq p, \quad t \in [\sigma,\gamma).$$

By the monotonicity of vector norms,

$$\|x(t)\| \leq \|q\| \leq \frac{\delta}{2}, \quad \forall t \in [\sigma,\gamma).$$

Step II. We show that

$$\|x(t;\sigma,\varphi)\| \leq K\mathrm{e}^{-\beta(t-\sigma)}, \forall t \in [\sigma,\gamma), \forall \varphi \in \mathcal{C}_r, \tag{3.36}$$

where $\beta > 0$ satisfies (3.13), (3.14) and r is determined in Step I and K depends on β, r.

Taking into account $p, q \gg 0$, we are able to choose a positive number K such that

$$|\varphi(t)| \ll Kq,\ t \in [-h,0],\ \varphi \in \mathcal{C}_r, \tag{3.37}$$

and

$$|\varphi(0)| + \sum_{i=1}^{m} D_i|\varphi(-h_i)| + \int_{-h}^{0} E_0(s)|\varphi(s)|\mathrm{d}s \ll Kp, \quad \forall \varphi \in \mathcal{C}_r. \tag{3.38}$$

By Step I, $\|x(t)\| \leq \frac{\delta}{2},\ t \in [\sigma,\gamma)$, where $x(t) := x(t;\sigma,\varphi),\ \varphi \in \mathcal{C}_r$. This implies $\|x_t\| \leq \frac{\delta}{2}$, for any $t \in [\sigma,\gamma)$. It follows from (3.7) and (3.9) that

$$b_i(t) \leq \frac{\partial g_i}{\partial x_i}(t;x(t),x_t) \leq a_{ii}(t) < 0, \quad t \in [\sigma,\gamma), \quad i \in \underline{n},$$

$$\left|\frac{\partial g_i}{\partial x_j}(t;x(t),x_t)\right| \leq a_{ij}(t), \quad t \in [\sigma,\gamma);\ i \neq j,\ i,j \in \underline{n},$$

and

$$|g(t;0,x_t)| \leq |\mathcal{L}(t;x_t)|, \qquad t \in [\sigma,\gamma).$$

Define $u(t) := K\mathrm{e}^{-\beta(t-\sigma)}q, t \in [\sigma-h,\infty)$ and $v(t) := K\mathrm{e}^{-\beta(t-\sigma)}p, t \in [\sigma,\infty)$. Then (3.37), (3.38) yields $|x(t)| \ll u(t), t \in [\sigma-h,\sigma]$ and $|y(\sigma)| \ll v(\sigma)$. We claim that

$$|x(t)| \leq u(t),\ t \in [\sigma,\gamma); \qquad |y(t)| \leq v(t),\ t \in [\sigma,\gamma).$$

The proof is similar to that of Step I. Assume on the contrary that there exists $t_\theta \in (\sigma, \gamma)$ such that either $|x(t_\theta)| \not\leq u(t_\theta)$ or $|y(t_\theta)| \not\leq v(t_\theta)$. Set $t_c := \inf\{t \in [\sigma, \gamma) : (|x(t)|, |y(t)|) \not\leq (u(t), v(t))\}$. By continuity, $t_1 > \sigma$ and one of the following statements holds:

(C3) $|x(t)| \leq u(t), t \in [\sigma, t_c]$ and there is $i_0 \in \underline{n}$ such that

$$|y(t)| \leq v(t), \forall t \in [\sigma, t_c); |y_{i_0}(t_c)| = v_{i_0}(t_c), |y_{i_0}(\tau_k)| > v_{i_0}(\tau_k), \tag{3.39}$$

for some $\tau_k \in (t_c, t_c + \frac{1}{k}), k \in \mathbb{N}$.

(C4) $|y(t)| \leq v(t), t \in [\sigma, t_c]$ and there is $k_0 \in \underline{n}$ such that

$$|x(t)| \leq u(t),\ \forall t \in [\sigma, t_c); |x_{k_0}(t_c)| = u_{k_0}(t_c), |x_{k_0}(\xi_k)| > u_{k_0}(\xi_k), \tag{3.40}$$

for some $\xi_k \in (t_c, t_c + \frac{1}{k}), k \in \mathbb{N}$.

If (C3) holds then

$$|z(t_c)| \leq \sum_{i=1}^{m} |D_i(t_c)||x(t_c - h_i)| + \int_{-h}^{0} |E(t_c, s)||x(t_c + s)|\mathrm{d}s$$

$$\leq \sum_{i=1}^{m} |D_i(t_c)|u(t_c - h_i) + \int_{-h}^{0} |E(t_c, s)|u(t_c + s)\mathrm{d}s$$

$$\leq K\mathrm{e}^{-\beta(t_c - \sigma)}\mathrm{e}^{\beta h}\left(\sum_{i=1}^{m} |D_i(t_c)| + \int_{-h}^{0} |E(t_c, s)|\mathrm{d}s\right) q$$

and thus,

$$A_1(t_c)|z(t_c)| \leq K\mathrm{e}^{-\beta(t_c - \sigma)}\mathrm{e}^{\beta h}\left(\sum_{i=1}^{m} A_1(t_c)|D_i(t_c)| + \int_{-h}^{0} A_1(t_c)|E(t_c, s)|\mathrm{d}s\right) q.$$

On the other hand, it follows from (3.9) and (2.2) that

$$|g(t_c, 0, x_{t_c})| \leq \Big| \int_{-h}^{0} \mathrm{d}[\eta(t_c, \theta)]x(t_c + \theta)\Big| \leq K\mathrm{e}^{-\beta(t_c - \sigma)}\mathrm{e}^{\beta h}(\mathrm{Var}_{[-h,0]}\eta_{ij}(t_c, \cdot))q$$

$$= K\mathrm{e}^{-\beta(t_c - \sigma)}\mathrm{e}^{\beta h}V(t_c)q.$$

Therefore,

$$A_1(t_c)|z(t_c)| + |g(t_c, 0, x_{t_c})| \leq K\mathrm{e}^{-\beta(t_c - \sigma)}\mathrm{e}^{\beta h}\left(\sum_{i=1}^{m} A_1(t_c)|D_i(t_c)| + \int_{-h}^{0} A_1(t_c)|E(t_c, s)|\mathrm{d}s + V(t_c)\right) q$$

$$= K\mathrm{e}^{-\beta(t_c - \sigma)}\mathrm{e}^{\beta h}V_0(t_c)q.$$

Thus,

$$\begin{aligned}D^+|y_{i_0}(t_c)| &\le \sum_{j=1}^n a_{i_0j}(t_c)|y_j(t_c)| + \sum_{j=1}^n a^{(1)}_{i_0j}(t_c)|z_j(t_c)| + |g_{i_0}(t_c,0,x_{t_c})|\\
&\le \sum_{j=1}^n a_{i_0j}(t_c)v_j(t_c) + \sum_{j=1}^n a^{(1)}_{i_0j}(t_c)|z_j(t_c)| + |g_{i_0}(t_c,0,x_{t_c})|\\
&\le \sum_{j=1}^n a_{i_0j}(t_c)K\mathrm{e}^{-\beta(t_c-\sigma)}p_j + K\mathrm{e}^{-\beta(t_c-\sigma)}\mathrm{e}^{\beta h}\sum_{j=1}^n v^{(0)}_{i_0j}(t_c)q_j\\
&= K\mathrm{e}^{-\beta(t_c-\sigma)}\left(\sum_{j=1}^n a_{i_0j}(t_c)p_j + \mathrm{e}^{\beta h}\sum_{j=1}^n v^{(0)}_{i_0j}(t_c)q_j\right)\\
&\overset{(3.13)}{<} K\mathrm{e}^{-\beta(t_c-\sigma)}(-\beta p_{i_0}) = D^+v_{i_0}(t_c).\end{aligned}$$

On the other hand, (3.39) implies that

$$\begin{aligned}D^+|y_{i_0}(t_c)| &= \limsup_{t\to t_c^+}\frac{|y_{i_0}(t)|-|y_{i_0}(t_c)|}{t-t_c} \ge \overline{\lim}_{k\to\infty}\frac{|y_{i_0}(\tau_k)|-|y_{i_0}(t_c)|}{\tau_k-t_c}\\
&\ge \overline{\lim}_{k\to\infty}\frac{v_{i_0}(\tau_k)-v_{i_0}(t_c)}{\tau_k-t_c} = \lim_{k\to\infty}\frac{v_{i_0}(\tau_k)-v_{i_0}(t_c)}{\tau_k-t_c} = \frac{\mathrm{d}v_{i_0}}{\mathrm{d}t}(t_c) = D^+v_{i_0}(t_c).\end{aligned}$$

This is a contradition.

The remainder of this step is similar to that of Step I and so it is omitted here.

Step III. Let $x(t;\sigma,\varphi), t\in[\sigma,\gamma)$, be the unique noncontinuable solution of (3.6) though (σ,φ) with $\varphi\in\mathcal{C}_r$. We claim that $\gamma=\infty$ and so the zero solution of (3.6) is ES.

By the mean value theorem, we have for each $i\in\underline{n}$,

$$|g_i(t,\varphi(0),\varphi)| \le |g_i(t,\varphi(0),\varphi)-g_i(t,0,\varphi)| + |g_i(t,0,\varphi)| \le \sum_{j=1}^n\left(\int_0^1\left|\frac{\partial g_i}{\partial x_j}(t,\xi\varphi(0),\varphi)\right|\mathrm{d}\xi\right)|\varphi_j(0)| + |g_i(t,0,\varphi)|.$$

From (3.7), (3.9), it follows that

$$|g_i(t,\varphi(0),\varphi)| \le \sum_{j=1}^n a^{(1)}_{ij}(t)|\varphi_j(0)| + |\mathcal{L}_i(t,\varphi)|,$$

for each $i\in\underline{n}$. Thus, $g_i(t,\varphi(0),\varphi)$ is bounded on $W\subset\mathbb{R}\times\mathcal{C}$ if W is a closed bounded set in $\mathbb{R}\times\mathcal{C}$. On the other hand, (3.36) implies that $x(\cdot;\sigma,\varphi)$ is bounded on $[\sigma,\gamma)$. Thus γ must be equal to ∞, by ([11], Thm. 8.5, p. 65). This completes the proof.

4. Discussion and illustrative examples

We first present an analogue of Theorem 3.3. Consider the nonlinear differential equation of neutral type

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}(t;x_t) = f(t,x(t)) + g\big(t;x_t\big), \tag{4.1}$$

where:

- $\mathcal{D}(\cdot;\cdot)$ is defined by (3.3), (3.4);
- $f(\cdot;\cdot) : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ is continuous and is (locally) Lipschitz continuous with respect to the second argument on each compact subset of $\mathbb{R}\times\mathbb{R}^n$ and $f(t;0)=0, t\in\mathbb{R}$;
- $g(\cdot;\cdot) : \mathbb{R}\times\mathcal{C} \to \mathbb{R}^n$ is continuous and is Lipschitz continuous in the second argument on compact subsets of $\mathbb{R}\times\mathcal{C}$ and $g(t;0)=0, t\in\mathbb{R}$.

Furthermore, we assume that
(H_3) $f(t;\cdot)$ is continuously differentiable on B_δ for any $t\in\mathbb{R}$, for some $\delta>0$ and

$$b_i(t) \le \frac{\partial f_i}{\partial x_i}(t;x) \le a_{ii}(t) < 0; \quad \left|\frac{\partial f_i}{\partial x_j}(t;x)\right| \le a_{ij}(t), \quad i\neq j,\ i,j\in \underline{n}, \tag{4.2}$$

for any $t\in\mathbb{R}$, any $x\in B_\delta$ and

(H_4)

$$|g(t;\varphi)| \le |\mathcal{L}(t;\varphi)|, \quad \forall t\in\mathbb{R},\ \forall\varphi\in\mathcal{C}_\delta, \tag{4.3}$$

where $a_{ij}(\cdot), i,j\in\underline{n}$ and $b_i(\cdot)$ and $\mathcal{L}(\cdot;\cdot)$ are as in (H_1)–(H_2).

Using the same method of proof as for Theorem 3.3, we can prove the following theorem.

Theorem 4.1. *Suppose (H_3)–(H_4) hold. If (3.13), (3.14) hold then the zero solution of (4.1) is ES.*

Proof. The proof of Theorem 4.1 is almost the same of that of Theorem 3.3. Thus it is omitted here. □

Furthermore, a similar result to Corollary 3.5 for (4.1) can be stated and proven easily.

We now make a brief comparison between existing results and the stability criteria of this paper. Consider the linear neutral time-invariant differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(x(t)-cx(t-h)) = ax(t)+bx(t-h), \tag{4.4}$$

where a,b,c,h are given real numbers and $h>0$.

By Corollary 3.5 (ii) (see also [14]), (4.4) is ES if

$$|c|<1 \qquad \text{and} \qquad a+\frac{|a||c|+|b|}{1-|c|}<0. \tag{4.5}$$

Consider the perturbed equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(x(t)-cx(t-h)) = ax(t)+bx(t-h)+q(x(t),x(t-h)), \tag{4.6}$$

where $q(\cdot,\cdot):\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a Lipschitz continuous function such that

$$\lim_{|u|+|v|\to 0}\frac{|q(u,v)|}{|u|+|v|}=0. \tag{4.7}$$

We show that the zero solution of (4.6) is ES provided (4.5) and (4.7) hold. This result can be seen as an "extension" of the famous Poincaré–Lyapunov theorem (see *e.g.* [3], Thm. 11.2, p. 336) to time-delay differential equations of neutral type.

Clearly, (4.6) is of the form (4.1) where

$$f(t;u) := au, \quad t,u\in\mathbb{R}; \qquad g(t;\varphi) := b\varphi(-h) + q(\varphi(0),\varphi(-h)), \quad t\in\mathbb{R}, \varphi\in\mathcal{C}.$$

It follows from (4.5) that

$$|c|<1 \qquad \text{and} \qquad a+\frac{|a||c|+|b|+2\epsilon}{1-|c|}<0, \tag{4.8}$$

for $\epsilon>0$ sufficiently small. On the other hand, (4.7) implies that there exists $\delta>0$ such that

$$|q(u,v)|\le \epsilon|u|+\epsilon|v|, \quad \forall u,v\in\mathbb{R},\ |u|\le\delta,\ \ |v|\le\delta.$$

It follows that

$$|g(t;\varphi)|\le \mathcal{L}(\varphi) := \epsilon|\varphi(0)| + (|b|+\epsilon)|\varphi(-h)|, \quad \forall t\in\mathbb{R}, \varphi\in\mathcal{C}_\delta. \tag{4.9}$$

Then (4.8), (4.9) ensures that the zero solution of (4.6) is ES, by Theorem 4.1.

Next, we consider a shunted power transmission line described by the equation

$$\dot{x}(t) = -h(x(t)) + c\dot{x}(t-\tau), \tag{4.10}$$

where $c\in\mathbb{R}, \tau>0$ and $h(\cdot):\mathbb{R}\to\mathbb{R}, h(0)=0$, is a continuous function, see *e.g.* [11].

Using a Lyapunov function, it has been shown in ([11], Thm. 8.5, p. 296) that the zero solution of (4.10) is uniformly asymptotically stable if $|c|<\frac{1}{2}$ and

$$xh(x)>0,\ \forall x\neq 0; \qquad \lim_{|x|\to\infty}|h(x)|=\infty. \tag{4.11}$$

Assume that $h(\cdot)$ is Lipschitz continuous with the Lipschitz constant b:

$$|h(x)-h(y)|\le b|x-y|, \quad \forall x,y\in\mathbb{R},$$

which ensures that (4.10) has a unique solution for each initial value function $\varphi\in\mathcal{C}^1([-h,0],\mathbb{R})$. Furthermore, we now consider for (4.10), a slightly stronger condition than (4.11):

$$h'(x)\ge a>0, \quad \forall x\in\mathbb{R}. \tag{4.12}$$

Clearly, (4.10) can be represented in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(x(t)-cx(t-\tau)) = -h(x(t)).$$

By Corollary 3.5 (ii), (4.10) is ES if

$$-a+\frac{b|c|}{1-|c|}<0, \quad \text{or equivalently} \quad |c|<\frac{a}{a+b}\cdot \tag{4.13}$$

Note that (4.12) is slightly stronger than (4.11), but our result (the zero solution of (4.10) is ES) is stronger than ([11], Thm. 8.5, p. 296) (the zero solution of (4.10) is uniformly asymptotically stable).

On the other hand, by Corollary 3.5 (i), (4.10) is ES provided there exist positive numbers p,q,β so that

$$-ap+b|c|q\mathrm{e}^{\beta\tau}<-\beta p; \qquad p+|c|\mathrm{e}^{\beta\tau}q<q. \tag{4.14}$$

It is important to note that (4.14) implies that solutions of (4.10) exponentially decay with the rate $\beta>0$. That is,

$$\|x(t;\varphi)\|\le K\mathrm{e}^{-\beta t}\|\varphi\|, \quad \forall t\ge 0.$$

Furthermore, if (4.14) holds for $\tau>0$ then by continuity, it still holds for any $\tau_*\in[\underline{\tau},\bar{\tau}]$ for some $\underline{\tau},\bar{\tau}$ with $0<\underline{\tau}<\tau<\bar{\tau}$. Thus, the zero solution of (4.10) is ES for any $\tau\in[\underline{\tau},\bar{\tau}]$. This gives a delay-dependent stability condition of (4.10).

Finally, we illustrate the main results by a couple of examples.

Example 4.2. Consider the nonlinear time-varying neutral delay differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(x(t)-\frac{1}{9}x(t-h)\right)=-(2+\cos t)\sin x(t)+a\sin t\sin\big(x(t-h)\big), \tag{4.15}$$

where $h>0$ is a constant delay and $a\in\mathbb{R}$ is a parameter. Obviously, (4.15) can be represented in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(x(t)-\frac{1}{9}x(t-h))=g(t;x(t),x_t),$$

where $g(t;x,\varphi):=-(2+\cos t)\sin x+a\sin t\sin\big(\varphi(-h)\big),\ t,x\in\mathbb{R},\ \varphi\in C([-h,0],\mathbb{R})$. Clearly,

$$-3\le\frac{\partial g}{\partial x}(t;x,\varphi)=-(2+\cos t)\cos x\le-\cos\left(\frac{\pi}{n}\right)<0,\quad x\in\left[-\frac{\pi}{n},\frac{\pi}{n}\right],\quad n\in\mathbb{N},n\ge3,$$

for any $t\in\mathbb{R}$ and any $\varphi\in C([-h,0],\mathbb{R})$. Furthermore, we have

$$|g(t;0,\varphi)|=|a||\sin t||\sin\big(\varphi(-h)\big)|\le|a||\varphi(-h)|.$$

Thus, by Corollary 3.5 (i), (4.15) is ES if for some $n\in\mathbb{N},n\ge3$, there are positive numbers p,q,β so that

$$-\cos\left(\frac{\pi}{n}\right)p+\left(|a|+\frac{1}{3}\right)\mathrm{e}^{\beta h}q<-\beta p;\qquad p+\frac{1}{9}\mathrm{e}^{\beta h}q<q.$$

Once again, this gives a dependent-delay stability condition for (4.15).

Theorems 3.3 and 4.1 can be applied to study behavior of solutions of neutral delay logistic equations [23] and the exponential stability of equilibria of various classes of neural networks of neutral type [5]. We present here an application to neural networks.

Example 4.3. Consider a neural network described by the following nonlinear neutral delay differential equation

$$\dot u_i(t)=-a_iu_i(t)+\sum_{j=1}^{n}w_{ij}g_j(u_j(t-h))+\sum_{j=1}^{n}d_{ij}\dot u_j(t-h)+I_i,i\in\underline{n} \tag{4.16}$$

$$u_i(t)=\varphi_i(t)\in\mathcal{C}^1([-h,0],\mathbb{R}),\quad t\in[-h,0],i\in\underline{n}, \tag{4.17}$$

where $u_i(t)$ denotes the state of the ith neuron at time t; the scalar $a_i>0$ is the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs at time t, $w_{ij},d_{ij},i,j\in\underline{n}$, are known scalars; the scalar $h>0$ represents the transmission delay; g_j and I_i are the activation function of the neurons and external constant inputs, respectively.

Assume that the activation function g_i is bounded, which satisfies

$$0\le\frac{g_i(\xi_1)-g_i(\xi_2)}{\xi_1-\xi_2}\le\sigma_i,\quad\forall\xi_1,\xi_2\in\mathbb{R},\ \xi_1\ne\xi_2, \tag{4.18}$$

for some $\sigma_i>0$ and each $i\in\underline{n}$. Now, let $u^*=(u_1^*,u_2^*,\dots,u_n^*)^T$ be the equilibrium point of (4.16) and let $x(t)=u(t)-u^*$. Define $A:=-\mathrm{diag}(a_1,a_2,\dots,a_n),W:=(w_{ij}),D:=(d_{ij})\in\mathbb{R}^{n\times n}$. Under this transformation, (4.16) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(x(t)-Dx(t-h)\big)=Ax(t)+Wf(x(t-h)), \tag{4.19}$$

where $x(t)=(x_1(t),x_2(t),\dots,x_n(t))^T$ is the state vector of the transformed system, and $f(x(t))=(f_1(x_1(t)),f_2(x_2(t)),...,f_n(x_n(t)))^T$ with $f_i(x_i(t))=g_i(x_i(t)+u_i^*)-g_i(u_i^*)$, $i\in\underline{n}$. Then, it is easy to see that $f_i(0)=0,i\in\underline{n}$ and

$$0\le\frac{f_i(\xi_1)-f_i(\xi_2)}{\xi_1-\xi_2}\le\sigma_i,\quad\forall\xi_1,\xi_2\in\mathbb{R},\ \xi_1\ne\xi_2, \tag{4.20}$$

for $i\in\underline{n}$. In particular, (4.20) yields $|f_i(\xi)|\le\sigma_i|\xi|,\ \forall\xi\in\mathbb{R}$, for $i\in\underline{n}$.

Applying Corollary 3.5 (ii) to (4.19), we conclude that (4.19) is ES, or equivalently, the equilibrium point $u^* = (u_1^*, u_2^*, \ldots, u_n^*)^T$ of (4.16) is exponentially stable if $\rho(|D|) < 1$ and the matrix

$$-\mathrm{diag}(a_1, a_2, \ldots, a_n) + \left(\mathrm{diag}(a_1, a_2, \ldots, a_n)|D| + |W|\mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)\right)\left(I_n - |D|\right)^{-1},$$

is Hurwitz stable.

References

[1] R.P. Agarwal and S.R. Grace, Asymptotic stability of certain neutral differential equations. *Math. Comput. Model.* **31** (2000) 9–15.

[2] A. Bellen and N. Guglielmi and A.E. Ruehli, Methods for linear systems of circuit delay differential equations of neutral type. *IEEE Transactions on Circuits and Systems: Fundamental theory and Applications* **46** (1999) 212–216.

[3] R. Bellman and K.L. Cooke, Differential Difference Equations. The Rand Corporation USA (1963).

[4] A. Berman and R.J. Plemmons, Nonnegative Matrices in Mathematical Sciences. Acad. Press, New York (1979).

[5] C.J. Cheng, T.L. Liao, J.J. Yan and C.C. Hwang, Globally asymptotic stability of a class of neutral-type with delays. *IEEE Trans. Syst. Man Cybern. B* **36** (2006) 1191–1195.

[6] M.A. Cruz and J.K. Hale, Stability of functional differential equations of neutral type. *J. Differ. Equ.* **7** (1970) 334–355.

[7] J. Dieudonné, Foundations of Modern Analysis. Academic Press (1988).

[8] E. Fridman, New Lyapunov-Krasovskii functionals for stability of linear retarded and neutral type. *Syst. Control Lett.* **43** (2001) 309–319.

[9] M. Gil, Exponential stability of nonlinear neutral type systems. *Arch. Control Sci.* **22** (2012) 125–143.

[10] J.K. Hale and P.M. Amorez, Stability in neutral equations. *Nonlin. Anal. Theory, Methods Appl.* **1** (1977) 161–173.

[11] J.K. Hale and S.M.V. Lunel, Introduction to Functional Differential Equations. Springer-Verlag Berlin, Heidelberg, New york (1993).

[12] V. Kharitonov, S. Mondie and J. Collado, Exponential estimates for neutral time-delay systems: An LMI Approach. *IEEE Trans. Autom. Control* **50** (2005) 666–670.

[13] V.B. Kolmanovskii and A. Myshkis, *Appl. Theory Funct. Differ. Equ.* Dordrecht, the Neitherlands: Kluwer (1996).

[14] L.M. Li, Stability of linear neutral delay-differential systems. *Bull. Austr. Math. Soc.* **38** (1988) 339–344.

[15] Y. Liu, S.M. Lee, O.M. Kwon and Park H. Ju, Delay-dependent exponential stability criteria for neutral systems with interval time-varying delays and nonlinear perturbations. *J. Franklin Inst.* **350** (2013) 3313–3327.

[16] H. Li and K. Gu, Discretized LyapunovKrasovskii functional for coupled differential difference equations with multiple delay channels. *Automatica* **46** (2010) 902–909.

[17] S.I. Niculescu and B. Brogliato, Force measurement time-delays and contact instability phenomenon. *Eur. J. Control* **5** (1999) 279–289.

[18] P.H.A. Ngoc and H. Trinh, Novel criteria for exponential stability of linear neutral time-varying differential systems. *IEEE Trans. Autom. Control* **61** (2016) 1590–1594.

[19] J. Shen and W.X. Zheng, Positivity and stability of coupled differential difference equations with time-varying delays. *Automatica* **57** (2015) 123–127.

[20] W. Rudin, Principles of Mathematical Analysis. McGraw-Hill Science (1976).

[21] A.P. Tchangani, M. Dambrine, J.P. Richard and V. Kolmanovskii, Stability of Nonlinear Differential Equations with Distributed Delay. *Nonlin. Anal. Theory, Methods Appl.* **34** (1998) 1081–1095.

[22] A.P.Tchangani, M. Dambrine and J.P. Richard, Stability, attraction domains, and ultimate boundedness for nonlinear neutral systems. *Math. Comput. Simul.* **45** (1998) 291–298.

[23] Z. Yi, Stability of neutral Lotka-Volterra systems. *J. Math. Anal. Appl.* **199** (1996) 391–402.

[24] K. Zhang and D.Q. Cao, Further results on asymptotic stability of linear neutral systems with multiple delays. *J. Franklin Inst.* **344** (2017) 858–866.