ESAIM: COCV 23 (2017) 937–976
DOI: 10.1051/cocv/2016019

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

ON COUPLED SYSTEMS OF KOLMOGOROV EQUATIONS WITH APPLICATIONS TO STOCHASTIC DIFFERENTIAL GAMES *

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Abstract. We prove that a family of linear bounded evolution operators $(\mathbf{G}(t,s))_{t\geq s\in I}$ can be associated, in the space of vector-valued bounded and continuous functions, to a class of systems of elliptic operators $\boldsymbol{\mathcal{A}}$ with unbounded coefficients defined in $I\times\mathbb{R}^d$ (where I is a right-halfline or $I=\mathbb{R}$) all having the same principal part. We establish some continuity and representation properties of $(\mathbf{G}(t,s))_{t\geq s\in I}$ and a sufficient condition for the evolution operator to be compact in $C_b(\mathbb{R}^d;\mathbb{R}^m)$. We prove also a uniform weighted gradient estimate and some of its more relevant consequence.

Mathematics Subject Classification. 35K45, 35K58, 47B07, 60H10, 91A15.

Received June 16, 2015. Accepted April 13, 2016.

1. Introduction

In recent years, the study of second-order elliptic operators with unbounded coefficients has been object of increasing interest since they appear in many models of probability and mathematical finance. Whereas the theory of Kolmogorov equations is well developed in the scalar case, as the considerable literature shows (see *e.g.*, [9] and the references therein), the case of systems of equations is nowadays at a preliminary level.

In this paper, we consider a family of nonautonomous second-order uniformly elliptic operators $\boldsymbol{\mathcal{A}}$ (having all the same principal part), defined on smooth functions $\mathbf{w}:\mathbb{R}^d\to\mathbb{R}^m$ by

$$(\boldsymbol{\mathcal{A}}\mathbf{w})(t,x)=\sum_{i,j=1}^d Q_{ij}(t,x)D^2_{ij}\mathbf{w}(x)+\sum_{j=1}^d B_j(t,x)D_j\mathbf{w}(x)+C(t,x)\mathbf{w}(x), \tag{1.1}$$

for any $t\in I$ (I being a right-halfline or $I=\mathbb{R}$) and $x\in\mathbb{R}^d$. In (1.1), the entries Q_{ij} of the matrix-valued function Q are smooth functions, possibly unbounded, and $\inf_{I\times\mathbb{R}^d}\lambda_Q(t,x)$ is positive, where $\lambda_Q(t,x)$ is the

Keywords and phrases. Nonautonomous parabolic systems, unbounded coefficients, evolution operators, compactness, gradient estimates, semilinear systems, stochastic games.

* *Work partially supported by the M.I.U.R. Research Project PRIN 2010-11 "Problemi differenziali di evoluzione: approcci deterministici e stocastici e loro interazioni" and by the INdAM-GNAMPA Project 2014 "Equazioni ellittiche e paraboliche a coefficienti illimitati".*

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minimum eigenvalue of the matrix $Q(t,x)$. As far as B_j and C are concerned, they are $m\times m$ matrices whose elements are smooth enough and possibly unbounded real valued functions (see Hypotheses 2.1).

We deal with the parabolic Cauchy problem associated to $\boldsymbol{\mathcal{A}}$ in $C_b(\mathbb{R}^d;\mathbb{R}^m)$ the space of vector-valued bounded and continuous functions, *i.e.*, we look for a locally in time bounded classical solution $\mathbf{u}=(u_1,u_2,\dots,u_m)$ of the system

$$\begin{cases} D_t\mathbf{u}(t,x)=(\boldsymbol{\mathcal{A}}\mathbf{u})(t,x), & (t,x)\in(s,+\infty)\times\mathbb{R}^d,\\ \mathbf{u}(s,x)=\mathbf{f}(x), & x\in\mathbb{R}^d, \end{cases} \tag{1.2}$$

where $s\in I$ and $\mathbf{f}\in C_b(\mathbb{R}^d;\mathbb{R}^m)$. By a locally in time bounded classical solution of (1.2) we mean a function $\mathbf{u}\in C^{1,2}((s,+\infty)\times\mathbb{R}^d;\mathbb{R}^m)\cap C_b([s,T]\times\mathbb{R}^d;\mathbb{R}^m)$ satisfying (1.2) and bounded in each strip $[s,T]\times\mathbb{R}^d$.

In [12, 21] the simpler case of weakly coupled equations (*i.e.*, $B_j(t,x)=b_j(x)I_m$ for any $(t,x)\in I\times\mathbb{R}^d$, $j=1,\dots,d$ and some real valued function b_j) is considered. More precisely, in [12] the analysis is carried over in $C_b(\mathbb{R}^d;\mathbb{R}^m)$, whereas in [21] also the L^p-setting is studied, assuming that the diffusion coefficients are bounded. Very recently, taking advantage of some results contained in this paper, the L^p-theory has been extended also to first-order coupled systems as in (1.1) (see [6]).

This paper is devoted to keep on the analysis started in [12] aiming at considering more general systems than those studied in [12]. The presence of a first order term as in (1.1), where the first partial derivatives of all the components of $\mathbf{u}$ are mixed together, makes the problem quite involved and already the unique solvability of the problem (1.2) is a not trivial question. Indeed, the method used in [12] takes strongly into account the special structure of the equations and can not be immediately adapted to our situation. To overcome this difficulty, we provide two sets of assumptions (see Hypoyheses 2.2 and 2.3) which yield uniqueness of a locally in time bounded solution to problem (1.2). Under Hypotheses 2.2, we extend to our situation the method used by Kresin and Maz'ia in [23] in the case of bounded coefficients. Such assumptions reduce to those assumed in [12] in the case of weakly coupled equations and represent the natural generalization to the vector case of those typically assumed in the case of a single equation. On the other hand, Hypotheses 2.3 allows to get uniqueness of the solution $\mathbf{u}$ as above by a comparison argument: we show that $\mathbf{u}$ can be estimated pointwise from above by $G(t,s)|\mathbf{f}|^2$, where $G(t,s)$ is the evolution operator associated to a suitable nonautonomous elliptic operator $\mathcal{A}$. Once uniqueness is guaranteed, the existence of a classical solution of the problem (1.2) is then proved by some compactness and localization argument based on interior Schauder estimates recalled in the Appendix. Hence, under the previous assumptions, we can associate an evolution operator $\mathbf{G}(t,s)$ to $\boldsymbol{\mathcal{A}}$ in $C_b(\mathbb{R}^d;\mathbb{R}^m)$, *i.e.*, for any $\mathbf{f}\in C_b(\mathbb{R}^d;\mathbb{R}^m)$, $\mathbf{G}(\cdot,s)\mathbf{f}$ represents the unique locally in time bounded classical solution to (1.2).

The next (natural) step in our investigation consists in proving some continuity properties of the evolution operator, which hold in the scalar case. In particular, an integral representation formula, in terms of some finite Borel measures, is available for $\mathbf{G}(t,s)$. More precisely, for any $i=1,\dots,m$, $t>s\in I$ and $x\in\mathbb{R}^d$, there exists a family $\{p_{ij}(t,s,x,\mathrm{d}y):j=1,\dots,m\}$ of finite Borel measures such that

$$(\mathbf{G}(t,s)\mathbf{f})_i(x)=\sum_{j=1}^m\int_{\mathbb{R}^d}f_j(y)p_{ij}(t,s,x,\mathrm{d}y),\qquad \mathbf{f}\in C_b(\mathbb{R}^d;\mathbb{R}^m). \tag{1.3}$$

Formula (1.3) allows to extend the evolution operator $\mathbf{G}(t,s)$ to the space of bounded Borel vector-valued functions and to prove the strong Feller property for $\mathbf{G}(t,s)$. With some considerable efforts, we prove that the measures $p_{ij}(t,s,x,\mathrm{d}y)$ are absolutely continuous with respect the Lebesgue measure. One of the main difficulty is represented by the fact that, differently from the scalar case, $p_{ij}(t,s,x,\mathrm{d}y)$ are signed measures.

In Section 4, we use the pointwise estimate of $|\mathbf{G}(t,s)\mathbf{f}|^2$ in terms of $G(t,s)|\mathbf{f}|^2$ to prove that the compactness of $G(t,s)$ in $C_b(\mathbb{R}^d)$ is a sufficient condition for the compactness of $\mathbf{G}(t,s)$ in $C_b(\mathbb{R}^d;\mathbb{R}^m)$. To prove the quoted estimate, besides the standard regularity assumptions on the coefficients of $\boldsymbol{\mathcal{A}}$, the just mentioned assumption on the drift matrices $B_j=b_jI_m+\tilde{B}_j$ $(j=1,\dots,d)$ and the existence of a Lyapunov function for the operator $\mathcal{A}(t)$, we impose that all the entries of the matrices $\tilde{B}_j$ $(j=1,\dots,d)$ can grow no faster than λ_Q^σ, for some $\sigma\in(0,1)$, and that the quadratic form associated to the potential C is bounded from above.

Section 5 is devoted to prove a weighted gradient estimate for the function $\mathbf{G}(t,s)\mathbf{f}$. More precisely, under suitable assumptions (see Hypotheses 5.1), which are essentially growth and algebraic conditions on the coefficients of the operator $\boldsymbol{\mathcal{A}}$, their derivatives and the positive definite (t,x)-dependent matrix M, we show that the map $M(J_x\mathbf{G}(\cdot,s)\mathbf{f})^T$ is continuous and bounded in $(s,T]\times\mathbb{R}^d$ and that for any $T>s\in I$ there exists a positive constant $K_{s,T}$ such that

$$\|M(J_x\mathbf{G}(t,s)\mathbf{f})^T\|_\infty \le K_{s,T}(t-s)^{-1/2}\|\mathbf{f}\|_\infty, \qquad t\in(s,T),\ \ \mathbf{f}\in C_b(\mathbb{R}^d,\mathbb{R}^m). \tag{1.4}$$

Unweighted uniform gradient estimates are classical when the coefficients of $\boldsymbol{\mathcal{A}}$ are bounded and have been recently extended to scalar elliptic operators with unbounded coefficients (see *e.g.*, [3, 4, 8, 25, 30]) to the case of unbounded coefficients. On the other hand, weighted gradient estimates seem to be new also in the scalar case. We stress that we can allow $M(t,\cdot)$ to grow at most linearly at infinity. This condition could seem too restrictive, but as the classical case of bounded coefficients shows, in general the gradient of the solution to problem (1.2) does not vanish with polynomial rate as $|x|\to+\infty$. Moreover, in view of the applications to stochastic differential games, this bound on the growth of M is not restrictive at all. Estimate (1.4) is obtained by adopting the Bernstein method (see [7]), which has been already used in the case of a single elliptic equation with unbounded coefficients (see [25,29]) and in the case of domains with sufficiently smooth boundaries under Dirichlet and first-order, non tangential homogeneous boundary conditions (see [3,4]). Differently from the scalar case, additional technical difficulties arise, due to the presence of the matrix M. As a first application of (1.4), we provide a sufficient condition for the compactness of the vector evolution operator $\mathbf{G}(t,s)$ in $C_b(\mathbb{R}^d;\mathbb{R}^m)$ to imply the compactness of $G(t,s)$ in $C_b(\mathbb{R}^d)$. The main step in this direction is the proof of the following representation formula of a component of $\mathbf{G}(t,s)\mathbf{f}$ in terms of $G(t,s)$:

$$(\mathbf{G}(t,s)\mathbf{f})_{\bar k}(x) = (G(t,s)f_{\bar k})(x) + \int_s^t (G(t,r)(\mathcal{S}\mathbf{G}(\cdot,s)\mathbf{f}(r,\cdot))(x)\mathrm{d}r, \qquad t>s\in I,\ \ x\in\mathbb{R}^d, \tag{1.5}$$

where $\mathcal{S}\mathbf{G}(\cdot,s)\mathbf{f}=\sum_{i=1}^d\langle row_{\bar k}\tilde B_i, D_i\mathbf{G}(\cdot,s)\mathbf{f}\rangle + \langle row_{\bar k}C,\mathbf{G}(\cdot,s)\mathbf{f}\rangle$ and $row_{\bar k}\tilde B_i$, $row_{\bar k}C$ denote the $\bar k$- row of the matrices $\tilde B_i$ and C, respectively. To make formula (1.5) meaningful, we need to guarantee that the integral term is well defined. We prove this fact assuming that $row_{\bar k}C$ is bounded for some $\bar k\in\{1,\dots,m\}$ and using (1.4).

Other (and more relevant for the applications) consequences of the gradient estimate are an existence result for the system of forward backward stochastic differential equations

$$\begin{cases} -\mathrm{d}\mathbf{Y}_\tau = \mathbf{H}(\tau,X_\tau,\mathbf{Z}_\tau)\mathrm{d}\tau - \mathbf{Z}_\tau\mathrm{d}W_\tau,\ \tau\in[t,T],\\ \mathrm{d}X_\tau = \mathbf{b}(\tau,X_\tau)\mathrm{d}\tau + G(\tau,X_\tau)\mathrm{d}W_\tau,\ \tau\in[t,T],\\ \mathbf{Y}_T=\mathbf{g}(X_T),\\ X_t = x, \qquad\qquad\qquad\qquad x\in\mathbb{R}^d,\end{cases} \tag{1.6}$$

and identification formulae for the pair $\mathbf{Y}$ and $\mathbf{Z}$ in terms of the mild solution to a semilinear problem associated with an autonomous elliptic operator of the type (1.1). The main novelty lies in the fact that we do not assume that $\mathbf{H}$ is (globally) Lipschitz continuous as assumed in [17, 34]: we just assume β-Hölder continuity (for some $\beta\in(0,1)$) with respect to the second set of variables, which is not uniform with respect to the other variables.

The above identification formulae are then used to prove the existence of a Nash equilibrium for a nonzero-sum stochastic differential game. We follow the approach of [19], where the coefficients of the controlled system are assumed to be bounded, and, as in our case, the diffusion is assumed to be independent of the control. The results in [19] have been extended to the infinite dimensional setting in [16] still assuming the coefficients of the controlled system to be bounded. Very recently, in [20] the authors have proved the existence of a Nash equilibrium, relaxing the boundedness of the drift of the controlled system but still assuming the diffusion to be bounded. We stress that, in our situation we can allow the diffusion of the controlled system to be unbounded.

Notation

Functions with values in $\mathbb{R}^m$ are displayed in bold style. Given a function $\mathbf{f}$ (resp. a sequence $(\mathbf{f}_n)$) as above, we denote by f_i (resp. $f_{n,i}$) its ith component (resp. the ith component of the function $\mathbf{f}_n$). By $B_b(\mathbb{R}^d;\mathbb{R}^m)$ we denote the set of all the bounded Borel measurable functions $\mathbf{f}:\mathbb{R}^d\to\mathbb{R}^m$. For any $k\geq 0$, $C^k_b(\mathbb{R}^d;\mathbb{R}^m)$ is the space of all the functions whose components belong to $C^k_b(\mathbb{R}^d)$, where the notation $C^k(\mathbb{R}^d)$ $(k\geq 0)$ is standard and we use the subscripts "c", "0" and "b", respectively, for spaces of functions with compact support, vanishing at infinity and bounded. Similarly, when $k\in(0,1)$, we use the subscript "loc" to denote the space of all $f\in C(\mathbb{R}^d)$ which are Hölder continuous in any compact set of $\mathbb{R}^d$. We assume that the reader is familiar also with the parabolic spaces $C^{\alpha/2,\alpha}(I\times\mathbb{R}^d)$ $(\alpha\in(0,1))$ and $C^{1,2}(I\times\mathbb{R}^d)$, and we use the subscript "loc" with the same meaning as above. By $J_x\mathbf{u}$ we denote the Jacobian matrix of $\mathbf{u}$ with respect to the spatial variables.

Square matrices of size m are thought as elements of $\mathbb{R}^{m^2}$. For any $M\in\mathbb{R}^{m^2}$, we denote by M_{ij}, and row_jM, the ijth element and the jth row vector of the matrix M. For any $k\in\mathbb{N}$, by I_k we denote the identity matrix of size k. Finally, λ_M and Λ_M indicate the minimum and the maximum eigenvalue of the (symmetric) matrix M. When M depends on x (resp. (t,x)) we write $\lambda_M(x)$ and $\Lambda_M(x)$ (resp. $\lambda_M(t,x)$ and $\Lambda_M(t,x)$) instead of $\lambda_{M(x)}$ and $\Lambda_{M(x)}$ (resp. $\lambda_{M(t,x)}$ and $\Lambda_{M(t,x)}$).

By $\mathbf{e}_j$ and $\mathbb{1}$, we denote, respectively, the jth vector of the Euclidean basis of $\mathbb{R}^m$ and the function identically equal to 1 in $\mathbb{R}^d$. The open disk with center at 0 and radius $R>0$ is denoted by D_R.

For any $R>0$ we denote by $\mathbf{G}^{\mathcal{D}}_R(t,s)$ (resp. $\mathbf{G}^{\mathcal{N}}_R(t,s)$) and $G^{\mathcal{D}}_R(t,s)$ (resp. $G^{\mathcal{N}}_R(t,s)$) the evolution operator associated with the realization of the operators $\boldsymbol{\mathcal{A}}$ and $\tilde{\mathcal{A}}$ (see Hypotheses 2.3) in $C_b(D_R;\mathbb{R}^m)$ and $C_b(D_R)$, respectively, with homogeneous Dirichlet (resp. Neumann) boundary conditions on ∂D_R. Finally, $G(t,s)$ denotes the evolution operator associated to the operator $\tilde{\mathcal{A}}$ in $C_b(\mathbb{R}^d)$, whose existence has been proved in [25].

2. Existence and uniqueness of locally in time bounded classical solutions to (1.2)

2.1. Assumptions, remarks and examples

Let I be an open right-halfline or $I=\mathbb{R}$ and let $\boldsymbol{\mathcal{A}}$ be the system of elliptic operators defined in (1.1). In this section we prove that, for any $s\in I$ and any $\mathbf{f}\in C_b(\mathbb{R}^d;\mathbb{R}^m)$ there exists a unique locally in time bounded classical solution to the Cauchy problem (1.2).

The following are standing assumptions that we will not mention anymore.

Hypotheses 2.1. *The coefficients $Q_{ij}=Q_{ji}$, $(B_i)_{hk}$ and C_{hk} belong to $C^{\alpha/2,\alpha}_{\rm loc}(I\times\mathbb{R}^d)$, for any $i,j=1,\dots,d$, $h,k=1,\dots,m$. Moreover, $\lambda_0:=\inf_{I\times\mathbb{R}^d}\lambda_Q>0$.*

In what follows we will consider, alternatively, two additional sets of assumptions.

Hypotheses 2.2.

(i) *There exist $\varepsilon>0$ and a function $\kappa:I\times\mathbb{R}^d\to\mathbb{R}$, bounded from above by a constant κ_0, such that the function*

$$\mathcal{K}_{\eta,\varepsilon}:=\sum_{i,j=1}^d(Q^{-1})_{ij}[\langle B_i\eta,\eta\rangle\langle B_j\eta,\eta\rangle-\langle B_i^*\eta,B_j^*\eta\rangle]-4\langle C\eta,\eta\rangle+4\varepsilon\kappa$$

is nonnegative in $I\times\mathbb{R}^d$ for any $\eta\in\partial D_1\subset\mathbb{R}^m$;

(ii) *for any bounded interval $J\subset I$ there exist $\mu_J\in\mathbb{R}$ and a positive (Lyapunov) function $\varphi_J\in C^2(\mathbb{R}^d)$ blowing up as $|x|\to+\infty$ such that $\sup_{\eta\in\partial D_1}\sup_{J\times\mathbb{R}^d}(\mathcal{A}_\eta\varphi_J-\mu_J\varphi_J)<+\infty$, where $\mathcal{A}_\eta={\rm Tr}(QD^2)+\langle\mathbf{b}_\eta,\nabla_x\rangle+2\varepsilon\kappa$ and $b_{\eta,j}=\langle B_j\eta,\eta\rangle$ for $j=1,\dots,d$.*

Hypotheses 2.3.

(i) *There exist functions* $b_i \in C^{\alpha/2,\alpha}_{\rm loc}(I\times\mathbb{R}^d)$ *and* $\tilde{B}_i \in C^{\alpha/2,\alpha}_{\rm loc}(I\times\mathbb{R}^d;\mathbb{R}^{m^2})$ *such that* $B_i := b_i I_m + \tilde{B}_i$ *in* $I\times\mathbb{R}^d$ *for any* $i = 1,\dots,d$ *and* $|(\tilde{B}_i)_{jk}| \le \xi\lambda_Q^\sigma$ *in* $I\times\mathbb{R}^d$, *for any* $j,k = 1,\dots,m$, $i = 1,\dots,d$ *and some locally bounded function* $\xi : I \to (0,+\infty)$ *and* $\sigma \in (0,1)$;
(ii) $H_J := \sup_{J\times\mathbb{R}^d}(\Lambda_C + 4^{-1}m^2 d\xi^2\lambda_Q^{2\sigma-1}) < +\infty$ *for any bounded interval* $J \subset I$;
(iii) *Hypothesis* 2.2(ii) *is satisfied with* $\mathcal{A}_\eta$ *replaced by* $\tilde{\mathcal{A}} = {\rm Tr}(QD^2) + \langle \mathbf{b}, \nabla\rangle$, *where* $\mathbf{b} = (b_1,\dots,b_d)$.

Remark 2.4.

(i) Hypotheses 2.2 and 2.3 are technical assumptions used to prove the uniqueness of the classical solution $\mathbf{u}$ to the Cauchy problem (1.2). Hypothesis 2.2(i) is obtained adapting a similar condition used in ([23], Thm. 8.7) in the case of bounded coefficients. In literature there are also different technical conditions which lead to a maximum principle for solutions to systems of partial differential equations. We refer the reader *e.g.*, to [28] where a weak maximum principle is proved for weak solutions to strongly coupled elliptic systems with bounded coefficients in bounded domains.
(ii) Hypothesis 2.2(i) can be replaced with the weaker requirement that $\mathcal{K}_{\eta,\varepsilon}$ is bounded from below in $J\times\mathbb{R}^d$, uniformly with respect to $\eta \in \partial D_1$, for any bounded interval $J \subset I$. Indeed, in this case, for any J as above, let $c_J > 0$ be such that $\mathcal{K}_{\eta,\varepsilon} \ge -c_J$ in $J\times\mathbb{R}^d$ for any $\eta \in \partial D_1$. The change of unknowns $\mathbf{v}(t,x) := {\rm e}^{-c_J(t-s)/4}\mathbf{u}(t,x)$ transforms the elliptic operator $\boldsymbol{\mathcal{A}}$ into the operator $\boldsymbol{\mathcal{A}} - c_J/4$, which satisfies Hypothesis 2.2(i) and, clearly, the uniqueness of $\mathbf{v}$ is equivalent to the uniqueness of $\mathbf{u}$.
(iii) In the scalar case when the elliptic operator in (1.1) is $\mathcal{A} = {\rm Tr}(QD^2) + \langle \mathbf{b}, \nabla\rangle + c$ and c is bounded from above (otherwise, Proposition 2.7 fails in general), taking $\varepsilon = 1$ and $\kappa = c$, one easily realizes that Hypothesis 2.2(i) is trivially satisfied. Moreover, Hypothesis 2.2(ii) reduces to require the existence of a Lyapunov function for the operator $\mathcal{A} + c$, for any bounded interval $J \subset I$. This condition seems to be much more general than that typically assumed (*i.e.*, the existence of a Lyapunov function for the operator $\mathcal{A}$) and, as the proof of Proposition 2.7 shows, it appears naturally when one considers the problem solved by u^2, where u is the solution to the Cauchy problem (1.2), with $\mathcal{A}$ and $f \in C_b(\mathbb{R}^d)$ instead of $\boldsymbol{\mathcal{A}}$ and $\mathbf{f}$. In view of this fact, Proposition 2.7 and Theorem 2.8 below hold true provided there exists $\gamma \ge 1$ and a Lyapunov function for the operator $\mathcal{A} + \gamma c$.
(iv) Hypotheses 2.2 and 2.3 are independent in general. Indeed Hypotheses 2.3(i)–2.3(ii) imply Hypothesis 2.2(i), whereas Hypothesis 2.2(ii) is stronger than Hypothesis 2.3(iii). Indeed, if Hypothesis 2.3(i) holds, then $\sum_{i,j=1}^d (Q^{-1})_{ij}[\langle \tilde{B}_i\eta,\eta\rangle\langle \tilde{B}_j\eta,\eta\rangle - \langle \tilde{B}_i\eta, \tilde{B}_j\eta\rangle]$ is negative and of order $\lambda_Q^{2\sigma-1}$. This fact together with Hypothesis 2.3(ii) implies Hypothesis 2.2(i) (taking the first point of this remark into account). On the other hand, assuming Hypothesis 2.3(i), the function $\mathcal{K}_{\boldsymbol{\eta},\varepsilon}$ which, clearly, depends only upon $\tilde{B}_i$ (since the diagonal parts cancel), can be of order less than $\lambda_Q^{1-2\sigma}$. For instance, assume $d = m = 2$, $Q = {\rm diag}(\lambda_Q, \Lambda_Q)$, $B_1 = b_1 I_2$ diagonal and $\tilde{B}_2 \ne 0$. Then, $\mathcal{K}_{\eta,0} = \Lambda_Q^{-1}(\langle \tilde{B}_2\eta,\eta\rangle^2 - |\tilde{B}_2\eta|^2) - 4\langle C\eta,\eta\rangle$ is bounded from below if $\Lambda_C + \xi^2\lambda_Q^{2\sigma}\Lambda_Q^{-1} < +\infty$, which is weaker than the condition in Hypothesis 2.3(ii) if $\lambda_Q = o(\Lambda_Q)$. Finally, concerning Hypotheses 2.2(ii) and 2.3(iii), the latter requires the existence of a Lyapunov function for *one* decomposition of each drift matrix, while the former requires the existence of a Lyapunov function for *any* decomposition $B_i = \mathbf{b}_\eta I_m + \tilde{B}_{\eta,i}$ $(\eta \in \partial D_1)$.

Now, we exhibit two classes of elliptic systems which satisfy Hypotheses 2.2 (see Example 2.5) or Hypotheses 2.3 (see Example 2.6).

Example 2.5. Let the coefficients of the operator $\boldsymbol{\mathcal{A}}$ be given by

$$Q_{ij} \equiv \delta_{ij}, \qquad B_i(t,x) = -x_i(1+|x|^2)^r g(t)\hat{B}_i, \qquad C(t,x) = -(1+|x|^2)^\gamma h(t)\hat{C}$$

for any $(t,x) \in I \times \mathbb{R}^d$, $i,j = 1,\ldots,d$, where

- $\hat{B}_i$ $(i = 1,\ldots,d)$ and $\hat{C}$ are constant and positive definite matrices;
- $g,h \in C^{\alpha/2}_{\rm loc}(I)$ have positive infima and g is bounded in I;
- $\gamma > 2r+1$, $r \geq 0$.

Observe that

$$\mathcal{K}_{\eta,0}(t,x) =(1+|x|^2)^{2r}(g(t))^2 \sum_{i=1}^{\mathrm{d}} x_i^2(|\langle \hat{B}_i\eta,\eta\rangle|^2 - |\hat{B}_i^*\eta|^2) + 4(1+|x|^2)^{\gamma} h(t)\langle \hat{C}\eta,\eta\rangle,$$

for any $(t,x) \in I \times \mathbb{R}^d$ and $\eta \in \partial D_1$. Hence,

$$\mathcal{K}_{\eta,0}(t,x) \geq -(1+|x|^2)^{2r}\|g\|_\infty^2 \sum_{i=1}^{d} x_i^2|\hat{B}_i|^2 + 4(1+|x|^2)^{\gamma} h_0\lambda_{\hat{C}},$$

for any t, x and η as above, where h_0 denotes the positive infimum of the function h. Since $\gamma > 2r+1$, the function $\mathcal{K}_{\eta,0}(t,\cdot)$ tends to $+\infty$ as $|x| \to +\infty$, uniformly with respect to $t \in I$ and $\eta \in \partial D_1$. Therefore, Hypothesis 2.2(i) is satisfied with $\varepsilon = 1$ and with a suitable choice of the constant κ. On the other hand, in this case the operator $\mathcal{A}_{\boldsymbol{\eta}}$ in Hypothesis 2.2(ii) is given by

$$\mathcal{A}_{\boldsymbol{\eta}} = \varDelta - g(t)(1+|x|^2)^r \sum_{j=1}^{d} \langle \hat{B}_j\eta,\eta\rangle x_j D_j + 2\kappa$$

and the function φ, defined by $\varphi(x) = 1+|x|^2$, for any $x \in \mathbb{R}^d$, satisfies Hypothesis 2.2(ii), with $\varepsilon = 1$ and $\mu_J = 2\kappa$.

Example 2.6. Let the coefficients of the operator $\boldsymbol{\mathcal{A}}$ be given by

$$\begin{aligned} &Q_{ij}(t,x) = q(t)(1+|x|^2)^k Q_0, \qquad B_i(t,x) = -b(t)x_i(1+|x|^2)^p I_m + \tilde{b}(t)(1+|x|^2)^r \tilde{B}_{0,i},\\ &C(t,x) = -h(t)(1+|x|^2)^{\gamma}\hat{C}, \end{aligned}$$

for any $(t,x) \in I \times \mathbb{R}^d$ and $i,j = 1,\ldots,d$. Here,

- $q \in C^{\alpha/2}_{\rm loc}(I) \cap C_b(I)$ has positive infimum and Q_0 is a constant positive definite matrix;
- the functions $b,\tilde{b}$ and h belong to $C^{\alpha/2}_{\rm loc}(I)$. Moreover, b has positive infimum;
- $\tilde{B}_{0,i}$ $(i = 1,\ldots,d)$ and $\hat{C}$ are constant, with $\hat{C}$ positive definite;
- the exponents k,p,r,γ are nonnegative, $p > (k-1) \vee 0$ and there exists $\sigma \in (0,1)$ such that $0 \leq r \leq k\sigma$ and $\gamma > k(2\sigma - 1)$.

Clearly Hypothesis 2.3(i) is satisfied. Moreover, the condition $\gamma > k(2\sigma-1)$ yields the boundedness from above of the function $\varLambda_C + 4^{-1}m^2 d\xi^2\lambda_Q^{2\sigma-1}$ so that the constant H_J in Hypothesis 2.3(ii) is finite. Further, since $\tilde{\mathcal{A}} = q(t)(1+|x|^2)^k \mathrm{Tr}(Q_0 D_x^2) - b(t)(1+|x|^2)^p\langle x,\nabla_x\rangle$ taking $\varphi(x) = 1+|x|^2$ we can estimate

$$\begin{aligned} (\tilde{\mathcal{A}}\varphi)(t,x) &= 2q(t)\mathrm{Tr}(Q_0)(1+|x|^2)^k - 2b(t)(1+|x|^2)^p|x|^2\\ &\leq 2\|q\|_\infty \mathrm{Tr}(Q_0)(1+|x|^2)^k - \inf_{t\in I} b(t)(1+|x|^2)^{p+1}, \end{aligned} \tag{2.1}$$

for any $t \in I$ and any $x \in \mathbb{R}^d \setminus D_1$. Due to the choice of p and k, the right-hand side of the previous inequality diverges to $-\infty$ as $|x| \to +\infty$, uniformly with respect to $t \in I$. Hence, Hypothesis 2.3(iii) is satisfied.

2.2. Existence and uniqueness under Hypotheses 2.2

The uniqueness of the classical solution to problem (1.2) which is bounded in any strip $[s,T]\times\mathbb{R}^d$, $T>s\in I$, is a straightforward consequence of the following result.

Proposition 2.7. *Fix $s\in I$, $\mathbf{f}\in C_b(\mathbb{R}^d;\mathbb{R}^m)$ and let $\mathbf{u}$ be a locally in time bounded classical solution to problem (1.2). If Hypotheses 2.2 hold true, then $\|\mathbf{u}(t,\cdot)\|_\infty\le e^{\varepsilon\kappa_0(t-s)}\|\mathbf{f}\|_\infty$ for any $t>s$.*

Proof. Fix $T>s\in I$ and let $\mu=\mu_{[s,T]}$ and $\varphi=\varphi_{[s,T]}$. Up to replacing μ with a larger constant if needed, we can assume that $\sup_{\eta\in\partial D_1}\sup_{[s,T]\times\mathbb{R}^d}(\mathcal{A}_\eta\varphi-\mu\varphi)<0$ and $\mu>2\varepsilon\kappa_0$.

For any $n\in\mathbb{N}$, we set $v_n(t,x):=e^{-\mu(t-s)}|\mathbf{u}(t,x)|^2-e^{-(\mu-2\varepsilon\kappa_0)(t-s)}\|\mathbf{f}\|_\infty^2-n^{-1}\varphi(x)$ for any $(t,x)\in[s,T]\times\mathbb{R}^d$. As it is immediately seen,

$$\begin{aligned}D_tv_n(t,\cdot)=&e^{-\mu(t-s)}[(\mathcal{A}_0|\mathbf{u}|^2)(t,\cdot)+(2\varepsilon\kappa(t,\cdot)-\mu)|\mathbf{u}(t,\cdot)|^2]+(\mu-2\varepsilon\kappa_0)e^{-(\mu-2\varepsilon\kappa_0)(t-s)}\|\mathbf{f}\|_\infty^2\\&-2e^{-\mu(t-s)}V(t,\cdot,D_1\mathbf{u}(t,\cdot),\dots,D_d\mathbf{u}(t,\cdot),\mathbf{u}(t,\cdot)),\end{aligned}$$

for any $t\in(s,T]$, where $V(\cdot,\cdot,\xi^1,\dots,\xi^d,\zeta):=\sum_{i,j=1}^dQ_{ij}\langle\xi^i,\xi^j\rangle-\sum_{j=1}^d\langle B_j\xi^j,\zeta\rangle-\langle(C-\varepsilon\kappa)\zeta,\zeta\rangle$ for any $\xi^1,\dots,\xi^d,\zeta\in\mathbb{R}^m$ and $\mathcal{A}_0=\mathrm{Tr}(QD^2)$. Since $\mu>2\varepsilon\kappa_0$ we can estimate

$$\begin{aligned}&D_tv_n(t,\cdot)-(\mathcal{A}_0v_n)(t,\cdot)-(2\varepsilon\kappa(t,\cdot)-\mu)v_n(t,\cdot)-2\varepsilon(\kappa(t,\cdot)-\kappa_0)e^{-(\mu-2\varepsilon\kappa_0)(t-s)}\|\mathbf{f}\|_\infty\\&<n^{-1}[(\mathcal{A}_0\varphi)(t,\cdot)+2\varepsilon\kappa(t,\cdot)\varphi-\mu\varphi]-2e^{-\mu(t-s)}V(t,\cdot,D_1\mathbf{u}(t,\cdot),\dots,D_d\mathbf{u}(t,\cdot),\mathbf{u}(t,\cdot)),\end{aligned}\tag{2.2}$$

for any $t\in(s,T]$ and $x\in\mathbb{R}^d$.

Our aim consists in proving that $v_n\le0$ in $[s,T]\times\mathbb{R}^d$ for any $n\in\mathbb{N}$. Once this property is checked, letting $n\to+\infty$ we will obtain $e^{-2\varepsilon\kappa_0(t-s)}|\mathbf{u}(t,x)|^2-\|\mathbf{f}\|_\infty^2\le0$ for any $t\in[s,T]$ and $x\in\mathbb{R}^d$. The estimate in the statement will follow from the arbitrariness of $T>s$.

Since v_n tends to $-\infty$ as $|x|\to+\infty$, uniformly with respect to $t\in[s,T]$, it has a maximum attained at some point $(t_0,x_0)\in[s,T]\times\mathbb{R}^d$. If $t_0=s$, then we are done since $v_n(s,\cdot)<0$. Suppose that $t_0>s$ and assume, by contradiction, that $v_n(t_0,x_0)>0$. Since $2\varepsilon\kappa(t_0,x_0)-\mu\le2\varepsilon\kappa_0-\mu<0$, the left-hand side of (2.2) is strictly positive at (t_0,x_0).

Let us prove that the right-hand side of (2.2) is nonpositive at (t_0,x_0). This will lead us to a contradiction and we will conclude that $v_n\le0$ in $[s,T]\times\mathbb{R}^d$.

Since ∇_xv_n vanishes at (t_0,x_0), $\langle D_j\mathbf{u}(t_0,x_0),\mathbf{u}(t_0,x_0)\rangle=e^{\mu(t_0-s)}D_j\varphi(x_0)/(2n)$ for any $j=1,\dots,d$. Hence, it is enough to show that the maximum of the function

$$F_{n,\zeta}(\xi^1,\dots,\xi^d):=n^{-1}[(\mathcal{A}_0\tilde{\varphi})(t_0,x_0)+2\varepsilon\kappa(t_0,x_0)\tilde{\varphi}(x_0)-\mu\tilde{\varphi}(x_0)]-2V(t_0,x_0,\xi^1,\dots,\xi^d,\zeta)$$

in the set $\varSigma=\{(\xi^1,\dots,\xi^d)\in\mathbb{R}^{md}:\langle\xi^j,\zeta\rangle=(2n)^{-1}D_j\tilde{\varphi}(x_0),\ j=1,\dots,d\}$ is nonpositive, where $\tilde{\varphi}=e^{\mu(t_0-s)}\varphi$. Note that the function $F_{n,\zeta}$ has a maximum in $\varSigma$ attained at some point $(\xi_0^1,\dots,\xi_0^d)$, since it tends to $-\infty$ as $\|(\xi^1,\dots,\xi^d)\|\to+\infty$. Applying the Lagrange multipliers theorem, we easily see that $(\xi_0^1,\dots,\xi_0^d)$ satisfies the conditions

$$2\sum_{k=1}^dQ_{jk}(t_0,x_0)\xi_{0,i}^k-\sum_{k=1}^m(B_j(t_0,x_0))_{ki}\zeta_k-\gamma_j\zeta_i=0,\qquad i=1,\dots,d,\ \ j=1,\dots,m,\tag{2.3}$$

for some real numbers $\gamma_1,\dots,\gamma_d$, where $\xi_{0,i}^k$ and ζ_i $(i=1,\dots,m)$ denote, respectively, the components of the vectors ξ_0^k and ζ. Multiplying both sides of (2.3) by ζ_i and summing over i, we get $\gamma_j=|\zeta|^{-2}[n^{-1}(Q(t_0,x_0)\nabla\tilde{\varphi}(x_0))_j-\langle B_j(t_0,x_0)\zeta,\zeta\rangle]$ for any $j=1,\dots,m$. Replacing the expression of γ_j in (2.3), we deduce that

$$\xi_0^j=\frac{1}{2n}|\zeta|^{-2}\zeta D_j\tilde{\varphi}(x_0)+\frac12\sum_{k=1}^d(Q^{-1})_{jk}(t_0,x_0)[B_k^*(t_0,x_0)\zeta-|\zeta|^{-2}\langle B_k(t_0,x_0)\zeta,\zeta\rangle\zeta],$$

for $j=1,\dots,d$. Hence, a direct computation shows that

$$V(t_0,x_0,\xi_0^1,\dots,\xi_0^d) = \frac{1}{4n^2|\zeta|^2}|\sqrt{Q}(t_0,x_0)\nabla\tilde{\varphi}(x_0)|^2 + \frac{1}{4}|\zeta|^2\mathcal{K}_{\zeta/|\zeta|}(t_0,x_0) - \frac{1}{2n|\zeta|^2}\sum_{j=1}^d D_j\tilde{\varphi}(x_0)\langle B_j(t_0,x_0)\zeta,\zeta\rangle$$

and, consequently,

$$\max_{\Sigma} F_{n,\boldsymbol{\zeta}} = \frac{1}{n}(\mathcal{A}_{\zeta/|\zeta|}(t_0)\tilde{\varphi}(x_0)-\mu\tilde{\varphi}(x_0)) - \frac{1}{2n^2|\zeta|^2}|\sqrt{Q}(t_0,x_0)\nabla\tilde{\varphi}(x_0)|^2 - \frac{1}{2}|\zeta|^2\mathcal{K}_{\zeta/|\zeta|,\varepsilon}(t_0,x_0).$$

By Hypothesis 2.2(ii) and the choice of μ, the right-hand side of the previous formula is nonpositive. The proof is complete. □

Now, we turn to show the existence of a unique locally in time bounded classical solution $\mathbf{u}$ to problem (1.2).

Theorem 2.8. *Under Hypotheses 2.2, for any $\mathbf{f}\in C_b(\mathbb{R}^d;\mathbb{R}^m)$ and $s\in I$, the Cauchy problem (1.2) admits a unique locally in time bounded classical solution $\mathbf{u}$. Moreover, $\mathbf{u}\in C^{1+\alpha/2,2+\alpha}_{\rm loc}((s,+\infty)\times\mathbb{R}^d)$ and satisfies*

$$\|\mathbf{u}(t,\cdot)\|_\infty \le \mathrm{e}^{\varepsilon\kappa_0(t-s)}\|\mathbf{f}\|_\infty, \qquad t>s. \tag{2.4}$$

Proof. Fix $\mathbf{f}\in C_b(\mathbb{R}^d;\mathbb{R}^m)$ and let $\mathbf{u}_n$ be the unique classical solution to the Cauchy−Dirichlet problem

$$\begin{cases} D_t\mathbf{u}_n(t,x) = (\boldsymbol{\mathcal{A}}\mathbf{u}_n)(t,x), & t\in(s,+\infty),\ x\in D_n,\\ \mathbf{u}_n(t,x)=\mathbf{0}, & t\in(s,+\infty),\ x\in\partial D_n,\\ \mathbf{u}_n(s,x)=\mathbf{f}(x), & x\in D_n,\end{cases} \tag{2.5}$$

(see [27], Thm. IV.5.5). By classical solution we mean a function which belongs to $C^{1,2}((s,+\infty)\times D_n)$ and is continuous in $([s,+\infty)\times\overline{D_n})\setminus(\{s\}\times\partial D_n)$.

Let us prove that the sequence $(\mathbf{u}_n)$ converges to a solution to problem (1.2) which satisfies the properties in the statement. The same arguments as in the proof of Proposition 2.7 show that

$$\|\mathbf{u}_n(t,\cdot)\|_\infty \le \mathrm{e}^{\varepsilon\kappa_0(t-s)}\|\mathbf{f}\|_\infty, \qquad t>s. \tag{2.6}$$

Hence, the interior Schauder estimates in Theorem A.2 guarantee that, for any compact set $E\subset(s,+\infty)\times\mathbb{R}^d$ and large n, the sequence $\|\mathbf{u}_n\|_{C^{1+\alpha/2,2+\alpha}(E;\mathbb{R}^m)}$ is bounded by a constant independent of n. By the Ascoli–Arzelà Theorem, a diagonal argument and the arbitrariness of E, we can determine a subsequence $(\mathbf{u}_{n_j})$ which converges to a function $\mathbf{u}\in C^{1+\alpha/2,2+\alpha}_{\rm loc}((s,+\infty)\times\mathbb{R}^d;\mathbb{R}^m)$ in $C^{1,2}(E;\mathbb{R}^m)$ for any E as above. Clearly, $\mathbf{u}$ satisfies the differential equation in (1.2) as well as the estimate (2.4), as it is easily seen letting $n\to+\infty$ in (2.6); we just need to show that $\mathbf{u}$ is continuous in $t=s$ and it therein equals the function $\mathbf{f}$. As a byproduct, we will deduce that the whole sequence $(\mathbf{u}_n)$ converges in $C^{1,2}(E;\mathbb{R}^m)$, for any compact set $E\subset(s,+\infty)\times\mathbb{R}^d$, since any subsequence of $(\mathbf{u}_n)$ has a subsequence which converges in $C^{1,2}(E;\mathbb{R}^m)$.

Fix $R\in\mathbb{N}$ and let ϑ be any smooth function such that $\chi_{D_{R-1}}\le\vartheta\le\chi_{D_R}$. For any $n_j>R$ the function $\mathbf{v}_k:=\vartheta\mathbf{u}_{n_j}$ belongs to $C([s,T]\times\overline{D_R};\mathbb{R}^m)\cap C^{1,2}((s,T]\times D_R;\mathbb{R}^m)$, it vanishes on $(s,T]\times\partial D_R$, $\mathbf{v}_j(s,\cdot)=\vartheta\mathbf{f}$ and $D_t\mathbf{v}_j-\boldsymbol{\mathcal{A}}\mathbf{v}_j=\mathbf{g}_j$ in $(s,T]\times D_R$, where $\mathbf{g}_j=-\mathrm{Tr}(QD^2\vartheta)\mathbf{u}_{n_j}-2(J_x\mathbf{u}_{n_j})Q\nabla\vartheta-\sum_{k=1}^d(B_k\mathbf{u}_{n_j})D_k\vartheta$, for any j such that $n_j>R$. Since the function $t\mapsto(t-s)\|\mathbf{u}_{n_j}(t,\cdot)\|_{C^2_b(D_{n_j})}$ is bounded in $(s,s+1)$ (by a constant depending on k) we can apply Proposition A.1 and, taking (2.6) into account, we can estimate $|\mathbf{g}_j(t,x)|\le K_R(1+(t-s)^{-1/2})\|\mathbf{f}\|_\infty$, for any $(t,x)\in(s,s+1)\times D_R$ and any $n_j>R$, where K_R is a positive constant independent of j. Let us represent $\mathbf{v}_j$ by means of the variation-of-constants formula

$$\mathbf{v}_j(t,x)=(\mathbf{G}_R^{\mathcal{D}}(t,s)(\vartheta\mathbf{f}))(x)+\int_s^t(\mathbf{G}_R^{\mathcal{D}}(t,r)\mathbf{g}_j(r,\cdot))(x)\mathrm{d}r,\qquad t\in[s,T],\ \ x\in D_R.$$

Recalling that $\mathbf{v}_j\equiv\mathbf{u}_{n_j}$ in D_{R-1} and taking the previous estimate into account, it follows that $|\mathbf{u}_{n_j}(t,\cdot)-\mathbf{f}|\le|\mathbf{G}_R^{\mathcal{D}}(t,s)(\vartheta\mathbf{f})-\mathbf{f}|+K_R'\sqrt{t-s}\|\mathbf{f}\|_\infty$ in D_{R-1}, for any $t\in(s,s+1)$ and some positive constant K_R' independent of j. Letting first j tend to $+\infty$ and, then, t tend to s^+, we deduce that $\mathbf{u}$ is continuous at $t=s$ for any $x\in D_{R-1}$. Since $R\in\mathbb{N}$ is arbitrary, we conclude that $\mathbf{u}\in C([s,T]\times\mathbb{R}^d;\mathbb{R}^m)$ and $\mathbf{u}(s,\cdot)=\mathbf{f}$. ☐

2.3. Existence and uniqueness under Hypotheses 2.3

Now we show that the assertions in Theorem 2.8 continue to hold if, as an alternative to Hypotheses 2.2, we consider Hypotheses 2.3. Note that, since we no longer assume Hypotheses 2.2, we can not apply Proposition 2.7 to guarantee the uniqueness of the solution to the Cauchy problem (1.2). The role of the following theorem is twofold. First, it replaces Proposition 2.7, and, combined with Theorem 2.8, it shows that the Cauchy problem (1.2) admits a unique locally in time bounded classical solution $\mathbf{u}$. Secondly, it shows that, for any $\mathbf{f}\in C_b(\mathbb{R}^d;\mathbb{R}^m)$, the function $|\mathbf{u}|^2$ can be estimated pointwise in terms of $G(t,s)|\mathbf{f}|^2$.

Theorem 2.9. *Let us assume that Hypotheses 2.3 are satisfied. Then, for any $s\in I$ and any $\mathbf{f}\in C_b(\mathbb{R}^d;\mathbb{R}^m)$, the Cauchy problem (1.2) admits a unique locally in time bounded classical solution $\mathbf{u}$. Moreover, $\mathbf{u}$ belongs to $C^{1+\alpha/2,2+\alpha}_{\rm loc}((s,+\infty)\times\mathbb{R}^d)$ and, for any $T>s\in I$,*

$$|\mathbf{u}(t,x)|^2\le \mathrm{e}^{2H_{[s,T]}(t-s)}(G(t,s)|\mathbf{f}|^2)(x),\qquad (t,x)\in[s,T]\times\mathbb{R}^d,\ \ \mathbf{f}\in C_b(\mathbb{R}^d;\mathbb{R}^m),\tag{2.7}$$

where $H_{[s,T]}$ is the constant in Hypothesis 2.3(ii).

Proof. Clearly, estimate (2.7) yields the uniqueness of the solution to problem (1.2). Moreover, the arguments here below can also be applied to prove that the solution $\mathbf{u}_n$ to the Cauchy problem (2.5) satisfies (2.7), with $\mathbb{R}^d$ replaced by D_n. This estimate replaces (2.6) and allows us to repeat verbatim the proof of Theorem 2.8 to get the existence of a classical solution $\mathbf{u}$ to the problem (1.2).

Fix $\mathbf{f}\in C_b(\mathbb{R}^d;\mathbb{R}^m)$, $T>s\in I$. To prove (2.7) we need to show that the function $v:[s,T]\times\mathbb{R}^d\to\mathbb{R}$, defined by $v(t,x):=\mathrm{e}^{-2H(t-s)}|\mathbf{u}(t,x)|^2-(G(t,s)|\mathbf{f}|^2)(x)$ for any $(t,x)\in[s,T]\times\mathbb{R}^d$, where $H=H_{[s,T]}$, is nonpositive. Clearly, it belongs to $C_b([s,T]\times\mathbb{R}^d)\cap C^{1,2}((s,T)\times\mathbb{R}^d)$ and $v(s,\cdot)\equiv 0$. Moreover, taking Hypothesis 2.3(i) into account, by a straightforward computation we get $D_tv(t,x)=\tilde{\mathcal{A}}v(t,x)+2\mathrm{e}^{-2H(t-s)}g(t,x)$ for any $(t,x)\in(s,T]\times\mathbb{R}^d$, where $g=\sum_{i=1}^d\langle\tilde{B}_iD_i\mathbf{u},\mathbf{u}\rangle-\mathrm{Tr}(J_x\mathbf{u}Q(J_x\mathbf{u})^T)+\langle C\mathbf{u},\mathbf{u}\rangle-H|\mathbf{u}|^2$. From Hypothesis 2.1, the Young and Cauchy-Schwarz inequalities and Hypothesis 2.3(i) we get

$$\begin{aligned}g\le&-\lambda_Q|J_x\mathbf{u}|^2+2m\xi\lambda_Q^\sigma|\mathbf{u}|\sum_{i=1}^d|D_i\mathbf{u}|+(\Lambda_C-H)|\mathbf{u}|^2\\ \le&(\varepsilon dm^2\xi^2-1)\lambda_Q|J_x\mathbf{u}|^2+[(4\varepsilon)^{-1}\lambda_Q^{2\sigma-1}+\lambda_C-H]|\mathbf{u}|^2\end{aligned}$$

in $[s,T]\times\mathbb{R}^d$, where $\varepsilon=\varepsilon(t)$ is an arbitrary positive function. Choosing $\varepsilon=(m^2\xi^2d)^{-1}$ and taking Hypothesis 2.3(ii) into account, we get $D_tv-\tilde{\mathcal{A}}v\le 0$ in $(s,T]\times\mathbb{R}^d$. The maximum principle in ([25], Thm. 2.1) shows that $v\le 0$ in $[s,T]\times\mathbb{R}^d$, which is the claim. ☐

Remark 2.10. We stress that the arguments used in the proof of Theorem 2.8 (and, hence, in the proof of Thm. 2.9) to guarantee the existence of a solution to problem (1.2) work as well if we approximate this problem by homogeneous Neumann–Cauchy problems in the ball D_n. We will use this approximation in the proof of Theorem 6.3.

3. The evolution operator and its main properties

As a consequence of Theorem 2.8 (resp. Thm. 2.9) we can associate an evolution operator $\{\mathbf{G}(t,s)\}_{t\geq s\in I}$ to $\boldsymbol{\mathcal{A}}$ in $C_b(\mathbb{R}^d,\mathbb{R}^m)$, by setting $\mathbf{G}(\cdot,s)\mathbf{f}:=\mathbf{u}$, where $\mathbf{u}$ is the unique locally in time bounded classical solution to the Cauchy problem (1.2). The uniqueness statement in Proposition 2.7 and Theorem 2.9 yield the evolution property of the family $\{\mathbf{G}(t,s)\}_{t\geq s\in I}$. Moreover, (2.4) and (2.7) (together with the estimate $\|G(t,s)\|_{\mathcal{L}(C_b(\mathbb{R}^d))}\leq 1$), imply that this family is an evolution operator in $C_b(\mathbb{R}^d;\mathbb{R}^m)$. Moreover, for any $T>s\in I$,

$$\|\mathbf{G}(t,s)\mathbf{f}\|_\infty\leq \mathrm{e}^{\ell(t-s)}\|\mathbf{f}\|_\infty,\qquad t\in[s,T],\ \ \mathbf{f}\in C_b(\mathbb{R}^d;\mathbb{R}^m),\tag{3.1}$$

with $\ell=\varepsilon\kappa_0$ (resp. $\ell=H_{[s,T]}$).

Remark 3.1. Note that, under Hypotheses 2.2, estimate (3.1) holds true for any $t>s\in I$. The same is true even if Hypotheses 2.3 are satisfied with $J=I$. In this latter case $\ell=H$ for any $t>s$ and some positive constant H, independent of t and s.

In this subsection we investigate the main properties of the evolution operator (which from now on we simply denote by $\mathbf{G}(t,s)$). All the results contained in this section hold true when at least one between Hypotheses 2.2 and 2.3 are satisfied.

3.1. Continuity properties

We start by proving some continuity properties of $\mathbf{G}(t,s)$.

Proposition 3.2. *Let $(\mathbf{f}_n)$ be a bounded sequence of functions in $C_b(\mathbb{R}^d;\mathbb{R}^m)$. Then, the following properties are satisfied:*

(i) *If $\mathbf{f}_n$ converges pointwise to $\mathbf{f}\in C_b(\mathbb{R}^d;\mathbb{R}^m)$, then $\mathbf{G}(\cdot,s)\mathbf{f}_n$ converges to $\mathbf{G}(\cdot,s)\mathbf{f}$ in $C^{1,2}(E)$ for any compact set $E\subset(s,+\infty)\times\mathbb{R}^d$;*
(ii) *If $\mathbf{f}_n$ converges to $\mathbf{f}$ locally uniformly in $\mathbb{R}^d$, then $\mathbf{G}(\cdot,s)\mathbf{f}_n$ converges to $\mathbf{G}(\cdot,s)\mathbf{f}$ locally uniformly in $[s,+\infty)\times\mathbb{R}^d$.*

Proof.

(i) From the inequality (3.1) and the interior Schauder estimates in Theorem A.2, we deduce that $\sup_{n\in\mathbb{N}}\|\mathbf{G}(\cdot,s)\mathbf{f}_n\|_{C^{1+\alpha/2,2+\alpha}(E)}<+\infty$ for any compact set $E\subset(s,+\infty)\times\mathbb{R}^d$. Therefore, using the same arguments as in the proof of Theorem 2.8, we can prove that there exists a function $\mathbf{v}\in C^{1+\alpha/2,2+\alpha}_{\mathrm{loc}}((s,+\infty)\times\mathbb{R}^d)$ and a subsequence $(\mathbf{G}(\cdot,s)\mathbf{f}_{n_k})$ which converges to $\mathbf{v}$ in $C^{1,2}(E)$ as $k\to+\infty$, for any E as above. Clearly, $D_t\mathbf{v}=\boldsymbol{\mathcal{A}}\mathbf{v}$ in $(s,+\infty)\times\mathbb{R}^d$.
To complete the proof, we need to show that $\mathbf{v}$ can be extended by continuity on $\{s\}\times\mathbb{R}^d$ and $\mathbf{v}(s,\cdot)\equiv\mathbf{f}$. Indeed, once this property is proved, we can conclude that $\mathbf{v}$ is a local in time bounded classical solution to problem (1.2). Hence, by uniqueness, we conclude that $\mathbf{v}\equiv\mathbf{G}(\cdot,s)\mathbf{f}$. Since this argument can be applied to any subsequence of $(\mathbf{G}(\cdot,s)\mathbf{f}_n)$ which converges in $C^{1,2}((s,+\infty)\times\mathbb{R}^d)$, and the limit is $\mathbf{G}(\cdot,s)\mathbf{f}$, we conclude that the whole sequence $(\mathbf{G}(\cdot,s)\mathbf{f}_n)$ converges to $\mathbf{G}(\cdot,s)\mathbf{f}$ locally uniformly in $(s,+\infty)\times\mathbb{R}^d$.
To prove that $\mathbf{v}$ can be extended by continuity at $t=s$, we fix $m,r\in\mathbb{N}$, with $r<m$. From the proof of Theorem 2.8 and recalling that $\sup_{n\in\mathbb{N}}\|\mathbf{f}_n\|_\infty<+\infty$, we deduce that

$$|(\mathbf{G}^{\mathcal{D}}_m(t,s)\mathbf{f}_n)(x)-\mathbf{f}_n(x)|\leq|\mathbf{G}^{\mathcal{D}}_r(t,s)(\vartheta\mathbf{f}_n)(x)-\vartheta(x)\mathbf{f}_n(x)|+K_r\sqrt{t-s}$$

for any $(t,x)\in(s,s+1)\times D_{r-1}$ and some positive constant K_r independent of m. Thus, letting $m\to+\infty$ we conclude that

$$|(\mathbf{G}(t,s)\mathbf{f}_n)(x)-\mathbf{f}_n(x)|\leq|\mathbf{G}^{\mathcal{D}}_r(t,s)(\vartheta\mathbf{f}_n)(x)-\vartheta(x)\mathbf{f}_n(x)|+K_r\sqrt{t-s},\tag{3.2}$$

for any $(t,x) \in (s,s+1)\times D_{r-1}$. Next step consists in letting $n \to +\infty$. Clearly, the left-hand side of (3.2) converges to $|\mathbf{v}(t,x)-\mathbf{f}(x)|$ for any $(t,x)\in(s,+\infty)\times\mathbb{R}^d$. As far as the right-hand side is concerned, we observe that Riesz's representation theorem (see [1], Rem. 1.57) shows that there exists a family $\{p_{ij}^r(t,s,x,\mathrm{d}y): t>s,\ x\in D_r,\ i,j=1,\dots m\}$ of Borel finite measures such that

$$(\mathbf{G}_r^{\mathcal{D}}(t,s)\mathbf{g})_i(x)=\sum_{j=1}^m\int_{\mathbb{R}^d} g_j(y)p_{ij}^r(t,s,x,\mathrm{d}y),\qquad \mathbf{g}\in C_c(D_r;\mathbb{R}^m), \tag{3.3}$$

for any $t>s$, $x\in\mathbb{R}^d$, $i=1,\dots,m$. Since each function $\vartheta\mathbf{f}_n$ is compactly supported in D_r, from (3.3) it follows that $\mathbf{G}_r^{\mathcal{D}}(\cdot,s)(\vartheta\mathbf{f}_n)$ converges to $\mathbf{G}_r^{\mathcal{D}}(\cdot,s)(\vartheta\mathbf{f})$ pointwise in $[s,+\infty)\times\mathbb{R}^d$, as $n\to+\infty$. Hence, we can take the limit in (3.2) and conclude that $|\mathbf{v}(t,\cdot)-\mathbf{f}|\le|\mathbf{G}_r^{\mathcal{D}}(t,s)(\vartheta\mathbf{f})-\vartheta\mathbf{f}|+K_r\sqrt{t-s}$ in D_{r-1}, for any $t\in(s,s+1)$. Since the function $\mathbf{G}_r^{\mathcal{D}}(\cdot,s)(\vartheta\mathbf{f})$ is continuous in $[s,+\infty)\times D_r$, this implies that $\mathbf{v}$ can be extended by continuity to $\{s\}\times D_{r-1}$ by setting $\mathbf{v}(s,\cdot)=\mathbf{f}$. The arbitrariness of r allows us to complete the proof.

(ii) Fix $T>s\in I$. In view of property (i), we just need to prove that, for any compact set $E\subset\mathbb{R}^d$ and $\varepsilon>0$, there exists $\delta\in(0,1)$ such that

$$\limsup_{n\to+\infty}\|\mathbf{G}(\cdot,s)\mathbf{f}_n-\mathbf{G}(\cdot,s)\mathbf{f}\|_{C([s,s+\delta]\times E;\mathbb{R}^m)}\le\varepsilon. \tag{3.4}$$

Fix $r\in\mathbb{N}$ such that $E\subset D_{r-1}$. Taking first the supremum over D_{r-1} and, then, the limsup as $n\to+\infty$ in both the sides of (3.2), we conclude that

$$\limsup_{n\to+\infty}\|\mathbf{G}(\cdot,s)\mathbf{f}_n-\mathbf{f}_n\|_{C([s,s+\delta]\times\overline{D_{r-1}};\mathbb{R}^m)}\le\|\mathbf{G}_r^{\mathcal{D}}(\cdot,s)(\vartheta\mathbf{f})-\vartheta\mathbf{f}\|_{C([s,s+\delta]\times\overline{D_{r-1}};\mathbb{R}^m)}+K_r\sqrt{\delta}.$$

Indeed, since $\vartheta\mathbf{f}_n$ tends to $\vartheta\mathbf{f}$, uniformly in D_r, from (2.6) it follows that $\mathbf{G}_r^{\mathcal{D}}(\cdot,s)(\vartheta\mathbf{f}_n)$ converges to $\mathbf{G}_r^{\mathcal{D}}(\cdot,s)(\vartheta\mathbf{f})$ uniformly in $[s,s+1]\times\overline{D_{r-1}}$. Finally, splitting

$$\mathbf{G}(\cdot,s)\mathbf{f}_n-\mathbf{G}(\cdot,s)\mathbf{f}=\mathbf{G}(\cdot,s)\mathbf{f}_n-\mathbf{f}_n+(\mathbf{f}_n-\mathbf{f})+\mathbf{f}-\mathbf{G}(\cdot,s)\mathbf{f},$$

and using the above estimate, we deduce that

$$\limsup_{n\to+\infty}\|\mathbf{G}(\cdot,s)\mathbf{f}_n-\mathbf{G}(\cdot,s)\mathbf{f}\|_{C([s,s+\delta]\times\overline{D_{r-1}};\mathbb{R}^m)}\le \|\mathbf{G}_r^{\mathcal{D}}(\cdot,s)(\vartheta\mathbf{f})-\vartheta\mathbf{f}\|_{C([s,s+\delta]\times\overline{D_{r-1}};\mathbb{R}^m)}+K_r\sqrt{\delta}+\|\mathbf{G}(\cdot,s)\mathbf{f}-\mathbf{f}\|_{C([s,s+\delta]\times\overline{D_{r-1}};\mathbb{R}^m)}.$$

Since the functions $\mathbf{G}_r^{\mathcal{D}}(\cdot,s)(\vartheta\mathbf{f})$ and $\mathbf{G}(\cdot,s)\mathbf{f}$ are continuous in $[s,s+1]\times\overline{D_{r-1}}$, estimate (3.4) follows immediately. □

3.2. A representation formula for $\mathbf{G}(t,s)$ and strong Feller property

In the following theorem we prove that the evolution operator $\mathbf{G}(t,s)$ can be extended to the set of all the bounded Borel measurable functions $\mathbf{f}:\mathbb{R}^d\to\mathbb{R}^m$. This is a consequence of the fact that, for any $\mathbf{f}\in C_b(\mathbb{R}^d;\mathbb{R}^m)$, each component of $\mathbf{G}(t,s)\mathbf{f}$ admits an integral representation formula in terms of some finite Borel measures. These measures are absolutely continuous with respect to the Lebesgue measure but, in general, differently from the scalar case, they are signed measures.

Theorem 3.3. *There exists a family $\{p_{ij}(t,s,x,\mathrm{d}y): t>s\in I,\ x\in\mathbb{R}^d,\ i,j=1,\dots,m\}$ of finite Borel measures, which are absolutely continuous with respect to the Lebesgue measure, such that formula (1.3) holds true for any $t>s$, $x\in\mathbb{R}^d$, $i=1,\dots,m$. Moreover, through formula (1.3), the evolution operator $\mathbf{G}(t,s)$ extends to $B_b(\mathbb{R}^d;\mathbb{R}^m)$ with a strong Feller evolution operator. Actually, $\mathbf{G}(\cdot,s)\mathbf{f}\in C_{\rm loc}^{1+\alpha/2,2+\alpha}((s,+\infty)\times\mathbb{R}^d;\mathbb{R}^m)$ for any $\mathbf{f}\in B_b(\mathbb{R}^d;\mathbb{R}^m)$ and $s\in I$.*

Proof. Throughout the proof, s is arbitrarily fixed in I. Since, for any $(t,x) \in (s,+\infty)\times\mathbb{R}^d$, the map $\mathbf{f}\mapsto (\mathbf{G}(t,s)\mathbf{f})(x)$ is bounded from $C_0(\mathbb{R}^d;\mathbb{R}^m)$ into $\mathbb{R}^m$, from the Riesz's Representation Theorem (see *e.g.*, [1], Rem. 1.57) it follows that there exists a family $\{p_{ij}(t,s,x,\mathrm{d}y) : t>s\in I,\, x\in\mathbb{R}^d,\, i,j=1,\dots m\}$ of finite Borel measures such that (1.3) is satisfied by any $\mathbf{f}\in C_0(\mathbb{R}^d;\mathbb{R}^m)$. To extend the previous formula to any $\mathbf{f}\in C_b(\mathbb{R}^d;\mathbb{R}^m)$, it suffices to approximate such an $\mathbf{f}$, locally uniformly in $\mathbb{R}^d$, by a sequence $(\mathbf{f}_n)\subset C_0(\mathbb{R}^d;\mathbb{R}^m)$, write (1.3), with $\mathbf{f}$ replaced by $\mathbf{f}_n$, and use both Proposition 3.2(*ii*) and the dominated convergence theorem, applied to the positive and negative parts of the measures $p_{ij}(t,s,x,\mathrm{d}y)$, to let n tend to $+\infty$. Clearly, formula (1.3) allows us to extend the evolution operator $\mathbf{G}(t,s)$ to $B_b(\mathbb{R}^d;\mathbb{R}^m)$.

Let us now prove that each measure $p_{ij}(t,s,x,\mathrm{d}y)$ is absolutely continuous with respect to the Lebesgue measure. Equivalently, we prove that, for any $(t,x)\in(s,+\infty)\times\mathbb{R}^d$ and any $i,j=1,\dots,m$, the positive and negative parts of $p_{ij}(t,s,x,\mathrm{d}y)$ are absolutely continuous with respect to the Lebesgue measure. For this purpose, we recall that, by the Hahn decomposition theorem (see *e.g.*, [35], Thm. 6.14), for any $(t,x)\in(s,+\infty)\times\mathbb{R}^d$ there exist two Borel sets $P=P(t,s,x)$ and $N=N(t,s,x)$ such that the maps $p^+_{ij}(t,s,x,\mathrm{d}y)$ and $p^-_{ij}(t,s,x,\mathrm{d}y)$, defined, respectively, by $p^+_{ij}(t,s,x,A)=p_{ij}(t,s,x,A\cap P)$ and $p^-_{ij}(t,s,x,A)=-p_{ij}(t,s,x,A\cap N)$ for any Borel set $A\subset\mathbb{R}^d$, are positive measures and $p_{ij}(t,s,x,\mathrm{d}y)=p^+_{ij}(t,s,x,\mathrm{d}y)-p^-_{ij}(t,s,x,\mathrm{d}y)$.

Being rather long, we split the proof into several steps.

Step 1. We claim that, for any $f\in B_b(\mathbb{R}^d)$ and $j=1,\dots,m$, the function $\mathbf{G}(\cdot,s)(f\mathbf{e}_j)$ belongs to $C^{1+\alpha/2,2+\alpha}_{\rm loc}((s,+\infty)\times\mathbb{R}^d;\mathbb{R}^m)$, $D_t\mathbf{G}(\cdot,s)(f\mathbf{e}_j)=\boldsymbol{\mathcal{A}}\mathbf{G}(\cdot,s)(f\mathbf{e}_j)$ in $(s,+\infty)\times\mathbb{R}^d$ and $\|\mathbf{G}(t,s)(f\mathbf{e}_j)\|_\infty\le \mathrm{e}^{(\ell\vee 0)(T-s)}\|f\|_\infty$ for any $t\in(s,T]$ and any $T>s$, where ℓ is the constant in (3.1). Clearly, since $\mathbf{G}(\cdot,s)\mathbf{f}=\sum_{j=1}^m\mathbf{G}(\cdot,s)(f_j\mathbf{e}_j)$ for any $\mathbf{f}\in B_b(\mathbb{R}^d,\mathbb{R}^m)$, the claim implies that the function $\mathbf{G}(\cdot,s)\mathbf{f}$ enjoys the regularity properties in the statement and $\|\mathbf{G}(t,s)\mathbf{f}\|_\infty\le\sqrt{m}\mathrm{e}^{(\ell\vee 0)(T-s)}\|\mathbf{f}\|_\infty$ for any $t\in(s,T]$[4].

To prove the claim, we begin by recalling that the space $B(\mathbb{R}^d)$ of all the real valued Borel functions coincides with the set $B^{\omega_1}(\mathbb{R}^d)=\bigcup_{\eta<\omega_1}B^\eta(\mathbb{R}^d)$, where, throughout this step, we denote by η the ordinal numbers and ω_1 is the first nonnumerable ordinal number. The sets $B^\eta(\mathbb{R}^d)$ are defined as follows: $B^0(\mathbb{R}^d)=C(\mathbb{R}^d)$ and, if $\eta>0$, then the definition of $B^\eta(\mathbb{R}^d)$ depends on the fact that $\eta+1$ is a successor ordinal or not. In the first case, $B^\eta(\mathbb{R}^d)$ is the set of the pointwise limits, everywhere in $\mathbb{R}^d$, of sequences of functions in $B^{\eta-1}(\mathbb{R}^d)$; in the second one, $B^\eta(\mathbb{R}^d)=\bigcup_{\eta_0<\eta}B^{\eta_0}(\mathbb{R}^d)$. Hence, any Borel function belongs to $B^\eta(\mathbb{R}^d)$ for some ordinal less than ω_1. We refer the reader to ([26], Chap. 30) and [36] for further details.

We fix $j\in\{1,\dots,m\}$ and, for any ordinal $\eta<\omega_1$, we denote by $\mathcal{P}_j(\eta)$ the set of all the functions $f\in B^\eta_b(\mathbb{R}^d)$ which satisfy the claim. We use the transfinite induction to prove that $\mathcal{P}_j(\eta)=B^\eta_b(\mathbb{R}^d)$ for any ordinal less than ω_1. In view of Theorem 2.8, $\mathcal{P}_j(0)=B^0_b(\mathbb{R}^d)=C_b(\mathbb{R}^d)$. Fix now an ordinal η and suppose that $\mathcal{P}_j(\beta)=B^\eta_b(\mathbb{R}^d)$ for any ordinal $\beta\le\eta$. We first assume that $\eta+1$ is a successor ordinal. In such a case, f is the pointwise limit, everywhere in $\mathbb{R}^d$, of a sequence $(f_n)\in B^\eta_b(\mathbb{R}^d)$. By assumptions, f is bounded; hence, up to replacing f_n by $f_n\wedge\|f\|_\infty$, which still belongs to $B^\eta_b(\mathbb{R}^d)$, we can assume that $\|f_n\|_\infty\le\|f\|_\infty$ for any $n\in\mathbb{N}$. Since $\mathbf{G}(\cdot,s)(f_n\mathbf{e}_j)\in C^{1+\alpha/2,2+\alpha}_{\rm loc}((s,+\infty)\times\mathbb{R}^d;\mathbb{R}^m)$ for any $n\in\mathbb{N}$, using the interior Schauder estimates in Theorem A.2, as in the proof of Theorem 2.8, we can prove that, up to a subsequence, $\mathbf{G}(t,s)(f_n\mathbf{e}_j)$ converges to a function $\mathbf{v}$ in $C^{1,2}(E;\mathbb{R}^m)$, for any compact set $E\subset(s,+\infty)\times\mathbb{R}^d$. The function $\mathbf{v}$ belongs to $C^{1+\alpha/2,2+\alpha}_{\rm loc}((s,+\infty)\times\mathbb{R}^d;\mathbb{R}^m)$, solves the equation $D_t\mathbf{v}=\boldsymbol{\mathcal{A}}\mathbf{v}$ in $(s,+\infty)\times\mathbb{R}^d$ and satisfies the estimate $\|\mathbf{v}(t,\cdot)\|_\infty\le\mathrm{e}^{\ell(T-s)}\|f\|_\infty$ for any $t\in(s,T]$ and any $T>s$. The representation formula (1.3) reveals that $\mathbf{v}=\mathbf{G}(\cdot,s)(f\mathbf{e}_j)$; hence, $f\in B^{\eta+1}_b(\mathbb{R}^d)$. Suppose now that $\eta+1$ is a limit ordinal. Then, $f\in B^\beta_b(\mathbb{R}^d)$ for some ordinal β less than $\eta+1$. Since $\mathcal{P}_j(\beta)=B^\beta_b(\mathbb{R}^d)$, it is clear that $f\in\mathcal{P}_j(\eta+1)$ and we are done also in this case.

Step 2. Now, we prove that for any $r>0$ there exists a positive constant K_0, depending on M, but being independent of t and $f\in C_c(D_r)$, such that

$$|(\mathbf{G}^{\mathcal{D}}_r(t,s)(f\mathbf{e}_j))(x)|\le|(G^{\mathcal{D}}_{r,0}(t,s)f)(x)|+K_0\sqrt{t-s}\,\|f\|_\infty,\qquad (t,x)\in(s,s+1)\times D_r. \tag{3.5}$$

[4] This estimate will be improved in Corollary 3.4, removing the constant $\sqrt{m}$.

for any $j \in \{1, \dots, m\}$. Here, $G^{\mathcal{D}}_{r,0}(t,s)$ denotes the evolution operator associated with the realization of the operator $\mathcal{A}_0 = \mathrm{Tr}(QD^2)$ in $C_b(D_r)$, with homogeneous Dirichlet boundary conditions.

In the rest of the proof, we denote by K a positive constant, which is independent of $t \in (s, s+1)$, $x \in D_r$ and may vary from line to line.

Fix $f \in C_c(D_r)$, $j \in \{1, \dots, m\}$, set $\mathbf{u} = \mathbf{G}^{\mathcal{D}}_r(\cdot, s)(f\mathbf{e}_j)$ and observe that

$$\mathbf{u}(t,x) = (G^{\mathcal{D}}_{r,0}(t,s)f)(x)\mathbf{e}_j + \int_s^t (\mathbf{G}^{\mathcal{D}}_{r,0}(r,s)\mathbf{g}(r,\cdot))(x)\mathrm{d}r, \qquad (t,x) \in (s,s+1) \times D_r, \tag{3.6}$$

where $(\mathbf{G}^{\mathcal{D}}_{r,0}(t,s)\mathbf{h})_k = G^{\mathcal{D}}_{r,0}(t,s)h_k$, for any $k = 1, \dots, m$ and any vector valued function $\mathbf{h}$, and $\mathbf{g} = \sum_{i=1}^d B_j D_j \mathbf{u}$ $+C\mathbf{u}$. Differentiating both sides of (3.6), taking the norms and using the estimate $\|G^{\mathcal{D}}_{r,0}(t,s)\psi\|_{C^1_b(D_r)} \le K(t-s)^{-1/2}\|\psi\|_\infty$, which holds true for any $\psi \in C_b(D_r)$ and any $t \in (s,s+1)$ (see [13], Thm. 4.6.3), we can estimate

$$\|J_x\mathbf{u}(t,\cdot)\|_{C_b(D_r;\mathbb{R}^m)} \le \frac{c}{\sqrt{t-s}}\|f\|_\infty + K\int_s^t \frac{1}{\sqrt{\sigma - s}}\|J_x\mathbf{u}(\sigma,\cdot)\|_{C_b(D_r;\mathbb{R}^m)}\mathrm{d}\sigma, \qquad t \in (s,s+1).$$

To get this estimate we also took advantage of the fact that $\|\mathbf{u}(t,\cdot)\|_\infty \le K\|f\|_\infty$ for any $t \in (s,s+1)$. The generalized Gronwall lemma (see [18]) shows that $\|J_x\mathbf{u}(t,\cdot)\|_{C_b(D_r;\mathbb{R}^m)} \le K(t-s)^{-1/2}\|f\|_\infty$ for any $t \in (s,s+1)$. We thus deduce that $\|\mathbf{g}(t,\cdot)\|_{C_b(D_r;\mathbb{R}^m)} \le K(t-s)^{-1/2}\|f\|_\infty$ for any $t \in (s,s+1)$ and, from (3.6), estimate (3.5) follows at once.

Step 3. Here, we prove that, for any Borel set $\mathcal{O} \subset \mathbb{R}^d$ with zero Lebesgue measure, any $r > 0$, $j \in \{1, \dots, m\}$ and $t \in (s, s+1)$, it holds that

$$|(\mathbf{G}(t,s)\chi_{\mathcal{O}}\mathbf{e}_j)(x)| \le K\sqrt{t-s}, \qquad t \in (s,s+1),\ \ x \in D_{r/2}. \tag{3.7}$$

This inequality follows once we prove that

$$|(\mathbf{G}(t,s)(f\mathbf{e}_j))(x)| \le (G^{\mathcal{D}}_{r,0}(t,s)|f|)(x) + K\sqrt{t-s}\,\|f\|_\infty, \tag{3.8}$$

for any t, x, f and j as above. Indeed, it is well known that $G^{\mathcal{D}}_{r,0}(t,s)$ admits an integral representation (see [13], Thm. 3.16) and this implies that $G^{\mathcal{D}}_{r,0}(t,s)$ can be extended to any function $f \in B_b(D_r)$ and $G^{\mathcal{D}}_{r,0}(t,s)\chi_{\mathcal{O}} = 0$, if $\mathcal{O}$ has null Lebesgue measure.

We first prove (3.8) for functions $f \in C_b(\mathbb{R}^d)$. For this purpose, we fix $M > 0$ and a function $\vartheta \in C^\infty_c(\mathbb{R}^d)$ such that $\chi_{D_{r/4}} \le \vartheta \le \chi_{D_{r/2}}$. By the proof of Theorem 2.8, $\mathbf{G}(\cdot,s)(f\mathbf{e}_j)$ is the local uniform limit in $(s,+\infty) \times \mathbb{R}^d$ of the unique classical solution $\mathbf{u}_n$ to the Cauchy problem (2.5), with $\mathbf{f}$ replaced by $\vartheta f\mathbf{e}_j$, Moreover,

$$\|\mathbf{G}^{\mathcal{D}}_M(t,\sigma)\mathbf{g}_n(\sigma,\cdot)\|_{L^\infty(D_r)} \le K\|\mathbf{g}_n(\sigma,\cdot)\|_{L^\infty(D_{r/2})} \le K_r\|f\|_\infty(1 + (\sigma - s)^{-\frac{1}{2}})$$

for any $\sigma \in (s,t)$. Letting n tend to $+\infty$, estimate (3.8) follows recalling that $|G^{\mathcal{D}}_{r,0}(t,s)(\vartheta f)| \le G^{\mathcal{D}}_{r,0}(t,s)|\vartheta f| \le G^{\mathcal{D}}_{r,0}(t,s)|f|$ in D_r.

By transfinite induction, arguing as in Step 1, we extend (3.8) to any $f \in B_b(\mathbb{R}^d)$. To make the induction work, it suffices to observe that, if $f \in B_b(\mathbb{R}^d)$ is the pointwise limit everywhere in $\mathbb{R}^d$ of a sequence $(f_n) \subset B_b(\mathbb{R}^d)$ of functions which satisfy (3.8) and $\|f_n\|_\infty \le \|f\|_\infty$ for any $n \in \mathbb{N}$, then, f satisfies (3.8) as well. This can be seen, writing (3.8) with f replaced by f_n, taking (1.3) into account and letting $n \to +\infty$.

Step 4. We can now complete the proof. We fix $i,j \in \{1, \dots, m\}$, $t_0 > s$, $x_0 \in \mathbb{R}^d$ and a Borel set A with null Lebesgue measure. Then, estimate (3.7) shows that $|\mathbf{G}(t,s)(\chi_{A\cap R}\mathbf{e}_j)| \le K\sqrt{t-s}$ in $D_{r/2}$ for any $r > 0$, $t \in (s,s+1)$ and some $K = K_r$, where $R = P$ or $R = N$. By the arbitrariness of r, it thus follows that $\mathbf{G}(t,s)(\chi_{A\cap R}\mathbf{e}_j)$ vanishes, locally uniformly in $\mathbb{R}^d$, as $t \to s^+$. Step 1 shows that the function $\mathbf{v}$, defined by $\mathbf{v}(s,\cdot) = \mathbf{0}$ and $\mathbf{v}(t,\cdot) = \mathbf{G}(t,s)(\chi_{A\cap R}\mathbf{e}_j)$, if $t > s$, belongs to $C^{1+\alpha/2,2+\alpha}_{\mathrm{loc}}((s,+\infty) \times \mathbb{R}^d;\mathbb{R}^m) \cap C([s,+\infty) \times \mathbb{R}^d;\mathbb{R}^m)$ and is bounded in each strip $[s,T] \times \mathbb{R}^d$. Moreover, $D_t\mathbf{v} = \boldsymbol{\mathcal{A}}\mathbf{v}$ in $(s,+\infty) \times \mathbb{R}^d$ and $\mathbf{v}(s,\cdot) = \mathbf{0}$ in $\mathbb{R}^d$. By Proposition 2.7, $\mathbf{v}$ identically vanishes in $(s,+\infty) \times \mathbb{R}^d$. Thus, we conclude that $(\mathbf{G}(\cdot,s)(\chi_{A\cap R}\mathbf{e}_j))(x_0) = \mathbf{0}$, which implies that $0 = (\mathbf{G}(t_0,s)(\chi_{A\cap P}\mathbf{e}_j))_i(x_0) = p^+_{ij}(t_0,s,x_0,A)$ and, similarly, $p^-_{ij}(t_0,s,x_0,A) = 0$. $\square$

Corollary 3.4. *The following properties are satisfied.*

(i) *Estimate* (3.1) *is satisfied by any* $\mathbf{f} \in B_b(\mathbb{R}^d;\mathbb{R}^m)$.
(ii) *Proposition* 3.2(i) *holds true for any bounded sequence* $(\mathbf{f}_n)$ *of Borel functions which converges pointwise* (*almost everywhere in* $\mathbb{R}^d$) *to a Borel measurable function* $\mathbf{f}$.

Proof. (i) Fix $\mathbf{f} \in B_b(\mathbb{R}^d;\mathbb{R}^m)$ and let the sequence $(\mathbf{f}_n) \in C_b(\mathbb{R}^d;\mathbb{R}^m)$ converge to $\mathbf{f}$ almost everywhere in $\mathbb{R}^d$ and satisfy the estimate $\|\mathbf{f}_n\|_\infty \le \|\mathbf{f}\|_\infty$. Since the measures $p_{ij}(t,s,x,\mathrm{d}y)$ are absolutely continuous with respect to the Lebesgue measure for any $t > s \in I$ and any $x \in \mathbb{R}^d$, by formula (1.3) and the dominated convergence theorem, $\mathbf{G}(t,s)\mathbf{f}_n$ converges to $\mathbf{G}(t,s)\mathbf{f}$ pointwise everywhere in $\mathbb{R}^d$, as $n \to +\infty$. Using (3.1), we can estimate $|\mathbf{G}(\cdot,s)\mathbf{f}_n| \le \mathrm{e}^{(\ell\vee 0)(T-s)}\|\mathbf{f}_n\|_\infty \le \mathrm{e}^{(\ell\vee 0)(T-s)}\|\mathbf{f}\|_\infty$ in $[s,T]\times\mathbb{R}^d$ and, letting n tend to $+\infty$, we conclude the proof.

(ii) Fix $s \in I$ and $(\mathbf{f}_n)$, $\mathbf{f}$ as in the statement. For any $\varepsilon > 0$, the functions $\mathbf{G}(s+\varepsilon,s)\mathbf{f}_n$ and $\mathbf{G}(s+\varepsilon,s)\mathbf{f}$ are bounded and continuous in $\mathbb{R}^d$, thanks to property (i) and Theorem 3.3. Moreover, the proof of property (i) shows that $\mathbf{G}(s+\varepsilon,s)\mathbf{f}_n$ converges pointwise in $\mathbb{R}^d$ to $\mathbf{G}(s+\varepsilon,s)\mathbf{f}$ as $n \to +\infty$. Splitting $\mathbf{G}(t,s)\mathbf{f}_n = \mathbf{G}(t,s+\varepsilon)\mathbf{G}(s+\varepsilon)\mathbf{f}_n$ for any $n \in \mathbb{N}$ and using Proposition 3.2(i) we conclude the proof. □

4. Compactness of $\mathbf{G}(t,s)$ in $C_b(\mathbb{R}^d;\mathbb{R}^m)$

To begin with, we show that the compactness of the evolution operator $\mathbf{G}(t,s)$ in $C_b(\mathbb{R}^d;\mathbb{R}^m)$ is equivalent to the tightness of the total variation of the measures $\{p_{ij}(t,s,x,\mathrm{d}y) : x \in \mathbb{R}^d\}$ introduced in Theorem 3.3.

Theorem 4.1. *Let us assume that either Hypotheses* 2.2 *or Hypotheses* 2.3 *are satisfied and fix* $b > a \in I$. *The evolution operator* $\mathbf{G}(t,s)$ *is compact in* $C_b(\mathbb{R}^d;\mathbb{R}^m)$ *for any* $a \le s < t \le b$ *if and only if the families* $\{|p_{ij}|(t,s,x,\mathrm{d}y) : x \in \mathbb{R}^d\}$ *are tight for any* s,t *as above and* $i,j \in \{1,\dots,m\}$.

Proof. Let us suppose that $\mathbf{G}(t,s)$ is compact in $C_b(\mathbb{R}^d;\mathbb{R}^m)$ for any $a \le s < t \le b$. We fix $i,j \in \{1,\dots,m\}$, $a \le s < t \le b$, $n \in \mathbb{N}$, $x \in \mathbb{R}^d$ and recall that

$$\begin{aligned}|p_{ij}|(t,s,x,\mathbb{R}^d\setminus D_n) &= \sup\left\{\int_{\mathbb{R}^d} f(y)p_{ij}(t,s,x,\mathrm{d}y) :\ f \in C_c(\mathbb{R}^d\setminus D_n),\ \|f\|_\infty \le 1\right\} \\ &\le \sup\{\|\mathbf{G}(t,s)(f\mathbf{e}_j)\|_\infty :\ f \in C_c(\mathbb{R}^d\setminus D_n),\ \|f\|_\infty \le 1\},\end{aligned} \tag{4.1}$$

(see [1], Prop. 1.4.7). Then, for any $n \in \mathbb{N}$, there exist functions $f_n \in C_c(\mathbb{R}^d\setminus D_n)$ with $\|f_n\|_\infty \le 1$ such that

$$\sup_{x\in\mathbb{R}^d}|p_{ij}|(t,s,x,\mathbb{R}^d\setminus D_n) \le \|\mathbf{G}(t,s)(f_n\mathbf{e}_j)\|_\infty + \frac{1}{n}\cdot \tag{4.2}$$

Clearly, $f_n\mathbf{e}_j$ vanishes pointwise in $\mathbb{R}^d$ as $n \to +\infty$ and $\|f_n\mathbf{e}_j\|_\infty \le 1$ for any $n \in \mathbb{N}$. By compactness and Proposition 3.2, we can extract a subsequence $(f_{n_h}\mathbf{e}_j)$ such that $\mathbf{G}(t,s)(f_{n_h}\mathbf{e}_j)$ vanishes uniformly in $\mathbb{R}^d$, as $h \to +\infty$. Now, writing (4.2) with n replaced by n_h and letting $h \to +\infty$ the tightness of the family $\{|p_{ij}|(t,s,x,\mathrm{d}y) :\ x \in \mathbb{R}^d\}$ follows.

Vice versa, let us suppose that the families $\{|p_{ij}|(t,s,x,\mathrm{d}y) : x \in \mathbb{R}^d\}$ are tight for any $a \le s < t \le b$ and $i,j \in \{1,\dots,m\}$. We fix $a \le s < r < t \le b$ and consider the operators $\mathbf{R}_n := \mathbf{G}(t,r)(\chi_{D_n}\mathbf{G}(r,s))$ in $C_b(\mathbb{R}^d;\mathbb{R}^m)$. Since $\mathbf{G}(t,r)$ is strong Feller (see Thm. 3.3), each operator $\mathbf{R}_n$ is bounded in $C_b(\mathbb{R}^d;\mathbb{R}^m)$. We claim that $\mathbf{R}_n$ is compact in $C_b(\mathbb{R}^d;\mathbb{R}^m)$ for any $n \in \mathbb{N}$. To this aim, let $(\mathbf{f}_k)$ be a bounded sequence in $C_b(\mathbb{R}^d;\mathbb{R}^m)$. From the interior Schauder estimates in Theorem A.2 it follows that the sequence $(\mathbf{G}(r,s)\mathbf{f}_k)$ is bounded in $C^{2+\alpha}(D_n;\mathbb{R}^m)$. Hence, there exists a subsequence $(\mathbf{G}(r,s)\mathbf{f}_{k_j})$ converging uniformly in D_n to some function $\mathbf{g}$ as $j \to +\infty$. As a byproduct, $\chi_{D_n}\mathbf{G}(r,s)\mathbf{f}_{k_j}$ converges to $\chi_{D_n}\mathbf{g}$ uniformly in $\mathbb{R}^d$ as $j \to +\infty$. Since the estimate (3.1) holds true also for bounded Borel functions (see Cor. 3.4(i)), we conclude that $\mathbf{R}_n\mathbf{f}_{k_j}$ converges uniformly in $\mathbb{R}^d$ to $\mathbf{G}(t,r)(\chi_{D_n}\mathbf{g})$ as $j \to +\infty$. Hence, $\mathbf{R}_n$ is compact in $C_b(\mathbb{R}^d;\mathbb{R}^m)$.

To complete the proof, we show that $\mathbf{R}_n$ converges to $\mathbf{G}(t,s)$ as $n\to+\infty$ in $\mathcal{L}(C_b(\mathbb{R}^d;\mathbb{R}^m))$. For this purpose we fix $\mathbf{f}\in C_b(\mathbb{R}^d,\mathbb{R}^m)$, $i\in\{1,\dots,m\}$. Using formula (1.3) we can write

$$(\mathbf{G}(t,s)\mathbf{f})_i(x)-(\mathbf{R}_n\mathbf{f})_i(x)=(\mathbf{G}(t,r)(\chi_{\mathbb{R}^d\setminus D_n}\mathbf{G}(r,s)\mathbf{f})_i(x)=\sum_{k=1}^m\int_{\mathbb{R}^d\setminus D_n}(\mathbf{G}(r,s)\mathbf{f}(y))_k p_{ik}(t,r,x,\mathrm{d}y)$$

for any $x\in\mathbb{R}^d$. Hence, taking (3.1) into account, we can estimate

$$\begin{aligned}\|\mathbf{G}(t,s)\mathbf{f}-\mathbf{R}_n\mathbf{f}\|_\infty\leq&\sum_{i,k=1}^m\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d\setminus D_n}|(\mathbf{G}(r,s)\mathbf{f}(y))_k||p_{ik}|(t,r,x,\mathrm{d}y)\\ \leq&\mathrm{e}^{(\ell\vee 0)(b-s)}\|\mathbf{f}\|_\infty\sum_{i,k=1}^m\sup_{x\in\mathbb{R}^d}|p_{ik}|(t,r,x,\mathbb{R}^d\setminus D_n).\end{aligned}$$

Letting n tend to $+\infty$ and using the tightness of the family $\{|p_{ij}|(t,r,x,\mathrm{d}y):x\in\mathbb{R}^d\}$, we conclude that $\mathbf{R}_n$ converges to $\mathbf{G}(t,s)$ in $\mathcal{L}(C_b(\mathbb{R}^d;\mathbb{R}^m)$. ☐

In the following theorem we provide sufficient conditions for the compactness of $\mathbf{G}(t,s)$ when Hypotheses 2.3 are satisfied. Actually, these assumptions guarantee the compactness of $G(t,s)$ in $C_b(\mathbb{R}^d)$, which has been already studied in [2, 31] extending the results in the autonomous case proved in [33].

Theorem 4.2. *Let $J\subset I$ be a bounded interval. Under Hypotheses 2.3, if $G(t,s)$ is compact in $C_b(\mathbb{R}^d)$ for every $(t,s)\in\Sigma_J:=\{(t,s)\in J\times J:\ t>s\}$, then $\mathbf{G}(t,s)$ is compact in $C_b(\mathbb{R}^d;\mathbb{R}^m)$ for every $(t,s)\in\Sigma_J$. In particular, if there exist a function $W\in C^2(\mathbb{R}^d)$ such that $\lim_{|x|\to\infty}W(x)=+\infty$, a number $R>0$ and a convex increasing function $g:[0,+\infty)\to\mathbb{R}$ with $1/g\in L^1((a,+\infty))$ for large a and $\tilde{\mathcal{A}}W\leq -g\circ W$ in $I\times(\mathbb{R}^d\setminus D_R)$, then $\mathbf{G}(t,s)$ is compact in $C_b(\mathbb{R}^d;\mathbb{R}^m)$ for any $t>s\in I$.*

Proof. In view of Theorem 4.1, we show that the family $\{|p_{ij}|(t,s,x,\mathrm{d}y):x\in\mathbb{R}^d\}$ is tight for any $(t,s)\in\Sigma_J$ and $i,j\in\{1,\dots,m\}$. From (4.1), (2.7) and the positivity of the evolution operator $G(t,s)$, it follows that

$$\begin{aligned}|p_{ij}|(t,s,x,\mathbb{R}^d\setminus D_n)\leq&\mathrm{e}^{H_J(t-s)}\sup\{((G(t,s)|f|^2)(x))^{\frac12}:\ f\in C_c(\mathbb{R}^d\setminus D_n),\ \|f\|_\infty\leq 1\}\\ \leq&\mathrm{e}^{H_J(t-s)}((G(t,s)\chi_{\mathbb{R}^d\setminus D_n})(x))^{\frac12}=\mathrm{e}^{H_J(t-s)}(g_{t,s}(x,\mathbb{R}^d\setminus D_n))^{\frac12}\end{aligned}$$

for any $\mathbb{R}^d$, any $(t,s)\in\Sigma_J$ and $i,j\in\{1,\dots,m\}$, where $g_{t,s}(\cdot,\mathrm{d}y)$ are the transition kernels associated with the evolution operator $G(t,s)$. The assertion now follows from ([2], Prop. 4.2), which shows that the compactness of the scalar evolution operator $G(t,s)$ in $C_b(\mathbb{R}^d)$, for any $(t,s)\in\Sigma_J$, is equivalent to the tightness of the family $\{g_{t,s}(x,\mathrm{d}y):x\in\mathbb{R}^d\}$ for any $(t,s)\in\Sigma_J$.

The last assertion follows from ([31], Thm. 3.3) ☐

Example 4.3. Let $\boldsymbol{\mathcal{A}}$ be as in Example 2.6. Using (2.1) we can estimate

$$\begin{aligned}(\tilde{\mathcal{A}}\varphi)(t,x)\leq&2\|q\|_\infty\mathrm{Tr}(Q_0)(1+|x|^2)^k-\inf_{t\in I}b(t)(1+|x|^2)^{p+1}\\ \leq&-\frac12\inf_{t\in I}b(t)(1+|x|^2)^{p+1}+K:=-g(\varphi(x)),\end{aligned}$$

for any $t\in I$, any $x\in\mathbb{R}^d\setminus\partial D_1$ and some positive constant K. Then, the assumptions of Theorem 4.2 are satisfied, with $W=\varphi$ and g as above. We thus conclude that $\mathbf{G}(t,s)$ is compact in $C_b(\mathbb{R}^d;\mathbb{R}^m)$.

5. Uniform gradient estimates

In this section, we prove a (weighted) uniform gradient estimate satisfied by the function $\mathbf{G}(t,s)\mathbf{f}$, which, besides, its own interest, leads to some remarkable consequences that we illustrate in the next section. We assume that the operator $\boldsymbol{\mathcal{A}}$ is given by (1.1) with $B_i = b_i I_m + \tilde{B}_i$ for any $i = 1,\dots,d$. As usually, we set $\mathbf{b} = (b_1,\dots,b_d)$. We further assume the following additional assumptions.

Hypotheses 5.1.

(i) *The coefficients Q_{ij}, b_j and the entries of the matrices B_i and C belong to $C^{0,1+\alpha}_{\rm loc}(\overline{J}\times\mathbb{R}^d)$ for any $i,j = 1,\dots,d$, some $\alpha\in(0,1)$ and some bounded interval J with $\overline{J}\subset I$; further, $\langle \mathbf{b}(t,x),x\rangle \le b_0(t,x)|x|$ for any $t\in J$, $x\in\mathbb{R}^d$ and some negative function b_0;*

(ii) *there exists a $(m\times m)$-matrix-valued function M such that $M_{ij} = M_{ji}\in C^{1,2+\alpha}_{\rm loc}(J\times\mathbb{R}^d)$ for any $i,j = 1,\dots,m$ and both $\inf_{J\times\mathbb{R}^d}\lambda_M$ and $\inf_{J\times\mathbb{R}^d}\lambda_{M^{-1}QM^{-1}}$ are positive;*

(iii) *the function $2\Lambda_C + \Lambda_{\mathcal{H}+\mathcal{H}^T}$ is bounded from above in $J\times\mathbb{R}^d$, where*

$$\mathcal{H} := M(J_x b)^T M^{-1} - \sum_{j=1}^d b_j (D_j M)M^{-1} - \sum_{i,j=1}^d Q_{ij}(D_{ij}M)M^{-1},$$

(iv) *there exist positive functions $\psi_h : J\times\mathbb{R}^d\to\mathbb{R}$ $(h = 1,\dots,6)$ and constants K_1, K_2, K_3 and K_4 such that*

$$\begin{aligned}
&|\tilde{B}_i(t,x)| \le \psi_1(t,x), && |D_k\tilde{B}_i(t,x)| \le \psi_2(t,x), && |D_kC(t,x)| \le \psi_3(t,x),\\
&|M^{-1}D_tMM^{-1}(t,x)| \le \psi_4(t,x), && |D_kM(t,x)| \le \psi_5(t,x), && |D_kQ(t,x)| \le \psi_6(t,x),
\end{aligned}$$

in $J\times\mathbb{R}^d$, for any $i,k = 1,\dots,d$ and

$$\sup_{(t,x)\in J\times\mathbb{R}^d}\frac{(\psi_1(t,x))^2}{\lambda_{M^{-1}QM^{-1}}(t,x)(\lambda_M(t,x))^2} < +\infty, \tag{5.1}$$

$$\sup_{(t,x)\in J\times\mathbb{R}^d}\frac{(\Lambda_Q(t,x))^2}{(1+|x|^2)\lambda_{M^{-1}QM^{-1}}(t,x)\lambda_Q(t,x)} < +\infty \tag{5.2}$$

$$\frac{(\Lambda_Q(t,x))^2}{1+|x|^4} \le K_1|2\Lambda_C(t,x) + \Lambda_{\mathcal{H}+\mathcal{H}^T}(t,x)| + K_2, \tag{5.3}$$

$$(\Lambda_M(t,x))^2(\psi_3(t,x))^2 \le K_3|2\Lambda_C(t,x) + \Lambda_{\mathcal{H}+\mathcal{H}^T}(t,x)| + K_4, \tag{5.4}$$

for any $(t,x)\in J\times\mathbb{R}^d$. Further,

$$\lim_{|x|\to+\infty}\sup_{t\in J}\frac{(\Lambda_Q(t,x)\psi_5(t,x))^2 + (\Lambda_M(t,x)\psi_6(t,x))^2 + \lambda_Q(t,x)(\psi_1(t,x))^2}{\lambda_Q(t,x)(\lambda_M(t,x))^2|2\Lambda_C(t,x) + \Lambda_{\mathcal{H}+\mathcal{H}^T}(t,x)|} = 0, \tag{5.5}$$

$$\lim_{|x|\to+\infty}\sup_{t\in J}\frac{\Lambda_M(t,x)\psi_2(t,x) + \lambda_M(t,x)\psi_4(t,x)}{\lambda_M(t,x)|2\Lambda_C(t,x) + \Lambda_{\mathcal{H}+\mathcal{H}^T}(t,x)|} = 0, \tag{5.6}$$

$$\lim_{|x|\to+\infty}\inf_{t\in J}\frac{\Lambda_Q(t,x)\psi_5(t,x)}{b_0(t,x)} = 0. \tag{5.7}$$

Example 5.2. Let $\boldsymbol{\mathcal{A}}$ be the operator in (1.1), with the potential C which satisfies the following conditions:

- C is a symmetric matrix-valued function with entries in $C^{\alpha/2,\alpha}_{\rm loc}(I\times\mathbb{R}^d)\cap C^{0,1+\alpha}_{\rm loc}(I\times\mathbb{R}^d)$;

- $\sup_{I\times\mathbb{R}^d}\varLambda_C<+\infty$ and $|D_iC(t,x)|\le K(1+|x|^2)^\tau$ for any $(t,x)\in I\times\mathbb{R}^d$, $i=1,\dots,d$ and some $K>0$ and $\tau\ge 0$,

and with Q_{ij} and B_i $(i,j=1,\dots,d)$ as in Example 2.6, *i.e.*, $Q_{ij}(t,x)=q(t)(1+|x|^2)^kQ_0$, $B_i(t,x)=-b(t)x_i(1+|x|^2)^pI_m+\tilde{b}(t)(1+|x|^2)^r\tilde{B}_{0,i}$ for any $i=1,\dots,d$, any $(t,x)\in I\times\mathbb{R}^d$, where

- $q\in C^{\alpha/2}_{\rm loc}(I)\cap C_b(I)$ has positive infimum and Q_0 is a constant positive definite matrix;
- the functions $b,\tilde{b}$ and h belong to $C^{\alpha/2}_{\rm loc}(I)$. Moreover, b has positive infimum;
- $\tilde{B}_{0,i}$ $(i=1,\dots,d)$ and $\hat{C}$ are constant, with $\hat{C}$ positive definite;

Take $M(t,x)=\mu(t)(1+|x|^2)^sI_m$ for any $(t,x)\in I\times\mathbb{R}^d$, some constant $0<s<1/2$ and some positive function $\mu\in C^1(I)$. Further, assume that the nonnegative parameters k, p, r, τ satisfy the following conditions:

$$\text{(a) } k\ge 2(r\vee s),\qquad \text{(b) } 2k-2\le p,\qquad \text{(c) } 2s+2\tau\le p,\qquad \text{(d) } 2r<2s+p,\qquad \text{(e) } k+s<p+1. \tag{5.8}$$

The smoothness assumptions in Hypotheses 2.1, 2.3 and 5.1 are obviously satisfied. To check the other conditions in Hypothesis 5.1, we need to compute b_0, $\varLambda_M$, $\varLambda_{M^2}$, λ_Q, $\varLambda_Q$ and $\lambda_{M^{-1}QM^{-1}}$. As it is immediately seen,

$$\begin{aligned}
&b_0(t,x)=-b(t)(1+|x|^2)^p|x|, && \varLambda_{M^2}(t,x)=(\varLambda_M(t,x))^2=(\mu(t))^2(1+|x|^2)^{2s},\\
&\lambda_Q(t,x)=\lambda_{Q_0}q(t)(1+|x|^2)^k, && \varLambda_Q(t,x)=\varLambda_{Q_0}q(t)(1+|x|^2)^k,\\
&\lambda_{M^{-1}QM^{-1}}(t,x)=\lambda_{Q_0}\frac{q(t)}{(\mu(t))^2}(1+|x|^2)^{k-2s},
\end{aligned}$$

for any $(t,x)\in I\times\mathbb{R}^d$. In particular, for any bounded interval $J\subset I$, the infimum over $J\times\mathbb{R}^d$ of the function $\lambda_{M^{-1}QM^{-1}}$ is positive due to conditions on μ, q and (5.8)(a). Finally, the matrix $\mathcal{H}(t,x)$ is symmetric at any $(t,x)\in I\times\mathbb{R}^d$ and

$$\begin{aligned}
\mathcal{H}(t,x)=&-b(t)(1+|x|^2)^pI_m-2pb(t)(1+|x|^2)^{p-1}x\otimes x+2sb(t)(1+|x|^2)^{p-1}|x|^2I_m\\
&-2sq(t)(1+|x|^2)^{k-2}\big(\mathrm{Tr}(Q_0)(1+|x|^2)+2(s-1)\langle Q_0x,x\rangle\big)I_m.
\end{aligned}$$

Hence, $\varLambda_{\mathcal{H}+\mathcal{H}^T}(t,x)=2\varLambda_{\mathcal{H}}(t,x)$ and using the fact that b has positive infimum, that the second and fourth matrices in the previous formula are nonpositive definite, and taking the conditions $s>0$ and (5.8)(e) into account, we conclude that

$$\varLambda_{\mathcal{H}}(t,x)\le -b(t)(1+|x|^2)^{p-1}(1+|x|^2-2s|x|^2)\le -(1-2s)\inf_{t\in I}b(t)(1+|x|^2)^p, \tag{5.9}$$

for any $(t,x)\in I\times\mathbb{R}^d$, where b_0 denotes the infimum of b. Hence, the function $2\varLambda_C+\varLambda_{\mathcal{H}+\mathcal{H}^T}$ is bounded from above in $I\times\mathbb{R}^d$ and the boundness from above of $\varLambda_C$ implies that $|2\varLambda_C+\varLambda_{\mathcal{H}+\mathcal{H}^T}|\ge K|x|^{2p}$ as $|x|\to+\infty$. Moreover, the functions ψ_j satisfy the following conditions: $\psi_1(t,x)=O(|x|^{2r})$, $\psi_2(t,x)=O(|x|^{2r-1})$, $\psi_3(t,x)=O(|x|^{2\tau})$, $\psi_4(t,x)=O(|x|^{-2s})$, $\psi_5(t,x)=O(|x|^{2s-1})$ and $\psi_6(t,x)=O(|x|^{2k-1})$, as $|x|\to+\infty$, uniformly with respect to $t\in J\subset I$, where J is an arbitrary bounded interval as above. Looking at the asymptotic behaviour as $|x|\to+\infty$, (5.1)–(5.7) lead to the following conditions on the parameters k, r, s, τ and p:

$$\begin{cases}
2r-k\le 0, & \text{from (5.1)}, \qquad (k+2s-1)\vee(2r)-2s-p<0, \quad \text{from (5.5)},\\
2s-1\le 0, & \text{from (5.2)}, \qquad (2r-1)\vee(-2s)-2p<0, \quad \text{from (5.6)},\\
2k-2-p\le 0, & \text{from (5.3)}, \qquad k+s-p-1<0, \quad \text{from (5.7)}.\\
2s+2\tau-p\le 0, & \text{from (5.4)},
\end{cases} \tag{5.10}$$

All the conditions in (5.10) are easily satisfied due to (5.8)(a)−(e) and the choice of s.

As it is easily seen, also Hypothesis 2.3 are satisfied with $\sigma = 1/2$ and $\varphi(x) = 1+|x|^2$ for any $x \in \mathbb{R}^d$. Indeed, the operator $\tilde{\mathcal{A}}$ is defined by $\tilde{\mathcal{A}} = q(1+|x|^2)^k \mathrm{Tr}(Q_0 D^2) - b(t)(1+|x|^2)^p \langle x, \nabla_x \rangle$. Taking condition (5.8)(e) into account, it can be easily checked that $\tilde{\mathcal{A}}\varphi$ diverges to $-\infty$ as $|x| \to +\infty$, uniformly with respect to $t \in I$.

We stress that also the case $s = 1/2$ can be considered. In this situation, from the first part of (5.9) we can infer that $\Lambda_{\mathcal{H}}(t,x) \le -K(1+|x|^2)^{p-1}$ for some positive constant K and the conditions to impose on the parameters k, p, r and τ are (5.8)(a) and

$$(b')\ 2k \le p+1, \qquad (c')\ 2\tau \le p-2, \qquad (d')\ 2r < p-1, \qquad (e')\ k < p. \tag{5.11}$$

We finally note that the conditions (5.8)(c) and (5.11)(c′) can be skipped if the potential C is constant in $I \times \mathbb{R}^d$.

Remark 5.3. We stress that (5.2) forces M to grow no faster than linearly as $|x| \to +\infty$. This condition might seem a bit strong but, already in the classical scalar case when the coefficients are bounded, in general the gradient of $G(t,s)f$ does not vanish as $|x| \to +\infty$. For instance, consider the one-dimensional autonomous operator $\mathcal{A} = u'' + u$ and take $f(x) = \sin(x)$ for any $x \in \mathbb{R}$. Then, $G(t,s)f = f$ for any $t \ge s \in \mathbb{R}$ and, clearly, $D_x T(t)f$ does not vanish at infinity.

Theorem 5.4. *Under Hypotheses 2.3 and 5.1, the function $M(J_x\mathbf{G}(\cdot,s)\mathbf{f})^T$ is bounded and continuous in $(J \cap (s,+\infty)) \times \mathbb{R}^d$ for any $s \in J$ and there exists a positive constant K_J such that*

$$\sqrt{t-s}\,\|M(t,\cdot)(J_x\mathbf{G}(t,s)\mathbf{f})^T\|_\infty \le K_J\|\mathbf{f}\|_\infty, \qquad t \in J\cap(s,+\infty),\ \ \mathbf{f} \in C_b(\mathbb{R}^d;\mathbb{R}^m). \tag{5.12}$$

Proof. To simplify the notation we set $\mathbf{u} := \mathbf{G}(\cdot,s)\mathbf{f}$, $\mathbf{u}_n = \mathbf{G}_n^{\mathcal{D}}(t,s)\mathbf{f}$, $\mathcal{F}_n := \sum_{j=1}^m |M\nabla_x u_{n,j}|^2$ and $\mathcal{G}_n := \sum_{i=1}^d \sum_{j=1}^m |M\nabla_x D_i u_{n,j}|^2$. Moreover, throughout the proof, we denote by K a positive constant, which may vary from line to line, may depend on s and $T = \sup J$, but is independent of n. Let us consider the function $v_n = |\mathbf{u}_n|^2 + a(\cdot - s)\vartheta_n^2\mathcal{F}_n$, where a is positive parameter to be fixed later on, $\vartheta_n(x) = \vartheta(|x|/n)$, for any $x \in \mathbb{R}^d$, and $\vartheta \in C_c^\infty(\mathbb{R})$ satisfies the condition $\chi_{(-\infty,1/2]} \le \vartheta \le \chi_{(-\infty,1]}$. From now on, we do not stress the dependence on n of the component of $\mathbf{u}_n$. Moreover, to ease the notation, we set $\phi_s(t) := t - s$. The results in ([13], Thms. 9.7 and 9.11) and straightforward computations show that v_n is smooth, vanishes on $(s,T] \times \partial D_n$, $v_n(s,\cdot) = |\mathbf{f}|^2$ and $D_t v_n - \tilde{\mathcal{A}} v_n = g_n$ in $(s,T] \times D_n$, where $g_n = \sum_{i=1}^5 g_{i,n}$ with

$$g_{1,n} = -2\sum_{j=1}^m |\sqrt{Q}\nabla_x u_j|^2 - 2a\phi_s\vartheta_n^2 \sum_{i,k=1}^d \sum_{j=1}^m Q_{ik}\langle M\nabla_x D_i u_j, M\nabla_x D_k u_j\rangle,$$

$$\begin{aligned} g_{2,n} = {} & a\phi_s\vartheta_n^2 \sum_{j=1}^m \langle(\mathcal{H}+\mathcal{H}^T)M\nabla_x u_j, M\nabla_x u_j\rangle + 2a\phi_s\vartheta_n^2 \sum_{j,k=1}^m C_{jk}\langle M\nabla_x u_k, M\nabla_x u_j\rangle \\ & - 2a\phi_s\vartheta_n^2 \sum_{i,k=1}^d \sum_{j=1}^m Q_{ik}\langle D_i M\nabla_x u_j, D_k M\nabla_x u_j\rangle + 2a\phi_s\vartheta_n^2 \sum_{j=1}^m \langle D_t M\nabla_x u_j, \nabla_x u_j\rangle, \end{aligned}$$

$$\begin{aligned} g_{3,n} = {} & -2a\phi_s\langle Q\nabla\vartheta_n, \nabla\vartheta_n\rangle\mathcal{F}_n - 2a\phi_s\vartheta_n(\tilde{\mathcal{A}}\vartheta_n)\mathcal{F}_n \\ & - 8a\phi_s\vartheta_n \sum_{k=1}^d \sum_{j=1}^m (Q\nabla\vartheta_n)_k\big(\langle M\nabla_x D_k u_j, M\nabla_x u_j\rangle + \langle D_k M\nabla_x u_j, M\nabla_x u_j\rangle\big), \end{aligned}$$

$$g_{4,n} = 2a\phi_s\vartheta_n^2 \sum_{i,k=1}^d \sum_{j=1}^m D_{ik}^2 u_j\langle M\nabla_x Q_{ik}, M\nabla_x u_j\rangle - 4a\phi_s\vartheta_n^2 \sum_{i,k=1}^d \sum_{j=1}^m Q_{ik}\langle D_i M\nabla_x D_k u_j, M\nabla_x u_j\rangle$$

$$
- 4a\phi_s\vartheta_n^2 \sum_{i,k=1}^{d}\sum_{j=1}^{m} Q_{ik}\langle D_iM\nabla_x u_j, M\nabla_x D_k u_j\rangle,
$$

$$
\begin{aligned}
g_{5,n} =& 2\sum_{i=1}^{d}\langle \mathbf{u}_n, \tilde{B}_iD_i\mathbf{u}_n\rangle + 2\langle C\mathbf{u}_n,\mathbf{u}_n\rangle + a\vartheta_n^2\mathcal{F}_n + 2a\phi_s\vartheta_n^2\sum_{j,k=1}^{m} u_k\langle M\nabla_x C_{jk}, M\nabla_x u_j\rangle\\
&+2a\phi_s\vartheta_n^2\sum_{i=1}^{d}\sum_{j,k=1}^{m}(\tilde{B}_i)_{jk}\langle M\nabla_x D_iu_k, M\nabla_x u_j\rangle + 2a\phi_s\vartheta_n^2\sum_{i=1}^{d}\sum_{j,k=1}^{m} D_iu_k\langle M\nabla_x(\tilde{B}_i)_{jk}, M\nabla_x u_j\rangle,
\end{aligned}
$$

and $\tilde{\mathcal{A}}$ is the operator in Hypothesis 2.3(iii).

Our aim consists in proving that $g_n \le Kv_n$ in $[s,T]\times\mathbb{R}^d$. This estimate and the classical maximum principle imply that $v_n \le K\|\mathbf{f}\|_\infty$, *i.e.*,

$$
\|\mathbf{G}_n^{\mathcal{D}}(t,s)\mathbf{f}\|_\infty + \sqrt{t-s}\|\vartheta_n M(J_x\mathbf{G}_n^{\mathcal{D}}(t,s)\mathbf{f})^T\|_\infty \le K\|\mathbf{f}\|_\infty, \qquad t\in(s,T]. \tag{5.13}
$$

Taking advantage of the proof of Theorem 2.8, which shows that $\mathbf{G}_n^{\mathcal{D}}(t,s)\mathbf{f}$ converges to $\mathbf{G}(t,s)\mathbf{f}$ in $C^2(D_r;\mathbb{R}^m)$ for any $r>0$, we can let $n\to+\infty$ in (5.13) and obtain the assertion. So, let us estimate the function g_n.

The term $g_{1,n}$. Recalling that for any pair of nonnegative definite matrices M_1 and M_2 it holds that $\mathrm{Tr}(M_1M_2)\ge\lambda_{M_1}\mathrm{Tr}(M_2)$ and $|M\xi|^2\le\Lambda_{M^2}|\xi|^2$ in $J\times\mathbb{R}^d$, for any $\xi\in\mathbb{R}^d$, we conclude that

$$
g_{1,n}\le -2\sum_{j=1}^{m}\langle M^{-1}QM^{-1}\,M\nabla_x u_j, M\nabla_x u_j\rangle - 2a\phi_s\vartheta_n^2\lambda_Q\mathcal{G}_n \le -2\lambda_{M^{-1}QM^{-1}}\mathcal{F}_n - 2a\phi_s\vartheta_n^2\lambda_Q\mathcal{G}_n.
$$

The term $g_{2,n}$. The assumptions on $\mathcal{H}$, M and C allow us to estimate

$$
g_{2,n}\le a\phi_s\vartheta_n^2(2\Lambda_C + 2\psi_4 + \Lambda_{\mathcal{H}+\mathcal{H}^T})\mathcal{F}_n.
$$

The term $g_{3,n}$. Its first term is negative; hence, we disregard it. As far as the other terms are concerned, using the estimates $|Q\nabla\vartheta_n|\le Kn^{-1}\Lambda_Q\chi_{D_{2n}\setminus D_n}|\vartheta'(|\cdot|n^{-1})|$ and $\mathrm{Tr}(QD^2\vartheta_n)\le K\Lambda_Qn^{-2}\chi_{D_{2n}\setminus D_n}$, which hold in $I\times\mathbb{R}^d$, and the Young inequality $Kxy\le a^{-1/2}K^2x^2 + a^{1/2}y^2$ we get

$$
\begin{aligned}
|g_{3,n}| \le& aK\phi_s\Lambda_Qn^{-2}\vartheta_n\chi_{D_{2n}\setminus D_n}\mathcal{F}_n + 2a\phi_s|\vartheta'(|\cdot|n^{-1})|\vartheta_n n^{-1}\langle\mathbf{b},x\rangle|x|^{-1}\chi_{D_{2n}\setminus D_n}\mathcal{F}_n\\
&+ aK\phi_s\vartheta_n n^{-1}\Lambda_Q\mathcal{F}_n^{1/2}\mathcal{G}_n^{1/2}\chi_{D_{2n}\setminus D_n} + aK\phi_s|\vartheta'(|\cdot|n^{-1})|\vartheta_n n^{-1}\Lambda_Q\lambda_M^{-1}\psi_5\chi_{D_{2n}\setminus D_n}\mathcal{F}_n\\
\le& \sqrt{a}K\phi_s\mathcal{F}_n + a^{3/2}K\phi_sn^{-4}\Lambda_Q^2\vartheta_n^2\chi_{D_{2n}\setminus D_n}\mathcal{F}_n + \sqrt{a}K\phi_s\lambda_Q^{-1}\Lambda_Q^2n^{-2}\chi_{D_{2n}\setminus D_n}\mathcal{F}_n\\
&+ a\phi_s|\vartheta'(|\cdot|n^{-1})|\vartheta_n n^{-1}(2b_0 + K\Lambda_Q\psi_5)\chi_{D_{2n}\setminus D_n}\mathcal{F}_n + a^{3/2}\phi_s\lambda_Q\vartheta_n^2\mathcal{G}_n.
\end{aligned}
$$

The term $g_{4,n}$. We denote by $g_{4,j,n}$ $(j=1,2,3)$ the terms which constitute $g_{4,n}$. To estimate $g_{4,1,n}$, we observe that

$$
\begin{aligned}
\Big|\sum_{i,k=1}^{d}\sum_{j=1}^{m} D_{ik}^2u_j\langle M\nabla_x Q_{ik}, M\nabla_x u_j\rangle\Big| =& \Big|\sum_{i,h,k=1}^{d}\sum_{j=1}^{m}\langle M\nabla_x Q_{ik}M_{kh}^{-1}(M\nabla_x D_iu_j)_h, M\nabla_x u_j\rangle\Big|\\
&\le\sqrt{d}\mathcal{F}_n^{1/2}\mathcal{G}_n^{1/2}\Lambda_M\lambda_M^{-1}\psi_6 \le \varepsilon^{-1/2}K\Lambda_M^2\lambda_M^{-2}\lambda_Q^{-1}\psi_6^2\mathcal{F}_n + \varepsilon^{1/2}\lambda_Q\mathcal{G}_n.
\end{aligned}
$$

Using the Young inequality, we conclude that $|g_{4,1,n}|\le a\varepsilon^{-1}K\phi_s\Lambda_M^2\lambda_M^{-2}\lambda_Q^{-1}\vartheta_n^2\psi_6^2\mathcal{F}_n + 2a\varepsilon\phi_s\vartheta_n^2\lambda_Q\mathcal{G}_n$ for any $\varepsilon>0$.

The other two terms in the definition of $g_{4,n}$ can be estimated in a similar way, splitting $D_iM\nabla_x = (D_iMM^{-1})M\nabla_x$. We obtain $|g_{4,2,n}+g_{4,3,n}|\le a\varepsilon^{-1}K\phi_s\vartheta_n^2\psi_5^2\lambda_M^{-2}\lambda_Q^{-1}\Lambda_Q^2\mathcal{F}_n + a\varepsilon\phi_s\vartheta_n^2\lambda_Q\mathcal{G}_n$, for any $\varepsilon>0$. Collecting everything together, and choosing $\varepsilon=1/6$ we get

$$
|g_{4,n}|\le aK\phi_s\lambda_Q^{-1}\vartheta_n^2\lambda_M^{-2}(\Lambda_Q^2\psi_5^2 + \Lambda_M^2\psi_6^2)\mathcal{F}_n + \frac{1}{2}a\phi_s\vartheta_n^2\lambda_Q\mathcal{G}_n.
$$

The term $g_{5,n}$. We denote by $g_{5,j,n}$ $(j = 1,\dots,6)$ its terms and observe that

$$g_{5,1,n} = 2\sum_{i=1}^{d}\sum_{j,h,k=1}^{m}(M^{-1})_{ih}(\tilde{B}_i)_{jk}(M\nabla_x u_k)_h u_j,$$

$$g_{5,6,n} = 2a\phi_s\vartheta_n^2\sum_{i=1}^{d}\sum_{j,h,k=1}^{m}(M\nabla_x u_k)_h\langle M\nabla_x(\tilde{B}_i)_{jk}(M^{-1})_{ih}, M\nabla_x u_{j,n}\rangle.$$

Using Hypothesis 5.1(iii) and, again, Young inequality, we obtain

$$\begin{aligned}|g_{5,n}| \le{}& (a^{-1/2}K + aK\phi_s + 2\Lambda_C)|\mathbf{u}_n|^2 + (a^{1/2}\lambda_M^{-2}\psi_1^2 + a)\mathcal{F}_n\\ &+ K\phi_s\vartheta_n^2(a\lambda_M^{-1}\Lambda_M\psi_2 + a^{3/2}\Lambda_M^2\psi_3^2 + a\lambda_M^{-2}\psi_1^2)\mathcal{F}_n + 2a^{3/2}\phi_s\vartheta_n^2\lambda_Q\mathcal{G}_n.\end{aligned}$$

Final estimate for g_n. Collecting all the terms together, we get

$$g_n \le (a^{-1/2}K + K\phi_s + 2\Lambda_C)|\mathbf{u}_n|^2 + \mathcal{I}_n\mathcal{F}_n + 3a\phi_s\vartheta_n^2\lambda_Q\left(\sqrt{a} - \frac{1}{2}\right)\mathcal{G}_n,$$

where

$$\begin{aligned}\mathcal{I}_n(t,x) ={}& \frac{a}{n}\phi_s(t)|\vartheta'(|x|n^{-1})|\vartheta_n(x)b_0(t,x)\left(2 + K\frac{\Lambda_Q(t,x)\psi_5(t,x)}{b_0(t,x)}\right)\chi_{D_{2n}\setminus D_n}(x) + a + \sqrt{a}K(T-s)\\ &- \lambda_{M^{-1}QM^{-1}}(t,x)\left[2 - \frac{\sqrt{a}(\psi_1(t,x))^2}{\lambda_{M^{-1}QM^{-1}}(t,x)(\lambda_M(t,x))^2} - \frac{(T-s)\sqrt{a}K(\Lambda_Q(t,x))^2}{(1+|x|^2)\lambda_{M^{-1}QM^{-1}}(t,x)\lambda_Q(t,x)}\right]\\ &+ a\phi_s(t)(\vartheta_n(x))^2\mathcal{J}_n(t,x),\end{aligned}$$

and

$$\begin{aligned}\mathcal{J}_n(t,x) ={}& 2\Lambda_C(t,x) + \Lambda_{\mathcal{H}+\mathcal{H}^T}(t,x) + K\frac{\Lambda_M(t,x)\psi_2(t,x) + \lambda_M(t,x)\psi_4(t,x)}{\lambda_M(t,x)}\\ &+ K\frac{(\Lambda_Q(t,x)\psi_5(t,x))^2 + (\Lambda_M(t,x)\psi_6(t,x))^2 + \lambda_Q(t,x)(\psi_1(t,x))^2}{\lambda_Q(t,x)(\lambda_M(t,x))^2}\\ &+ K\sqrt{a}\left(\frac{(\Lambda_Q(t,x))^2}{1+|x|^4} + (\Lambda_M(t,x))^2(\psi_3(t,x))^2\right),\end{aligned}$$

for any $(t,x) \in [s,T]\times\mathbb{R}^d$. Clearly, the coefficients in front of $|\mathbf{u}_n|^2$ and $\mathcal{G}_n$ are, respectively, bounded in $[s,T]$, for any choice of a, and negative, if $a < 1/4$. Let us consider the term $\mathcal{I}_n$. Since b_0 is a negative function, using condition (5.7) we conclude that the first line of $\mathcal{I}_n$ can be estimated from above by $\sqrt{a}K$ if $a < 1/4$. As far as the term in the second line of the definition of $\mathcal{I}_n$ is concerned, using the conditions in (5.1) and (5.2) we conclude that the term in square brackets is bounded from below by $2 - \sqrt{a}K$, so that, choosing a sufficiently small we can make the second line of $\mathcal{I}_n$ less that $-\lambda_{M^{-1}QM^{-1}}(t,x)$. As far as $\mathcal{J}_n$ is concerned, we observe that, by conditions (5.3)–(5.6), for any $\varepsilon > 0$, there exists $R > 0$, independent of n, such that the sum of the last three terms in $\mathcal{J}_n$ can be bounded from above by $K\{[2\varepsilon + \sqrt{a}(K_1 + K_3)]|2\Lambda_C(t,x) + \Lambda_{\mathcal{H}+\mathcal{H}^T}(t,x)| + K_2 + K_4\}$ for any $t \in J$, $x \in \mathbb{R}^d \setminus D_R$ and $n \in \mathbb{N}$. Since the function $2\Lambda_C + \Lambda_{\mathcal{H}+\mathcal{H}^T}$ is bounded from above in $J\times\mathbb{R}^d$, choosing $\varepsilon = 1/4$ and a sufficiently small we can thus make $\mathcal{J}_n$ bounded from above in $J\times(\mathbb{R}^d\setminus D_R)$ and hence in $J\times\mathbb{R}^d$. Putting everything together, we conclude that

$$\mathcal{I}_n(t,x) \le \sqrt{a}K - \inf_{(s,y)\in J\times\mathbb{R}^d}\lambda_{M^{-1}QM^{-1}}(s,y) + aK, \qquad (t,x)\in J\times\mathbb{R}^d,$$

for a small enough. Taking a smaller value of a (independent of n), if needed, we can easily make $\mathcal{I}_n$ negative and thus conclude that $g_n \le Kv_n$ as it has been claimed. □

Remark 5.5. The conditions in Hypotheses 5.1(iii), (iv) are technical and, as the proof of Theorem 5.4 shows, are used to estimate the function g_n ($n \in \mathbb{N}$), which appears in the nonhomogeneous part of the Cauchy problem solved by the function v_n, in terms of a constant (independent of n) times the function v_n itself. Also in the scalar case, technical conditions are assumed to prove gradient estimates. Here, things are much more complicated due to the fact that the single equation is replaced by a system of equations and we are interested in weighted gradient estimates. We stress that the condition on the matrix $\mathcal{H}$ in Hypothesis 5.1(iii) is more than just a technical assumption. Indeed, in the scalar case, if $M = 1$ and $C = 0$, then $\mathcal{H}$ is the transpose of the Jacobian matrix of b. Hence, the condition on $\mathcal{H}$ is just a dissipativity assumption on b. Without assuming this condition one can provide examples of elliptic operators such that the associated evolution operator does not satisfy the unweighted gradient estimate (5.12). We refer the reader to ([8], Example 6.1.11). Finally, we note that, in genuine vector valued case, if the matrices B_i identically vanish for any $i = 1, \ldots, d$, then the operator $\boldsymbol{\mathcal{A}}$ reduces to the one considered in [12]. The conditions assumed here slightly differ from the ones considered in the quoted paper since under our assumptions also the Jacobian matrix of b may help to control the growth of both the diffusion coefficients and the gradient of the potential matrix C.

Remark 5.6. In the particular case when $M = I_m$, we can relax a bit Hypotheses 5.1. Indeed, we introduced the function ϑ_n in the definition of v_n in the proof of Theorem 5.12 to guarantee that v_n vanishes on $(s,T)\times\partial D_n$ and thus apply the maximum principle. By Remark 2.10 we can approximate the evolution operator $\mathbf{G}(t,s)$ with the evolution operator $\mathbf{G}_n^{\mathcal{N}}(t,s)$ associated to the realization of $\boldsymbol{\mathcal{A}}$ in D_n with homogeneous Neumann boundary conditions. We can thus define the function v_n by setting $v_n(t,\cdot) = |\mathbf{G}_n^{\mathcal{N}}(t,s)\mathbf{f}|^2 + a(t-s)\sum_{j=1}^m |M\nabla_x(\mathbf{G}_n^{\mathcal{N}}(t,s)\mathbf{f})_j|^2$ for any $t \in (s,T)$. Since the normal derivative of $|J_x\mathbf{G}_n^{\mathcal{N}}(t,s)\mathbf{f}|^2$ is nonpositive on $(s,T)\times\partial D_n$, the normal derivative of v_n is nonpositive on $(s,T)\times\partial D_n$ as well. With this choice of v_n, the term $g_{3,n}$ disappears and we do not need to assume anymore the conditions (5.2), (5.3) and (5.7). Moreover, in this case the matrix $\mathcal{H}$ reduces to the matrix $J_x b$. Therefore, we are assuming a bound on the growth as $|x| \to +\infty$ of the quadratic form associated with the matrix $J_x\mathbf{b}$. In the scalar case, this is a typical assumption used to prove gradient estimates both in the autonomous and nonautonomous setting (see *e.g.*, [5, 8, 9, 25, 29, 30]). Finally, if the operator $\boldsymbol{\mathcal{A}}$ satisfies Hypotheses 5.1, then the scalar operator $\tilde{\mathcal{A}} = \mathrm{Tr}(QD^2) + \langle \mathbf{b}, \nabla\rangle$ satisfies the same conditions, and therefore, the scalar evolution operator $G(t,s)$ satisfies (5.12) as well.

Example 5.7. Coming back to Example 5.2, if we assume that $M \equiv I$ *i.e.*, if we are interested in unweighted gradient estimates, then, in view of Remark 5.6, we have to assume the condition (5.1), (5.4), (5.5) and (5.6), where $\psi_4 = \psi_5 = 0$ and $\lambda_M \equiv \Lambda_M \equiv 1$. Then, the system (5.10) provides us with the following conditions on p, k, r and τ:

$$2r - k \le 0, \qquad 2\tau - p \le 0, \qquad (k-1)\vee(2r) - p < 0, \qquad 2r - 1 - 2p < 0.$$

Noting that the last condition is implied by the third one, we conclude that the conditions on p, k, r and τ are the following: $2r \le k < p+1$, $2\tau \le p$, $p > 2r$.

6. Some applications of the gradient estimate (5.12)

6.1. The converse of Theorem 4.2

As Theorem 4.2 shows, the compactness of the evolution operator $G(t,s)$ in $C_b(\mathbb{R}^d)$ implies the compactness of the evolution operator $\mathbf{G}(t,s)$ in $C_b(\mathbb{R}^d;\mathbb{R}^m)$. Now, we are interested in finding out sufficient condition for the converse. The main step in this direction, consists in proving formula (1.5). Once it is proved, we can adapt to our situation the arguments in the proof of ([12], Thm. 3.6).

Since, in general, the evolution operator $G(t,s)$ is well defined only on bounded (and Borel measurable) functions, to make formula (1.5) meaningful we need to guarantee that the Borel measurable function $(\mathcal{S}\mathbf{G}(\cdot,s)\mathbf{f})(r,\cdot)$ is bounded in $\mathbb{R}^d$ for any $r \in (s,t)$ and that the integral in the right-hand side of (1.5) is finite. Note that in the weakly coupled case considered in [12], $\tilde{B}_i \equiv 0$ for any $i = 1, \ldots, d$. Hence, the boundedness of $row_{\bar{k}}C$ and the uniform estimate (3.1) were enough to guarantee the existence of the integral term in (1.5). In our situation things are much more difficult since we have to guarantee that also the function $\sum_{i=1}^d \langle row_{\bar{k}}\tilde{B}_i(r,\cdot), D_i\mathbf{G}(r,s)\mathbf{f}\rangle$ is bounded in $\mathbb{R}^d$ for any $r\in(s,t)$. As we will see in the following proposition, thanks to the estimate (5.12), the boundedness of the function $(\mathcal{S}\mathbf{G}(\cdot,s)\mathbf{f})(r,\cdot)$ can be guaranteed under the following two different additional assumptions.

Hypotheses 6.1. *Hypotheses 2.3 and 5.1 are satisfied. Further, there exist a bounded interval J and a positive constant K_* such that $|\tilde{B}_i|\le K_*\lambda_M$ in $J\times\mathbb{R}^d$ and $row_{\bar{k}}C\in C_b(J\times\mathbb{R}^d;\mathbb{R}^m)$ for any $i=1,\dots,d$ and some $\bar{k}\in\{1,\dots,m\}$.*

Hypotheses 6.2. *Hypotheses 2.3 and 5.1 with $M=I_m$ are satisfied. Further, there exist $\bar{k}\in\{1,\dots,m\}$ and a bounded interval J such that $row_{\bar{k}}C$ and $row_{\bar{k}}\tilde{B}_i$ belong to $C_b(J\times\mathbb{R}^d;\mathbb{R}^m)$ for any $i=1,\dots,d$.*

Theorem 6.3. *Assume that Hypotheses 6.1 (resp. Hypotheses 6.2) are satisfied. If $\mathbf{G}(t,s)$ is compact in $C_b(\mathbb{R}^d;\mathbb{R}^m)$ for any $(t,s)\in\Sigma_J=\{(t,s)\in J\times J:\ \ t>s\}$, then $G(t,s)$ is compact in $C_b(\mathbb{R}^d)$ for the same values of t and s.*

Proof. To simplify the notation, we set $\mathbf{u}:=\mathbf{G}(\cdot,s)\mathbf{f}$ and $T:=\sup J$. Moreover, by K we will denote a positive constant, which is independent of j, n, t and $\mathbf{f}$, and may vary from line to line.

As a first step, we show that the integral in the right-hand side of (1.5) is well defined when $\mathbf{f}\in B_b(\mathbb{R}^d;\mathbb{R}^m)$. Since $\|G(t,s)\|_{\mathcal{L}(C_b(\mathbb{R}^d))}\le 1$ for any $I\ni s\le t$, we just prove that the function $r\mapsto\|(\mathcal{S}\mathbf{G}(\cdot,s)\mathbf{f})(r,\cdot)\|_\infty$ belongs to $L^1((s,t))$ for any $t\in(s,T)$. The boundedness of $row_{\bar{k}}C$ and (3.1) yield that $\langle row_{\bar{k}}C,\mathbf{u}\rangle\in C_b([s,T]\times\mathbb{R}^d)$ and $\|\langle row_{\bar{k}}C,\mathbf{u}\rangle\|_\infty\le K\|row_{\bar{k}}C\|_\infty\|\mathbf{f}\|_\infty$. Moreover, if Hypotheses 6.1 are satisfied, then from (5.12), we get

$$\begin{aligned}\left|\sum_{i=1}^d\langle row_{\bar{k}}\tilde{B}_i(r,\cdot),D_i\mathbf{u}(r,\cdot)\rangle\right| &\le\sqrt{md}\bigg(\sum_{i=1}^d\sum_{j=1}^m|\tilde{B}_i(r,\cdot)_{\bar{k}j}|^2(D_iu_j(r,\cdot))^2\bigg)^{1/2}\\ &\le\sqrt{md}K_*\lambda_M(r,\cdot)|J_x\mathbf{u}(r,\cdot)|\le\sqrt{md}K_*\|M(r,\cdot)(J_x\mathbf{u}(r,\cdot))^T\|_\infty\\ &\le K(r-s)^{-1/2}\|\mathbf{f}\|_\infty,\end{aligned}$$

for any $r\in(s,T)$. On the other hand, if Hypotheses 6.2 hold true, then, from (5.12), with $M=I_m$, it follows that

$$\left|\sum_{i=1}^d\langle row_{\bar{k}}\tilde{B}_i(r,\cdot),D_i\mathbf{u}(r,\cdot)\rangle\right|\le K\max_{1\le i\le d}\|row_{\bar{k}}B_i\|_\infty(r-s)^{-1/2}\|\mathbf{f}\|_\infty.$$

In both the cases, the function $(\mathcal{S}\mathbf{G}(\cdot,s)\mathbf{f})(r,\cdot)$ is bounded in $\mathbb{R}^d$ for any $r\in(s,T)$ and

$$\|(\mathcal{S}\mathbf{G}(\cdot,s)\mathbf{f})(r,\cdot)\|_\infty\le K_0(r-s)^{-1/2}\|\mathbf{f}\|_\infty,\qquad r\in(s,+\infty)\cap J,\tag{6.1}$$

where K_0 depends on m, d, s, J, $\|row_{\bar{k}}C\|_\infty$, and also on $\|row_{\bar{k}}\tilde{B}_i\|_\infty$, $(i=1,\dots,d)$ when Hypotheses 6.2 hold true. This, in particular, yields that the integral in the right-hand side of formula (1.5) is well defined.

Now, we split the rest of the proof into three steps. The first two steps are devoted to prove formula (1.5) for functions $\mathbf{f}\in C_c^{2+\alpha}(\mathbb{R}^d;\mathbb{R}^m)$. Once it is proved for such functions, this formula can be easily extended to functions in $B_b(\mathbb{R}^d;\mathbb{R}^m)$ by density. Indeed, if $\mathbf{f}\in B_b(\mathbb{R}^d;\mathbb{R}^m)$, then we fix a sequence $(\mathbf{f}_n)\subset C_c^{2+\alpha}(\mathbb{R}^d;\mathbb{R}^m)$, bounded with respect to the sup-norm and converging to $\mathbf{f}$ almost everywhere in $\mathbb{R}^d$. Writing (1.5) with $\mathbf{f}$ replaced by $\mathbf{f}_n$ and letting $n\to+\infty$, from Corollary 3.4(ii) we conclude that $\mathbf{G}(t,s)\mathbf{f}_n$ and $G(t,s)f_{n,\bar{k}}$ converge to $\mathbf{G}(t,s)\mathbf{f}$ and $G(t,s)f_{\bar{k}}$, respectively, locally uniformly in $\mathbb{R}^d$, as $n\to+\infty$. Similarly, $\mathcal{S}\mathbf{G}(\cdot,s)\mathbf{f}_n$ converges locally uniformly in $(s,+\infty)\times\mathbb{R}^d$ to $\mathcal{S}\mathbf{G}(\cdot,s)\mathbf{f}$, as $n\to+\infty$. Taking (6.1) (with $\mathbf{f}$ replaced by $\mathbf{f}_n$) and the contractivity of the evolution operator $G(t,s)$ into account, we can apply the dominated convergence theorem to infer that the integral term in (1.5) converges to the corresponding one with $\mathbf{f}_n$ replaced by $\mathbf{f}$, and formula (1.5) follows.

Step 1. Here, we assume Hypotheses 6.2 and denote by $\mathbf{G}_n^{\mathcal{N}}(t,s)$ the evolution operator associated with the operator $\boldsymbol{\mathcal{A}}$ in $C_b(D_n;\mathbb{R}^m)$, with homogeneous Neumann boundary conditions, which satisfies the gradient estimate $\|\mathbf{G}_n^{\mathcal{N}}(t,s)\mathbf{f}\|_{C_b(D_n;\mathbb{R}^m)}+\sqrt{t-s}\|J_x\mathbf{G}_n^{\mathcal{N}}(t,s)\mathbf{f}\|_{C_b(D_n;\mathbb{R}^m)}\le K\|\mathbf{f}\|_\infty$ for any $n\in\mathbb{N}$, by Remark 5.6.

As it has been stressed in Remark 2.10, $G_n^{\mathcal{N}}(\cdot,s)\mathbf{f}$ tends to $\mathbf{u}$ locally uniformly in $\mathbb{R}^d$, as $n\to+\infty$. If we split $(\boldsymbol{\mathcal{A}}\tilde{\mathbf{u}}_n)_{\bar{k}}=\tilde{\mathcal{A}}u_{n,k}+\Phi_{n,\bar{k}}$, where $\Phi_{n,\bar{k}}=\sum_{j=1}^d\langle row_{\bar{k}}\tilde{B}_j,D_j\tilde{\mathbf{u}}_n\rangle+\langle row_{\bar{k}}C,\tilde{\mathbf{u}}_n\rangle$, then we can write

$$\tilde{u}_{\bar{k}}(t,x)=(G_n^{\mathcal{N}}(t,s)f_{\bar{k}})(x)+\int_s^t(G_n^{\mathcal{N}}(t,\sigma)\Phi_{n,\bar{k}}(\sigma,\cdot))(x)\mathrm{d}\sigma,\qquad t\in(s,T),\ \ x\in\overline{D_n}. \tag{6.2}$$

Clearly, $\tilde{u}_{\bar{k}}$ and $G_n^{\mathcal{N}}(\cdot,s)f_{\bar{k}}$ converge to $(\mathbf{G}(t,s)\mathbf{f})_{\bar{k}}$ and $G(\cdot,s)f_{\bar{k}}$, respectively, locally uniformly in $(s,+\infty)\times\mathbb{R}^d$. As far as the integral term in (6.2) is concerned, the boundedness of $row_{\bar{k}}C$ and $row_{\bar{k}}B_i$ $(i=1,\dots,d)$ and the above gradient estimate imply, first, that $\|\Phi_{n,\bar{k}}(\sigma,\cdot)\|_{C_b(D_n)}\le K(\sigma-s)^{-1/2}\|\mathbf{f}\|_\infty$ and, then, that $\|G_n^{\mathcal{N}}(t,\sigma)\Phi_{n,\bar{k}}(\sigma,\cdot)\|_{C_b(D_n)}\le K(\sigma-s)^{-1/2}\|\mathbf{f}\|_\infty$ for any $\sigma\in(s,T)$ and any $n\in\mathbb{N}$, since each operator $G_n^{\mathcal{N}}(t,\sigma)$ is contractive. Next, we estimate

$$\begin{aligned}&|G_n^{\mathcal{N}}(t,\sigma)\Phi_{n,\bar{k}}(\sigma,\cdot)-G^{\mathcal{N}}(t,\sigma)(\mathcal{S}\mathbf{G}(\cdot,s)\mathbf{f})(\sigma,\cdot)|\le\\&\qquad G_n^{\mathcal{N}}(t,\sigma)|\Phi_{n,\bar{k}}(\sigma,\cdot)-(\mathcal{S}\mathbf{G}(\cdot,s)\mathbf{f})(\sigma,\cdot)|+|G_n^{\mathcal{N}}(t,\sigma)(\mathcal{S}\mathbf{G}(\cdot,s)\mathbf{f})(\sigma,\cdot)-G(t,\sigma)(\mathcal{S}\mathbf{G}(\cdot,s)\mathbf{f})(\sigma,\cdot)|.\end{aligned} \tag{6.3}$$

As $n\to+\infty$, the last term in (6.3) vanishes, locally uniformly in $\mathbb{R}^d$, due to Remark 2.10. To show that also the first term vanishes, we prove the following claim.

Claim: For any $r>0$, $\varepsilon>0$ and $\sigma\in(s,t)$, there exists $h\in\mathbb{N}$ such that $G_n^{\mathcal{N}}(t,\sigma)\chi_{\mathbb{R}^d\setminus D_h}\le\varepsilon$ in D_r, for any n sufficiently large.

Once the claim is proved, we estimate

$$\begin{aligned}\|G_n^{\mathcal{N}}(t,\sigma)g_n(\sigma,\cdot)\|_{C_b(D_r)}\le&\|G_n^{\mathcal{N}}(t,\sigma)(\chi_{D_h}g_n(r,\cdot))\|_{C_b(D_r)}+\|G_n^{\mathcal{N}}(t,\sigma)(\chi_{\mathbb{R}^d\setminus D_h}g_n(r,\cdot))\|_{C_b(D_r)}\\\le&\|g_n(\sigma,\cdot)\|_{C_b(D_h)}+\|G_n^{\mathcal{N}}(t,\sigma)\chi_{\mathbb{R}^d\setminus D_h}\|_{C_b(D_r)}\sup_{n\in\mathbb{N}}\|g_n(\sigma,\cdot)\|_{C_b(D_n)}\\\le&\|g_n(\sigma,\cdot)\|_{C_b(D_h)}+\varepsilon K(\sigma-s)^{-1/2}\|f\|_\infty,\end{aligned}$$

where $g_n=\Phi_{n,\bar{k}}-\mathcal{S}\mathbf{G}(\cdot,s)\mathbf{f}$. Since g_n vanishes locally uniformly in $(s,+\infty)\times\mathbb{R}^d$ as $n\to+\infty$,

$$\limsup_{n\to+\infty}\|G_n^{\mathcal{N}}(t,\sigma)g_n(\sigma,\cdot)\|_{C_b(D_r)}\le\varepsilon K(\sigma-s)^{-1/2}\|f\|_\infty$$

and, letting $\varepsilon\to0^+$, we conclude that $G_n^{\mathcal{N}}(t,\sigma)g_n(\sigma,\cdot)$ vanishes uniformly in D_r as $n\to+\infty$, for any $\sigma\in(s,T)$.

To prove the claim, we fix a sequence $(\psi_n)\subset C_b(\mathbb{R}^d)$ satisfying $\chi_{\mathbb{R}^d\setminus D_n}\le\psi_n\le\chi_{\mathbb{R}^d\setminus D_{n-1}}$. Since ψ_n vanishes locally uniformly in $\mathbb{R}^d$, $G(t,\sigma)\psi_n$ tends to 0 locally uniformly in $\mathbb{R}^d$, for any $\sigma\in(s,t)$, by ([25], Prop. 3.1). Therefore, for any fixed $\varepsilon,R>0$, there exists $k=k(r)>0$ such that $G(t,r)\psi_k\le\varepsilon/2$ in D_r. Since $G_n^{\mathcal{N}}(t,\sigma)\psi_k$ converges to $G(t,\sigma)\psi_k$ in D_r, we can determine $n_0=n_0(r)\in\mathbb{N}$ such that $\|G(t,\sigma)\psi_k-G_n^{\mathcal{N}}(t,\sigma)\psi_k\|_{C_b(D_r)}\le\varepsilon/2$ for any $n\ge n_0$, which yields the claim.

Step 2. Here, we assume Hypotheses 6.1. In such a case, the argument in Step 1 does not work, since from the proof of Theorem 5.4 we now just infer that

$$\sqrt{t-s}\|\vartheta_n\sqrt{Q}(t,\cdot)J_x\mathbf{u}_n(t,\cdot)\|_{C_b(D_n;\mathbb{R}^m)}+\|\mathbf{u}_n(t,\cdot)\|_{C_b(D_n;\mathbb{R}^m)}\le K\|\mathbf{f}\|_\infty,\qquad n\in\mathbb{N}, \tag{6.4}$$

where (ϑ_n) is a sequence of cut-off functions, such that $\mathrm{supp}(\vartheta_n)\subset D_n$ for any $n\in\mathbb{N}$, and $\mathbf{u}_n:=\mathbf{G}_n^{\mathcal{D}}(\cdot,s)\mathbf{f}$. From (6.4) we can not deduce the crucial estimate $\|\Phi_{n,\bar{k}}(r,\cdot)\|_{C_b(D_n)}\le K(r-s)^{-1/2}\|\mathbf{f}\|_\infty$. To overcome this difficulty, we use a slightly different approximation argument. We denote by $\mathbf{\Psi}_n$ the function whose components are $\Psi_{n,i} = \vartheta_n \sum_{j=1}^d \langle row_i \tilde{B}_j, D_j \mathbf{u}_n\rangle + \langle row_i C, \mathbf{u}_n\rangle$, for any $i = 1, \dots, m$. Since $\mathbf{u}_n \in C^{1+\alpha/2,2+\alpha}((s,T)\times D_n;\mathbb{R}^m)$ (see [27], Thm. IV.5.5), the function $\mathbf{\Psi}_n$ belongs to $C^{\alpha/2,\alpha}((s,T)\times D_n;\mathbb{R}^m)$ and is compactly supported in $[s,T]\times D_n$. Hence ([27], Thm. IV.5.5) shows that, for any $n\in\mathbb{N}$ such that $\mathrm{supp}(\mathbf{f})\subset D_n$, there exists a unique function $\mathbf{w}_n \in C^{1+\alpha/2,2+\alpha}((s,T)\times D_n;\mathbb{R}^m)$ which satisfies $D_t\mathbf{w}_n = \tilde{\mathcal{A}}\mathbf{w}_n + \mathbf{\Psi}_n$ in $(s,T)\times\mathbb{R}^d$, $\mathbf{w}_n(s,\cdot) = \mathbf{f}$ and it vanishes on $(s,T)\times\partial D_n$. Here, $\tilde{\mathcal{A}}$ is the diagonal vector-valued operator whose m-components coincide with the operator $\tilde{\mathcal{A}}$. For any $i = 1,\dots,m$, the component $w_{n,i}$ of $\mathbf{w}_n$ can be represented through the formula

$$w_{n,i}(t,x) = (G_n^{\mathcal{D}}(t,s)f_i)(x) + \int_s^t (G_n^{\mathcal{D}}(t,r)\Psi_{n,i}(r,\cdot))(x)\mathrm{d}r, \qquad t\in(s,T),\ x\in\overline{D_n}. \tag{6.5}$$

We claim that $\mathbf{u}$ is the limit of the sequence $(\mathbf{w}_n)$. By Theorem A.2, $\|\mathbf{w}_n\|_{C^{1+\alpha/2,2+\alpha}(K;\mathbb{R}^m)} \le K$ for any compact set $E\subset(s,T)\times\mathbb{R}^d$ and n large enough. Hence, we can extract a subsequence $(\mathbf{w}_{n_j})$ which, as $j\to+\infty$, converges in $C^{1,2}([s+\tau^{-1},T]\times\overline{D_\tau};\mathbb{R}^m)$, for any $\tau > (T-s)^{-1}$, to some function $\mathbf{w}\in C^{1+\alpha/2,2+\alpha}_{\mathrm{loc}}((s,T)\times\mathbb{R}^d;\mathbb{R}^m)$. Since $\mathbf{u}_n$ converges to $\mathbf{u}$ in $C^{1,2}([s+\tau^{-1},T]\times\overline{D_\tau};\mathbb{R}^m)$ (see the proof of Thm. 2.8), we conclude that $D_t\mathbf{w} = \tilde{\mathcal{A}}\mathbf{w} + \mathcal{S}\mathbf{u}$ in $(s,T)\times\mathbb{R}^d$ where $\mathcal{S}_k\mathbf{u} = \sum_{i=1}^d\langle row_k\tilde{B}_i, D_i\mathbf{u}\rangle + \langle row_k C,\mathbf{u}\rangle$ for any $k=1,\dots,m$. To conclude that $\mathbf{w}=\mathbf{u}$, it suffices to show that $\mathbf{w}$ can be extended by continuity at $t=s$, where it equals $\mathbf{f}$. For this purpose, we follow the same strategy as in the proof of Theorem 2.8, localizing the problem in the ball D_R for any $R>0$. To make the arguments therein contained work, we need to show that $|\mathbf{g}_{n_j}(t,x)| \le K(t-s)^{-1/2}\|\mathbf{f}\|_\infty$ for any $t\in(s,s+1)$, $x\in D_R$, where $\mathbf{g}_{n_j} = -\mathbf{w}_{n_j}\tilde{\mathcal{A}}\vartheta - 2J_x\mathbf{w}_{n_j}(Q\nabla\vartheta) + \vartheta\mathbf{\Psi}_{n_j}$ $(n_j > M)$ and ϑ is a smooth function such that $\chi_{D_{R-1}} \le \vartheta \le \chi_{D_R}$. The term $\mathbf{\Psi}_{n_j}$ can be estimated using the proof of Theorem 5.4, which shows that $\|\mathbf{u}_n(t,\cdot)\|_{C_b(D_n;\mathbb{R}^m)} + \sqrt{t-s}\|\vartheta_n J_x\mathbf{u}_n(t,\cdot)\|_{C_b(D_n;\mathbb{R}^m)} \le K\|\mathbf{f}\|_\infty$ for any $t\in(s,T)$ and $n\in\mathbb{N}$, and implies that $\|\mathbf{\Psi}_{n_j}(t,\cdot)\|_{C_b(D_{n_j};\mathbb{R}^m)} \le K(t-s)^{-1/2}\|\mathbf{f}\|_\infty$ for any $j\in\mathbb{N}$. As far as the function $\mathbf{w}_{n_j}$ is concerned, we observe that the operator $\tilde{\mathcal{A}}$ satisfies Hypotheses 5.1. Therefore, $\sqrt{t-s}\|\vartheta_n\nabla_x G^{\mathcal{D}}_{n_j}(t,s)g\|_{C_b(D_n;\mathbb{R}^m)} \le K\|g\|_{C_b(D_n;\mathbb{R}^m)}$ for any $t\in(s,T)$ and any $g\in C_c(D_n)$. Differentiating formula (6.5) with respect to x, taking the sup norm in D_R of both sides, and using the previous two estimates, we conclude that

$$\begin{aligned}\|J_x\mathbf{w}_{n_j}(t,\cdot)\|_{C_b(B_R)} &\le \sum_{i=1}^d \|\nabla_x G^{\mathcal{D}}_{n_j}(t,s)f_i\|_{C_b(D_R)} + \sum_{i=1}^d\int_s^t \|\nabla_x G^{\mathcal{D}}_{n_j}(t,r)\Psi_{n_j,i}(r,\cdot)\|_{C_b(D_R)}\mathrm{d}r\\ &\le K\bigg(\frac{\|\mathbf{f}\|_\infty}{\sqrt{t-s}} + \int_s^t\frac{\|\mathbf{\Psi}_{n_j}(r,\cdot)\|_{C_b(D_{n_j};\mathbb{R}^m)}}{\sqrt{t-r}}\mathrm{d}r\bigg) \le K\frac{\|\mathbf{f}\|_\infty}{\sqrt{t-s}}\end{aligned}$$

for any $t\in(s,T]$ and $n_j>R$. The wished estimate on $\mathbf{g}_{n_j}$ follows. Since the above arguments can be applied to any convergent subsequence of $(\mathbf{w}_n)$, the sequence $(\mathbf{w}_n)$ itself converges to $\mathbf{u}$.

We now fix $i=\bar{k}$ in (6.5) and let $n\to+\infty$. Since $\Psi_{n,i}$ converges to $\mathcal{S}\mathbf{G}(\cdot,s)\mathbf{f}$, locally uniformly in $(s,+\infty)\times\mathbb{R}^d$, as $n\to+\infty$, and $\|\Psi_{n,i}(t,\cdot)\|_{C_b(D_n)} \le K(t-s)^{-1/2}\|\mathbf{f}\|_\infty$ for any $t\in(s,T)$, we can repeat the same arguments as in Step 1, with $G_n^{\mathcal{N}}(t,s)$ replaced by the operator $G_n^{\mathcal{D}}(t,s)$, and complete the proof of (1.5).

Step 3. Let $(f_j)\subset C_b(\mathbb{R}^d)$ be a bounded sequence and set $\mathbf{f}_j = f_j\mathbf{e}_{\bar{k}}$ for any $j\in\mathbb{N}$. Without loss of generality, we assume that $\|f_j\|_\infty\le 1$ for any $j\in\mathbb{N}$. We fix $(t,s)\in\Sigma_J$ and $s_0\in[s,t]$ satisfying $s_0-s\le(8K)^{-2}$, where K_0 is the constant in (6.1). Since $\mathbf{G}(s_0,s)$ is compact in $C_b(\mathbb{R}^d;\mathbb{R}^m)$, there exists a subsequence $(\mathbf{G}(s_0,s)\mathbf{f}_{j_n^0})$ converging uniformly in $\mathbb{R}^d$, as $n\to+\infty$, to some function $\mathbf{g}_{s_0}\in C_b(\mathbb{R}^d;\mathbb{R}^m)$. Clearly, $(\mathbf{G}(t,s)\mathbf{f}_{j_n^0})_{\bar{k}}$ converges uniformly to the $\bar{k}$th component of $G(t,s_0)\mathbf{g}_{s_0}$. Moreover, recalling that $G(t,s)$ is a contractive evolution operator, we can estimate

$$\begin{aligned}\sup_{x\in\mathbb{R}^d}\bigg|\int_s^t(G(t,r)(\mathcal{S}\mathbf{G}(\cdot,s)(\mathbf{f}_{j_n^0}-\mathbf{f}_{j_m^0}))(r,\cdot))(x)\mathrm{d}r\bigg| &\le \int_s^{s_0}\|(\mathcal{S}\mathbf{G}(\cdot,s)(\mathbf{f}_{j_n^0}-\mathbf{f}_{j_m^0}))(r,\cdot)\|_\infty\mathrm{d}r\\ &\quad+\int_{s_0}^t\|(\mathcal{S}\mathbf{G}(\cdot,s)(\mathbf{f}_{j_n^0}-\mathbf{f}_{j_m^0}))(r,\cdot)\|_\infty\mathrm{d}r.\end{aligned}\tag{6.6}$$

Estimate (6.1) and our choice of s_0 imply that the first term in the right-hand side of (6.6) does not exceed $1/2$. On the other hand, $\mathcal{S}\mathbf{f}(r,\cdot) = \mathcal{S}\mathbf{G}(\cdot,s_0)\mathbf{G}(s_0,s)(\mathbf{f}_{j_n^0} - \mathbf{f}_{j_m^0})$ and applying (6.1), with $\mathbf{G}(\cdot,s)$ replaced by $\mathbf{G}(\cdot,s_0)$, we estimate the last term in the right-hand side of (6.6) from above by $K\|G(s_0,s)(\mathbf{f}_{j_n^0} - \mathbf{f}_{j_m^0})\|_\infty$. Putting everything together, we conclude that

$$\sup_{x\in\mathbb{R}^d}\left|\int_s^t (G(t,r)(\mathcal{S}\mathbf{G}(\cdot,s)(\mathbf{f}_{j_n^0} - \mathbf{f}_{j_m^0}))(r,\cdot))(x)\mathrm{d}r\right| \leq \frac{1}{2} + K\|G(s_0,s)(\mathbf{f}_{j_n^0} - \mathbf{f}_{j_m^0})\|_\infty, \tag{6.7}$$

which, combined with (1.5), (3.1), shows that $\|G(t,s)(f_{j_n^0} - f_{j_m^0})\|_\infty \leq \frac{1}{2} + K\|\mathbf{G}(s_0,s)(\mathbf{f}_{j_n^0} - \mathbf{f}_{j_m^0})\|_\infty$. The last term in (6.7) vanishes as $n,m\to+\infty$. Therefore, there exists $N_0\in\mathbb{N}$ such that $\|G(t,s)(f_{j_n^0} - f_{j_m^0})\|_\infty \leq 1$ for any $n,m\geq N_0$.

Now, we fix $s_1\in(s,t)$ such that $s_1 - s \leq (16K)^{-2}$ and repeat the same construction as above with s_1 replacing s_0. We thus determine $N_1\in\mathbb{N}$ and a subsequence $(f_{j_n^1})$ of $(f_{j_n^0})$ such that $\|G(t,s)(f_{j_n^1} - f_{j_m^1})\|_\infty \leq 1/2$ for any $m,n\geq N_1$. Iterating this argument, for any $h\in\mathbb{N}$ we can determine a subsequence $(f_{j_n^h})\subset(f_{j_n^{h-1}})$ and an integer N_h such that

$$\|G(t,s)(f_{j_n^h} - f_{j_m^h})\|_\infty \leq 2^{-h}, \qquad m,n\geq N_h. \tag{6.8}$$

To conclude the proof, we consider the diagonal sequence (ψ_n) with $\psi_n = f_{j_n^n}$ for any $n\in\mathbb{N}$. We claim that $G(t,s)\psi_n$ converges uniformly in $\mathbb{R}^d$. For this purpose, we fix $\varepsilon>0$ and $h\in\mathbb{N}$ such $2^{-h}\leq\varepsilon$. We also set $N=\max\{h,N_h\}$. Recalling that $\psi_n,\psi_m\in(f_{j_p^h})$ if $n,m\geq h$, from (6.8) we deduce that $\|G(t,s)(\psi_n-\psi_m)\|_\infty\leq\varepsilon$ for any $m,n\geq N$, and we are done. □

6.2. Semilinear systems and systems of markovian forward backward stochastic differential equations

At first, we consider a Cauchy problem for a semilinear system of parabolic equations and we show that, under suitable assumptions, it admits a mild solution $\mathbf{u}$. The special form of the nonlinear part allows us to connect the Cauchy problem with a system of Markovian backward stochastic differential equations (BSDEs for short); in particular, we prove that a solution $(\mathbf{Y},\mathbf{Z})$ to the system of BDSEs exists and it can be written in terms of the function $\mathbf{u}$. Hereafter, we assume Hypotheses 2.3.

6.2.1. Semilinear systems

In this subsection we deal with the backward semilinear Cauchy problem

$$\begin{cases} D_t\mathbf{u}(t,x) + (\boldsymbol{\mathcal{A}}\mathbf{u})(t,x) = (\boldsymbol{\Psi}(\mathbf{u}))(t,x), \ t\in[0,T),\ x\in\mathbb{R}^d, \\ \mathbf{u}(T,x) = \mathbf{g}(x), \qquad x\in\mathbb{R}^d, \end{cases} \tag{SL-CP}$$

where $\boldsymbol{\mathcal{A}}$ is the operator in (1.1) and $(\boldsymbol{\Psi}(\mathbf{u}))(t,x) = \boldsymbol{\psi}(t,x,\sqrt{Q}(t,x)J_x\mathbf{u}(t,x))$ for any $t\in[0,T)$ and $x\in\mathbb{R}^d$. Throughout this subsection, we assume Hypotheses 5.1 with $M=\sqrt{Q}$ and the following conditions on $\mathbf{g}$ and $\boldsymbol{\psi}$.

Hypotheses 6.4. *The functions* $\mathbf{g}$ *and* $\boldsymbol{\psi}$ *belong to* $C_b(\mathbb{R}^d;\mathbb{R}^m)$ *and* $C([0,T]\times\mathbb{R}^d\times\mathbb{R}^{dm};\mathbb{R}^m)$ *respectively. Further, there exist positive constants* Θ *and* $\beta\in(0,1)$ *such that*

$$|\boldsymbol{\psi}(t,x_1,z_1) - \boldsymbol{\psi}(t,x_2,z_2)| \leq \Theta(1+|z_1|+|z_2|)(|x_1-x_2|^\beta + |z_1-z_2|^\beta), \tag{6.9}$$

$$|\boldsymbol{\psi}(t,x,z)| \leq \Theta(1+|z|), \tag{6.10}$$

for any $t\in[0,T]$, $x,x_1,x_2\in\mathbb{R}^d$ *and* $z,z_1,z_2\in\mathbb{R}^{md}$.

Example 6.5. The conditions (6.9) and (6.10) are satisfied, *e.g.*, when:

(i) the function $\boldsymbol{\psi}$ belongs to $C([0,T];\mathrm{Lip}(\mathbb{R}^d\times\mathbb{R}^{md};\mathbb{R}^m))$ and $|\boldsymbol{\psi}(\cdot,\cdot,0)|$ is bounded in $[0,T]\times\mathbb{R}^d$. Indeed, we can estimate

$$|\boldsymbol{\psi}(t,x,z)|\le|\boldsymbol{\psi}(t,x,z)-\boldsymbol{\psi}(t,x,0)|+|\boldsymbol{\psi}(t,x,0)|\le K_0|z|+\sup_{(s,y)\in[0,T]\times\mathbb{R}^d}|\boldsymbol{\psi}(s,y,0)|, \tag{6.11}$$

for any $(t,x,z)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^{md}$ where $K_0=\sup_{t\in[0,T]}[\boldsymbol{\psi}(t,\cdot,\cdot)]_{\mathrm{Lip}(\mathbb{R}^d\times\mathbb{R}^{md})}$, and (6.10) follows. Moreover, for any $\beta\in(0,1)$,

$$|\boldsymbol{\psi}(t,x,z_2)-\boldsymbol{\psi}(t,x,z_1)|\le K_0|z_2-z_1|^{1-\beta}|z_2-z_1|^{\beta}\le K_0(1+|z_1|+|z_2|)|z_2-z_1|^{\beta}, \tag{6.12}$$

$$\begin{aligned}|\boldsymbol{\psi}(t,x_2,z)-\boldsymbol{\psi}(t,x_1,z)|&=|\boldsymbol{\psi}(t,x_2,z)-\boldsymbol{\psi}(t,x_1,z)|^{1-\beta}|\boldsymbol{\psi}(t,x_2,z)-\boldsymbol{\psi}(t,x_1,z)|^{\beta}\\&\le K_0^{\beta}(1+|\boldsymbol{\psi}(t,x_1,z)|+|\boldsymbol{\psi}(t,x_1,z)|)|x_2-x_1|^{\beta},\end{aligned} \tag{6.13}$$

for any $t\in[0,T]$, $x,x_1,x_2\in\mathbb{R}^d$, $z,z_1,z_2\in\mathbb{R}^{md}$. Using (6.11), from (6.12) and (6.13), estimate (6.9) follows. In this case, we recover the regularity assumptions considered in [34];

(ii) the components of $\boldsymbol{\psi}$ are defined by $\psi_i(t,x,z):=(1+|z|)h_i(t)f_i(x)g_i(z)$ for any $(t,x,z)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^{md}$, $i=1,\dots,m$ and some functions $f_i\in C^\beta_b(\mathbb{R}^d)$, $g_i\in C^\beta_b(\mathbb{R}^{md})$ and $h_i\in C([0,T])$. In this case,

$$\begin{aligned}&|\psi_i(t,x_2,z_2)-\psi_i(t,x_1,z_1)|\\
&\le|\psi_i(t,x_2,z_2)-\psi_i(t,x_1,z_2)|+|\psi_i(t,x_1,z_2)-\psi_i(t,x_1,z_1)|\\
&\le\|h_i\|_\infty\|g_i\|_\infty[f_i]_{C^\beta(\mathbb{R}^d)}(1+|z_2|)|x_2-x_1|^\beta+\|h_i\|_\infty\|f_i\|_\infty|(1+|z_2|)g_i(z_2)-(1+|z_2|)g_i(z_1)|\\
&\quad+\|h_i\|_\infty\|f_i\|_\infty|(1+|z_2|)g_i(z_1)-(1+|z_1|)g_i(z_1)|\\
&\le\|h_i\|_\infty\|g_i\|_\infty[f_i]_{C^\beta_b(\mathbb{R}^d)}(1+|z_2|)|x_2-x_1|^\beta+(1+|z_2|)\|h_i\|_\infty\|f_i\|_\infty[g_i]_{C^\beta_b(\mathbb{R}^{md})}|z_2-z_1|^\beta\\
&\quad+\|h_i\|_\infty\|f_i\|_\infty\|g_i\|_\infty|z_2-z_1|\\
&\le\|h_i\|_\infty\|g_i\|_\infty[f_i]_{C^\beta_b(\mathbb{R}^d)}(1+|z_2|)|x_2-x_1|^\beta+(1+|z_2|)\|h_i\|_\infty\|f_i\|_\infty[g_i]_{C^\beta_b(\mathbb{R}^{md})}|z_2-z_1|^\beta\\
&\quad+\|h_i\|_\infty\|f_i\|_\infty\|g_i\|_\infty(1+|z_1|+|z_2|)|z_2-z_1|^\beta,\end{aligned}$$

for any $t\in[0,T]$, $x_1,x_2\in\mathbb{R}^d$, $z_1,z_2\in\mathbb{R}^{md}$ and $i=1,\dots,m$. Hence, condition (6.9) is satisfied with $\Theta^2=\sum_{i=1}^m\|h_i\|_\infty^2\|f_i\|^2_{C^\beta_b(\mathbb{R}^d)}\|g_i\|^2_{C^\beta_b(\mathbb{R}^{md})}$. Clearly, also condition (6.10) is satisfied.

Without loss of generality, we can suppose that α (see Hypotheses 2.1) and β coincide, and hereafter we simply denote them by α.

Let us prove the existence of a mild solution $\mathbf{u}$ of (SL-CP), *i.e.*, a function $\mathbf{u}\in\boldsymbol{\mathcal{K}}_T$ which satisfies the equation

$$\mathbf{u}(t,x)=(\widehat{\mathbf{G}}(T-t,0)\mathbf{g})(x)-\int_t^T(\widehat{\mathbf{G}}(T-t,T-s)(\boldsymbol{\Psi}(\mathbf{u}))(s,\cdot))(x)\mathrm{d}s,\qquad (t,x)\in[0,T]\times\mathbb{R}^d, \tag{6.14}$$

Here, $\boldsymbol{\mathcal{K}}_T$ is the set of all functions $\mathbf{u}\in C_b([0,T]\times\mathbb{R}^d;\mathbb{R}^m)\cap C^{0,1}([0,T)\times\mathbb{R}^d;\mathbb{R}^m)$ such that $\|\mathbf{u}\|_{\boldsymbol{\mathcal{K}}_T}:=\|\mathbf{u}\|_\infty+[\mathbf{u}]_{\boldsymbol{\mathcal{K}}_T}:=\|\mathbf{u}\|_\infty+\sup_{t\in[0,T)}\sqrt{T-t}\|\sqrt{Q}(t,\cdot)(J_x\mathbf{u}(t,\cdot))^T\|_\infty<+\infty$. Moreover, $\widehat{\mathbf{G}}(t,s)$ is the evolution operator associated with the elliptic operators $\sum_{i,j=1}^d Q_{ij}(T-\cdot,\cdot)D^2_{ij}+\sum_{j=1}^d B_j(T-\cdot,\cdot)D_j+C(T-\cdot,\cdot)$.

We first approximate $\mathbf{g}$ and $\boldsymbol{\psi}$ by two sequences $(\mathbf{g}_n)$ and $(\boldsymbol{\psi}^{(n)})$ of globally Lipschitz continuous functions with respect to x and the pair (x,z), respectively, defined as follows: $\mathbf{g}_n(x):=(\varrho_n\star\mathbf{g})(x)$ and $\boldsymbol{\psi}^{(n)}(t,x,z):=\vartheta(n^{-1}|z|)(\varrho_n\star\boldsymbol{\psi}(t,\cdot,\cdot))(x,z)$ for any $n\in\mathbb{N}$, $t\in[0,T]$, $x\in\mathbb{R}^d$ and $z\in\mathbb{R}^{md}$, where $\star$ denotes the convolution operator, $\varrho_n(x,z)=\vartheta(n^{-1}|x|)\vartheta(n^{-1}|z|)$ for any $(x,z)\in\mathbb{R}^d\times\mathbb{R}^{md}$, and $\vartheta:\mathbb{R}\to\mathbb{R}$ is smooth and satisfy $\chi_{D_1}\le\vartheta\le\chi_{D_2}$.

Remark 6.6.

(i) Since we are assuming Hypotheses 5.1 with $M = \sqrt{Q}$, the evolution operator $\widehat{\mathbf{G}}(t,s)$ satisfies the estimate $\sqrt{t-s}\|\sqrt{Q}(T-t,\cdot)(J_x\widehat{\mathbf{G}}(t,s)\mathbf{f})^T\|_\infty \le K_T\|\mathbf{f}\|_\infty$ for any $0 \le s < t \le T$, $\mathbf{f} \in C_b(\mathbb{R}^d;\mathbb{R}^m)$ and some positive constant K_T. From Hypothesis 2.1, we also deduce that $\sqrt{t-s}\|(J_x\widehat{\mathbf{G}}(t,s)\mathbf{f})^T\|_\infty \le K_T\lambda_0^{-1/2}\|\mathbf{f}\|_\infty$ for the same s and $\mathbf{f}$.

(ii) The choices of $\mathbf{g}_n$ and $\boldsymbol{\psi}^{(n)}$ are also connected with the application to systems of forward backward stochastic differential equations (FBSDEs in short) of the next subsection, where we show that the convergence of the subsequence $(\mathbf{u}_{k_n})$ implies that a sequence of solutions $(\mathbf{Y}_{k_n}, \mathbf{Z}_{k_n})$ to a family of approximated systems of FBSDEs converges to a pair of processes $(\mathbf{Y}, \mathbf{Z})$.

Proposition 6.7. *For any $n \in \mathbb{N}$ there exists a unique mild solution $\mathbf{u}_n \in \mathcal{K}_T$ to the Cauchy problem (SL-CP), with $(\boldsymbol{\psi}, \mathbf{g})$ replaced by $(\boldsymbol{\psi}^{(n)}, \mathbf{g}_n)$. Moreover, there exists a positive constant L, independent of n, such that $\|\mathbf{u}_n\|_{\mathcal{K}_T} \le L$ for any $n \in \mathbb{N}$.*

Proof. The proof is classical, hence we do not enter too much in details. To enlighten the notation, throughout the proof, we denote by K a positive constant, depending at most on T, which may vary from line to line. Since each function $\boldsymbol{\psi}^{(n)}(t,x,\cdot)$ is Lipschitz continuous in $\mathbb{R}^{md}$, uniformly with respect to $(t,x) \in [0,T]\times\mathbb{R}^d$, classical arguments, based on the Banach fixed-point theorem and the generalized Gronwall lemma, allow us to show that equation (6.14) admits a unique solution, which is defined in $[0,T]\times\mathbb{R}^d$. To prove that $\sup_{n\in\mathbb{N}}\|\mathbf{u}_n\|_{\mathcal{K}_T} < +\infty$, we observe that $|\boldsymbol{\psi}^{(n)}(t,x,z)| \le K(1+|z|)$ and $\|\mathbf{g}_n\|_\infty \le K\|\mathbf{g}\|_\infty$ for any $t \in [0,T]$, $x \in \mathbb{R}^d$, $z \in \mathbb{R}^{md}$ and $n \in \mathbb{N}$. From equation (6.14), with $\boldsymbol{\psi}^{(n)}$ replacing $\boldsymbol{\psi}$, and Remark 6.6(i) we thus deduce that

$$\sqrt{T-t}\|\sqrt{Q}(t,\cdot)(J_x\mathbf{u}_n(t,\cdot))^T\|_\infty \le K_T(T-t)^{-\frac{1}{2}}\|\mathbf{g}\|_\infty + K + K\int_t^T \frac{\|\sqrt{Q}(t,\cdot)(J_x\mathbf{u}_n(t,\cdot))^T\|_\infty}{\sqrt{s-t}}\mathrm{d}s$$

for any $t \in [0,T)$. The generalized Gronwall lemma shows that $\sqrt{T-t}\|\sqrt{Q}(t,\cdot)(J_x\mathbf{u}_n(t,\cdot))^T\|_\infty$ is uniformly bounded for any $t \in [0,T)$ by a positive constant independent of $n \in \mathbb{N}$. Now, taking the sup-norm (with respect to $x \in \mathbb{R}^d$) of both the sides of the equation

$$\mathbf{u}_n(t,x) = (\widehat{\mathbf{G}}(T-t,0)\mathbf{g}_n)(x) - \int_t^T (\widehat{\mathbf{G}}(T-t,T-s)(\boldsymbol{\Psi}^{(n)}(\mathbf{u}_n))(s,\cdot))(x)\mathrm{d}s, \qquad (t,x) \in [0,T)\times\mathbb{R}^d \tag{6.15}$$

and using this last estimate, we deduce that the sup-norm of $\mathbf{u}_n$ in $[0,T]\times\mathbb{R}^d$ is bounded from above by a positive constant independent of $n \in \mathbb{N}$. This completes the proof. $\square$

To go further and prove that a subsequence $(\mathbf{u}_{k_n}) \subset (\mathbf{u}_n)$ converges to a mild solution to (SL-CP), we need an intermediate result. For any $n \in \mathbb{N}$, we introduce the space $\mathcal{X}_{T,n} := C_b([0,T-n_T^{-1}]\times\overline{B}_{n_T};\mathbb{R}^m)\cap C^{0,1}([0,T-n_T^{-1}]\times B_{\hat{n}};\mathbb{R}^m)$, where $n_T := [1/T]+n$, and the operator $\boldsymbol{\Gamma}^{(n)}$, defined by

$$(\boldsymbol{\Gamma}^{(n)}(\mathbf{u}))(t,x) := (\widehat{\mathbf{G}}(T-t,0)\mathbf{g}_n)(x) - \int_{t+n_T^{-1}}^T (\widehat{\mathbf{G}}(T-t,T-s)(\boldsymbol{\Psi}^{(n)}(\mathbf{u}))(s,\cdot))(x)\mathrm{d}s$$

for any $k,n \in \mathbb{N}$, any $(t,x) \in [0,T-n_T^{-1}]\times\mathbb{R}^d$ and any $\mathbf{u} \in \mathcal{K}_T$.

Proposition 6.8. *The operator $\boldsymbol{\Gamma}^{(n)}$ is compact from $\mathcal{K}_T$ in $\mathcal{X}_{T,n}$.*

Proof. Fix $n \in \mathbb{N}$ and prove that the integral term $\tilde{\boldsymbol{\Gamma}}^{(n)}$ in the definition of $\boldsymbol{\Gamma}^{(n)}$ is a compact operator. For this purpose, let $\mathcal{W}$ be a bounded subset of $\mathcal{K}_T$. We claim that the families $\mathfrak{J}_{h,i} = \{D_i^h\tilde{\boldsymbol{\Gamma}}^{(n)}(\mathbf{w}) : \mathbf{w} \in \mathcal{W}\}$ ($h = 0,1$, $i = 1,\ldots,d$) are equibounded and equicontinuous. Throughout the proof, we denote by K_n positive constants, which may vary from line to line and depend on n. The equiboundedness of the families $\mathfrak{J}_{h,i}$ follows easily from estimates (3.1) and Remark 6.6(i), taking into account that $|\boldsymbol{\psi}^{(n)}(t,x,z)| \le K(1+|z|)$ for any $t \in [0,T]$, $x \in \mathbb{R}^d$,

$z \in \mathbb{R}^{md}$ and some positive constant K independent of n. To prove the equicontinuity of $\mathfrak{I}_{h,i}$, we fix $\mathbf{w} \in \mathcal{W}$, $r, t \in [0, T - n_T^{-1}]$, with $t > r$, $x, y \in D_{n_T}$ and observe that

$$\begin{aligned}|(D_i^h \tilde{\boldsymbol{\Gamma}}^{(n)}(\mathbf{w}))(t,x) - (D_i^h \tilde{\boldsymbol{\Gamma}}^{(n)}(\mathbf{w}))(r,y)| &\le |(D_i^h \tilde{\Gamma}^{(n)}(\mathbf{w}))(r,x) - (D_i^h \tilde{\Gamma}^{(n)}(\mathbf{w}))(r,y)| \\ &+ \left| \int_{r+n_T^{-1}}^{t+n_T^{-1}} (D_i^h (\widehat{\mathbf{G}}(T-r, T-s)(\boldsymbol{\Psi}^{(n)}(\mathbf{w}))(s,\cdot)))(x) \mathrm{d}s \right| \\ &+ \left| \int_{t+n_T^{-1}}^{T} (D_i^h (\widehat{\mathbf{G}}(T-t, T-s) - \widehat{\mathbf{G}}(T-r, T-s))(\boldsymbol{\Psi}^{(n)}(\mathbf{w})))(s,\cdot))(x) \mathrm{d}s \right| \end{aligned} \tag{6.16}$$

for $h = 0, 1$ and $i = 1, \ldots, d$. To estimate the right-hand side of (6.16) we use several times the inequality

$$\|\mathbf{G}(\cdot, t_1)(\Psi^{(n)}(\mathbf{w}))(s,\cdot)\|_{C^{1+\alpha/2, 2+\alpha}([t_2,T]\times E_0;\mathbb{R}^m)} \le K_{\delta, E_0}(T-s)^{-1/2}, \tag{6.17}$$

which holds true for any $t_1, t_2 \in [0, T)$ such that $t_2 - t_1 \ge \delta$ and any convex compact set $E_0 \subset \mathbb{R}^d$, with proper choices of t_1 and t_2. This estimate follows from Theorem A.2 which shows that $\|\mathbf{G}(\cdot, t_1)\mathbf{f}\|_{C^{1+\alpha/2,2+\alpha}([t_2,T]\times E_0;\mathbb{R}^m)} \le K'_{E_0,\delta}\|\mathbf{f}\|_\infty$ for any $\mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$ and any t_1, t_2 as above and the estimate $\|(\Psi^{(n)}(\mathbf{w}))(s,\cdot)\|_{C_b(\mathbb{R}^d;\mathbb{R}^m)} \le K(T-s)^{-1/2}$, which holds true for any $s \in [0, T)$, $\mathbf{w} \in \mathcal{W}$ and follows from (6.10) and the definition of the function $\Psi^{(n)}(\mathbf{w})$ (see the beginning of this subsection). Using (6.17) we can estimate

$$\begin{aligned}&|(D_i^h \tilde{\Gamma}^{(n)}(\mathbf{w}))(r,x) - (D_i^h \tilde{\Gamma}^{(n)}(\mathbf{w}))(r,y)| \le \\ &\qquad K_{D_{n_T}, n_T^{-1}}|x-y| \int_{r+n_T^{-1}}^{T} \|D_x^2 \widehat{\mathbf{G}}(T-r, T-s)(\boldsymbol{\Psi}^{(n)}(\mathbf{u}))(s,\cdot)\|_\infty \mathrm{d}s \le 2K_{D_{n_T}, n_T^{-1}}\sqrt{T}|x-y|.\end{aligned}$$

The other two terms in the right-hand side of (6.16) can be estimated likewise. Summing up, we thus conclude that $|(D_i^h \tilde{\Gamma}^{(n)}(\mathbf{w}))(t,x) - (D_i^h \tilde{\Gamma}^{(n)}(\mathbf{w}))(r,y)| \le K_n((t-r)^{\alpha/2} + |x-y|)$. This estimate shows that the families $\mathfrak{I}_{h,i}$ ($h = 0, 1$, $i = 1, \ldots, d$) are equicontinuous in $[0, T - n_T^{-1}] \times D_{n_T}$. Arzelà–Ascoli Theorem allows us to conclude that $\tilde{\boldsymbol{\Gamma}}^{(n)}$ is compact from $\mathcal{K}_T$ to $\mathcal{X}_{T,n}$. □

Proposition 6.9. *Up to a subsequence, $(\mathbf{u}_n)$ and $(J_x\mathbf{u}_n)$ converge locally uniformly in $[0,T]\times\mathbb{R}^d$ and $[0,T)\times\mathbb{R}^d$, respectively. If we denote by $\mathbf{u}$ the limit of $(\mathbf{u}_n)$, then $\mathbf{u}$ belongs to $\mathcal{K}_T$ and is a mild solution of (SL-CP).*

Proof. Since each operator $\boldsymbol{\Gamma}^{(n)}$ is compact and the sequence $(\mathbf{u}_n)$, defined in Proposition 6.7 is bounded in $\mathcal{K}_T$ by a positive constant K, using a diagonal argument, we can define a subsequence $(\mathbf{u}_{k_n}) \subset (\mathbf{u}_n)$ such that, for any $m \in \mathbb{N}$, $\boldsymbol{\Gamma}^{(m)}(\mathbf{u}_{k_n})$ converges to some function $\boldsymbol{\zeta}^{(m)}$ in $\mathcal{X}_{T,m}$, as $n \to +\infty$.

Being rather long, we split the rest of the proof in some steps.

Step 1. Here, we prove that the sequences $(\boldsymbol{\Gamma}^{(k_n)}(\mathbf{u}_{k_n}))$ and $(J_x\boldsymbol{\Gamma}^{(k_n)}(\mathbf{u}_{k_n}))$ converge. Fix $h \in \mathbb{N}$, $(t,x) \in [0, T - h_T^{-1}) \times B_{h_T}$, and $l, n, m \in \mathbb{N}$ such that $n, m \ge l > h$. Then, for $\gamma = 0, 1$ and $i = 1, \ldots, d$ we estimate $|D_i^\gamma \boldsymbol{\Gamma}^{(k_n)}(\mathbf{u}_{k_n}) - D_i^\gamma \boldsymbol{\Gamma}^{(k_m)}(\mathbf{u}_{k_m})| \le \mathfrak{B}_{m,n,i}^{(l,\gamma)} + \mathfrak{C}_{i,\gamma}^{(l,m)} + \mathfrak{C}_{i,\gamma}^{(l,n)} + \mathfrak{D}_{i,\gamma}^{(l,m)} + \mathfrak{D}_{i,\gamma}^{(l,n)} + \mathfrak{E}_{i,\gamma}^{(m,n)}$, where

$$\begin{aligned}\mathfrak{B}_{m,n,i}^{(l,\gamma)}(t,x) &= |(D_i^\gamma \boldsymbol{\Gamma}^{(k_l)}(\mathbf{u}_{k_n}))(t,x) - (D_i^\gamma \boldsymbol{\Gamma}^{(k_l)}(\mathbf{u}_{k_m}))(t,x)|, \\ \mathfrak{C}_{i,\gamma}^{(l,p)}(t,x) &= \left| \int_{t+1/\hat{k}_l}^{T} (D_i^\gamma \widehat{\mathbf{G}}(T-t, T-s)[\boldsymbol{\Psi}^{(k_l)}(\mathbf{u}_{k_p}) - \boldsymbol{\Psi}^{(k_p)}(\mathbf{u}_{k_p})])(s,x) \mathrm{d}s \right|, \\ \mathfrak{D}_{i,\gamma}^{(l,p)}(t,x) &= \left| \int_{t+1/\hat{k}_p}^{t+1/\hat{k}_l} (D_i^\gamma \widehat{\mathbf{G}}(T-t, T-s)\boldsymbol{\Psi}^{(k_p)}(\mathbf{u}_{k_p}))(s,x) \mathrm{d}s \right|, \\ \mathfrak{E}_{i,\gamma}^{(m,n)}(t,x) &= |(\widehat{\mathbf{G}}(T-t, 0)(\mathbf{g}_{k_n} - \mathbf{g}_{k_m}))(x)|.\end{aligned}$$

Fix $\varepsilon > 0$. Clearly, from the first part of the proof, we conclude that, for any $l \in \mathbb{N}$, there exists $N_l \in \mathbb{N}$ such that $\mathfrak{B}^{(l,\gamma)}_{m,n,i}(t,x) \le \varepsilon/5$, for any $n, m \ge N_l$.

As far as the term $\mathfrak{C}^{(l,p)}_{i,\gamma}$ ($p = m, n$) is concerned, we observe that from Remark 6.6(i) we obtain

$$\begin{aligned}\mathfrak{C}^{(l,p)}_{i,\gamma} \le & K_T \int_{T-\delta}^{T} \frac{\|(\boldsymbol{\Psi}^{(k_l)}(\mathbf{u}_{k_p}))(s,\cdot) - (\boldsymbol{\Psi}^{(k_p)}(\mathbf{u}_{k_p}))(s,\cdot)\|_\infty}{(s-t)^{\gamma/2}} \mathrm{d}s \\ & + K_T \int_t^{T-\delta} \frac{\|(\boldsymbol{\Psi}^{(k_l)}(\mathbf{u}_{k_p}))(s,\cdot) - (\boldsymbol{\Psi}^{(k_p)}(\mathbf{u}_{k_p}))(s,\cdot)\|_\infty}{(s-t)^{\gamma/2}} \mathrm{d}s \end{aligned} \tag{6.18}$$

for any $\delta > 0$. Using the estimate $|\boldsymbol{\psi}^{(n)}(t,x,z)| \le K(1+|z|)$, which holds for any $t \in [0,T]$, $x \in \mathbb{R}^d$, $z \in \mathbb{R}^{md}$, $n \in \mathbb{N}$ and some positive constant K, we get $\|\boldsymbol{\Psi}^{(k_l)}(\mathbf{u}_{k_p}(s,\cdot)) - \boldsymbol{\Psi}^{(k_p)}(\mathbf{u}_{k_p}(s,\cdot))\|_\infty \le K(1+\sqrt{T-s})$. Therefore,

$$K_T \int_{T-\delta}^{T} \frac{\|(\boldsymbol{\Psi}^{(k_l)}(\mathbf{u}_{k_p}))(s,\cdot) - (\boldsymbol{\Psi}^{(k_p)}(\mathbf{u}_{k_p}))(s,\cdot)\|_\infty}{(s-t)^{\gamma/2}} \mathrm{d}s \le K(h_T^{-1} - \delta)^{-\frac{\gamma}{2}} \sqrt{\delta}(2+\sqrt{\delta}),$$

for any $t \in [0, T - h_T^{-1}]$, provided that $\delta < h_T^{-1}$.

As far as the second term in the right-hand side of (6.18) is concerned, we first observe that

$$|\boldsymbol{\psi}^{(n)}(t,x,z) - \boldsymbol{\psi}(t,x,z)| \le K(1+|z|)n^{-\alpha}, \qquad t \in [0,T],\ x \in \mathbb{R}^d,\ z \in D_n, \tag{6.19}$$

for any $n \in \mathbb{N}$, as it easily follows from the equality

$$\boldsymbol{\psi}^{(n)}(t,x,z) - \boldsymbol{\psi}(t,x,z) = \int_{\mathbb{R}^d} \mathrm{d}y_1 \int_{\mathbb{R}^{md}} \varrho_n(y_1,y_2)(\boldsymbol{\psi}(x-y_1, z-y_2) - \boldsymbol{\psi}(x,z))\mathrm{d}y_2,$$

for any $t \in [0,T]$, $x \in \mathbb{R}^d$, $z \in D_n$, and Hypotheses 6.4. Splitting $\boldsymbol{\Psi}^{(k_p)} - \boldsymbol{\Psi}^{(k_l)} = (\boldsymbol{\Psi}^{(k_p)} - \boldsymbol{\Psi}) - (\boldsymbol{\Psi}^{(k_l)} - \boldsymbol{\Psi})$ and using (6.19), we can estimate

$$\|(\boldsymbol{\Psi}^{(k_l)}(\mathbf{u}_{k_p}))(s,\cdot) - (\boldsymbol{\Psi}^{(k_p)}(\mathbf{u}_{k_p}))(s,\cdot)\|_\infty \le K k_l^{-\alpha}(1+\delta^{-1/2}),$$

for any $s \in [t, T-\delta]$ and $k_p \ge k_l \ge K\delta^{-\gamma/2}$, and we thus conclude that

$$\int_t^{T-\delta} \frac{\|(\boldsymbol{\Psi}^{(k_l)}(\mathbf{u}_{k_p})(s,\cdot) - (\boldsymbol{\Psi}^{(k_p)}(\mathbf{u}_{k_p}))(s,\cdot)\|_\infty}{(s-t)^{\gamma/2}} \mathrm{d}s \le K k_l^{-\alpha}(1+\delta^{-\frac{1}{2}})(T-\delta-t)^{1-\frac{\gamma}{2}}.$$

It is easy to check that we can fix δ small and l_1 large such that $\mathfrak{C}^{(l,m)}_{i,\gamma} \le \varepsilon/5$ and $\mathfrak{C}^{(l,n)}_{i,\gamma} \le \varepsilon/5$ for $m, n > l \ge l_1$.

Now we consider $\mathfrak{D}^{(l,p)}_{i,\gamma}$, which, thanks again to Remark 6.6(i), we estimate as follows:

$$\mathfrak{D}^{(l,p)}_{i,\gamma}(t,x) \le K_T \lambda_0^{-\gamma/2} \int_{t+\frac{1}{k_p}}^{t+\frac{1}{k_l}} (t-s)^{-\frac{\gamma}{2}}(1+K(T-s)^{-1/2})\mathrm{d}s \le K\lambda_0^{-\frac{\gamma}{2}}(1+\sqrt{h_T})k_l^{\frac{\gamma}{2}-1}$$

for any $p > l$. Hence, there exists $l_2 \in \mathbb{N}$ such that $\mathfrak{C}^{(l,p)}_{i,\gamma} \le \varepsilon/5$ in $[0, T-h_T^{-1}) \times B_{h_T}$ for any $p > l_2$.

As far as $\mathfrak{E}^{(m,n)}_{i,\gamma}$ is concerned, we observe that, since $\mathbf{g}_{k_n}$ converges to $\mathbf{g}$ locally uniformly in $\mathbb{R}^d$, from Proposition 3.2(ii) and Theorem A.2, we conclude that there exists $l_3 \in \mathbb{N}$ such that $\mathfrak{E}^{(m,n)}_{i,\gamma} \le \varepsilon$ for any $n, m \ge l_3$. Summing up, if $m, n \ge \max\{N_{l_1 \vee l_2}, l_1, l_2, l_3\}$, then $\|\boldsymbol{\Gamma}^{(k_n)}(\mathbf{u}_{k_n}) - \boldsymbol{\Gamma}^{(k_m)}(\mathbf{u}_{k_m})\|_{\mathfrak{X}_h} \le \varepsilon$, and we are done.

Step 2. Here, we show that also $(\mathbf{u}_{k_n})$ converges in $\mathfrak{X}_{T,h}$, for any $h \in \mathbb{N}$. For this purpose, we observe that, from (6.15) it follows that $\mathbf{u}_{k_n} = \boldsymbol{\Gamma}^{(k_n)}(\mathbf{u}_{k_n}) - \mathfrak{F}_n$, where

$$\mathfrak{F}_n(t,x) := \int_t^{t+1/k_n} (\widehat{\mathbf{G}}(T-t, T-s)(\boldsymbol{\Psi}^{(k_n)}(\mathbf{u}_{k_n})))(s,x)\mathrm{d}s, \qquad t \in [0, T-h_T^{-1}] \times B_{h_T}.$$

Arguing as we did to estimate the term $\boldsymbol{\mathfrak{D}}_{i,\gamma}^{(l,p)}$, we conclude that $\boldsymbol{\mathcal{F}}_n$ and $J_x\boldsymbol{\mathcal{F}}_n$ vanish uniformly in $[0,T-h_T^{-1}]\times B_{h_T}$, as $n\to+\infty$. By the arbitrariness of h, we conclude that there exists a function $\mathbf{u}$ such that $\mathbf{u}_{k_n}$ and $J_x\mathbf{u}_{k_n}$ converges locally uniformly in $[0,T)\times\mathbb{R}^d$ to $\mathbf{u}$ and $J_x\mathbf{u}$, respectively, as $n\to+\infty$.

Step 3. Here, we conclude the proof, by showing that $\mathbf{u}$ is a mild solution of (SL-CP) and $\mathbf{u}\in\boldsymbol{\mathcal{K}}_T$. Since the sequence $(\mathbf{u}_n)$ is bounded in $\boldsymbol{\mathcal{K}}_T$, $\mathbf{u}$ belongs to $\boldsymbol{\mathcal{K}}_T$ as well. On the other hand, since

$$\mathbf{u}_{k_n}(t,x)=(\widehat{\mathbf{G}}(T-t,0)\mathbf{g}_{k_n})(x)-\int_t^T(\widehat{\mathbf{G}}(T-t,T-s)(\boldsymbol{\Psi}^{(k_n)}(\mathbf{u}_{k_n})))(s,x)\mathrm{d}s \tag{6.20}$$

and $J_x\mathbf{u}_{k_n}(s,\cdot)$ converges to $J_x\mathbf{u}(s,\cdot)$ locally uniformly in $[0,T)\times\mathbb{R}^d$, using Proposition 3.2 and the properties of $\boldsymbol{\psi}^{(k_n)}$ we deduce that $\widehat{\mathbf{G}}(T-t,T-s)((\boldsymbol{\Psi}^{k_n}(\mathbf{u}_{k_n}))(s,\cdot))$ converges pointwise in $[0,T)\times\mathbb{R}^d$ to $\widehat{\mathbf{G}}(T-t,T-s)((\boldsymbol{\Psi}(\mathbf{u}))(s,\cdot))$. Moreover, due again to the estimate $|\boldsymbol{\psi}^{(n)}(t,x,z)|\le K(1+|z|)$, we can let n tend to $+\infty$ in (6.20) and conclude that $\mathbf{u}$ satisfies (SL-CP) in $[0,T)\times\mathbb{R}^d$. Finally, we extend $\mathbf{u}$ by continuity in T setting $\mathbf{u}(T,\cdot)=\mathbf{g}$. This completes the proof. □

6.2.2. Systems of markovian FBSDEs

Here, we study a system of forward-backward stochastic differential equations and show that its solution can be written in terms of the mild solution to a semilinear Cauchy problem of the type considered in the previous subsection.

Let $(\Omega,\mathcal{E},\mathbb{P})$ be a complete probability space and let (W_t) be a d-dimensional Wiener process. By $(\mathcal{F}_t^W)$ we denote its natural filtration augmented with the negligible sets in $\mathcal{E}$.

For any $p\in[1,+\infty)$, we denote by $\mathbb{H}^p$ and $\mathbb{K}$, respectively, the space of progressively measurable with respect to $\mathcal{F}_t^W$ random processes (X_t) such that $\|X\|_{\mathbb{H}^p}:=\mathbb{E}\sup_{t\in[0,T]}|X_t|^p<+\infty$, and the space of all the $\mathcal{F}_t^W$-progressively measurable processes $(\mathbf{Y},\mathbf{Z})$ such that

$$\|(\mathbf{Y},\mathbf{Z})\|_{cont}^2:=\mathbb{E}\sup_{t\in[0,T]}|\mathbf{Y}_t|^2+\mathbb{E}\int_0^T|\mathbf{Z}_\sigma|^2\mathrm{d}\sigma<+\infty.$$

We consider the system (1.6), with $\mathbf{H}=(H_1,\dots,H_m)$ and

$$H_j(t,x,z):=\sum_{i=1}^d\sum_{k=1}^m(\tilde{B}_i(t,x)(P(t,x))^{-1})_{jk}z_{ki}+\psi_j(t,x,z),\qquad j=1,\dots,m,$$

under the following assumptions.

Hypotheses 6.10.

(i) $P_{ij}=P_{ji}\in C_{\rm loc}^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)\cap C_{\rm loc}^{1,2+\alpha}([0,T]\times\mathbb{R}^d)$ *for some* $\alpha\in(0,1)$ *and any* $i,j=1,\dots,d$*; moreover, the function* λ_P *is bounded from below by a positive constant;*

(ii) *The entries of the vector-valued function* $\mathbf{b}=(b_1,\dots,b_d)$ *in Hypothesis 2.3 and of the matrices-valued functions* $\tilde{B}_i$ $(i=1,\dots,d)$ *belong to* $C_{\rm loc}^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^d)\cap C_{\rm loc}^{0,1+\alpha}([0,T]\times\mathbb{R}^d)$*; moreover* $\mathbf{g}\in C_b(\mathbb{R}^d;\mathbb{R}^m)$*;*

(iii) $\mathbf{b}$ *and* P *grow at most linearly as* $|x|\to+\infty$*, uniformly with respect to* $t\in[0,T]$*. Moreover,*

- *the function* $\Lambda_{\mathcal{H}+\mathcal{H}^T}$ *is bounded from above in* $[0,T]\times\mathbb{R}^d$*, where*

$$\mathcal{H}:=P(J_xb)^TP^{-1}-\sum_{j=1}^db_j(D_jP)P^{-1}-\frac{1}{2}\sum_{i,j=1}^d(P^2)_{ij}(D_{ij}P)P^{-1},$$

and $\langle\mathbf{b}(t,x),x\rangle\le b_0(t,x)|x|$ *for any* $(t,x)\in[0,T]\times\mathbb{R}^d$ *and some negative function* b_0*;*

- $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}|\tilde{B}_i(t,x)|(\lambda_P(t,x))^{-1}<+\infty$*;*

- *The functions*

$$\sup_{t\in[0,T]}\frac{|\Lambda_{P^2}(t,\cdot)|D_kP(t,\cdot)|^2+(\Lambda_P(t,\cdot)|D_kP^2(t,\cdot)|)^2+(\lambda_{P^2}(t,\cdot)|\tilde{B}_i(t,\cdot)|)^2}{\lambda_{P^2}(t,\cdot)(\lambda_P(t,\cdot))^2|\Lambda_{\mathcal{H}+\mathcal{H}^T}(t,\cdot)|}$$

$$\sup_{t\in[0,T]}\frac{\Lambda_P(t,\cdot)|D_k\tilde{B}_i(t,\cdot)|+\lambda_P(t,x)|(P(t,\cdot))^{-1}D_tP(t,\cdot)(P(t,\cdot))^{-1}|}{\lambda_P(t,\cdot)|\Lambda_{\mathcal{H}+\mathcal{H}^T}(t,\cdot)|}$$

$$\inf_{t\in[0,T]}\frac{\Lambda_{P^2}(t,\cdot)|D_kP(t,\cdot)|}{b_0(t,\cdot)}$$

vanish as $|x|$ *tends to* $+\infty$ *for any* $i,k=1,\dots,d$*;*

(iv) *The function* $\boldsymbol{\psi}:[0,T]\times\mathbb{R}^d\times\mathbb{R}^{md}\to\mathbb{R}^m$ *satisfies* (6.9) *and* (6.10) *with* $\beta=\alpha$.

Remark 6.11. Hypotheses 6.10(i)–(iii) guarantee that the operator $\boldsymbol{\mathcal{A}}=\frac{1}{2}\sum_{i,j=1}^d(P^2)_{ij}(t,x)D_x^2+\sum_{j=1}^dB_j(t,x)D_j$, where $B_j(t,x)=-b_j(t,x)I_m+\tilde{B}_i$ satisfies the assumptions of Theorem 5.4 with $M=P$.

Example 6.12. Hypotheses 6.10 are satisfied for instance if we consider an arbitrary function $\mathbf{g}\in C_b(\mathbb{R}^d;\mathbb{R}^m)$, the functions $\boldsymbol{\psi}$ in Example 6.5 and the operator $\boldsymbol{\mathcal{A}}$ in Example 5.2, where now we take $p=0$, $C\equiv 0$ and $M(t,x)=P(t,x)=\sqrt{2q(t)}\sqrt{Q_0}(1+|x|^2)^{k/2}$ for any $(t,x)\in I\times\mathbb{R}^d$. In this case the conditions (5.8) (where we disregard (c) since $C\equiv 0$) reduce to requiring that $2r<k<2/3$.

We now denote by $\mathbf{u}$ the mild solution to (SL-CP) with $C\equiv 0$, provided by Proposition 6.9, and state the main result of this subsection.

Theorem 6.13. *For any* $0\le t\le\tau\le T$ *and* $x\in\mathbb{R}^d$*, set* $\mathbf{Y}(\tau,t,x):=\mathbf{u}(\tau,X(\tau,t,x))$ *and* $\mathbf{Z}(\tau,t,x):=P(\tau,X(\tau,t,x))(J_x\mathbf{u}(\tau,X(\tau,t,x)))^T$*. Then,* $(\mathbf{Y},\mathbf{Z})\in\mathbb{K}$ *and* $(X,\mathbf{Y},\mathbf{Z})$ *is an adapted solution to* (1.6)*.*

Proof. Throughout the proof, $(\mathbf{u}_{k_n})$ is the subsequence of $(\mathbf{u}_n)$ provided by Proposition 6.9, *i.e.*, $\mathbf{u}_{k_n}$ is the unique mild solution to (SL-CP), with $C\equiv 0$ and $\mathbf{g}$ and $\boldsymbol{\Psi}$ replaced by $\mathbf{g}_{k_n}$ and $\boldsymbol{\Psi}^{(k_n)}$, respectively.

Now we fix $t\in[0,T]$, $x\in\mathbb{R}^d$, set $X_\tau:=X(\tau,t,x)$ and, for any $\tau\in[t,T]$, define $\mathbf{Y}_\tau=\mathbf{u}(\tau,X_\tau)$, $\mathbf{Y}_\tau^{k_n}=\mathbf{u}_{k_n}(\tau,X_\tau)$, $\mathbf{Z}_\tau=P(\tau,X_\tau)(J_x\mathbf{u}(\tau,X_\tau))^T$ and $\mathbf{Z}_\tau^{k_n}=P(\tau,X_\tau)(J_x\mathbf{u}_{k_n}(\tau,X_\tau))^T$. Under Hypotheses 6.10, there exists a unique adapted process (X_t) with continuous trajectories such that $X_\tau=x+\int_t^\tau\mathbf{b}(\sigma,X_\sigma)\mathrm{d}\sigma+\int_t^\tau P(\sigma,X_\sigma)\mathrm{d}W_t$ (see [24], Thm. 3.4.6). By [17], $(\mathbf{Y}^{k_n},\mathbf{Z}^{k_n})$ satisfies the equation (recall that $\mathbf{g}_{k_n}$ and $\boldsymbol{\psi}^{(k_n)}$ are smooth approximations of $\mathbf{g}$ and $\boldsymbol{\psi}$)

$$\mathbf{Y}_\tau^{k_n}+\int_\tau^T\mathbf{Z}_\sigma^{k_n}\mathrm{d}W_\sigma=\mathbf{g}_{k_n}(X_T)+\int_\tau^T\mathbf{H}_{k_n}(\sigma,X_\sigma,\mathbf{Z}_\sigma^{k_n})\mathrm{d}\sigma,\qquad\mathbb{P}\text{-}a.s.,\ \ \tau\in[t,T].\tag{6.21}$$

Our aim consists in showing that we can let n tend to $+\infty$ in both the sides of (6.21) obtaining

$$\mathbf{Y}_\tau+\int_\tau^T\mathbf{Z}_\sigma\mathrm{d}W_\sigma=\mathbf{g}(X_T)+\int_\tau^T\mathbf{H}(\sigma,X_\sigma,\mathbf{Z}_\sigma)\mathrm{d}\sigma\qquad\mathbb{P}\text{-}a.s.,\ \ \tau\in[t,T].\tag{6.22}$$

Clearly, $\mathbf{g}_{k_n}$ converges to $\mathbf{g}$ locally uniformly in $\mathbb{R}^d$. Moreover, since $\mathbf{u}_{k_n}$ and $J_x\mathbf{u}_{k_n}$ converge locally uniformly in $[0,T)\times\mathbb{R}^d$ to $\mathbf{u}$ and $J_x\mathbf{u}$, respectively, $\mathbf{Y}_\tau^{k_n}$ and $\mathbf{Z}_\tau^{k_n}$ converge almost surely in Ω to $\mathbf{Y}_\tau$ and $\mathbf{Z}_\tau$, respectively, as $n\to+\infty$. As far as the integral term in the right-hand side of (6.21) is concerned, we begin by observing that

$$|\boldsymbol{\psi}_{k_n}(t,x,z_1)-\boldsymbol{\psi}(t,x,z_2)|\le K(1+|z_1|+|z_2|)\big(k_n^{-\alpha}+|z_1-z_2|^\alpha\big),\tag{6.23}$$

for any $t \in [0,T]$, $x \in \mathbb{R}^d$, $z_1 \in D_{k_n}$ and $z_2 \in \mathbb{R}^{md}$. Moreover, by (5.1) it follows that $|\tilde{B}_i| \le K\lambda_P$. Hence, from the identification formula for $\mathbf{Z}_\sigma^{k_n}$, the definition of $\mathbf{Z}$, Proposition 6.7 and (6.23) we conclude that

$$\begin{aligned}
&|H_{k_n,i}(\sigma, X_\sigma, \mathbf{Z}_\sigma^{k_n}) - H_i(\sigma, X_\sigma, \mathbf{Z}_\sigma)|\\
\le\ & \sum_{j=1}^d \sum_{k=1}^m |(\tilde{B}_j(t,x)(P(t,x))^{-1})_{ik}(\mathbf{Z}_\sigma^{k_n}) - \mathbf{Z}_\sigma)_{kj}| + |\psi_i(t,x,\mathbf{Z}_\sigma^{k_n}) - \psi_i(t,x,\mathbf{Z}_\sigma)|\\
\le\ & \sum_{j=1}^d |(\tilde{B}_j(t,x)(J_x(\mathbf{u}_{k_n}(\sigma,X_\sigma) - \mathbf{u}(\sigma,X_\sigma))^T)_{ij}| + |\psi_i(t,x,\mathbf{Z}_\sigma^{k_n}) - \psi_i(t,x,\mathbf{Z}_\sigma)|\\
\le\ & K\lambda_P(t,x)|J_x(\mathbf{u}_{k_n}(\sigma,X_\sigma) - \mathbf{u}(\sigma,X_\sigma))|\\
& + K(1+|\mathbf{Z}_\sigma^{k_n}|+|\mathbf{Z}_\sigma|)\big(k_n^{-\alpha}+|P(\sigma,X_\sigma)[(J_x(\mathbf{u}_{k_n}-\mathbf{u}))(\sigma,X_\sigma)]^T|^\alpha\big)\\
\le\ & K|\mathbf{Z}_\sigma^{k_n} - \mathbf{Z}_\sigma| + K(1+|\mathbf{Z}_\sigma^{k_n}|+|\mathbf{Z}_\sigma|)\big(k_n^{-\alpha}+|\mathbf{Z}_\sigma^{k_n} - \mathbf{Z}_\sigma|^\alpha\big)
\end{aligned}$$

$\mathbb{P}$-almost surely, for almost every $\sigma \in (\tau,T)$ and any $i = 1,\ldots,m$ and that the last side of the previous inequality vanishes $\mathrm{d}\sigma \otimes \mathbb{P}$-almost surely, as $n \to +\infty$. Moreover, $|\boldsymbol{\psi}_{k_n}(t,x,z)| \le K(1+|z|)$ for any $t \in [0,T]$, $x \in \mathbb{R}^d$, any $z \in \mathbb{R}^{md}$ and some positive constant K, independent of n, and this shows that

$$|\mathbf{H}_{k_n}(\sigma, X_\sigma, \mathbf{Z}_\sigma^{k_n})| \le K|\mathbf{Z}_\sigma^{k_n}| + |\psi_{k_n}(\sigma, X_\sigma, \mathbf{Z}_\sigma^{k_n})| \le K(T-\sigma)^{-\frac{1}{2}}.$$

By dominated convergence, we conclude that the integral term in the right-hand side of (6.21) converges $\mathbb{P}$-almost surely to $\int_\tau^T \mathbf{H}(\sigma, X_\sigma, \mathbf{Z}_\sigma)\mathrm{d}\sigma$.

It remains to prove the convergence of $\int_\tau^T \mathbf{Z}_\sigma^{k_n}\mathrm{d}W_\sigma$ to $\int_\tau^T \mathbf{Z}_\sigma\mathrm{d}W_\sigma$. First, we prove that $\int_\tau^T \mathbf{Z}_\sigma\mathrm{d}W_\sigma$ makes sense, since this is not guaranteed by the previous estimates, which show only that the growth of $\mathbf{Z}_\sigma$ can be estimated by $(T-\sigma)^{-1/2}$, which is not square integrable in (τ,T). For this purpose, we show that $(\mathbf{Z}_\tau^{k_n})$ is a Cauchy sequence $L^2((0,T)\times\Omega)$. This is enough to conclude that $\mathbf{Z}_\tau$ is a square integrable process since $\mathbf{Z}_\tau^{k_n}$ pointwise tends to $\mathbf{Z}_\tau$. Let us set $\overline{\mathbf{Y}}_\sigma^{n,m} := \mathbf{Y}_\sigma^{k_n} - \mathbf{Y}_\sigma^{k_m}$, $\overline{\mathbf{Z}}_\sigma^{n,m} := \mathbf{Z}_\sigma^{k_n} - \mathbf{Z}_\sigma^{k_m}$, $\overline{\mathbf{g}}_T^{n,m} := \mathbf{g}_{k_n}(X_T) - \mathbf{g}_{k_m}(X_T)$, $\overline{\mathbf{H}}_\sigma^{n,m} := \mathbf{H}_{k_n}(\sigma, X_\sigma, \mathbf{Z}_\sigma^{k_n}) - \mathbf{H}_{k_m}(\sigma, X_\sigma, \mathbf{Z}_\sigma^{k_m})$ for any $n,m \in \mathbb{N}$ and $\sigma \in [0,T]$. Integrating the Itô formula $d|\overline{\mathbf{Y}}_\tau^{n,m}|^2 = -2\langle\overline{\mathbf{Y}}_\tau^{n,m}, \overline{\mathbf{H}}_\tau^{n,m}\rangle\mathrm{d}\tau + 2\langle\overline{\mathbf{Y}}_\tau^{n,m}, \overline{\mathbf{Z}}_\tau^{n,m}\rangle\mathrm{d}W_\tau + |\overline{\mathbf{Z}}_\tau^{n,m}|^2\mathrm{d}\tau$, we obtain

$$|\overline{\mathbf{Y}}_\tau^{n,m}|^2 + \int_\tau^T |\overline{\mathbf{Z}}_\sigma^{n,m}|^2\mathrm{d}\sigma = |\overline{\mathbf{g}}_T^{n,m}|^2 + 2\int_\tau^T \langle\overline{\mathbf{Y}}_\sigma^{n,m}, \overline{\mathbf{H}}_\sigma^{n,m}\rangle\mathrm{d}\sigma - 2\int_\tau^T \langle\overline{\mathbf{Y}}_\sigma^{n,m}, \overline{\mathbf{Z}}_\sigma^{n,m}\rangle\mathrm{d}W_\sigma. \tag{6.24}$$

Since $(\mathbf{Y}^{k_n}, \mathbf{Z}^{k_n}), (\mathbf{Y}^{k_m}, \mathbf{Z}^{k_m}) \in \mathbb{K}$, the processes $\{\int_0^\tau \langle\overline{\mathbf{Y}}_\sigma^{n,m}, \overline{\mathbf{Z}}_\sigma^{n,m}\rangle\mathrm{d}W_\sigma : \tau \in [0,T]\}$ are martingales. Hence, from (6.24) it follows that

$$\mathbb{E}\int_\tau^T |\overline{\mathbf{Z}}_\sigma^{n,m}|^2\mathrm{d}\sigma = \mathbb{E}|\overline{\mathbf{g}}_T^{n,m}|^2 - \mathbb{E}|\overline{\mathbf{Y}}_\tau^{n,m}|^2 + 2\mathbb{E}\int_\tau^T \langle\overline{\mathbf{Y}}_\sigma^{n,m}, \overline{\mathbf{H}}_\sigma^{n,m}\rangle\mathrm{d}\sigma. \tag{6.25}$$

Since the sequence $(\mathbf{Y}_\tau^{k_n})$ is bounded and converges $\mathbb{P}$-almost surely to $\mathbf{Y}_\tau$, it is a Cauchy sequence in $L^2(\Omega,\mathbb{P})$. A similar argument can be applied to the sequence $(\mathbf{g}_{k_n})$. Finally, we note that

$$\left|\mathbb{E}\int_\tau^T \langle\overline{\mathbf{Y}}_\sigma^{n,m}, \overline{\mathbf{H}}_\sigma^{n,m}\rangle\mathrm{d}\sigma\right| \le \mathbb{E}\bigg(\sup_{\tau\in[0,T]}|\overline{\mathbf{Y}}_\tau^{n,m}|\int_\tau^T |\overline{\mathbf{H}}_\sigma^{n,m}|\mathrm{d}\sigma\bigg) \le 2L\mathbb{E}\int_\tau^T |\overline{\mathbf{H}}_\sigma^{n,m}|\mathrm{d}\sigma$$

and, since $\mathbf{H}_{k_n}(\sigma, X_\sigma, \mathbf{Z}_\sigma^{k_n})$ converges to $\mathbf{H}(\sigma, X_\sigma, \mathbf{Z}_\sigma)$ $(\mathrm{d}\sigma\otimes\mathbb{P})$-almost everywhere in $(0,T)\times\Omega$ and

$$\mathbf{H}_{k_n}(\sigma, X_\sigma, \mathbf{Z}_\sigma^{k_n}) \le K(T-\sigma)^{-1/2}$$

for any $\sigma \in (0,T)$, $(\mathbf{H}_{k_n}(\sigma, X_\sigma, \mathbf{Z}_\sigma^{k_n}))$ is a Cauchy sequence in $L^1((\tau,T)\times\Omega, \mathrm{d}\sigma\otimes\mathbb{P})$. From (6.25) it now follows that $(\mathbf{Z}^{k_n})$ is a Cauchy sequence in $L^2((\tau,T)\times\Omega, \mathrm{d}\sigma\otimes\mathbb{P})$. This implies that $\int_\tau^T \mathbf{Z}_\sigma\mathrm{d}W_\sigma$ makes sense and that

$\mathbf{Z}^{k_n}$ converges to $\mathbf{Z}$ in $L^2((0,T)\times\Omega)$. Thus, $\mathbb{E}|\int_\tau^T(\mathbf{Z}_\sigma^{k_n}-\mathbf{Z}_\sigma)\mathrm{d}W_\sigma|^2$ tends to 0 as $n\to+\infty$. We conclude that $\int_\tau^T\mathbf{Z}_\sigma^{k_n}\mathrm{d}W_\sigma$ tends to $\int_\tau^T\mathbf{Z}_\sigma\mathrm{d}W_\sigma$ in $\mathbb{P}$-almost surely. Hence, we can let $n\to+\infty$ in (6.21) and deduce that $(\mathbf{Y},\mathbf{Z})$ is a solution to (6.22). Clearly, we have also proved that $(\mathbf{Y},\mathbf{Z})\in\mathbb{K}$. □

Corollary 6.14. *For any $t\in[0,T]$, the law of the process $\{\mathbf{Y}_\tau\}_{\tau\in[t,T]}$, obtained as the limit of the sequence $\{\mathbf{Y}_\tau^{k_n}\}_{\tau\in[t,T]}$, is uniquely determined,* i.e., *if $(\Omega,\mathcal{F},\{\mathcal{F}_t\},\mathbb{P})$ and $(\tilde\Omega,\tilde{\mathcal{F}},\{\tilde{\mathcal{F}}_t\},\tilde{\mathbb{P}})$ are two probability spaces, and $\{\mathbf{Y}_\tau\}_{\tau\in[t,T]}$, $\{\tilde{\mathbf{Y}}_\tau\}_{\tau\in[t,T]}$ are the random processes of Theorem 6.13, related respectively to $(\Omega,\mathcal{F},\{\mathcal{F}_t\},\mathbb{P})$ and $(\tilde\Omega,\tilde{\mathcal{F}},\{\tilde{\mathcal{F}}_t\},\tilde{\mathbb{P}})$, then $\{\mathbf{Y}_\tau\}_{\tau\in[t,T]}$, $\{\tilde{\mathbf{Y}}_\tau\}_{\tau\in[t,T]}$ have the same law.*

Proof. The result is a straightforward consequence of the uniqueness in law of the solutions $(\mathbf{Y}^{k_n})$ of the approximated problems (6.21), and the $\mathbb{P}$-a.s. convergence of $(\mathbf{Y}^{k_n})$ to $\mathbf{Y}$. □

6.3. Nash Equilibrium for a non-zero stochastic differential game

In this last subsection, we adapt our results to obtain the existence of a Nash equilibrium to a non-zero sum stochastic differential game. Let $(\Omega,\mathcal{E},\mathbb{P})$ be a complete probability space and let (W_t) be a d-dimensional Wiener process. By $(\mathcal{F}_t^W)$ we denote its natural filtration augmented with the negligible sets in $\mathcal{E}$. Let $T>0$, $t\in[0,T)$ and $x\in\mathbb{R}^d$. As in the previous subsection, by $(X_\tau=X(\tau,t,x))$ we denote the unique adapted and continuous solution of the stochastic equation

$$X_\tau=x+\int_t^T\mathbf{b}(\sigma,X_\sigma)\mathrm{d}\sigma+\int_t^T P(\sigma,X_\sigma)\mathrm{d}W_\sigma,\qquad \mathbb{P}-a.s.,\ \ \tau\in[t,T].$$

We notice that, exactly as in the previous subsection, the assumptions on $\mathbf{b}=(b_1,\dots,b_d)$ (see Hypotheses 2.3) and P, formulated below (see Hypotheses 6.16), imply existence and uniqueness of the solution to the above equation.

We suppose that m players intervene on a system and, for any player $i=1,\dots,m$, we introduce the space of admissible controls $U_i:=\{u:[0,T]\times\Omega\to V_i : u \text{ is a predictable process}\}$, where $V_i\subset\mathbb{R}$ are prescribed sets. Moreover, we introduce the space of admissible strategies $\mathfrak{U}:=\otimes_{i=1}^m U_i$. Given $\mathbf{u}\in\mathfrak{U}$, we define

$$W_\tau^{(\mathbf{u})}:=W_\tau-\int_t^\tau\mathbf{r}_1(\sigma,X_\sigma)\mathrm{d}\sigma-\int_t^\tau\mathbf{r}_2(X_\sigma,\mathbf{u}_\sigma)\mathrm{d}\sigma,\qquad \mathbb{P}^{(\mathbf{u})}:=\rho^{(\mathbf{u})}\mathbb{P},\tag{6.26}$$

where $\rho^{(\mathbf{u})}=\exp\big(\int_t^\tau(\mathbf{r}_1(\sigma,X_\sigma)+\mathbf{r}_2(X_\sigma,\mathbf{u}_\sigma))\mathrm{d}\sigma-\frac{1}{2}\int_t^\tau(\mathbf{r}_1(\sigma,X_\sigma)+\mathbf{r}_2(X_\sigma,\mathbf{u}_\sigma))^2\mathrm{d}\sigma\big)$. Note that $W^{(\mathbf{u})}$ is a d-dimensional Wiener process with respect to $\mathbb{P}^{(\mathbf{u})}$ and that X satisfies

$$\begin{aligned}X_\tau=&x+\int_t^T\mathbf{b}(\sigma,X_\sigma)\mathrm{d}\sigma+\int_t^T P(\sigma,X_\sigma)\mathbf{r_1}(\sigma,X_\sigma)\mathrm{d}\sigma\\&+\int_t^T P(\sigma,X_\sigma)\mathbf{r}_2(X_\sigma,\mathbf{u}_\sigma)\mathrm{d}\sigma+\int_t^T P(\sigma,X_\sigma)\mathrm{d}W_\sigma^{(\mathbf{u})},\end{aligned}\tag{6.27}$$

$\mathbb{P}$-almost surely. For any $\mathbf{u}\in\mathfrak{U}$, to any player i, a cost functional J_i is associated, which depends on the strategies of the whole players. More precisely,

$$J_i(\mathbf{u})=\mathbb{E}^{(\mathbf{u})}\bigg[\int_0^T h_i(X_s,\mathbf{u}_s)\mathrm{d}s+g_i(X_T)\bigg],\qquad i=1,\dots,m,\tag{6.28}$$

where $\mathbb{E}^{(\mathbf{u})}$ is the expectation with respect to $\mathbb{P}^{(\mathbf{u})}$ and $\mathbf{h}:\mathbb{R}^d\times\mathfrak{U}\to\mathbb{R}^m$, the running cost, and $\mathbf{g}:\mathbb{R}^d\to\mathbb{R}^m$, the terminal cost, are bounded Borel measurable functions.

Definition 6.15. The strategy $\mathfrak{U}\ni\tilde{\mathbf{u}}=(\tilde u_1,\dots,\tilde u_m)$ is a Nash equilibrium to problem (6.27)–(6.28), if $J_i(\tilde{\mathbf{u}})\le J_i(\tilde u_1,\dots,\tilde u_{i-1},u_i,\tilde u_{i+1},\dots,\tilde u_m)$ for any $u_i\in U_i$ and any $i=1,\dots,m$.

If $\tilde{\mathbf{u}}$ is a Nash equilibrium for $i=1,\dots,m$, then the player i has no earn changing its control $\tilde{u}_i$, if the other $m-1$ players choose the strategy $(\tilde{u}_1,\dots,\tilde{u}_{i-1},\tilde{u}_{i+1},\dots,\tilde{u}_m)$. The Hamiltonian function $\tilde{\mathbf{H}}$ associated to this system is defined by $\tilde{H}_i(t,x,z,\mathbf{u}) := \langle z^i, \mathbf{r}_1(t,x)+\mathbf{r}_2(x,\mathbf{u})\rangle + h_i(x,\mathbf{u})$ for any $x\in\mathbb{R}^d$, $z\in\mathbb{R}^{md}$, $\mathbf{u}\in U$ (U being the set of all the controls) and $i=1,\dots,m$. We assume the following assumptions on $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{h}=(h_1,\dots,h_m)$ and $\tilde{\mathbf{H}}$.

Hypotheses 6.16.

(i) $\mathbf{h}\in C_b^\gamma(\mathbb{R}^d\times\mathfrak{U};\mathbb{R}^m)$ *for some* $\gamma\in(0,1)$, $\mathbf{r}_1\in C^{0,1+\gamma}_{\rm loc}([0,T]\times\mathbb{R}^d;\mathbb{R}^d)\cap C_b([0,T]\times\mathbb{R}^d;\mathbb{R}^d)$ *and* $\mathbf{r}_2\in C_b^\gamma(\mathbb{R}^d\times\mathfrak{U};\mathbb{R}^d)$*;*
(ii) *The functions* P, $\mathbf{b}$, $\tilde{B}_i := (P\mathbf{r}_1)_i I_m$ $(i=1,\dots,d)$ *and* $Q=\frac{1}{2}P^2$ *satisfy Hypotheses 6.10;*
(iii) $\tilde{\mathbf{H}}$ *satisfies the generalized minimax condition,* i.e., *for any* $i=1,\dots,m$ *there exists a function* $\tilde{\mathbf{u}}\in C([0,T]\times\mathbb{R}^d\times\mathbb{R}^{md};\mathfrak{U})$, *with* $\sup_{t\in[0,T]}\|\tilde{\mathbf{u}}(t,\cdot)\|_{C_b^\beta(\mathbb{R}^d\times\mathbb{R}^{md};\mathfrak{U})}<+\infty$ *for some* $\beta\in(0,1)$, *such that for any* $t\in[0,T]$, $x\in\mathbb{R}^d$, $z\in\mathbb{R}^{md}$ *and* $u_i\in U_i$, *with* $(i=1,\dots,m)$, *it holds that*

$$\tilde{H}_i(t,x,z,\tilde{\mathbf{u}}(t,x,z)) \le \tilde{H}_i(t,x,z,\tilde{u}_1(t,x,z),\dots,\tilde{u}_{i-1}(t,x,z),u_i,\tilde{u}_{i+1}(t,x,z),\dots,\tilde{u}_m(t,x,z)),$$

$\mathbb{P}$-*almost surely in* Ω.

Remark 6.17.

(i) Since $\tilde{B}_i := (P\mathbf{r_1})_i I_m$, the differential operator $\boldsymbol{\mathcal{A}}$ in (SL-CP) is uncoupled. We stress that this is the classical setting: indeed, comparing our situation with the classical literature (see *e.g.*, [10,11,14,15,32], it is possible to see that, in view of applications to differential games, the equations of the system of PDE's are coupled only in the semilinear term, while the linear part is the same for any component.
(ii) The function $\boldsymbol{\psi}$, whose components are $\boldsymbol{\psi}_i(t,x,z) := \langle z^i, \mathbf{r}_2(x,\tilde{\mathbf{u}}(t,x,z))\rangle + h_i(x,\tilde{\mathbf{u}}(t,x,z))$ for any $t\in[0,T]$, $x\in\mathbb{R}^d$, $z\in\mathbb{R}^{md}$ and $i=1,\dots,m$, satisfies Hypothesis 6.10(iv) with $\alpha:=\beta\gamma$.
(iii) Under Hypotheses 6.16, Theorem 6.13 holds true.

Example 6.18. Suppose that the operator $\boldsymbol{\mathcal{A}}$ is as in Example 6.12 with $\tilde{b}\equiv 0$, *i.e.*, $(\boldsymbol{\mathcal{A}}\mathbf{u})_h = q(t)(1+|x|^2)^k \mathrm{Tr}(Q_0 D_x^2 u_h) - b(t)\langle x, \nabla_x u_h\rangle$ for any $h=1,\dots,m$ and any smooth enough function $\mathbf{u}$, where $q,b\in C^{\alpha/2}([0,T])$, q_0 has positive infimum, Q_0 is a constant positive definite matrix, $0<k<2/3$. Further, assume, for simplicity that $\mathbf{r}_1=\mathbf{0}$ and

$$r_{2,j}(t,x,\mathbf{u}) = \sum_{s=1}^m f_{js}(u_s), \qquad h_i(x,\mathbf{u}) = h_i(x,u_i), \quad i=1,\dots,m, \ j=1,\dots,d,$$

where $f_{is}:U_s\to\mathbb{R}$ and $h_i:\mathbb{R}^d\times U_i\to\mathbb{R}$ are smooth functions, for any $i=1,\dots,d$ and $s=1,\dots,m$. Then, Hypothesis 6.16(i), (ii) are satisfied and the minimax condition reduces to

$$\sum_{j=1}^d z_j^i f_{ji}(\tilde{u}_i(t,x,z)) + h_i(t,x,\tilde{u}_i(t,x,z)) \le \sum_{j=1}^d z_j^i f_{ji}(u_i) + h_i(t,x,u_i(t,x,z)),$$

for any $u_i\in U_i$, $t\in[0,T]$, $x\in\mathbb{R}^d$, $z\in\mathbb{R}^{md}$ and $i=1,\dots,m$.
If we take $f_{is}(u_s)=u_s$, $h_i(x,\mathbf{u})=|u_i|^2$, $U_i=\overline{D_1}\subset\mathbb{R}^d$ and we define $\phi(\eta) := |\eta|\eta^{-1}((|\eta|/2)\wedge 1)$ and $w_i := \sum_{j=1}^d z_j^i$ for any $i,s=1,\dots,m$, then the function $\tilde{\mathbf{u}} = (-\phi(w_1),\dots,-\phi(w_m))$ has the claimed smoothness and satisfies the minimax condition. Indeed, let us fix $i\in\{1,\dots,m\}$. If $|w_i|\ge 2$, then $\tilde{u}_i(t,x,z) = -|w_i|/w_i$ and the minimax condition becomes $-|w_i|+1-w_iu_i-(u_i)^2\le 0$ which is satisfied since

$$-|w_i|+1-w_iu_i-u_i^2 \le -|w_i|+|w_i||u_i|+1-u_i^2 = (1-|u_i|)(1+|u_i|-|w_i|)\le 0.$$

On the other hand, if $|w_1|\le 2$, then $\tilde{u}_i(t,x,z) = -w_i/2$ and the minimax condition is $-(w_i)^2/4 - w_iu_i - u_i^2\le 0$, which is clearly satisfied. Note that ϕ is a Lipschitz continuous map.

Theorem 6.19. *There exists a Nash equilibrium for problem* (6.27)–(6.28).

Proof. We begin by proving that $J_1(\tilde{\mathbf{u}}) \le J_1(u_1, \tilde{u}_2, \dots, \tilde{u}_m)$ for any $u_1 \in U_1$, where $\tilde{\mathbf{u}}$ is as in Hypothesis 6.16. For this purpose, we fix $(u_1)_t$ arbitrarily in U_i, set $\widehat{\mathbf{u}}^1_t := ((u_1)_t, (\tilde{u}_2)_t, \dots, (\tilde{u}_m)_t)$ and observe that X_τ satisfies the equation

$$\begin{aligned}X_\tau =& x + \int_t^\tau \mathbf{b}(\sigma, X_\sigma)\mathrm{d}\sigma + \int_t^\tau P(\sigma, X_\sigma)\mathbf{r}_1(\sigma, X_\sigma)\mathrm{d}\sigma \\ &+ \int_t^\tau P(\sigma, X_\sigma)\mathbf{r}_2(X_\sigma, \widehat{\mathbf{u}}^1_\sigma)\mathrm{d}\sigma + \int_t^\tau P(\sigma, X_\sigma)\mathrm{d}W^{(\widehat{\mathbf{u}}^1)}_\sigma,\end{aligned}$$

for any $\tau \in [t, T]$, where $\tilde{W}^{(\hat{\mathbf{u}})}_\tau$ is as in (6.26), with $\mathbf{u}$ replaced by $\widehat{\mathbf{u}}^1$. We now introduce the backward system

$$\mathbf{Y}_\tau + \int_t^\tau \mathbf{Z}_\sigma d\tilde{W}_\sigma = \mathbf{g}(X_T) + \int_t^\tau \tilde{\mathbf{H}}(\sigma, X_\sigma, \mathbf{Z}_\sigma, \widehat{\mathbf{u}}^1_\sigma)\mathrm{d}\sigma,$$

where $\tilde{\mathbf{H}}$ has been introduced after Definition 6.15. By Theorem 6.13, this system admits a solution $(X, \mathbf{Y}, \mathbf{Z})$. Writing the backward system with respect to $W^{(\widehat{\mathbf{u}}^1)}$, we get

$$\mathbf{Y}_\tau + \int_\tau^T \mathbf{Z}_\sigma \mathrm{d}W^{(\widehat{\mathbf{u}}^1)}_\sigma + \int_\tau^T \mathbf{Z}_\sigma \mathbf{r}_1(\sigma, X_\sigma)\mathrm{d}\sigma + \int_\tau^T \mathbf{Z}_\sigma \mathbf{r}_2(X_\sigma, \widehat{\mathbf{u}}^1_\sigma)\mathrm{d}\sigma = \mathbf{g}(X_T) + \int_\tau^T \mathbf{H}(\sigma, X_\sigma, \mathbf{Z}_\sigma)\mathrm{d}\sigma. \tag{6.29}$$

Note that $\mathbb{E}^{(\widehat{\mathbf{u}}^1)}\left(\int_0^T |\mathbf{Z}_t|^2 dt\right)^{1/2} < +\infty$. Indeed,

$$\left|\int_\tau^T \mathbf{Z}_\sigma \mathrm{d}W^{(\widehat{\mathbf{u}}^1)}_\sigma\right| \le 2 \sup_{\tau\in[0,T]} |\mathbf{Y}_\tau| + \int_\tau^T \big(|\mathbf{Z}_\sigma| + |\mathbf{H}(\sigma, X_\sigma, \mathbf{Z}_\sigma)|\big)\mathrm{d}\sigma.$$

Taking into account the Burkholder–Davis–Gundy inequalities (see [22], Thm. 3.28) and using the estimate $|\mathbf{H}(\sigma, X_\sigma, \mathbf{Z}_\sigma)| \le |\mathbf{H}(\sigma, X_\sigma, 0)| + |\mathbf{Z}_\sigma|^{1+\alpha}$, which follows from Hypothesis 6.16 and Remark 6.17, we get

$$\begin{aligned}\mathbb{E}^{(\widehat{\mathbf{u}}^1)}\bigg(\int_0^T |\mathbf{Z}_t|^2 dt\bigg)^{\frac{1}{2}} &\le K\mathbb{E}^{(\widehat{\mathbf{u}}^1)} \sup_{\tau\in[0,T]} |\mathbf{Y}_\tau| + K\mathbb{E}^{(\widehat{\mathbf{u}}^1)}\int_\tau^T (|\mathbf{Z}_\sigma| + |\mathbf{H}(\sigma, X_\sigma, 0)|)\mathrm{d}\sigma + K\mathbb{E}^{(\widehat{\mathbf{u}}^1)}\int_\tau^T |\mathbf{Z}_\sigma|^{1+\alpha}\mathrm{d}\sigma \\ &\le K(\tilde{\mathbb{E}}\varrho^2)^{\frac{1}{2}}\bigg(\tilde{\mathbb{E}} \sup_{\tau\in[0,T]} |\mathbf{Y}_\tau| + \tilde{\mathbb{E}}\int_\tau^T \big(|\mathbf{Z}_\sigma|^2 + |\mathbf{H}(\sigma, X_\sigma, 0)|^2\big)\mathrm{d}\sigma\bigg)^{\frac{1}{2}} \\ &\quad + K(\tilde{\mathbb{E}}\varrho^{\frac{2}{1-\alpha}})^{\frac{1-\alpha}{2}}\bigg(\tilde{\mathbb{E}}\int_\tau^T |\mathbf{Z}_\sigma|^2\mathrm{d}\sigma\bigg)^{\frac{2}{1+\alpha}},\end{aligned}$$

for some positive constant K, which may vary from line to line, and the last side of the previous chain of inequalities is finite. Hence, taking the conditional expectation in (6.29) with respect to $\mathbb{P}^{(\widehat{\mathbf{u}}^1)}$ and $\tau = t$, we obtain

$$\mathbf{Y}_t = \mathbb{E}^{(\widehat{\mathbf{u}}^1)}[\varphi(X_T)|\mathcal{F}_t] + \mathbb{E}^{(\widehat{\mathbf{u}}^1)}\bigg[\int_t^T [\mathbf{H}(\sigma, X_\sigma, \mathbf{Z}_\sigma, \widehat{\mathbf{u}}^1_\sigma) - \mathbf{Z}_\sigma(\mathbf{r}_1(\sigma, X_\sigma) + \mathbf{r}_2(X_\sigma, \widehat{\mathbf{u}}^1_\sigma))]\mathrm{d}\sigma\bigg|\mathcal{F}_t\bigg].$$

Adding and subtracting $\mathbb{E}^{(\widehat{\mathbf{u}}^1)}\left[\int_t^T \mathbf{h}(X_\sigma, \widehat{\mathbf{u}}^1_\sigma)\mathrm{d}\sigma \Big| \mathcal{F}_t\right]$ and setting $t=0$, we get

$$\begin{aligned}\mathbf{Y}_0 =&\ \mathbf{J}(\widehat{\mathbf{u}}^1) + \mathbb{E}^{(\widehat{\mathbf{u}}^1)}\left[\int_0^T [\mathbf{H}(\sigma, X_\sigma, \mathbf{Z}_\sigma,) - \mathbf{Z}_\sigma(\mathbf{r}_1(\sigma, X_\sigma) + \mathbf{r}_2(X_\sigma, \widehat{\mathbf{u}}^1_\sigma)) - \mathbf{h}(X_\sigma, \widehat{\mathbf{u}}^1_\sigma)]\mathrm{d}\sigma\right]\\ =&\ \mathbf{J}(\widehat{\mathbf{u}}^1) + \mathbb{E}^{(\widehat{\mathbf{u}}^1)}\left[\int_0^T (\mathbf{H}(\sigma, X_\sigma, \mathbf{Z}_\sigma) - \tilde{\mathbf{H}}(\sigma, X_\sigma, \mathbf{Z}_\sigma, \widehat{\mathbf{u}}^1_\sigma))\mathrm{d}\sigma\right],\end{aligned} \tag{6.30}$$

where $\mathbf{J} = (J_1, \ldots, J_m)$. Considering the first component of the first and last side of (6.30) and using the minimax condition in Hypothesis 6.16(iii), we conclude that $Y_{0,1} \le J_1(\widehat{\mathbf{u}}^1)$. Applying the same argument with $\widehat{\mathbf{u}}^1$ replaced by $\tilde{\mathbf{u}}$, we get $Y_{0,1} = J_1(\tilde{\mathbf{u}})$. Moreover, the same procedure yields that $J_k(\tilde{\mathbf{u}}) = Y_{0,k} \le J_k(\widehat{\mathbf{u}}^k)$ for any $k = 1, \ldots, m$. □

Appendix A. *A priori* estimates for solutions to parabolic systems

In this appendix we prove some *a priori* estimates for classical solutions to nonautonomous parabolic equations associated with systems of elliptic operators as in (1.1) which satisfy Hypotheses 2.1. To enlighten the notation, we set $\|\cdot\|_{h,R} = \|\cdot\|_{C^h_b(D_R;\mathbb{R}^k)}$ for any $h \in \mathbb{N}\cup\{0\}$ and $k \in \mathbb{N}$, $R > 0$. Moreover, for any $\alpha > 0$, we denote by $\|\cdot\|_\alpha$ the Euclidean norm in $C^\alpha_b(\mathbb{R}^h)$ when $h \in \{1, d\}$. Finally, the open disk with center at x_0 and radius $R > 0$ is denoted by $D_R(x_0)$.

We recall that, for any $0 \le \alpha < \theta$ and any bounded domain Ω of class C^θ, there exists a positive constant K such that

$$\|\mathbf{f}\|_{C^\alpha(\overline{\Omega};\mathbb{R}^k)} \le K\|\mathbf{f}\|_\infty^{1-\frac{\alpha}{\theta}}\|\mathbf{f}\|^{\frac{\alpha}{\theta}}_{C^\theta(\overline{\Omega};\mathbb{R}^k)}, \tag{A.1}$$

for any $\mathbf{f} \in C^\theta(\overline{\Omega};\mathbb{R}^k)$ ($k \ge 1$). (The same estimate holds true with $\Omega = \mathbb{R}^k$.) Moreover, if $\mathcal{T}$ belongs to $\mathcal{L}(C(\overline{\Omega};\mathbb{R}^k); C^\beta(\overline{\Omega};\mathbb{R}^k)) \cap \mathcal{L}(C^\theta_0(\overline{\Omega};\mathbb{R}^k); C^\beta(\overline{\Omega};\mathbb{R}^k))$ for some $\beta, \theta > 0$, then $\mathcal{T}$ is bounded from $C^\alpha_0(\overline{\Omega};\mathbb{R}^k)$ to $C^\beta(\overline{\Omega};\mathbb{R}^k)$, for any $\alpha \in (0,\theta)\setminus\mathbb{N}$, and

$$\|\mathcal{T}\|_{\mathcal{L}(C^\theta_0(\overline{\Omega};\mathbb{R}^k);C^\beta(\overline{\Omega};\mathbb{R}^k))} \le \|\mathcal{T}\|^{1-\frac{\alpha}{\theta}}_{\mathcal{L}(C(\overline{\Omega};\mathbb{R}^k);C^\beta(\overline{\Omega};\mathbb{R}^k))}\|\mathcal{T}\|_{\mathcal{L}(C^\theta_0(\overline{\Omega};\mathbb{R}^k);C^\beta(\overline{\Omega};\mathbb{R}^k))}. \tag{A.2}$$

Proposition A.1. *Let $\Omega \subset \mathbb{R}^d$ be an open set, $T > s \in I$ and $\mathbf{u} \in C_b([s,T]\times\overline{\Omega};\mathbb{R}^m) \cap C^{1,2}((s,T)\times\Omega;\mathbb{R}^m)$ satisfy the equation $D_t\mathbf{u} = \mathcal{A}\mathbf{u} + \mathbf{g}$ in $(s,T)\times\Omega$ for some $\mathbf{g} \in C^{\alpha/2,\alpha}((s,T)\times\Omega;\mathbb{R}^m)$. Further, assume that the function $t \mapsto (t-s)\|\mathbf{u}(t,\cdot)\|_{C^2_b(\Omega;\mathbb{R}^m)}$ is bounded in (s,T). Then, for any $R_1 > 0$ and any $x_0 \in \Omega$, such that $D_{R_1}(x_0) \Subset \Omega$, there exists a positive constant $K_0 = K_0(R_1, \lambda_0, s, T)$ such that, for any $t \in (s,T)$,*

$$\begin{aligned}(t-s)\|D^2_x\mathbf{u}(t,\cdot)\|_{L^\infty(D_{R_1}(x_0);\mathbb{R}^m)} + \sqrt{t-s}\,\|J_x\mathbf{u}(t,\cdot)\|_{L^\infty(D_{R_1}(x_0);\mathbb{R}^m)} \le&\\ K_0(\|\mathbf{u}\|_{C_b([s,T]\times\overline{\Omega};\mathbb{R}^m)} + \|\mathbf{g}\|_{C^{\alpha/2,\alpha}((s,T)\times\Omega;\mathbb{R}^m)}).&\end{aligned} \tag{A.3}$$

Proof. Throughout the proof, we denote by K a positive constant, which can vary from line to line, may depend on R_1 and T, but it is independent of n. Up to a translation, we can assume that $x_0 = 0$. We fix R_1 as in the statement and R_2 such that $D_{R_2} \Subset \Omega$. Then, we define $r_n := 2R_1 - R_2 + (2-2^{-n})(R_2 - R_1)$ for any $n \in \mathbb{N}\cup\{0\}$. Further, we consider a sequence $(\vartheta_n) \subset C^\infty_c(\mathbb{R}^d)$ of functions satisfying $\chi_{D_{r_n}} \le \vartheta_n \le \chi_{D_{r_{n+1}}}$ for any $n \in \mathbb{N}$ and $\|\vartheta_n\|_{C^k_b(\mathbb{R}^d)} \le 2^{kn}c$ for any $k = 0,1,2,3$. Let us set $\mathbf{u}_n := \vartheta_n\mathbf{u}$ and observe that each function $\mathbf{u}_n$ vanishes on $[s,T]\times\partial D_{r_{n+1}}$ and $D_t\mathbf{u}_n = \mathcal{A}\mathbf{u}_n + \mathbf{g}_n$ in $(s,T)\times D_{r_{n+1}}$, where $\mathbf{g}_n = \vartheta_n\mathbf{g} - \mathrm{Tr}(QD^2\vartheta_n)\mathbf{u} - 2(J_x\mathbf{u})Q\nabla\vartheta_n - \sum_{j=1}^d D_j\vartheta_n B_j\mathbf{u}$. In view of the variation-of-constants-formula it thus follows that

$$\mathbf{u}(t,x) = (\mathbf{G}^{\mathcal{D}}_{n+1}(t,s)\vartheta_n\mathbf{u}(s,\cdot))(x) + \int_s^t (\mathbf{G}^{\mathcal{D}}_{n+1}(t,r)\mathbf{g}_n(r,\cdot))(x)\mathrm{d}r, \qquad t \in [s,T],\ x \in D_{r_n}.$$

It is well known that $(t-s)\|\mathbf{G}^{\mathcal{D}}_{n+1}(t,s)\mathbf{h}\|_{2,r_{n+1}}\le K\|\mathbf{h}\|_{0,r_{n+1}}$ and $\sqrt{t-s}\|\mathbf{G}^{\mathcal{D}}_{n+1}(t,s)\mathbf{k}\|_{2,r_{n+1}}\le K\|\mathbf{k}\|_{1,r_{n+1}}$ for any $t\in(s,T)$, any $n\in\mathbb{N}$ and any $\mathbf{h}\in C(D_{r_{n+1}};\mathbb{R}^m)$, $\mathbf{k}\in C^1_0(D_{r_{n+1}};\mathbb{R}^m)$, respectively. Note that the constant K in the previous two estimates is independent of n since it depends on the ellipticity constant of the operator $\boldsymbol{\mathcal{A}}$ and on the Hölder norms of its coefficients in $(s,T)\times D_{r_{n+1}}$, which can be estimated in terms of the same norms taken in $(s,T)\times D_{R_2}$.

Estimate (A.2), with $\theta=1$, $\beta=2$, $\mathcal{T}=\mathbf{G}^{\mathcal{D}}_{n+1}(t,s)$, yields that $(t-s)^{1-\frac{\alpha}{2}}\|\mathbf{G}^{\mathcal{D}}_{n+1}(t,s)\boldsymbol{\varphi}\|_{2,r_{n+1}}\le K\|\boldsymbol{\varphi}\|_{\alpha,r_{n+1}}$ for any $\boldsymbol{\varphi}\in C^\alpha_0(D_{r_{n+1}};\mathbb{R}^m)$ and $t\in(s,T)$. Since $\mathbf{g}_n(\sigma,\cdot)\in C^\alpha_0(D_{r_{n+1}};\mathbb{R}^m)$ for any $\sigma\in(s,s+1)$, we can estimate

$$(t-s)\|\mathbf{u}_n(t,\cdot)\|_{2,r_n}\le K\|\mathbf{u}\|_\infty+K\int_s^t(t-\sigma)^{-1+\frac{\alpha}{2}}\|\mathbf{g}_n(\sigma,\cdot)\|_{\alpha,r_{n+1}}\mathrm{d}\sigma,\qquad t\in(s,T).\tag{A.4}$$

Note that $\|\mathbf{g}_n(\sigma,\cdot)\|_{\alpha,r_{n+1}}\le K\|\vartheta_n\|_{2+\alpha,r_{n+1}}(\|\mathbf{u}(\sigma,\cdot)\|_{1+\alpha,r_{n+1}}+\|\mathbf{g}(\sigma,\cdot)\|_{\alpha,r_{n+1}})$ for any $\sigma\in(s,T)$. Using (A.1) and Young inequality, for any $\sigma\in(s,T)$, $\varepsilon>0$ and $n\in\mathbb{N}$ we get

$$\begin{aligned}\|\mathbf{u}(\sigma,\cdot)\|_{1,r_{n+1}}&\le K(\sigma-s)^{-\frac12}\|\mathbf{u}\|_\infty^{\frac12}\zeta_{n+1}^{\frac12}\le(\sigma-s)^{-\frac12}(K\varepsilon^{-1}\|\mathbf{u}\|_\infty+\varepsilon\zeta_{n+1}),\\ \|J_x\mathbf{u}(\sigma,\cdot)\|_{\alpha,r_{n+1}}&\le K(\sigma-s)^{-\frac{\alpha+1}{2}}\|\mathbf{u}\|_\infty^{\frac{1-\alpha}{2}}\zeta_{n+1}^{\frac{1+\alpha}{2}}\le(\sigma-s)^{-\frac{\alpha+1}{2}}(K\varepsilon^{-\frac{1+\alpha}{1-\alpha}}\|\mathbf{u}\|_\infty+\varepsilon\zeta_{n+1}),\end{aligned}$$

where $\zeta_n:=\sup_{\sigma\in(s,T)}(\sigma-s)\|\mathbf{u}(\sigma,\cdot)\|_{2,r_n}$. Since $\|\vartheta_n\|_{C^{2+\alpha}_b(\mathbb{R}^d)}\le K8^n$ for any $n\in\mathbb{N}$, from these last three estimates we conclude that

$$\|\mathbf{g}_n(\sigma,\cdot)\|_{\alpha,r_{n+1}}\le 8^n(\sigma-s)^{-\frac{\alpha+1}{2}}(K\varepsilon^{-\frac{1+\alpha}{1-\alpha}}\|\mathbf{u}\|_\infty+\varepsilon\zeta_{n+1})+K8^n\|\mathbf{g}(\sigma,\cdot)\|_{\alpha,r_{n+1}},\tag{A.5}$$

for any $\sigma\in(s,s+1)$ and $\varepsilon>0$. Replacing (A.5) into (A.4) yields

$$\zeta_n\le K\|\mathbf{u}\|_\infty+8^nK\left(\varepsilon^{-\frac{1+\alpha}{1-\alpha}}\|\mathbf{u}\|_\infty+\varepsilon\zeta_{n+1}+\|\!|\mathbf{g}|\!\|_{\alpha/2,\alpha}\right),\qquad n\in\mathbb{N},\ \ \varepsilon\in(0,1),\tag{A.6}$$

where $\|\!|\mathbf{g}|\!\|_{\alpha/2,\alpha}=\|\mathbf{g}\|_{C^{\alpha/2,\alpha}((s,T)\times\Omega;\mathbb{R}^m)}$. Let us fix $\eta\in(0,64^{-1/(1-\alpha)})$ and $\varepsilon=8^{-n}K^{-1}\eta$. Multiplying both sides of (A.6) by η^n and summing from 1 to $N\in\mathbb{N}$, we get

$$\zeta_0-\eta^{N+1}\zeta_{N+1}\le K\|\mathbf{u}\|_\infty\sum_{j=0}^N 64^{\frac{j}{1-\alpha}}\eta^j+K\|\!|\mathbf{g}|\!\|_{\alpha/2,\alpha}\sum_{j=0}^N(8\eta)^j\le K(\|\mathbf{u}\|_\infty+\|\!|\mathbf{g}|\!\|_{\alpha/2,\alpha}),$$

since both the two series converge, due to the choice of η. To conclude, we observe that $\eta^{N+1}\zeta_{N+1}$ tends to 0 as $N\to+\infty$. Indeed, by assumptions, ζ_{N+1} is bounded, uniformly with respect to N. It thus follows that $\eta^{n+1}\zeta_{n+1}$ vanishes as $n\to+\infty$. We have so proved that $(t-s)\|\mathbf{u}(t,\cdot)\|_{2,R_1}\le K(\|\mathbf{u}\|_\infty+\|\!|\mathbf{g}|\!\|_{\alpha/2,\alpha})$ for any $t\in(s,T)$. Again, estimate (A.1) implies that $\|J_x\mathbf{u}(t,\cdot)\|_{0,R_1}\le K\|\mathbf{u}(t,\cdot)\|_{0,R_1}^{1/2}\|\mathbf{u}(t,\cdot)\|_{2,R_1}^{1/2}$ for any $t\in(s,T)$, and this allows us to complete the proof of (A.3). □

Theorem A.2. *Fix $T>s\in I$ and let $\mathbf{u}\in C^{1+\alpha/2,2+\alpha}_{\mathrm{loc}}((s,T]\times\mathbb{R}^d;\mathbb{R}^m)$ satisfy the differential equation $D_t\mathbf{u}=\boldsymbol{\mathcal{A}}\mathbf{u}+\mathbf{g}$ in $(s,T]\times\mathbb{R}^d$, for some $\mathbf{g}\in C^{\alpha/2,\alpha}_{\mathrm{loc}}((s,T]\times\mathbb{R}^d;\mathbb{R}^m)$. Then, for any $\tau\in(0,T-s)$ and any pair of bounded open sets Ω_1 and Ω_2 such that $\Omega_1\Subset\Omega_2$, there exists a positive constant K, depending on $\Omega_1,\Omega_2,\tau,s,T$, but being independent of $\mathbf{u}$, such that*

$$\|\mathbf{u}\|_{C^{1+\alpha/2,2+\alpha}((s+\tau,T)\times\Omega_1;\mathbb{R}^m)}\le K(\|\mathbf{u}\|_{C_b((s+\tau/2,T)\times\Omega_2;\mathbb{R}^m)}+\|\mathbf{g}\|_{C^{\alpha/2,\alpha}((s+\tau/2,T)\times\Omega_2;\mathbb{R}^m)}).\tag{A.7}$$

Moreover, for any bounded set $J\subset I$ and any $\delta_0>0$, the constant K depends on δ_0 but is independent of $s,T\in J$ such that $T-s\ge\delta_0$.

Proof. The main step of the proof consists in showing that, for any $x_0 \in \overline{\Omega_1}$ and any $r > 0$ such that $D_{2r}(x_0) \Subset \Omega_2$, estimate (A.7) is satisfied with $\Omega_1 = D_r(x_0)$ and $\Omega_2 = D_{2r}(x_0)$. Indeed, once this latter estimate is proved, a covering argument will allow us to obtain easily estimate (A.7). So, let us prove (A.7) with Ω_1 and Ω_2 as above and $x_0 = 0$ (indeed, up to a translation, we can reduce to this case). Throughout the proof, we denote by K a positive constant, independent of $\mathbf{u}$, $\mathbf{g}$ and n, which may vary from line to line. Moreover, for any $\alpha, \beta \geq 0$, we denote by $\|\cdot\|_{\alpha,\beta}$ the Euclidean norm of the space $C^{\alpha,\beta}([s,T]\times\mathbb{R}^d;\mathbb{R}^m)$.

For any $n \in \mathbb{N}$, $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$, set

$$\varphi_n(t) = \varphi\left(1 + \frac{t - s - t_{n+1}}{t_n - t_{n+1}}\right), \qquad \vartheta_n(x) = \vartheta\left(1 + \frac{|x| - r_n}{r_{n+1} - r_n}\right),$$

where $\varphi, \vartheta \in C^\infty(\mathbb{R})$ satisfy $\chi_{[2,\infty)} \leq \varphi \leq \chi_{[1,\infty)}$ and $\chi_{(-\infty,1]} \leq \vartheta \leq \chi_{(-\infty,2]}$, $r_n = (2 - 2^{-n})r$ and $t_n = (2^{-1} + 2^{-n-1})\tau$, for any $n \in \mathbb{N}$. Each function $\mathbf{v}_n := \mathbf{u}\varphi_n\vartheta_n$ vanishes at $t = s$ and satisfies $D_t\mathbf{v}_n = \hat{\boldsymbol{\mathcal{A}}}\mathbf{v}_n + \mathbf{g}_n$, where $\mathbf{g}_n = \varphi_n\vartheta_n\mathbf{g} - \varphi_n\mathbf{u}\mathrm{Tr}(QD^2\vartheta_n) - \vartheta_n\sum_{j=1}^d D_j\vartheta_n B_j\mathbf{u} - 2\varphi_n(J_x\mathbf{u})Q\nabla\vartheta_n + \varphi_n'\vartheta_n\mathbf{u}$ and $\hat{\boldsymbol{\mathcal{A}}}$ is a nonautonomous elliptic operator with bounded and smooth coefficients, which coincide with the coefficients of the operator $\boldsymbol{\mathcal{A}}$ in $[s,T]\times D_{2r}$ (recall that $\mathbf{v}_n$ is compactly supported in $[s,T]\times D_{2r}$).

By well known results, $\|\mathbf{v}_n\|_{1+\alpha/2,2+\alpha} \leq K\|\mathbf{g}_n\|_{\alpha/2,\alpha}$ and, for any J and δ_0 as in the statement, the constant K is independent of $s, T \in J$ such that $T - s \geq \delta_0$. Estimating $\|\mathbf{g}_n\|_{\alpha/2,\alpha}$ we get

$$\begin{aligned}\|\mathbf{v}_n\|_{1+\alpha/2,2+\alpha} \leq& K\|\vartheta_n\|_3\|\varphi\|_2\big(\|\mathbf{u}\|_{C^{\alpha/2,1+\alpha}((s+t_{n+1},T)\times D_{r_{n+1}};\mathbb{R}^m)} + \|\mathbf{g}\|_{C^{\alpha/2,\alpha}((s+t_{n+1},T)\times D_{r_{n+1}};\mathbb{R}^m)}\big)\\ \leq& 2^{5n}K\big(\|\mathbf{v}_{n+1}\|_{\alpha/2,\alpha} + \|J_x\mathbf{v}_{n+1}\|_{\alpha/2,\alpha} + \|\mathbf{g}\|_{C^{\alpha/2,\alpha}((s+\tau/2,T)\times D_{2r};\mathbb{R}^m)}\big).\end{aligned} \tag{A.8}$$

Using (A.1) we get $\|\boldsymbol{\zeta}\|_\alpha \leq K(\varepsilon^{-j-\frac{(j+1)\alpha}{2}}\|\boldsymbol{\zeta}\|_\infty + \varepsilon\|\boldsymbol{\zeta}\|_{2+\alpha})$ $(j = 1,2)$ and $\|\boldsymbol{\varphi}\|_{\alpha/2} \leq K(\varepsilon^{-\frac{\alpha}{2}}\|\boldsymbol{\varphi}\|_\infty + \varepsilon\|\boldsymbol{\varphi}\|_{1+\alpha/2})$ for any $\boldsymbol{\zeta} \in C^{2+\alpha}_b(\mathbb{R}^d;\mathbb{R}^m)$, $\boldsymbol{\varphi} \in C^{1+\alpha/2}_b([s,T];\mathbb{R}^m)$, $\varepsilon(0,1)$ and some positive constant K independent of ε. Applying these estimates to $\mathbf{v}_{n+1}$, we deduce that

$$\|\mathbf{v}_{n+1}\|_{\alpha/2,\alpha} + \|J_x\mathbf{v}_{n+1}\|_{0,\alpha} \leq K(\varepsilon\|\mathbf{v}_{n+1}\|_{1+\alpha/2,2+\alpha} + \varepsilon^{-(1+\alpha)}\|\mathbf{v}_{n+1}\|_\infty), \qquad \varepsilon \in (0,1)\ \ n \in \mathbb{N}. \tag{A.9}$$

To estimate the $\alpha/2$-Hölder norm of the function $J_x\mathbf{v}_{n+1}(\cdot,x)$ for any $x \in \mathbb{R}^d$, we observe that $\mathbf{v}_{n+1} \in \mathrm{Lip}([s,T], C^\alpha_b(\mathbb{R}^n;\mathbb{R}^m))$ and $\|\mathbf{v}_{n+1}\|_{\mathrm{Lip}} \leq K\|\mathbf{v}_{n+1}\|_{1+\alpha/2,2+\alpha}$. Indeed, writing

$$\mathbf{v}_{n+1}(t_2,x) - \mathbf{v}_{n+1}(t_1,x) = \int_{t_1}^{t_2} D_t\mathbf{v}_{n+1}(\sigma,x)\mathrm{d}\sigma, \qquad t_1,t_2 \in [s,T],\ \ x \in \mathbb{R}^d,$$

we deduce that $\|\mathbf{v}_{n+1}(t_2,\cdot) - \mathbf{v}_{n+1}(t_1,\cdot)\|_\alpha \leq K\|D_t\mathbf{v}_{n+1}\|_{0,\alpha}|t_2 - t_1|$ for any $t_1, t_2 \in [s,T]$. Since the function $\mathbf{v}_{n+1}$ is bounded in $[s,T]$ with values in $C^{2+\alpha}_b(\mathbb{R}^d)$, by interpolation we get

$$\begin{aligned}\|\mathbf{v}_{n+1}(t_2,\cdot) - \mathbf{v}_{n+1}(t_1,\cdot)\|_1 \leq& K\|\mathbf{v}_{n+1}(t_1,\cdot) - \mathbf{v}_{n+1}(t_2,\cdot)\|_\alpha^{\frac{1+\alpha}{2}}\|\mathbf{v}_{n+1}(t_2,\cdot) - \mathbf{v}_{n+1}(t_1,\cdot)\|_{2+\alpha}^{\frac{1-\alpha}{2}}\\ \leq& K\|\mathbf{v}_{n+1}\|_{1+\alpha/2,2+\alpha}|t_2 - t_1|^{\frac{1+\alpha}{2}}\end{aligned}$$

for any $t_1, t_2 \in [s,T]$. This shows that $J_x\mathbf{v}_{n+1} \in C^{(1+\alpha)/2,0}((s,T)\times\mathbb{R}^d;\mathbb{R}^{md})$ and

$$\|J_x\mathbf{v}_{n+1}\|_{(1+\alpha)/2,0} \leq K\|\mathbf{v}_{n+1}\|_{1+\alpha/2,2+\alpha}. \tag{A.10}$$

Using the interpolative estimate $\|\boldsymbol{\varphi}\|_{\alpha/2} \leq K(\varepsilon\|\boldsymbol{\varphi}\|_{(1+\alpha)/2} + \varepsilon^{-\alpha}\|\boldsymbol{\varphi}\|_\infty)$, which holds for any $\varepsilon \in (0,1)$ and any $\boldsymbol{\varphi} \in C^{(1+\alpha)/2}_b([s,T];\mathbb{R}^m)$, (A.9) and (A.10), we obtain that

$$\|J_x\mathbf{v}_{n+1}\|_{\alpha/2,\alpha} \leq K(\varepsilon\|\mathbf{v}_{n+1}\|_{1+\alpha/2,2+\alpha} + \varepsilon^{-(1+\alpha)}\|\mathbf{v}_{n+1}\|_\infty + \varepsilon^{-\alpha}\|J_x\mathbf{v}_{n+1}\|_\infty). \tag{A.11}$$

Now, $\|J_x \mathbf{v}_{n+1}\|_\infty \le \delta \|\mathbf{v}_{n+1}\|_{0,2+\alpha} + \delta^{-\frac{1}{1+\alpha}}\|\mathbf{v}_{n+1}\|_\infty$ for any $\delta \in (0,1)$. Choosing $\delta = \varepsilon^{1+\alpha}$ and replacing this estimate in (A.11), we get $\|J_x \mathbf{v}_{n+1}\|_{\alpha/2,\alpha} \le K(\varepsilon \|\mathbf{v}_{n+1}\|_{1+\alpha/2,2+\alpha} + \varepsilon^{-(1+\alpha)}\|\mathbf{v}_{n+1}\|_\infty)$. This estimate, (A.8) and (A.9) yield

$$\|\mathbf{v}_n\|_{1+\alpha/2,2+\alpha} \le 2^{5n} K(\varepsilon \|\mathbf{v}_{n+1}\|_{1+\alpha/2,2+\alpha} + \varepsilon^{-(1+\alpha)}\|\mathbf{v}_{n+1}\|_\infty + \|\mathbf{g}\|_{C^{\alpha/2,\alpha}((s+\tau/2,T)\times D_{2r};\mathbb{R}^m)}),$$

for any $\varepsilon \in (0,1)$. We fix $\eta \in (0, 2^{-5(2+\alpha)})$ and choose $\varepsilon = \varepsilon_n = 2^{-5n}K^{-1}\eta$; from the previous estimate we obtain

$$\zeta_n \le (\eta \zeta_{n+1} + 2^{5n(2+\alpha)}K\|\mathbf{u}\|_{C_b((s+\tau/2)\times D_{2r})} + 2^{5n}K\|\mathbf{g}\|_{C^{\alpha/2,\alpha}((s+\tau/2,T)\times D_{2r};\mathbb{R}^m)}),$$

where $\zeta_n = \|\mathbf{v}_n\|_{1+\alpha/2,2+\alpha}$. Now, we can proceed as in the proof of Proposition A.1 and complete the proof. □

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