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THE INVERSE PROBLEM IN CONVEX OPTIMIZATION WITH LINEAR CONSTRAINTS *, **

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Abstract. In this paper, we solve an inverse problem arising in convex optimization. We consider a maximization problem under m linear constraints. We characterize the solutions of this kind of problems. More precisely, we give necessary and sufficient conditions for a given function in $\mathbb{R}^n$ to be the solution of a multi-constraint maximization problem. The conditions we give here extend well-known results in microeconomic theory.

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1. Introduction

In this paper, we consider a multi-constraint maximization problem of the form

$$\max_{x} \ f(x)$$
$$Ax \leq C(A)$$

where $x \in \mathbb{R}^n$, A is an $m \times n$ matrix and f and C are some functions that satisfy certain conditions which will be specified later. Hence, we are dealing with a multi-constraint maximization problem with linear constraints. The solution of this problem is a function of the parameters $A = (a_j^i)$. We assume certain conditions on the functions f and C that guarantee the differentiability of the solutions which we require to be at least of class C^2. Our main objective is to characterize the solutions of this type of optimization problems. We rely on the first order conditions and optimality conditions to achieve our objective. Moreover, we make use of the envelope theorem and the value function, $V(A) = f(x(A))$, of the above problem.

Such kind of problems arise in many applications especially in some economic contexts in microeconomic theory. Economic applications to this problem will be given in the sequel. Moreover, we will show that the results we get here generalize well-known results in consumer theory, see [6] for a recent survey. An inverse problem arising from economic theory was also solved by Ekeland and Djitté [8].

We use the indirect approach to deal with this problem. This approach depends on the value function, $V(A)$. The necessary and sufficient conditions on a given function $x(A) \in \mathbb{R}^n$ for the existence of a value function will

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be given. It turns out that the necessary and sufficient conditions will include a set of function λ_{ij}, $i, j = 1, \dots, m$ that can be computed from $x(A)$. The problem then is to find the objective function. This is a duality problem. We consider a class of functions introduced by Epstein [10] that is stable under duality.

Our problem will be split into mathematical integration problem and economic integration problem. The mathematical and the economic integration problems can be stated as follows:

- **Mathematical integration.** Given a function $x(A)$ and a family of functions λ_{ik}, $1 \le i, k \le m$, what are the necessary and sufficient conditions for the existence of $m + 1$ functions $\lambda_1, \dots, \lambda_m$ and V that satisfy equation (2.2) with $\lambda_{ik} = \lambda_i/\lambda_k$ and $C^i(a^i) = (a^i)^T x(A)$.
- **Economic integration.** In addition to the mathematical integration, we impose the following additional conditions on the functions that satisfy (2.2): the functions λ_i are strictly positive and the function V is quasi-convex with respect to each a^i for all $i = 1, \dots, m$.

Both of these problems will be solved. The duality problem will then be solved.

In this model, the objective function is assumed to satisfy a set of conditions that will be specified later. One of these conditions requires f to be strictly increasing in each of its arguments. This condition permits us to write the inequality constraints as equalities.

To get the necessary and sufficient conditions for mathematical integration, we use the techniques of exterior differential calculus that showed to be powerful for the treatment of such problems. A good reference to these techniques is the book by Bryant *et al.* [4]. We get local results; that is, the functions involved in the integration problem are defined in a neighbourhood of some given point. We define a family of differential forms and set up an integration problem using these forms. The solution of this integration problem, then, requires solving a nonlinear system of partial differential equations. The integration problem will be solved using Darboux Theorem [4].

The rest of the article is organized in the following way: in the next section, we set up the model and present its basic assumptions. Then, the main results that include the necessary and sufficient conditions for mathematical integration are given in Sections 3 and 4. In Section 5, the economic integration problem is solved. Then, duality problem is considered. The necessary and sufficient conditions for the 2-constraint case are given in Section 7. The geometry of the problem and some economic applications are finally discussed. Proofs of main results are gathered in the appendix.

2. Setting up the model

We consider a multi-constraint maximization problem of the form

$$(\mathcal{P}) \begin{cases} \max\limits_x \ f(x) \\ Ax = C(A) \end{cases}$$

Where f is a function that satisfies certain regularity and convexity conditions that will be specified later, A is an $m \times n$ matrix of rank m and $C : \mathbb{R}^{m\times n}_{++} \to \mathbb{R}^m_{++}$ is a given mapping. The ith constraint takes the form $(a^i)^T x = C^i(A)$ where a^i is the ith row of the matrix A. Define the Lagrangian function

$$L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_k \left(C^k(A) - \sum_{l=1}^{n} a^k_l x^l \right)$$

with $x \in \mathbb{R}^n_{++}$ and $\lambda \in \mathbb{R}^m_{++}$. The first order conditions for interior maximum are

$$\begin{aligned} \frac{\partial f}{\partial x^j} &= \sum_{k=1}^{m} \lambda_k a^k_j, \quad j = 1, \dots, n \\ Ax &= C(A) \end{aligned}$$

Define the value function of this problem by

$$V(A) = \max_x \left\{ f(x) + \sum_{k=1}^{m} \lambda_k \left(C^k(A) - \sum_{l=1}^{n} a_l^k x^l \right) \right\}$$

If the functions $C^1(a^1), \dots, C^m(a^m)$ are convex on $\mathbb{R}^n_{++}$ then the value function $V(a^1, \dots, a^m)$ is quasi-convex with respect to each a^i for $i = 1, \dots, m$, see [2].

Differentiating the function $V(A)$ with respect to a_j^i and using the envelope theorem we get

$$\frac{\partial V}{\partial a_j^i} = \sum_{k=1}^{m} \lambda_k \frac{\partial C^k}{\partial a_j^i} - \lambda_i x^j. \tag{2.1}$$

We suppose that C^k is a function of the vector $a^k \in \mathbb{R}^n_{++}$ only, where a^k is the kth row of the matrix A. Moreover, we assume that each component of the mapping $C(A)$ is not homogeneous of degree one because this entails division by zero. This implies, in particular that, the function $x(A)$ is not homogeneous of degree zero and the Lagrange multiplier corresponding to the ith constraint, $\lambda_i(A)$ is not homogeneous of degree -1 in a^i. The case of homogeneous mapping $C(A)$ will not be treated here. We adopt the following assumptions on the mapping C:

Assumption 2.1. For each $i \in \{1, \dots, m\}$, we assume that the function C^i has the following properties:

(a) $C^i : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ is a function of a^i only.
(b) C^i is a convex function of a^i.
(c) C^i is of class C^2.
(d) C^i is not homogeneous of degree one in a^i; that is, $(a^i)^T D_{a^i} C^i - C^i(a^i) \neq 0$.

We consider the following assumptions on the objective function f:

Assumption 2.2. Assume the function f satisfies the following conditions:

(1) f is strictly increasing in each of its arguments.
(2) the Hessian matrix $D_x^2 f$ is negative definite on the subspace $\{D_x f\}^\perp$.
(3) f is of class C^2.

By applying the implicit function theorem, one can show that the solution of the above maximization problem as well as the associated vector of Lagrange multipliers are of class C^2, we refer to [3] for details. Assumption 2.1(a) implies that $D_{a^i} C^k = 0$ if $i \neq k$ which reduces equation (2.1) to

$$\frac{\partial V}{\partial a_j^i} = \lambda_i \left(\frac{\partial C^i}{\partial a_j^i} - x^j \right). \tag{2.2}$$

Define a family of differential 1-forms $\omega^1, \dots, \omega^m$ by

$$\omega^i = \sum_{j=1}^{n} \left(\frac{\partial C^i}{\partial a_j^i} - x^j \right) \mathrm{d}a_j^i. \tag{2.3}$$

It follows that the differential of V, $\mathrm{d}V$, can be written as:

$$\mathrm{d}V = \sum_{i=1}^{m} \lambda_i \omega^i. \tag{2.4}$$

Notice that

$$d\omega^i = \sum_{j,l} \frac{\partial^2 C^i}{\partial a_l^i \partial a_j^i} da_l^i \wedge da_j^i - \sum_{j,k,l} \frac{\partial x^j}{\partial a_l^k} da_l^k \wedge da_j^i.$$

The coefficients in the first summation are symmetric, so we end up with

$$d\omega^i = -\sum_{j,k,l} \frac{\partial x^j}{\partial a_l^k} da_l^k \wedge da_j^i. \tag{2.5}$$

The ith constraint is $(a^i)^T x(A) = C^i(a^i)$. Differentiating both sides of this equality with respect to a_j^i and rearranging, we get:

$$\frac{\partial C^i}{\partial a_j^i} - x^j = \sum_{r=1}^{n} \frac{\partial x^r}{\partial a_j^i} a_r^i. \tag{2.6}$$

Using this result, the 1-form ω^i can be written as:

$$\omega^i = \sum_{r,j=1}^{n} \frac{\partial x^r}{\partial a_j^i} a_r^i da_j^i \tag{2.7}$$

Now, our inverse problem can be stated as follows:

- We observe the functions $x^j(A)$, $j = 1, \ldots, n$ from $\mathbb{R}_{++}^{mn}$ to $\mathbb{R}_{++}$.
- Then we define the functions $C^i(a^i) \equiv (a^i)^T x(A)$.
- We observe also a family of positive functions λ_{ik} using symmetry conditions that will be given below.
- Our objective is to find a function $f(x)$, by first finding the value function $V(A)$, such that $x(A) \in \operatorname{argmax}\{f(x) | Ax = C(A)\}$ and $V(A) = f(x(A))$.

The inverse problem will be solved in three steps. In the first step, we identify a set of necessary conditions. Then, we find sufficient conditions by solving the following problem: given a family of m differential 1-forms $\Omega_1, \ldots, \Omega_m$ that satisfy the conditions

$$\Omega_i \wedge \Omega_k = 0, \quad \text{for any } i, k$$

can we find $m+1$ functions $\mu_1, \ldots, \mu_m$ and V such that $\mu_k dV = \Omega_k$. Notice that the function V is independent of k. Finally, we solve for functions that have the required curvature and positivity conditions. Notice that the family of m 1-forms, $\{\Omega_1, \ldots, \Omega_m\}$ generates a vector space of dimension one; that is, $\operatorname{span}\{\Omega_1, \ldots, \Omega_m\} = \operatorname{span}\{\Omega_1\}$.

Henceforth, we set $\eta_i(A) = ((a^i)^T (D_{a^i} x) a^i)^{-1}$. Note that it is an observed quantity (it can be computed from $x(A)$ and A). We wrote $\eta_i(A)$ to emphasize the fact that η_i is a function of A. In fact, η_i can be written as:

$$\eta_i^{-1} = (a^i)^T (D_{a^i} x) a^i = \sum_{j=1}^{n} \left(\frac{\partial C^i}{\partial a_j^i} - x^j \right) a_j^i = (a^i)^T (D_{a^i} C^i) - C^i(a^i)$$

where η_i^{-1} is the reciprocal of η_i. Using equation (2.2), we find that

$$(a^i)^T (D_{a^i} V) = \lambda_i ((a^i)^T (D_{a^i} C^i) - C^i(a^i)).$$

Remark 2.3. Let us suppose, for a moment, that $C^i(a^i)$ is homogeneous of degree ρ; that is, $(a^i)^T (D_{a^i} C^i) = \rho C^i(a^i)$. It follows from this equation that

$$(a^i)^T (D_{a^i} V) = \lambda_i (\rho - 1) C^i(a^i).$$

We conclude that $(a^i)^T(D_{a^i}V)$ is negative, zero or positive if $\rho < 1$, $\rho = 1$ or $\rho > 1$, respectively. If $\rho = 1$ then $C^i(a^i)$ is homogeneous of degree one in which case the value function V is homogeneous of degree zero, hence $(a^i)^T(D_{a^i}V) = 0$. Moreover, we conclude that $0 < C^i(a^i) = (a^i)^T(D_{a^i}C^i - (D_{a^i}x)a^i)$. Notice that if $C^i(a^i) = c^i$ (C^i is independent of a^i) then $\eta_i^{-1} = -c^i$.

Using equations (2.2) and (2.6) we find that

$$\lambda_i = \eta_i \sum_j \frac{\partial V}{\partial a_j^i} a_j^i.$$

Consequently, η_i and $(a^i)^T D_{a^i}V$ should have the same sign since $\lambda_i > 0$. Moreover, we have

$$\frac{\partial C^i}{\partial a_j^i} - x^j = \frac{\partial V/\partial a_j^i}{\eta_i \sum_j \frac{\partial V}{\partial a_j^i} a_j^i}. \tag{2.8}$$

We will come back to this equation in the applications section as this equation has an important counterpart in economics.

To allow for better follow up of our exposition, we will restrict the ranges of the subscripts and superscripts used in the sequel as follows, $1 \le i, k, k', s, t \le m$ and $1 \le j, j', l, l', r \le n$. In what follows, δ_k^i denotes the Kronecker symbol which equals one if $i = k$ and zero otherwise.

Now we are ready to give our main results. We first identify a set of necessary conditions satisfied by the function $x(A)$ as well as the vector of Lagrange multipliers. Then, the necessary and sufficient conditions for mathematical and economic integration will be given.

3. Mathematical integration: necessary conditions

In the following sections we give the main results of the paper. We first give a set of symmetry conditions satisfied by the function $x(A)$. Then, we give the necessary and sufficient conditions for mathematical integration. Necessary conditions permit us to specify (proportionality) functions $\lambda_{ik} > 0$. As we will see, sufficient conditions involve a system of partial differential equations that should be satisfied by these functions as well as x. Consider the following result

Theorem 3.1. *Let $x(A)$ be a solution of problem $(\mathcal{P})$ and $\lambda(A)$ be the corresponding vector of Lagrange multipliers. Then, the following symmetry conditions are satisfied:*

$$\lambda_k \left(\frac{\partial x^l}{\partial a_j^i} - \eta_i \sum_{j'=1}^n \frac{\partial x^l}{\partial a_{j'}^i} a_{j'}^i \sum_{r=1}^n \frac{\partial x^r}{\partial a_j^i} a_r^i \right) = \lambda_i \left(\frac{\partial x^j}{\partial a_l^k} - \eta_k \sum_{j'=1}^n \frac{\partial x^j}{\partial a_{j'}^k} a_{j'}^k \sum_{r=1}^n \frac{\partial x^r}{\partial a_l^k} a_r^k \right) \tag{3.1}$$

for all $1 \le i, k \le m$ and $1 \le j, l \le n$.

Proof. See Appendix. □

Some remarks are in order:

Remark 3.2.

(a) As the function $x(A)$ is observable, we can use symmetry conditions (3.1) that we write as $\Sigma_{ij}^{kl} = \Sigma_{kl}^{ij}$ to determine the proportionality functions $\lambda_{ik} := \lambda_i/\lambda_k$. It is important to point out that we do not observe the Lagrange multipliers $\lambda_1, \ldots, \lambda_m$. We observe, however, the functions λ_{ik}. The above necessary conditions can be written as $\lambda_i S_k = \lambda_k S_i^T$ where S_k is the $n \times n$ matrix whose ij-entry is given by

$$S_k^{jl} = \frac{\partial x^j}{\partial a_l^k} - \eta_k \sum_{j'=1}^n \frac{\partial x^j}{\partial a_{j'}^k} a_{j'}^k \sum_{r=1}^n \frac{\partial x^r}{\partial a_l^k} a_r^k$$

(b) Conditions (3.1) mean that there is a symmetric matrix corresponding to each constraint and these matrices are proportional.

Theorem 3.3. *Let $x(A)$ be a solution of problem $(\mathcal{P})$ and $\lambda_1, \dots, \lambda_m$ are the corresponding Lagrange multipliers. Then the following conditions are equivalent*

(a) $\lambda_i S_k = \lambda_k S_i^T$, *for all* $i, k = 1, \dots, m$.

(b) $\sum\limits_{i=1}^{m} \lambda_i \mathrm{d}\omega^i \wedge \omega^1 \wedge \ldots \wedge \omega^m = 0$.

Proof. See Appendix. □

As a consequence of the last theorem, we can conclude that the conditions given in Theorem 3.1 are necessary but not sufficient for the decomposition $\mathrm{d}V = \sum_i \lambda_i \omega^i$. Moreover, if there is only one constraint then $S = S^T$ if and only if $\mathrm{d}\omega \wedge \omega = 0$. Consequently, the condition $\mathrm{d}\omega \wedge \omega = 0$ is both necessary and sufficient for mathematical integration in the single constraint case. In the multi-constraint case, however, we need additional conditions on the proportionality functions $\lambda_{ik} := \lambda_i/\lambda_k$ as well as on the function $x(A)$. Symmetry conditions can also be interpreted as follows: for any given i and k, we fix all variables except a^i and a^k. Optimality conditions imply that $\lambda_i\omega^i + \lambda_k\omega^k = \mathrm{d}V$. Consequently, we have $d(\lambda_i\omega^i) = -d(\lambda_k\omega^k)$ which implies that $\lambda_i S_k = \lambda_k S_i^T$. However, the above theorem proves that $\lambda_i S_k = \lambda_k S_i^T$, for all i and k, are equivalent to $\sum\limits_{i=1}^{m} \lambda_i \mathrm{d}\omega^i \in \mathrm{span}\{\omega^1, \dots, \omega^m\}$ which means that $\sum_{i=1}^m \lambda_i \mathrm{d}\omega^i + \sum_{i=1}^m \beta_i \wedge \omega^i = 0$ for some 1-forms $\beta_1, \dots, \beta_m$. Obviously, this result is not sufficient, we need $\beta_i = \mathrm{d}\lambda_i$, compare equations (A.8) and (A.11) below.

4. Mathematical integration: Necessary and sufficient conditions

The conditions given so far are not sufficient for mathematical integration. Our objective now is to give sufficient conditions and to express them as a system of partial differential equations that have to be satisfied by the coefficient functions λ_{ik} and the function $x(A)$.

Notice that $\lambda_{ii} = 1$ for every $i = 1, \dots, m$ and $\lambda_{ik}\lambda_{ki} = 1$. Equation (2.2) implies that

$$\frac{1}{\lambda_k}\frac{\partial V}{\partial a_j^i} = \lambda_{ik}\left(\frac{\partial C^i}{\partial a_j^i} - x^j\right). \tag{4.1}$$

Define a family of 1-forms Ω_k, $k = 1, \dots, m$, by

$$\Omega_k = \sum_{s=1}^{m} \lambda_{sk}\omega^s \tag{4.2}$$

where ω^s is the 1-form defined by (2.3) or the equivalent form (2.7). Notice that $\Omega_1, \dots, \Omega_m$ are defined using observable functions only. Then equation (4.1) can be written as $\mu_k \mathrm{d}V = \Omega_k$ which is equivalent to $\Omega_k \wedge \mathrm{d}\Omega_k = 0$. Clearly, the family of 1-forms defined by (4.2) are collinear to the same gradient $\mathrm{d}V$. The last equation gives us the necessary and sufficient conditions for mathematical integration. This result stems from the underlying structure of the optimization problem. The following result proves that the 1-forms $\Omega_1, \dots, \Omega_m$ are proportional.

Lemma 4.1. *Let $\Omega_1, \dots, \Omega_m$ be the family of 1-forms defined by (4.2) with $\lambda_{ik} = \frac{\lambda_i}{\lambda_k}$ then $\Omega_i \wedge \Omega_k = 0$ for all $i, k = 1, \dots, m$.*

Proof. Using the definition of Ω_k in (4.2) we have

$$\Omega_i \wedge \Omega_k = \sum_{s,t=1}^{m} (\lambda_{ti}\lambda_{sk})\omega^t \wedge \omega^s = \sum_{t<s} (\lambda_{ti}\lambda_{sk} - \lambda_{si}\lambda_{tk})\omega^t \wedge \omega^s$$

The coefficients $\lambda_{ti}\lambda_{sk} - \lambda_{si}\lambda_{tk}$ are identically zero since

$$\frac{\lambda_{ti}\lambda_{sk}}{\lambda_{si}\lambda_{tk}} = \frac{\lambda_t\lambda_s}{\lambda_i\lambda_k}\frac{\lambda_i\lambda_k}{\lambda_s\lambda_t} = 1.$$

This proves the result. $\square$

This is a general result that is true for any 1-forms defined by equation (4.2) with coefficients $\lambda_{ik} = \lambda_i/\lambda_k$. This result is obvious if $\Omega_k = \mu_k \mathrm{d}V$.

Theorem 4.2. *Given the family of 1-forms $\Omega_1, \ldots, \Omega_m$ defined above, then there exist $m+1$ functions $\mu_1, \ldots, \mu_m$ and V, defined in a neighbourhood $\mathcal{U}$ of some point $\bar{A} \in \mathbb{R}^{mn}_{++}$, such that $\mu_k \mathrm{d}V = \Omega_k$ for $k = 1, \ldots, m$ if and only if the condition $\Omega_k \wedge \mathrm{d}\Omega_k = 0$ holds in a neighbourhood $\mathcal{V}$ of $\bar{A}$ with $\mathcal{U} \subset \mathcal{V}$.*

Proof. Using Darboux Theorem [4], $\Omega_k \wedge \mathrm{d}\Omega_k = 0$ if and only if there exist two functions μ_k and V_k such that $\mu_k \mathrm{d}V_k = \Omega_k$. Lemma 4.1 implies that

$$\Omega_i \wedge \Omega_k = \mu_i\mu_k \mathrm{d}V_i \wedge \mathrm{d}V_k = 0.$$

Therefore, $\mathrm{d}V_k = \phi_{ik}(A)\mathrm{d}V_i$, $\forall i, k = 1, \ldots, m$ for some function ϕ_{ik}. So we can set $\mathrm{d}V_1 = \cdots = \mathrm{d}V_m = \mathrm{d}V$. $\square$

We also need the following lemma.

Lemma 4.3. *Let $\Omega_1, \ldots, \Omega_m$ be the family of differential 1-forms defined in (4.2). Then, if $\Omega_i \wedge \mathrm{d}\Omega_i = 0$ for some i, then $\Omega_k \wedge \mathrm{d}\Omega_k = 0$ for any $k \in \{1, \ldots, m\}$.*

Proof. Let $i, k \in \{1, \ldots, m\}$. Assume that $\Omega_i \wedge \mathrm{d}\Omega_i = 0$. Note that $\Omega_i \wedge \Omega_k = 0$ if and only if $\Omega_k = \varphi\Omega_i$ for some function φ. Taking the exterior derivative we get $\mathrm{d}\Omega_k = \varphi \mathrm{d}\Omega_i + d\varphi \wedge \Omega_i$. Multiply both sides of the last equation by Ω_k and using the fact that $\Omega_k = \varphi\Omega_i$, we find that $\Omega_k \wedge \mathrm{d}\Omega_k = \varphi^2\Omega_i \wedge \mathrm{d}\Omega_i + \varphi\Omega_i \wedge d\varphi \wedge \Omega_i = 0$. This completes the proof. $\square$

Clearly, the 1-forms $\Omega_1, \ldots, \Omega_m$ belong to the space of 1-forms spanned by $\omega^1, \ldots, \omega^m$. Moreover, it follows from the definition of $\omega^1, \ldots, \omega^m$ that they are linearly independent since $\omega^1 \wedge \ldots \wedge \omega^m \neq 0$. Let us consider the following result.

Lemma 4.4. *Let $\beta_1, \ldots, \beta_m$ belong to the subspace of 1-forms spanned by $\alpha^1, \ldots, \alpha^m$. Suppose that $\alpha^1, \ldots, \alpha^m$ are linearly independent; that is, $\alpha^1 \wedge \ldots \wedge \alpha^m \neq 0$. Then $\beta_i \wedge \beta_k = 0$ if and only if there exist C^m_2 rank-one symmetric $m \times m$ matrices $M_{ik} = (b_{is}b_{kt})$, such that $\beta_i = \sum_{s=1}^m b_{is}\alpha^s$.*

Proof. Since $\beta_1, \ldots, \beta_m$ belong to the linear span of $\alpha^1, \ldots, \alpha^m$ then for any i there exist m functions $b_{i1}, \ldots, b_{im}$ such that $\beta_i = \sum_{j=1}^n b_{is}\alpha^s$ Therefore, $\beta_i \wedge \beta_k = \sum_{s,t} b_{is}b_{kt}\alpha^s \wedge \alpha^t = \sum_{s<t}(b_{is}b_{kt} - b_{it}b_{ks})\alpha^s \wedge \alpha^t$. Thus, $\beta_i \wedge \beta_k = 0$ if and only if $b_{is}b_{kt} = b_{it}b_{ks}$. $\square$

Our objective now is to explicit the necessary and sufficient conditions for mathematical integration given in Theorem 4.2.

Theorem 4.5. *Given the family of 1-forms $\Omega_1, \ldots, \Omega_m$. Then $\Omega_k \wedge \mathrm{d}\Omega_k = 0$ if and only if for any $k' \in \{1, \ldots, m\}$ the following conditions are satisfied for all $1 \le i, s \le m$, $1 \le j, l \le n$.*

$$\frac{\partial \lambda_{ik}}{\partial a^s_l}\sum_r \frac{\partial x^r}{\partial a^i_j}a^i_r - \lambda_{ik}\frac{\partial x^j}{\partial a^s_l} + \frac{\eta_{k'}}{\lambda_{kk'}}\left(\sum_{j'}\frac{\partial \lambda_{sk}}{\partial a^{k'}_{j'}}a^{k'}_{j'}\sum_r \frac{\partial x^r}{\partial a^s_l}a^s_r - \lambda_{sk}\sum_{j'}\frac{\partial x^l}{\partial a^{k'}_{j'}}a^{k'}_{j'}\right.$$

$$-\frac{1}{\eta_{k'}}\frac{\partial \lambda_{k'k}}{\partial a_l^s}+\lambda_{k'k}\sum_j \frac{\partial x^{j'}}{\partial a_l^s}a_{j'}^{k'}\Bigg)\lambda_{ik}\sum_{r=1}^n \frac{\partial x^r}{\partial a_j^i}a_r^i$$

$$=\frac{\partial \lambda_{sk}}{\partial a_j^i}\sum_r \frac{\partial x^r}{\partial a_l^s}a_r^s-\lambda_{sk}\frac{\partial x^l}{\partial a_j^i}+\frac{\eta_{k'}}{\lambda_{kk'}}\Bigg(\sum_{j'}\frac{\partial \lambda_{ik}}{\partial a_{j'}^{k'}}a_{j'}^{k'}\sum_r \frac{\partial x^r}{\partial a_j^i}a_r^i-\lambda_{ik}\sum_{j'}\frac{\partial x^j}{\partial a_{j'}^{k'}}a_{j'}^{k'}$$

$$-\frac{1}{\eta_{k'}}\frac{\partial \lambda_{k'k}}{\partial a_j^i}+\lambda_{k'k}\sum_{j'} \frac{\partial x^{j'}}{\partial a_j^i}a_{j'}^{k'}\Bigg)\lambda_{sk}\sum_{r=1}^n \frac{\partial x^r}{\partial a_l^s}a_r^s \tag{4.3}$$

Proof. See Appendix. □

Remark 4.6. It is clear that the necessary and sufficient conditions are imposed on observable functions. Moreover, when we get the functions $\mu_1,\ldots,\mu_k$ and V by setting $\lambda_i=\mu_k\lambda_{ik}$ so as to get (2.2) as required.

The next result proves that (4.3) includes the conditions given in Theorem (3.1).

Corollary 4.7. *Suppose that conditions (4.3) are satisfied then*

(a) $S_i=S_i^T$, *for all* $i=1,\ldots,m$.
(b) $S_i=\lambda_{ik}S_k^T$ *for all* $i,k=1,\ldots,m$.

Proof. If $s=k'=i=k$ then, using the fact that $\lambda_{ii}=1$, relations (4.3) boil down to the following symmetry conditions

$$\frac{\partial x^j}{\partial a_l^i}-\eta_i\left(\sum_{j'}\frac{\partial x^l}{\partial a_{j'}^i}a_{j'}^i-\sum_{j'}\frac{\partial x^{j'}}{\partial a_l^i}a_{j'}^i\right)\sum_{r=1}^n\frac{\partial x^r}{\partial a_j^i}a_r^i$$

$$=\frac{\partial x^l}{\partial a_j^i}-\eta_i\left(\sum_{j'}\frac{\partial x^j}{\partial a_{j'}^i}a_{j'}^i-\sum_{j'}\frac{\partial x^{j'}}{\partial a_j^i}a_{j'}^i\right)\sum_{r=1}^n\frac{\partial x^r}{\partial a_l^i}a_r^i$$

so we get (a). To prove (b), it suffices to take $k'=s=k$ and $i\neq k$ in (4.3) which writes down in this case as

$$\frac{\partial \lambda_{ik}}{\partial a_l^k}\sum_r\frac{\partial x^r}{\partial a_j^i}a_r^i-\lambda_{ik}\frac{\partial x^j}{\partial a_l^k}+\eta_k\lambda_{ik}\left(-\sum_{j'}\frac{\partial x^l}{\partial a_{j'}^k}a_{j'}^k+\sum_j\frac{\partial x^{j'}}{\partial a_l^k}a_{j'}^k\right)\sum_{r=1}^n\frac{\partial x^r}{\partial a_j^i}a_r^i$$

$$=-\frac{\partial x^l}{\partial a_j^i}+\eta_k\left(\sum_{j'}\frac{\partial \lambda_{ik}}{\partial a_{j'}^k}a_{j'}^k\sum_r\frac{\partial x^r}{\partial a_j^i}a_r^i-\lambda_{ik}\sum_{j'}\frac{\partial x^j}{\partial a_{j'}^k}a_{j'}^k\right)\sum_{r=1}^n\frac{\partial x^r}{\partial a_l^k}a_r^k. \tag{4.4}$$

Now, multiply both sides by a_j^i, summing over j and solving to get the following formula

$$\frac{\partial \lambda_{ik}}{\partial a_l^k}=\lambda_{ik}\eta_k\left(\sum_{j'}\frac{\partial x^l}{\partial a_{j'}^k}a_{j'}^k-\sum_j\frac{\partial x^{j'}}{\partial a_l^k}a_{j'}^k\right)-\eta_i\sum_{j'}\frac{\partial x^l}{\partial a_{j'}^i}a_{j'}^i+\eta_k\sum_{j'}\frac{\partial \lambda_{ik}}{\partial a_{j'}^k}a_{j'}^k\sum_{r=1}^n\frac{\partial x^r}{\partial a_l^k}a_r^k$$

substitute back into (4.4) to get the conditions $S_i=\lambda_{ik}S_k^T$. □

Now, we have the following theorem that solves the mathematical integration problem.

Theorem 4.8. *Given a function $x(A) \in \mathbb{R}^n_{++}$ and a family of strictly positive functions λ_{ik}, $1 \leq i, k \leq m$ all of class C^2 defined in a neighbourhood $\mathcal{V}$ of some point $\bar{A}$ such that $\lambda_{ti}\lambda_{sk} = \lambda_{si}\lambda_{tk}$ for all $1 \leq i, k, s, t \leq m$. Define the functions $C^1(a^1), \dots, C^m(a^m)$ by $C^i(a^i) = (a^i)^T x(A)$. Then, there exist $m + 1$ functions $\mu_1, \dots, \mu_k$ and V, defined in a possibly smaller neighbourhood $\mathcal{U} \subset \mathcal{V}$, such that $\mu_k \mathrm{d}V = \Omega_k$ if and only if conditions (4.3) are satisfied in $\mathcal{V}$.*

Proof. Given the functions $x(A)$ and λ_{ik}, $1 \leq i, k \leq m$ as in the statement of the theorem, define a family of 1-forms Ω_k, $k = 1, \dots, m$ as in (4.2). Symmetry conditions (4.3) are equivalent to $\Omega_k \wedge \mathrm{d}\Omega_k = 0$ for all k. Now, $\Omega_k \wedge \mathrm{d}\Omega_k = 0$ if and only if there exist two functions μ_k and V_k such that $\mu_k \mathrm{d}V_k = \Omega_k$. Symmetry conditions on the coefficients λ_{ik} guarantee that $\mathrm{d}V_1 = \cdots = \mathrm{d}V_k = \mathrm{d}V$ using Lemma (4.4). The proof is complete. □

5. Economic integration

In this section, we give the necessary and sufficient conditions for the existence of $m+1$ functions $\lambda_1, \dots, \lambda_m$ and V such that $\mathrm{d}V = \lambda(dC - x)$ where $\lambda_1, \dots, \lambda_m$ are strictly positive and V is quasi-convex with respect to each a^i, $i = 1, \dots, m$. Such a result solves the economic integration problem. The following theorem relates the matrix $\lambda_i S_k$, for any i, k, to the value function V and the mapping C.

Theorem 5.1. *Let $x(A)$ be a solution of a problem of type $(\mathcal{P})$, $\lambda_1, \dots, \lambda_m$ be the associated Lagrange multipliers and $V(A)$ be the value function then*

$$\lambda_i \left(\frac{\partial x^j}{\partial a^k_l} - \eta_k \sum_{j'=1}^n \frac{\partial x^j}{\partial a^k_{j'}} a^k_{j'} \sum_{r=1}^n \frac{\partial x^r}{\partial a^k_l} a^k_r \right) = -\frac{\partial^2 V}{\partial a^k_l \partial a^i_j} + \lambda_i \frac{\partial^2 C^i}{\partial a^k_l \partial a^i_j} \delta^i_k$$

$$+ \frac{\eta_i}{\lambda_i} \left(\sum_{j'} \frac{\partial^2 V}{\partial a^k_l \partial a^i_{j'}} a^i_{j'} - \left(\lambda_i \sum_{j'} \frac{\partial^2 C^i}{\partial a^k_l \partial a^i_{j'}} a^i_{j'} - \frac{\partial V}{\partial a^k_l} \right) \delta^i_k \right) \frac{\partial V}{\partial a^i_j}$$

$$+ \frac{\eta_k}{\lambda_k} \left(\sum_{l'} \frac{\partial^2 V}{\partial a^k_{l'} \partial a^i_j} a^k_{l'} - \frac{\eta_i}{\lambda_i} \sum_{j',l'} \frac{\partial^2 V}{\partial a^k_{l'} \partial a^i_{j'}} a^k_{l'} a^i_{j'} \frac{\partial V}{\partial a^i_j} + \eta_i \sum_{j',l'} \frac{\partial^2 C^i}{\partial a^k_{l'} \partial a^i_{j'}} a^k_{l'} a^i_{j'} \frac{\partial V}{\partial a^i_j} \delta^i_k \right.$$

$$\left. - \frac{\partial V}{\partial a^i_j} \delta^i_k - \lambda_i \sum_{l'} \frac{\partial^2 C^i}{\partial a^k_{l'} \partial a^i_j} a^k_{l'} \delta^i_k \right) \frac{\partial V}{\partial a^k_l}$$

The proof of this theorem is given in the appendix.

Notice that if $i \neq k$ then the Hessian matrix of the mapping $C(A)$ drops out of this formula because of the assumption that C^i depends on a^i only.

Remark 5.2. Set $i = k$ in the previous equality. Then

$$\lambda_i S_i = -D^2_{a^i} V + \lambda_i D^2_{a^i} C^i + R_i$$

Where R_i is a rank one symmetric matrix that takes the form $Q(D_{a^i}V)^T$ for some matrix Q. Let $\zeta \in \{D_{a^i}V\}^\perp$ then we have

$$\zeta^T S_i \zeta = -\frac{1}{\lambda_i} \zeta^T (D^2_{a^i} V) \zeta + \zeta^T (D^2_{a^i} C^i) \zeta.$$

It follows that the $n \times n$ matrix S_i has no specific negativity properties since the first term is negative while the second one is positive. In fact, this holds true for any vector $\zeta \in \mathbb{R}^n$ since any such vector can be written as $\zeta = \bar{\zeta} + t a^i$ where $\bar{\zeta}$ is orthogonal to $D_{a^i}V$. This follows from the fact that $(a^i)^T D_{a^i} V = \frac{\lambda_i}{\eta_i} \neq 0$ and Theorem 5.1.

If the function $C^i(a^i)$ is an affine function then the matrix S_i is indeed negative semi-definite. Moreover, if we pre-multiply both sides of the equality in Theorem 5.1 by $(a^i)^T$ and post-multiply both sides by a^i then both sides of the equality are identically zero. To see this, it suffices to use the fact that $\eta_i = \lambda_i (a^i)^T D_{a^i} V$.

Although the matrix S_i is not necessarily negative semi-definite, there do exist negativity conditions related to the function $x(A)$. We have the following result.

Lemma 5.3. *Suppose that $V(A)$ is the value function, $x(A)$ is a solution and $\lambda(A)$ is the associated vector of Lagrange multipliers for problem $(\mathcal{P})$, $C(A) = Ax(A)$. Then we have*

$$D^2_{a^i} V(A) = \lambda_i(A)(D^2_{a^i} C^i(a^i) - D_{a^i} x(A)) + D_{a^i}\lambda(A)(D_{a^i} C^i(a^i) - x)^T. \tag{5.1}$$

Moreover, the $n \times n$ matrix $D^2_{a^i} C^i(a^i) - D_{a^i} x(A)$ is symmetric and positive semi-definite on $(D_{a^i} V)^{\perp}$.

Proof. Equation (5.1) follows by differentiating the first order conditions $D_{a^i} V = \lambda_i(D_{a^i} C^i(a^i) - x)$ and the positivity result follows from the fact that the value function V is quasi-convex with respect to a^i. □

We have also the following results

Lemma 5.4. *Let $x(A)$ be a solution of problem $(\mathcal{P})$ and $C(A) = Ax(A)$. Then*

$$\sum_r \frac{\partial^2 x^r}{\partial a^k_l \partial a^i_j} a^s_r + \frac{\partial x^l}{\partial a^i_j}\delta^s_k + \frac{\partial x^j}{\partial a^k_l}\delta^i_s = \frac{\partial^2 C^s}{\partial a^k_l \partial a^i_j}\delta^i_s \delta^k_s. \tag{5.2}$$

Moreover, if $C^i(a^i)$ is a convex function then the $n \times n$ matrix M^i where

$$M^i_{jl} = \sum_r \frac{\partial^2 x^r}{\partial a^i_l \partial a^i_j} a^i_r + \frac{\partial x^l}{\partial a^i_j} + \frac{\partial x^j}{\partial a^i_l}$$

is symmetric and positive semi-definite.

Proof. Differentiating the sth constraint with respect to a^i_j and a^k_l

$$\sum_{r=1}^n \frac{\partial^2 x^r}{\partial a^k_l \partial a^i_j} a^s_r + \frac{\partial x^l}{\partial a^i_j}\delta^s_k + \frac{\partial x^j}{\partial a^k_l}\delta^i_s = \frac{\partial^2 C^s}{\partial a^k_l \partial a^i_j}\delta^i_s \delta^k_s. \tag{5.3}$$

Thus, we have equation (5.2). Positivity follows from the convexity of $C^i(a^i)$. □

Lemma 5.5. *Let $x(A)$ and $C(A)$ be as above. Then the matrix T^i defined by*

$$T^i_{jl} = \sum_r \frac{\partial^2 x^r}{\partial a^i_l \partial a^i_j} a^i_r + \frac{\partial x^l}{\partial a^i_j}$$

is symmetric and positive semi-definite on the subspace $\{(a^i)^T D_{a^i} x\}^{\perp}$.

Proof. It follows from the above calculations that $T^i + D_{a^i} x = D^2_{a^i} C^i$. Using equation (5.1) and the fact that $D_{a^i} C^i - x = \frac{1}{\lambda_i} D_{a^i} V$, we get

$$D^2_{a^i} V = \lambda_i T^i + \frac{1}{\lambda_i}(D_{a^i}\lambda_i)(D_{a^i} V)^T.$$

The result follows from the last equality, the quasi-convexity of V with respect to a^i and the fact that $D_{a^i} V = \lambda_i((a^i)^T D_{a^i} x)$. □

The following theorem solves the economic integration problem.

Theorem 5.6. *Let $x(A) \in \mathbb{R}^n_{++}$, $\lambda_{ik}(A) > 0$ be given functions defined on a neighbourhood $\mathcal{U}$ of some point $\bar{A} \in \mathbb{R}^{mn}_{++}$. Define $C(A) = Ax(A)$. Suppose that the following conditions are satisfied in $\mathcal{U}$ for all $i, k = 1, \ldots, m$*

(a) $\lambda_{ti}\lambda_{sk} = \lambda_{si}\lambda_{tk}$ *for all* $1 \le i, k, s, t \le m$.
(b) *Conditions* (4.3).
(c) *The matrix* M^i *is positive semi-definite.*
(d) *The restriction of the matrix* T^i *to* $\{(a^i)^T D_{a^i} x\}^\perp$ *is positive definite.*

Then, there exist positive functions $\lambda_1, \dots, \lambda_m$ *and a function* V *which is quasi-convex with respect to* a^i *for each* i*, defined in a neighbourhood* $\mathcal{V} \subset \mathcal{U}$ *such that* $D_{a^i} V = \lambda_i (D_{a^i} C^i - x)$.

Proof. Notice first that condition (c) implies that the function $C^i(a^i)$ is convex. Consider the family of 1-forms $\Omega_1, \dots, \Omega_m$ defined by

$$\Omega_k = \sum_{i,j} \lambda_{ik} \left(\frac{\partial C^i}{\partial a^i_j} - x^j \right) \mathrm{d}a^i_j = \sum_{i=1}^m \lambda_{ik} \sum_{r,j=1}^n \frac{\partial x^r}{\partial a^i_j} a^i_r \mathrm{d}a^i_j.$$

Conditions (4.3) are equivalent to $\Omega_k \wedge \mathrm{d}\Omega_k = 0$. Using Darboux theorem, the last equation is satisfied if and only if there exist two functions μ_k and V such that $\mu_k \mathrm{d}V = \Omega_k$. Note that V is independent of k. Therefore, we have

$$\mu_k \mathrm{d}V = \sum_{i=1}^m \lambda_{ik} \sum_{r,j=1}^n \frac{\partial x^r}{\partial a^i_j} a^i_r \mathrm{d}a^i_j. \tag{5.4}$$

Apply the previous 1-form to the vector field ξ^s to get

$$\mu_k ((a^s)^T D_{a^s} V) = \lambda_{sk} \frac{1}{\eta_s}.$$

It follows that $\eta_s(A)\mu_k(A)(a^s)^T D_{a^s} V(A) = \lambda_{sk}(A) > 0$, for all A in sufficiently small neighbourhood of some point $\bar{A}$. We can assume that $\eta_s (a^s)^T D_{a^s} V > 0$ and $\mu_k > 0$. Substitute for λ_{ik} in (5.4), we get

$$\mu_k \mathrm{d}V = \sum_i \mu_k \eta_i ((a^i)^T D_{a^i} V) \left(\frac{\partial C^i}{\partial a^i_j} - x^j \right) \mathrm{d}a^i_j.$$

Canceling μ_k and setting $\lambda_i = \eta_i((a^i)^T D_{a^i} V) > 0$, we obtain

$$\mathrm{d}V = \sum_{i=1}^m \lambda_i \left(\frac{\partial C^i}{\partial a^i_j} - x^j \right) \mathrm{d}a^i_j.$$

It remains to prove that the function V has the required positivity conditions. Note that

$$\frac{\partial^2 V}{\partial a^s_l \partial a^i_j} = \sum_{i=1}^m \lambda_i \left(\sum_{r=1}^n \frac{\partial^2 x^r}{\partial a^s_l \partial a^i_j} a^i_r + \frac{\partial x^l}{\partial a^i_j} \delta^i_s \right) + \frac{\partial \lambda_i}{\partial a^s_l} \sum_{r=1}^n \frac{\partial x^r}{\partial a^i_j} a^i_r.$$

Using relations (5.3), we can write $D^2_p V$ as

$$\frac{\partial^2 V}{\partial a^s_l \partial a^s_j} = \lambda_s T^s_{jl} + \frac{\partial \lambda_s}{\partial a^s_l} \sum_{r=1}^n \frac{\partial x^r}{\partial a^s_j} a^s_r.$$

Take a vector $\varrho \in \{D_{a^s} V\}^\perp$; that is, ϱ satisfies the condition

$$\sum_{j=1}^n \sum_{r=1}^n \frac{\partial x^r}{\partial a^s_j} a^s_r \varrho_j = 0.$$

It follows that

$$\sum_{j,l=1}^n \frac{\partial^2 V}{\partial a^s_l \partial a^s_j} \varrho_j \varrho_l = \lambda_s \sum_{j,l=1}^n T^s_{jl} \varrho_j \varrho_l > 0.$$

We conclude that the matrix $D^2_{a^s} V$ is positive definite on $\{D_{a^s} V\}^\perp$; that is, V is quasi-convex with respect to a^s. The proof is complete. $\square$

6. Duality

After solving the mathematical and economic integration problems, we get functions $\lambda_1, \ldots, \lambda_m$ and V that have the required properties. The question now is how to get a concave (or quasi-concave) objective function. In the single constraint case, if $V(a)$ is strongly convex (meaning that the Hessian is positive), then $f(x) = \min_a \{V(a) \mid a'x \le c(a)\}$ is quasi-convex (see [5], Prop. 11).

The objective function can be obtained from the value function using the duality relation

$$f(x) = \min\left\{V(A) | (a^i)^T x(A) = C^i(A)\right\}.$$

The function f is not necessarily quasi-concave. However, we can introduce a class of functions that is stable under duality, see [3, 10]. We need to define the following space

$$\mathcal{E}(A) = \{\nu = (\nu^1, \ldots, \nu^m) \in \mathbb{R}^{mn} | (\nu^i)^T D_{a^i} V = 0, i = 1, \ldots, m\}.$$

We now recall the definitions of QE-convex and QE-concave introduced by Epstein [10].

Definition 6.1. Let $\mathcal{U} \subset \mathbb{R}^n_{++}$ and $\mathcal{V} \subset \mathbb{R}^{mn}_{++}$. Suppose that $C(A)$ is a convex mapping. Then,

- We say that a function $f(x)$ is locally QE-concave if

$$\forall x^* \in \mathcal{U}, \exists A^* \in \mathcal{V} \text{ such that } f(x^*) = \max_{x \in \mathcal{U}} \{f(x) | A^* x = C(A^*)\}.$$

- We say that a function $V(A)$ is locally QE-convex if

$$\forall A^* \in \mathcal{V}, \exists x^* \in \mathcal{U} \text{ such that } V(A^*) = \min_{A \in \mathcal{V}} \{V(A) | A x^* = C(A)\}.$$

We have the following theorems:

Theorem 6.2. *The value function $V(A)$ is locally QE-convex if $D^2_A V$ is positive definite on $\mathcal{E}(A)$.*

Proof. Let $\mathcal{V}$ be a neighbourhood of a point $\bar{A}$ in which the function V is defined. The assumption that $D^2_A V$ is positive definite on $\mathcal{E}(A)$ for all $A \in \mathcal{V}$ implies that if $\nu = (\nu^1, \ldots, \nu^m) \in \mathcal{E}$ such that $(a^1 + \nu^1, \ldots, a^m + \nu^m) \in \mathcal{V}$ then

$$V(a^1 + \nu^1, \ldots, a^m + \nu^m) > V(a^1, \ldots, a^m) \tag{6.1}$$

To show that V is locally QE-convex, suppose that A^* is given. Let x^* be such that

$$V(A^*) = \min_A \{V(A) | A x^* = C(A)\}.$$

Take

$$x^*(A) = D_{a^i} C^i(a^{i*}) - \frac{1}{\lambda_i(A^*)} D_{a^i} V(A^*)$$

and

$$\lambda_i(A^*) = \eta_i(A^*)(a^{i*})^T D_{a^i} V(A^*)$$

where $\eta_i^{-1} = (a^i)^T (D_{a^i} C^i) - C^i(a^i)$. The point A^* satisfies the first order optimality conditions. Its clear that $A^* x^*(A^*) = C(a^{i*})$. The point A^* satisfies the second order condition for minimum which is the positive definiteness of $D^2_A V$ on $\mathcal{E}(A^*)$. This completes the proof. $\square$

Now, we need to show that the function

$$f(x) = \min_{A \in \mathcal{V}} \{V(A) | A x = C(A)\}$$

is locally QE-concave if V is locally QE-convex. Let $f(x)$ be a given locally QE-concave function. Define a function $V : \mathcal{V} \subset \mathbb{R}^{mn}_{++} \to \mathbb{R}$ by

$$V(A) = \max_{x \in \mathcal{U}} \{f(x) | Ax = C(A)\}$$

define also the function $f^*(x) = \min_{A \in \mathcal{V}} \{V(A) | Ax = C(A)\}$.

Suppose that the function $V(A)$ is defined in a neighbourhood of some point $\bar{A} \in \mathbb{R}^{mn}_{++}$, then $\mathcal{U} = \{x \in \mathbb{R}^n_{++} | Ax = C(A), \forall A \in \mathcal{V}\}$. The following theorem establishes duality between f and V.

Theorem 6.3. *If V is locally QE-convex then f^* is locally QE-concave. Moreover, $f^* = f$ throughout $\mathcal{U}$ if f is locally QE-concave.*

Proof. See [1]. □

Theorem (5.1) implies that, on the space $\mathcal{E}$, we have for any fixed $k_0 \in \{1, \dots, m\}$

$$\lambda_{k_0} \frac{\partial^2 V}{\partial a^k_l \partial a^i_j} = \lambda_{ik_0} \left(\sum_r \frac{\partial^2 x^r}{\partial a^k_l \partial a^i_j} a^i_r (1 - \delta^i_k) + T^i_{jl} \delta^i_k \right) := K^{ik}_{jl}.$$

Clearly, the assumption of positive definiteness of $D^2 V$ on the subspace $\mathcal{E}$ can now be stated in terms of observable functions, namely λ_{ik_0} and x. Moreover, it is a stronger condition than the assumption of positive definiteness of T^i on $\{(a^i) D_{a^i} x\}^\perp$ as required in theorem (5.6). To put all pieces of the puzzle together, we state the following theorem that gives the solution of the inverse problem:

Theorem 6.4. *Let $x(A) \in \mathbb{R}^n_{++}$, $\lambda_{ik}(A) > 0$ be given functions defined on a neighbourhood $\mathcal{U}$ of some point $\bar{A} \in \mathbb{R}^{mn}_{++}$. Define $C(A) = Ax(A)$. Suppose that the following conditions are satisfied throughout $\mathcal{U}$*

(a) *$\lambda_{ti} \lambda_{sk} = \lambda_{si} \lambda_{tk}$ for all $1 \leq i, k, s, t \leq m$.*
(b) *Conditions (4.3).*
(c) *The matrix M^i is positive semi-definite.*
(d) *The restriction of the tensor K to the subspace $\mathcal{E}$ is positive definite.*

Then, there exists a locally QE-concave function $f(x)$ such that

$$x(A) \in arg\max \{f(x) | Ax = C(A)\}.$$

7. Particular case: $m = 2$

In the 2-constraint particular case, we have $i, s, k, k' \in \{1, 2\}$ which gives 16 cases to consider in Theorem 4.5. Fortunately, some of these cases are redundant. The first type of redundancy comes from the proportionality of Ω_1 and Ω_2 and Lemma 4.3. Consequently, we can take only one value for k, say $k = 1$, and $k' \in \{1, 2\}$.

The second kind of redundancy arises from symmetry with respect to i and s, the case $s = 1, i = 2$ gives the same conditions as $s = 2, i = 1$. This reduces the number of cases to be considered to 6. Let $\lambda = \lambda_{21}$, note that $\lambda_{11} = \lambda_{22} = 1$ and $\lambda_{12} = 1/\lambda_{21}$. We consider each case in turn:

(a) First we set $k = k' = 1$. We consider the following 3 subcases:
   (i) $s = i = 1$, in this case $\lambda_{11} = 1$. After canceling identical terms from both sides we get

$$\frac{\partial x^j}{\partial a^1_l} + \eta_1 \sum_{j'} \frac{\partial x^l}{\partial a^1_{j'}} a^1_{j'} \sum_{r=1}^n \frac{\partial x^r}{\partial a^1_j} a^1_r . = \frac{\partial x^l}{\partial a^1_j} + \eta_1 \sum_{j'} \frac{\partial x^j}{\partial a^1_{j'}} a^1_{j'} \sum_{r=1}^n \frac{\partial x^r}{\partial a^1_l} a^1_r . \tag{7.1}$$

So we have $S_1 = S_1^T$.

(ii) $s=1, i=2$ we get in this case the following conditions

$$\frac{\partial\lambda}{\partial a_l^1}\sum_r \frac{\partial x^r}{\partial a_j^2}a_r^2 - \lambda\frac{\partial x^j}{\partial a_l^1} + \eta_1\left(-\sum_{j'}\frac{\partial x^l}{\partial a_{j'}^1}a_{j'}^1 + \sum_j \frac{\partial x^{j'}}{\partial a_l^1}a_{j'}^1\right)\lambda\sum_{r=1}^n \frac{\partial x^r}{\partial a_j^2}a_r^2.$$

$$= -\frac{\partial x^l}{\partial a_j^2} + \eta_1\left(\sum_{j'}\frac{\partial\lambda}{\partial a_{j'}^1}a_{j'}^1\sum_r\frac{\partial x^r}{\partial a_j^2}a_r^2 - \lambda\sum_{j'}\frac{\partial x^j}{\partial a_{j'}^1}a_{j'}^1\right)\sum_{r=1}^n\frac{\partial x^r}{\partial a_l^1}a_r^1. \tag{7.2}$$

Multiply both sides of the previous equation by a_j^2 and summing over j we find the following formula

$$\frac{\partial\lambda}{\partial a_l^1} = \lambda\eta_1\sum_{j'}\left(\frac{\partial x^l}{\partial a_{j'}^1} - \frac{\partial x^{j'}}{\partial a_l^1}\right)a_{j'}^1 - \eta_2\sum_{j'}\frac{\partial x^l}{\partial a_{j'}^2}a_{j'}^2 + \eta_1\sum_{j'}\frac{\partial\lambda}{\partial a_{j'}^1}a_{j'}^1\sum_r\frac{\partial x^r}{\partial a_l^1}a_r^1.$$

Substitute in the previous equation for $\frac{\partial\lambda}{\partial a_l^1}$ we get the condition $S_2 = \lambda S_1$.

(iii) $s=i=2$

$$\frac{\partial\lambda}{\partial a_l^2}\sum_r\frac{\partial x^r}{\partial a_j^2}a_r^2 - \lambda\frac{\partial x^j}{\partial a_l^2} - \eta_1\lambda^2\sum_{j'}\frac{\partial x^l}{\partial a_{j'}^1}a_{j'}^1\sum_{r=1}^n\frac{\partial x^r}{\partial a_j^2}a_r^2$$

$$= \frac{\partial\lambda}{\partial a_j^2}\sum_r\frac{\partial x^r}{\partial a_l^2}a_r^2 - \lambda\frac{\partial x^l}{\partial a_j^2} - \eta_1\lambda^2\sum_{j'}\frac{\partial x^j}{\partial a_{j'}^1}a_{j'}^1\sum_{r=1}^n\frac{\partial x^r}{\partial a_l^2}a_r^2 \tag{7.3}$$

(b) $k=1, k'=2$. We have another 3 subcases to consider:

(i) $i=s=1$

$$\frac{\partial x^j}{\partial a_l^1} + \lambda\eta_2\left(\sum_{j'}\frac{\partial x^l}{\partial a_{j'}^2}a_{j'}^2 + \frac{1}{\eta_2}\frac{\partial\lambda}{\partial a_l^1}\right)\sum_{r=1}^n\frac{\partial x^r}{\partial a_j^1}a_r^1$$

$$= \frac{\partial x^l}{\partial a_j^1} + \lambda\eta_2\left(\sum_{j'}\frac{\partial x^j}{\partial a_{j'}^2}a_{j'}^2 + \frac{1}{\eta_2}\frac{\partial\lambda}{\partial a_j^1}\right)\sum_{r=1}^n\frac{\partial x^r}{\partial a_l^1}a_r^1 \tag{7.4}$$

(ii) $i=s=2$

$$\frac{\partial\lambda}{\partial a_l^2}\sum_r\frac{\partial x^r}{\partial a_j^2}a_r^2 - \lambda\frac{\partial x^j}{\partial a_l^2} - \lambda^2\eta_2\left(\lambda\sum_{j'}\frac{\partial x^l}{\partial a_{j'}^2}a_{j'}^2 + \frac{1}{\eta_2}\frac{\partial\lambda}{\partial a_l^2}\right)\sum_{r=1}^n\frac{\partial x^r}{\partial a_j^2}a_r^2$$

$$= \frac{\partial\lambda}{\partial a_j^2}\sum_r\frac{\partial x^r}{\partial a_l^2}a_r^2 - \lambda\frac{\partial x^l}{\partial a_j^2} - \lambda^2\eta_2\left(\lambda\sum_{j'}\frac{\partial x^j}{\partial a_{j'}^2}a_{j'}^2 + \frac{1}{\eta_2}\frac{\partial\lambda}{\partial a_j^2}\right)\sum_{r=1}^n\frac{\partial x^r}{\partial a_l^2}a_r^2. \tag{7.5}$$

Multiply both sides of the last equation by p_j^2 and summing over j, we get

$$\frac{\partial\lambda}{\partial a_l^2} = \lambda\eta_2\sum_r\left(\frac{\partial x^r}{\partial a_l^2} - \frac{\partial x^l}{\partial a_r^2}\right)a_r^2.$$

Substituting back in the above equation we get $S_2 = S_2^T$.

(iii) $s = 1, i = 2$

$$\frac{\partial\lambda}{\partial a_l^1}\sum_r \frac{\partial x^r}{\partial a_j^2}a_r^2 - \lambda\frac{\partial x^j}{\partial a_l^1} - \lambda^2\left(\eta_2\sum_{j'}\frac{\partial x^l}{\partial a_{j'}^2}a_{j'}^2 + \frac{\partial\lambda}{\partial a_l^1}\right)\sum_{r=1}^n \frac{\partial x^r}{\partial a_j^2}a_r^2 = -\frac{\partial x^l}{\partial a_j^2}$$

$$+\lambda\eta_2\left(\sum_{j'}\frac{\partial\lambda}{\partial a_{j'}^2}a_{j'}^2\sum_r\frac{\partial x^r}{\partial a_j^2}a_r^2 - \lambda\sum_{j'}\frac{\partial x^j}{\partial a_{j'}^2}a_{j'}^2 - \frac{1}{\eta_2}\frac{\partial\lambda}{\partial a_j^2} + \lambda\sum_{j'}\frac{\partial x^{j'}}{\partial a_j^2}a_{j'}^2\right)\sum_{r=1}^n\frac{\partial x^r}{\partial a_l^1}a_r^1. \tag{7.6}$$

So far, we have proved the following theorem.

Theorem 7.1. *Given functions x and λ of class C^2. There exist three functions V, λ_1 and λ_2 such that $D_{a^i}V = \lambda_i(D_{a^i}C^i - x)$, $i = 1, 2$ if and only if conditions (7.1)–(7.6) are fulfilled.*

8. Geometry of the problem

Consider the following pair of dual problems

$$v(a) = \max u(x) \quad \text{subject to} \quad a^T x = c(a)$$

and

$$f(x) = \min v(a) \quad \text{subject to} \quad a^T x = c(a).$$

The first order conditions for those problems are, respectively

$$\frac{\partial v}{\partial p_i} = \lambda\left(\frac{\partial c}{\partial a_i} - x^i\right)$$

and

$$\frac{\partial f}{\partial x^i} = \mu a_i.$$

Define the 1-form

$$\omega(a) = \sum_{i=1}^n\left(\frac{\partial c}{\partial a_i} - x^i\right)da_i.$$

Then $\omega(a)$ vanishes on the tangent space to the $(n-1)$-dimensional manifold defined by

$$M(a) = \{a \in \mathbb{R}^n | v(a) = \text{const.}\}$$

That is, for any $\xi \in T_aM$, $\langle\omega(a), \xi\rangle = 0$. This is our integration problem: given $\omega(a)$, can we find a $(n-1)$-dimensional manifold M such that ω vanishes on the tangent space of M. The existence of this manifold is guaranteed by the symmetry of the matrix

$$S = D_a x + \frac{1}{a'(D_a x)a}((D_a x) - (D_a x)')aa'(D_a x).$$

Following the argument in [7], take a mapping $x(a)$ that we assume to be invertible with inverse $a(x)$. Define the 1-form $\pi(x) = \sum_{i=1}^n a_i \mathrm{d}x^i$. Integrating π means that we want to find a $(n-1)$-dimensional manifold such that the form π vanishes on its tangent space.

In the general m-constraint case, we have the following dual problems

$$V(A) = \max f(x) \quad \text{subject to} \quad Ax = C(A)$$

and

$$f(x) = \min V(A) \quad \text{subject to} \quad Ax = C(A).$$

The first order conditions for these problems are, respectively

$$\frac{\partial V}{\partial a^i_j} = \sum_k \lambda_k \frac{\partial C^k}{\partial a^i_j} - \lambda_i x^j$$

and

$$\frac{\partial f}{\partial x^i} = \sum_{k=1}^m \mu_k a^k_j.$$

Analogously, we define a family of 1-forms $\omega^1, \ldots, \omega^m$ by $\omega^i = dC^i - \sum_j x^i \mathrm{d}a^i_j$. The symmetry of the matrix S_i guarantees the existence of a $(n-1)$-dimensional manifold

$$M_i = \left\{a^i \in \mathbb{R}^n | V_i(a^i; a^{-i}) = c^i\right\}$$

where a^{-i} denotes the set of row vectors of the matrix A except the ith row, such that ω^i vanishes over its tangent space. Clearly, this is not sufficient for our purpose. This is reflected in the fact that these symmetry conditions are not sufficient for mathematical integration.

9. Applications

In this section, the dependent variables will be denoted by P instead of A as they represent prices. The inverse problem we considered in this article has interesting applications in microeconomics. The results we got here extend basic results in microeconomic theory. This kind of problems, maximization under several constraints, arise in many economic contexts; *e.g.*, rationing, choice under uncertainty and other applications such as models of uncertainty with production. The objective function f is called the individual's utility function. This function represents the tastes (or preferences) of the consumer on the set of affordable goods. The solution of the optimization problem is called, in such models, the individual demand function. The value function is called the indirect utility function which gives the maximum utility achieved by the consumer under budget constraints. This function has many interesting properties in the basic individual model. These properties include zero-homogeneity and quasi-convexity. In our setting, however, the indirect utility function is quasi-convex with respect to each a^i if each component of the mapping C is convex. It is not zero-homogeneous unless income mapping is one-homogeneous.

9.1. The basic consumer's problem

The basic individual problem in consumer theory takes the form of a maximization problem of the utility function $U(x)$ under one budget constraint in which the income y is price independent; that is,

$$\max_x U(x) \quad \text{subject to} \quad p^T x = y$$

where $p \in \mathbb{R}^n_{++}$ is the price vector. The solution to this problem, $x(p, y)$ that is called the individual demand function is, characterized by the following conditions:

- $p^T x(p, y) = y$ (Walras law).
- $x(tp, ty) = x(p, y)$ (zero-homogeneity).
- Symmetry and negative semi-definiteness of the Slutsky matrix S where

$$S_{ij} = \frac{\partial x^i}{\partial p_j} + \frac{\partial x^i}{\partial y} x^j.$$

9.2. The consumer's problem when income is price dependent

Now, if the consumer's income depends on the price vector p, then we have a maximization problem of the utility function $U(x)$ under the linear constraint $p^T x = c(p)$. In this case, the necessary and sufficient conditions of theorem (4.5) boil down to

$$\frac{\partial x^j}{\partial p_l} - \eta \left(\sum_{j'} \frac{\partial x^j}{\partial p_{j'}} p_{j'} - \sum_{j'} \frac{\partial x^{j'}}{\partial p_j} p_{j'} \right) \sum_{r=1}^{n} \frac{\partial x^r}{\partial p_l} p_r$$

$$= \frac{\partial x^l}{\partial p_j} - \eta \left(\sum_{j'} \frac{\partial x^l}{\partial p_{j'}} p_{j'} - \sum_{j} \frac{\partial x^{j'}}{\partial p_l} p_{j'} \right) \sum_{r=1}^{n} \frac{\partial x^r}{\partial p_j} p_r.$$

This is indeed the extended Slutsky matrix given in [2] that characterizes individual demand functions in the single constrain case. Moreover, if the function c is independent of a then we get the Slutsky matrix of the standard individual model. It is important to point out that in the single constraint case the symmetry of this matrix is both necessary and sufficient for mathematical integration.

Remark 9.1. Equation (2.8) is a generalized Roy's identity in consumer theory. This relation can be used, as in the classical individual model, to find the demand function from the indirect utility function V as the income mapping C is given

$$x^j = \frac{\partial C^i}{\partial p^i_j} - ((p^i)^T (D_{p^i} C^i) - C^i(p^i)) \frac{\partial V / \partial p^i_j}{\sum_j \frac{\partial V}{\partial p^i_j} p^i_j}.$$

It can be readily verified that this formula reduces to the classical Roy's identity if C^i is independent of the price vector a^i.

We also get generalization of the results in [3] when the mapping C is independent of A, see Corollary (4.7).

9.3. Point rationing

There are two types of rationing: simple rationing and points rationing. Simple rationing consists of exogenous restrictions on certain consumption goods whereas points rationing means that the consumer has a certain number of rationing coupons. Points rationing could be considered as replacing a systems with one currency by a system of multiple currencies. Under points rationing, the consumer's problem takes the form

$$\max_{(x^1,\dots,x^n)} U(x^1, \dots, x^n)$$

subject to the constraints $\sum_{j=1}^n p^i_j x^j = c^i(p^i)$, $i = 1, \dots, m$. In this model, p^i_j refers to the price of good j in currency i. It is assumed here that the individual's income is price dependent.

10. Concluding remarks

In this paper,we have solved the inverse problem $\max_x f(x)$ subject to the linear constraints $Ax = C(A)$. We assumed that $m < n$ and that the rank of the matrix A is m. If $m > n$, then $\omega^1 \wedge \ldots \wedge \omega^m = 0$; that is, $\omega^1, \dots, \omega^m$ are linearly dependent. Consequently, the condition $\sum_i \lambda_i \mathrm{d}\omega^i \wedge \omega^1 \wedge \ldots \wedge \omega^m = 0$ is fulfilled.

In fact, we have treated the problem in its most general form. In some cases, however, we need to deal with some problems in which there is few number of parameters. More precisely, we are given a mapping $q \to A(q)$ so that the problem depends, ultimately, on the parameters q. This would, rather, simplify the necessary and sufficient conditions. To get an idea of this case, we give a simple example from microeconomic theory. Let us consider a consumer whose utility function is $U(x^1, \dots, x^n)$ and his income is normalized to 1. Suppose that the prices of the n consumption goods $p_1, \dots, p_n$ are determined by the prices of capital and labor used in the production process, namely, wage rate w and capital rental price v. The consumer maximizes $U(x^1, \dots, x^n)$ under the budget constraint $p_1(v,w)x^1 + \cdots + p_n(v,w)x^n = 1$. Let $V(v,w)$ be the value function of this problem. Then, the envelope theorem implies that

$$-\frac{1}{\lambda}\mathrm{d}V = x'(D_v p)\mathrm{d}v + x'(D_w p)\mathrm{d}w := \omega.$$

The necessary and sufficient condition for this decomposition is $\omega \wedge \mathrm{d}\omega = 0$ is always fulfilled in the parameter (v,w)-space; this is a 3-form in a 2-dimensional space[2].

Appendix A. Proofs of main results

A.1. Proof of Theorem 3.1

We use the following preliminary results in the proof of theorem

Lemma A.1. *Let $x(A)$ be a solution of a multi-constraint maximization problem of the above type then*

(a) $\sum_{l=1}^{n} \frac{\partial x^l}{\partial a_j^i} a_l^k = 0$ *if* $i \neq k$.

(b) $\sum_{j=1}^{n} \left(\frac{\partial C^i}{\partial a_j^i} - x^j\right) a_j^i = (a^i)^T (D_{a^i} x) a^i$.

Proof. Differentiate the kth constraint $(a^k)^T x = C^k(a^k)$ with respect to a_j^i we get

$$\sum_{j=1}^{n} \frac{\partial x^l}{\partial a_j^i} a_l^k + x^j \delta_k^i = \frac{\partial C^k}{\partial a_j^i} \delta_k^i.$$

Condition (a) follows when $i \neq k$. If $i = k$ then multiply both sides of the last equality by a_j^i, summing over j and rearranging to get (b). This completes the proof. □

We need also the following lemma.

Lemma A.2. *Let λ_i, $i = 1, \dots, m$ be the Lagrange multiplier corresponding to the ith constraint. Then, the $m \times m$ matrix $\Lambda = (\Lambda_{ik})$, $i, k = 1, \dots, m$ is symmetric where*

$$\Lambda_{ik} = \eta_k \sum_{l=1}^{n} \frac{\partial \lambda_i}{\partial a_l^k} a_l^k.$$

Moreover, let $q^i = \eta_i a^i$ then

$$\Lambda_{ik} = (q^k)^T (D^2_{a^k a^i} V) q^i + (q^k)^T (D^2_{a^k a^i} C^i) q^i \delta_k^i - \eta_k \lambda_i \delta_k^i \tag{A.1}$$

Proof. Let $i, k \in \{1, \dots, m\}$ with $i \neq k$. By differentiating equality (2.2) with respect to a_l^k we get

$$\frac{\partial^2 V}{\partial a_l^k \partial a_j^i} = \frac{\partial \lambda_i}{\partial a_l^k}\left(\frac{\partial C^i}{\partial a_j^i} - x^j\right) - \lambda_i \frac{\partial x^j}{\partial a_l^k}.$$

[2] I would like to thank an anonymous referee for pointing out the issues discussed in this section and for other constructive comments that contributed to improving the presentation of this article. Errors are mine.

Multiply both sides of the last equality by $a^i_j a^k_l$ and summing up to get

$$\sum_{j,l=1}^{n} \frac{\partial^2 V}{\partial a^k_l \partial a^i_j} a^i_j a^k_l = \sum_{l=1}^{n} \frac{\partial \lambda_i}{\partial a^k_l} a^k_l \sum_{j=1}^{n} \left(\frac{\partial C^i}{\partial a^i_j} - x^j \right) a^i_j - \lambda_i \sum_{j,l=1}^{n} \frac{\partial x^j}{\partial a^k_l} a^i_j a^k_l.$$

Using Lemma (A.1), we end up with

$$\sum_{j,l=1}^{n} \frac{\partial^2 V}{\partial a^k_l \partial a^i_j} a^i_j a^k_l = \eta_i^{-1} \sum_{l=1}^{n} \frac{\partial \lambda_i}{\partial a^k_l} a^k_l.$$

Multiply both sides by η_k, we find that

$$\Lambda_{ik} = \eta_k \eta_i \sum_{j,l=1}^{n} \frac{\partial^2 V}{\partial a^k_l \partial a^i_j} a^i_j a^k_l.$$

This proves the symmetry of the matrix Λ. $\square$

Equation (A.1) can be considered as a generalization of the homogeneity condition for Lagrange multiplier when C^i is independent of a^i, see [3]. Now, we start the proof of Theorem 3.1.

Proof. Let $x(A)$ be a solution of problem $(\mathcal{P})$ and $\lambda(A) = (\lambda_1(A), \dots, \lambda_m(A))$ be the corresponding vector of Lagrange multipliers. These functions are related to the value function through decomposition (2.4). Taking the exterior derivative of both sides of equation (2.4), gives the 2-form

$$\sum_{k=1}^{m} \left(\lambda_k \mathrm{d}\omega^k + \mathrm{d}\lambda_k \wedge \omega^k \right) = 0. \tag{A.2}$$

Introduce a family of vector fields $\xi^1, \dots, \xi^m$ defined by

$$\xi^i = \sum_{j=1}^{n} a^i_j \frac{\partial}{\partial a^i_j}. \tag{A.3}$$

Equation (2.7) implies that

$$\langle \omega^i, \xi^i \rangle = \sum_{r,j=1}^{n} \frac{\partial x^r}{\partial a^i_j} a^i_r a^i_j = (a^i)^T (D_{a^i} x) a^i = \eta_i^{-1}.$$

Notice that $\langle \omega^i, \xi^k \rangle = 0$ if $i \neq k$. Applying equation (A.2) to the vector field ξ^i

$$\sum_{k=1}^{m} \lambda_k \langle \mathrm{d}\omega^k, (\xi^i, .) \rangle + \sum_{k=1}^{m} \langle \mathrm{d}\lambda_k, \xi^i \rangle \omega^k - \mathrm{d}\lambda_i \langle \omega^i, \xi^i \rangle = 0.$$

Since $\langle \omega^i, \xi^i \rangle = \eta_i^{-1}$, solving for $\mathrm{d}\lambda_i$, we get

$$\mathrm{d}\lambda_i = \eta_i \left(\sum_{k=1}^{m} \lambda_k \langle \mathrm{d}\omega^k, (\xi^i, .) \rangle + \sum_{k=1}^{m} \langle \mathrm{d}\lambda_k, \xi^i \rangle \omega^k \right). \tag{A.4}$$

Substituting this value of $\mathrm{d}\lambda_i$ into $\sum_{i=1}^{m} \left(\lambda_i \mathrm{d}\omega^i + \mathrm{d}\lambda_i \wedge \omega^i \right) = 0$, we get

$$\sum_{i=1}^{m} \left(\lambda_i \mathrm{d}\omega^i + \eta_i \sum_{k=1}^{m} \left(\lambda_k \langle \mathrm{d}\omega^k, (\xi^i, .) \rangle + \langle \mathrm{d}\lambda_k, \xi^i \rangle \omega^k \right) \wedge \omega^i \right) = 0. \tag{A.5}$$

Expanding each summation on the right-hand side of this equality, we can write

$$\begin{aligned}\langle \mathrm{d}\omega^k, (\xi^i, .)\rangle &= -\sum_{j,s,l} \frac{\partial x^j}{\partial a_l^s} \mathrm{d}a_l^s \wedge \mathrm{d}a_j^k \left(\sum_{j=1}^{n} a_j^i \frac{\partial}{\partial a_j^i}, . \right) \\ &= -\sum_{j,l} \frac{\partial x^j}{\partial a_l^i} a_l^i \mathrm{d}a_j^k + \sum_{j,s,l} \frac{\partial x^j}{\partial a_l^s} a_j^i \mathrm{d}a_l^s \delta_k^i.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_k \lambda_k \langle \mathrm{d}\omega^k, (\xi^i, .)\rangle &= -\sum_{j,k,l} \lambda_k \frac{\partial x^j}{\partial a_l^i} a_l^i \mathrm{d}a_j^k + \sum_{j,k,s,l} \lambda_k \frac{\partial x^j}{\partial a_l^s} a_j^i \mathrm{d}a_l^s \delta_k^i \\ &= -\sum_{j,k,l} \lambda_k \frac{\partial x^j}{\partial a_l^i} a_l^i \mathrm{d}a_j^k + \lambda_i \sum_{j,k,l} \frac{\partial x^j}{\partial a_l^k} a_j^i \mathrm{d}a_l^k.\end{aligned}$$

Using Lemma (A.1), we get

$$\sum_k \lambda_k \langle \mathrm{d}\omega^k, (\xi^i, .)\rangle = -\sum_{j,k,l} \lambda_k \frac{\partial x^l}{\partial a_j^i} a_j^i \mathrm{d}a_l^k + \lambda_i \omega^i. \tag{A.6}$$

Similarly,

$$\langle \mathrm{d}\lambda_k, \xi^i\rangle = \left\langle \sum_{s,l} \frac{\partial \lambda_k}{\partial a_l^s} \mathrm{d}a_l^s, \sum_{j=1}^{n} a_j^i \frac{\partial}{\partial a_j^i} \right\rangle = \sum_{l=1}^{n} \frac{\partial \lambda_k}{\partial a_l^i} a_l^i = \frac{1}{\eta_i} \Lambda_{ki}. \tag{A.7}$$

Therefore,

$$\sum_{k=1}^{m} \langle \mathrm{d}\lambda_k, \xi^i\rangle \omega^k = \sum_{k=1}^{m} \frac{1}{\eta_i} \Lambda_{ki} \omega^k.$$

It follows, from the above calculations, that

$$\mathrm{d}\lambda_i = -\eta_i \sum_{k,l} \sum_{j=1}^{n} \lambda_k \frac{\partial x^l}{\partial a_j^i} a_j^i \mathrm{d}a_l^k + \sum_k \Lambda_{ki} \omega^k + \eta_i \lambda_i \omega^i. \tag{A.8}$$

Using Lemma (A.2), we conclude that

$$\mathrm{d}\lambda_i \wedge \omega^i = -\eta_i \sum_{k,l} \sum_{j=1}^{n} \lambda_k \frac{\partial x^l}{\partial a_j^i} a_j^i \mathrm{d}a_l^k \wedge \omega^i.$$

Then, equation (A.5) becomes

$$\sum_{i,j,k,l} \left(\lambda_i \frac{\partial x^j}{\partial a_l^k} + \eta_i \sum_{j'=1}^{n} \lambda_k \frac{\partial x^l}{\partial a_{j'}^i} a_{j'}^i \sum_{r,j=1}^{n} \frac{\partial x^r}{\partial a_j^i} a_r^i \right) \mathrm{d}a_l^k \wedge \mathrm{d}a_j^i = 0. \tag{A.9}$$

The result follows. $\square$

A.2. Proof of Theorem 3.3

Proof. Firstly, we prove that (a) implies (b). Define a family of 1-forms $\beta_1, \dots, \beta_m$ by

$$\beta_i = \eta_i \sum_{l,j',k} \lambda_k \frac{\partial x^l}{\partial a^i_{j'}} a^i_{j'} \mathrm{d}a^k_l.$$

Symmetry conditions in (a) read as

$$\lambda_i \left(\frac{\partial x^j}{\partial a^k_l} - \eta_k \sum_{j'=1}^{n} \frac{\partial x^j}{\partial a^k_{j'}} a^k_{j'} \sum_{r=1}^{n} \frac{\partial x^r}{\partial a^k_l} a^k_r \right) = \lambda_k \left(\frac{\partial x^l}{\partial a^i_j} - \eta_i \sum_{j'=1}^{n} \frac{\partial x^l}{\partial a^i_{j'}} a^i_{j'} \sum_{r=1}^{n} \frac{\partial x^r}{\partial a^i_j} a^i_r \right).$$

Recall the definition of ω^i in (2.7) and equation (2.5), the last symmetry conditions are equivalent to

$$\sum_{i=1}^{n} (\lambda_i \mathrm{d}\omega^i + \beta_i \wedge \omega^i) = 0.$$

Multiply by $\omega^1 \wedge \ldots \wedge \omega^m$ to get condition (b).

Conversely, condition (b) means that there exist m differential 1-forms $\gamma_1, \dots, \gamma_m$ such that

$$\sum_{i=1}^{m} (\lambda_i \mathrm{d}\omega^i + \gamma_i \wedge \omega^i) = 0. \tag{A.10}$$

Then, write γ_i as

$$\gamma_i = \sum_{k,l} \varphi^i_{kl}(A) \mathrm{d}a^k_l$$

for some smooth functions $\varphi^i_{kl}(A)$. Now, apply the 2-form in equation (A.10) to the vector field ξ^k for some $1 \le k \le m$, the first summation gives, using equation (A.6),

$$\sum_{i=1}^{m} \lambda_i \langle \mathrm{d}\omega^i, (\xi^k, .)\rangle = -\sum_{i,j,l} \lambda_i \frac{\partial x^l}{\partial a^k_j} a^k_j \mathrm{d}a^i_l + \lambda_k \omega^k.$$

While the second summation gives us

$$\sum_{i=1}^{m} \langle \gamma_i \wedge \omega^i, (\xi^k, .)\rangle = \left(\sum_{l=1}^{n} \varphi^i_{kl} a^k_l \right) \omega^k - \eta_k^{-1} \gamma_k.$$

It follows from the two previous equations that,

$$-\sum_{j,k,l} \lambda_k \frac{\partial x^l}{\partial a^i_j} a^i_j \mathrm{d}a^k_l + \lambda_i \omega^i + \left(\sum_{l=1}^{n} \varphi^k_{il} a^i_l \right) \omega^i - \eta_i^{-1} \gamma_i = 0.$$

Solving for γ_i

$$\gamma_i = -\eta_i \sum_{j,k,l} \lambda_k \frac{\partial x^l}{\partial a^i_j} a^i_j \mathrm{d}a^k_l + \eta_i \sum_{l=1}^{n} \varphi^k_{il} a^i_l \omega^i + \eta_i \lambda_i \omega^i. \tag{A.11}$$

So, we conclude that

$$\gamma_i = -\eta_i \sum_{j,k,l} \lambda_k \frac{\partial x^l}{\partial a^i_j} a^i_j \mathrm{d}a^k_l \qquad \bmod \omega^i$$

Now, plug this value of γ_i into (A.10) and expand to get

$$\sum_{i,j,k,l}\left(\lambda_i \frac{\partial x^j}{\partial a_l^k} + \eta_i\lambda_k \sum_{j'=1}^{n} \frac{\partial x^l}{\partial a_{j'}^i} a_{j'}^i \sum_{r=1}^{n} \frac{\partial x^r}{\partial a_j^i} a_r^i\right) \mathrm{d}a_l^k \wedge \mathrm{d}a_j^i = 0.$$

Symmetry conditions (a) follow from the last equality. This completes the proof. □

A.3. Proof of Theorem 4.5

Proof. Recall that $\Omega_k \wedge \mathrm{d}\Omega_k = 0$ if and only if there exists a 1-form α_k such that

$$\mathrm{d}\Omega_k = \alpha_k \wedge \Omega_k \tag{A.12}$$

The 1-form α_k can be identified $(\bmod\ \Omega_k)$. Notice that

$$\Omega_k = \sum_{i=1}^{m} \lambda_{ik} \sum_{r,j=1}^{n} \frac{\partial x^r}{\partial a_j^i} a_r^i \mathrm{d}a_j^i.$$

Let $\xi^{k'}$ be a vector field defined as in (A.3), then

$$\langle \Omega_k, \xi^{k'} \rangle = \lambda_{k'k} (a^{k'})^T (D_{a^{k'}} x) a^{k'} = \lambda_{k'k} \eta_{k'}^{-1}.$$

To find a 1-form α_k that satisfies equation (A.12), we apply both sides of that equation to the vector field $\xi^{k'}$, so we have

$$\langle \mathrm{d}\Omega_k, (\xi^{k'}, .) \rangle = \langle \alpha_k, \xi^{k'} \rangle \Omega_k - \alpha_k \langle \Omega_k, \xi^{k'} \rangle.$$

Using the fact that $\langle \Omega_k, \xi^{k'} \rangle = \lambda_{k'k} \eta_{k'}^{-1}$ and solving for α_k, we get

$$\alpha_k = \frac{\eta_{k'}}{\lambda_{k'k}} \left[\langle \alpha_k, \xi^{k'} \rangle \Omega_k - \langle \mathrm{d}\Omega_k, (\xi^{k'}, .) \rangle \right].$$

Substitute for α_k in equation (A.12) that becomes

$$\mathrm{d}\Omega_k = -\frac{\eta_{k'}}{\lambda_{k'k}} \langle \mathrm{d}\Omega_k, (\xi^{k'}, .) \rangle \wedge \Omega_k. \tag{A.13}$$

Now

$$\Omega_k = \sum_{i=1}^{m} \lambda_{ik} \left(\frac{\partial C^i}{\partial a_j^i} - x^j \right) \mathrm{d}a_j^i.$$

Taking the exterior derivative and using the budget constraint we find that

$$\mathrm{d}\Omega_k = \sum_{i,j,s,l} \left(\frac{\partial \lambda_{ik}}{\partial a_l^s} \sum_r \frac{\partial x^r}{\partial a_j^i} a_r^i - \lambda_{ik} \frac{\partial x^j}{\partial a_l^s} \right) \mathrm{d}a_l^s \wedge \mathrm{d}a_j^i.$$

Now, we apply the 2-form $\mathrm{d}\Omega_k$ to the vector field $\xi^{k'}$, we conclude that

$$\begin{aligned}\langle \mathrm{d}\Omega_k, (\xi^{k'}, .) \rangle = &\sum_{i,j,l} \left(\frac{\partial \lambda_{ik}}{\partial a_l^{k'}} \sum_r \frac{\partial x^r}{\partial a_j^i} a_r^i - \lambda_{ik} \frac{\partial x^j}{\partial a_l^{k'}} \right) a_l^{k'} \mathrm{d}a_j^i \\ &- \sum_{j,s,l} \left(\frac{\partial \lambda_{k'k}}{\partial a_l^s} \sum_r \frac{\partial x^r}{\partial a_j^{k'}} a_r^{k'} - \lambda_{k'k} \frac{\partial x^j}{\partial a_l^s} \right) a_j^{k'} \mathrm{d}a_l^s\end{aligned}$$

Rewrite equation (A.13) as

$$\mathrm{d}\Omega_k + \frac{\eta_{k'}}{\lambda_{k'k}} \langle \mathrm{d}\Omega_k, (\xi^{k'}, .) \rangle \wedge \Omega_k = 0.$$

Depending on the above formulas of $\mathrm{d}\Omega_k$ and $\langle \mathrm{d}\Omega_k, (\xi^{k'}, .)\rangle$ the last equation can be expanded as

$$\sum_{i,j,s,l} \left(\frac{\partial \lambda_{ik}}{\partial a_l^s} \sum_r \frac{\partial x^r}{\partial a_j^i} a_r^i - \lambda_{ik} \frac{\partial x^j}{\partial a_l^s} + \frac{\eta_{k'}}{\lambda_{kk'}} \left(\sum_{j'} \frac{\partial \lambda_{sk}}{\partial a_{j'}^{k'}} a_{j'}^{k'} \sum_r \frac{\partial x^r}{\partial a_l^s} a_r^s - \lambda_{sk} \sum_{j'} \frac{\partial x^l}{\partial a_{j'}^{k'}} a_{j'}^{k'} \right.\right.$$
$$\left.\left. - \frac{1}{\eta_{k'}} \frac{\partial \lambda_{k'k}}{\partial a_l^s} + \lambda_{k'k} \sum_{j'} \frac{\partial x^{j'}}{\partial a_l^s} a_{j'}^{k'} \right) \lambda_{ik} \sum_{r=1}^n \frac{\partial x^r}{\partial a_j^i} a_r^i \right) \mathrm{d}a_l^s \wedge \mathrm{d}a_j^i = 0. \tag{A.14}$$

Write the previous equation as

$$\sum_{i,s,j,l} (\Gamma_{k'k})_{ij}^{sl} \mathrm{d}a_l^s \wedge \mathrm{d}a_j^i = 0,$$

where

$$(\Gamma_{k'k})_{ij}^{sl} = \frac{\partial \lambda_{ik}}{\partial a_l^s} \sum_r \frac{\partial x^r}{\partial a_j^i} a_r^i - \lambda_{ik} \frac{\partial x^j}{\partial a_l^s} + \frac{\eta_{k'}}{\lambda_{kk'}} \left(\sum_{j'} \frac{\partial \lambda_{sk}}{\partial a_{j'}^{k'}} a_{j'}^{k'} \sum_r \frac{\partial x^r}{\partial a_l^s} a_r^s - \lambda_{sk} \sum_{j'} \frac{\partial x^l}{\partial a_{j'}^{k'}} a_{j'}^{k'} \right.$$
$$\left. - \frac{1}{\eta_{k'}} \frac{\partial \lambda_{k'k}}{\partial a_l^s} + \lambda_{k'k} \sum_{j} \frac{\partial x^{j'}}{\partial a_l^s} a_{j'}^{k'} \right) \lambda_{ik} \sum_{r=1}^n \frac{\partial x^r}{\partial a_j^i} a_r^i.$$

Then equation (A.14) is satisfied if and only if $(\Gamma_{k'k})_{ij}^{sl} = (\Gamma_{k'k})_{sl}^{ij}$ for any given $k', k \in \{1, \dots, m\}$ and all $1 \le i, s \le m$ and $1 \le j, l \le n$. So, we get the required symmetry conditions. This completes the proof. □

A.4. Proof of Theorem 5.1

First, we have from the envelope theorem

$$\frac{\partial V}{\partial a_j^i} = \lambda_i \left(\frac{\partial C^i}{\partial a_j^i} - x^j \right) = \lambda_i \sum_r \frac{\partial x^r}{\partial a_j^i} a_r^i. \tag{A.15}$$

Differentiate this equation with respect to a_l^k, we get

$$\frac{\partial^2 V}{\partial a_l^k \partial a_j^i} = \frac{\partial \lambda_i}{\partial a_l^k} \left(\frac{\partial C^i}{\partial a_j^i} - x^j \right) + \lambda_i \left(\frac{\partial^2 C^i}{\partial a_l^k \partial a_j^i} \delta_k^i - \frac{\partial x^j}{\partial a_l^k} \right). \tag{A.16}$$

Multiply both sides by a_j^i, summing over j, and solving to get the following formula

$$\frac{\partial \lambda_i}{\partial a_l^k} = \eta_i \sum_{j'} \frac{\partial^2 V}{\partial a_l^k \partial a_{j'}^i} a_{j'}^i - \lambda_i \eta_i \sum_{j'} \left(\frac{\partial^2 C^i}{\partial a_l^k \partial a_{j'}^i} \delta_k^i - \frac{\partial x^{j'}}{\partial a_l^k} \right) a_{j'}^i.$$

Note that

$$\lambda_i \sum_{j'} \frac{\partial x^{j'}}{\partial a_l^k} a_{j'}^i = \frac{\partial V}{\partial a_l^k} \delta_k^i.$$

Using the last two equations together with (A.15) and (A.16) we find that

$$\lambda_i \frac{\partial x^j}{\partial a_l^k} = -\frac{\partial^2 V}{\partial a_l^k \partial a_j^i} + \lambda_i \frac{\partial^2 C^i}{\partial a_l^k \partial a_j^i} \delta_k^i + \frac{\eta_i}{\lambda_i} \left(\sum_{j'} \frac{\partial^2 V}{\partial a_l^k \partial a_{j'}^i} a_{j'}^i - \left(\lambda_i \sum_{j'} \frac{\partial^2 C^i}{\partial a_l^k \partial a_{j'}^i} a_{j'}^i - \frac{\partial V}{\partial a_l^k} \right) \delta_k^i \right) \frac{\partial V}{\partial a_j^i}. \tag{A.17}$$

Now, multiply both sides of (A.16) by a_l^k and add up with respect to l to get the following formula after rearrangement

$$\lambda_i \sum_{l'} \frac{\partial x^j}{\partial a_{l'}^k} a_{l'}^k = \sum_{l'} \frac{\partial^2 V}{\partial a_{l'}^k \partial a_j^i} a_{l'}^k + \frac{\eta_i}{\lambda_i} \sum_{j',l'} \frac{\partial^2 V}{\partial a_{l'}^k \partial a_{j'}^i} a_{l'}^k a_{j'}^i \frac{\partial V}{\partial a_j^i} - \eta_i \sum_{j',l'} \frac{\partial^2 C^i}{\partial a_{l'}^k \partial a_{j'}^i} a_{l'}^k a_{j'}^i \frac{\partial V}{\partial a_j^i} \delta_k^i \tag{A.18}$$
$$+ \frac{\partial V}{\partial a_j^i} \delta_k^i + \lambda_i \sum_{l'} \frac{\partial^2 C^i}{\partial a_{l'}^k \partial a_j^i} a_{l'}^k \delta_k^i.$$

To get this formula we used also

$$\sum_{l'=1}^n \frac{\partial \lambda_i}{\partial a_{l'}^k} a_{l'}^k = \eta_i \sum_{j',l'} \frac{\partial^2 V}{\partial a_{l'}^k \partial a_{j'}^i} a_{l'}^k a_{j'}^i - \lambda_i \eta_i \sum_{j',l'} \frac{\partial^2 C^i}{\partial a_{l'}^k \partial a_{j'}^i} a_{l'}^k a_{j'}^i \delta_k^i + \lambda_i \delta_k^i.$$

Recall that

$$\lambda_i S_k = \lambda_i \left(\frac{\partial x^j}{\partial a_l^k} - \eta_k \sum_{j'=1}^n \frac{\partial x^j}{\partial a_{j'}^k} a_{j'}^k \sum_{r=1}^n \frac{\partial x^r}{\partial a_l^k} a_r^k \right).$$

It suffices to substitute from (A.15), (A.17) and (A.18) into this equation to get the required formula. The proof is complete.

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