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ALMOST CONVEX VALUED PERTURBATION TO TIME OPTIMAL CONTROL SWEEPING PROCESSES

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Abstract. In this work, we study the existence of solutions of a perturbed sweeping process and of a time optimal control problem under a condition on the perturbation that is strictly weaker than the usual assumption of convexity.

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1. Introduction

The existence of solutions for the following first order differential inclusion governed by the sweeping process

$$(P)\begin{cases} \dot{u}(t)\in -N_{K(t)}(u(t))+F(t,u(t)), \ \text{ a.e } \ t\in[0,T],\\ u(t)\in K(t), \ \forall t\in[0,T],\\ u(0)=u_0, \end{cases}$$

where $N_{K(t)}(.)$ denotes the normal cone to $K(t)$ ($K(t)$ are convex or non-convex sets) and $F:[0,T]\times\mathbb{R}^d\rightrightarrows\mathbb{R}^d$ is a convex compact valued multifunction, Lebesgue-mesurable on $[0,T]$ and upper semicontinuous on $\mathbb{R}^d$, has been studied by many authors, see for example [5–7], and their references. Our aim in this paper is to provide existence results for the problem

$$(P_F)\begin{cases} \dot{u}(t)\in -N_{K}(u(t))+F(u(t)), \ \text{ a.e } \ t\in[0,T],\\ u(t)\in K, \ \forall t\in[0,T],\\ u(0)=u_0, \end{cases}$$

where K is a non-nempty closed and ρ-prox regular subset of $\mathbb{R}^d$ and $F:\mathbb{R}^d\rightrightarrows\mathbb{R}^d$ is an upper semicontinuous multifunction with almost-convex values, which is a strictly weaker condition than the convexity. Note that in [9], Cellina and Ornelas studied the first order Cauchy problem $\dot{u}(t)\in F(u(t)),\ u(0)=u_0$, with F an upper semicontinuous multifunction with non-empty compact and almost convex values, and in [1] we have extended this result to a second order differential inclusion with boundary conditions. Moreover, we prove the existence of solutions to the time optimal control problem $\dot{u}(t)\in -N_K(u(t))+f(u(t),\nu(t)),\ \nu(t)\in U(u(t))$, when the

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set $F(x)=f(x,U(x))$ is compact and almost convex. Filippov in [12], proved the first general theorem on the existence of solutions to a minimum time control problem of the form $\dot{u}(t)=f(u(t),\nu(t)),\ \nu(t)\in U(u(t))$, the classical assumption of convexity of the images of the map $F(x)=f(x,U(x))$ was replaced in [9] by the weaker assumption of almost convexity of the same images.

This paper is organized as follows. In Section 2 we present some notation and preliminaries, in Section 3 we prove the existence of solutions of (P_F) and of a time optimal control problem where F is a multifunction with non-convex values, using the convexified problem.

2. Notation and preliminaries

We denote by $\overline{\mathbf{B}}$ the unit closed ball of $\mathbb{R}^d$. $\mathbf{L}^1_{\mathbb{R}^d}([0,T])$ is the space of all Lebesgue integrable $\mathbb{R}^d$-valued mappings defined on $[0,T]$. By $\mathbf{C}_{\mathbb{R}^d}([0,T])$ we denote the Banach space of all continuous mappings $u:[0,T]\to\mathbb{R}^d$ endowed with the sup-norm.

For a subset $A\subset\mathbb{R}^d$, $co(A)$ denotes the convex hull of A and $\overline{co}(A)$ denotes its closed convex hull.

For a nonempty closed subset S of $\mathbb{R}^d$, we denote by $d_S(.)$ the usual distance function associated with S, *i.e.*, $d_S(u)=\inf\limits_{y\in S}\|u-y\|$, $\mathrm{Proj}_S(u)$ the projection of u onto S defined by

$$\mathrm{Proj}_S(u)=\{y\in S:\quad d_S(u)=\|u-y\|\},$$

and $\delta^*(x',S)=\sup\limits_{y\in S}\langle x',y\rangle$ the support function of S at $x'\in\mathbb{R}^d$.

Let X be a vector space, a set $D\subset X$ is called almost convex if for every $\xi\in co(D)$ there exist λ_1 and λ_2, $0\le\lambda_1\le 1\le\lambda_2$, such that $\lambda_1\xi\in D$, $\lambda_2\xi\in D$.

Every convex set is almost convex. If a set D is almost convex and $0\in co(D)$, then $0\in D$. Typical cases of almost convex sets are $D=\partial C$, with C a convex set not containing the origin, or $D=\{0\}\cup\partial C$, C a convex set containing the origin. Other notions of almost convexity exist in the literature (sometimes, a subset $D\subset\mathbb{R}^d$ is called almost convex if $cl(D)$ is convex and $ri(cl(D))\subset D$).

The following results are needed in the proof of our theorems.

Theorem 2.1 (see [2]). *Let us consider a sequence of absolutely continuous mappings $x_k(.)$ from an interval I of $\mathbb{R}$ to $\mathbb{R}^d$ satisfying*

(a) $\forall t\in I,\ (x_k(t))$ *is a relatively compact subset of* $\mathbb{R}^d$;
(b) *there exists a positive function* $\delta(.)\in\mathbf{L}^1_{\mathbb{R}}(I)$ *such that, for almost all* $t\in I,\ \|\dot{x}_k(t)\|\le\delta(t)$.

Then, there exists a subsequence (again denoted by) $(x_k(.))$ converging to an absolutely continuous mapping $x(.)$ from I to $\mathbb{R}^d$ in the sense that:

(i) $(x_k(.))$ *converges uniformly to* $x(.)$ *over compact subsets of* I;
(ii) $(\dot{x}_k(.))$ *converges weakly to* $\dot{x}(.)$ *in* $\mathbf{L}^1_{\mathbb{R}^d}(I)$.

Theorem 2.2 (see [4]). *Let U be a topological space and let Φ be a multifunction from $[0,T]\times U$ with non empty convex compact values in a Hausdorff locally convex space E such that for every $t\in[0,T]$, $\Phi(t,.)$ is upper semicontinuous and for every $x\in U$, $\Phi(.,x)$ is Lebesgue-mesurable. Let (x_n) and x defined from $[0,T]$ to U and (y_n) and y be scalarly Lebesgue-integrable mappings from $[0,T]$ to E. We assume the following hypotheses*

(a) *there exists a sequence* (e'_n) *in* E' *which separates the points of* E
(b) $\lim\limits_{n\to\infty}x_n(t)=x(t)$, a.e.
(c) *for every fixed* $x'\in E'$, *the sequence* $(\langle x',y_n(.)\rangle)$ *converges to* $\langle x',y(.)\rangle$ *with respect to the weak topology* $\sigma(\mathbf{L}^1_E([0,T]),\mathbf{L}^\infty_{E'}([0,T]))$
(d) $y_n(t)\in\Phi(t,x_n(t))$, a.e.
Then $y(t)\in\Phi(t,x(t))$, a.e.

We need in the sequel to recall some definitions and results that will be used throughout the paper. Let G be an open subset of a Hilbert space H and $h : G \to (-\infty, +\infty]$ be a lower semicontinuous function. The proximal subdifferential $\partial^P h(x)$, of h at x (see [11]) is defined by $\xi \in \partial^P h(x)$ iff there exist positive numbers σ and ς such that

$$h(y) - h(x) + \sigma\|y - x\|^2 \geq \langle \xi, y - x\rangle, \ \forall y \in x + \varsigma\overline{\mathbf{B}}_H.$$

Let x be a point of $S \subset H$. We recall (see [11]) that the proximal normal cone to S at x is defined by $N^P_S(x) = \partial^P \delta(x, S)$, where $\delta(., S)$ denotes the indicator function of S, *i.e.*, $\delta(x, S) = 0$ if $x \in S$ and $+\infty$ otherwise. Note that the proximal normal cone is also given by:

$$N^P_S(x) = \{\xi \in H : \ \exists\alpha > 0 \text{ s.t } x \in \text{Proj}_S(x + \alpha\xi)\}.$$

If h is a real-valued locally-Lipschitz function defined on H, the Clarke subdifferential $\partial^C h(x)$, of h at x (see [10]) is the nonempty convex compact subset of H given by:

$$\partial^C h(x) = \{\xi \in H : \ h^\circ(x; v) \geq \langle \xi, v\rangle, \forall v \in H\},$$

where

$$h^\circ(x; v) = \lim_{y\to x,}\sup_{t\downarrow 0} \frac{h(y + tv) - f(y)}{t}$$

is the generalized directional derivative of h at x in the direction v. The Clarke normal cone $N^C_S(x)$ to S at $x \in S$ is defined by polarity with $T^C_S(x)$, that is,

$$N^C_S(x) = \{\xi \in H : \ \langle \xi, v\rangle \leq 0, \ \forall v \in T^C_S(x)\},$$

where $T^C_S(x)$ denotes the clarke tangent cone and is given by

$$T^C_S(x) = \{v \in H : \ d^\circ_S(x; v) = 0\}.$$

Recall now, that for a given $\rho \in]0, +\infty]$ the subset S is uniformly ρ-prox-regular (see [13]) or equivalently ρ-proximally smooth (see [11]) if and only if every nonzero proximal normal to S can be realized by a ρ-ball, this means that for all $\overline{x} \in S$ and all $0 \neq \xi \in N^P_S(\overline{x})$ one has

$$\left\langle \frac{\xi}{\|\xi\|}, x - \overline{x} \right\rangle \leq \frac{1}{2\rho}\|x - \overline{x}\|^2,$$

for all $x \in S$. We make the convention $\frac{1}{\rho} = 0$ for $\rho = +\infty$. Recall that for $\rho = +\infty$ the uniform ρ-prox-regularity of S is equivalent to the convexity of S.

The following proposition summarizes some important consequences of the uniform prox-regularity needed in the sequel. For the proof of these results we refer the reader to [13].

Proposition 2.3. *Let S be a non-empty closed subset of H. The following assertions hold:*

1) *for all $x \in H$, $\partial d^P_S(x) = N^P_S(x) \cap \overline{\mathbf{B}}_H$;*
2) i) *all (usual) normal cones coincide for a uniformly prox-regular set S, and they are denoted by the usual notation N_S. The same holds for the subdifferential of $d_S(.)$;*
   ii) *$\partial d_S(x)$ is a weakly compact set;*
   iii) *for all $x \in \mathbb{R}^d$ with $d_S(x) < \rho$, $\text{Proj}_S(x)$ is a singleton of H.*

The following is an important closedness property of the subdifferential of the distance function associated with a multifunction (see [3]).

Theorem 2.4. *let $\rho \in]0, +\infty]$, Ω be an open subset of H, and $K : \Omega \rightrightarrows H$ be a Hausdorff-continuous multifunction. Assume that $K(z)$ is uniformly ρ-prox-regular for all $z \in \Omega$. Then for a given $0 < \sigma < \rho$, the following holds: for any $\bar{z} \in \Omega$, $\bar{x} \in K(\bar{z}) + (\rho - \sigma)\overline{\mathbf{B}}_H$, $x_n \to \bar{x}, z_n \to \bar{z}$ with $z_n \in \Omega$ (x_n not necessarily in $K(z_n)$) and $\xi_n \in \partial d_{K(z_n)}(x_n)$ with $\xi_n \to^w \bar{\xi}$ one has $\bar{\xi} \in d_{K(z)}(\bar{x})$.*
Here $\to^w$ means the weak convergence in H.

Remark 2.5. As a direct consequence of this theorem we have for every $\rho \in]0, +\infty]$, for a given $0 < \sigma < \rho$, and for every multifunction $K : \Omega \rightrightarrows H$ with uniformly ρ-prox regular values, the multifunction $(z, x) \mapsto \partial d_{K(z)}(x)$ is upper semicontinuous from $\{(z, x) \in \Omega \times H : x \in K(z) + (\rho - \sigma)\overline{\mathbf{B}}_H\}$ into H, which is equivalent to the upper semicontinuity of the function $(z, x) \mapsto \delta^*(p, \partial d_{K(z)}(x))$, on $\{(z, x) \in \Omega \times H : x \in K(z) + (\rho - \sigma)\overline{\mathbf{B}}_H\}$ for any $p \in H$.

Let $\bar{t} \in [0, T]$. We denote by $A_{u_0}(\bar{t}) = \{u(\bar{t}) : \ u(.) \in \mathfrak{T}_{\bar{t}}(u_0)\}$ the attainable set at $\bar{t}$ for the problem (P_F), where $\mathfrak{T}_{\bar{t}}(u_0)$ is the set of the trajectories of the differential inclusion (P_F) on the interval $[0, \bar{t}]$.

3. Existence results

First, we present an existence result of solutions of the problem (P) where $F : [0, T] \times \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ is a convex compact valued multifunction (see Thm. 1.5 in [7]), and we prove that the set of the trajectories is compact.

Theorem 3.1. *Let $T > 0$, and let $K : [0, T] \rightrightarrows \mathbb{R}^d$ be a nonempty closed valued multifunction satisfying the following assumptions:*

(H_1) *for each $t \in [0, T]$, $K(t)$ is ρ-prox regular for some fixed $\rho \in]0, +\infty]$,*
(H_2) *K varies in an absolutely continuous way, that is, there exists a nonnegative absolutely continuous function $v : [0, T] \to \mathbb{R}$ such that*

$$|d(x, K(t)) - d(y, K(s))| \leq \|x - y\| + |v(t) - v(s)|$$

for all $x, y \in \mathbb{R}^d$ and all $s, t \in [0, T]$. Let $F : [0, T] \times \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ be a convex compact valued multifunction such that:
(i) *for every $t \in [0, T]$, $F(t, .)$ is upper semicontinuous on $\mathbb{R}^d$,*
(ii) *for every $x \in \mathbb{R}^d$, $F(., x)$ is Lebesgue-mesurable on $[0, T]$,*
(iii) *there are two nonnegative constants p and q such that*

$$F(t, x) \subset (p + q\|x\|)\overline{\mathbf{B}}, \ \forall (t, x) \in [0, T] \times \mathbb{R}^d.$$

Then, for each $u_0 \in K(0)$:

1) *there is an absolutely continuous solution $u : [0, T] \to \mathbb{R}^d$ of the problem (P) satisfying*

$$\|\dot{u}(t)\| \leq \alpha(t) + \beta(t), \ \text{a.e. } t \in [0, T],$$

where

$$\alpha(t) = |\dot{v}(t)| + 2(p + q\|u_0\|),$$

and

$$\beta(t) = 2q \int_0^t [\alpha(s)\exp(2q(t - s)]\mathrm{d}s;$$

2) *for all $\bar{t} \in [0, T]$, the set of the trajectories $\mathfrak{T}_{\bar{t}}(u_0)$, is compact.*

Proof.

1) See the proof of Theorem 1.5 in [7].
2) a) Fix any $\bar{t} \in [0,T]$, and let us prove that the set

$$\mathfrak{T}_{\bar{t}}(u_0) = \{u \in \mathbf{C}_{\mathbb{R}^d}([0,\bar{t}]) : \ u \text{ is an absolutely continuous solution of } (P)\},$$

is compact. Let (u_n) be a sequence in $\mathfrak{T}_{\bar{t}}(u_0)$. Then, for each $n \in \mathbb{N}$, u_n is an absolutely continuous solution of (P), and

$$\|\dot{u}_n(t)\| \leq \alpha(t) + \beta(t), \text{ a.e. } t \in [0,\bar{t}]. \tag{3.1}$$

We get, for almost every $t \in [0,\bar{t}]$,

$$\|u_n(t)\| \leq \|u_0 + \int_0^t \dot{u}_n(s)\mathrm{d}s\| \leq \|u_0\| + \int_0^t (\alpha(s)+\beta(s))\mathrm{d}s,$$

so,

$$\|u_n(t)\| \leq \|u_0\| + \int_0^T (\alpha(s)+\beta(s))\mathrm{d}s = \|u_0\| + \|\alpha+\beta\|_{\mathbf{L}^1_{\mathbb{R}}([0,T])}. \tag{3.2}$$

We conclude that $(u_n(t))$ is relatively compact. On the other hand, for all $t_1, t_2 \in [0,\bar{t}]$ such that $t_1 \leq t_2$ we have

$$\|u_n(t_1) - u_n(t_2)\| \leq \int_{t_1}^{t_2} \|\dot{u}_n(s)\|\mathrm{d}s \leq \int_{t_1}^{t_2} (\alpha(s)+\beta(s))\mathrm{d}s.$$

Since $(\alpha+\beta) \in \mathbf{L}^{\mathbf{1}}_{\mathbb{R}}([0,\bar{t}])$, we get the equicontinuity of the sequence $(u_n(.))$. By the Ascoli–Arzelà theorem we conclude that $(u_n(.))$ is relatively compact in $\mathbf{C}_{\mathbb{R}^d}([0,\bar{t}])$, and since $\|\dot{u}_n(t)\| \leq \alpha(t)+\beta(t)$, a.e. on $[0,\bar{t}]$, we conclude by Theorem 2.1, that there exists a subsequence (again denoted by) $(u_n(.))$ converging to an absolutely continuous mapping $u(.)$ from $[0,\bar{t}]$ to $\mathbb{R}^d$ in the sense that, $(u_n(.))$ converges uniformly to $u(.)$ and $(\dot{u}_n(.))$ converges $\sigma(\mathbf{L}^1_{\mathbb{R}^d}([0,\bar{t}]), \mathbf{L}^\infty_{\mathbb{R}^d}([0,\bar{t}]))$ to $\dot{u}(.)$. Then

$$u(t) = \lim_{n\to\infty} u_n(t) = u_0 + \lim_{n\to\infty}\int_0^t \dot{u}_n(s)\mathrm{d}s = u_0 + \int_0^t \dot{u}(s)\mathrm{d}s, \ \forall t \in [0,\bar{t}].$$

Now, for each $n \in \mathbb{N}$, since $u_n(.)$ is a solution of (P), there exists a measurable mapping $f_n : [0,\bar{t}] \to \mathbb{R}^d$ such that for almost every $t \in [0,\bar{t}]$, $f_n(t) \in F(t, u_n(t))$, and

$$\dot{u}_n(t) - f_n(t) \in -N_{K(t)}(u_n(t)).$$

As

$$\|f_n(t)\| \leq p + q\|u_n(t)\|, \text{ a.e. } t \in [0,\bar{t}],$$

using the relation (3.2) we get

$$\|f_n(t)\| \leq p + q[\|u_0\| + \|\alpha+\beta\|_{\mathbf{L}^1_{\mathbb{R}}([0,T])}] = m_2. \tag{3.3}$$

It is clear that (f_n) is bounded in $\mathbf{L}^\infty_{\mathbb{R}^d}([0,\bar{t}])$, taking a subsequence if necessary, we may conclude that (f_n) weakly* or $\sigma(\mathbf{L}^\infty_{\mathbb{R}^d}([0,\bar{t}]), \mathbf{L}^1_{\mathbb{R}^d}([0,\bar{t}])]$-converges to some mapping $f \in \mathbf{L}^\infty_{\mathbb{R}^d}([0,\bar{t}])$. Consequently, for all $v(.) \in \mathbf{L}^1_{\mathbb{R}^d}([0,\bar{t}])$, we have

$$\lim_{n\to\infty} \langle f_n(.), v(.)\rangle = \langle f(.), v(.)\rangle.$$

Let $z(.) \in \mathbf{L}^\infty_{\mathbb{R}^d}([0,\bar{t}]) \subset \mathbf{L}^1_{\mathbb{R}^d}([0,\bar{t}])$, then

$$\lim_{n\to\infty} \langle f_n(.), z(.)\rangle = \langle f(.), z(.)\rangle.$$

This shows that $(f_n(.))$ weakly or $\sigma(\mathbf{L}^1_{\mathbb{R}^d}([0,\bar{t}]),\mathbf{L}^\infty_{\mathbb{R}^d}([0,\bar{t}]))$-converges to $f(.)$, by Theorem 2.2 we conclude that $f(t)\in F(t,u(t))$ a.e. on $[0,\bar{t}]$.

Let us prove now that u is a solution of the problem (P). By the relation (3.1) and (3.3), we get for almost every $t\in[0,\bar{t}]$

$$\|\dot{u}_n(t)-f_n(t)\|\leq\|\dot{u}_n(t)\|+\|f_n(t)\|\leq\alpha(t)+\beta(t)+m_2:=\gamma(t),$$

that is,

$$\dot{u}_n(t)-f_n(t)\in\gamma(t)\overline{\mathbf{B}},$$

since

$$\dot{u}_n(t)-f_n(t)\in -N_{K(t)}(u_n(t))$$

we get by (1) of Proposition 2.3

$$\dot{u}_n(t)-f_n(t)\in -\gamma(t)\partial d_{K(t)}(u_n(t)). \tag{3.4}$$

Remark that $(\dot{u}_n-f_n)$ weakly converges in $\mathbf{L}^1_{\mathbb{R}^d}([0,\bar{t}])$ to $\dot{u}-f$. An application of the Mazur's trick to $(\dot{u}_n-f_n)$ provides a sequence (z_n) with $z_n\in co\{\dot{u}_k-f_k:\ k\geq n\}$ such that (z_n) converges strongly in $\mathbf{L}^1_{\mathbb{R}^d}([0,\bar{t}])$ to $\dot{u}-f$. We can extract from (z_n) a subsequence which converges a.e. to $\dot{u}-f$. Then, for almost every $t\in[0,\bar{t}]$

$$\dot{u}(t)-f(t)\in\bigcap_{n\geq 0}\overline{\{z_k(t):\ k\geq n\}}\subset\bigcap_{n\geq 0}\overline{co}\{\dot{u}_k(t)-f_k(t):\ k\geq n\}.$$

Fix any $t\in[0,\bar{t}]$ and $\mu\in\mathbb{R}^d$, then the last relation gives

$$\begin{aligned}\langle\mu,\dot{u}(t)-f(t)\rangle&\leq\limsup_{n\to\infty}\delta^*(\mu,-\gamma(t)\partial d_{K(t)}(u_n(t)))\\&\leq\delta^*(\mu,-\gamma(t)\partial d_{K(t)}(u(t))),\end{aligned}$$

where the second inequality follows from Theorem 2.4 and Remark 2.5. Taking the supremum over $\mu\in\mathbb{R}^d$, we deduce that

$$\delta(\dot{u}(t)-f(t),-\gamma(t)\partial d_{K(t)}(u(t))=\delta^{**}(\dot{u}(t)-f(t),-\gamma(t)\partial d_{K(t)}(u(t))\leq 0,$$

which entails

$$\dot{u}(t)-f(t)\in-\gamma(t)\partial d_{K(t)}(u(t)))\subset -N_{K(t)}(u(t)),$$

where the last set is well defined since $u_n(t)\in K(t)$, $K(t)$ is closed and then $u(t)\in K(t)$. This shows that $\mathfrak{T}_{\bar{t}}(u_0)$ is compact.

b) With the same arguments, one can prove that $A_{u_0}(\bar{t})$ is compact.

Now we are able to give an existence result and a property of the attainable set for the problem (P_F) where F has almost convex compact values. For the proof of our Theorem we need the following result.

Theorem 3.2. *Let K be a non-empty closed and ρ-prox regular set. Let $F:\mathbb{R}^d\rightrightarrows\mathbb{R}^d$ be a compact valued multifunction, upper semicontinuous on $\mathbb{R}^d$. Suppose that there are nonnegative constants p and q such that*

$$F(x)\subset(p+q\|x\|)\overline{\mathbf{B}},\ \forall x\in\mathbb{R}^d.$$

Let $u_0\in K$ and let $x:[0,T]\to\mathbb{R}^d$ be an absolutely continuous solution of the problem

$$(P_{coF})\begin{cases}\dot{u}(t)\in -N_K(u(t))+co(F(u(t))),\ \text{a.e. } t\in[0,T],\\ u(t)\in K,\ \forall t\in[0,T],\\ u(0)=u_0.\end{cases}$$

Assume that there are two integrable functions $\lambda_1(.)$ and $\lambda_2(.)$ defined on $[0,T]$, satisfying $0\leq\lambda_1(t)\leq 1\leq\lambda_2(t)$ and such that, for almost every $t\in[0,T]$, we have

$$\lambda_1(t)f(t)\in F(x(t))\ \ \textit{and}\ \ \lambda_2(t)f(t)\in F(x(t)),$$

where $f : [0,T] \to \mathbb{R}^d$ *is a measurable mapping satisfying* $\dot{x}(t) \in -N_K(x(t)) + f(t)$ a.e. *and* $f(t) \in co(F(x(t)))$, *for all* $t \in [0,T]$. *Then there exists a nondecreasing absolutely continuous map* $t(.)$ *of the interval* $[0,T]$ *into itself, such that the map* $\tilde{x}(\tau) = x(t(\tau))$ *is a solution of the problem* (P_F). *Moreover,* $\tilde{x}(0) = x(0)$ *and* $\tilde{x}(T) = x(T)$.

Proof.
Step 1. Let $[a,b] \subset [0,T]$ be an interval, and assume that, on this interval, there exist two integrable functions $\lambda_1(.)$ and $\lambda_2(.)$, with the properties stated above. In addition, assume that $\lambda_1(\tau) > 0$ a.e. We claim that there exist two measurable subsets of $[a,b]$, having characteristic functions $\mathcal{X}_1$ and $\mathcal{X}_2$ such that $\mathcal{X}_1 + \mathcal{X}_2 = \mathcal{X}_{[a,b]}$, and an absolutely continuous function $s : [a,b] \to [a,b]$ with $s(a) - s(b) = a - b$, such that

$$\dot{s}(\tau) = \mathcal{X}_1(\tau)\frac{1}{\lambda_1(\tau)} + \mathcal{X}_2(\tau)\frac{1}{\lambda_2(\tau)}.$$

Set

$$\psi(\tau) = \begin{cases} \dfrac{1}{2} & \text{when } \lambda_1(\tau) = \lambda_2(\tau) = 1, \\ \dfrac{\lambda_2(\tau) - 1}{\lambda_2(\tau) - \lambda_1(\tau)} & \text{otherwise.} \end{cases}$$

With this definition we have that $0 \leq \psi(\tau) \leq 1$ and both equalities hold true

$$1 = \psi(\tau) + \big(1 - \psi(\tau)\big) = \psi(\tau)\lambda_1(\tau) + \big(1 - \psi(\tau)\big)\lambda_2(\tau).$$

In particular, we have

$$\int_a^b 1\mathrm{d}\tau = \int_a^b \Big(\psi(\tau) + \big(1 - \psi(\tau)\big)\Big)\mathrm{d}\tau = \int_a^b \Big(\frac{\psi(\tau)\lambda_1(\tau)}{\lambda_1(\tau)} + \frac{\big(1 - \psi(\tau)\big)\lambda_2(\tau)}{\lambda_2(\tau)}\Big)\mathrm{d}\tau.$$

We wish to apply Liapunov's theorem on the range of measures, to infer the existence of two measurable subsets having characteristic functions $\mathcal{X}_1(.), \mathcal{X}_2(.)$ such that $\mathcal{X}_1 + \mathcal{X}_2 = \mathcal{X}_{[a,b]}$ and with the property

$$\int_a^b 1\mathrm{d}\tau = \int_a^b \Big(\mathcal{X}_1(\tau)\frac{1}{\lambda_1(\tau)} + \mathcal{X}_2(\tau)\frac{1}{\lambda_2(\tau)}\Big)\mathrm{d}\tau. \tag{3.5}$$

However, it is not obvious that the function $\dfrac{1}{\lambda_1(\tau)}$ is integrable. For this purpose, we shall use a device already used in [8]. Consider the sequence of disjoint sets

$$E^n = \Big\{\tau \in [a,b] : \ n < \frac{1}{\lambda_1(\tau)} \leq n+1\Big\}.$$

We have that $\displaystyle\bigcup_{n\in\mathbb{N}} E^n = [a,b]$. Applying Liapunov's theorem to each E^n, we infer the existence of two sequences of measurable subsets E_1^n, E_2^n, having characteristic functions $\mathcal{X}_1^n, \mathcal{X}_2^n$, such that for every n

$$\int_{E^n} 1\mathrm{d}\tau = \int_{E^n} \Big(\mathcal{X}_1^n(\tau)\frac{1}{\lambda_1(\tau)} + \mathcal{X}_2^n(\tau)\frac{1}{\lambda_2(\tau)}\Big)\mathrm{d}\tau.$$

For each k, the function

$$\sigma_k(\tau) = \sum_{n=0}^{k} \Big(\mathcal{X}_1^n(\tau)\frac{1}{\lambda_1(\tau)} + \mathcal{X}_2^n(\tau)\frac{1}{\lambda_2(\tau)}\Big)$$

is positive, and the sequence $(\sigma_k(.))$ converges pointwise monotonically increasing to

$$\sigma(\tau) = \mathcal{X}_1(\tau)\frac{1}{\lambda_1(\tau)} + \mathcal{X}_2(\tau)\frac{1}{\lambda_2(\tau)}. \tag{3.6}$$

Moreover, the sequence of sets $V^k = \bigcup_{n=0}^{k} E^n$ is monotonically increasing to $[a,b]$ so that

$$\int_a^b 1\mathrm{d}\tau = \int_{\bigcup_k V^k} 1\mathrm{d}\tau = \int_{\bigcup_n E^n} 1\mathrm{d}\tau,$$

and

$$\int_{\bigcup_k V^k} 1\mathrm{d}\tau = \lim_{k\to\infty} \int_{V^k} 1\mathrm{d}\tau,$$

and since the sets E^n are disjoint we get

$$\int_a^b 1\mathrm{d}\tau = \lim_{k\to\infty} \int_{V^k} 1\mathrm{d}\tau = \lim_{k\to\infty} \int_{\bigcup_{n=0}^{k} E^n} 1\mathrm{d}\tau = \lim_{k\to\infty} \sum_{n=0}^{k} \int_{E^n} 1\mathrm{d}\tau,$$

then

$$\begin{aligned}
\int_a^b 1\mathrm{d}\tau &= \lim_{k\to\infty} \sum_{n=0}^{k} \int_{E^n} \left(\mathcal{X}_1^n(\tau)\frac{1}{\lambda_1(\tau)} + \mathcal{X}_2^n(\tau)\frac{1}{\lambda_2(\tau)} \right)\mathrm{d}\tau \\
&= \lim_{k\to\infty} \sum_{n=0}^{k} \int_{E^n} \sum_{n=0}^{k} \left(\mathcal{X}_1^n(\tau)\frac{1}{\lambda_1(\tau)} + \mathcal{X}_2^n(\tau)\frac{1}{\lambda_2(\tau)} \right)\mathrm{d}\tau = \lim_{k\to\infty} \sum_{n=0}^{k} \int_{E^n} \sigma_k(\tau)\mathrm{d}\tau \\
&= \lim_{k\to\infty} \int_{\bigcup_{n=0}^{k} E^n} \sigma_k(\tau)\mathrm{d}\tau = \lim_{k\to\infty} \int_{\bigcup_n E^n} \sigma_k(\tau)\mathrm{d}\tau = \int_{\bigcup_n E^n} \lim_{k\to\infty} \sigma_k(\tau)\mathrm{d}\tau,
\end{aligned}$$

we conclude that

$$\int_a^b 1\mathrm{d}\tau = \int_a^b \sigma(\tau)\mathrm{d}\tau = \int_a^b \left(\mathcal{X}_1(\tau)\frac{1}{\lambda_1(\tau)} + \mathcal{X}_2(t)\frac{1}{\lambda_2(\tau)} \right)\mathrm{d}\tau.$$

Set $\dot{s}(\tau) = \sigma(\tau)$. Then $\int_a^b \dot{s}(\tau)\mathrm{d}\tau = b - a$.

Step 2.

(a) Consider the set

$$C = \{\tau \in [0,T] : \quad 0 \in F(x(\tau))\}.$$

It is clear that C is closed. Indeed, let (τ_n) be a sequence in C converging to $\tau \in [0,T]$. Then, for each $n \in \mathbb{N}$, $0 \in F(x(\tau_n))$. Since $x(.)$ is continuous and F is upper semicontinuous with compact values, we conclude that $0 \in F(x(\tau))$, that is, C is closed.

(b) Consider the case in which C is empty. In this case, it cannot be that $\lambda_1(\tau) = 0$ on a set of positive measure, and the Step 1 can be applied to the interval $[0,T]$. Set $s(\tau) = \int_0^\tau \dot{s}(\omega)\mathrm{d}\omega$, s is increasing and we have $s(0) = 0$ and $s(T) = \int_0^T \dot{s}(\omega)\mathrm{d}\omega = T$, that is, s maps $[0,T]$ into itself. Let $t : [0,T] \to [0,T]$ be its inverse, then $t(0) = 0;\ t(T) = T$ and we have $\frac{\mathrm{d}}{\mathrm{d}\tau} s(t(\tau)) = \dot{s}(t(\tau))\dot{t}(\tau) = 1$. Then,

$$\dot{t}(\tau) = \frac{1}{\dot{s}\big(t(\tau)\big)} = \frac{1}{\sigma\big(t(\tau)\big)} = \Big(\lambda_1\big(t(\tau)\big)\mathcal{X}_1\big(t(\tau)\big) + \lambda_2\big(t(\tau)\big)\mathcal{X}_2\big(t(\tau)\big) \Big).$$

Consider the map $\tilde{x} : [0,T] \to \mathbb{R}^d$ defined by $\tilde{x}(\tau) = x\big(t(\tau)\big)$. We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\tilde{x}(\tau) &= \dot{t}(\tau)\dot{x}\big(t(\tau)\big) = \frac{1}{\dot{s}\big(t(\tau)\big)}\dot{x}\big(t(\tau)\big)\\ &= \Big(\lambda_1\big(t(\tau)\big)\mathcal{X}_1\big(t(\tau)\big) + \lambda_2\big(t(\tau)\big)\mathcal{X}_2\big(t(\tau)\big)\Big)\dot{x}\big(t(\tau)\big)\\ &\in \Big(\lambda_1\big(t(\tau)\big)\mathcal{X}_1\big(t(\tau)\big) + \lambda_2\big(t(\tau)\big)\mathcal{X}_2\big(t(\tau)\big)\Big)\Big(- N_K\big(x\big(t(\tau)\big)\big) + f\big(t(\tau)\big)\Big),\end{aligned}$$

by the properties of the normal cone and the assumption on f we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\tilde{x}(\tau) &\in -N_K(x\big(t(\tau)\big)) + f\big(t(\tau)\big)\Big(\lambda_1\big(t(\tau)\big)\mathcal{X}_1\big(t(\tau)\big) + \lambda_2\big(t(\tau)\big)\mathcal{X}_2\big(t(\tau)\big)\Big)\\ &\subset -N_K\big(x\big(t(\tau)\big)\big) + F\big(x\big(t(\tau)\big)\big)\\ &= -N_K\big(\tilde{x}\big(t(\tau)\big)\big) + F\big(\tilde{x}(\tau)\big).\end{aligned}$$

(c) Now we shall assume that C is nonempty. Let $c = \sup\{\tau :\ \tau \in C\}$, there is a sequence (τ_n) in C such that $\lim_{n\to\infty} \tau_n = c$. Since C is closed we get $c \in C$. The complement of C is open relative to $[0,T]$, it consists of at most countably many nonoverlapping open intervals $]a_i, b_i[$, with the possible exception of one of the form $[a_{i_i}, b_{i_i}[$ with $a_{i_i} = 0$ and one of the form $]a_{i_f}, b_{i_f}]$ with $a_{i_f} = c$. For each i, apply Step 1 to the interval $]a_i, b_i[$ to infer the existence of A_1^i and A_2^i, two subsets of $]a_i, b_i[$ with characteristic functions $\mathcal{X}_1^i(.)$, $\mathcal{X}_2^i(.)$ such that $\mathcal{X}_1^i(.) + \mathcal{X}_2^i(.) = \mathcal{X}_{]a_i,b_i[}(.)$. Setting

$$\dot{s}(\tau) = \frac{1}{\lambda_1(\tau)}\mathcal{X}_1^i(\tau) + \frac{1}{\lambda_2(\tau)}\mathcal{X}_2^i(\tau),$$

we obtain

$$\int_{a_i}^{b_i} \dot{s}(\omega)\mathrm{d}\omega = b_i - a_i.$$

(d) On $[0,c]$, set

$$\dot{s}(\tau) = \frac{1}{\lambda_2(\tau)}\mathcal{X}_C(\tau) + \sum_i \left(\frac{1}{\lambda_1(\tau)}\mathcal{X}_1^i(\tau) + \frac{1}{\lambda_2(\tau)}\mathcal{X}_2^i(\tau)\right),$$

where the sum is over all intervals contained in $[0,c]$. We have that

$$\int_0^c \dot{s}(\omega)\mathrm{d}\omega = \kappa \le c$$

since $\lambda_2(\tau) \ge 1$, and $\int_{a_i}^{b_i} \dot{s}(\omega)\mathrm{d}\omega = b_i - a_i$. Setting $s(\tau) = \int_0^\tau \dot{s}(\omega)\mathrm{d}\omega$, we obtain that $s(.)$ is an invertible map from $[0,c]$ to $[0,\kappa]$.

(e) Define $t : [0,\kappa] \to [0,c]$ to be the inverse of $s(.)$. Extend $t(.)$ as an absolutely continuous map $\tilde{t}(.)$ on $[0,c]$, setting $\dot{\tilde{t}}(\tau) = 0$ for $\tau \in]\kappa, c]$. We claim that the mapping $\tilde{x}(\tau) = x(\tilde{t}(\tau))$ is a solution of the problem (P_F) on the interval $[0,c]$. Moreover, we claim that it satisfies $\tilde{x}(c) = x(c)$.

Observe that, as in (b), we have that for $\tau \in [0,\kappa]$, $\tilde{t}(\tau) = t(\tau)$ is invertible and

$$\dot{t}(\tau) = \lambda_2\big(t(\tau)\big)\mathcal{X}_C\big(t(\tau)\big) + \sum_i \Big(\lambda_1\big(t(\tau)\big)\mathcal{X}_1^i\big(t(\tau)\big) + \lambda_2\big(t(\tau)\big)\mathcal{X}_2^i\big(t(\tau)\big)\Big).$$

Since $\frac{\mathrm{d}}{\mathrm{d}\tau}\tilde{x}(\tau) = \dot{t}(\tau)\dot{x}(t(\tau))$ we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\tilde{x}(\tau) &= \dot{x}\big(t(\tau)\big)\Big(\lambda_2\big(t(\tau)\big)\mathcal{X}_C\big(t(\tau)\big) + \sum_i \big(\lambda_1\big(t(\tau)\big)\mathcal{X}_1^i\big(t(\tau)\big) + \lambda_2\big(t(\tau)\big)\mathcal{X}_2^i\big(t(\tau)\big)\big)\Big)\\ &\in \Big(-N_K\big(x\big(t(\tau)\big)\big) + f\big(t(\tau)\big)\Big)\Big(\lambda_2\big(t(\tau)\big)\mathcal{X}_C\big(t(\tau)\big)\\ &+ \sum_i \big(\lambda_1(t(\tau))\mathcal{X}_1^i(t(\tau)) + \lambda_2\big(t(\tau)\big)\mathcal{X}_2^i\big(t(\tau)\big)\big)\Big)\\ &\subset -N_K\big(x\big(t(\tau)\big)\big) + F\big(x\big(t(\tau)\big)\big) = -N_K\big(\tilde{x}(\tau)\big) + F\big(\tilde{x}(\tau)\big).\end{aligned}$$

In particular, from $t(\kappa) = c$ and $\dot{\tilde{t}}(\tau) = 0$ for all $\tau \in]\kappa, c]$ we obtain

$$\tilde{t}(\tau) = \tilde{t}(\kappa) = t(\kappa), \forall \tau \in]\kappa, c],$$

then

$$\tilde{x}(\kappa) = x(\tilde{t}(\kappa)) = x(t(\kappa)) = x(\tilde{t}(\tau)) = \tilde{x}(\tau),\ \forall \tau \in]\kappa, c],$$

so, $\tilde{x}(c) = x(c)$, $\tilde{x}$ is constant on $]\kappa, c]$, and we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\tilde{x}(\tau) = 0 \in F(x(c)) = F(\tilde{x}(\tau)) \subset co\big(F(\tilde{x}(\tau)\big), \forall \tau \in]\kappa, c]. \tag{3.7}$$

As $0 \in -N_K(\tilde{x}(\tau))$, using (3.7) we conclude that for $\tau \in]\kappa, c]$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\tilde{x}(\tau) = 0 \in -N_K(\tilde{x}(\tau)) + F(\tilde{x}(\tau)).$$

This proves our claim.
(f) It remains to define the solution on $[c, T]$. On it, $\lambda_1(\tau) > 0$ and the construction of Step 1 and (b) can be repeated to find a solution to the problem (P_F) on $[c, T]$. This completes the proof of the theorem. □

Theorem 3.3. *Let K be a nonempty closed and ρ-prox regular set. Let $F : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ be an almost convex compact valued multifunction, upper semicontinuous on $\mathbb{R}^d$. Suppose that there are nonnegative constants p and q such that*

$$F(x) \subset (p + q\|x\|)\overline{\mathbf{B}},\ \forall x \in \mathbb{R}^d.$$

Then, for each $u_0 \in K$:

1) *the problem (P_F) has at least an absolutely continuous solution;*
2) *for every $\bar{t} \in [0, T]$, the attainable set at $\bar{t}$, $A_{u_0}(\bar{t})$, coincides with $A^{co}_{u_0}(\bar{t})$, the attainable set at $\bar{t}$ of the convexified problem (P_{coF}).*

Proof.
1) In view of Theorem 3.1, as $co(F) : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ is a multifunction with convex compact values, upper semicontinuous on $\mathbb{R}^d$ since F is upper semicontinuous on $\mathbb{R}^d$, and for all $x \in \mathbb{R}^d$,

$$co(F(x)) \subset (p + q\|x\|)co(\overline{\mathbf{B}}) = (p + q\|x\|)\overline{\mathbf{B}},$$

we conclude the existence of an absolutely continuous solution $u(.)$ of the problem $(P_{co(F)})$ satisfying

$$\|\dot{u}(t)\| \leq \alpha(t) + \beta(t),\ \text{a.e. } t \in [0, T].$$

Then,

$$\|u(t)\| \leq \|u_0\| + \int_0^t (\alpha(s)+\beta(s))\mathrm{d}s \leq \|u_0\| + \|\alpha+\beta\|_{\mathbf{L}^1_{\mathbb{R}}([0,T])},$$

and

$$co(F(u(t))) \subset (p + q(\|u_0\| + \|\alpha+\beta\|_{\mathbf{L}^1_{\mathbb{R}}([0,T])}))\overline{\mathbf{B}} = m_2\overline{\mathbf{B}}.$$

Let $f(.)$ be a Lebesgue-measurable selection of $co(F(u(.)))$, *i.e.* $f(t) \in co(F(u(t)))$, for all $t \in [0,T]$ and such that $\dot{u}(t) \in -N_K(u(t)) + f(t)$, a.e. Let us prove that there exist two integrable functions $\lambda_1(.)$, $\lambda_2(.)$ defined on $[0,T]$ and satisfying $0 \leq \lambda_1(t) \leq 1 \leq \lambda_2(t)$, such that for almost every $t \in [0,T]$, $\lambda_1(t)f(t) \in F(u(t))$ and $\lambda_2(t)f(t) \in F(u(t))$.

Since for every $t \in [0,T]$ $F(u(t))$ is almost convex, there exist two nonempty sets $\Lambda_1(t)$ and $\Lambda_2(t)$ such that

$$\Lambda_1(t) = \{\lambda_1 \in [0,1] : \lambda_1 f(t) \in F(u(t))\}$$

and

$$\Lambda_2(t) = \{\lambda_2 \in [1,+\infty[: \lambda_2 f(t) \in F(u(t))\}.$$

Set $Z = \{t : f(t) = 0\}$. There is no loss of generality in assuming that, for $t \in Z$, $\Lambda_1(t) = \Lambda_2(t) = \{1\}$.

We must show that the multifunction $\Lambda_1 : [0,T] \rightrightarrows [0,1]$ is measurable. Applying Lusin's theorem to f, we can write $[0,T] \setminus Z$ as $(\bigcup_{i\in I} B_i) \cup \mathcal{N}$, where I is countable, each B_i is compact, the measure of $\mathcal{N}$ is 0, and the restriction of f to each B_i is continuous. We need to prove that the graph of Λ_1 ($gph(\Lambda_1)$) is closed on $B_i \times [0,1]$. Let (t_n, λ_1^n) be a sequence in $gph(\Lambda_1)/_{B_i\times[0,1]}$ which converges to $(t,\lambda_1) \in B_i \times [0,1]$. Then, for each $n \in \mathbb{N}$, $\lambda_1^n f(t_n) \in F(u(t_n))$. Since F is upper semicontinuous with compact values and since $f(.)$ and $u(.)$ are continuous on B_i, we get $\lambda_1 f(t) \in F(u(t))$, then $\lambda_1 \in \Lambda_1(t)$. It follows that Λ_1 has a closed graph on $B_i \times [0,1]$. In addition, its values are closed subsets of $[0,1]$ because the values of F are closed. Then, we conclude that Λ_1 is upper semicontinuous and consequently it is measurable on $[0,T]$.

The proof that $\Lambda_2 : [0,T] \rightrightarrows [1,+\infty[$ is measurable is similar, with the difference that the values of Λ_2 need not be bounded. In this case, we write $[0,T] \setminus Z$ as the countable union of the sets $M_n = \{t : \|f(t)\| \geq \frac{1}{n}\}$. On each M_n, and for all $\lambda_2 \in \Lambda_2(t)$ we have $\lambda_2 f(t) \in F(u(t)) \subset m_2\overline{\mathbf{B}}$. So, Λ_2 has an upper bound on M_n, and the same reasoning as in the previous point can be applied.

Consequently, by the existence of measurable selection theorem, there are measurable selections $\lambda_1(.)$ and $\lambda_2(.)$, of Λ_1 and Λ_2 respectively, satisfying $0 \leq \lambda_1(t) \leq 1 \leq \lambda_2(t)$, and such that, for every $t \in [0,T]$, we have

$$\lambda_1(t)f(t) \in F(u(t)) \text{ and } \lambda_2(t)f(t) \in F(u(t)).$$

Using Theorem 3.2, we conclude the existence of a solution $\tilde{u}(.)$ of the problem (P_F) such that $u(T) = \tilde{u}(T)$.
2) For every $\bar{t} \in [0,T]$ the attainable set at $\bar{t}, A_{u_0}(\bar{t})$, is contained in the attainable set at $\bar{t}$ of the convexified problem, $A^{co}_{u_0}(\bar{t})$, it is enough to show that $A^{co}_{u_0}(\bar{t}) \subset A_{u_0}(\bar{t})$.

Let $x(\bar{t}) \in A^{co}_{u_0}(\bar{t})$, so $x(.)$ is an absolutely continuous solution of the problem $(P_{co(F)})$. The point **1)** of Theorem 3.2, can be repeated on $[0,\bar{t}]$ to find a solution $\tilde{x}(.)$ of the problem (P_F) such that $x(\bar{t}) = \tilde{x}(\bar{t}) \in A_{u_0}(\bar{t})$. Consequently $A^{co}_{u_0}(\bar{t}) \subset A_{u_0}(\bar{t})$. This finishes the proof. □

In the following corollary we prove the existence of solutions to the minimum time problem for the differential inclusion

$$(P_f)\begin{cases} \dot{u}(t) \in -N_K(u(t)) + f(u(t),\nu(t)), \text{ a.e. } t \in [0,T], \\ \nu(t) \in U(u(t)), \ \forall t \in [0,T], \\ u(t) \in K, \ \forall t \in [0,T], \\ u(0) = u_0, \end{cases}$$

under the almost convexity assumption on the set $F(x) = f(x, U(x))$.

Corollary 3.4. *Let $T > 0$ and K be a nonempty closed and ρ-prox regular set. Let $U : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$, be a compact valued multifunction, upper semicontinuous on $\mathbb{R}^d$ and $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be a mapping satisfying the following assumptions:*

(i) *for any $y \in \mathbb{R}^d$, $f(., y)$ is continuous on $\mathbb{R}^d$;*
(ii) *there are nonnegative constants p and q such that*

$$\|f(x, y)\| \leq p + q\|x\|, \ \forall (x, y) \in \mathbb{R}^d \times \mathbb{R}^d;$$

(iii) *the set $F(x) = f(x, U(x))$ is compact and almost convex for every $x \in \mathbb{R}^d$.*
Let u_0, u_1 be given in $\mathbb{R}^d$, and assume that for some $0 \leq \tilde{t} \leq T$, $u_1 \in A_{u_0}(\tilde{t})$. Then, the problem of reaching u_1 from u_0 in a minimum time admits a solution.

Proof. Let $\hat{t} = \inf\{t \in [0, \tilde{t}] : \ u_1 \in A_{u_0}(t)\}$. Let (t_n) be decreasing to $\hat{t}$ and for each n let $u_n(.)$ be a solution of the problem

$$\begin{cases} \dot{u}(t) \in -N_K(u(t)) + F(u(t)), \text{ a.e. } t \in [0, t_n], \\ u(t) \in K, \forall t \in [0, t_n], \\ u(0) = u_0 \end{cases}$$

such that $u_n(t_n) = u_1$. We define the sequence $(\hat{u}_n(.))$ by $\hat{u}_n(t) = u_n(t)$, for all $t \in [0, \hat{t}]$. Then $(\hat{u}_n(t)) \subset A_{u_0}(t) = A^{co}_{u_0}(t)$. Since $A^{co}_{u_0}(t)$ is compact, by extracting a subsequence if necessary we may conclude that $(\hat{u}_n(t))$ converges to $\hat{u}(t) \in A^{co}_{u_0}(t)$, clearly $\hat{u}(\hat{t}) = u_1 \in A^{co}_{u_0}(\hat{t})$. By Theorem 3.3 we have $A^{co}_{u_0}(\hat{t}) = A_{u_0}(\hat{t})$. Consequently, $\hat{u}$ is the solution of the problem (P_f) that reaches u_1 in the minimum time, and $\hat{t}$ is the value of the minimum time for the problem in consideration. This finishes the proof. □

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