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THE LEBEAU–ROBBIANO INEQUALITY FOR THE ONE-DIMENSIONAL FOURTH ORDER ELLIPTIC OPERATOR AND ITS APPLICATION

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Abstract. In this paper, we establish the Lebeau–Robbiano inequality for the one-dimensional fourth order elliptic operator by using a point-wise estimate. Based on this inequality, we obtain the null controllability of one-dimensional stochastic fractional order Cahn–Hilliard equation.

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1. Introduction

In this paper, we investigate the one-dimensional fourth order elliptic operator A on $L^2(I)$ as follows

$$\begin{cases} \mathcal{D}(A) = H_0^2(I) \cap H^4(I), \\ Ay = y_{xxxx} \quad \forall y \in \mathcal{D}(A), \end{cases}$$

where $I = (0, 1)$.

Let $\{\lambda_i\}_{i=1}^{\infty}, 0 < \lambda_1 \leq \lambda_2 \leq \ldots$ be the eigenvalues of A and $\{e_i\}_{i=1}^{\infty}$ be the corresponding real eigenfunctions such that $\|e_i\|_{L^2(I)} = 1$ $(i = 1, 2, 3, \ldots)$, which serves as an orthonormal basis of $L^2(I)$ (see [13], Thm. 8.94).

The main result in this paper is an observability estimate on partial sums of eigenfunctions for the eigenfunctions of A, *i.e.* the Lebeau–Robbiano inequality:

Theorem 1.1. *Let ω be a nonempty open subset of I. There exist two positive constants C_1, C_2 such that*

$$\sum_{\lambda_i \leq r} |a_i|^2 \leq C_1 \mathrm{e}^{C_2\sqrt{r}} \int_\omega \left| \sum_{\lambda_i \leq r} a_i e_i \right|^2 \mathrm{d}x$$

for every finite $r > 0$ and every choice of $\{a_i\}_{\lambda_i \leq r}$ with $a_i \in \mathbb{C}$.

Keywords and phrases. Lebeau–Robbiano inequality, null controllability, stochastic fractional order Cahn–Hilliard equation.

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The Lebeau–Robbiano inequality was first established in [5] for the Laplacian with homogeneous Dirichlet boundary condition, then the same result for more general boundary conditions and second order elliptic operators were established in [7, 11].

In the study on the controllability of PDE, the inequality of this type is the foundation of Lebeau–Robbiano iteration technology. By this inequality, [5] obtained the null controllability of a linear system of thermoelasticity in a compact, C^∞, n-dimensional connected Riemannian manifold. Then, this inequality was used to establish a certain L^∞-null controllability for the internal controlled heat equation when the control functions are restricted in arbitrary subset of positive measure in time variable in [14]. Later, in [10], *via* the Lebeau–Robbiano inequality, the authors presented a time optimal control problem with control constraints of the rectangular type for internal controlled heat equations. Recently, the similar result as in [14] was extended to forward stochastic heat equations in [9] and second order parabolic equation with equivalued surface boundary conditions in [11]. In [3], the Lebeau–Robbiano inequality can also be used to depict nodal sets of sums of functions.

However, to the best of our knowledge, the Lebeau–Robbiano inequality for fourth order elliptic operator has not been established. It should be noted that Theorem 1.1 in this paper is not a straightforward generalization of the Lebeau–Robbiano inequality in [4], although the inequality has the same formal. To prove Theorem 1.1, a point-wise estimate for the operator $\partial_t^2 - \partial_x^4$ in Proposition 2.1 plays a crucial role. In order to obtain the point-wise estimate, we need to choose suitable grouping for the terms of $\theta(y_{tt} - y_{xxxx})$ (see I_1 and I_2 in the proof of Prop. 2.1) since the high order of the operator. Due to the special properties of the operator $\partial_t^2 - \partial_x^4$ and the boundary terms in the point-wise estimate, we need to construct several weight functions.

Based on Theorem 1.1, we can obtain the second result in this paper: the null controllability of one-dimensional stochastic fractional order Cahn–Hilliard equation.

First, we give some assumptions.

(H1) Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $\{B(t)\}_{t\geq 0}$ is defined such that $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration generated by $B(\cdot)$, augmented by all the P-null sets in $\mathcal{F}$. Let H be a Banach space, and let $C([0,T];H)$ be the Banach space of all H-valued strongly continuous functionals defined on $[0,T]$. We denote by $L^2_{\mathcal{F}}(0,T;H)$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes $X(\cdot)$ such that $\mathbb{E}(\|X(\cdot)\|^2_{L^2(0,T;H)}) < \infty$, with the canonical norm; by $L^\infty_{\mathcal{F}}(0,T;H)$ the Banach space consisting of all H-valued $\{F_t\}_{t\geq 0}$-adapted bounded processes; and by $L^2_{\mathcal{F}}(\Omega; C[0,T];H))$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted continuous processes $X(\cdot)$ such that $\mathbb{E}(\|X(\cdot)\|^2_{C([0,T];H)}) < \infty$, with the canonical norm; by $L^r_{\mathcal{F}}(0,T;L^2(\Omega;H))$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes X such that $\|\mathbb{E}\|X\|^2_H\|_{L^r(0,T)} < \infty (1 \leq r \leq \infty)$, with the canonical norm.

(H2) $E \subset [0,T]$ with positive measure. χ_ω and χ_E denote the characteristic function of ω and E, respectly.

(H3) We define the operator A^α as follows

$$\begin{cases} \mathcal{D}(A^\alpha) = \left\{ u \in L^2(I) | u = \sum\limits_{i\geq 1} a_i e_i \text{ and } \sum\limits_{i\geq 1} |a_i|^2 \lambda_i^{2\alpha} < +\infty \right\}, \\ A^\alpha u = \sum\limits_{i\geq 1} a_i \lambda_i^\alpha e_i \text{ where } u = \sum\limits_{i\geq 1} a_i e_i. \end{cases}$$

(H4) $a(t) \in L^\infty_{\mathcal{F}}(0,T;\mathbb{R})$, $\xi = \|a\|^2_{L^\infty_{\mathcal{F}}(0,T;\mathbb{R})}$.

(H5) For each $r > 0$, we set $X_r = \text{span}\{e_i(x)\}_{\lambda_i \leq r}$ and denote by P_r the orthogonal projection from $L^2(I)$ to X_r.

(H6) Throughout this paper, C(and sometimes $\widetilde{C}$) denotes various positive constants. $C(\ldots)$ stands for a positive constant that depends on what are enclosed in the brackets.

Our second main result in this paper is the following theorem.

Theorem 1.2. *Let $\alpha > \frac{1}{2}$. For any $y_0 \in L^2(\Omega, \mathcal{F}_0, P; L^2(I))$, there is a control $f \in L^\infty_{\mathcal{F}}(0,T; L^2(\Omega; L^2(I)))$ such that the solution y of system*

$$\begin{cases} \mathrm{d}y + A^\alpha y = a(t)y\mathrm{d}B + \chi_\omega \chi_E f \mathrm{d}t & \text{in } I \times (0,T), \\ y(0,t) = 0 = y(1,t) & \text{in } (0,T), \\ y_x(0,t) = 0 = y_x(1,t) & \text{in } (0,T), \\ y(x,0) = y_0(x) & \text{in } I, \end{cases} \tag{1.1}$$

satisfies $y(T) = 0$ in I, P-a.s. Moreover, the control f satisfies the following estimate

$$\|f\|^2_{L^\infty_{\mathcal{F}}(0,T;L^2(\Omega;L^2(I)))} \le L\mathbb{E}\|y_0\|^2_{L^2(I)},$$

where L is a constant independent of y_0.

Remark 1.3. We refer to ([1], Chap. 6) for the well-posedness of system (1.1) in the class $L^2_{\mathcal{F}}(\Omega; C([0,T]; L^2(I)))$.

The rest of this paper is organized as follows. In Section 2, we establish a point-wise estimate. Theorem 1.1 is proved in Section 3 according to two important lemmas. Section 4 is devoted to the proof of Theorem 1.2.

2. A point-wise estimate

In this section, we give a point-wise estimate which will be used in the proof of Theorem 1.1 and the estimate itself has independent interest.

Proposition 2.1. *Set $\theta = \mathrm{e}^l$, $l = \lambda\varphi$, $\varphi = \mathrm{e}^{\mu\psi}$ and $u = \theta y$ in $Q = I \times (0,T)$, where $\psi \in C^\infty(Q)$ and $y \in C^4(Q)$. Assume that $\mu \ge 1$ and $\lambda \ge \lambda_0$, where $\lambda_0 = \lambda_0(\mu,\psi)$ such that $\lambda\mu^{-1}\varphi \ge 1$. Then we have the following point-wise estimate*

$$\begin{aligned} &\lambda^7\mu^8\varphi^7\psi_x^8 u^2 + \lambda^5\mu^6\varphi^5\psi_x^6 u_x^2 + \lambda^3\mu^4\varphi^3\psi_x^4 u_{xx}^2 + \lambda\mu^2\varphi\psi_x^2 u_{xxx}^2 \\ &\quad + \lambda^3\mu^4\varphi^3\psi_x^4 u_t^2 + \lambda\mu^2\varphi\psi_x^2 u_{xt}^2 + \{\dots\}_x + \{\dots\}_{xx} + \{\dots\}_{xxx} + \{\dots\}_{xxxx} + \{\dots\}_t + \{\dots\}_{tt} \\ &\le C(\psi)\left(\lambda^7\mu^7\varphi^7 u^2 + \lambda^5\mu^5\varphi^5 u_x^2 + \lambda^3\mu^3\varphi^3 u_{xx}^2 + \lambda\mu\varphi u_{xxx}^2 + \lambda^3\mu^3\varphi^3 u_t^2 + \lambda\mu\varphi u_{xt}^2 + \theta^2|y_{tt} - y_{xxxx}|^2\right), \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} \{\dots\}_x = \Big\{ &-\frac{3}{2}B_{2xx}u_x^2 + \frac{3}{2}B_2 u_{xx}^2 - \frac{1}{2}B_4 u_{xxx}^2 - 4a_x u_x^2 + 2a_{xxx}u^2 + b_x u_{xx}^2 - B_0 u_t u_{xxx} + \frac{1}{2}B_1 B_2 u^2 \\ &+ \frac{3}{2}(B_1B_4)_{xx}u^2 - \frac{3}{2}B_1B_4u_x^2 - (bB_1)_x u^2 + \frac{1}{2}B_2B_3u_x^2 + \frac{1}{2}B_3B_4u_{xx}^2 - (aB_3)_x u^2 + B_0B_3u_tu_x \\ &- \frac{1}{2}B_2u_t^2 - \frac{3}{2}B_{4xx}u_t^2 + \frac{3}{2}B_4u_{xt}^2 - b_tu_tu_x + b_xu_t^2 - (B_4c)_xu_x^2 + \frac{1}{2}acu^2 + \frac{1}{2}bcu_x^2 \Big\}_x, \end{aligned}$$

$$\{\dots\}_{xx} = \left\{\frac{3}{2}B_{2x}u_x^2 - 3a_{xx}u^2 + 2au_x^2 - \frac{b}{2}u_{xx}^2 - \frac{3}{2}(B_1B_4)_xu^2 + \frac{1}{2}bB_1u^2 + \frac{1}{2}aB_3u^2 + \frac{3}{2}B_{4x}u_t^2 - \frac{1}{2}bu_t^2 + \frac{1}{2}B_4cu_x^2\right\}_{xx},$$

$$\{\dots\}_{xxx} = \left\{-\frac{1}{2}B_2u_x^2 + 2a_xu^2 + \frac{1}{2}B_1B_4u^2 - \frac{1}{2}B_4u_t^2\right\}_{xxx},$$

$$\{\dots\}_{xxxx} = \left\{-\frac{1}{2}au^2\right\}_{xxxx},$$

$$\{\dots\}_t = \left\{\frac{1}{2}B_0B_1u^2 - \frac{1}{2}B_0B_3u_x^2 + B_2u_tu_x + \frac{1}{2}B_0u_t^2 - a_tu^2 + B_4u_tu_{xxx} + bu_tu_{xx} + \frac{1}{2}b_tu_x^2\right\}_t,$$

$$\{\dots\}_{tt} = \left\{\frac{1}{2}au^2\right\}_{tt},$$

$$a = 4l_x^2l_{xx}, \quad b = 8l_{xx}, \quad c = -10l_xl_{xx}, \quad B_0 = -2l_t, \quad B_1 = -l_x^4, \quad B_2 = 4l_x^3, \quad B_3 = -6l_x^2, \quad B_4 = 4l_x.$$

Remark 2.2. The key points in the proof of Proposition 2.1 are suitable grouping and the choice of a, b and c.

Proof. From the assumptions of λ, μ, φ and l, we can obtain that

$$\begin{cases} \lambda^n \mu^m \varphi^n \le \lambda^{n+i}\mu^{m-i}\varphi^{n+i} \le \lambda^{n+i}\mu^{m+i}\varphi^{n+i}, \ \forall\ m,n \in \mathbb{Z} \ \text{ and } \ i \in \mathbb{N}, \\ |l_{\underbrace{x\dots x}_{k}\underbrace{t\dots t}_{j}}| \le C(\psi)\lambda\mu^{k+j}\varphi, \qquad \forall\ k,j \in \mathbb{N}, \end{cases} \tag{2.2}$$

where $l_{\underbrace{x\dots x}_{k}\underbrace{t\dots t}_{j}} = \frac{\partial^{k+j} l}{\partial^k x \partial^j t}$.

Direct computation shows that

$$\theta(y_{tt} - y_{xxxx}) = u_{tt} + A_0 u_t + A_1 u + A_2 u_x + A_3 u_{xx} + A_4 u_{xxx} - u_{xxxx}$$

where

$$\begin{aligned} &A_0 = -2l_t, \\ &A_1 = l_t^2 - l_{tt} - l_x^4 - 4l_x l_{xxx} + l_{xxxx} + 6l_x^2 l_{xx} - 3l_{xx}^2, \\ &A_2 = -12 l_x l_{xx} + 4l_x^3 + 4l_{xxx}, \\ &A_3 = -6l_x^2 + 6l_{xx}, \\ &A_4 = 4l_x. \end{aligned}$$

Set

$$\begin{aligned} &I_1 = -u_{xxxx} + B_1 u + B_3 u_{xx} + u_{tt} + c u_x, \\ &I_2 = B_2 u_x + B_4 u_{xxx} + B_0 u_t + a u + b u_{xx}, \\ &R = \theta(y_{tt} - y_{xxxx}) - I_1 - I_2 = S_0 u + S_1 u_x + S_2 u_{xx}, \end{aligned}$$

where

$$\begin{aligned} &S_0 = l_t^2 - l_{tt} - 4l_x l_{xxx} + l_{xxxx} - 3l_{xx}^2 + 2l_x^2 l_{xx}, \\ &S_1 = 4l_{xxx} - 2l_x l_{xx}, S_2 = -2l_{xx}. \end{aligned}$$

Step 1. We shall prove the following equality

$$\begin{aligned} I_1 \cdot I_2 = {}& u^2\{\dots\} + u_x^2\{\dots\} + u_{xx}^2\{\dots\} + u_{xxx}^2\{\dots\} + u_t^2\{\dots\} + u_{xt}^2\{\dots\} + u_{xt}u_{xxx}\{\dots\} \\ &+ u_t u_x\{\dots\} + u_t u_{xxx}\{\dots\} + \{\dots\}_x + \{\dots\}_{xx} + \{\dots\}_{xxx} + \{\dots\}_{xxxx} + \{\dots\}_t + \{\dots\}_{tt}, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} u^2\{\dots\} &= u^2\Big\{-\frac{1}{2}a_{xxxx} - \frac{1}{2}(B_1B_2)_x - \frac{1}{2}(B_1B_4)_{xxx} + \frac{1}{2}(bB_1)_{xx} - \frac{1}{2}(B_0B_1)_t + \frac{1}{2}(aB_3)_{xx} + \frac{1}{2}a_{tt} + aB_1 - \frac{1}{2}(ac)_x\Big\}, \\ u_x^2\{\dots\} &= u_x^2\Big\{\frac{1}{2}B_{2xxx} + 2a_{xx} + \frac{3}{2}(B_1B_4)_x - bB_1 - \frac{1}{2}(B_2B_3)_x - aB_3 + \frac{1}{2}(B_0B_3)_t - \frac{1}{2}b_{tt} + B_2c + \frac{1}{2}(B_4c)_{xx} - \frac{1}{2}(bc)_x\Big\}, \\ u_{xx}^2\{\dots\} &= u_{xx}^2\left\{-\frac{3}{2}B_{2x} - a - \frac{1}{2}b_{xx} - \frac{1}{2}(B_3B_4)_x + bB_3 - B_4c\right\}, \\ u_{xxx}^2\{\dots\} &= u_{xxx}^2\left\{\frac{1}{2}B_{4x} + b\right\}, \\ u_t^2\{\dots\} &= u_t^2\left\{\frac{1}{2}B_{2x} - \frac{1}{2}B_{0t} - a + \frac{1}{2}B_{4xxx} - \frac{1}{2}b_{xx}\right\}, \\ u_{xt}^2\{\dots\} &= u_{xt}^2\left\{-\frac{3}{2}B_{4x} + b\right\}, \\ u_{xt}u_{xxx}\{\dots\} &= u_{xt}u_{xxx}\{B_0\}, \\ u_t u_x\{\dots\} &= u_t u_x\{-(B_0B_3)_x - B_{2t} + b_{xt} + B_0 c\}, \\ u_t u_{xxx}\{\dots\} &= u_t u_{xxx}\{B_{0x} - B_{4t}\}, \end{aligned}$$

and $\{\dots\}_x, \{\dots\}_{xx}, \{\dots\}_{xxx}, \{\dots\}_{xxxx}, \{\dots\}_t, \{\dots\}_{tt}$ are the same as in Proposition 2.1.

Indeed, (2.3) can be obtained from the following equations

$$\begin{aligned}
u_{xxxx}B_2u_x &= \frac{1}{2}\left(B_2u_x^2\right)_{xxx} - \frac{3}{2}\left(B_{2x}u_x^2\right)_{xx} + \frac{3}{2}\left(B_{2xx}u_x^2 - B_2u_{xx}^2\right)_x + \frac{3}{2}B_{2x}u_{xx}^2 - \frac{1}{2}B_{2xxx}u_x^2,\\
u_{xxxx}B_4u_{xxx} &= \frac{1}{2}\left(\left(B_4u_{xxx}^2\right)_x - B_{4x}u_{xxx}^2\right),\\
u_{xxxx}au &= \frac{1}{2}\left(au^2\right)_{xxxx} - 2\left(a_xu^2\right)_{xxx} + \left(3a_{xx}u^2 - 2au_x^2\right)_{xx}\\
&\quad + \left(4a_xu_x^2 - 2a_{xxx}u^2\right)_x + au_{xx}^2 - 2a_{xx}u_x^2 + \frac{1}{2}a_{xxxx}u^2,\\
u_{xxxx}bu_{xx} &= \frac{1}{2}\left(bu_{xx}^2\right)_{xx} - \left(b_xu_{xx}^2\right)_x - bu_{xxx}^2 + \frac{1}{2}b_{xx}u_{xx}^2,\\
u_{xxxx}B_0u_t &= (B_0u_tu_{xxx})_x - B_{0x}u_tu_{xxx} - B_0u_{xt}u_{xxx},\\
B_1uB_2u_x &= \frac{1}{2}\left(\left(B_1B_2u^2\right)_x - (B_1B_2)_xu^2\right),\\
B_1uB_4u_{xxx} &= \frac{1}{2}\left(B_1B_4u^2\right)_{xxx} - \frac{3}{2}\left((B_1B_4)_xu^2\right)_{xx} + \frac{3}{2}(B_1B_4)_xu_x^2\\
&\quad + \frac{3}{2}\left((B_1B_4)_{xx}u^2 - B_1B_4u_x^2\right)_x - \frac{1}{2}(B_1B_4)_{xxx}u^2,\\
B_1ubu_{xx} &= \frac{1}{2}\left(bB_1u^2\right)_{xx} - \left[(bB_1)_xu^2\right]_x - bB_1u_x^2 + \frac{1}{2}(bB_1)_{xx}u^2,\\
B_1uB_0u_t &= \frac{1}{2}\left(\left(B_0B_1u^2\right)_t - (B_0B_1)_tu^2\right),\\
B_3u_{xx}B_2u_x &= \frac{1}{2}\left(\left(B_2B_3u_x^2\right)_x - (B_2B_3)_xu_x^2\right),\\
B_3u_{xx}B_4u_{xxx} &= \frac{1}{2}\left(\left(B_3B_4u_{xx}^2\right)_x - (B_3B_4)_xu_{xx}^2\right),\\
B_3u_{xx}au &= \frac{1}{2}\left(aB_3u^2\right)_{xx} - \left((aB_3)_xu^2\right)_x - aB_3u_x^2 + \frac{1}{2}(aB_3)_{xx}u^2,\\
B_3u_{xx}B_0u_t &= \frac{1}{2}(B_0B_3)_tu_x^2 - (B_0B_3)_xu_tu_x - \frac{1}{2}\left(B_0B_3u_x^2\right)_t + (B_0B_3u_xu_t)_x,\\
u_{tt}B_2u_x &= (B_2u_tu_x)_t - B_{2t}u_tu_x - \frac{1}{2}\left(\left(B_2u_t^2\right)_x - B_{2x}u_t^2\right),\\
u_{tt}B_0u_t &= \frac{1}{2}\left(\left(B_0u_t^2\right)_t - B_{0t}u_t^2\right),\\
u_{tt}au &= \frac{1}{2}\left(au^2\right)_{tt} - \left(a_tu^2\right)_t - au_t^2 + \frac{1}{2}a_{tt}u^2,\\
u_{tt}B_4u_{xxx} &= (B_4u_tu_{xxx})_t - B_{4t}u_tu_{xxx} - \frac{1}{2}\left(B_4u_t^2\right)_{xxx} + \frac{3}{2}\left(B_{4x}u_t^2\right)_{xx}\\
&\quad - \frac{3}{2}\left(B_{4xx}u_t^2 - B_4u_{xt}^2\right)_x - \frac{3}{2}B_{4x}u_{xt}^2 + \frac{1}{2}B_{4xxx}u_t^2,\\
u_{tt}bu_{xx} &= (bu_tu_{xx})_t + b_{xt}u_tu_x - \frac{1}{2}b_{tt}u_x^2 + \frac{1}{2}\left(b_tu_x^2\right)_t - (b_tu_tu_x)_x - \frac{1}{2}\left(bu_t^2\right)_{xx} + \left(b_xu_t^2\right)_x + bu_{xt}^2 - \frac{1}{2}b_{xx}u_t^2,\\
B_4u_{xxx}cu_x &= \frac{1}{2}\left(B_4cu_x^2\right)_{xx} - \left((B_4c)_xu_x^2\right)_x - B_4cu_{xx}^2 + \frac{1}{2}(B_4c)_{xx}u_x^2,\\
aucu_x &= \frac{1}{2}\left(\left(acu^2\right)_x - (ac)_xu^2\right),\\
bu_{xx}cu_x &= \frac{1}{2}\left(\left(bcu_x^2\right)_x - (bc)_xu_x^2\right).
\end{aligned}$$

Step 2. We shall prove estimate (2.1).

Indeed,

$$
\begin{aligned}
u^2\{\dots\} &= u^2\{10\lambda^7\mu^8\varphi^7\psi_x^8 + R_0\},\\
u_x^2\{\dots\} &= u_x^2\{22\lambda^5\mu^6\varphi^5\psi_x^6 + R_1\},\\
u_{xx}^2\{\dots\} &= u_{xx}^2\{6\lambda^3\mu^4\varphi^3\psi_x^4 + R_2\},\\
u_{xxx}^2\{\dots\} &= u_{xxx}^2\{10\lambda\mu^2\varphi\psi_x^2 + R_3\},\\
u_t^2\{\dots\} &= u_t^2\{2\lambda^3\mu^4\varphi^3\psi_x^4 + R_4\},\\
u_{xt}^2\{\dots\} &= u_{xt}^2\{2l_{xx}\},\\
u_{xt}u_{xxx}\{\dots\} &= u_{xt}u_{xxx}\{2l_t\},\\
u_tu_x\{\dots\} &= u_tu_x\{-24l_{xt}l_x^2 - 4l_tl_xl_{xx} + 8l_{xxxt}\},\\
u_tu_{xxx}\{\dots\} &= u_tu_{xxx}\{-6l_{xt}\},
\end{aligned}
\tag{2.4}
$$

where

$$
\begin{aligned}
&|R_0| \le C\lambda^7\mu^7\varphi^7, \quad |R_1| \le C\lambda^5\mu^5\varphi^5, \quad |R_2| \le C\lambda^3\mu^3\varphi^3,\\
&|R_3| \le C\lambda\mu\varphi, \quad |R_4| \le C\lambda^3\mu^3\varphi^3.
\end{aligned}
\tag{2.5}
$$

Here we have used (2.2). Similarly, we have

$$
\begin{aligned}
|u_{xt}u_{xxx}\{\dots\}| &\le C\left(\lambda\mu\varphi u_{xxx}^2 + \lambda\mu\varphi u_{xt}^2\right),\\
|u_tu_x\{\dots\}| &\le C\left(\lambda^5\mu^5\varphi^5u_x^2 + \lambda^3\mu^3\varphi^3u_t^2\right),\\
|u_tu_{xxx}\{\dots\}| &\le C\left(\lambda\mu\varphi u_{xxx}^2 + \lambda^3\mu^3\varphi^3u_t^2\right),\\
S_0^2u^2 &\le C\lambda^7\mu^7\varphi^7u^2,\\
S_1^2u_x^2 &\le C\lambda^5\mu^5\varphi^5u_x^2,\\
S_2^2u_{xx}^2 &\le C\lambda^3\mu^3\varphi^3u_{xx}^2,
\end{aligned}
\tag{2.6}
$$

this implies

$$
\begin{aligned}
I_1I_2 &\le \frac{1}{2}(I_1+I_2)^2\\
&= \frac{1}{2}\left[\theta(y_{tt} - y_{xxxx}) - R\right]^2\\
&\le \theta^2|y_{tt} - y_{xxxx}|^2 + |R|^2\\
&\le C\left(\theta^2|y_{tt} - y_{xxxx}|^2 + S_0^2u^2 + S_1^2u_x^2 + S_2^2u_{xx}^2\right)\\
&\le C\left(\theta^2|y_{tt} - y_{xxxx}|^2 + \lambda^7\mu^7\varphi^7u^2 + \lambda^5\mu^5\varphi^5u_x^2 + \lambda^3\mu^3\varphi^3u_{xx}^2\right).
\end{aligned}
\tag{2.7}
$$

Thanks to (2.4)–(2.7) all together with (2.3), we can obtain (2.1). □

3. Proof of Theorem 1.1

Through this section, we make the following assumptions:

(A1) Let $T_0, T', T'', \gamma, b_0$ satisfy

$$
\begin{cases}
0 < \gamma < T' < \frac{T_0}{2} < T'' < T_0 - \gamma,\\
\left|\frac{T_0}{2} - T'\right| = \left|T'' - \frac{T_0}{2}\right|,\\
\max\left\{T'' - \frac{T_0}{2}, \left(1 + \left(\frac{T_0}{2} - T'\right)^2\right)^{\frac{1}{2}}\right\} < b_0 < \frac{T_0}{2} - \gamma.
\end{cases}
$$

(A2) For a measurable $F \subset Q_0 \triangleq I \times (0, T_0)$ and a function g satisfies $g, g_x, g_{xx}, g_{xxx}, g_t, g_{xt} \in L^2(F)$, we set

$$\|g\|_{\star,F} = \Big(\|g\|^2_{L^2(F)} + \|g_x\|^2_{L^2(F)} + \|g_{xx}\|^2_{L^2(F)} + \|g_{xxx}\|^2_{L^2(F)} + \|g_t\|^2_{L^2(F)} + \|g_{xt}\|^2_{L^2(F)}\Big)^{\frac{1}{2}}.$$

(A3) Define $\mathcal{G} \triangleq \{g \in L^2(Q_0) \mid g_x, g_{xx}, g_{xxx}, g_{xxxx}, g_t, g_{xt} \in L^2(Q_0)\}$.

Remark 3.1. The existence of T_0, T', T'', γ and b_0 is easy to obtain, for example, $T_0 = 4, T' = 1, T'' = 3, \gamma = 0.5, b_0 = 1.42$ satisfies (A1).

We borrow some idea from [11].

In order to prove Theorem 1.1, we investigate the following system

$$\begin{cases} y_{tt} - y_{xxxx} = 0 & \text{in } Q_0, \\ y(0,t) = 0 = y(1,t) & \text{in } (0,T_0), \\ y_x(0,t) = 0 = y_x(1,t) & \text{in } (0,T_0). \end{cases} \tag{3.1}$$

First, we establish two important lemmas.

Lemma 3.2. *There exists a constant $\alpha_1 \in (0,1)$ such that we have the following estimate*

$$\|y\|_{L^2(I\times(T',T''))} \le C\|y\|^{\alpha_1}_{\star,\omega\times(\gamma,T_0-\gamma)}\|y\|^{1-\alpha_1}_{\star,Q_0}, \tag{3.2}$$

where $y \in \mathcal{G}$ solves (3.1).

Proof. Introduce $\phi \in C_0^\infty(\gamma, T_0-\gamma)$ and $\psi_0 \in C^\infty(\overline{I})$ which enjoy the following properties

$$\begin{cases} 0 \le \phi(t) \le 1 & t \in (\gamma, T_0-\gamma), \\ \phi(t) = 1 & t \in (\frac{T_0}{2} - b_0, \frac{T_0}{2} + b_0), \\ \psi_0(x) > 0 & x \in I, \\ |\psi_{0x}| > 0 & x \in \overline{I}\backslash\omega, \\ \psi_0(0) = \psi_0(1) = 0, \quad \psi_{0x}(0) > 0, \quad \psi_{0x}(1) < 0, \quad \|\psi_0\|_{C(\overline{I})} = 1. \end{cases} \tag{3.3}$$

The existence of ψ_0 can be found in [15].

Set $\psi_1(x,t) = \psi_0(x) - (t - \frac{T_0}{2})^2 + \widetilde{C}_1$, where $\widetilde{C}_1$ is chosen to be so large to make $\psi_1 > 0$. According to (A1), it is obvious that $(T', T'') \subset (\frac{T_0}{2} - b_0, \frac{T_0}{2} + b_0)$.

We here let $\psi(x,t) = \psi_1(x,t)$ in Proposition 2.1 and apply (2.1) to $\overline{y} = \phi y$ and $w = \theta\overline{y}$, some straightforward calculation gives that

$$\begin{aligned} \int_{Q_0} \Big(&\lambda^7\mu^8\varphi^7\psi_{1x}^8 w^2 + \lambda^5\mu^6\varphi^5\psi_{1x}^6 w_x^2 + \lambda^3\mu^4\varphi^3\psi_{1x}^4 w_{xx}^2 + \lambda\mu^2\varphi\psi_{1x}^2 w_{xxx}^2 \\ &+ \lambda^3\mu^4\varphi^3\psi_{1x}^4 w_t^2 + \lambda\mu^2\varphi\psi_{1x}^2 w_{xt}^2 + \{\ldots\}_x + \{\ldots\}_{xx} + \{\ldots\}_{xxx} + \{\ldots\}_{xxxx} + \{\ldots\}_t + \{\ldots\}_{tt}\Big)\mathrm{d}x\mathrm{d}t \\ &\le C(\psi)\int_{Q_0} (\lambda^7\mu^7\varphi^7 w^2 + \lambda^5\mu^5\varphi^5 w_x^2 + \lambda^3\mu^3\varphi^3 w_{xx}^2 + \lambda\mu\varphi w_{xxx}^2 \\ &\qquad + \lambda^3\mu^3\varphi^3 w_t^2 + \lambda\mu\varphi w_{xt}^2 + \theta^2|\overline{y}_{tt} - \overline{y}_{xxxx}|^2)\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.4}$$

It follows from (3.3) that

$$w(0,t) = w(1,t) = w_x(0,t) = w_x(1,t) = 0 \quad \forall\ t \in (0,T_0)$$

and

$$w(x,0)=w_t(x,0)=w(x,T_0)=w_t(x,T_0)=0 \quad \forall\, x\in I.$$

Thus

$$\int_{Q_0}(\{\dots\}_t+\{\dots\}_{tt})\mathrm{d}x\mathrm{d}t=0,$$

$$\begin{aligned}\int_{Q_0}(\{\dots\}_x+\{\dots\}_{xx}+\{\dots\}_{xxx}+\{\dots\}_{xxxx})\mathrm{d}x\mathrm{d}t &= \int_0^{T_0}\big(w_{xx}^2(-10l_x^3+4l_{xxx})-8l_{xx}w_{xx}w_{xxx}\\ &\quad+w_{xxx}^2(-2l_x)\big)(\cdot,t)\big|_0^1\mathrm{d}t\\ &\triangleq V(1)-V(0).\end{aligned}\tag{3.5}$$

If we choose $\lambda\ge\lambda_0$ with λ_0 large enough and note that $\psi_{0x}(1)<0,\ \psi_{0x}(0)>0$, then it holds that

$$-V(0)\ge 0, V(1)\ge 0,\tag{3.6}$$

In view of (3.4)–(3.6), there holds

$$\begin{aligned}&\int_{Q_0}\big(\lambda^7\mu^8\varphi^7\psi_{1x}^8w^2+\lambda^5\mu^6\varphi^5\psi_{1x}^6w_x^2+\lambda^3\mu^4\varphi^3\psi_{1x}^4w_{xx}^2+\lambda\mu^2\varphi\psi_{1x}^2w_{xxx}^2+\lambda^3\mu^4\varphi^3\psi_{1x}^4w_t^2+\lambda\mu^2\varphi\psi_{1x}^2w_{xt}^2\big)\,\mathrm{d}x\mathrm{d}t\\ &\quad\le C(\psi)\int_{Q_0}\big(\lambda^7\mu^7\varphi^7w^2+\lambda^5\mu^5\varphi^5w_x^2+\lambda^3\mu^3\varphi^3w_{xx}^2+\lambda\mu\varphi w_{xxx}^2\\ &\qquad+\lambda^3\mu^3\varphi^3w_t^2+\lambda\mu\varphi w_{xt}^2+\theta^2|\overline{y}_{tt}-\overline{y}_{xxxx}|^2\big)\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{3.7}$$

Set $Q_0^\omega=\omega\times(0,T_0)$. Recall that $|\psi_{1x}|>0$ in $\overline{I}\setminus\omega$, it follows that

$$\begin{aligned}&\int_{Q_0\setminus Q_0^\omega}\big(\lambda^7\mu^8\varphi^7w^2+\lambda^5\mu^6\varphi^5w_x^2+\lambda^3\mu^4\varphi^3w_{xx}^2+\lambda\mu^2\varphi w_{xxx}^2+\lambda^3\mu^4\varphi^3w_t^2+\lambda\mu^2\varphi w_{xt}^2\big)\,\mathrm{d}x\mathrm{d}t\\ &\quad\le C_1(\psi)\int_{Q_0}\big(\lambda^7\mu^7\varphi^7w^2+\lambda^5\mu^5\varphi^5w_x^2+\lambda^3\mu^3\varphi^3w_{xx}^2+\lambda\mu\varphi w_{xxx}^2\\ &\qquad+\lambda^3\mu^3\varphi^3w_t^2+\lambda\mu\varphi w_{xt}^2+\theta^2|\overline{y}_{tt}-\overline{y}_{xxxx}|^2\big)\mathrm{d}x\mathrm{d}t,\end{aligned}$$

from which if we choose $\mu=\mu_0=C_1(\psi)+1$, then it holds that

$$\begin{aligned}&\int_{Q_0\setminus Q_0^\omega}\big(\lambda^7\mu^7\varphi^7w^2+\lambda^5\mu^5\varphi^5w_x^2+\lambda^3\mu^3\varphi^3w_{xx}^2+\lambda\mu\varphi w_{xxx}^2+\lambda^3\mu^3\varphi^3w_t^2+\lambda\mu\varphi w_{xt}^2\big)\mathrm{d}x\mathrm{d}t\\ &\quad\le C_2(\psi)\bigg(\int_{Q_0}\theta^2|\overline{y}_{tt}-\overline{y}_{xxxx}|^2\mathrm{d}x\mathrm{d}t+\int_{Q_0^\omega}\big(\lambda^7\mu^7\varphi^7w^2+\lambda^5\mu^5\varphi^5w_x^2\\ &\qquad+\lambda^3\mu^3\varphi^3w_{xx}^2+\lambda\mu\varphi w_{xxx}^2+\lambda^3\mu^3\varphi^3w_t^2+\lambda\mu\varphi w_{xt}^2\big)\mathrm{d}x\mathrm{d}t\bigg).\end{aligned}$$

Then

$$\begin{aligned}&\int_{Q_0\setminus Q_0^\omega}\big(\lambda^7\mu^7\varphi^7w^2+\lambda^5\mu^5\varphi^5w_x^2+\lambda^3\mu^3\varphi^3w_{xx}^2+\lambda\mu\varphi w_{xxx}^2+\lambda^3\mu^3\varphi^3w_t^2+\lambda\mu\varphi w_{xt}^2\big)\,\mathrm{d}x\mathrm{d}t\\ &\quad+\int_{Q_0^\omega}\big(\lambda^7\mu^7\varphi^7w^2+\lambda^5\mu^5\varphi^5w_x^2+\lambda^3\mu^3\varphi^3w_{xx}^2+\lambda\mu\varphi w_{xxx}^2+\lambda^3\mu^3\varphi^3w_t^2+\lambda\mu\varphi w_{xt}^2\big)\,\mathrm{d}x\mathrm{d}t\\ &\quad\le C\bigg(\int_{Q_0}\theta^2|\overline{y}_{tt}-\overline{y}_{xxxx}|^2\mathrm{d}x\mathrm{d}t+\int_{Q_0^\omega}\big(\lambda^7\mu^7\varphi^7w^2+\lambda^5\mu^5\varphi^5w_x^2\\ &\qquad+\lambda^3\mu^3\varphi^3w_{xx}^2+\lambda\mu\varphi w_{xxx}^2+\lambda^3\mu^3\varphi^3w_t^2+\lambda\mu\varphi w_{xt}^2\big)\mathrm{d}x\mathrm{d}t\bigg),\end{aligned}$$

and thus

$$\begin{aligned}\int_{Q_0}&\left(\lambda^7\mu^7\varphi^7 w^2+\lambda^5\mu^5\varphi^5 w_x^2+\lambda^3\mu^3\varphi^3 w_{xx}^2+\lambda\mu\varphi w_{xxx}^2+\lambda^3\mu^3\varphi^3 w_t^2+\lambda\mu\varphi w_{xt}^2\right)\mathrm{d}x\mathrm{d}t\\ &\le C\Bigg(\int_{Q_0}\theta^2|\overline{y}_{tt}-\overline{y}_{xxxx}|^2\mathrm{d}x\mathrm{d}t+\int_{Q_0^\omega}\big(\lambda^7\mu^7\varphi^7 w^2+\lambda^5\mu^5\varphi^5 w_x^2\\ &\qquad+\lambda^3\mu^3\varphi^3 w_{xx}^2+\lambda\mu\varphi w_{xxx}^2+\lambda^3\mu^3\varphi^3 w_t^2+\lambda\mu\varphi w_{xt}^2\big)\mathrm{d}x\mathrm{d}t\Bigg),\end{aligned}$$

from which it holds that

$$\begin{aligned}\int_{Q_0}&\left(\lambda^7\varphi^7 w^2+\lambda^5\varphi^5 w_x^2+\lambda^3\varphi^3 w_{xx}^2+\lambda\varphi w_{xxx}^2+\lambda^3\varphi^3 w_t^2+\lambda\varphi w_{xt}^2\right)\mathrm{d}x\mathrm{d}t\\ &\le C(\mu)\Bigg(\int_{Q_0}\theta^2|\overline{y}_{tt}-\overline{y}_{xxxx}|^2\mathrm{d}x\mathrm{d}t+\int_{Q_0^\omega}(\lambda^7\varphi^7 w^2+\lambda^5\varphi^5 w_x^2\\ &\qquad+\lambda^3\varphi^3 w_{xx}^2+\lambda\varphi w_{xxx}^2+\lambda^3\varphi^3 w_t^2+\lambda\varphi w_{xt}^2)\mathrm{d}x\mathrm{d}t\Bigg).\end{aligned}$$

Returning w to $\theta\overline{y}$ ($\mu=\mu_0$ is now fixed), we can obtain that

$$\begin{aligned}\int_{Q_0}&\left(\lambda^7\varphi^7\theta^2\overline{y}^2+\lambda^5\varphi^5\theta^2\overline{y}_x^2+\lambda^3\varphi^3\theta^2\overline{y}_{xx}^2+\lambda\varphi\theta^2\overline{y}_{xxx}^2+\lambda^3\varphi^3\theta^2\overline{y}_t^2+\lambda\varphi\theta^2\overline{y}_{xt}^2\right)\mathrm{d}x\mathrm{d}t\\ &\le C\Bigg(\int_{Q_0}\theta^2|\overline{y}_{tt}-\overline{y}_{xxxx}|^2\mathrm{d}x\mathrm{d}t+\int_{Q_0^\omega}(\lambda^7\varphi^7\theta^2\overline{y}^2+\lambda^5\varphi^5\theta^2\overline{y}_x^2\\ &\qquad+\lambda^3\varphi^3\theta^2\overline{y}_{xx}^2+\lambda\varphi\theta^2\overline{y}_{xxx}^2+\lambda^3\varphi^3\theta^2\overline{y}_t^2+\lambda\varphi\theta^2\overline{y}_{xt}^2)\mathrm{d}x\mathrm{d}t\Bigg).\end{aligned}\tag{3.8}$$

Taking into account the construction of ϕ and ψ_0, we have

$$\begin{aligned}\int_{Q_0}&\left(\lambda^7\varphi^7\theta^2\overline{y}^2+\lambda^5\varphi^5\theta^2\overline{y}_x^2+\lambda^3\varphi^3\theta^2\overline{y}_{xx}^2+\lambda\varphi\theta^2\overline{y}_{xxx}^2+\lambda^3\varphi^3\theta^2\overline{y}_t^2+\lambda\varphi\theta^2\overline{y}_{xt}^2\right)\mathrm{d}x\mathrm{d}t\\ &\ge\int_{Q_0}\lambda^7\varphi^7\theta^2\overline{y}^2\\ &\ge\int_{I\times(T',T'')}\lambda^7\varphi^7\theta^2\overline{y}^2\\ &\ge\lambda^7\mathrm{e}^{7\mu\left(\widetilde{C}_1-\left(\frac{T_0}{2}-T'\right)^2\right)}\mathrm{e}^{2\lambda\mathrm{e}^{\mu\left(\widetilde{C}_1-\left(\frac{T_0}{2}-T'\right)^2\right)}}\int_{I\times(T',T'')}y^2\mathrm{d}x\mathrm{d}t,\end{aligned}$$

$$\begin{aligned}\int_{Q_0^\omega}&\left(\lambda^7\varphi^7\theta^2\overline{y}^2+\lambda^5\varphi^5\theta^2\overline{y}_x^2+\lambda^3\varphi^3\theta^2\overline{y}_{xx}^2+\lambda\varphi\theta^2\overline{y}_{xxx}^2+\lambda^3\varphi^3\theta^2\overline{y}_t^2+\lambda\varphi\theta^2\overline{y}_{xt}^2\right)\mathrm{d}x\mathrm{d}t\\ &\le C\lambda^7\mathrm{e}^{7\mu\left(\widetilde{C}_1+1\right)}\mathrm{e}^{2\lambda\mathrm{e}^{\mu(\widetilde{C}_1+1)}}\int_\gamma^{T_0-\gamma}\int_\omega\left(y^2+y_x^2+y_{xx}^2+y_{xxx}^2+y_t^2+y_{xt}^2\right)\mathrm{d}x\mathrm{d}t\\ &=C\lambda^7\mathrm{e}^{7\mu\left(\widetilde{C}_1+1\right)}\mathrm{e}^{2\lambda\mathrm{e}^{\mu(\widetilde{C}_1+1)}}\|y\|^2_{\star,\omega\times(\gamma,T_0-\gamma)}\end{aligned}$$

and

$$\int_{Q_0}\theta^2|\overline{y}_{tt}-\overline{y}_{xxxx}|^2\mathrm{d}x\mathrm{d}t=\int_{Q_0}\theta^2|\phi_{tt}y+2\phi_t y_t|^2\mathrm{d}x\mathrm{d}t=\int_{(0,T_0)\setminus(\frac{T_0}{2}-b_0,\frac{T_0}{2}+b_0)}\int_I\theta^2|\phi_{tt}y+2\phi_t y_t|^2\mathrm{d}x\mathrm{d}t$$
$$\leq C\mathrm{e}^{2\lambda\mathrm{e}^{\mu(1-b_0^2+\tilde{C}_1)}}\|y\|^2_{\star,Q_0}.$$

From the above estimates and (3.8), we know that

$$\lambda^7\mathrm{e}^{7\mu\left(\tilde{C}_1-\left(\frac{T_0}{2}-T'\right)^2\right)}\mathrm{e}^{2\lambda\mathrm{e}^{\mu\left(\tilde{C}_1-\left(\frac{T_0}{2}-T'\right)^2\right)}}\|y\|^2_{L^2(I\times(T',T''))}$$
$$\leq C\left(\mathrm{e}^{2\lambda\mathrm{e}^{\mu\left(1-b_0^2+\tilde{C}_1\right)}}\|y\|^2_{\star,Q_0}+\lambda^7\mathrm{e}^{7\mu\left(\tilde{C}_1+1\right)}\mathrm{e}^{2\lambda\mathrm{e}^{\mu\left(\tilde{C}_1+1\right)}}\|y\|^2_{\star,\omega\times(\gamma,T_0-\gamma)}\right),$$

namely,

$$\|y\|^2_{L^2(I\times(T',T''))}\leq C\left(\lambda^{-7}\mathrm{e}^{2\lambda\left(\mathrm{e}^{\mu\left(1-b_0^2+\tilde{C}_1\right)}-\mathrm{e}^{\mu\left(\tilde{C}_1-\left(\frac{T_0}{2}-T'\right)^2\right)}\right)-7\mu\left(\tilde{C}_1-\left(\frac{T_0}{2}-T'\right)^2\right)}\|y\|^2_{\star,Q_0}\right.$$
$$\left.+\,\mathrm{e}^{7\mu\left(1+\left(\frac{T_0}{2}-T'\right)^2\right)}\mathrm{e}^{2\lambda\left(\mathrm{e}^{\mu(\tilde{C}_1+1)}-\mathrm{e}^{\mu\left(\tilde{C}_1-\left(\frac{T_0}{2}-T'\right)^2\right)}\right)}\|y\|^2_{\star,\omega\times(\gamma,T_0-\gamma)}\right). \tag{3.9}$$

Set

$$\begin{cases}\varepsilon=\lambda^{-7}\mathrm{e}^{2\lambda\left(\mathrm{e}^{\mu\left(1-b_0^2+\tilde{C}_1\right)}-\mathrm{e}^{\mu\left(\tilde{C}_1-\left(\frac{T_0}{2}-T'\right)^2\right)}\right)-7\mu\left(\tilde{C}_1-\left(\frac{T_0}{2}-T'\right)^2\right)},\\ \varepsilon_0=\lambda_0^{-7}\mathrm{e}^{2\lambda_0\left(\mathrm{e}^{\mu\left(1-b_0^2+\tilde{C}_1\right)}-\mathrm{e}^{\mu\left(\tilde{C}_1-\left(\frac{T_0}{2}-T'\right)^2\right)}\right)-7\mu\left(\tilde{C}_1-\left(\frac{T_0}{2}-T'\right)^2\right)},\\ k=\frac{\mathrm{e}^{\mu(\tilde{C}_1+1)}-\mathrm{e}^{\mu\left(\tilde{C}_1-\left(\frac{T_0}{2}-T'\right)^2\right)}}{\mathrm{e}^{\mu\left(\tilde{C}_1-\left(\frac{T_0}{2}-T'\right)^2\right)}-\mathrm{e}^{\mu(1-b_0^2+\tilde{C}_1)}},\end{cases}$$

where $\mu=\mu_0$.

(3.9) implies that for any $\varepsilon\in(0,\varepsilon_0]$ the following inequality holds:

$$\|y\|_{L^2(I\times(T',T''))}\leq C\left(\varepsilon^{-k}\|y\|_{\star,\omega\times(\gamma,T_0-\gamma)}+\varepsilon\|y\|_{\star,Q_0}\right). \tag{3.10}$$

Therefore, (3.10) holds for all $\varepsilon>0$. Further, if we let

$$\alpha_1=\frac{1}{1+k},\quad \varepsilon=\left(\frac{\|y\|_{\star,\omega\times\left(\frac{T_0}{2}-b,\frac{T_0}{2}+b\right)}}{\|y\|_{\star,Q_0}}\right)^{\alpha_1},$$

(3.10) implies (3.2). □

Lemma 3.3. *There exists a constant $\alpha_2\in(0,1)$ such that we have the following estimate*

$$\|y\|_{\star,\omega\times(\gamma,T_0-\gamma)}\leq C\left(\|y_t(0)\|_{L^2(\omega)}+\|y(0)\|_{H^3(\omega)}\right)^{\alpha_2}\|y\|^{1-\alpha_2}_{\star,Q_0}, \tag{3.11}$$

where $y\in\mathcal{G}$ solves (3.1).

Proof. Since ω is a nonempty open set, there exists an interval $\omega_0 \subset \omega$. Set $a = \inf \omega_0, b = \sup \omega_0$. We introduce the set

$$N(\tau) = \{(x,t) \mid x \in (a+\tau, b-\tau), t \in (0, T_0 - \tau)\}.$$

Let τ_i $(i = 1,2,3)$ be such that $0 < \tau_3 < \tau_2 < \tau_1 < \min\{\frac{b-a}{2}, T_0\}$, thus $N(\tau_1) \subset N(\tau_2) \subset N(\tau_3) \subset \omega_0 \times (0, T_0)$. We take a function $h = h(x)$ such that

$$h(x) = \begin{cases} \frac{2}{b-a}\left(x - \frac{a+b}{2}\right) + 1 & x \in \left(a, \frac{a+b}{2}\right), \\ \frac{2}{a-b}\left(x - \frac{a+b}{2}\right) + 1 & x \in \left[\frac{a+b}{2}, b\right). \end{cases} \tag{3.12}$$

Let $\chi \in C^\infty(N(\tau_3))$ with the properties

$$\begin{cases} 0 \le \chi \le 1 & (x,t) \in N(\tau_3) \\ \chi \equiv 1 & (x,t) \in N(\tau_2) \\ \chi \equiv 0 & (x,t) \in N(\tau_3) \cap U(\partial N(\tau_3) \backslash \omega_0 \times \{0\}) \end{cases}$$

where $U(\partial N(\tau_3) \backslash \omega_0 \times \{0\})$ is a neighborhood of $\partial N(\tau_3) \backslash \omega_0 \times \{0\}$, which is very small.

Step 1. We shall prove that there exists a constant $\beta_1 \in (0,1)$ such that

$$\|y\|_{\star, N(\tau_1)} \le C \left(\|y_t(0)\|_{L^2(\omega)} + \|y(0)\|_{H^3(\omega)}\right)^{\beta_1} \|y\|_{\star, Q_0}^{1-\beta_1}. \tag{3.13}$$

In order to prove (3.13), set $\psi_2(x,t) = h(x)$. We here let $\psi(x,t) = \psi_2(x,t)$ in Proposition 2.1. Since h is piecewise smooth in $N(\tau_3)$, *i.e.* h is smooth in $(a+\tau_3, \frac{a+b}{2}) \times (0, T_0 - \tau_3)$ and $(\frac{a+b}{2}, b - \tau_3) \times (0, T_0 - \tau_3)$, we can apply (2.1) to $\tilde{y} = \chi y$ and $u = \theta \tilde{y}$ in $(a+\tau_3, \frac{a+b}{2}) \times (0, T_0 - \tau_3)$ and $(\frac{a+b}{2}, b-\tau_3) \times (0, T_0 - \tau_3)$, respectively and then add the two estimates together to obtain that

$$\begin{aligned} &\int_{N(\tau_3)} \left(\lambda^7\mu^8\varphi^7 u^2 + \lambda^5\mu^6\varphi^5 u_x^2 + \lambda^3\mu^4\varphi^3 u_{xx}^2 + \lambda\mu^2\varphi u_{xxx}^2 + \lambda^3\mu^4\varphi^3 u_t^2 + \lambda\mu^2\varphi u_{xt}^2\right) \mathrm{d}x\mathrm{d}t \\ &\qquad \le C \int_{N(\tau_3)} (\theta^2 |\tilde{y}_{tt} - \tilde{y}_{xxxx}|^2 - \{\ldots\}_x - \{\ldots\}_{xx} - \{\ldots\}_{xxx} - \{\ldots\}_{xxxx} - \{\ldots\}_t - \{\ldots\}_{tt}) \mathrm{d}x\mathrm{d}t, \end{aligned}$$

where μ is large enough with respect to $C(\psi_2)$.

Taking into the construction of ψ_2 and χ, we have

$$\begin{aligned} &\int_{N(\tau_3)} \left(\lambda^7\mu^8\varphi^7 u^2 + \lambda^5\mu^6\varphi^5 u_x^2 + \lambda^3\mu^4\varphi^3 u_{xx}^2 + \lambda\mu^2\varphi u_{xxx}^2 + \lambda^3\mu^4\varphi^3 u_t^2 + \lambda\mu^2\varphi u_{xt}^2\right) \mathrm{d}x\mathrm{d}t \\ &\quad \ge \int_{N(\tau_1)} \left(\lambda^7\mu^8\varphi^7 u^2 + \lambda^5\mu^6\varphi^5 u_x^2 + \lambda^3\mu^4\varphi^3 u_{xx}^2 + \lambda\mu^2\varphi u_{xxx}^2 + \lambda^3\mu^4\varphi^3 u_t^2 + \lambda\mu^2\varphi u_{xt}^2\right) \mathrm{d}x\mathrm{d}t \\ &\quad \ge C \int_{N(\tau_1)} \left(\lambda^7\mu^8\varphi^7\theta^2 \tilde{y}^2 + \lambda^5\mu^6\varphi^5\theta^2 \tilde{y}_x^2 + \lambda^3\mu^4\varphi^3\theta^2 \tilde{y}_{xx}^2 + \lambda\mu^2\varphi\theta^2 \tilde{y}_{xxx}^2 + \lambda^3\mu^4\varphi^3\theta^2 \tilde{y}_t^2 + \lambda\mu^2\varphi\theta^2 \tilde{y}_{xt}^2\right) \mathrm{d}x\mathrm{d}t \\ &\quad = C \int_{N(\tau_1)} \left(\lambda^7\mu^8\varphi^7\theta^2 y^2 + \lambda^5\mu^6\varphi^5\theta^2 y_x^2 + \lambda^3\mu^4\varphi^3\theta^2 y_{xx}^2 + \lambda\mu^2\varphi\theta^2 y_{xxx}^2 + \lambda^3\mu^4\varphi^3\theta^2 y_t^2 + \lambda\mu^2\varphi\theta^2 y_{xt}^2\right) \mathrm{d}x\mathrm{d}t \\ &\quad \ge C \int_{N(\tau_1)} \lambda\mu^2\varphi\theta^2 \left(y^2 + y_x^2 + y_{xx}^2 + y_{xxx}^2 + y_t^2 + y_{xt}^2\right) \mathrm{d}x\mathrm{d}t \\ &\quad \ge C\lambda\mu^2 \mathrm{e}^{h(a+\tau_1)\mu} \mathrm{e}^{2\lambda \mathrm{e}^{h(a+\tau_1)\mu}} \|y\|_{\star, N(\tau_1)}^2 \end{aligned}$$

and

$$\begin{aligned}
&\int_{N(\tau_3)} (-\{\dots\}_x - \{\dots\}_{xx} - \{\dots\}_{xxx} - \{\dots\}_{xxxx} - \{\dots\}_t - \{\dots\}_{tt})\,\mathrm{d}x\mathrm{d}t \\
&\qquad = \int_{N(\tau_3)} (-\{\dots\}_t - \{\dots\}_{tt})\mathrm{d}x\mathrm{d}t \\
&\qquad = \int_{a+\tau_3}^{b-\tau_3} ((B_2 u_t u_x)(x,0) + (B_4 u_t u_{xxx})(x,0) + (b u_t u_{xx})(x,0) + (a u u_t)(x,0))\,\mathrm{d}x \\
&\qquad \le C\int_{\omega_0} \left(\lambda^3\mu^3\left(\varphi^3 u_t^2\right)(x,0) + \lambda^3\mu^3\left(\varphi^3 u_x^2\right)(x,0) + \lambda^{-1}\mu^{-1}\left(\varphi^{-1}u_{xxx}^2\right)(x,0)\right. \\
&\qquad\quad \left. + \lambda^{-1}\mu\left(\varphi^{-1}u_{xx}^2\right)(x,0) + \lambda^3\mu^5\left(\varphi^3 u^2\right)(x,0)\right)\mathrm{d}x \\
&\qquad \le C\lambda^3\mu^5\mathrm{e}^{3\mu}\mathrm{e}^{2\lambda\mathrm{e}^{\mu}} \int_{\omega_0} \left(y_t^2(x,0) + y_x^2(x,0) + y_{xxx}^2(x,0) + y_{xx}^2(x,0) + y^2(x,0)\right)\mathrm{d}x \\
&\qquad \le C\lambda^3\mu^5\mathrm{e}^{3\mu}\mathrm{e}^{2\lambda\mathrm{e}^{\mu}} \left(\|y_t(0)\|^2_{L^2(\omega)} + \|y(0)\|^2_{L^2(\omega)} + \|y_x(0)\|^2_{L^2(\omega)} + \|y_{xx}(0)\|^2_{L^2(\omega)} + \|y_{xxx}(0)\|^2_{L^2(\omega)}\right) \\
&\qquad = C\lambda^3\mu^5\mathrm{e}^{3\mu}\mathrm{e}^{2\lambda\mathrm{e}^{\mu}} \left(\|y_t(0)\|^2_{L^2(\omega)} + \|y(0)\|^2_{H^3(\omega)}\right).
\end{aligned}$$

Noting that $\tilde{y}$ solves

$$\tilde{y}_{tt} - \tilde{y}_{xxxx} = y(\chi_{tt} - \chi_{xxxx}) + 2y_t\chi_t - 4y_x\chi_{xxx} - 6y_{xx}\chi_{xx} - 4y_{xxx}\chi_x,$$

this leads to

$$\begin{aligned}
\int_{N(\tau_3)} \theta^2|\tilde{y}_{tt} - \tilde{y}_{xxxx}|^2\mathrm{d}x\mathrm{d}t &= \int_{N(\tau_3)} \theta^2|y(\chi_{tt} - \chi_{xxxx}) + 2y_t\chi_t - 4y_x\chi_{xxx} - 6y_{xx}\chi_{xx} - 4y_{xxx}\chi_x|^2\mathrm{d}x\mathrm{d}t \\
&= \int_{N(\tau_3)\setminus N(\tau_2)} \theta^2|y(\chi_{tt} - \chi_{xxxx}) + 2y_t\chi_t - 4y_x\chi_{xxx} - 6y_{xx}\chi_{xx} - 4y_{xxx}\chi_x|^2\mathrm{d}x\mathrm{d}t \\
&\le C\mathrm{e}^{2\lambda\mathrm{e}^{h(a+\tau_2)\mu}} \int_{N(\tau_3)\setminus N(\tau_2)} (y^2 + y_t^2 + y_x^2 + y_{xx}^2 + y_{xxx}^2)\mathrm{d}x\mathrm{d}t \\
&\le C\mathrm{e}^{2\lambda\mathrm{e}^{h(a+\tau_2)\mu}} \int_{N(\tau_3)} \left(y^2 + y_t^2 + y_x^2 + y_{xx}^2 + y_{xxx}^2\right)\mathrm{d}x\mathrm{d}t \\
&= C\mathrm{e}^{2\lambda\mathrm{e}^{h(a+\tau_2)\mu}} \|y\|^2_{\star,N(\tau_3)}.
\end{aligned}$$

From the above estimates, we conclude that

$$\lambda\mu^2\mathrm{e}^{h(a+\tau_1)\mu}\mathrm{e}^{2\lambda\mathrm{e}^{h(a+\tau_1)\mu}}\|y\|^2_{\star,N(\tau_1)} \le C\left(\lambda^3\mu^5\mathrm{e}^{3\mu}\mathrm{e}^{2\lambda\mathrm{e}^{\mu}}(\|y_t(0)\|^2_{L^2(\omega)} + \|y(0)\|^2_{H^3(\omega)}) + \mathrm{e}^{2\lambda\mathrm{e}^{h(a+\tau_2)\mu}}\|y\|^2_{\star,N(\tau_3)}\right),$$

namely,

$$\begin{aligned}
\|y\|^2_{\star,N(\tau_1)} \le C\Big(&\lambda^2\mu^3\mathrm{e}^{3\mu - h(a+\tau_1)\mu}\mathrm{e}^{2\lambda\left(\mathrm{e}^{\mu} - \mathrm{e}^{h(a+\tau_1)\mu}\right)}\left(\|y_t(0)\|^2_{L^2(\omega)} + \|y(0)\|^2_{H^3(\omega)}\right) \\
&+\lambda^{-1}\mu^{-2}\mathrm{e}^{-h(a+\tau_1)\mu}\mathrm{e}^{2\lambda\left(\mathrm{e}^{h(a+\tau_2)\mu} - \mathrm{e}^{h(a+\tau_1)\mu}\right)}\|y\|^2_{\star,N(\tau_3)}\Big).
\end{aligned}$$

Set

$$\begin{cases}
\varepsilon = \lambda^{-1}\mu^{-2}\mathrm{e}^{-h(a+\tau_1)\mu}\mathrm{e}^{2\lambda\left(\mathrm{e}^{h(a+\tau_2)\mu} - \mathrm{e}^{h(a+\tau_1)\mu}\right)}, \\
\varepsilon_0 = \lambda_0^{-1}\mu^{-2}\mathrm{e}^{-h(a+\tau_1)\mu}\mathrm{e}^{2\lambda_0\left(\mathrm{e}^{h(a+\tau_2)\mu} - \mathrm{e}^{h(a+\tau_1)\mu}\right)}, \\
k = \frac{\mathrm{e}^{\mu} - \mathrm{e}^{h(a+\tau_1)\mu}}{\mathrm{e}^{h(a+\tau_1)\mu} - \mathrm{e}^{h(a+\tau_2)\mu}}.
\end{cases}$$

It is clear that $1 > h(a+\tau_1) > h(a+\tau_2)$, therefore, we can choose μ large enough such that $k > 2$. Then for any $\varepsilon \in (0, \varepsilon_0]$ the following inequality holds:

$$\|y\|_{\star,N(\tau_1)} \leq C\varepsilon\|y\|_{\star,Q_0} + C\varepsilon^{-k}\left(\|y_t(0)\|_{L^2(\omega)} + \|y(0)\|_{H^3(\omega)}\right). \tag{3.14}$$

Therefore, (3.14) holds for all $\varepsilon > 0$. Further, if we let

$$\beta_1 = \frac{1}{1+k}, \varepsilon = \left(\frac{\|y_t(0)\|_{L^2(\omega)} + \|y(0)\|_{H^3(\omega)}}{\|y\|_{\star,Q_0}}\right)^{\beta_1},$$

(3.13) follows immediately from (3.14).

There must be some open ball $B \subset N(\tau_1)$, it is easy to see that

$$\|y\|_{\star,B} \leq C\left(\|y_t(0)\|_{L^2(\omega)} + \|y(0)\|_{H^3(\omega)}\right)^{\beta_1}\|y\|_{\star,Q_0}^{1-\beta_1}.$$

Step 2. We shall prove that there exists a constant $\beta_2 \in (0,1)$ such that

$$\|y\|_{\star,\omega\times(\gamma,T_0-\gamma)} \leq C\|y\|_{\star,B}^{\beta_2}\|y\|_{\star,Q_0}^{1-\beta_2}.$$

Indeed, let $B_i,\ i=1,2,3$ be three open balls with the properties $B_1 \subset\subset B_2 \subset\subset B_3 \subset\subset Q_0$. Take $\eta \in C_0^\infty(Q_0)$ be valued in $(0,1)$ and $\eta \equiv 1$ in B_3.

Set

$$\begin{aligned}
\psi_3(x,t) &= (x-x_0)^2 - \widetilde{C}_2\left(t - \frac{T_0}{2}\right)^2 + \widetilde{C}_3,\\
r_1 &= \sup_{(x,t)\in B_1}\left(|x-x_0|^2 + |t-t_0|^2\right)^{\frac{1}{2}},\\
r_2 &= \sup_{(x,t)\in B_2}|t-t_0|,\\
r_3 &= \inf_{(x,t)\in Q_0\setminus B_3}|t-t_0|,\\
r_{Q_0} &= \sup_{(x,t)\in Q_0\setminus B_3}|x-x_0|,
\end{aligned}$$

where (x_0,t_0) is the center of B_1 and $\widetilde{C}_2, \widetilde{C}_3$ are large enough such that $\widetilde{C}_2(r_3^2 - r_2^2) \geq r_{Q_0}^2$ and $\psi_3 \geq 0$.

We here let $\psi(x,t) = \psi_3(x,t)$ in Proposition 2.1 and apply (2.1) to $\hat{y} = \eta y$ and $z = \theta\hat{y}$, then it follows that

$$\begin{aligned}
&\int_{Q_0}\left(\lambda^3\mu^4\varphi^3\psi_{3x}^4 z_t^2 + \lambda^7\mu^8\varphi^7\psi_{3x}^8 z^2 + \lambda^5\mu^6\varphi^5\psi_{3x}^8 z_x^2 + \lambda^3\mu^4\varphi^3\psi_{3x}^4 z_{xx}^2 + \lambda\mu^2\varphi\psi_{3x}^2 z_{xxx}^2 + \lambda\mu^2\varphi\psi_{3x}^2 z_{xt}^2\right)\mathrm{d}x\mathrm{d}t\\
&\quad\leq C\Bigg(\int_{Q_0}\theta^2|\hat{y}_{tt} - \hat{y}_{xxxx}|^2\mathrm{d}x\mathrm{d}t + \int_{Q_0}\big(\lambda^3\mu^3\varphi^3 z_t^2 + \lambda^7\mu^7\varphi^7 z^2\\
&\qquad\qquad + \lambda^5\mu^5\varphi^5 z_x^2 + \lambda^3\mu^3\varphi^3 z_{xx}^2 + \lambda\mu\varphi z_{xxx}^2 + \lambda\mu\varphi z_{xt}^2\big)\mathrm{d}x\mathrm{d}t\Bigg).
\end{aligned}$$

By the same argument as in the proof of Lemma 3.2, there exists $\mu_1 > 0$ such that for $\mu \geq \mu_1$, we have

$$\begin{aligned}
&\int_{Q_0}\left(\lambda^3\mu^3\varphi^3 z_t^2 + \lambda^7\mu^7\varphi^7 z^2 + \lambda^5\mu^5\varphi^5 z_x^2 + \lambda^3\mu^3\varphi^3 z_{xx}^2 + \lambda\mu\varphi z_{xxx}^2 + \lambda\mu\varphi z_{xt}^2\right)\mathrm{d}x\mathrm{d}t\\
&\quad\leq C\Bigg(\int_{Q_0}\theta^2|\hat{y}_{tt} - \hat{y}_{xxxx}|^2\mathrm{d}x\mathrm{d}t + \int_{B_1}\big(\lambda^3\mu^3\varphi^3 z_t^2 + \lambda^7\mu^7\varphi^7 z^2\\
&\qquad\qquad + \lambda^5\mu^5\varphi^5 z_x^2 + \lambda^3\mu^3\varphi^3 z_{xx}^2 + \lambda\mu\varphi z_{xxx}^2 + \lambda\mu\varphi z_{xt}^2\big)\mathrm{d}x\mathrm{d}t\Bigg).
\end{aligned}$$

Taking into account the construction of ψ_3, we have

$$\begin{aligned}
&\int_{Q_0}\left(\lambda^3\mu^3\varphi^3 z_t^2+\lambda^7\mu^7\varphi^7 z^2+\lambda^5\mu^5\varphi^5 z_x^2+\lambda^3\mu^3\varphi^3 z_{xx}^2+\lambda\mu\varphi z_{xxx}^2+\lambda\mu\varphi z_{xt}^2\right)\mathrm{d}x\mathrm{d}t\\
&\quad\geq C\int_{Q_0}\left(\lambda^3\mu^3\varphi^3\theta^2\hat{y}_t^2+\lambda^7\mu^7\varphi^7\theta^2\hat{y}^2+\lambda^5\mu^5\varphi^5\theta^2\hat{y}_x^2+\lambda^3\mu^3\varphi^3\theta^2\hat{y}_{xx}^2+\lambda\mu\varphi\theta^2\hat{y}_{xxx}^2+\lambda\mu\varphi\theta^2\hat{y}_{xt}^2\right)\mathrm{d}x\mathrm{d}t\\
&\quad\geq C\int_{B_2}\left(\lambda^3\mu^3\varphi^3\theta^2 y_t^2+\lambda^7\mu^7\varphi^7\theta^2 y^2+\lambda^5\mu^5\varphi^5\theta^2 y_x^2+\lambda^3\mu^3\varphi^3\theta^2 y_{xx}^2+\lambda\mu\varphi\theta^2 y_{xxx}^2+\lambda\mu\varphi\theta^2 y_{xt}^2\right)\mathrm{d}x\mathrm{d}t\\
&\quad\geq C\lambda\mu\mathrm{e}^{\mu\left(\widetilde{C}_3-\widetilde{C}_2 r_2^2\right)}\mathrm{e}^{2\lambda\mathrm{e}^{\mu\left(\widetilde{C}_3-\widetilde{C}_2 r_2^2\right)}}\int_{B_2}\left(y_t^2+y^2+y_x^2+y_{xx}^2+y_{xxx}^2+y_{xt}^2\right)\mathrm{d}x\mathrm{d}t\\
&\quad= C\lambda\mu\mathrm{e}^{\mu\left(\widetilde{C}_3-\widetilde{C}_2 r_2^2\right)}\mathrm{e}^{2\lambda\mathrm{e}^{\mu\left(\widetilde{C}_3-\widetilde{C}_2 r_2^2\right)}}\|y\|_{\star,B_2}^2
\end{aligned}$$

and

$$\begin{aligned}
&\int_{B_1}\left(\lambda^3\mu^3\varphi^3 z_t^2+\lambda^7\mu^7\varphi^7 z^2+\lambda^5\mu^5\varphi^5 z_x^2+\lambda^3\mu^3\varphi^3 z_{xx}^2+\lambda\mu\varphi z_{xxx}^2+\lambda\mu\varphi z_{xt}^2\right)\mathrm{d}x\mathrm{d}t\\
&\quad\leq C\int_{B_1}\left(\lambda^3\mu^3\varphi^3\theta^2 y_t^2+\lambda^7\mu^7\varphi^7\theta^2 y^2+\lambda^5\mu^5\varphi^5\theta^2 y_x^2+\lambda^3\mu^3\varphi^3\theta^2 y_{xx}^2+\lambda\mu\varphi\theta^2 y_{xxx}^2+\lambda\mu\varphi\theta^2 y_{xt}^2\right)\mathrm{d}x\mathrm{d}t\\
&\quad\leq C\lambda^7\mu^7\mathrm{e}^{7\mu\left(r_1^2+\widetilde{C}_3\right)}\mathrm{e}^{2\lambda\mathrm{e}^{\mu\left(r_1^2+\widetilde{C}_3\right)}}\int_{B_1}\left(y_t^2+y^2+y_x^2+y_{xx}^2+y_{xxx}^2+y_{xt}^2\right)\mathrm{d}x\mathrm{d}t\\
&\quad= C\lambda^7\mu^7\mathrm{e}^{7\mu\left(r_1^2+\widetilde{C}_3\right)}\mathrm{e}^{2\lambda\mathrm{e}^{\mu\left(r_1^2+\widetilde{C}_3\right)}}\|y\|_{\star,B_1}^2.
\end{aligned}$$

Noting that $\hat{y}$ solves

$$\hat{y}_{tt}-\hat{y}_{xxxx}=y(\eta_{tt}-\eta_{xxxx})+y_t\cdot 2\eta_t-y_x\cdot 4\eta_{xxx}-y_{xx}\cdot 6\eta_{xx}-y_{xxx}\cdot 4\eta_x,$$

we have

$$\begin{aligned}
\int_{Q_0}\theta^2|\hat{y}_{tt}-\hat{y}_{xxxx}|^2\mathrm{d}x\mathrm{d}t &\leq \int_{Q_0}\theta^2\left|y(\eta_{tt}-\eta_{xxxx})+y_t\cdot 2\eta_t-y_x\cdot 4\eta_{xxx}-y_{xx}\cdot 6\eta_{xx}-y_{xxx}\cdot 4\eta_x\right|^2\mathrm{d}x\mathrm{d}t\\
&= \int_{Q_0\setminus B_3}\theta^2\left|y(\eta_{tt}-\eta_{xxxx})+y_t\cdot 2\eta_t-y_x\cdot 4\eta_{xxx}-y_{xx}\cdot 6\eta_{xx}-y_{xxx}\cdot 4\eta_x\right|^2\mathrm{d}x\mathrm{d}t\\
&\leq C\mathrm{e}^{2\lambda\mathrm{e}^{\mu(r_{Q_0}^2-\widetilde{C}_2 r_3^2+\widetilde{C}_3)}}\|y\|_{\star,Q_0}^2.
\end{aligned}$$

From the above estimates, we know that

$$\lambda\mu\mathrm{e}^{\mu\left(\widetilde{C}_3-\widetilde{C}_2 r_2^2\right)}\mathrm{e}^{2\lambda\mathrm{e}^{\mu\left(\widetilde{C}_3-\widetilde{C}_2 r_2^2\right)}}\|y\|_{\star,B_2}^2\leq C\left(\mathrm{e}^{2\lambda\mathrm{e}^{\mu\left(r_{Q_0}^2-\widetilde{C}_2 r_3^2+\widetilde{C}_3\right)}}\|y\|_{\star,Q_0}^2+\lambda^7\mu^7\mathrm{e}^{7\mu\left(r_1^2+\widetilde{C}_3\right)}\mathrm{e}^{2\lambda\mathrm{e}^{\mu\left(r_1^2+\widetilde{C}_3\right)}}\|y\|_{\star,B_1}^2\right),$$

namely,

$$\begin{aligned}
\|y\|_{\star,B_2}^2\leq C\Bigg(&\lambda^{-1}\mu^{-1}\mathrm{e}^{-\left(\widetilde{C}_3-\widetilde{C}_2 r_2^2\right)\mu}\mathrm{e}^{2\lambda\left(\mathrm{e}^{\mu\left(r_{Q_0}^2-\widetilde{C}_2 r_3^2+\widetilde{C}_3\right)}-\mathrm{e}^{\mu\left(\widetilde{C}_3-\widetilde{C}_2 r_2^2\right)}\right)}\|y\|_{\star,Q_0}^2\\
&+\lambda^6\mu^6\mathrm{e}^{7\mu\left(r_1^2+\widetilde{C}_3\right)-\mu\left(\widetilde{C}_3-\widetilde{C}_2 r_2^2\right)}\mathrm{e}^{2\lambda\left(\mathrm{e}^{\mu\left(r_1^2+\widetilde{C}_3\right)}-\mathrm{e}^{\mu\left(\widetilde{C}_3-\widetilde{C}_2 r_2^2\right)}\right)}\|y\|_{\star,B_1}^2\Bigg).
\end{aligned}$$

Set

$$\begin{cases} \varepsilon = \lambda^{-1}\mu^{-1}\mathrm{e}^{-\left(\widetilde{C}_3-\widetilde{C}_2 r_2^2\right)\mu}\mathrm{e}^{2\lambda\left(\mathrm{e}^{\mu\left(r_{Q_0}^2-\widetilde{C}_2 r_3^2+\widetilde{C}_3\right)}-\mathrm{e}^{\mu\left(\widetilde{C}_3-\widetilde{C}_2 r_2^2\right)}\right)}, \\ \varepsilon_0 = \lambda_0^{-1}\mu^{-1}\mathrm{e}^{-\left(\widetilde{C}_3-\widetilde{C}_2 r_2^2\right)\mu}\mathrm{e}^{2\lambda_0\left(\mathrm{e}^{\mu\left(r_{Q_0}^2-\widetilde{C}_2 r_3^2+\widetilde{C}_3\right)}-\mathrm{e}^{\mu\left(\widetilde{C}_3-\widetilde{C}_2 r_2^2\right)}\right)}, \\ k = \frac{\mathrm{e}^{\mu\left(r_1^2+\widetilde{C}_3\right)}-\mathrm{e}^{\left(\widetilde{C}_3-\widetilde{C}_2 r_2^2\right)\mu}}{\mathrm{e}^{\left(\widetilde{C}_3-\widetilde{C}_2 r_2^2\right)\mu}-\mathrm{e}^{\left(r_Q^2-\widetilde{C}_2 r_3^2+\widetilde{C}_3\right)\mu}} = \frac{\mathrm{e}^{r_1^2\mu}-\mathrm{e}^{\left(-\widetilde{C}_2 r_2^2\right)\mu}}{\mathrm{e}^{\left(-\widetilde{C}_2 r_2^2\right)\mu}-\mathrm{e}^{\left(r_Q^2-\widetilde{C}_2 r_3^2\right)\mu}}. \end{cases}$$

Note $\widetilde{C}_2(r_3^2-r_2^2) > r_Q^2$, therefore, we can choose μ large enough such that $k > 6$.

By the same method as in Step 1, it is easy to see that

$$\|y\|_{\star,B_2} \le C\|y\|_{\star,B_1}^{\delta}\|y\|_{\star,Q_0}^{1-\delta}, \tag{3.15}$$

where $\delta \in (0,1)$.

For any open ball $B' \subset\subset Q_0$, there is finite natural number m and two sequences of open balls $\{B^i\}_{i=1}^n$ and $\{\widetilde{B}^i\}_{i=1}^m$ such that

$$\begin{cases} B' \subset B^1, \widetilde{B}^i \subset\subset B^i \cap B^{i+1}, i = 1,\dots,m-1, \\ \widetilde{B}^m \subset\subset B^m, \widetilde{B}^m \subset B. \end{cases}$$

In view of (3.15), there must be a sequence $\{\delta_i\}_{i=1}^m$ such that

$$\begin{aligned} \|y\|_{\star,B'} &\le \|y\|_{\star,B^1} \le C\|y\|_{\star,\widetilde{B}^1}^{\delta_1}\|y\|_{\star,Q_0}^{1-\delta_1} \le C\|y\|_{\star,B^2}^{\delta_1}\|y\|_{\star,Q_0}^{1-\delta_1} \\ &\le C\|y\|_{\star,\widetilde{B}^2}^{\delta_1\delta_2}\|y\|_{\star,Q_0}^{1-\delta_1\delta_2} \le \dots \le C\|y\|_{\star,\widetilde{B}^m}^{\delta_1\delta_2\dots\delta_m}\|y\|_{\star,Q_0}^{1-\delta_1\delta_2\dots\delta_m}. \end{aligned}$$

Setting $\delta' = \delta_1\delta_2\dots\delta_m$, it follows that

$$\|y\|_{\star,B'} \le C\|y\|_{\star,B}^{\delta'}\|y\|_{\star,Q_0}^{1-\delta'}. \tag{3.16}$$

Since $\omega\times(\gamma, T_0-\gamma) \subset\subset Q_0$, there must be a finite subcover of open balls, then from (3.16) we know there is a constant $0 < \beta_2 < 1$ such that

$$\|y\|_{\star,\omega\times(\gamma,T_0-\gamma)} \le C\|y\|_{\star,B}^{\beta_2}\|y\|_{\star,Q_0}^{1-\beta_2}.$$

Define $\alpha_2 = \beta_1\beta_2$, we can deduce that

$$\|y\|_{\star,\omega\times(\gamma,T_0-\gamma)} \le C\left(\|y_t(0)\|_{L^2(\omega)} + \|y(0)\|_{H^3(\omega)}\right)^{\alpha_2}\|y\|_{\star,Q_0}^{1-\alpha_2}. \qquad \square$$

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. Following Lemma 3.2 and 3.3, there exists two constants $\alpha_1, \alpha_2 \in (0,1)$ such that for any $y \in \mathcal{G}$ which solves (3.1), satisfy

$$\|y\|_{L^2(I\times(T',T''))} \le C\|y\|_{\star,\omega\times(\gamma,T_0-\gamma)}^{\alpha_1}\|y\|_{\star,Q_0}^{1-\alpha_1}$$

and

$$\|y\|_{\star,\omega\times(\gamma,T_0-\gamma)} \le C\left(\|y_t(0)\|_{L^2(\omega)} + \|y(0)\|_{H^3(\omega)}\right)^{\alpha_2}\|y\|_{\star,Q_0}^{1-\alpha_2},$$

which conclude that

$$\|y\|_{L^2(I\times(T',T''))} \le C\left(\|y_t(0)\|_{L^2(\omega)} + \|y(0)\|_{H^3(\omega)}\right)^{\alpha_1\alpha_2}\|y\|_{\star,Q_0}^{1-\alpha_1\alpha_2}. \tag{3.17}$$

Set $y(x,t) = \sum\limits_{\lambda_i \le r} \frac{shtb_i}{b_i} a_i e_i$, where $b_i = \sqrt{\lambda_i}$ and $\frac{shtb}{b} = t$ for $b = 0$. Direct computation shows that y given as above solves (3.1), which vanishes when $(x,t) \in \omega \times \{0\}$. It is obvious that both Rey and Imy satisfy (3.1). Applying (3.17) to Rey gives that

$$\|\mathrm{Re}y\|_{L^2(I\times(T',T''))} \le C\left(\|\mathrm{Re}y_t(0)\|_{L^2(\omega)} + \|\mathrm{Re}y(0)\|_{H^3(\omega)}\right)^{\alpha_1\alpha_2} \|\mathrm{Re}y\|_{\star,Q_0}^{1-\alpha_1\alpha_2}. \tag{3.18}$$

Since

$$\begin{aligned}
\|\mathrm{Re}y\|_{\star,Q_0}^2 &\le C\mathrm{e}^{C\sqrt{r}} \sum_{\lambda_i\le r} |\mathrm{Re}a_i|^2,\\
(\mathrm{Re}y)_t(x,0) &= \sum_{\lambda_i\le r} \mathrm{Re}a_i e_i,\\
(\mathrm{Re}y)(x,0) &= (\mathrm{Re}y)_x(x,0) = (\mathrm{Re}y)_{xx}(x,0) = (\mathrm{Re}y)_{xxx}(x,0) = 0
\end{aligned}$$

and

$$\begin{aligned}
\|\mathrm{Re}y\|_{L^2(I\times(T',T''))}^2 &= \int_{T'}^{T''}\int_I \left|\sum_{\lambda_i\le r} \frac{shtb_i}{b_i}\mathrm{Re}a_i e_i\right|^2 \mathrm{d}x\mathrm{d}t\\
&= \sum_{\lambda_i\le r} |\mathrm{Re}a_i|^2 \int_{T'}^{T''} \left|\frac{shtb_i}{b_i}\right|^2 \mathrm{d}t\\
&\ge \sum_{\lambda_i\le r} |\mathrm{Re}a_i|^2 \int_{T'}^{T''} t^2 \mathrm{d}t\\
&= C(T',T'') \sum_{\lambda_i\le r} |\mathrm{Re}a_i|^2,
\end{aligned}$$

it is shown that

$$\sum_{\lambda_i\le r} |\mathrm{Re}a_i|^2 \le C\mathrm{e}^{C\sqrt{r}} \int_\omega \left|\sum_{\lambda_i\le r} \mathrm{Re}a_i e_i\right|^2 \mathrm{d}x.$$

By the same manner, we have

$$\sum_{\lambda_i\le r} |\mathrm{Im}a_i|^2 \le C\mathrm{e}^{C\sqrt{r}} \int_\omega \left|\sum_{\lambda_i\le r} \mathrm{Im}a_i e_i\right|^2 \mathrm{d}x.$$

This complete the proof with

$$\sum_{\lambda_i\le r} |a_i|^2 \le C_1\mathrm{e}^{C_2\sqrt{r}} \int_\omega \left|\sum_{\lambda_i\le r} a_i e_i\right|^2 \mathrm{d}x. \qquad \square$$

4. Proof of Theorem 1.2

The proof of Theorem 1.2 combines the ideas in [8,9]. First we give several lemmas.

Lemma 4.1. ([6], pp. 256–257) *For almost all* $\tilde{t} \in E$, *there exists a sequence of numbers* $\{t_i\}_{i=1}^{\infty} \subset (0, T)$ *such that*

$$t_1 < t_2 < \ldots t_i < t_{i+1} < \ldots < \tilde{t}, \quad t_i \to \tilde{t} \ \text{ as } \ i \to \infty, \tag{4.1}$$

$$m(E \cap [t_i, t_{i+1}]) \geq \rho(t_{i+1} - t_i), \quad i = 1, 2, \ldots, \tag{4.2}$$

$$\frac{t_{i+1} - t_i}{t_{i+2} - t_{i+1}} \leq C_0, \quad i = 1, 2, \ldots, \tag{4.3}$$

where ρ *and* C_0 *are two positive constants which are independent of* i.

Lemma 4.2.

(a) *For each* $r \geq \lambda_1$, *there exists a control* $f_r \in L^\infty_{\mathcal{F}}(0, T; L^2(\Omega; X_r))$ *such that the solution* y *of system* (1.1) *with* $f = f_r$ *satisfies* $P_r(y(\cdot, T)) = 0$ *in* I, P*-a.s. Moreover,* f_r *verifies: If* $2\lambda_1^\alpha > \xi$, *then*

$$\|f_r\|^2_{L^\infty_{\mathcal{F}}(0,T;L^2(\Omega;X_r))} \leq \frac{C_1 \mathrm{e}^{C_2\sqrt{r}}}{(m(E))^2} \mathbb{E}\|y_0\|^2_{L^2(I)}.$$

For the general case, it holds that

$$\|f_r\|^2_{L^\infty_{\mathcal{F}}(0,T;L^2(\Omega;X_r))} \leq \frac{C_1 \mathrm{e}^{C_2\sqrt{r}+\xi T}}{(m(E))^2} \mathbb{E}\|y_0\|^2_{L^2(I)}.$$

(b) *If* $f \equiv 0$ *in system* (1.1), *then for any* $y_0 \in L^2(\Omega, P, \mathcal{F}_0; L^2(I))$ *with* $P_{\lambda_{k-1}}(y_0) = 0$ *for some* $k = 2, 3, \ldots$, *the corresponding solution* y *of system* (1.1) *satisfies*

$$\mathbb{E}\|y(t)\|^2_{L^2(I)} \leq \mathrm{e}^{-(2\lambda_k^\alpha - \xi)t} \mathbb{E}\|y_0\|^2_{L^2(I)}. \tag{4.4}$$

Proof.
(a) We first introduce the following backward stochastic fractional order Cahn–Hilliard equation

$$\begin{cases} \mathrm{d}z - A^\alpha z = -a(t) Z \mathrm{d}t + Z \mathrm{d}B & \text{in } Q, \\ z(0,t) = 0 = z(1,t) & \text{in } (0,T), \\ z_x(0,t) = 0 = z_x(1,t) & \text{in } (0,T), \\ z(x,T) = z_T(x) & \text{in } I, \end{cases} \tag{4.5}$$

where $z_T \in L^2(\Omega, P, \mathcal{F}_T; X_r)$. According to [2, 12], (4.5) admits one and only one solution $(z, Z) \in L^2_{\mathcal{F}}(\Omega; C([0,T]; L^2(I))) \times L^2_{\mathcal{F}}(0, T; L^2(I)))$.

Since $z_T \in L^2(\Omega, P, \mathcal{F}_T; X_r)$, z_T can be written as $z_T = \sum_{\lambda_i \leq r} z_T^i e_i$ for a sequence of $\mathcal{F}_T$-measurable random variable $\{z_T^i\}_{\lambda_i \leq r}$. Then the solution (z, Z) of (4.5) can be expressed as

$$z = \sum_{\lambda_i \leq r} z_i(t) e_i, Z = \sum_{\lambda_i \leq r} Z_i(t) e_i,$$

where $z_i(\cdot) \in L^2_{\mathcal{F}}(\Omega; C([0,T]))$ and $Z_i(\cdot) \in L^2_{\mathcal{F}}(0,T)(\lambda_i \leq r)$ satisfy the following equation

$$\begin{cases} \mathrm{d}z_i - \lambda_i^\alpha z_i \mathrm{d}t = -a(t) Z_i \mathrm{d}t + Z_i \mathrm{d}B \\ z_i(T) = z_T^i. \end{cases} \text{ in } (0,T), \tag{4.6}$$

It follows from Theorem 1.1 that

$$\mathbb{E}\sum_{\lambda_i\leq r}|z_i(t)|^2 \leq C_1\mathrm{e}^{C_2\sqrt{r}}\mathbb{E}\int_\omega\left|\sum_{\lambda_i\leq r}z_i(t)e_i\right|^2\mathrm{d}x = C_1\mathrm{e}^{C_2\sqrt{r}}\mathbb{E}\int_\omega|z(t)|^2\mathrm{d}x \tag{4.7}$$

for any $t\in[0,T]$.

By Itô's formula, we see that $d|z|^2 = 2z\mathrm{d}z + (\mathrm{d}z)^2$. Hence we obtain that

$$\begin{aligned}\mathbb{E}\int_I|z(t)|^2\mathrm{d}x - \mathbb{E}\int_I|z(0)|^2\mathrm{d}x &= 2\mathbb{E}\int_0^t\sum_{\lambda_i\leq r}\lambda_i^\alpha|z_i(s)|^2\mathrm{d}s + \mathbb{E}\int_0^t\int_I(-2a(s)zZ + Z^2)\mathrm{d}x\mathrm{d}s\\ &\geq 2\mathbb{E}\int_0^t\sum_{\lambda_i\leq r}\lambda_i^\alpha|z_i(s)|^2\mathrm{d}s - \mathbb{E}\int_0^t\int_I|a(s)z|^2\mathrm{d}x\mathrm{d}s\\ &\geq \mathbb{E}\int_0^t\sum_{\lambda_i\leq r}(2\lambda_i^\alpha - \xi)|z_i(s)|^2\mathrm{d}s \geq 0. \end{aligned}\tag{4.8}$$

In view of (4.7)) and (4.8), we obtain that

$$\mathbb{E}\int_I z^2(x,0)\mathrm{d}x \leq C_1\mathrm{e}^{C_2\sqrt{r}}\mathbb{E}\int_\omega z^2(x,t)\mathrm{d}x$$

for $t\in[0,T]$. Therefore,

$$\int_E\left(\mathbb{E}\int_I z^2(x,0)\mathrm{d}x\right)^{\frac{1}{2}}\mathrm{d}t \leq (C_1\mathrm{e}^{C_2\sqrt{r}})^{\frac{1}{2}}\int_E\left(\mathbb{E}\int_\omega z^2(x,t)\mathrm{d}x\right)^{\frac{1}{2}}\mathrm{d}t.$$

Hence we deduce that for each $z_T\in L^2(\Omega,P,\mathcal{F}_T;X_r)$,

$$\begin{aligned}\mathbb{E}\int_I z^2(x,0)\mathrm{d}x &\leq \frac{C_1\mathrm{e}^{C_2\sqrt{r}}}{(m(E))^2}\left(\int_0^T\left(\mathbb{E}\int_I\chi_E\chi_\omega z^2(x,t)\mathrm{d}x\right)^{\frac{1}{2}}\mathrm{d}t\right)^2\\ &= \frac{C_1\mathrm{e}^{C_2\sqrt{r}}}{(m(E))^2}\|\chi_E\chi_\omega z\|^2_{L^1_{\mathcal{F}}(0,T;L^2(\Omega;L^2(I)))}.\end{aligned}$$

According to dual argument and a Riesz-type Representation Theorem, one can get the first result in part (a) by similar argument as in [9].

Applying Itô's formula to $\mathrm{e}^{\tau t}|z|^2$, the second result in part a) can be obtained following the above methods with minor changes.

(b) Since $y_0\in L^2(\Omega,\mathcal{F}_0,P;L^2(I))$ with $P_{\lambda_{k-1}}(y_0)=0$, y_0 can be written as $y_0=\sum_{i=k}^\infty y_0^ie_i$ for suitable $y_0^i\in L^2(\Omega,\mathcal{F}_0,P)$. Thus, the solution y of system (4.7) can be expressed as

$$y=\sum_{i=k}^\infty y^i(t)e_i,$$

where $y^i\in L^2_{\mathcal{F}}(\Omega;C([0,T]))$ solves the following equation

$$\begin{cases}\mathrm{d}y^i + \lambda_i^\alpha y^i\mathrm{d}t = a(t)y^i\mathrm{d}B & \text{in } [0,T],\\ y^i(0)=y_0^i.\end{cases}\tag{4.9}$$

Applying Itô's formula to $\mathrm{e}^{(2\lambda_k^\alpha-\xi)t}|y(t)|^2$, we have that

$$\mathrm{d}\left(\mathrm{e}^{(2\lambda_k^\alpha-\xi)t}|y(t)|^2\right)=\mathrm{e}^{(2\lambda_k^\alpha-\xi)t}2y\mathrm{d}y+\mathrm{e}^{(2\lambda_k^\alpha-\xi)t}(\mathrm{d}y)^2+(2\lambda_k^\alpha-\xi)\,\mathrm{e}^{(2\lambda_k^\alpha-\xi)t}y^2\mathrm{d}t.$$

Hence it is clear that

$$\begin{aligned}\mathbb{E}\int_I \mathrm{e}^{(2\lambda_k^\alpha-\xi)t}|y(t)|^2\mathrm{d}x-\mathbb{E}\int_I |y(0)|^2\mathrm{d}x=\mathbb{E}\int_0^T \mathrm{e}^{(2\lambda_k^\alpha-\xi)s}\sum_{i=k}^{\infty}\left(-2\lambda_i^\alpha\right)|y^i|^2\mathrm{d}s\\ +\mathbb{E}\int_0^T\int_I \mathrm{e}^{(2\lambda_k^\alpha-\xi)s}a^2(s)|y|^2\mathrm{d}x\mathrm{d}s\\ +(2\lambda_k^\alpha-\xi)\,\mathbb{E}\int_0^T\int_I \mathrm{e}^{(2\lambda_k^\alpha-\xi)s}|y|^2\mathrm{d}x\mathrm{d}s\le 0,\end{aligned}$$

which gives the desired estimate (4.4) immediately. □

Proof of Theorem 1.2. By same idea in [9], without loss of generality, in what follows we assume that $\lambda_1^\alpha\ge\xi$ and $C_1\ge 1$.

By Lemma 4.1, we can take a number $\tilde t\in E$ with $\tilde t<T$ and a sequence $\{t_N\}_{N=1}^{\infty}\subseteq(0,T)$ such that (4.1)–(4.3) hold for some positive numbers ρ and C_0, and such that $\tilde t-t_1\le\min\{\lambda_1^\alpha,1\}$.

First, we take $\tilde y_0(x)$ to be $q(x,t_1)$, where q is the solution to the following equation

$$\begin{cases}\mathrm{d}q+A^\alpha q\mathrm{d}t=a(t)q\mathrm{d}B & \text{in } I\times(0,t_1),\\ q(0,t)=0=q(1,t) & \text{in } (0,t_1),\\ q_x(0,t)=0=q_x(1,t) & \text{in } (0,t_1),\\ q(x,0)=y_0(x) & \text{in } I.\end{cases}\tag{4.10}$$

Next, we will show that there exists a control $\tilde f\in L^\infty_{\mathcal{F}}(t_1,\tilde t;L^2(\Omega;L^2(I)))$ such that the solution $\tilde y$ of

$$\begin{cases}\mathrm{d}\tilde y+A^\alpha\tilde y\mathrm{d}t=a(t)\tilde y\mathrm{d}B+\chi_\omega\chi_E\tilde f\mathrm{d}t & \text{in } I\times(t_1,\tilde t),\\ \tilde y(0,t)=0=\tilde y(1,t) & \text{in } (t_1,\tilde t),\\ \tilde y_x(0,t)=0=\tilde y_x(1,t) & \text{in } (t_1,\tilde t),\\ \tilde y(x,t_1)=\tilde y_0(x) & \text{in } I,\end{cases}\tag{4.11}$$

satisfying $\tilde y(\tilde t)=0$ in I, P-a.s and

$$\left\|\tilde f\right\|^2_{L^\infty_{\mathcal{F}}(t_1,\tilde t;L^2(\Omega;L^2(I)))}\le L\mathbb{E}\|\tilde y_0\|^2_{L^2(I)},\tag{4.12}$$

where L is a constant independent of y_0.

Indeed, set $K_N=[t_{2N-1},t_{2N}]$ and $J_N=[t_{2N},t_{2N+1}]$ for $N=1,2,\dots$, then

$$[t_1,\tilde t)=\bigcup_{N=1}^{\infty}\left(K_N\cup J_N\right).$$

It is clear that $m(E\cap K_N)>0$ and $m(E\cap J_N)>0$ for all $N\ge 1$.

According to Lemmas 4.1 and 4.2 and the same argument in proof of Theorem 1.1 in [9], we control equation (4.11) on K_N corresponding a control f_N such that $P_{\bar r_N}(y_N(\cdot,t_{2N}))=0$ and

$$\|f_N\|^2_{L^\infty_{\mathcal{F}}(t_{2N-1},t_{2N};L^2(\Omega;L^2(I)))}\le 2^{N-1}\left(\frac{C_1}{\rho^2(t_2-t_1)^2}\right)^N C_0^4C_0^{4\times 2}\dots C_0^{4\times(N-1)}\alpha_1\alpha_2\dots\alpha_N\mathbb{E}\|\tilde y_0\|^2_{L^2(I)},$$

where

$$\alpha_N = \begin{cases} e^{C_2\sqrt{\bar{r}_1}}, & N=1, \\ e^{C_2\sqrt{\bar{r}_N}} e^{\left(-2\bar{r}_{N-1}^{\alpha}+\xi\right)(t_3-t_2)C_0^{-2(N-2)}}, & N \geq 2, \end{cases}$$
$$\bar{r}_N = \left(\overline{C}\tilde{C}^{N-1}+\xi\right)^{\frac{2}{b}}, \qquad N \geq 1,$$
$$\overline{C} = \frac{2}{t_3-t_2}, \qquad \tilde{C} = \frac{2C_1}{\rho^2(t_2-t_1)^2}C_0^2, \qquad b = \alpha - \frac{1}{2}.$$

On the other hand, we let the equation evolve freely on J_N.

Now, we prove the existence of the constant L in (4.12).

It is clear that

$$\|f_N\|^2_{L^\infty_{\mathcal{F}}(t_{2N-1},t_{2N};L^2(\Omega;L^2(I)))} \leq \tilde{C}^{N(N-1)}\alpha_1\alpha_2\ldots\alpha_N\mathbb{E}\|\tilde{y}_0\|^2_{L^2(I)},$$

where $N > 1$, and $2^{\frac{2}{b}} \leq \bar{r}_1 < \bar{r}_2 < \ldots < \bar{r}_N \to \infty$ as $N \to \infty$. Moreover,we have

$$(\bar{r}_{N-1})^{\frac{b}{2}}(t_3-t_2)C_0^{-2(N-2)} - \xi(t_3-t_2)C_0^{-2(N-2)} \geq 2, \quad N \geq 2,$$

therefore

$$e^{-\left(2\bar{r}_{N-1}^{\alpha}-\xi\right)(t_3-t_2)C_0^{-2(N-2)}} \leq e^{-4\bar{r}_{N-1}^{\frac{b+1}{2}}}, \quad N \geq 2. \tag{4.13}$$

Note that

$$\tilde{C}^{N(N-1)}e^{-\bar{r}_{N-1}^{\frac{b+1}{2}}} = \frac{\tilde{C}^{N(N-1)}}{\left(e^{\bar{r}_{N-1}^{\frac{b}{2}}}\right)^{\bar{r}_{N-1}^{\frac{1}{2}}}} \leq \frac{\tilde{C}^{N(N-1)}}{\left(e^{2\tilde{C}^{N-2}}\right)^{\bar{r}_{N-1}^{\frac{1}{2}}}} \leq \frac{\tilde{C}^{N(N-1)}}{(\tilde{C}^{(N-2)2\bar{r}_{N-1}^{\frac{1}{2}}}}$$

and

$$e^{C_2\sqrt{\bar{r}_N}}e^{-\bar{r}_{N-1}^{\frac{b+1}{2}}} \leq C(b)e^{\tilde{C}^{\frac{N}{b}}-\tilde{C}^{\frac{N(b+1)}{b}}}$$

for each $N \geq 2$, where $C(b)$ is a positive constant depend on b. We derive from the definition of $\bar{r}_N$ that there exists a natural number N_0, such that for all $N \geq N_0$,

$$\tilde{C}^{N(N-1)}e^{-\bar{r}_{N-1}^{\frac{b+1}{2}}} \leq 1 \quad \text{and} \quad e^{C_2\sqrt{\bar{r}_N}}e^{-\bar{r}_{N-1}^{\frac{b+1}{2}}} \leq 1. \tag{4.14}$$

Combining (4.13) and (4.14), we see that for all $N \geq N_0$,

$$\begin{aligned} \tilde{C}^{N(N-1)}\alpha_N &= \tilde{C}^{N(N-1)}e^{C_2\sqrt{\bar{r}_N}}e^{(-2\bar{r}_{N-1}^{\alpha}+\xi)(t_3-t_2)C_0^{-2(N-2)}} \\ &\leq \tilde{C}^{N(N-1)}e^{C_2\sqrt{\bar{r}_N}}e^{-4\bar{r}_{N-1}^{\frac{b+1}{2}}} \\ &= \tilde{C}^{N(N-1)}e^{-\bar{r}_{N-1}^{\frac{b+1}{2}}} \cdot e^{C_2\sqrt{\bar{r}_N}}e^{-\bar{r}_{N-1}^{\frac{b+1}{2}}} \cdot e^{-2\bar{r}_{N-1}^{\frac{b+1}{2}}} \\ &\leq 1. \end{aligned}$$

Moreover, it is obviously that $\alpha_N \leq 1$ for any $N \geq N_0$. We set

$$L = \max\left\{\tilde{C}^{N(N-1)}\alpha_1\alpha_2\ldots\alpha_N, 1 \leq N \leq N_0\right\} < \infty.$$

Then we can obtain that

$$\|f_N\|^2_{L^\infty_{\mathcal{F}}(t_{2N-1},t_{2N};L^2(\Omega;L^2(I)))} \leq L\mathbb{E}\|\tilde{y}_0\|^2_{L^2(I)},$$

where $N \geq 1$. We now contruct $\tilde{f}$ by setting

$$\tilde{f} = \begin{cases} f_N(x,t) & x \in I, t \in K_N, N \geq 1, \\ 0 & x \in I, t \in J_N, N \geq 1. \end{cases}$$

It is clear that $\tilde{f} \in L^\infty_{\mathcal{F}}(t_1, \tilde{t}; L^2(\Omega; L^2(I)))$ and satisfies the estimate

$$\|\tilde{f}\|^2_{L^\infty_{\mathcal{F}}(t_1,\tilde{t};L^2(\Omega;L^2(I)))} \leq L\mathbb{E}\|\tilde{y}_0\|^2_{L^2(I)}.$$

Arguing as in [9,10], we can obtain $\tilde{y}(x,\tilde{t}) = 0$ in I, P-a.s.

Finally, construct a control f by setting

$$f = \begin{cases} 0 & \text{in } I \times (0, t_1), \\ \tilde{f}(x,t) & \text{in } I \times (t_1, \tilde{t}), \\ 0 & \text{in } I \times (\tilde{t}, T). \end{cases}$$

It is clear that $f \in L^\infty_{\mathcal{F}}(0,T; L^2(\Omega; L^2(I)))$ and that the corresponding solution y of (1.1) satisfies $y(T) = 0$ in I, P-a.s. Moreover, the control f satisfies the estimate

$$\|f\|^2_{L^\infty_{\mathcal{F}}(0,T;L^2(\Omega;L^2(I)))} \leq L\mathbb{E}\|y_0\|^2_{L^2(I)}. \qquad \square$$

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References

[1] G. Da Prato and J. Zabczyk, Stochastic Equations in Infinite Dimensions. Cambridge University Press, Cambridge (1992).

[2] Y. Hu and S. Peng, Adapted solution of backward stochastic evolution equations. *Stoch. Anal. Appl.* **9** (1991) 445–459.

[3] D. Jerison and G. Lebeau, Nodal sets of sums of eigenfunctions, in: Harmonic Analysis and Partial Differential Equations. *Chicago Lectures in Math.* Univ. Chicago Press, Chicago, IL (1999) 223–239.

[4] G. Lebeau and L. Robbiano, Contrôle exact de l'équation de la chaleur. *Commun. Partial Differ. Equ.* **20** (1995) 335–336.

[5] G. Lebeau and E. Zuazua, Null controllability of a system of linear thermoelasticity. *Arch. Rational Mech. Anal.* **141** (1998) 297–329.

[6] J.L. Lions, Optimal Control of Systems Governed by Partial Differential Equations. Springer-Verlag, Berlin, Heidelberg, New York (1971).

[7] Q. Lü, *Observation and control for stochastic partial differential equations.* Ph.D. thesis, School of Mathematics. Sichuan University, Chengdu (2010).

[8] Q. Lü, Bang-bang principle of time optimal controls and null controllability of fractional order parabolic equations. *Acta Math. Sin.* (*Engl. Ser.*) **26** (2010) 2377–2386.

[9] Q. Lü, Some results on the controllability of forward stochastic heat equations with control on the drift. *J. Funct. Anal.* **260** (2011) 832–851.

[10] Q. Lü and G. Wang, On the existence of time optimal controls with constraints of the rectangular type for heat equations. *SIAM J. Control Optim.* **49** (2011) 1124–1149.

[11] Q. Lü and Z. Yin, The L^∞-null controllability of parabolic equation with equivalued surface boundary conditions. *Asymptot. Anal.* **83** (2013) 355–378.

[12] N.I. Mahmudov and M.A. McKibben, On backward stochastic evolution equations in Hilbert spaces and optimal control. *Nonlinear Anal.* **67** (2007) 1260–1274.

[13] M. Renardy and R.C. Rogers, An Introduction to Partial Differential Equations, 2nd edn. Vol. 13 of *Texts in Applied Mathematics*. Springer-Verlag, New York (2004).

[14] G. Wang, L^∞-null controllability for the heat equation and its consequences for the time optimal control problem. *SIAM J. Control Optim.* **47** (2008) 1701–1720.

[15] Z.C. Zhou, Observability estimate and null controllability for one-dimensional fourth order parabolic equation. *Taiwanese J. Math.* **16** (2012) 1991–2017.