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ZERO DYNAMICS AND FUNNEL CONTROL OF GENERAL LINEAR DIFFERENTIAL-ALGEBRAIC SYSTEMS *

Thomas Berger[1]

Abstract. We study linear differential-algebraic multi-input multi-output systems which are not necessarily regular and investigate the zero dynamics and tracking control. We introduce and characterize the concept of autonomous zero dynamics as an important system theoretic tool for the analysis of differential-algebraic systems. We use the autonomous zero dynamics and (E, A, B)-invariant subspaces to derive the so called zero dynamics form – which decouples the zero dynamics of the system – and exploit it for the characterization of system invertibility and asymptotic stability of the zero dynamics. A refinement of the zero dynamics form is then used to show that the funnel controller (that is a static nonlinear output error feedback) achieves – for a special class of right-invertible systems with asymptotically stable zero dynamics – tracking of a reference signal by the output signal within a pre-specified performance funnel. It is shown that the results can be applied to a class of passive electrical networks.

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1. Introduction

Differential-algebraic equations (DAEs) are a combination of differential equations along with algebraic constraints. They have been discovered as an appropriate tool for modeling many problems *e.g.* in mechanical multibody dynamics [19], electrical networks [39], and chemical engineering [30]. These problems indeed have in common that the dynamics are algebraically constrained, for instance by tracks, Kirchhoff laws, or conservation laws. As a result of the power in application, DAEs are nowadays an established field in applied mathematics and subject of various monographs and textbooks, see *e.g.* [15,31,32]. In the present work, we consider questions related to the zero dynamics, system inversion, and closed-loop control of linear constant coefficient DAEs with special emphasis on the non-regular case. The concepts of (E, A, B)-invariance, autonomous and asymptotically stable zero dynamics, left- and right-invertibility are considered for the DAE case. We further show that the "funnel controller" (developed in [26] for minimum-phase ordinary differential equation systems with strict relative degree one) achieves, for all right-invertible DAE systems with asymptotically stable zero dynamics which satisfy a certain relative degree assumption, tracking of a reference signal by the output signal within

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[1] Universität Hamburg, Fachbereich Mathematik, Bundesstraße 55, 20146 Hamburg, Germany. thomas.berger@uni-hamburg.de

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a pre-specified performance funnel. We stress that knowledge of specific system parameters is not required for feasibility of funnel control.

We consider linear constant coefficient DAEs of the form

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} Ex(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t), \end{aligned} \tag{1.1}$$

where $E, A \in \mathbb{R}^{l\times n}$, $B \in \mathbb{R}^{l\times m}$, $C \in \mathbb{R}^{p\times n}$. The set of these systems is denoted by $\Sigma_{l,n,m,p}$ and we write $[E, A, B, C] \in \Sigma_{l,n,m,p}$. In the present paper, we put special emphasis on the non-regular case, *i.e.*, we do not assume that $sE - A$ is *regular*, that is $l = n$ and $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$. Note that non-regular DAE systems actually appear in several "real world" applications, see *e.g.* [18].

The functions $u : \mathbb{R} \to \mathbb{R}^m$ and $y : \mathbb{R} \to \mathbb{R}^p$ are called *input* and *output* of the system, resp. A trajectory $(x, u, y) : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$ is said to be a *solution* of (1.1) if, and only if, it belongs to the *behavior* of (1.1):

$$\mathfrak{B}_{(1.1)} := \left\{ (x, u, y) \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p) \;\middle|\; \begin{array}{l} Ex \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^l) \text{ and } (x, u, y) \\ \text{solves (1.1) for almost all } t \in \mathbb{R} \end{array} \right\}.$$

Recall that any function $z \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^l)$ is in particular continuous. More smoothness of u and y is required for some results such as funnel control in Section 5.

In the present paper, we provide, in particular, a unified framework for two important classes of differential-algebraic systems which have been investigated in earlier work [10, 11]. These two classes encompass regular systems $[E, A, B, C] \in \Sigma_{n,n,m,m}$ for which the transfer function is defined by

$$G(s) = C(sE - A)^{-1}B \in \mathbb{R}(s)^{m\times m}.$$

The notions of properness and strict relative degree are required in the following.

Definition 1.1 (Properness and strict relative degree)**.** A rational matrix function $G(s) \in \mathbb{R}(s)^{p\times m}$ is called *proper* if $\lim_{s\to\infty} G(s) = D$ for some $D \in \mathbb{R}^{p\times m}$.

We say that a square matrix function $G(s) \in \mathbb{R}(s)^{m\times m}$ has *strict relative degree* ρ if

$$\rho \;=\; \operatorname{sr\,deg} G(s) := \sup \left\{ k \in \mathbb{Z} \;\middle|\; \lim_{s\to\infty} s^k\, G(s) \in \mathbf{Gl}_m(\mathbb{R}) \right\} \in \mathbb{Z}.$$

Note that by ([11], Prop. 1.2) any transfer function with nonnegative strict relative degree has a proper inverse over $\mathbb{R}(s)$, but the converse is false in general.

In the earlier work [10, 11] the following two system classes have been considered: The class of regular systems with proper inverse transfer function has been investigated in [11]; the class of regular systems with strict relative degree one has been investigated in [10]. These two classes are distinct and have been treated by different approaches in [10, 11]. In the present paper, we consider the more general class Σ of DAE systems which in particular contains the aforementioned two classes of systems: Σ contains all right-invertible systems (see Def. 4.1) with autonomous zero dynamics (see Def. 3.1), which satisfy a certain relative degree assumption (see (5.3)).

The main difference between the class Σ and the classes considered in [10, 11] is that the pencil $sE - A$ is not necessarily regular and hence a transfer function $G(s)$ does not exist in general. Therefore, a new approach is required since the work in [10, 11] is based on the existence of a transfer function. Within the framework of the class Σ introduced in the present paper it is possible to obtain a generalization of the *inverse* transfer function, see Remark A.4. We also show that the class Σ includes all regular systems with a vector relative degree which is componentwise smaller or equal to 1, see Appendix B. This in particular encompasses systems with a "mixed relative degree", *i.e.*, a vector relative degree with possibly different components. Remark 5.5 also shows that a class of passive electrical networks is encompassed: systems with invertible and positive real transfer function are included in Σ. We use the class Σ to show that funnel control is feasible for a much larger class of systems than considered in [26] for ODEs and in [10, 11] for DAEs.

The paper is organized as follows: in Section 2 we collect some preliminary results on matrix pencils, in particular the quasi-Kronecker form. In Section 3 we recall the crucial concept of (autonomous) zero dynamics and derive characterizations of autonomous zero dynamics in terms of a rank condition and the maximal (E, A, B)-invariant subspace included in $\ker C$. The latter also allows to derive the so called zero dynamics form in Theorem 3.6 - one of the main results of the paper - which decouples the zero dynamics of the system. Asymptotic stability of zero dynamics is defined and characterized as well in Section 3. The zero dynamics form is then refined in Section 4 and exploited for the characterization of system invertibility. The refinement of the zero dynamics form is also used to show feasibility of the funnel controller in Section 5, which is proved to work for the class Σ (where additionally asymptotically stable zero dynamics are required) in Theorem 5.3 – the second main result of the present paper. In Section 6 we illustrate Theorem 5.3 by a simulation of the funnel controller for a system (1.1). Finally, in Appendix A some results on polynomial matrices and the zero dynamics form are derived, which are crucial for the proof of Theorem 5.3, and in Appendix B systems with a vector relative degree are related to the findings of the paper.

We close the introduction with the nomenclature used in the present paper.

Nomenclature

$\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$		set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, set of all integers, resp.
$\ell(\alpha), \lvert\alpha\rvert$		length $\ell(\alpha) = l$ and absolute value $\lvert\alpha\rvert = \sum_{i=1}^{l} \alpha_i$ of a multi-index $\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{N}^l$
$\mathbb{R}_{\geq 0}$	$=$	$[0, \infty)$
$\mathbb{C}_+$, $\mathbb{C}_-$		the open set of complex numbers with positive, negative real part, resp.
$\mathbf{Gl}_n(\mathbb{R})$		the set of invertible real $n \times n$ matrices
$\mathbb{R}[s]$		the ring of polynomials with coefficients in $\mathbb{R}$
$\mathbb{R}(s)$		the quotient field of $\mathbb{R}[s]$
$R^{n \times m}$		the set of $n \times m$ matrices with entries in a ring R
$\sigma(A)$		the spectrum of a matrix $A \in \mathbb{R}^{n \times n}$
$\lVert x \rVert$	$=$	$\sqrt{x^\top x}$, the Euclidean norm of $x \in \mathbb{R}^n$
$\lVert A \rVert$	$=$	$\max\{\, \lVert A x \rVert \mid x \in \mathbb{R}^m, \lVert x \rVert = 1 \,\}$, induced matrix norm of $A \in \mathbb{R}^{n \times m}$
$A^{-1}\mathcal{S}$	$=$	$\{\, x \in \mathbb{R}^m \mid Ax \in \mathcal{S} \,\}$, the pre-image of the set $\mathcal{S} \subseteq \mathbb{R}^n$ under $A \in \mathbb{R}^{n \times m}$
$\mathcal{L}^1_{\mathrm{loc}}(I; \mathbb{R}^n)$		the set of locally Lebesgue integrable functions $f : I \to \mathbb{R}^n$, where $\int_K \lVert f(t) \rVert \, \mathrm{d}t < \infty$ for all compact $K \subseteq I$ and $I \subseteq \mathbb{R}$ is an interval
$\dot f$ $(f^{(i)})$		the (ith) weak derivative of $f \in \mathcal{L}^1_{\mathrm{loc}}(I; \mathbb{R}^n)$, $i \in \mathbb{N}_0$ (see [1], Chap. 1)
$\mathcal{W}^{k,1}_{\mathrm{loc}}(I; \mathbb{R}^n)$	$=$	$\left\{\, f \in \mathcal{L}^1_{\mathrm{loc}}(I; \mathbb{R}^n) \;\middle|\; f^{(i)} \in \mathcal{L}^1_{\mathrm{loc}}(I; \mathbb{R}^n) \text{ for } i = 0, \ldots, k \,\right\}$, $k \in \mathbb{N}_0 \cup \{\infty\}$
$\mathcal{L}^\infty(I; \mathbb{R}^n)$		the set of essentially bounded functions $f : I \to \mathbb{R}^n$ (see [1], Chap. 2)
$\text{ess-sup}_J \lVert f \rVert$		the essential supremum of the measurable function $f : I \to \mathbb{R}^n$ over $J \subseteq I$
$\mathcal{C}^k(I; \mathbb{R}^n)$		the set of k-times continuously differentiable functions $f : I \to \mathbb{R}^n$, $k \in \mathbb{N}_0 \cup \{\infty\}$
$\mathcal{B}^k(I; \mathbb{R}^n)$	$=$	$\left\{\, f \in \mathcal{C}^k(I; \mathbb{R}^n) \;\middle|\; \frac{\mathrm{d}^i}{\mathrm{d}t^i} f \in \mathcal{L}^\infty(I; \mathbb{R}^n) \text{ for } i = 0, \ldots, k \,\right\}$, $k \in \mathbb{N}_0 \cup \{\infty\}$
$f\vert_J$		the restriction of the function $f : I \to \mathbb{R}^n$ to $J \subseteq I$

2. Preliminaries

For convenience we call the extended matrix pencil $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ the *system pencil* of $[E,A,B,C]\in\Sigma_{l,n,m,p}$. In Section 3 we will derive a so called "zero dynamics form" of $[E,A,B,C]$ within the equivalence class defined by:

Definition 2.1 (System equivalence)**.** Two systems $[E_i,A_i,B_i,C_i]\in\Sigma_{l,n,m,p}$, $i=1,2$, are called *system equivalent* if

$$\exists\, S\in \mathbf{Gl}_l(\mathbb{R}), T\in\mathbf{Gl}_n(\mathbb{R}):\ \begin{bmatrix} S & 0\\ 0 & I_p\end{bmatrix}\begin{bmatrix} sE_1-A_1 & B_1\\ C_1 & 0\end{bmatrix}\begin{bmatrix} T & 0\\ 0 & I_m\end{bmatrix} = \begin{bmatrix} sE_2-A_2 & B_2\\ C_2 & 0\end{bmatrix};$$

we write

$$[\,E_1,\,A_1,\,B_1,\,C_1\,] \overset{S,T}{\sim} [\,E_2,\,A_2,\,B_2,\,C_2\,].$$

The notion of system equivalence goes back to Rosenbrock [40], see also the survey [9] and the references therein.

We introduce the following notation: For $k\in\mathbb{N}$, we define the matrices

$$N_k=\begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & \ddots \\ & & 1 \;\; 0\end{bmatrix}\in\mathbb{R}^{k\times k},\quad K_k=\begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0\end{bmatrix},\ L_k=\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1\end{bmatrix}\in\mathbb{R}^{(k-1)\times k}.$$

For some multi-index $\alpha=(\alpha_1,\ldots,\alpha_l)\in\mathbb{N}^l$, we define

$$N_\alpha=\operatorname{diag}(N_{\alpha_1},\ldots,N_{\alpha_l})\in\mathbb{R}^{|\alpha|\times|\alpha|},\quad K_\alpha=\operatorname{diag}(K_{\alpha_1},\ldots,K_{\alpha_l}),\ L_\alpha=\operatorname{diag}(L_{\alpha_1},\ldots,L_{\alpha_l})\in\mathbb{R}^{(|\alpha|-l)\times|\alpha|}. \quad (2.1)$$

We use the quasi-Kronecker form [6, 7], which is a version of the Kronecker canonical form [21] where all involved matrices are real-valued.

Proposition 2.2 (Quasi-Kronecker form)**.** *For any matrix pencil $s\widehat{E}-\widehat{A}\in\mathbb{R}[s]^{\widehat{l}\times\widehat{n}}$ there exist $S\in\mathbf{Gl}_{\widehat{l}}(\mathbb{R})$, $T\in\mathbf{Gl}_{\widehat{n}}(\mathbb{R})$, $A_s\in\mathbb{R}^{n_s\times n_s}$, and $\alpha\in\mathbb{N}^{n_\alpha}$, $\beta\in\mathbb{N}^{n_\gamma}$, $\gamma\in\mathbb{N}^{n_\gamma}$ such that*

$$S(s\widehat{E}-\widehat{A})T \;=\; \begin{bmatrix} sI_{n_s}-A_s & 0 & 0 & 0\\ 0 & sN_\alpha - I_{|\alpha|} & 0 & 0\\ 0 & 0 & sK_\beta - L_\beta & 0\\ 0 & 0 & 0 & sK_\gamma^\top - L_\gamma^\top\end{bmatrix}. \quad (2.2)$$

The multi-indices α,β,γ are unique up to a permutation of their respective entries. Further, the matrix A_s is unique up to similarity.

For a discussion of the blocks in (2.2) and their solution behavior we refer to the survey [45], see also the textbooks [31, 32].

Since each block in $sK_\beta-L_\beta$ ($sK_\gamma^\top-L_\gamma^\top$) causes a single drop of the column (row) rank of $sE-A$, resp., we have

$$\ell(\beta)=\widehat{n}-\operatorname{rk}_{\mathbb{R}(s)}\left(s\widehat{E}-\widehat{A}\right),\quad \ell(\gamma)=\widehat{l}-\operatorname{rk}_{\mathbb{R}(s)}\left(s\widehat{E}-\widehat{A}\right). \quad (2.3)$$

For later use we collect the following lemma.

Lemma 2.3 (Full column rank and quasi-Kronecker form)**.** *Let $s\widehat{E}-\widehat{A}\in\mathbb{R}[s]^{\widehat{l}\times\widehat{n}}$ and consider any quasi-Kronecker form (2.2) of $s\widehat{E}-\widehat{A}$. Then $\ell(\beta)=0$ if, and only if, $\operatorname{rk}_{\mathbb{R}[s]} s\widehat{E}-\widehat{A}=\widehat{n}$.*

Proof. The assertion is immediate from (2.3) and $\operatorname{rk}_{\mathbb{R}[s]} s\widehat{E}-\widehat{A}=\operatorname{rk}_{\mathbb{R}(s)} s\widehat{E}-\widehat{A}$. □

3. Zero dynamics

In this section we recall the crucial concept of zero dynamics for DAE systems (1.1), which has been introduced by Byrnes and Isidori [17], as well as the notion of autonomous zero dynamics, which seem to have their first appearance in [48]. We derive several important characterizations of autonomous zero dynamics and, as the main result of this section, the so called zero dynamics form in Theorem 3.6.

Definition 3.1 (Zero dynamics)**.** The *zero dynamics* of system (1.1) are defined as the set of trajectories

$$\mathcal{ZD}_{(1.1)} := \left\{ (x,u,y) \in \mathfrak{B}_{(1.1)} \;\middle|\; y \overset{\text{a.e.}}{=} 0 \right\}.$$

The zero dynamics $\mathcal{Z}D_{(1.1)}$ are called *autonomous* if

$$\forall\, w \in \mathcal{Z}D_{(1.1)}\ \forall\, I \subseteq \mathbb{R} \text{ open interval}: \quad w|_I \overset{\text{a.e.}}{=} 0 \quad \Longrightarrow \quad w \overset{\text{a.e.}}{=} 0. \tag{3.1}$$

The definition of autonomous zero dynamics is a special case of the definition of autonomy, as it has been introduced in ([36], Sect. 3.2) for general behaviors. In order to characterize (autonomous) zero dynamics we introduce the well-known concept of (E,A,B)-invariance, see [3,16,33,35].

Definition 3.2 ((E,A,B)-invariance)**.** Let $(E,A,B) \in \mathbb{R}^{l\times n} \times \mathbb{R}^{l\times n} \times \mathbb{R}^{l\times m}$ and $\mathcal{V} \subseteq \mathbb{R}^n$ be a linear subspace. Then $\mathcal{V}$ is called (E,A,B)*-invariant* if

$$A\mathcal{V} \subseteq E\mathcal{V} + \operatorname{im} B. \tag{3.2}$$

For a system $[E,A,B,C] \in \Sigma_{l,n,m,p}$, we denote the maximal (E,A,B)-invariant subspace included in $\ker C$ (with respect to subspace inclusion) with $\max(E,A,B;\ker C)$. This space can be derived from a sequence of subspaces which terminates after finitely many steps.

Lemma 3.3 (Subspace sequences leading to $\max(E,A,B;\ker C)$)**.** *Let* $[E,A,B,C] \in \Sigma_{l,n,m,p}$ *and define* $\mathcal{V}_0 := \ker C$ *and*

$$\forall\, i \in \mathbb{N}: \ \mathcal{V}_i := A^{-1}(E\mathcal{V}_{i-1} + \operatorname{im} B) \cap \ker C.$$

Then the sequence $(\mathcal{V}_i)$ *is nested, terminates and satisfies*

$$\exists\, k^* \in \mathbb{N}\ \forall\, j \in \mathbb{N}: \ \mathcal{V}_0 \supsetneq \mathcal{V}_1 \supsetneq \cdots \supsetneq \mathcal{V}_{k^*} = \mathcal{V}_{k^*+j} = A^{-1}(E\mathcal{V}_{k^*} + \operatorname{im} B) \cap \ker C. \tag{3.3}$$

Furthermore,

$$\mathcal{V}_{k^*} = \max(E,A,B;\ker C) \tag{3.4}$$

and, if $(x,u,y) \in \mathcal{ZD}_{(1.1)}$*, then* $x(t) \in \mathcal{V}_{k^*}$ *for almost all* $t \in \mathbb{R}$.

Proof. It is easy to see that (3.3) holds true and (3.4) follows from ([35], Lem. 2.1). For the last statement let $(x,u,y) \in \mathcal{ZD}_{(1.1)}$. Then we have

$$Ax(t) = \tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) - Bu(t) \quad \text{and} \quad x(t) \in \ker C$$

for almost all $t \in \mathbb{R}$. Since, for any subspace $\mathcal{S} \subseteq \mathbb{R}^n$, if $x(t) \in \mathcal{S}$ for almost all $t \in \mathbb{R}$, then $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) \in E\mathcal{S}$ for almost all $t \in \mathbb{R}$, we conclude

$$x(t) \in A^{-1}(\{\tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t)\} + \operatorname{im} B) \cap \ker C \subseteq \mathcal{V}_1 \text{ for almost all } t \in \mathbb{R}.$$

Inductively, we obtain $x(t) \in \mathcal{V}_{k^*}$ for almost all $t \in \mathbb{R}$. □

The following result is a general version of ([11], Prop. 4.3), which follows immediately from Lemma 3.3.

Proposition 3.4 (Characterization of zero dynamics)**.** *Let* $[E, A, B, C] \in \Sigma_{l,n,m,p}$*. If* $(x, u, y) \in \mathfrak{B}_{(1.1)}$*, then*

$$(x, u, y) \in \mathcal{ZD}_{(1.1)} \quad \iff \quad \Big[x(t) \in \max(E, A, B; \ker C) \quad \textit{for almost all } t \in \mathbb{R}\Big].$$

Next, we state some characterizations of autonomous zero dynamics in terms of a pencil rank condition (exploiting the quasi-Kronecker form) and some conditions involving the largest (E, A, B)-invariant subspace included in $\ker C$.

Proposition 3.5 (Characterization of autonomous zero dynamics)**.** *Let* $[E, A, B, C] \in \Sigma_{l,n,m,p}$*. Then the following three statements are equivalent:*

(i) $\mathcal{Z}D_{(1.1)}$ *is autonomous.*
(ii) $\operatorname{rk}_{\mathbb{R}[s]} \left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right] = n + m.$
(iii) **(A1)** $\operatorname{rk} B = m,$

(A2) $\ker E \cap \max(E, A, B; \ker C) = \{0\},$

(A3) $\operatorname{im} B \cap E \max(E, A, B; \ker C) = \{0\}.$

Proof. In view of Proposition 2.2, there exist $S \in \mathbf{Gl}_{l+p}(\mathbb{R}), T \in \mathbf{Gl}_{n+m}(\mathbb{R})$ such that (using the matrices defined in (2.1))

$$S \begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix} T = \begin{bmatrix} sI_{n_s} - A_s & 0 & 0 & 0 \\ 0 & sN_\alpha - I_{|\alpha|} & 0 & 0 \\ 0 & 0 & sK_\beta - L_\beta & 0 \\ 0 & 0 & 0 & sK_\gamma^\top - L_\gamma^\top \end{bmatrix} \tag{3.5}$$

(i)⇒(ii): Suppose that (ii) does not hold. Then Lemma 2.3 yields $\ell(\beta) > 0$. Therefore, we find $z \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{|\beta|}) \setminus \{0\}$ and $I \subseteq \mathbb{R}$ open interval such that $z|_I = 0$ and $(\frac{\mathrm{d}}{\mathrm{d}t} K_\beta - L_\beta)z = 0$. This implies that

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}E - A & -B \\ -C & 0 \end{bmatrix} T\,(0, 0, z^\top, 0)^\top = 0,$$

which contradicts autonomous zero dynamics.

(ii)⇒(i): By (ii) and Lemma 2.3 it follows that $\ell(\beta) = 0$ in (3.5). Let $w \in \mathcal{Z}D_{(1.1)}$ and $I \subseteq \mathbb{R}$ be an open interval such that $w|_I \overset{\text{a.e.}}{=} 0$. Then, with $(v_1^\top, v_2^\top, v_3^\top)^\top = T^{-1}w$, we have

$$S^{-1} \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} I_{n_s} - A_s & 0 & 0 \\ 0 & \frac{\mathrm{d}}{\mathrm{d}t} N_\alpha - I_{|\alpha|} & 0 \\ 0 & 0 & \frac{\mathrm{d}}{\mathrm{d}t} K_\gamma^\top - L_\gamma^\top \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \overset{\text{a.e.}}{=} \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}E - A & -B \\ -C & 0 \end{bmatrix} w \overset{\text{a.e.}}{=} 0,$$

and thus $(\frac{\mathrm{d}}{\mathrm{d}t} I_{n_s} - A_s)v_1 \overset{\text{a.e.}}{=} 0$, $(\frac{\mathrm{d}}{\mathrm{d}t} N_\alpha - I_{|\alpha|})v_2 \overset{\text{a.e.}}{=} 0$, and $(\frac{\mathrm{d}}{\mathrm{d}t} K_\gamma^\top - L_\gamma^\top)v_3 \overset{\text{a.e.}}{=} 0$. Then, successively solving each block in $(\frac{\mathrm{d}}{\mathrm{d}t} N_\alpha - I_{|\alpha|})v_2 \overset{\text{a.e.}}{=} 0$ and $(\frac{\mathrm{d}}{\mathrm{d}t} K_\gamma^\top - L_\gamma^\top)v_3 \overset{\text{a.e.}}{=} 0$ gives $v_2 \overset{\text{a.e.}}{=} 0$ and $v_3 \overset{\text{a.e.}}{=} 0$. Since $v_1|_I \overset{\text{a.e.}}{=} 0$ it follows that $v_1 \overset{\text{a.e.}}{=} 0$. So we may conclude that $w \overset{\text{a.e.}}{=} 0$, by which the zero dynamics are autonomous.

(i)⇒(iii): Let $V \in \mathbb{R}^{n \times k}$ with full column rank such that $\operatorname{im} V = \max(E, A, B; \ker C)$. By definition of $\max(E, A, B; \ker C)$ there exist $N \in \mathbb{R}^{k \times k}, M \in \mathbb{R}^{m \times k}$ such that $AV = EVN + BM$ and $CV = 0$. Therefore, we have

$$\left(s \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}\right) \begin{bmatrix} V & 0 \\ -M & -I_m \end{bmatrix} = \begin{bmatrix} EV(sI_k - N) & B \\ 0 & 0 \end{bmatrix}.$$

By (ii) we find $s_0 \in \mathbb{C}$ such that $\left[\begin{smallmatrix} s_0E-A & -B \\ -C & 0 \end{smallmatrix}\right]$ has full column rank and $s_0I_k - N$ is invertible. We show that $[EV, B]$ has full column rank. Let $x \in \mathbb{R}^k$, $u \in \mathbb{R}^m$ be such that $EVx + Bu = 0$. Then

$$\begin{bmatrix} s_0E - A & -B \\ -C & 0 \end{bmatrix} \begin{bmatrix} V & 0 \\ -M & -I_m \end{bmatrix} \begin{pmatrix} (s_0I_k - N)^{-1}x \\ u \end{pmatrix} = \begin{bmatrix} EV(s_0I_k - N) & B \\ 0 & 0 \end{bmatrix} \begin{pmatrix} (s_0I_k - N)^{-1}x \\ u \end{pmatrix} = 0,$$

thus $\begin{bmatrix} V & 0 \\ -M & -I_m \end{bmatrix} \begin{pmatrix} (s_0I_k - N)^{-1}x \\ u \end{pmatrix} = 0$ and full column rank of V gives $x = 0$ and $u = 0$. Now $\operatorname{rk}[EV, B] = k+m$ clearly implies (A1)–(A3).

(iii)⇒(i): Let $(x, u, y) \in \mathcal{ZD}_{(1.1)}$ and $I \subseteq \mathbb{R}$ an open interval such that $(x, u)|_I \overset{\text{a.e.}}{=} 0$. Choose $W \in \mathbb{R}^{n\times(n-k)}$ such that $[V, W] \in \mathbf{Gl}_n(\mathbb{R})$. Applying the coordinate transformation $(z_1^\top, z_2^\top)^\top = [V, W]^{-1}x$ and observing that, by Proposition 3.4, $x(t) \in \operatorname{im} V$ for almost all $t \in \mathbb{R}$, it follows $Wz_2(t) = x(t) - Vz_1(t) \in \operatorname{im} W \cap \operatorname{im} V = \{0\}$ for almost all $t \in \mathbb{R}$ and hence $z_2 \overset{\text{a.e.}}{=} 0$. By (A1)–(A3) we have that $\operatorname{rk}[EV, B] = k+m$, thus there exists $S \in \mathbf{Gl}_l(\mathbb{R})$ such that

$$S[EV, B] = S \begin{bmatrix} I_k & 0 \\ 0 & I_m \\ 0 & 0 \end{bmatrix}, \quad S[EW, AV] = \begin{bmatrix} E_1 & A_1 \\ E_2 & A_2 \\ E_3 & A_3 \end{bmatrix}.$$

Then $Ex \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R}; \mathbb{R}^l)$ implies that $w := z_1 + E_1z_2 \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R}; \mathbb{R}^k)$ and we have $w \overset{\text{a.e.}}{=} z_1$. Furthermore, $A_iz_1 \overset{\text{a.e.}}{=} A_iw$, $i = 1, 2, 3$, and since $\left[\begin{smallmatrix} E_2 \\ E_3 \end{smallmatrix}\right] z_2 \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R}; \mathbb{R}^{n-k})$ we find $\frac{\text{d}}{\text{d}t} \left[\begin{smallmatrix} E_2 \\ E_3 \end{smallmatrix}\right] z_2 \overset{\text{a.e.}}{=} 0$. Then we obtain

$$\begin{aligned} \tfrac{\text{d}}{\text{d}t}Ex(t) \overset{\text{a.e.}}{=} Ax(t) + Bu(t) \iff& \tfrac{\text{d}}{\text{d}t}S(EVz_1(t) + EWz_2(t)) - SBu(t) \overset{\text{a.e.}}{=} SA(Vz_1(t) + Wz_2(t)) \overset{\text{a.e.}}{=} SAVz_1(t) \\ \iff& \begin{pmatrix} \frac{\text{d}}{\text{d}t}w(t) \\ u(t) \\ 0 \end{pmatrix} \overset{\text{a.e.}}{=} \begin{pmatrix} \frac{\text{d}}{\text{d}t}w(t) \\ \frac{\text{d}}{\text{d}t}E_2z_2(t) + u(t) \\ \frac{\text{d}}{\text{d}t}E_3z_2(t) \end{pmatrix} \overset{\text{a.e.}}{=} \begin{pmatrix} A_1w(t) \\ A_2w(t) \\ A_3w(t) \end{pmatrix}. \end{aligned}$$

hence $w|_I \overset{\text{a.e.}}{=} z_1|_I$ and $Vz_1|_I \overset{\text{a.e.}}{=} x|_I \overset{\text{a.e.}}{=} 0$ together with $\operatorname{rk} V = k$ give $w = 0$ and therefore $(x, u) \overset{\text{a.e.}}{=} 0$. □

The characterization in Proposition 3.5 was observed for ODE systems $(I, A, B, C) \in \Sigma_{n,n,m,m}$ by Ilchmann and Wirth (personal communication, July 4, 2012). The following zero dynamics form for systems with autonomous zero dynamics in Theorem 3.6 was derived for ODE systems (I, A, B, C) by Isidori ([28], Rem. 6.1.3); however, in [28] it is not clear that the assumptions (A1), (A3) are equivalent to autonomous zero dynamics (note that (A2) is superfluous for ODEs). A zero dynamics form for time-varying ODE systems has been derived in [14] and it seems that a zero dynamics form for time-varying DAEs can be obtained from a combination of the results in [14] with the following theorem.

Theorem 3.6 (Zero dynamics form). *Consider $[E, A, B, C] \in \Sigma_{l,n,m,p}$ and suppose that the zero dynamics $\mathcal{ZD}_{(1.1)}$ are autonomous. Let $V \in \mathbb{R}^{n\times k}$ be such that $\operatorname{im} V = \max(E, A, B; \ker C)$ and $\operatorname{rk} V = k$. Then there exist $W \in \mathbb{R}^{n\times(n-k)}$ and $S \in \mathbf{Gl}_l(\mathbb{R})$ such that $[V, W] \in \mathbf{Gl}_n(\mathbb{R})$ and*

$$[E, A, B, C] \overset{S,[V,W]}{\sim} \left[\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}\right], \tag{3.6}$$

where

$$\tilde{E} = \begin{bmatrix} I_k & E_2 \\ 0 & E_4 \\ 0 & E_6 \end{bmatrix}, \ \tilde{A} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \\ 0 & A_6 \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} 0 \\ I_m \\ 0 \end{bmatrix}, \ \tilde{C} = [0, C_2] \tag{3.7}$$

such that

$$\max\left(\begin{bmatrix}E_4\\E_6\end{bmatrix},\begin{bmatrix}A_4\\A_6\end{bmatrix},\begin{bmatrix}I_m\\0\end{bmatrix},C_2\right)=\{0\},\tag{3.8}$$

and $A_1 \in \mathbb{R}^{k\times k}$, $E_2 \in \mathbb{R}^{k\times(n-k)}$, $A_2 \in \mathbb{R}^{k\times(n-k)}$, $A_3 \in \mathbb{R}^{m\times k}$ $E_4 \in \mathbb{R}^{m\times(n-k)}$, $A_4 \in \mathbb{R}^{m\times(n-k)}$, $E_6 \in \mathbb{R}^{(l-k-m)\times(n-k)}$, $A_6 \in \mathbb{R}^{(l-k-m)\times(n-k)}$, $C_2 \in \mathbb{R}^{p\times(n-k)}$.

For uniqueness we have: If $[E,A,B,C]$, $[\hat{E},\hat{A},\hat{B},\hat{C}] \in \Sigma_{l,n,m,p}$ *are in the form* (3.7) *such that* (3.8) *holds, and*

$$[E,A,B,C] \overset{S,T}{\sim} \left[\hat{E},\hat{A},\hat{B},\hat{C}\right] \quad \textit{for some } S\in \mathbf{Gl}_l(\mathbb{R}), T\in\mathbf{Gl}_n(\mathbb{R}),\tag{3.9}$$

then

$$S=\begin{bmatrix}S_1&0&S_3\\0&I_m&S_6\\0&0&S_9\end{bmatrix},\quad T=\begin{bmatrix}S_1^{-1}&T_2\\0&T_4\end{bmatrix},$$

where $S_1\in\mathbf{Gl}_k(\mathbb{R}), S_9\in\mathbf{Gl}_{l-k-m}(\mathbb{R}), T_4\in\mathbf{Gl}_{n-k}(\mathbb{R})$ *and* S_3,S_6,T_2 *are of appropriate sizes. In particular the dimensions of the matrices in* (3.7) *are unique and* A_1 *is unique up to similarity,* i.e., $\sigma(A_1)$ *is unique.*

Proof.

Step 1: We prove (3.6) and (3.7). By Proposition 3.5, autonomous zero dynamics are equivalent to the conditions (A1)–(A3). These conditions imply that $k+m\le l$. Then we may find $W\in\mathbb{R}^{n\times(n-k)}$ such that $[V,W]\in\mathbf{Gl}_n(\mathbb{R})$. Considering the transformed system $\big(E[V,W],A[V,W],B,C[V,W]\big)$, we find that $CV=0$, since $\operatorname{im} V\subseteq\ker C$. Further observe that EV has full column rank by (A2) and, since B has full column rank by (A1) and $\operatorname{im} EV\cap \operatorname{im} B=\{0\}$ by (A3), we obtain that $[EV,B]$ has full column rank. Hence, we find $S\in\mathbf{Gl}_l(\mathbb{R})$ such that $S\,[EV,B]=\left[\begin{smallmatrix}I_k&0\\0&I_m\\0&0\end{smallmatrix}\right]$. Therefore,

$$[E,A,B,C] \overset{S,[V,W]}{\sim} \big(SE[V,W],SA[V,W],SB,C[V,W]\big)=\left[\begin{bmatrix}I_k&E_2\\0&E_4\\0&E_6\end{bmatrix},\begin{bmatrix}A_1&A_2\\A_3&A_4\\A_5&A_6\end{bmatrix},\begin{bmatrix}0\\I_m\\0\end{bmatrix},[0,C_2]\right].$$

By (E,A,B)-invariance of $\operatorname{im} V$, there exist $N\in\mathbb{R}^{k\times k}$, $M\in\mathbb{R}^{m\times k}$ such that $AV=EVN+BM$, thus

$$S^{-1}\begin{bmatrix}A_1\\A_3\\A_5\end{bmatrix}=AV=EVN+BM=S^{-1}\begin{bmatrix}I_k\\0\\0\end{bmatrix}N+S^{-1}\begin{bmatrix}0\\I_m\\0\end{bmatrix}M,$$

which gives $A_5=0$.

Step 2: We show (3.8). Let $(\overline{E},\overline{A},\overline{B},\overline{C}):=\left(\left[\begin{smallmatrix}E_4\\E_6\end{smallmatrix}\right],\left[\begin{smallmatrix}A_4\\A_6\end{smallmatrix}\right],\left[\begin{smallmatrix}I_m\\0\end{smallmatrix}\right],C_2\right)$ and $q\in\mathbb{N}_0$, $Z\in\mathbb{R}^{(n-k)\times q}$, $X\in\mathbb{R}^{q\times q}$, $Y\in\mathbb{R}^{m\times q}$ be such that $\overline{A}Z=\overline{E}ZX+\overline{B}Y$ and $\overline{C}Z=0$. We show that $\mathcal{V}:=\operatorname{im}[V,WZ]$ is (E,A,B)-invariant and included in $\ker C$.

Step 2a: We show (E, A, B)-invariance of $\mathcal{V}$. Since $AV = EVA_1 + BA_3$, this follows from

$$\begin{aligned} A[V, WZ] &= \left[EVA_1 + BA_3, S^{-1}\begin{bmatrix} A_2 \\ A_4 \\ A_6 \end{bmatrix} Z\right] = \left[EVA_1 + BA_3, S^{-1}\begin{bmatrix} A_2 Z \\ E_4 ZX + Y \\ E_6 ZX \end{bmatrix}\right] \\ &= \left[EVA_1 + BA_3, S^{-1}\left(\begin{bmatrix} E_2 \\ E_4 \\ E_6 \end{bmatrix} ZX + \begin{bmatrix} A_2 Z - E_2 ZX \\ 0 \\ 0 \end{bmatrix} + \tilde{B}Y\right)\right] \\ &= [EVA_1 + BA_3, EWZX + EV(A_2 Z - E_2 ZX) + BY] \\ &= E[V, WZ]\begin{bmatrix} A_1 & A_2 Z - E_2 ZX \\ 0 & X \end{bmatrix} + B[A_3, Y]. \end{aligned}$$

Step 2b: We show that $\mathcal{V}$ is included in $\ker C$. This is immediate from $C[V, WZ] = [0, C_2 Z] = 0$.

Now, since $\operatorname{im} V$ is the largest (E, A, B)-invariant subspace included in $\ker C$, if follows that $\operatorname{im}[V, WZ] \subseteq \operatorname{im} V$ and hence, since $\operatorname{im} V \cap \operatorname{im} W = \{0\}$ and W has full column rank, $Z = 0$. This implies (3.8).

Step 3: We show the uniqueness property. First note that

$$\max(SET, SAT, SB; \ker CT) = T^{-1} \max(E, A, B; \ker C),$$

and hence

$$\dim \max(SET, SAT, SB; \ker CT) = \dim \max(E, A, B; \ker C).$$

Therefore, the block structures of $[E, A, B, C]$ and $[\hat{E}, \hat{A}, \hat{B}, \hat{C}]$ coincide. Now, we now show that

$$\max(E, A, B; \ker C) = \max(SET, SAT, SB; \ker CT) = \operatorname{im}\begin{bmatrix} I_k \\ 0 \end{bmatrix}. \tag{3.10}$$

First, we consider $\max(E, A, B; \ker C)$. Since

$$A\begin{bmatrix} I_k \\ 0 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_3 \\ 0 \end{bmatrix} = E\begin{bmatrix} I_k \\ 0 \end{bmatrix} A_1 + BA_3 \quad \wedge \quad C\begin{bmatrix} I_k \\ 0 \end{bmatrix} = 0,$$

we find that $\operatorname{im}\left[\begin{smallmatrix} I_k \\ 0 \end{smallmatrix}\right]$ is (E, A, B)-invariant and included in $\ker C$. In order to show maximality, let $V = \left[\begin{smallmatrix} V_1 \\ V_2 \end{smallmatrix}\right] \in \mathbb{R}^{n\times q}$, $N \in \mathbb{R}^{q\times q}$, $M \in \mathbb{R}^{m\times q}$ be such that $AV = EVN + BM$ and $CV = 0$. In particular, this implies that

$$\begin{bmatrix} A_4 \\ A_6 \end{bmatrix} V_2 = \begin{bmatrix} E_4 \\ E_6 \end{bmatrix} V_2 N + \begin{bmatrix} I_m \\ 0 \end{bmatrix} [M, A_3 V_1] \quad \wedge \quad C_2 V_2 = 0,$$

and hence (3.8) implies that $V_2 = 0$, thus $\operatorname{im} V \subseteq \operatorname{im}\left[\begin{smallmatrix} I_k \\ 0 \end{smallmatrix}\right]$. Since $[SET, SAT, SB, CT]$ has the same block structure as $[E, A, B, C]$, we have proved (3.10).

From (3.10) we obtain

$$\operatorname{im}\begin{bmatrix} I_k \\ 0 \end{bmatrix} = \max(SET, SAT, SB; \ker CT) = T^{-1}\max(E, A, B; \ker C) = \operatorname{im} T^{-1}\begin{bmatrix} I_k \\ 0 \end{bmatrix},$$

by which T takes the form $T = \left[\begin{smallmatrix} T_1 & T_2 \\ 0 & T_4 \end{smallmatrix}\right]$, $T_1 \in \mathbf{Gl}_k(\mathbb{R})$, $T_4 \in \mathbf{Gl}_{n-k}(\mathbb{R})$. Moreover,

$$\begin{bmatrix} 0 \\ I_m \\ 0 \end{bmatrix} = \hat{B} = SB = S \begin{bmatrix} 0 \\ I_m \\ 0 \end{bmatrix}, \quad \text{and hence} \quad S = \begin{bmatrix} S_1 & 0 & S_3 \\ S_4 & I_m & S_6 \\ S_7 & 0 & S_9 \end{bmatrix}.$$

Now,

$$\begin{bmatrix} I_k \\ 0 \\ 0 \end{bmatrix} = \hat{E} \begin{bmatrix} I_k \\ 0 \end{bmatrix} = SET \begin{bmatrix} I_k \\ 0 \end{bmatrix} = \begin{bmatrix} S_1 T_1 \\ S_4 T_1 \\ S_7 T_1 \end{bmatrix},$$

by which $T_1 = S_1^{-1}$, $S_4 = 0$ and $S_7 = 0$. This completes the proof of the theorem. □

Remark 3.7 (Zero dynamics form)**.** The name "zero dynamics form" for the form (3.7) may be justified since the zero dynamics are decoupled in this form. If $(x, u, y) \in \mathcal{ZD}_{(1.1)}$, then, applying the coordinate transformation $(z_1^\top, z_2^\top)^\top = [V, W]^{-1} x$ from Theorem 3.6, gives $x = V z_1 + W z_2$ and from Proposition 3.4 we obtain $x(t) \in \operatorname{im} V$ for almost all $t \in \mathbb{R}$. Then $\operatorname{im} V \cap \operatorname{im} W = \{0\}$ gives $z_2 \overset{\text{a.e.}}{=} 0$ and $w := z_1 + E_2 z_2 \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R}; \mathbb{R}^k)$ (which satisfies $w \overset{\text{a.e.}}{=} z_1$) and u solve

$$\tfrac{\mathrm{d}}{\mathrm{d}t} w \overset{\text{a.e.}}{=} A_1 w, \quad 0 \overset{\text{a.e.}}{=} A_3 w + u.$$

Therefore, w as the solution of an ODE characterizes the "dynamics" within the zero dynamics (almost everywhere) and z_2 and u are given by algebraic equations depending on w.

The next result is important for a further refinement of the zero dynamics form in Theorem 4.3.

Proposition 3.8 (Invariant subspace, trivial zero dynamics and system pencil)**.** *Let* $[E, A, B, C] \in \Sigma_{l,n,m,p}$. *Then the following statements are equivalent:*

(i) $\operatorname{rk} B = m$ *and* $\max(E, A, B; \ker C) = \{0\}$,

(ii) $\mathcal{ZD}_{(1.1)} \subseteq \left\{ w \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m) \;\middle|\; w \overset{a.e.}{=} 0 \right\}$,

(iii) $\left[\begin{smallmatrix} sE - A & -B \\ -C & 0 \end{smallmatrix}\right]$ *is left invertible over* $\mathbb{R}[s]$.

Proof.

(i)⇒(ii): Let $(x, u, y) \in \mathcal{ZD}_{(1.1)}$ and observe that by $\max(E, A, B; \ker C) = \{0\}$ and Proposition 3.4 we have $x \overset{\text{a.e.}}{=} 0$. Then $\operatorname{rk} B = m$ implies $u \overset{\text{a.e.}}{=} 0$ and (ii) is shown.

(ii)⇒(iii): This can be concluded from ([36], Thm. 3.6.2).

(iii)⇒(i): Clearly, (iii) implies $\operatorname{rk} B = m$. In order to show $\max(E, A, B; \ker C) = \{0\}$ we prove that for all $k \in \mathbb{N}$, $V \in \mathbb{R}^{n \times k}$, $N \in \mathbb{R}^{k \times k}$ and $M \in \mathbb{R}^{m \times k}$ the implication

$$\big(AV = EVN + BM \quad \wedge \quad CV = 0\big) \quad \Longrightarrow \quad V = 0$$

holds. If the left hand side holds true, then we have

$$\begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix} \begin{bmatrix} V \\ -M \end{bmatrix} = \begin{bmatrix} EV \\ 0 \end{bmatrix} (sI_k - N) \in \mathbb{R}[s]^{(l+p) \times k}.$$

By existence of a left inverse $L(s) \in \mathbb{R}[s]^{(n+m) \times (l+p)}$ of $\left[\begin{smallmatrix} sE - A & -B \\ -C & 0 \end{smallmatrix}\right]$ we find that

$$\begin{bmatrix} V \\ -M \end{bmatrix} = L(s) \begin{bmatrix} EV \\ 0 \end{bmatrix} (sI_k - N) = \begin{bmatrix} L_1(s) EV (sI_k - N) \\ L_2(s) EV (sI_k - N) \end{bmatrix}$$

with $L_1(s) = \sum_{i=0}^q s^i L_1^i \in \mathbb{R}[s]^{n \times l}$ and $L_2(s) \in \mathbb{R}[s]^{m \times l}$. Comparison of coefficients of the first equation gives

$$V = -L_1^0 EVN, \; L_1^0 EV = L_1^1 EVN, \; L_1^1 EV = L_1^2 EVN, \; \ldots \;, \; L_1^{q-1} EV = L_1^q EVN, \; L_1^q EV = 0,$$

and backward solution yields $V = 0$, which concludes the proof of the proposition. □

Remark 3.9 (Zero dynamics and system pencil/Kronecker form)**.** We stress the difference in the characterization of autonomous and trivial zero dynamics in terms of the system pencil as they arise from Propositions 3.5 and 3.8: the zero dynamics are autonomous if, and only if, the system pencil has full column rank over $\mathbb{R}[s]$; they are trivial if, and only if, the system pencil is left invertible over $\mathbb{R}[s]$.

Using the quasi-Kronecker form, it follows that the zero dynamics $\mathcal{ZD}_{(1.1)}$ are

(i) autonomous if, and only if, in a quasi-Kronecker form (3.5) of the system pencil no underdetermined blocks are present, *i.e.*, $\ell(\beta) = 0$. The dynamics within the zero dynamics are then characterized by the ODE $\dot{z} = A_s z$.

(ii) trivial if, and only if, in a quasi-Kronecker form (3.5) of the system pencil no underdetermined blocks and no ODE blocks are present, *i.e.*, $\ell(\beta) = 0$ and $n_s = 0$. The remaining nilpotent and overdetermined blocks then have trivial solutions only.

In the remainder of this section we consider asymptotically stable zero dynamics.

Definition 3.10 (Asymptotically stable zero dynamics)**.** For $[E, A, B, C] \in \Sigma_{l,n,m,p}$, the zero dynamics $\mathcal{Z}D_{(1.1)}$ are called *asymptotically stable* if

$$\forall\,(x, u, y) \in \mathcal{Z}D_{(1.1)} :\ \lim_{t\to\infty} \text{ess-sup}_{[t,\infty)} \|(x, u)\| = 0.$$

Lemma 3.11 (Characterization of asymptotically stable zero dynamics)**.** *For* $[E, A, B, C] \in \Sigma_{l,n,m,p}$

$$\mathcal{Z}D_{(1.1)} \text{ are asymptotically stable} \quad \Longleftrightarrow \quad \forall\,\lambda \in \overline{\mathbb{C}}_+ :\ \text{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A & -B \\ -C & 0 \end{bmatrix} = n + m.$$

Proof of Lemma 3.11.

$\Rightarrow$: This is straightforward.

$\Leftarrow$: The rank condition implies that the system pencil must have full column rank over $\mathbb{R}[s]$. Therefore, by Lemma 2.3, in a quasi-Kronecker form (3.5) of the system pencil it holds that $\ell(\beta) = 0$. It is also immediate that $\sigma(A_s) \subseteq \mathbb{C}_-$. The asymptotic stability of $\mathcal{Z}D_{(1.1)}$ then follows from a consideration of the solutions to each block in (3.5). □

It can be shown [5] that systems with asymptotically stable zero dynamics can be stabilized by a control in the behavioral sense, provided they are right-invertible. This concept is introduced, in the framework of system inversion, in the next section.

4. System inversion

In this section we investigate the properties of left-invertibility, right-invertibility, and invertibility of DAE systems. In order to treat these problems we derive a refinement of the zero dynamics form from Theorem 3.6.

In the following we give the definition of left- and right-invertibility of a system, which are from ([46], Sect. 8.2) – generalized to the DAE case. A detailed analysis of left- and right-invertibility of ODE systems can also be found in [38, 41]. For DAE systems, invertibility properties have been investigated by Geerts [22]. In contrast to [22], we do not use the distributional solution framework for the definition of left- and right-invertibility and mainly investigate the system decomposition in Theorem 4.3 and its properties.

Definition 4.1 (System invertibility)**.** $[E, A, B, C] \in \Sigma_{l,n,m,p}$ is called

(i) *left-invertible* if

$$\forall\,(x, u, y) \in \mathfrak{B}_{(1.1)} :\ \left[y \overset{\text{a.e.}}{=} 0 \ \wedge\ Ex(0) = 0\right] \implies u \overset{\text{a.e.}}{=} 0. \tag{4.1}$$

(ii) *right-invertible* if

$$\forall\, y \in \mathcal{C}^\infty(\mathbb{R}; \mathbb{R}^p)\ \exists\,(x, u) \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^n \times \mathbb{R}^m) :\ (x, u, y) \in \mathfrak{B}_{(1.1)}.$$

(iii) *invertible* if $[E, A, B, C]$ is left-invertible and right-invertible.

We show that a DAE system with autonomous zero dynamics is left-invertible, but the converse if false in general.

Lemma 4.2 (Autonomous zero dynamics imply left-invertibility)**.** *Let* $[E, A, B, C] \in \Sigma_{l,n,m,p}$. *Then*

$$\mathcal{ZD}_{(1.1)} \text{ is autonomous } \underset{\not\Leftarrow}{\Longrightarrow} [E, A, B, C] \text{ is left-invertible.}$$

Proof.
$\Longrightarrow$: We show that (4.1) is satisfied. To this end let $(x, u, y) \in \mathfrak{B}_{(1.1)}$ with $y \overset{\text{a.e.}}{=} 0$ and $Ex(0) = 0$. Hence, $(x, u, y) \in \mathcal{ZD}_{(1.1)}$ and applying the coordinate transformation $(z_1^\top, z_2^\top)^\top = [V, W]^{-1}x$ from Theorem 3.6 yields $Vz_1(t) + Wz_2(t) = x(t) \overset{\text{Prop. 3.4}}{\in} \operatorname{im} V$ for almost all $t \in \mathbb{R}$. Therefore, $z_2 \overset{\text{a.e.}}{=} 0$ and we have that $w := z_1 + E_2 z_2 \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R}; \mathbb{R}^k)$ satisfies $w \overset{\text{a.e.}}{=} z_1$ and $\frac{\mathrm{d}}{\mathrm{d}t} w \overset{\text{a.e.}}{=} A_1 w$. Since $Ex(0) = 0$ we obtain from (3.7) that $w(0) = 0$, and hence it follows that $w = 0$ and thus $z_1 \overset{\text{a.e.}}{=} 0$ and $u \overset{\text{a.e.}}{=} -A_3 z_1 \overset{\text{a.e.}}{=} 0$.
$\not\Longleftarrow$: That the converse does, in general, not hold true can be observed from the system $[E, A, B, C]$ with

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ C = [0, 0, 1].$$ □

If $sE - A$ is regular, then equivalence holds in the statement of Lemma 4.2, see Proposition 4.8. Note also that in ([22], Def. 4.10) the concept of "left invertibility in the strong sense" is introduced, which means that if $x(0) = 0$ and $y = 0$, then $u = 0$ and $Ex = 0$; however, it should be noted that in [22] distributional solutions are considered. Using the algebraic characterization in Proposition 3.5 and ([22], Cor. 4.15) it can then be concluded that autonomous zero dynamics are equivalent to "left invertibility in the strong sense", provided that $[E^\top, A^\top, C^\top]$ has full row rank.

In the following we investigate right-invertibility for systems with autonomous zero dynamics. In order for $[E, A, B, C] \in \Sigma_{l,n,m,p}$ to be right invertible it is necessary that C has full row rank (*i.e.*, $\operatorname{im} C = \mathbb{R}^p$). This additional assumption leads to the following form for $[E, A, B, C]$ specializing the form (3.7).

Theorem 4.3 (System inversion form)**.** *Let* $[E, A, B, C] \in \Sigma_{l,n,m,p}$ *with autonomous zero dynamics and* $\operatorname{rk} C = p$. *Then there exist* $S \in \mathbf{Gl}_l(\mathbb{R})$ *and* $T \in \mathbf{Gl}_n(\mathbb{R})$ *such that*

$$[E, A, B, C] \overset{S,T}{\sim} \left[\hat{E}, \hat{A}, \hat{B}, \hat{C}\right], \tag{4.2}$$

where

$$\hat{E} = \begin{bmatrix} I_k & 0 & 0 \\ 0 & E_{22} & E_{23} \\ 0 & E_{32} & N \\ 0 & E_{42} & E_{43} \end{bmatrix}, \ \hat{A} = \begin{bmatrix} Q & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & I_{n_3} \\ 0 & A_{42} & 0 \end{bmatrix}, \ \hat{B} = \begin{bmatrix} 0 \\ I_m \\ 0 \\ 0 \end{bmatrix}, \ \hat{C} = [0, I_p, 0], \tag{4.3}$$

$k = \dim \max(E, A, B; \ker C)$ *and* $N \in \mathbb{R}^{n_3 \times n_3}$, $n_3 = n - k - p$, *is nilpotent with* $N^\nu = 0$ *and* $N^{\nu-1} \neq 0$, $\nu \in \mathbb{N}$, $E_{22}, A_{22} \in \mathbb{R}^{m \times p}$ *and all other matrices are of appropriate sizes.*

Proof. By Theorem 3.6 system $[E, A, B, C]$ is equivalent to $[\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}]$ in (3.7). Since C and therefore C_2 has full row rank, there exists $\tilde{T} \in \mathbf{Gl}_{n-k}(\mathbb{R})$ such that $C_2 \tilde{T} = [I_p, 0]$. Hence,

$$\left[\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}\right] \overset{I, \left[\begin{smallmatrix} I & 0 \\ 0 & \tilde{T} \end{smallmatrix}\right]}{\sim} \left[\begin{bmatrix} I_k & \tilde{E}_{12} & \tilde{E}_{13} \\ 0 & \tilde{E}_{22} & \tilde{E}_{23} \\ 0 & \tilde{E}_{32} & \tilde{E}_{33} \end{bmatrix}, \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} \\ 0 & \tilde{A}_{32} & \tilde{A}_{33} \end{bmatrix}, \begin{bmatrix} 0 \\ I_m \\ 0 \end{bmatrix}, [0, I_p, 0] \right].$$

Now, since

$$\max\left(\begin{bmatrix}\tilde{E}_{22} & \tilde{E}_{23}\\ \tilde{E}_{32} & \tilde{E}_{33}\end{bmatrix}, \begin{bmatrix}\tilde{A}_{22} & \tilde{A}_{23}\\ \tilde{A}_{32} & \tilde{A}_{33}\end{bmatrix}, \begin{bmatrix}I_m\\ 0\end{bmatrix}; \ker\,[I_p, 0]\right) = \{0\} \quad \text{and} \quad \operatorname{rk}\begin{bmatrix}I_m\\ 0\end{bmatrix} = m$$

by Theorem 3.6, we may infer from Proposition 3.8 that there exists $X(s) \in \mathbb{R}[s]^{(n+m-k)\times(l+p-k)}$ such that

$$\begin{bmatrix}X_{11}(s) & X_{12}(s) & X_{13}(s)\\ X_{21}(s) & X_{22}(s) & X_{23}(s)\\ X_{31}(s) & X_{32}(s) & X_{33}(s)\end{bmatrix}\begin{bmatrix}s\tilde{E}_{22}-\tilde{A}_{22} & s\tilde{E}_{23}-\tilde{A}_{23} & I_m\\ s\tilde{E}_{32}-\tilde{A}_{32} & s\tilde{E}_{33}-\tilde{A}_{33} & 0\\ I_p & 0 & 0\end{bmatrix} = \begin{bmatrix}I_p & 0 & 0\\ 0 & I_{n-k-p} & 0\\ 0 & 0 & I_m\end{bmatrix}.$$

Obviously, $X_{21}(s) = 0$ and hence $X_{22}(s)(s\tilde{E}_{33} - \tilde{A}_{33}) = I_{n-k-p}$, *i.e.*, $s\tilde{E}_{33} - \tilde{A}_{33}$ is left invertible over $\mathbb{R}[s]$. This implies that in a QKF (2.2) of $s\tilde{E}_{33} - \tilde{A}_{33}$ it holds $n_s = 0$ and $\ell(\beta) = 0$. By a permutation of the rows in the block $sK_\gamma^\top - L_\gamma^\top$ we may achieve that there exists $\hat{S} \in \mathbf{Gl}_{l-k-m}(\mathbb{R})$, $\hat{T} \in \mathbf{Gl}_{n-k-p}(\mathbb{R})$ such that $\hat{S}(s\tilde{E}_{33} - \tilde{A}_{33})\hat{T} = \begin{bmatrix} sN - I_{n_3}\\ s\hat{E}_{43} - \hat{A}_{43}\end{bmatrix}$, where N is nilpotent. Hence,

$$\left[\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}\right] \overset{\left[\begin{smallmatrix} I & 0\\ 0 & \left[\begin{smallmatrix} I & 0\\ 0 & \hat{S}\end{smallmatrix}\right]\end{smallmatrix}\right], \left[\begin{smallmatrix} I & 0\\ 0 & \tilde{T}\cdot\left[\begin{smallmatrix} I & 0\\ 0 & \hat{T}\end{smallmatrix}\right]\end{smallmatrix}\right]}{\sim} \left[\begin{bmatrix} I_k & \tilde{E}_{12} & \tilde{E}_{13}\\ 0 & \tilde{E}_{22} & \tilde{E}_{23}\\ 0 & \hat{E}_{32} & N\\ 0 & \hat{E}_{42} & \hat{E}_{43}\end{bmatrix}, \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13}\\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23}\\ 0 & \hat{A}_{32} & I_{n_3}\\ 0 & \hat{A}_{42} & \hat{A}_{43}\end{bmatrix}, \begin{bmatrix}0\\ I_m\\ 0\\ 0\end{bmatrix}, [0, I_p, 0]\right].$$

Applying additional elementary row and column operations we obtain that

$$[E, A, B, C] \overset{\overline{S},\overline{T}}{\sim} [\overline{E}, \overline{A}, \overline{B}, \overline{C}]$$

for some $\overline{S} \in \mathbf{Gl}_l(\mathbb{R})$ and $\overline{T} \in \mathbf{Gl}_n(\mathbb{R})$, where

$$\overline{E} = \begin{bmatrix} I_k & 0 & E_{13}\\ 0 & E_{22} & E_{23}\\ 0 & E_{32} & N\\ 0 & E_{42} & E_{43}\end{bmatrix}, \ \overline{A} = \begin{bmatrix} Q & A_{12} & 0\\ A_{21} & A_{22} & 0\\ 0 & 0 & I_{n_3}\\ 0 & A_{42} & 0\end{bmatrix}, \ \overline{B} = \begin{bmatrix}0\\ I_m\\ 0\\ 0\end{bmatrix}, \ \overline{C} = [0, I_p, 0].$$

It only remains to show that by an additional transformation we can obtain that $E_{13} = 0$. To this end consider

$$\check{S} := \begin{bmatrix} I & 0 & QL & 0\\ 0 & I & A_{21}L & 0\\ 0 & 0 & I & 0\\ 0 & 0 & 0 & I\end{bmatrix}, \quad \check{T} := \begin{bmatrix} I & -QLE_{32} & -L\\ 0 & I & 0\\ 0 & 0 & I\end{bmatrix}, \quad L := \sum_{i=0}^{\nu-1} Q^i E_{13} N^i,$$

and observe that $\check{S}\overline{B} = \overline{B} = \hat{B}$, $\overline{C}\check{T} = \overline{C} = \hat{C}$ and $\check{S}\overline{E}\check{T}$, $\check{S}\overline{A}\check{T}$ have the same block structure as $\hat{E}, \hat{A}$ and N is nilpotent. □

The form derived in Theorem 4.3 is a generalization of the zero dynamics form derived in ([11], Thm. 2.3) for system $[E, A, B, C] \in \Sigma_{n,n,m,m}$ with regular $sE - A$ and proper inverse transfer function. Uniqueness of the entries in the form (4.3) may be analyzed similar to the last statement in Theorem 3.6.

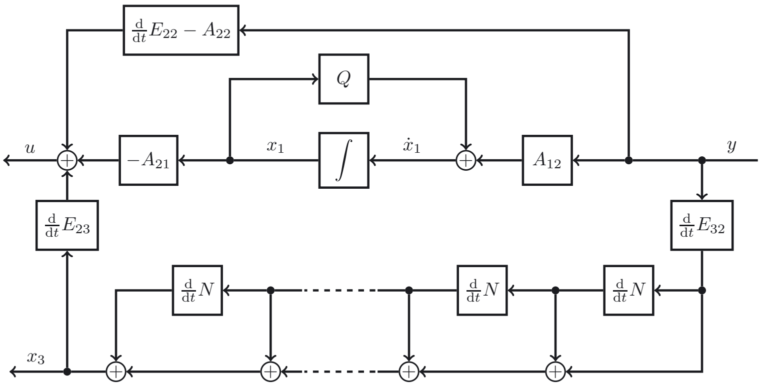

FIGURE 1. System $[E, A, B, C] \in \Sigma_{l,n,m,p}$ in form (4.3).

Remark 4.4 (DAE of system inversion form)**.** Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ with autonomous zero dynamics and $\operatorname{rk} C = p$. The behavior of the DAE (1.1) may be interpreted, in terms of the form (4.2), (4.3) in Theorem 4.3, as follows: $(x, u, y) \in \mathfrak{B}_{[E,A,B,C]} \cap \big(\mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^n) \times \mathcal{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^m) \times \mathcal{W}^{\nu+1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^p)\big)$ if, and only if, (Tx, u, y) solves

$$\boxed{\begin{aligned} \dot{x}_1 &= Qx_1 + A_{12}y \\ 0 &= -E_{22}\dot{y} - \textstyle\sum_{k=0}^{\nu-1} E_{23}N^kE_{32}y^{(k+2)} + A_{21}x_1 + A_{22}y + u \\ x_3 &= \textstyle\sum_{k=0}^{\nu-1} N^kE_{32}y^{(k+1)} \\ 0 &= -E_{42}\dot{y} - \textstyle\sum_{k=0}^{\nu-1} E_{43}N^kE_{32}y^{(k+2)} + A_{42}y, \end{aligned}} \tag{4.4}$$

where $Tx = (x_1^\top, y^\top, x_3^\top)^\top \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^{k+p+n_3})$; see also Figure 1.

The next corollary follows directly from Theorem 4.3 and the form (4.3).

Corollary 4.5 (Asymptotically stable zero dynamics)**.** *Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ with autonomous zero dynamics and $\operatorname{rk} C = p$. Then, using the notation from Theorem 4.3, the zero dynamics $\mathcal{ZD}_{(1.1)}$ are asymptotically stable if, and only if, $\sigma(Q) \subseteq \mathbb{C}_-$.*

The equations in (4.4) can be viewed as a realization of the inverse system of $[E, A, B, C]$ in the behavioral sense, *i.e.*, where inputs and outputs have been interchanged. The second equation in (4.4) can be solved explicitly for u – the output of the inverse system which is again a DAE. In contrast to classical approaches of system inversion for ODEs [42, 43], this does not involve the inversion of an input-output mapping, but can be treated solely within the behavioral framework.

For ODE systems ($E = I$), the Byrnes–Isidori form derived in [27], see also ([28], Sect. 5.1), is a special case of the form (4.4). Note also that in this case the form (4.4) does not simplify in general. If additionally $m = p$, then the last equation in (4.4) vanishes. But in general this equation is present and is one reason for $[E, A, B, C]$ not being right-invertible in general. Necessary and sufficient conditions for the latter are derived next.

Proposition 4.6 (System invertibility)**.** *Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ with autonomous zero dynamics. Then, in terms of the form (4.3) from Theorem 4.3,*

$$[E, A, B, C] \text{ is invertible} \quad \iff \quad \operatorname{rk} C = p, E_{42} = 0, A_{42} = 0 \text{ and } E_{43}N^jE_{32} = 0 \text{ for } j = 0, \dots, \nu - 1.$$

Proof. By Lemma 4.2 $[E, A, B, C]$ is left-invertible, so it remains to show the equivalence for right-invertibility.

$\Rightarrow$: It is is clear that $\operatorname{rk} C = p$, otherwise we might choose any constant $y \equiv y^0$ with $y^0 \notin \operatorname{im} C$, which cannot be attained by the output of the system. Now, by Theorem 4.3 we may assume, without loss of generality, that the system is in the form (4.3). Assume that $A_{42} \neq 0$. Hence, there exists $y^0 \in \mathbb{R}^p$ such that $A_{42}y^0 \neq 0$. Then, for $y \equiv y^0$ and all $x \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n)$, $u \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^m)$ it holds that $(x,u,y) \notin \mathfrak{B}_{(1.1)}$ (since the last equation in (4.4) is not satisfied), which contradicts right-invertibility. Therefore, we have $A_{42} = 0$. Repeating the argument for E_{42} and $E_{43}N^jE_{32}$ with $y(t) = ty^0$ and $y(t) = t^{j+2}y^0$, resp., yields that $E_{42} = 0$ and $E_{43}N^jE_{32} = 0$, $j = 0,\dots,\nu-1$.

$\Leftarrow$: This is immediate from (4.4) since $y \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R}^p)$. □

Remark 4.7. Let $[E,A,B,C] \in \Sigma_{l,n,m,p}$ with autonomous zero dynamics. If $l = n$, $p = m$ and $\operatorname{rk} C = m$, then $[E,A,B,C]$ is invertible. This can be seen using the form (4.3) from Theorem 4.3.

In the remainder of this section we restrict ourselves to regular systems and relate the invertibility properties to the transfer function of the system.

Proposition 4.8 (Invertibility for regular systems)**.** *Let $[E,A,B,C] \in \Sigma_{n,n,m,p}$ be such that $sE - A$ is regular and let $G(s) = C(sE-A)^{-1}B \in \mathbb{R}(s)^{p\times m}$ be its transfer function. Then we have the following statements:*

(i) *$\mathcal{ZD}_{(1.1)}$ is autonomous $\iff$ $[E,A,B,C]$ is left-invertible $\iff$ $\operatorname{rk}_{\mathbb{R}[s]} G(s) = m$.*

(ii) *$[E,A,B,C]$ is right-invertible $\iff$ $\operatorname{rk}_{\mathbb{R}[s]} G(s) = p$.*

Proof. The proof of (ii) and of the equivalence between autonomous zero dynamics and $\operatorname{rk}_{\mathbb{R}[s]} G(s) = m$ can be found in ([4], Prop. 5.3.1). In view of Lemma 4.2 it remains to show that left-invertibility implies $\operatorname{rk}_{\mathbb{R}[s]} G(s) = m$. Since $sE - A$ is regular, by ([12], Thm. 2.6) there exist $S,T \in \mathbf{Gl}_n(\mathbb{R})$ such that

$$S(sE-A)T = \begin{bmatrix} sI_{n_1} - J & 0 \\ 0 & sN - I_{n_2} \end{bmatrix}, \quad \text{where } N \in \mathbb{R}^{n_2\times n_2} \text{ is nilpotent and } J \in \mathbb{R}^{n_1 \times n_1}.$$

Let $SB = \left[\begin{smallmatrix} B_1 \\ B_2 \end{smallmatrix}\right]$ and $CT = [C_1, C_2]$. Let $(x,u,y) \in \mathfrak{B}_{(1.1)}$ be such that $y = Cx \overset{\text{a.e.}}{=} 0$ and $Ex(0) = 0$. Then $T^{-1}x = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)$ satisfies

$$\tfrac{\mathrm{d}}{\mathrm{d}t} x_1(t) = Jx_1(t) + B_1u(t), \qquad \tfrac{\mathrm{d}}{\mathrm{d}t} Nx_2(t) = x_2(t) + B_2u(t).$$

Invoking ([12], Prop. 2.20) we find that the solution of these equations with $x_1(0)$ and $u \in \mathcal{C}^{n_2-1}(\mathbb{R};\mathbb{R}^m)$ are given by

$$x_1(t) = \int_0^t e^{J(t-s)}B_1u(s)\,\mathrm{d}s\,, \quad x_2(t) = -\sum_{k=0}^{n_2-1} N^kB_2u^{(k)}(t), \quad t \in \mathbb{R}.$$

Since $Cx \overset{\text{a.e.}}{=} 0$ it follows that

$$\forall\, t \in \mathbb{R}: \int_0^t C_1e^{J(t-s)}B_1u(s)\,\mathrm{d}s - \sum_{k=0}^{n_2-1} C_2N^kB_2u^{(k)}(t) = 0. \tag{4.5}$$

By left-invertibility of $[E,A,B,C]$ we have that for all $u \in \mathcal{C}^{n_2-1}(\mathbb{R};\mathbb{R}^m)$ equation (4.5) can only be true if $u = 0$. Using Laplace transform this implies that

$$P(s) := C_1(sI_{n_1} - J)^{-1}B_1 - \sum_{k=0}^{n_2-1} C_2N^kB_2s^k$$

has full column rank over $\mathbb{R}[s]$. By ([11], (1.7)) we find that $P(s) = C(sE-A)^{-1}B = G(s)$ and hence $\operatorname{rk}_{\mathbb{R}[s]} G(s) = m$. □

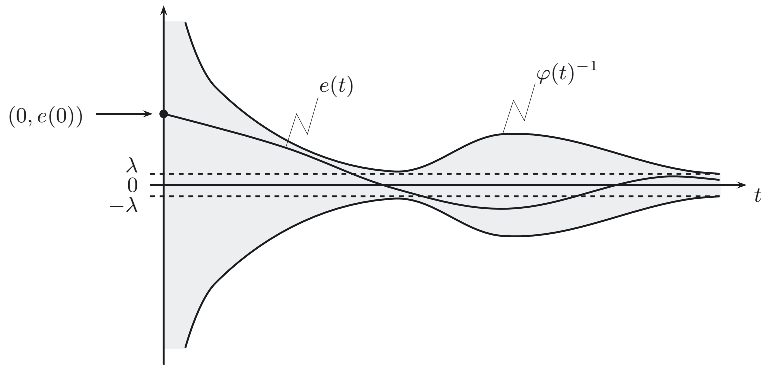

FIGURE 2. Error evolution in a funnel $\mathcal{F}_\varphi$ with boundary $\varphi(t)^{-1}$ which has a pole at $t = 0$.

5. Funnel control

In this section we consider funnel control for systems $[E, A, B, C] \in \Sigma_{l,n,m,m}$ with the same number of inputs and outputs.

Drawbacks of classical output feedback controllers, such as constant or adaptive high-gain control, are discussed in [24] for ODE systems. These drawbacks typically comprise that the input is very sensitive to output perturbations, tracking would require an internal model and, most importantly, transient behavior is not taken into account. To overcome these drawbacks, the concept of "funnel control" is introduced (see [24] and the references therein): For any function φ belonging to

$$\Phi^\mu := \left\{ \varphi \in \mathcal{C}^\mu(\mathbb{R}_{\geq 0}; \mathbb{R}) \cap \mathcal{B}^1(\mathbb{R}_{\geq 0}; \mathbb{R}) \;\middle|\; \varphi(0) = 0, \quad \varphi(s) > 0 \text{ for all } s > 0 \text{ and } \liminf_{s \to \infty} \varphi(s) > 0 \right\},$$

for $\mu \in \mathbb{N}$, we associate the *performance funnel*

$$\mathcal{F}_\varphi := \left\{ (t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \;\middle|\; \varphi(t)\|e\| < 1 \right\}, \tag{5.1}$$

see Figure 2. The control objective is feedback control so that the tracking error $e(\cdot) = y(\cdot) - y_{\text{ref}}(\cdot)$, where $y_{\text{ref}}(\cdot)$ is the reference signal, evolves within $\mathcal{F}_\varphi$ and all variables are bounded. More specific, the transient behavior is supposed to satisfy

$$\forall\, t > 0: \;\; \|e(t)\| < 1/\varphi(t),$$

and, moreover, if φ is chosen so that $\varphi(t) \geq 1/\lambda$ for all t sufficiently large, then the tracking error remains smaller than λ.

By choosing $\varphi(0) = 0$ we ensure that the width of the funnel is infinity at $t = 0$, see Figure 2. In the following we only treat "infinite" funnels for technical reasons, since if the funnel is finite, that is $\varphi(0) > 0$, then we need to assume that the initial error is within the funnel boundaries at $t = 0$, *i.e.*, $\varphi(0)\|Cx^0 - y_{\text{ref}}(0)\| < 1$, and this assumption suffices.

As indicated in Figure 2, we do not assume that the funnel boundary decreases monotonically. Certainly, in most situations it is convenient to choose a monotone funnel, however there are situations where widening the funnel at some later time might be beneficial, *e.g.*, when it is known that the reference signal varies strongly.

To ensure error evolution within the funnel, we introduce, for $\hat{k} > 0$, the *funnel controller*:

$$\boxed{\begin{aligned} u(t) &= -k(t)\, e(t), \qquad\qquad \text{where} \qquad e(t) = y(t) - y_{\text{ref}}(t) \\ k(t) &= \frac{\hat{k}}{1 - \varphi(t)^2 \|e(t)\|^2}. \end{aligned}} \tag{5.2}$$

If we assume asymptotically stable zero dynamics, we see intuitively that, in order to maintain the error evolution within the funnel, high gain values may only be required if the norm $\|e(t)\|$ of the error is close to the funnel boundary $\varphi(t)^{-1}$: $k(\cdot)$ increases if necessary to exploit the high-gain property of the system and decreases if a high gain is not necessary. This intuition underpins the choice of the gain $k(t)$ in (5.2), where the constant $\hat{k} > 0$ is only of technical importance, see Remark 5.1. The control design (5.2) has two advantages: $k(\cdot)$ is non-monotone and (5.2) is a static time-varying proportional output feedback of striking simplicity. We also stress that the system parameters need not be known.

In the following we show that funnel control for systems (1.1) is feasible under some appropriate assumptions. In [11] it is shown that funnel control works for DAE systems with regular pencil $sE - A$, proper inverse transfer function and asymptotically stable zero dynamics. In [10] it is then shown that funnel control is also feasible if the assumption of proper inverse transfer function is replaced by the existence of a positive strict relative degree, however a filter has to be incorporated in the feedback in this case. What we have presented in the present paper so far is a unified framework for the two *distinct* cases "proper inverse transfer function" and "positive strict relative degree one" and, furthermore, we do not need to assume that $sE - A$ is regular. In fact, we only need the following assumptions for funnel control being feasible for a system $[E, A, B, C] \in \Sigma_{l,n,m,m}$:

- $[E, A, B, C]$ has asymptotically stable zero dynamics,
- $[E, A, B, C]$ is right-invertible,
- the matrix
$$\Gamma = -\lim_{s\to\infty} s^{-1}[0, I_m]L(s)\begin{bmatrix}0\\ I_m\end{bmatrix} \in \mathbb{R}^{m\times m} \tag{5.3}$$
exists and satisfies $\Gamma = \Gamma^\top \geq 0$, where $L(s)$ denotes a left inverse of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0\end{smallmatrix}\right]$ over $\mathbb{R}(s)$,
- $\hat{k}$ in (5.2) satisfies
$$\hat{k} > \left\| \lim_{s\to\infty}\left([0, I_m]L(s)\begin{bmatrix}0\\ I_m\end{bmatrix} + s\Gamma\right)\right\|. \tag{5.4}$$

Note that the left inverse $L(s)$ exists by Proposition 3.5 and the autonomous zero dynamics of $[E, A, B, C]$; by Lemma A.1, Γ is independent of the choice of $L(s)$. Furthermore, for single-input, single-output systems with transfer function $g(s) = \frac{p(s)}{q(s)} \in \mathbb{R}(s)\setminus\{0\}$, the existence of Γ in (5.3) is equivalent to $\deg q(s) - \deg p(s) \leq 1$, *i.e.*, $g(s)$ has strict relative degree smaller or equal to one. Furthermore, any regular systems with a vector relative degree which is componentwise smaller or equal to 1 satisfies existence of Γ, see Appendix B. Therefore, the existence of Γ can be viewed as "some relative degree one condition".

The condition (5.4) on $\hat{k}$ seems undesirable, since at first glance it is not known how large $\hat{k}$ must be chosen; this is just the drawback of high-gain control that we seek to avoid by the introduction of funnel control. However, condition (5.4) turns out to be structural, since $-[0, I_m]L(s)[0, I_m]^\top$ is a generalization of the inverse transfer function (*cf*. Rem. A.4) and we only need a bound for the norm of the constant coefficient in its Laurent series. In several important cases it is indeed possible to calculate the bound explicitly, see Remark 5.1.

Another interpretation of condition (5.4) is that it simply guarantees that the subsystem of the closed-loop system that describes the input-output behavior (*i.e.*, the algebraic variables are omitted) is index-1, *cf*. [13].

Remark 5.1 (Initial gain)**.** The condition (5.4) is specific for DAEs and already appears in [10, 11], but not in the ODE case, see [26]. A careful inspection of the proof of Theorem 5.3 shows that we have to ensure that the matrix $\hat{A}_{22} - k(t)I_m$ is invertible for all $t \geq 0$, and in order to avoid singularities we choose, as a simple condition, the "minimal value" $\hat{k}$ of $k(\cdot)$ to be larger than $\|A_{22}\| \geq \|\hat{A}_{22}\|$. In most cases the lower bound for $\hat{k}$ in (5.4) can be calculated easily. We perform the calculation for some classes of ODEs: Consider the system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t),\end{aligned} \tag{5.5}$$

where $A \in \mathbb{R}^{n\times n}, B, C^\top \in \mathbb{R}^{n\times m}, D \in \mathbb{R}^{m\times m}$. System (5.5) can be rewritten in the form (1.1) as

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x(t) \\ w(t) \end{pmatrix} &= \begin{bmatrix} A & 0 \\ 0 & -I_m \end{bmatrix} \begin{pmatrix} x(t) \\ w(t) \end{pmatrix} + \begin{bmatrix} B \\ D \end{bmatrix} u(t) \\ y(t) &= [C, I] \begin{pmatrix} x(t) \\ w(t) \end{pmatrix}. \end{aligned}$$

Observe that $s \left[\begin{smallmatrix} I_n & 0 \\ 0 & 0 \end{smallmatrix}\right] - \left[\begin{smallmatrix} A & 0 \\ 0 & -I_m \end{smallmatrix}\right]$ is regular, and hence applying Remark A.4 gives

$$\Gamma = \lim_{s\to\infty} s^{-1} \left([C, I] \begin{bmatrix} sI - A & 0 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} B \\ D \end{bmatrix} \right)^{-1} = \lim_{s\to\infty} s^{-1} \left(C(sI - A)^{-1} B + D \right)^{-1}.$$

Assume now that $D \in \mathbf{Gl}_m(\mathbb{R})$, *i.e.*, the system has strict relative degree 0. Then

$$\Gamma = \lim_{s\to\infty} s^{-1} D^{-1} \sum_{k=0}^{\infty} \left(-D^{-1} C(sI - A)^{-1} B \right)^k = 0,$$

and

$$\lim_{s\to\infty} \left([0, I_m] L(s) \begin{bmatrix} 0 \\ I_m \end{bmatrix} + s\Gamma \right) = -\lim_{s\to\infty} D^{-1} \sum_{k=0}^{\infty} \left(-D^{-1} C(sI - A)^{-1} B \right)^k = -D^{-1}.$$

Therefore, (5.4) reads $\hat{k} > \|D^{-1}\|$. If $D = 0$ and $CB \in \mathbf{Gl}_m(\mathbb{R})$, *i.e.*, the system has strict relative degree 1, then similar calculations lead to $\Gamma = \left(CB\right)^{-1}$ and (5.4) simply reads $\hat{k} > 0$; the latter is a general condition compared to the choice $\hat{k} = 1$ in [26].

For single-input, single-output systems the above conditions can also be motivated by just looking at the output equation $y = cx + du$, $c^\top \in \mathbb{R}^n, d \in \mathbb{R}$. If a feedback $u = -ky$, $k > 0$ is applied, then $(1 + dk)y = cx$ and in order to solve this equation for y it is sufficient that either $k > 0$ (no further condition) if $d = 0$, or $k > |d^{-1}|$ if $d \neq 0$.

Before we state our main result, we need to define consistency of the initial value of the closed-loop system and solutions of the latter. Compared to the previous sections, here we require more smoothness of the trajectories.

Definition 5.2 (Consistent initial value)**.** Let $[E, A, B, C] \in \Sigma_{l,n,m,m}$, $\varphi \in \Phi^1$ and $y_{\mathrm{ref}} \in \mathcal{B}^1(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$. An initial value $x^0 \in \mathbb{R}^n$ is called *consistent* for the closed-loop system (1.1), (5.2), if there exists a solution of the initial value problem (1.1), (5.2), $x(0) = x^0$, *i.e.*, a function $x \in \mathcal{C}^1([0, \omega); \mathbb{R}^n)$ for some $\omega \in (0, \infty]$, such that $x(0) = x^0$ and x satisfies (1.1), (5.2) for all $t \in [0, \omega)$.

Note that, in practice, consistency of the initial state of the "unknown" system should be satisfied as far as the DAE $[E, A, B, C]$ is the correct model.

We are now in a position to state the main result of this section.

Theorem 5.3 (Funnel control)**.** *Let $[E, A, B, C] \in \Sigma_{l,n,m,m}$ be right-invertible and have asymptotically stable zero dynamics. Suppose that, for a left inverse $L(s)$ of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ over $\mathbb{R}(s)$, the matrix Γ in (5.3) exists and satisfies $\Gamma = \Gamma^\top \geq 0$. Let $\hat{k} > 0$ be such that (5.4) is satisfied and let, for $\nu \in \mathbb{N}$ as in Theorem 4.3, $\varphi \in \Phi^{\nu+1}$ define a performance funnel $\mathcal{F}_\varphi$.*

Then, for any reference signal $y_{\mathrm{ref}} \in \mathcal{B}^{\nu+2}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$, any consistent initial value $x^0 \in \mathbb{R}^n$, the application of the funnel controller (5.2) to (1.1) yields a closed-loop initial-value problem that has a solution and every solution can be extended to a global solution. Furthermore, for every global solution $x(\cdot)$,

(i) *$x(\cdot)$ is bounded and the corresponding tracking error $e(\cdot) = Cx(\cdot) - y_{\text{ref}}(\cdot)$ evolves uniformly within the performance funnel $\mathcal{F}_\varphi$; more precisely,*

$$\exists\, \varepsilon > 0 \;\forall\, t > 0: \;\; \|e(t)\| \le \varphi(t)^{-1} - \varepsilon. \tag{5.6}$$

(ii) *the corresponding gain function $k(\cdot)$ given by (5.2) is bounded:*

$$\forall\, t_0 > 0: \;\sup_{t\ge t_0} |k(t)| \le \frac{|\hat{k}|}{1-(1-\varepsilon\lambda_{t_0})^2},$$

where $\lambda_{t_0} := \inf_{t\ge t_0} \varphi(t) > 0$ for all $t_0 > 0$.

Proof. Note that Γ is well-defined by Lemma A.1. We proceed in several steps.
Step 1: By Lemma A.3, the closed-loop system (1.1), (5.2) is, without loss of generality, in the form

$$\begin{aligned} \dot{x}_1(t) &= Q\, x_1(t) + A_{12}\, e(t) + A_{12}\, y_{\text{ref}}(t) \\ \Gamma\, \dot{e}(t) &= (A_{22} - k(t) I_m)\, e(t) + A_{22}\, y_{\text{ref}}(t) - \Gamma\, \dot{y}_{\text{ref}}(t) + \tilde{\Psi}(x_1^0, e)(t) \\ x_3(t) &= \textstyle\sum_{k=0}^{\nu-1} N^k E_{32}\, e^{(k+1)}(t) + \sum_{k=0}^{\nu-1} N^k E_{32}\, y_{\text{ref}}^{(k+1)}(t) \\ k(t) &= \tfrac{\hat{k}}{1-\varphi(t)^2\|e(t)\|^2}, \end{aligned} \tag{5.7}$$

where $x_1^0 = [I_k, 0, 0] T^{-1} x^0$ and

$$\tilde{\Psi}(x_1^0, e)(t) = \Psi(x_1^0, e)(t) + \Psi(x_1^0, y_{\text{ref}})(t) - A_{21} e^{Qt} x_1^0, \quad t \ge 0.$$

Note that, as $[0, I_m] L(s) [0, I_m]^\top = X_{45}(s)$ for the representation in (A.2),

$$\hat{k} > \left\| \lim_{s\to\infty} \left([0, I_m] L(s) [0, I_m]^\top + s\Gamma \right) \right\| = \|A_{22}\|.$$

By consistency of the initial value x^0 there exists a local solution $(x_1, e, x_3, k) \in \mathcal{C}^1([0,\rho); \mathbb{R}^{n+1})$ of (5.7) for some $\rho > 0$ and initial data

$$(x_1, e, x_3, k)(0) = \left(T^{-1} x^0 - (0, y_{\text{ref}}(0), 0)^\top, \;\hat{k} \right),$$

where the differentiability follows since $y_{\text{ref}} \in \mathcal{B}^{\nu+2}(\mathbb{R}_{\ge 0}; \mathbb{R}^m)$ and $\varphi \in \mathcal{C}^{\nu+1}(\mathbb{R}_{\ge 0}; \mathbb{R})$. It is clear that $(t, e(t))$ belongs to the set $\mathcal{F}_\varphi$ for all $t \in [0, \rho)$. Even more so, we have that

$$\forall\, t \in [0,\rho): \;\; (t, x_1(t), e(t), x_3(t), k(t)) \in \tilde{\mathcal{D}} := \left\{ \, (t, x_1, e, x_3, k) \in \mathbb{R}_{\ge 0} \times \mathbb{R}^{n+1} \;\middle|\; \varphi(t)\|e\| < 1 \, \right\}.$$

We will now, for the time being, ignore the first and third equation in (5.7) and construct an integral-differential equation from the second and fourth equation, which is solved by (e, k). To this end, observe that by $\Gamma = \Gamma^\top \ge 0$, there exists an orthogonal matrix $V \in \mathbf{Gl}_m(\mathbb{R})$ and a diagonal matrix $D \in \mathbb{R}^{m_1 \times m_1}$ with only positive entries for some $0 \le m_1 \le m$, such that $\Gamma = V^\top \left[\begin{smallmatrix} D & 0 \\ 0 & 0 \end{smallmatrix}\right] V$.

In order to decouple the second equation in (5.7) into an ODE and an algebraic equation, we introduce the new variables $e_1(\cdot) = [I_{m_1}, 0] V e(\cdot)$ and $e_2(\cdot) = [0, I_{m-m_1}] V e(\cdot)$. Rewriting (5.7) and invoking $\|e(t)\|^2 = \|V e(t)\|^2 = \|e_1(t)\|^2 + \|e_2(t)\|^2$, this leads to the system

$$\begin{aligned} \dot{e}_1(t) &= [D^{-1}, 0](V A_{22} V^\top - k(t) I_m) \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} + [D^{-1}, 0] V\, \Theta_1(e_1, e_2)(t) \\ 0 &= [0, I_{m-m_1}] V A_{22} V^\top \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} - k(t) e_2(t) + [0, I_{m-m_1}] V\, \Theta_1(e_1, e_2)(t) \\ k(t) &= \tfrac{\hat{k}}{1-\varphi(t)^2(\|e_1(t)\|^2 + \|e_2(t)\|^2)}, \end{aligned} \tag{5.8}$$

on $\mathbb{R}_{\geq 0}$ where

$$\Theta_1 : \mathcal{C}^\nu(\mathbb{R}_{\geq 0};\mathbb{R}^{m_1}) \times \mathcal{C}^\nu(\mathbb{R}_{\geq 0};\mathbb{R}^{m-m_1}) \to \mathcal{C}^{\nu+1}(\mathbb{R}_{\geq 0};\mathbb{R}^m),$$
$$(e_1,e_2) \mapsto \left(t \mapsto A_{22}\, y_{\rm ref}(t) - \Gamma\, \dot y_{\rm ref}(t) + \tilde{\Psi}\left(x_1^0, V^\top \left(e_1^\top, e_2^\top\right)^\top\right)(t)\right).$$

Introduce the set

$$\mathcal{D} := \left\{ \, (t,k,e_1,e_2) \in \mathbb{R}_{\geq 0} \times [\hat k, \infty) \times \mathbb{R}^{m_1} \times \mathbb{R}^{m-m_1} \;\middle|\; \varphi(t)^2 \left(\|e_1\|^2 + \|e_2\|^2\right) < 1 \, \right\}$$

and define

$$f_1 : \mathcal{D} \times \mathbb{R}^m \to \mathbb{R}^{m_1},\ (t,k,e_1,e_2,\xi) \mapsto \left[D^{-1},0\right] \left(VA_{22}V^\top - kI_m\right) \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + \left[D^{-1},0\right] V\, \xi.$$

By differentiation of the second equation in (5.8), and using

$$\hat A_{22} = [0, I_{m-m_1}]VA_{22}V^\top \begin{bmatrix} 0 \\ I_{m-m_1} \end{bmatrix}, \quad \hat A_{21} = [0, I_{m-m_1}]VA_{22}V^\top \begin{bmatrix} I_{m_1} \\ 0 \end{bmatrix},$$

we get

$$0 = \hat A_{21}\dot e_1(t) + \hat A_{22}\dot e_2(t) - \dot k(t)e_2(t) - k(t)\dot e_2(t) + [0, I_{m-m_1}]V\, \tfrac{\rm d}{{\rm d}t}\Theta_1(e_1,e_2)(t). \tag{5.9}$$

Observe that the derivative of k is given by

$$\dot k(t) = 2\hat k \left(1 - \varphi(t)^2 \left(\|e_1(t)\|^2 + \|e_2(t)\|^2\right)\right)^{-2} \times \left(\varphi(t)\dot\varphi(t)\left(\|e_1(t)\|^2 + \|e_2(t)\|^2\right) + \varphi(t)^2\left(e_1(t)^\top \dot e_1(t) + e_2(t)^\top \dot e_2(t)\right)\right). \tag{5.10}$$

Now let

$$M : \mathcal{D} \to \mathbf{Gl}_{m-m_1}(\mathbb{R}),\ (t,k,e_1,e_2) \mapsto \left(\hat A_{22} - k\left(I_{m-m_1} + 2\varphi(t)^2\left(1 - \varphi(t)^2(\|e_1\|^2 + \|e_2\|^2)\right)^{-2} e_2e_2^\top\right)\right),$$

$$\Theta_2 : \mathcal{C}^\nu(\mathbb{R}_{\geq 0};\mathbb{R}^{m_1}) \times \mathcal{C}^\nu(\mathbb{R}_{\geq 0};\mathbb{R}^{m-m_1}) \to \mathcal{C}^{\nu}(\mathbb{R}_{\geq 0};\mathbb{R}^m),$$
$$(e_1,e_2) \mapsto \left(t \mapsto A_{22}\, \dot y_{\rm ref}(t) - \Gamma\, \ddot y_{\rm ref}(t) + \tfrac{\rm d}{{\rm d}t}\Psi(x_1^0, y_{\rm ref})(t) + \int_0^t A_{21}Qe^{Q(t-\tau)}A_{12}\, V^\top \begin{pmatrix} e_1(\tau) \\ e_2(\tau) \end{pmatrix} {\rm d}\tau \right)$$

and

$$f_2 : \mathcal{D} \times \mathbb{R}^{m_1} \times \mathbb{R}^m \to \mathbb{R}^{(m-m_1)},\ (t,k,e_1,e_2,\tilde e_1,\xi) \mapsto$$
$$2\hat k\left(1 - \varphi(t)^2\left(\|e_1\|^2 + \|e_2\|^2\right)\right)^{-2}\left(\varphi(t)\dot\varphi(t)\left(\|e_1\|^2 + \|e_2\|^2\right) + \varphi(t)^2\left(e_1^\top \tilde e_1\right)\right)e_2$$
$$-\hat A_{21}\tilde e_1 - [0, I_{m-m_1}]V\, \left(A_{21}A_{12}V^\top\left(e_1^\top, e_2^\top\right)^\top + \xi\right).$$

We show that M is well-defined. To this end let

$$G : \mathcal{D} \to \mathbb{R}^{(m-m_1)\times(m-m_1)},\ (t,k,e_1,e_2) \mapsto 2\varphi(t)^2\left(1 - \varphi(t)^2\left(\|e_1\|^2 + \|e_2\|^2\right)\right)^{-2} e_2e_2^\top$$

and observe that G is symmetric and positive semi-definite everywhere, hence there exist $\hat V : \mathcal{D} \to \mathbb{R}^{(m-m_1)\times(m-m_1)}$, $\hat V$ orthogonal everywhere, and $\hat D : \mathcal{D} \to \mathbb{R}^{(m-m_1)\times(m-m_1)}$, $\hat D$ a diagonal matrix with non-negative entries everywhere, such that $G = \hat V^{-1}\hat D\hat V$. Therefore, $(I+G)^{-1} = \hat V^{-1}(I+\hat D)^{-1}\hat V$ and $(I+\hat D)^{-1}$ is diagonal with entries in $(0,1]$ everywhere, which implies that $\|(I+G)^{-1}\| \leq 1$. Then, for all $(t,k,e_1,e_2) \in \mathcal{D}$, we obtain

$$\|k^{-1}(I + G(t,k,e_1,e_2))^{-1}\hat A_{22}\| \leq \hat k\|\hat A_{22}\| \leq \hat k\|A_{22}\| < 1.$$

and hence $k^{-1}(I + G(t, e_1, e_2))^{-1}\hat{A}_{22} - I$ is invertible, which gives invertibility of

$$M(t, k, e_1, e_2) = \hat{A}_{22} - k(I + G(t, k, e_1, e_2)).$$

Now, inserting $\dot{k}$ from (5.10) into (5.9) and rearranging according to $\dot{e}_2$ gives

$$M\big(t, k(t), e_1(t), e_2(t)\big)\,\dot{e}_2(t) = f_2\big(t, k(t), e_1(t), \dot{e}_1(t), e_2(t), \Theta_2(e_1, e_2)(t)\big).$$

With

$$\tilde{f}_2 : \mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^{(m-m_1)},\ (t, k, e_1, e_2, \xi_1, \xi_2) \mapsto M(t, k, e_1, e_2)^{-1} f_2(t, k, e_1, e_2, f_1(t, k, e_1, e_2, \xi_1), \xi_2),$$

and

$$\begin{aligned} f_3 : \mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R},\ (t, k, e_1, e_2, \xi_1, \xi_2) &\mapsto 2\hat{k}\left(1 - \varphi(t)^2(\|e_1\|^2 + \|e_2\|^2)\right)^{-2} \\ &\times \Big(\varphi(t)\dot{\varphi}(t)(\|e_1\|^2 + \|e_2\|^2) + \varphi(t)^2(e_1^\top f_1(t, k, e_1, e_2, \xi_1) + e_2^\top \tilde{f}_2(t, k, e_1, e_2, \xi_1, \xi_2))\Big) \end{aligned}$$

we get the system

$$\begin{aligned} \dot{e}_1(t) &= f_1(t, k(t), e_1(t), e_2(t), \Theta_1(e_1, e_2)(t)) \\ \dot{e}_2(t) &= \tilde{f}_2(t, k(t), e_1(t), e_2(t), \Theta_1(e_1, e_2)(t), \Theta_2(e_1, e_2)(t)) \\ \dot{k}(t) &= f_3(t, k(t), e_1(t), e_2(t), \Theta_1(e_1, e_2)(t), \Theta_2(e_1, e_2)(t)). \end{aligned} \tag{5.11}$$

$(k, e_1, e_2) \in \mathcal{C}^1([0, \rho); \mathbb{R}^{m+1})$ obtained from (e, k) is a local solution of (5.11) with

$$(k, e_1, e_2)(0) = \Big(\hat{k}, V([0, I_m, 0]T^{-1}x^0 - y_{\text{ref}}(0))\Big) =: \eta \tag{5.12}$$

and

$$\forall\, t \in [0, \rho) :\ (t, k(t), e_1(t), e_2(t)) \in \mathcal{D}.$$

Step 2: We show that the local solution (x_1, e, x_3, k) can be extended to a maximal solution, the graph of which leaves every compact subset of $\tilde{\mathcal{D}}$.
With $z = (k, e_1^\top, e_2^\top)^\top$ and appropriate $F : \mathcal{D} \times \mathbb{R}^{2m} \to \mathbb{R}^{m+1}$, we may write (5.11), (5.12) in the form

$$\dot{z}(t) = F(t, z(t), (Tz)(t)), \qquad z(0) = \eta, \tag{5.13}$$

where $Tz = (\Theta_1(e_1, e_2)^\top, \Theta_2(e_1, e_2)^\top)^\top$ and $T : \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^{m+1}) \to \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^{2m})$ is an operator with the properties as in ([25], Def. 2.1) (note that in [25] only operators with domain $\mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R})$ are considered, but the generalization to domain $\mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^q)$ is straightforward). It is immediate that T satisfies properties (i)–(iii) in ([25], Def. 2.1); (iv) follows from the fact that $\sigma(Q) \subseteq \mathbb{C}_-$ by the asymptotically stable zero dynamics (*cf.* also (A.5)) and $y_{\text{ref}} \in \mathcal{B}^{\nu+2}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$.

Furthermore, for $\mu := \max\{1, \nu\}$ and the functions defined in Step 1, we find that f_1 and f_2 are μ-times continuously differentiable (since $\varphi \in \mathcal{C}^{\nu+1}(\mathbb{R}_{\geq 0}; \mathbb{R})$). Furthermore, M is μ-times continuously differentiable and invertible on $\mathcal{D}$, hence M^{-1} is μ-times continuously differentiable as well. Finally, this gives that $\tilde{f}_2$ and f_3 are μ-times continuously differentiable and hence we have $F \in \mathcal{C}^\mu(\mathcal{D} \times \mathbb{R}^{2m}; \mathbb{R}^{m+1})$.

Let $\tilde{z} = (k, e_1^\top, e_2^\top)^\top \in \mathcal{C}^1([0, \rho); \mathbb{R}^{m+1})$ be the local solution of (5.11) obtained at the end of Step 1. Then $\tilde{z}$ solves (5.13). Observe that, since F is μ-times continuously differentiable and T is essentially an integral-operator, *i.e.*, it increments the degree of differentiability, we have $\tilde{z} \in \mathcal{C}^{\mu+1}([0, \rho); \mathbb{R}^{m+1})$. Then ([25], Thm. B.1)[2] is applicable to the system (5.13) and we may conclude that

(a) there exists a solution of (5.13), *i.e.*, a function $z \in \mathcal{C}([0, \rho); \mathbb{R}^{m+1})$ for some $\rho \in (0, \infty]$ such that z is locally absolutely continuous, $z(0) = \eta$, $(t, z(t)) \in \mathcal{D}$ for all $t \in [0, \rho)$ and (5.13) holds for almost all $t \in [0, \rho)$,

[2] In [25] a domain $\mathcal{D} \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R}$ is considered, but the generalization to the higher dimensional case is only a technicality.

(b) every solution can be extended to a maximal solution $z \in \mathcal{C}([0,\omega);\mathbb{R}^{m+1})$, *i.e.*, z has no proper right extension that is also a solution,
(c) if $z \in \mathcal{C}([0,\rho);\mathbb{R}^{m+1})$ is a maximal solution, then the closure of graph z is not a compact subset of $\mathcal{D}$.

Property (c) follows since F is locally essentially bounded, as it is at least continuously differentiable. Clearly $\tilde{z}$ is a solution (in the context of (a)) of (5.13), hence by (b) it can be extended to a maximal solution $\hat{z} \in \mathcal{C}([0,\omega);\mathbb{R}^{m+1})$. Similar to $\tilde{z}$, $\hat{z}$ is $(\mu+1)$-times continuously differentiable.

We show that the extended solution $\hat{z}$ leads to an extended solution of (5.7). Clearly, $\hat{z}$ is a solution of (5.11). Integrating the equations for k and e_2 in (5.11) and invoking consistency of the initial values gives that (k, e_1, e_2) also solve the problem (5.8) and this leads to a maximal solution $(x_1, e, x_3, k) \in \mathcal{C}^1([0,\omega);\mathbb{R}^{n+1})$, $\omega \in (0,\infty]$, of (5.7) (extension of the original local solution (x_1, e, x_3, k) - for brevity we use the same notation) with graph $(x_1, e, x_3, k) \subseteq \tilde{\mathcal{D}}$. Furthermore, by (c) we have

$$\text{the closure of graph}\,(x_1, e, x_3, k) \text{ is not a compact subset of } \tilde{\mathcal{D}}. \tag{5.14}$$

Step 3: We show that k is bounded. Seeking a contradiction, assume that $k(t) \to \infty$ for $t \to \omega$. Using $e_1(\cdot) = [I_{m_1}, 0]Ve(\cdot)$ and $e_2(\cdot) = [0, I_{m-m_1}]Ve(\cdot)$, we obtain from (5.8) that

$$\|e_2(t)\| \le \|\big(\hat{A}_{22} - k(t)I_{m-m_1}\big)^{-1}\| \Big(\|\hat{A}_{21}e_1(t)\| + \|[0, I_{m-m_1}]V\Theta_1(e_1,e_2)(t)\|\Big).$$

Observing that, since $\|\hat{A}_{22}\| \le \|A_{22}\| < \hat{k}$,

$$\left\|\big(\hat{A}_{22} - k(t)I_{m-m_1}\big)^{-1}\right\| = k(t)^{-1}\left\|\big(I_{m-m_1} - k(t)^{-1}\hat{A}_{22}\big)^{-1}\right\| \le k(t)^{-1}\frac{1}{1 - k(t)^{-1}\left\|\hat{A}_{22}\right\|} \le k(t)^{-1}\frac{\hat{k}}{\hat{k} - \left\|\hat{A}_{22}\right\|},$$

and invoking boundedness of e_1 (since e evolves within the funnel) and boundedness of $\Theta_1(e_1,e_2)$ (since $y_{\rm ref} \in \mathcal{B}^{\nu+2}(\mathbb{R}_{\ge 0};\mathbb{R}^m)$ and (A.5) holds) we obtain

$$\|e_2(t)\| \le k(t)^{-1}\frac{\hat{k}}{\hat{k} - \left\|\hat{A}_{22}\right\|}\left(\left\|\hat{A}_{21}e_1\right\|_\infty + \|[0, I_{m-m_1}]V\Theta_1(e_1,e_2)\|_\infty\right) \underset{t\to\omega}{\longrightarrow} 0. \tag{5.15}$$

Now, if $m_1 = 0$ then $e = e_2$ and we have $\lim_{t\to\omega}\|e(t)\| = 0$, which implies, by boundedness of φ, $\lim_{t\to\omega}\varphi(t)^2\|e(t)\|^2 = 0$, hence $\lim_{t\to\omega}k(t) = \hat{k}$, a contradiction. Hence, in the following we assume that $m_1 > 0$.

Let $\delta \in (0,\omega)$ be arbitrary but fix and $\lambda := \inf_{t\in(0,\omega)}\varphi(t)^{-1} > 0$. Since $\dot{\varphi}$ is bounded and $\liminf_{t\to\infty}\varphi(t) > 0$ we find that $\frac{\rm d}{{\rm d}t}\,\varphi|_{[\delta,\infty)}(\cdot)^{-1}$ is bounded and hence there exists a Lipschitz bound $L > 0$ of $\varphi|_{[\delta,\infty)}(\cdot)^{-1}$. Furthermore, let $\hat{A}_{11} := [I_{m_1}, 0]VA_{22}V^\top[I_{m_1}, 0]^\top$, $\hat{A}_{12} := [I_{m_1}, 0]VA_{22}V^\top[0, I_{m-m_1}]^\top$ and

$$\begin{aligned}
\alpha &:= \left\|D^{-1}\hat{A}_{11}\right\|\|e_1\|_\infty + \left\|\left[D^{-1}, 0\right]V\Theta_1(e_1,e_2)\right\|_\infty,\\
\beta &:= \tfrac{2}{\lambda\hat{k}}\left\|D^{-1}\hat{A}_{12}\right\|,\\
\gamma &:= \tfrac{\hat{k}}{\hat{k} - \|\hat{A}_{22}\|}\left(\left\|\hat{A}_{21}e_1\right\|_\infty + \|[0, I_{m-m_1}]V\Theta_1(e_1,e_2)\|_\infty\right),\\
\kappa &:= \tfrac{\lambda^2\hat{k}}{4\,\sigma_{\max}(\Gamma)} > 0,
\end{aligned}$$

where $\sigma_{\max}(\Gamma)$ denotes the largest eigenvalue of the positive semi-definite matrix Γ and $\sigma_{\max}(\Gamma) > 0$ since $m_1 > 0$. Choose $\varepsilon > 0$ small enough so that

$$\varepsilon \le \min\left\{\frac{\lambda}{2}, \min_{t\in[0,\delta]}\big(\varphi(t)^{-1} - \|e_1(t)\|\big)\right\}$$

and

$$L \leq -\alpha - \beta\gamma\varepsilon + \frac{\kappa}{\varepsilon}\cdot \tag{5.16}$$

We show that

$$\forall\, t \in (0, \omega):\ \ \varphi(t)^{-1} - \|e_1(t)\| \geq \varepsilon. \tag{5.17}$$

By definition of ε this holds on $(0, \delta]$. Seeking a contradiction suppose that

$$\exists\, t_1 \in [\delta, \omega):\ \ \varphi(t_1)^{-1} - \|e_1(t_1)\| < \varepsilon.$$

Then for

$$t_0 := \max\left\{\ t \in [\delta, t_1)\ \middle|\ \varphi(t)^{-1} - \|e_1(t)\| = \varepsilon\ \right\}$$

we have for all $t \in [t_0, t_1]$ that

$$\varphi(t)^{-1} - \|e_1(t)\| \leq \varepsilon \quad \text{and} \quad \|e_1(t)\| \geq \varphi(t)^{-1} - \varepsilon \geq \lambda - \varepsilon \geq \frac{\lambda}{2}$$

and

$$k(t) = \frac{\hat{k}}{1 - \varphi(t)^2\|e(t)\|^2} \geq \frac{\hat{k}}{1 - \varphi(t)^2\|e_1(t)\|^2} = \frac{\hat{k}}{(1 - \varphi(t)\|e_1(t)\|)(1 + \varphi(t)\|e_1(t)\|} \geq \frac{\hat{k}}{2\varepsilon\varphi(t)} \geq \frac{\lambda\hat{k}}{2\varepsilon}\cdot$$

Now we have, for all $t \in [t_0, t_1]$,

$$\begin{aligned}
\frac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|e_1(t)\|^2 &= e_1(t)^\top \dot{e}_1(t) \\
&= e_1(t)^\top \left(D^{-1}\left(\hat{A}_{11} - k(t)I_{m_1}\right)e_1(t) + D^{-1}\hat{A}_{12}e_2(t) + \left[D^{-1}, 0\right] V\,\Theta_1(e_1, e_2)(t)\right) \\
&\leq \alpha\|e_1(t)\| + \left\|D^{-1}\hat{A}_{12}\right\| \|e_2(t)\|\|e_1(t)\| - \frac{\lambda\hat{k}}{2\varepsilon} e_1(t)^\top D^{-1} e_1(t)^\top \\
&\leq \alpha\|e_1(t)\| + \left\|D^{-1}\hat{A}_{12}\right\| \|e_2(t)\|\|e_1(t)\| - \frac{\lambda\hat{k}}{2\varepsilon\,\sigma_{\max}(\Gamma)}\|e_1(t)\|^2.
\end{aligned}$$

Moreover, from the inequality in (5.15) we obtain that, for all $t \in [t_0, t_1]$,

$$\|e_2(t)\| \leq k(t)^{-1}\gamma \leq \frac{2}{\lambda\hat{k}}\gamma\varepsilon.$$

This yields that

$$\frac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|e_1(t)\|^2 \leq \left(\alpha + \beta\gamma\varepsilon - \frac{\kappa}{\varepsilon}\right)\|e_1(t)\| \overset{(5.16)}{\leq} -L\|e_1(t)\|.$$

Therefore, using $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e_1(t)\|^2 = \|e_1(t)\|\frac{\mathrm{d}}{\mathrm{d}t}\|e_1(t)\|$, we find that

$$\begin{aligned}
\|e_1(t_1)\| - \|e_1(t_0)\| &= \int_{t_0}^{t_1} \frac{1}{2}\|e_1(t)\|^{-1}\tfrac{\mathrm{d}}{\mathrm{d}t}\|e_1(t)\|^2\ \mathrm{d}t \\
&\leq -L(t_1 - t_0) \leq -|\varphi(t_1)^{-1} - \varphi(t_0)^{-1}| \leq \varphi(t_1)^{-1} - \varphi(t_0)^{-1},
\end{aligned}$$

and hence

$$\varepsilon = \varphi(t_0)^{-1} - \|e_1(t_0)\| \leq \varphi(t_1)^{-1} - \|e_1(t_1)\| < \varepsilon,$$

a contradiction.

Therefore, (5.17) holds and by (5.15) there exists $\tilde{t} \in [0, \omega)$ such that $\|e_2(t)\| \leq \varepsilon$ for all $t \in [\tilde{t}, \omega)$. Then, invoking $\varepsilon \leq \frac{\lambda}{2}$, we obtain for all $t \in [\tilde{t}, \omega)$

$$\|e(t)\|^2 = \|e_1(t)\|^2 + \|e_2(t)\|^2 \leq (\varphi(t)^{-1} - \varepsilon)^2 + \varepsilon^2 \leq \varphi(t)^{-2} - 2\varepsilon\lambda + 2\varepsilon^2 \leq \varphi(t)^{-2} - 2\varepsilon^2.$$

This implies boundedness of k, a contradiction.

Step 4: We show that x_1 and x_3 are bounded. To this end, observe that $z = (k, e_1^\top, e_2^\top)^\top$ solves (5.13) and, by Step 3, z is bounded. Using (A.5) and $y_{\mathrm{ref}} \in \mathcal{B}^{\nu+2}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$ we find that Tz is bounded as well. This implies, since F is continuously differentiable, that $\dot{z}$ is bounded. Then again, we obtain that $\frac{\mathrm{d}}{\mathrm{d}t}(Tz)$ is bounded and differentiating (5.13) gives boundedness of $\ddot{z}$. Iteratively, we have that

$$\forall\, j = 0, \dots, \nu+1 : \ \Big(\exists\, c_0, \dots, c_j > 0\ \forall\, t \in [0,\omega) : \ \|z(t)\| \leq c_0, \dots, \|z^{(j)}(t)\| \leq c_j\Big) \\ \implies \Big(\exists\, C > 0\ \forall\, t \in [0,\omega) : \ \|(Tz)^{(j)}(t)\| \leq C\Big)$$

and successive differentiation of (5.13) finally yields that $z, \dot{z}, \dots, z^{(\nu+1)}$ are bounded. This gives boundedness of $e, \dot{e}, \dots, e^{(\nu+1)}$. Then, from the first and third equation in (5.7) and the fact that $\sigma(Q) \subseteq \mathbb{C}_-$ and $y_{\mathrm{ref}} \in \mathcal{B}^{\nu+2}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$, it is immediate that x_1 and x_3 are bounded.

Step 5: We show that $\omega = \infty$. First note that by Step 3 and Step 4 we have that $(x_1, e, x_3, k) : [0,\omega) \to \mathbb{R}^{n+1}$ is bounded. Further noting that boundedness of k is equivalent to (5.6) (for $t \in [0,\omega)$), the assumption $\omega < \infty$ implies existence of a compact subset $\mathcal{K} \subseteq \tilde{\mathcal{D}}$ such that graph$\,(x_1, e, x_3, k) \subseteq \mathcal{K}$. This contradicts (5.14).

Step 6: It remains to show (ii). This follows from

$$\forall\, t > 0 : \ k(t) = \hat{k} + k(t)\varphi(t)^2\|e(t)\|^2 \overset{(5.6)}{\leq} \hat{k} + k(t)\varphi(t)^2(\varphi(t)^{-1} - \varepsilon)^2 = \hat{k} + k(t)(1 - \varphi(t)\varepsilon)^2. \qquad \square$$

Remark 5.4.

(i) Note that ν in Theorem 5.3 is in general not known explicitly. However, we have, by Theorem 4.3, the estimate $\nu \leq n_3 = n - k - m$, where $k = \dim \max(E, A, B; \ker C)$. Hence, choosing φ and y_{ref} to be $(n - m + 2)$-times continuously differentiable will always suffice.

(ii) Theorem 5.3 specifies ([11], Rem. 6.4 (i)): it is shown that, compared ([11], Thm. 6.2), regularity is not needed and the assumptions of ([11], Thm. 6.2) can be relaxed, while funnel control is still feasible.

(iii) The problem of finding a solution of (5.13) with the properties (a)–(c) as in the proof of Theorem 5.3 is not solved just by the consistency of the initial value, *i.e.*, existence of a local solution, since it is not clear that this solution can be extended to a maximal solution which leaves every compact subset of $\mathcal{D}$. Solvability for any other initial value (for (5.13)) is required for this.

Remark 5.5 (Passive electrical networks)**.** The findings of the present paper, in particular the application of the funnel controller, can also be applied to a class of passive electrical networks. A common way of modeling electrical networks is the *modified nodal analysis* (MNA), see *e.g.* [20, 23, 47]. This modeling procedure results in a description of the circuit by a system of the form (1.1), where the inputs and outputs are appropriately chosen and the matrices E, A, B, C have specific properties, see also [37]. Omitting the details of this procedure and the circuit theoretic background, we are only interested in the resulting system (1.1) and its properties.

From [37] we have that, in a MNA model of a passive electrical circuit,

$$\left.\begin{array}{l} sE - A \text{ is regular,} \\ G(s) := C(sE - A)^{-1}B \text{ has no poles in } \mathbb{C}_+, \\ \forall\, \lambda \in \mathbb{C}_+ : \ G(\lambda) + \overline{G(\lambda)}^\top \geq 0. \end{array}\right\} \tag{5.18}$$

The second and third property in (5.18) state that $G(s)$ is *positive real*. Note that in a MNA model we have even more structure than stated in (5.18), such as $C = B^\top$ and a special block structure of E, A, B, C. However, Condition (5.18) is sufficient for our purposes.

We consider the class of systems which satisfy (5.18) and

$$G(s) \text{ is invertible over } \mathbb{R}(s). \tag{5.19}$$

Property (5.19) implies (following the lines of the proof of Prop. B.3) that the zero dynamics are autonomous and $[E, A, B, C]$ is right-invertible. Since $G(s)$ is positive real and invertible over $\mathbb{R}(s)$, we may infer that $G(s)^{-1}$ is positive real as well. Then, by ([2], p. 216) (see also [37], Prop. 7) we obtain that $G(s)^{-1} = G_{\mathrm{p}}(s) + sM$, where $G_{\mathrm{p}}(s) \in \mathbb{R}(s)^{m\times m}$ is proper and $M \in \mathbb{R}^{m\times m}$ satisfies $M = M^\top \geq 0$. As in Remark A.4 we may now conclude that $[0, I_m]L(s)[0, I_m]^\top = -G(s)^{-1}$ and hence we obtain existence of

$$\Gamma = \lim_{s\to\infty} s^{-1}G(s)^{-1} = M.$$

Therefore, $[E, A, B, C]$ satisfies the assumptions of Lemma A.3. If furthermore asymptotically stable zero dynamics are assumed, then by Theorem 5.3 the funnel controller works for the class of systems satisfying (5.18) and (5.19). A larger class of circuits without the requirement (5.19) has been investigated in [8].

6. Simulations

For purposes of illustration we consider an example of a differential-algebraic system (1.1) and apply the funnel controller (5.2). The simulation of the funnel controller for a mechanical system with springs, masses and dampers which has a proper inverse transfer function is performed in ([11], Sect. 7.1). In ([11], Sect. 7.1) an academic example of a system with singular matrix pencil $sE - A$ is considered and it is shown that the funnel controller works for this system, however a proof was not included. This was the reason for the conjecture in ([11], Rem. 6.4) that the funnel controller works for a much larger class than systems with proper inverse transfer function. It is now clear that funnel control is feasible for this example since it satisfies the assumptions of Theorem 5.3. The simulation of the funnel controller for a differential-algebraic system with strict relative degree one can be found in ([10], Sect. 6). For all of the aforementioned systems, feasibility of funnel control has been proved in Theorem 5.3.

Due to the above reasons, and in order to point out the peculiarities, in the present section we only state an academic example which has neither proper inverse transfer function nor strict relative degree one, but satisfies the assumptions of Theorem 5.3. Consider system (1.1) with

$$[E, A, B, C] := \left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 & -2 \\ 3 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right]. \tag{6.1}$$

It is immediate that $[E, A, B, C]$ is right-invertible and in the form (4.3), has asymptotically stable zero dynamics, and the matrix

$$\Gamma = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

satisfies (5.3) and $\Gamma = \Gamma^\top \geq 0$. We set

$$\hat{k} := 2 > \sqrt{2} = \left\| \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\| = \|A_{22}\| = \left\| \lim_{s\to\infty} ([0, I_m]L(s)[0, I_m]^\top + s\Gamma) \right\|, \tag{6.2}$$

where $L(s)$ is an inverse of the system pencil, see also Step 1 in the proof of Theorem 5.3 for the latter equalities. The (consistent) initial value for the closed-loop system (6.1), (5.2) is chosen as

$$x^0 = (-4, 3, -2)^\top. \tag{6.3}$$

As reference signal $y_{\mathrm{ref}} : \mathbb{R}_{\geq 0} \to \mathbb{R}^2$, we take the first and second component of the (chaotic) solution of the following initial-value problem for the Lorenz system

$$\begin{aligned} \dot{\xi}_1(t) &= 10\,(\xi_2(t) - \xi_1(t)), & \xi_1(0) &= 5 \\ \dot{\xi}_2(t) &= 28\,\xi_1(t) - \xi_1(t)\,\xi_3(t) - \xi_2(t), & \xi_2(0) &= 5 \\ \dot{\xi}_3(t) &= \xi_1(t)\,\xi_2(t) - \tfrac{8}{3}\,t\xi_3(t), & \xi_3(0) &= 5. \end{aligned} \tag{6.4}$$

It is well-known that the unique global solution of (6.4) is bounded with bounded derivative on the positive real axis, see for example ([44], Appendix C). The solution of (6.4) is depicted in Figure 3.

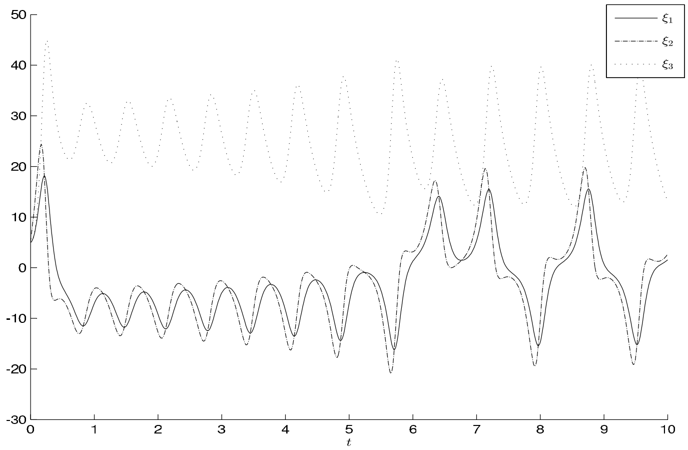

FIGURE 3. Components $\xi_i(\cdot)$ of the Lorenz system (6.4).

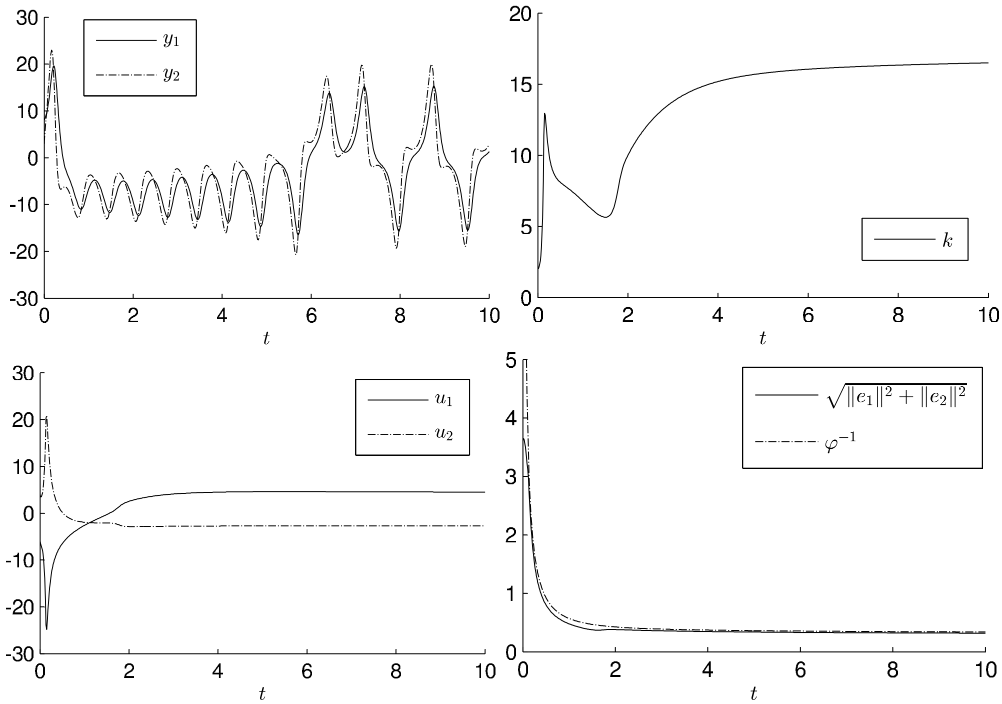

FIGURE 4. Simulation of the funnel controller (5.2) with funnel boundary specified in (6.5) and reference signal $y_{\rm ref}(\cdot) = (\xi_1(\cdot), \xi_2(\cdot))^\top$ given in (6.4) applied to system (6.1) with initial data (6.2), (6.3).

The funnel $\mathcal{F}_\varphi$ is determined by the function

$$\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \quad t \mapsto 0.5\ te^{-t} + 2\ \arctan t. \tag{6.5}$$

Note that this prescribes an exponentially (exponent 1) decaying funnel in the transient phase $[0, T]$, where $T \approx 3$, and a tracking accuracy quantified by $\lambda = 1/\pi$ thereafter, see Figure 4d.

The simulation has been performed in MATLAB (solver: ode15s, relative tolerance: 10^{-14}, absolute tolerance: 10^{-5}). In Figure 4 the simulation, over the time interval $[0, 10]$, of the funnel controller (5.2) with funnel boundary specified in (6.5) and reference signal $y_{\mathrm{ref}}(\cdot) = (\xi_1(\cdot), \xi_2(\cdot))^\top$ given in (6.4), applied to system (6.1) with initial data (6.2), (6.3) is depicted. Figure 4a shows the output components $y_1(\cdot)$ and $y_2(\cdot)$ tracking the rather "vivid" reference signal $y_{\mathrm{ref}}(\cdot)$ within the funnel shown in Figure 4d. Note that an action of the input components $u_1(\cdot)$ and $u_2(\cdot)$ in Figure 4c and the gain function $k(\cdot)$ in Figure 4b is required only if the error $\|e(t)\|$ is close to the funnel boundary $\varphi(t)^{-1}$. It can be seen that initially the error is very close to the funnel boundary and hence the gain rises sharply. Then, at approximately $t = 0.2$, the distance between error and funnel boundary gets larger and the gain drops accordingly. After $t = 2$, the error gets close to the funnel boundary again which causes the gain to rise again. This in particular shows that the gain function $k(\cdot)$ is non-monotone.

Appendix A. Polynomial matrices

The purpose of this section is, essentially, to derive a simplification of the form (4.3) under the condition that for a left inverse $L(s)$ of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ the matrix

$$\Gamma = -\lim_{s\to\infty} s^{-1}[0, I_m]L(s)\begin{bmatrix} 0 \\ I_p \end{bmatrix} \in \mathbb{R}^{m\times p} \tag{A.1}$$

exists. This simplified form then provides an operator differential-algebraic equation which is used in the proof of Theorem 5.3 to show feasibility of funnel control.

In the following we parameterize all left inverses of the system pencil for right-invertible systems with autonomous zero dynamics; this is important to read off some properties of the block matrices in the form (4.3). Furthermore, it is shown that the lower right block in any left inverse is well-defined and therefore Γ in (A.1) is well-defined, provided it exists. The existence of a left inverse of the system pencil over $\mathbb{R}(s)$ is clear, since by Proposition 3.5 autonomous zero dynamics lead to a full column rank of the system pencil over $\mathbb{R}[s]$.

Lemma A.1 (Left inverse of system pencil)**.** *Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ be right-invertible and have autonomous zero dynamics. Then $L(s) \in \mathbb{R}(s)^{(n+m)\times(l+p)}$ is a left inverse of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ if, and only if, using the notation from Theorem 4.3,*

$$L(s) = \begin{bmatrix} T & 0 \\ 0 & I_m \end{bmatrix}\begin{bmatrix} (sI_k - Q)^{-1} & 0 & 0 & X_{14}(s) & X_{15}(s) \\ 0 & 0 & 0 & X_{24}(s) & I_p \\ 0 & 0 & (sN - I_{n_3})^{-1} & X_{34}(s) & X_{35}(s) \\ X_{41}(s) & I_m & X_{43}(s) & X_{44}(s) & X_{45}(s) \end{bmatrix}\begin{bmatrix} S & 0 \\ 0 & I_p \end{bmatrix}, \tag{A.2}$$

where $[X_{14}(s)^\top, X_{24}(s)^\top, X_{34}(s)^\top, X_{44}(s)^\top]^\top \in \mathbb{R}(s)^{(n+m)\times(l+p-n-m)}$ and

$$\begin{aligned} X_{15}(s) &= (sI - Q)^{-1}A_{12}, & X_{35}(s) &= -s(sN - I)^{-1}E_{32}, \\ X_{41}(s) &= A_{21}(sI - Q)^{-1}, & X_{43}(s) &= -sE_{23}(sN - I)^{-1}, \\ X_{45}(s) &= -(sE_{22} - A_{22}) + A_{21}(sI - Q)^{-1}A_{12} + s^2E_{23}(sN - I)^{-1}E_{32}, \end{aligned}$$

and $L(s)$ is partitioned according to the block structure of (4.3).

If $L_1(s), L_2(s) \in \mathbb{R}(s)^{(n+m)\times(l+p)}$ are two left inverse matrices of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$, then

$$[0, I_m]L_1(s)\begin{bmatrix} 0 \\ I_p \end{bmatrix} = [0, I_m]L_2(s)\begin{bmatrix} 0 \\ I_p \end{bmatrix}.$$

Furthermore, if Γ in (A.1) exists, then it is well-defined.

Proof. By Proposition 4.6 we have $\operatorname{rk} C = p$ and hence the assumptions of Theorem 4.3 are satisfied. The statements can then be verified by a simple calculation. □

Now we investigate the consequences of the assumption of existence of Γ in (A.1).

Lemma A.2 (Consequences of existence of Γ)**.** *Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ be right-invertible and have autonomous zero dynamics. Suppose that, for a left inverse $L(s)$ of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ over $\mathbb{R}(s)$, the matrix Γ in (A.1) exists. Then, using the notation from Theorem 4.3, we have*

$$\forall\, k = 0, \ldots, \nu - 1: \;\; E_{23} N^k E_{32} = 0. \tag{A.3}$$

and, furthermore, $\Gamma = E_{22}$.

Proof. The left inverse $L(s)$ is given in (A.2) and Γ is independent of the choice of $L(s)$ by Lemma A.1. By existence of Γ the matrix $s^{-1}[0, I_m]L(s)[0, I_p]^\top$ is proper, which implies that

$$s^{-1}X_{45}(s) = -\left(E_{22} - s^{-1}A_{22}\right) + s^{-1}A_{21}(sI - Q)^{-1}A_{12} + sE_{23}(sN - I)^{-1}E_{32}$$

is proper. Hence, $sE_{23}(sN - I)^{-1}E_{32} = \sum_{k=0}^{\nu-1} E_{23}N^kE_{32}s^{k+1}$ has to be proper. This yields (A.3) and the last statement is then an immediate consequence of $\Gamma = -\lim_{s\to\infty} s^{-1}X_{45}(s) = E_{22}$. □

The final result of this section, the simplification of the form (4.3), relies on partially solving the equations (4.4) using the condition (A.3) derived in Lemma A.2.

Lemma A.3 (Behavior and underlying equations)**.** *Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ be right-invertible and have autonomous zero dynamics. Suppose that, for a left inverse $L(s)$ of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ over $\mathbb{R}(s)$, the matrix Γ in (A.1) exists. Then, using the notation from Theorem 4.3, for any $(x, u, y) \in \mathfrak{B}_{(1.1)} \cap \left(\mathcal{C}^1(\mathbb{R};\mathbb{R}^n) \times \mathcal{C}^0(\mathbb{R};\mathbb{R}^m) \times \mathcal{C}^{\nu+1}(\mathbb{R};\mathbb{R}^p)\right)$ and $Tx = (x_1^\top, y^\top, x_3^\top)^\top \in \mathcal{C}^1(\mathbb{R};\mathbb{R}^{k+p+n_3})$, (Tx, u, y) solves*

$$\boxed{\begin{aligned} \dot{x}_1(t) &= Q\, x_1(t) + A_{12}\, y(t) \\ \Gamma\, \dot{y}(t) &= A_{22}y(t) + \Psi(x_1(0), y)(t) + u(t) \\ x_3(t) &= \textstyle\sum_{k=0}^{\nu-1} N^k E_{32}\, y^{(k+1)}(t), \\ 0 &= 0, \end{aligned}} \tag{A.4}$$

where

$$\Psi : \mathbb{R}^k \times \mathcal{C}^\nu(\mathbb{R};\mathbb{R}^m) \to \mathcal{C}^{\nu+1}(\mathbb{R};\mathbb{R}^m),\;\; (x_1^0, y) \mapsto \left(t \mapsto A_{21}e^{Qt}x_1^0 + \int_0^t A_{21}e^{Q(t-\tau)}A_{12}\, y(\tau)\; \mathrm{d}\tau\right).$$

Ψ is linear in each argument and, if $\sigma(Q) \subseteq \mathbb{C}_-$, then Ψ has the property

$$\Psi\big(\mathbb{R}^k \times (\mathcal{L}^\infty(\mathbb{R};\mathbb{R}^p) \cap \mathcal{C}^\nu(\mathbb{R};\mathbb{R}^p))\big) \subseteq \mathcal{L}^\infty(\mathbb{R};\mathbb{R}^m) \cap \mathcal{C}^{\nu+1}(\mathbb{R};\mathbb{R}^m). \tag{A.5}$$

Proof. The assumptions of Theorem 4.3 are satisfied and, invoking Lemma A.2, it is clear that the respective first and third equations in (4.4) and (A.4) coincide. By Proposition 4.6 and right-invertibility of $[E, A, B, C]$, the fourth equation in (4.4) reads $0 = 0$. Therefore, it remains to show that under the additional assumption of existence of Γ, the second equation in (A.4) follows from (4.4). To this end, observe that by Lemma A.2, namely (A.3), the second equation in (4.4) reads

$$E_{22}\dot{y}(t) = A_{22}y(t) + A_{21}x_1(t) + u(t). \tag{A.6}$$

Insertion of the solution of the first equation in (4.4) into (A.6) then yields the assertion.

Statement (A.5) about Ψ is obvious from the representation of Ψ and the fact that if $\sigma(Q) \subseteq \mathbb{C}_-$, then there exist $\mu, M > 0$ such that

$$\forall\, t \geq 0: \;\; \|e^{Qt}\| \leq Me^{-\mu t}.$$

□

Remark A.4 (Regular systems)**.** Let $[E, A, B, C] \in \Sigma_{n,n,m,m}$ be such that $sE - A$ is regular. If $L(s)$ is a left inverse of the system pencil, then we have

$$\begin{aligned}\begin{bmatrix} I_n & (sE-A)^{-1}B \\ 0 & I_m \end{bmatrix} &= L(s) \begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix} \begin{bmatrix} I_n & (sE-A)^{-1}B \\ 0 & I_m \end{bmatrix} \\ &= L(s) \begin{bmatrix} sE-A & 0 \\ -C & -C(sE-A)^{-1}B \end{bmatrix},\end{aligned}$$

and therefore $C(sE - A)^{-1}B$ is invertible over $\mathbb{R}(s)$, and

$$H(s) := -[0, I_m]L(s) \begin{bmatrix} 0 \\ I_m \end{bmatrix} = \big(C(sE-A)^{-1}B\big)^{-1},$$

i.e., $H(s)$ is exactly the inverse transfer function of the system $[E, A, B, C]$. Note that, if $sE - A$ is not regular, then the transfer function $C(sE - A)^{-1}B$ does not exist, but $H(s)$ may still be defined in terms of the left inverse $L(s)$. Therefore, $H(s)$ can be viewed as a generalization of the *inverse* of the transfer function.

Appendix B. Relative degree

In this section we give the definition of vector relative degree for transfer functions of regular systems $[E, A, B, C] \in \Sigma_{n,n,m,p}$ and relate this property to the findings of the paper.

Definition B.1 (Vector relative degree)**.** We say that $G(s) \in \mathbb{R}(s)^{p\times m}$ has *vector relative degree* $(\rho_1, \ldots, \rho_p) \in \mathbb{Z}^{1\times p}$, if the limit

$$D := \lim_{s\to\infty} \operatorname{diag}(s^{\rho_1}, \ldots, s^{\rho_p})\, G(s) \in \mathbb{R}^{p\times m}$$

exists and satisfies $\operatorname{rk} D = p$.

Remark B.2 (Vector relative degree)**.**

(i) It is an easy calculation that if $G(s) \in \mathbb{R}(s)^{p\times m}$ has a vector relative degree, then the vector relative degree is unique. However, a vector relative degree does not necessarily exist, even if $G(s)$ is (strictly) proper; see Example B.4.

(ii) If $G(s) \in \mathbb{R}(s)^{m\times m}$ has vector relative degree $(\rho_1, \ldots, \rho_m) \in \mathbb{Z}^{1\times m}$, then $\rho = \rho_1 = \ldots = \rho_m$ if, and only if, $G(s)$ has strict relative degree ρ.

(iii) Isidori ([28], Sect. 5.1) introduced a local version of vector relative degree for nonlinear systems. Definition B.1 coincides with Isidori's definition if strictly proper transfer functions are considered. In this sense, Definition B.1 is a generalization to arbitrary rational transfer functions. For linear ODE systems a global version of the vector relative degree has been stated in [34]. It is straightforward to show that $[I_n, A, B, C] \in \Sigma_{n,n,m,m}$ has vector relative degree $(\rho_1, \ldots, \rho_p)$ in the sense of ([34], Def. 2.1) if, and only if, $C(sI - A)^{-1}B$ has vector relative degree $(\rho_1, \ldots, \rho_p)$.

In the following we show that a regular system with transfer function which has componentwise vector relative degree smaller or equal to one, is included in the class of systems investigated in this paper, *i.e.*, in particular satisfies the assumptions of Lemma A.3. If furthermore asymptotically stable zero dynamics and $\Gamma = \Gamma^\top \geq 0$ are assumed, then by Theorem 5.3 funnel control is feasible for this class of systems.

Proposition B.3 (Vector relative degree ≤ 1 implies existence of Γ)**.** *Let $[E, A, B, C] \in \Sigma_{n,n,m,m}$ be such that $sE - A$ is regular and $G(s) := C(sE - A)^{-1}B$ has vector relative degree $(\rho_1, \ldots, \rho_m)$ with $\rho_i \leq 1$ for all $i = 1, \ldots, m$. Then*

(i) $\mathcal{ZD}_{(1.1)}$ *are autonomous,*

(ii) $[E, A, B, C]$ *is right-invertible,*

(iii) $\begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix}$ *has inverse* $L(s)$ *over* $\mathbb{R}(s)$ *and the matrix* Γ *in* (5.3) *exists and satisfies*

$$\forall\, j = 1,\ldots,m:\ \ \Gamma e_j = \begin{cases} \left(\lim\limits_{s\to\infty} \operatorname{diag}\,(s^{\rho_1},\ldots,s^{\rho_p})\,G(s)\right)^{-1} e_j, & \text{if } \rho_j = 1, \\ 0, & \text{if } \rho_j < 1. \end{cases} \tag{B.1}$$

Proof.
Step 1: We show that $G(s)$ is invertible over $\mathbb{R}(s)$. To this end, let $F(s) := \operatorname{diag}\,(s^{\rho_1},\ldots,s^{\rho_p})\,G(s)$. Since

$$D := \lim_{s\to\infty} F(s) = \lim_{s\to\infty} \operatorname{diag}\,(s^{\rho_1},\ldots,s^{\rho_p})\,G(s) \in \mathbf{Gl}_m(\mathbb{R})$$

exists, $G_{\rm sp}(s) := F(s) - D \in \mathbb{R}(s)^{m\times m}$ is strictly proper, *i.e.*, $\lim_{s\to\infty} G_{\rm sp}(s) = 0$. Since D is invertible, $F(s)$ is invertible as well, as by the Sherman–Morrison–Woodbury formula

$$F(s)^{-1} = D^{-1} - D^{-1}G_{\rm sp}(s)(I + D^{-1}G_{\rm sp}(s))^{-1}D^{-1} \in \mathbb{R}(s)^{m\times m}. \tag{B.2}$$

It is then immediate that $G(s)$ has inverse $G(s)^{-1} = F(s)^{-1}\operatorname{diag}\,(s^{\rho_1},\ldots,s^{\rho_p})$ over $\mathbb{R}(s)$.

Step 2: We show (i). Using invertibility of $G(s)$ we calculate

$$\begin{aligned} &\begin{bmatrix} (sE-A)^{-1} & 0 \\ -G(s)^{-1}C(sE-A)^{-1} & -G(s)^{-1} \end{bmatrix} \begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix} \begin{bmatrix} I_n & (sE-A)^{-1}B \\ 0 & I_m \end{bmatrix} \\ &\qquad = \begin{bmatrix} (sE-A)^{-1} & 0 \\ -G(s)^{-1}C(sE-A)^{-1} & -G(s)^{-1} \end{bmatrix} \begin{bmatrix} sE-A & 0 \\ -C & -G(s) \end{bmatrix} = I_{n+m}, \end{aligned}$$

which gives invertibility of the system pencil and thus the zero dynamics are autonomous by Proposition 3.5.

Step 3: We show (ii). It is clear that $\operatorname{rk} C = m$, since otherwise there exists $x \in \mathbb{R}^n \setminus \{0\}$ such that $x^\top C = 0$ and hence $x^\top G(s) = 0$, which contradicts invertibility of $G(s)$. Therefore, we find that $[E,A,B,C]$ is right-invertible by virtue of Remark 4.7.

Step 4: We show (iii). As in Remark A.4 we may conclude that $[0, I_m]L(s)[0, I_m]^\top = -G(s)^{-1}$ and

$$\begin{aligned} s^{-1}G(s)^{-1} &= \big(\operatorname{diag}\,(s^{\rho_1},\ldots,s^{\rho_p})G(s)\big)^{-1}\operatorname{diag}\,(s^{\rho_1-1},\ldots,s^{\rho_p-1}) \\ &\overset{\text{(B.2)}}{=} \big(D^{-1} - D^{-1}G_{\rm sp}(s)(I + D^{-1}G_{\rm sp}(s))^{-1}D^{-1}\big)\operatorname{diag}\,(s^{\rho_1-1},\ldots,s^{\rho_p-1}). \end{aligned}$$

Hence, using $\rho_i \le 1$ for $i = 1,\ldots,m$, we obtain existence of

$$\Gamma = \lim_{s\to\infty} s^{-1}G(s)^{-1} \in \mathbb{R}^{m\times m},$$

where $\Gamma e_j = D^{-1}e_j$ if $\rho_j = 1$ and $\Gamma e_j = 0$ if $\rho_j < 1$, for all $j = 1,\ldots,m$. □

We illustrate the vector relative degree and Proposition B.3 by means of an example.

Example B.4. Consider system (1.1) with

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix},\ B = C = I_2.$$

It can be seen that $sE - A$ is regular and

$$G(s) = C(sE-A)^{-1}B = \begin{bmatrix} s-1 & -2 \\ 0 & -3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s-1} & -\frac{2}{3(s-1)} \\ 0 & -\frac{1}{3} \end{bmatrix}.$$

We calculate

$$D := \lim_{s\to\infty} \operatorname{diag}(s,1)\, G(s) = \begin{bmatrix} 1 & -\frac{2}{3} \\ 0 & -\frac{1}{3} \end{bmatrix} \in \mathbf{Gl}_2(\mathbb{R}),$$

and hence $G(s)$ has vector relative degree $(1,0)$. Proposition B.3 then implies that $\mathcal{ZD}_{(1.1)}$ are autonomous, $[E,A,B,C]$ is right-invertible, and Γ in (5.3) exists. In fact, it is easy to see that the zero dynamics are asymptotically stable and

$$\Gamma = \lim_{s\to\infty} s^{-1} G(s)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

satisfies (B.1): $\Gamma e_1 = D^{-1} e_1 = \left[\begin{smallmatrix} 1 & -2 \\ 0 & -3 \end{smallmatrix}\right] e_1$ and $\Gamma e_2 = 0$. Since $\Gamma = \Gamma^\top \geq 0$, the assumptions of Theorem 5.3 are satisfied.

We like to stress that, compared to the above, the regular system (6.1) from Section 6 does not have a vector relative degree: while its transfer function $G(s)$ is proper, the limit

$$\lim_{s\to\infty} G(s) = -\frac{1}{3}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

does not have full row rank. Nevertheless, as shown in Section 6, funnel control is feasible.

Remark B.5 (High-frequency gain matrix)**.** For systems $[E,A,B,C] \in \Sigma_{n,n,m,m}$ with regular $sE - A$ and strict relative degree $\rho \in \mathbb{N}$ the matrix

$$\lim_{s\to\infty} s^{\rho}\, C(sE-A)^{-1}B$$

is called the high-frequency gain matrix, see [10]. If $\rho = 1$, then by Proposition B.3, Γ in (5.3) exists and we have, from the proof of Proposition B.3, $\Gamma = \left(\lim_{s\to\infty} s\, C(sE-A)^{-1}B\right)^{-1}$, *i.e.*, Γ is exactly the inverse of the high-frequency gain matrix. Since, furthermore, Γ is also defined when no high-frequency gain matrix exists, we may view the definition of Γ an appropriate generalization of the high-frequency gain matrix to DAEs which do not have a strict relative degree. In particular, if $C(sE-A)^{-1}B$ has proper inverse, then $\Gamma = 0$.

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