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APPROXIMATION AND UNIFORM POLYNOMIAL STABILITY OF C_0-SEMIGROUPS

L. Maniar[1] and S. Nafiri[1,*]

Abstract. Consider the classical solutions of the abstract approximate problems

$$x_n'(t) = A_n x_n(t),\ t \geq 0, \quad x_n(0) = x_{0n},\ n \in \mathbb{N},$$

given by $x_n(t) = T_n(t)x_{0n},\ t \geq 0, x_{0n} \in D(A_n)$, where A_n generates a sequence of C_0-semigroups of operators $T_n(t)$ on the Hilbert spaces H_n. Classical solutions of this problem may converge to 0 polynomially, but not exponentially, in the following sense

$$\|T_n(t)x\| \leq C_n t^{-\beta} \|A_n^{\alpha} x\|,\ x \in D(A_n^{\alpha}), \quad t > 0,\ n \in \mathbb{N},$$

for some constants C_n, α and $\beta > 0$. This paper has two objectives. First, necessary and sufficient conditions are given to characterize the uniform polynomial stability of the sequence $T_n(t)$ on Hilbert spaces H_n. Secondly, approximation in control of a one-dimensional hyperbolic-parabolic coupled system subject to Dirichlet$-$Dirichlet boundary conditions, is considered. The uniform polynomial stability of corresponding semigroups associated with approximation schemes is proved. Numerical experimental results are also presented.

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1. Introduction

One of the main issues in the theory of approximation of partial differential equations is to determine whether the approximate solutions of these equations still converge to an equilibrium, when the continuous ones have this property and if yes how fast do the approximate solutions converge to it. Consider the classical solutions of the abstract approximate problems

$$x_n'(t) = A_n x_n(t),\ t \geq 0, \quad x_n(0) = x_{0n},\ n \in \mathbb{N} \tag{1.1}$$

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[1] Département de Mathématiques, Faculté des Sciences Semlalia, Laboratoire LMDP, UMMISCO (IRD-UPMC), Université Cadi Ayyad, B.P. 2390, 40000 Marrakesh, Morocco. maniar@uca.ma; nafirisalim@gmail.com

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given by $x_n(t) = T_n(t)x_{0n},\ t \geq 0,\ x_{0n} \in D(A_n)$, where the sequence A_n generates the family of C_0-semigroups $T_n(t)$ on the Hilbert spaces H_n. We assume that $0 \in \rho(A_n)$ and

$$\sup_{n\in\mathbb{N}} \|A_n^{-1}\| < \infty.$$

Recall that a sequence of C_0-semigroups $T_n(t)$ on the Hilbert spaces H_n is said to be uniformly exponentially stable if there are positive constants M and ω independent of n such that

$$\|T_n(t)\| \leq M\mathrm{e}^{-\omega t}. \tag{1.2}$$

Proving the existence of such constants M and ω is not easy in general, as the counterexample in [14] shows. Even for semigroups $T_n(t) = \mathrm{e}^{A_n t}$ with A_n being an $n \times n$ matrix, uniform negative boundedness away from zero of the spectrum $\sigma(A_n)$ of A_n does not guarantee the existence of such constants, that is (1.2) could be violated. To our knowledge, Banks *et al.* [3] were the first to investigate the lack of uniform exponential stability for weakly damped wave equations. In [3] numerical simulations suggest that the exponential decay of the discretized energy might not be uniform, with respect to the step size, for the classical finite difference, and finite element schemes. To remedy this situation, the authors propose to use the mixed finite element method. Later, several remedies have been proposed to overcome this difficulty: Tychonoff regularization in Glowinski *et al.* [12], filtering of high frequencies in Infante and Zuazua [16]. We refer to the review paper Zuazua [27] for more details and extensive references.

For some hyperbolic-parabolic coupled systems, the solutions may converge to 0 polynomially, and not exponentially. Moreover as far as numerical approximation of such systems is concerned, little is known about the uniform (w.r.t. the mesh size) polynomial decay of the approximate energy. This motivates us to study the question of uniform polynomial stability for the family of systems (1.1), *i.e.*

$$\|tT_n(t)A_n^{-\alpha}\| \leq C \tag{1.3}$$

for all $t > 0$ and n, for some positive constants C and α.

Our main interest in this paper is to find necessary and sufficient conditions under which (1.3) holds. Recently, in [1], the authors showed a similar result of uniform polynomial stability for a class of second order evolution equations.

As in [8–11, 21], our goal is to investigate the uniform polynomial stability of a family of semigroups, generalizing the characterization of polynomial stability of Borichev−Tomilov [6] for a single semigroup, see also [4].

The outline of this paper is the following. In Section 2, we recall some fractional powers properties and prove a uniform version of the moment inequality result, Lemma 2.4, that will be useful later. In Section 3, we prove the main result of the paper Theorem 3.2, which characterize the uniform polynomial stability for a family of semigroups on Hilbert spaces. In Section 4, the result is applied to the family of abstract thermoelastic system

$$\begin{cases} \ddot{u}_n + \rho B_n u_n - \mu B_n^\tau \theta_n = 0, \\ \dot{\theta}_n + \kappa B_n \theta_n + \sigma B_n^\tau \dot{u}_n = 0, \\ u_n(0) = u_{0n},\ \dot{u}_n(0) = u_{1n},\ \theta_n(0) = \theta_{0n}, \end{cases}$$

where ρ, μ, κ and σ are positive constants, $0 \leq \tau < \frac{1}{2}$. This system can be seen as a semi-discretization of the following coupled hyperbolic/parabolic system

$$\begin{cases} u_{tt} - u_{xx} + \gamma\theta = 0 & \text{in } \Omega, \\ \theta_t - \theta_{xx} - \gamma u_t = 0 & \text{in } \Omega, \\ u\mid_{\partial\Omega} = 0 = \theta\mid_{\partial\Omega}, & \\ u(0) = u_0,\ u_t(0) = u_1,\ \theta(0) = \theta_0 & \text{on } \Omega. \end{cases}$$

We introduce three semi-discrete schemes (finite difference, finite element and spectral element) and prove that the approximate semigroup of the coupled hyperbolic/parabolic system decays uniformly polynomially to zero, with $\alpha = 2$ in (1.3). A convergence result is also proved thanks to the Trotter−Kato theorem. Finally, in Section 5, we illustrate numerically the mathematical results.

2. Estimates of fractional powers of a sequence of operators

Proposition 2.1. *Let a sequence of closed operators $A_n, n \in \mathbb{N}$, with dense domains such that*

$$[0, +\infty) \subset \rho(A_n), \quad n \in \mathbb{N},$$

$$M := \sup_{r>0,\ n\in\mathbb{N}} \|rR(r, A_n)\| < \infty, \tag{2.1}$$

and

$$c_0 := \sup_{n\in\mathbb{N}} \|(-A_n)^{-1}\| < \infty. \tag{2.2}$$

Then, there is $0 < \phi < \frac{\pi}{4}$ and $0 < C := C(c_0, M, \phi)$ (independent of n) such that

$$\|R(\mu, A_n)\| \leq \frac{C}{1 + |\mu|} \tag{2.3}$$

for all $n \in \mathbb{N}$ and $\mu \in \Sigma := \{z \in \mathbb{C},\ z \neq 0 : |\arg(z)| < \phi\}$.

Proof. Let $\mu = r + is$ for $r > 0$ and $|s| \leq \frac{r}{2M}$. We have

$$\begin{aligned} \mu - A_n &= r - A_n + is \\ &= (I + isR(r, A_n))(r - A_n), \end{aligned}$$

and since $\|is\ R(r, A_n)\| \leq \frac{1}{2}$, one has

$$\|R(\mu, A_n)\| \leq \frac{2M}{r} \leq \frac{2M}{\cos\phi} \frac{1}{|\mu|} =: \frac{\hat{M}}{|\mu|} \tag{2.4}$$

for any $\phi \leq \frac{1}{2M} \leq \frac{\pi}{6}$. From this, we can conclude that $\Sigma \subset \rho(A_n)$ and (2.4) is satisfied for all $\mu \in \Sigma$ (in fact, let $\phi \leq \frac{1}{2M}$ and $\mu = r + is \in \Sigma$. Let $\alpha = \arg\mu$. Then $\frac{|s|}{r} = \mathrm{tg}|\alpha| \leq \mathrm{tg}\phi \leq \frac{1}{2M}$. If we consider now $\mu \in \Sigma$ such that $|\mu| > \frac{1}{2c_0}$, from (2.4), we have

$$\|R(\mu, A_n)\| \leq \frac{\hat{M}}{|\mu|} = \frac{\hat{M}}{1 + |\mu|} \frac{1 + |\mu|}{|\mu|} \leq \frac{\overline{M}}{1 + |\mu|},$$

where $\overline{M} := \hat{M} \sup_{t \geq \frac{1}{2c_0}} \dfrac{1+t}{t}$. Finally, let $\mu \in B(0, \frac{1}{2c_0})$. One has

$$\mu - A_n = (I + \mu(-A_n)^{-1})(-A_n),$$

and since $\|\mu(-A_n)^{-1}\| \leq |\mu| c_0 \leq \frac{1}{2}$, $\mu \in \rho(A_n)$ and

$$\|R(\mu, A_n)\| \leq 2c_0 \leq \frac{2c_0}{1 + |\mu|}(1 + |\mu|) \leq \frac{2c_0 + 1}{1 + |\mu|}.$$

This achieves the proof. $\square$

Definition 2.2. We define the negative fractional powers of the sequence of operators A_n, according to ([7], Def. 5.25, p. 137), by the formula

$$A_n^{-\alpha} = \frac{1}{2\pi i}\int_{\Gamma} \lambda^{-\alpha} R(\lambda, A_n)\mathrm{d}\lambda, \qquad 0 < \alpha < \infty, \quad n \in \mathbb{N} \tag{2.5}$$

where $\lambda^{-\alpha} = \mathrm{e}^{-\alpha \log \lambda}$ and $\mathbb{R}^+$ is taken as the cut branch of the complex log function and where the curve $\Gamma = \Gamma_1 \cup \Gamma_2$ is given by

$$\Gamma = \{-\epsilon + t\mathrm{e}^{i\phi},\ t \in [0, +\infty)\} \cup \{-\epsilon + t\mathrm{e}^{-i\phi},\ t \in [0, +\infty)\},$$

where $\epsilon = \frac{1}{2c_0} > 0$ and $\phi = \arctan(\frac{1}{2M})$.

For $\alpha > 0$, the operator A_n^{α} is defined as the inverse of $A_n^{-\alpha}$ with domain $D(A_n^{\alpha}) = rg(A_n^{-\alpha})$.

Remark 2.3. Throughout this section, whenever $A_n^{-\alpha}$ is mentioned, Proposition 2.1 is directly taken into consideration since otherwise $A_n^{-\alpha}$ is not well defined.

The sequence of operators $A_n^{-\alpha}$ is uniformly bounded. In fact, when α is an integer, *i.e.* $\alpha = p \in \mathbb{N}_0$, from (2.2) we have

$$\|A_n^{-\alpha}\| = \|A_n^{-p}\| \le \|A_n^{-1}\|^p \le c_0^p.$$

Let now $0 < \alpha < p+1$, $\alpha \notin \mathbb{N}$. In view of the estimate (2.3) and according to ([7], Thm. 5.27, p. 138), we have the formula

$$A_n^{-\alpha} = \frac{1}{2\pi i}\frac{p!}{(1-\alpha)\dots(p-\alpha)}(1-\mathrm{e}^{-2\pi i\alpha})\int_0^{\infty} s^{p-\alpha}R(s, A_n)^{p+1}\mathrm{d}s. \tag{2.6}$$

If $p = 0$, *i.e.*, $\alpha \in (0, 1)$, we obtain the following representation for $A_n^{-\alpha}$

$$A_n^{-\alpha} = \frac{1}{2\pi i}(1-\mathrm{e}^{-2\pi i\alpha})\int_0^{\infty} s^{-\alpha}R(s, A_n)\mathrm{d}s.$$

In view of (2.3) and using the same argument as in ([7], Thm. 5.29, p. 139), we conclude that $A_n^{-\alpha}$ is uniformly bounded.

Suppose now that $0 < \alpha < 1$. Consider the sequence of operators $A_n^{-\alpha}\lambda^{\alpha}A_nR(\lambda, A_n)$. For every $\lambda > 0$, we have

$$A_n^{-\alpha} = \frac{1}{2\pi i}(1-\mathrm{e}^{-2\pi i\alpha})\int_0^{\infty} s^{-\alpha}R(\lambda s, A_n)\mathrm{d}(\lambda s).$$

This implies

$$\begin{aligned} A_n^{-\alpha}\lambda^{\alpha}A_nR(\lambda, A_n) = &\frac{1}{2\pi i}(1-\mathrm{e}^{-2\pi i\alpha})\int_0^1 s^{-\alpha}A_nR(\lambda s, A_n)\lambda R(\lambda, A_n)\mathrm{d}s \\ &+ \frac{1}{2\pi i}(1-\mathrm{e}^{-2\pi i\alpha})\int_1^{\infty} s^{-\alpha-1}\lambda sR(\lambda s, A_n)A_nR(\lambda, A_n)\mathrm{d}s. \end{aligned}$$

For each $\lambda > 0$, we have

$$\|A_nR(\lambda, A_n)\| \le M + 1 \tag{2.7}$$

and due to (2.1) it follows that

$$\|A_n^{-\alpha}\lambda^{\alpha}A_nR(\lambda, A_n)\| \le M', \tag{2.8}$$

where $M' = \frac{M(1+M)}{\pi}\left[\frac{1}{1-\alpha} + \frac{1}{\alpha}\right]$. For $\alpha = 0$ or $\alpha = 1$, $M' = M$ or $M' = M + 1$ respectively. Thus the sequence of operators $A_n^{-\alpha}\lambda^\alpha A_n R(\lambda, A_n)$ is (uniformly) bounded.

At the end of this section, we use the above results to prove a uniform version of interpolation type inequalities, that we called uniform moment inequality. It allows us to estimate $\|A_n^\beta\|$ (uniformly) in terms of $\|A_n^\alpha\|$ and $\|A_n^\gamma\|$ if $\alpha < \beta < \gamma$. For a single operator A we refer to ([7], Thm. 5.34, p. 141).

Lemma 2.4. (Uniform moment inequality). *Let $\alpha < \beta < \gamma$, then there exists a constant $L = L(\alpha, \beta, \gamma)$ independent of n such that*

$$\|A_n^\beta x\| \le L\|A_n^\gamma x\|^{\frac{\beta-\alpha}{\gamma-\alpha}}\|A_n^\alpha x\|^{\frac{\gamma-\beta}{\gamma-\alpha}}, \quad \forall x \in D(A_n^\gamma),\ \forall n \in \mathbb{N}. \tag{2.9}$$

Proof. Consider first the special case when $\gamma = 0$, $\alpha = -\alpha_0$, $\beta = -\beta_0$ $(0 < \beta_0 < \alpha_0)$ and $p \le \alpha_0 < p + 1$. Then $\beta_0 \in (0, p + 1)$, and we know that

$$\|s^{p-\beta_0}R(s, A_n)^{p+1}x_0\| \le s^{\alpha_0-\beta_0-1}\|A^{-p-1+\alpha_0}s^{p+1-\alpha_0}A_n R(s, A_n)\|\|A_n^p R(s, A_n)^p\|\|A_n^{-\alpha_0}x_0\|.$$

It follows from (2.7) and (2.8) that

$$\|s^{p-\beta_0}R(s, A_n)^{p+1}x_0\| \le Ks^{\alpha_0-\beta_0-1}\|A_n^{-\alpha_0}x_0\|, \tag{2.10}$$

and

$$\|s^{p-\beta_0}R(s, A_n)^{p+1}x_0\| \le K's^{p-\beta_0}s^{-p-1} = K's^{-\beta_0-1}\|x_0\|, \tag{2.11}$$

where $K = \frac{M(1+M)^{p+1}}{\pi}\left[\frac{1}{1-(p+1-\alpha_0)} + \frac{1}{p+1-\alpha_0}\right]$ and $K' = M^{p+1}$. Using (2.6), (2.10) and (2.11), we obtain

$$\begin{aligned}
\|A_n^{-\beta_0}x_0\| &\le C\left\|\int_0^\infty s^{p-\beta_0}R(s, A_n)^{p+1}x_0\mathrm{d}s\right\| \\
&= C\left\|\int_0^\tau s^{p-\beta_0}R(s, A_n)^{p+1}x_0\mathrm{d}s + \int_\tau^\infty s^{p-\beta_0}R(s, A_n)^{p+1}x_0\mathrm{d}s\right\| \\
&\le KC\int_0^\tau s^{\alpha_0-\beta_0-1}\|A_n^{-\alpha_0}x_0\| + K'C\int_\tau^\infty s^{-\beta_0-1}\|x_0\|\mathrm{d}s \\
&= \frac{KC}{\alpha_0-\beta_0}\tau^{\alpha_0-\beta_0}\|A_n^{-\alpha_0}x_0\| + \frac{K'C}{\beta_0}\tau^{-\beta_0}\|x_0\|
\end{aligned}$$

for all $\tau > 0$. Taking $\tau := \|A_n^{-\alpha_0}x_0\|^{\frac{-1}{\alpha_0}}\|x_0\|^{\frac{1}{\alpha_0}}$ yields

$$\|A_n^{-\beta_0}x_0\| \le L(\alpha_0, \beta_0)\|x_0\|^{\frac{\alpha_0-\beta_0}{\alpha_0}}\|A_n^{-\alpha_0}x_0\|^{\frac{\beta_0}{\alpha_0}},$$

where $L(\alpha_0, \beta_0) = \left(\frac{KC}{\alpha_0-\beta_0} + \frac{K'C}{\beta_0}\right)$. We now turn to the general case. Suppose that $\alpha < \beta < \gamma$ and $x \in D(A_n^\gamma)$. We apply the last inequality to the element $A_n^\gamma x$ with $\alpha_0 = \gamma - \alpha$ and $\beta_0 = \gamma - \beta$. Then, we obtain

$$\|A_n^\beta x\| = \|A_n^{-\beta_0}A_n^\gamma x\| \le L(\gamma-\alpha, \gamma-\beta)\|A_n^\gamma x\|^{\frac{\beta-\alpha}{\gamma-\alpha}}\|A_n^\alpha x\|^{\frac{\gamma-\beta}{\gamma-\alpha}}, \quad \forall x \in D(A_n^\gamma),\ \forall n \in \mathbb{N}.$$

This proves the lemma. □

3. Uniform polynomial stability of C_0-semigroups

In this section we give a characterization of uniform polynomial stability in terms of the uniform growth of the sequence of resolvent operators on the imaginary axis.

Definition 3.1. Let $T_n(t)$, $n \in \mathbb{N}$, be a sequence of C_0-semigroups of operators on Hilbert spaces H_n and let A_n be their corresponding generators.

(i) The family $T_n(\cdot)$ is said to be uniformly bounded if there exists $M > 0$ (independent of n) such that

$$\|T_n(t)\|_{\mathscr{L}(H_n)} \leqslant M, \quad t \geq 0, \; n \in \mathbb{N}.$$

(ii) The family $T_n(\cdot)$ is said to be uniformly polynomially stable of order $\alpha > 0$ if it is uniformly bounded, if $i\mathbb{R} \subset \rho(A_n)$ and if

$$\|tT_n(t)A_n^{-\alpha}\| \leq C, \tag{3.1}$$

for all $t > 0$, $n \in \mathbb{N}$ and for some positive constant C (independent of n).

We give now the main result of the paper which gives necessary and sufficient conditions to characterize the uniform polynomial stability of $T_n(t)$, a sequence of C_0-semigroups on Hilbert spaces H_n. For a single semigroup $T(t)$, the characteristic condition of polynomial stability was given by Borichev and Tomilov in [6], see also [4,5].

Theorem 3.2. *Let $T_n(t)$, $n \in \mathbb{N}$, be a uniformly bounded sequence of C_0-semigroups on the Hilbert spaces H_n and let A_n be the corresponding infinitesimal generators, such that for all $n \in \mathbb{N}$, $i\mathbb{R} \subset \rho(A_n)$. Then for a fixed $\alpha > 0$ the following conditions are equivalent*

(1) $$\sup_{|s|\geq 1,\; n\in\mathbb{N}} \frac{1}{|s|^{\alpha}} \|R(is, A_n)\| < \infty.$$

(2) $$\sup_{t\geqslant 0,\; n\in\mathbb{N}} \|tT_n(t)A_n^{-\alpha}\| < \infty.$$

(3) $$\sup_{t\geqslant 0,\; n\in\mathbb{N}} \|t^{\frac{1}{\alpha}} T_n(t)A_n^{-1}\| < \infty.$$

Remark 3.3. The proof of Theorem 3.2 is based on the results found in [4–6] for a single semigroup. We adapt such results, to obtain the (uniform) polynomial stability of a family of semigroups of operators.

To prove Theorem 3.2, we need first to characterize the uniform boundedness of a family of C_0-semigroups, which is the object of the following lemma found in [6] for a single semigroup. Here we set

$$\mathbb{C}_{\pm} := \{z \in \mathbb{C} : \; \mathrm{Re}\, z \gtrless 0\}.$$

Lemma 3.4. *Let $T_n(t), n \in \mathbb{N}$, be a sequence of C_0-semigroups on the Hilbert spaces H_n and let A_n be the corresponding generators. Then $T_n(\cdot)$ is uniformly bounded if and only if*

(i) $\mathbb{C}_+ \subset \rho(A_n)$, $\forall n \in \mathbb{N}$.
(ii) *There exists $C > 0$ independent of n such that*

$$\sup_{\substack{\xi>0 \\ n\in\mathbb{N}}} \xi \int_{\mathbb{R}} \Big(\|R(\xi + i\eta, A_n)\|^2 + \|R(\xi + i\eta, A_n^*)\|^2\Big) \mathrm{d}\eta \leq C. \tag{3.2}$$

Proof. First, we assume that $T_n(t)$ is uniformly bounded, *i.e.*, there exist M, $\alpha > 0$ such that

$$\|T_n(t)\|_{\mathscr{L}(H_n)} \leqslant M, \quad t \geq 0, \; n \in \mathbb{N}.$$

Then (i) holds by Hille−Yosida theorem (see [7], Thm. 3.5, p. 73). For (ii), by the uniform boundedness principle, we only need to prove that

$$\sup_{\substack{\xi>0 \\ n\in\mathbb{N}, \|x_n\|\leq 1}} \xi \int_{\mathbb{R}} \Big(\|R(\xi + i\eta, A_n)x_n\|^2 + \|R(\xi + i\eta, A_n^*)x_n\|^2\Big) \mathrm{d}\eta \leq C.$$

To have this, we consider the rescaled semigroup $(T_n^{-\xi}(t))_{n\geq 1}$ with $T_n^{-\xi}(t) := \mathrm{e}^{-\xi t}T_n(t)$. Then by ([7], Thm 1.10.(i), p. 55) and for $x_n \in H_n, \eta \in \mathbb{R}$, we have

$$R(\xi + i\eta, A_n)x_n = R(i\eta, A_n - \xi) = \int_0^\infty \mathrm{e}^{-i\eta t}T_n^{-\xi}(t)x_n \mathrm{d}t.$$

Using the Fourier Transform $\mathscr{F}:\ L^2(\mathbb{R}, H_n) \to L^2(\mathbb{R}, H_n)$ we obtain

$$R(\xi + i\eta, A_n) = \mathscr{F}(T_n^{-\xi}(\cdot))(\eta),$$

where we extend $T_n^{-\xi}(\cdot)$ to $\mathbb{R}$ by setting $T_n^{-\xi}(t) := 0$ for $t < 0$. Since $T_n^{-\xi}(t)$ is uniformly bounded, we have $T_n^{-\xi}(\cdot)x_n \in L^2(\mathbb{R}, H_n)$. Now, we conclude, from Plancherel's Theorem, that

$$\int_{-\infty}^{+\infty} \|R(\xi + i\eta, A_n)x_n\|^2 \mathrm{d}\eta = 2\pi \int_0^\infty \|T_n^{-\xi}(t)x_n\|^2 \leqslant \frac{\pi M^2}{\xi}\|x_n\|^2,$$

thus

$$\sup_{\substack{\xi>0 \\ n\in\mathbb{N}, \|x_n\|\leq 1}} \xi \int_{\mathbb{R}} \|R(\xi + i\eta, A_n)x_n\|^2 \mathrm{d}\eta \leqslant \pi M^2. \tag{3.3}$$

Since $\|T_n(t)\| = \|T_n^*(t)\|$ for every $T_n(t) \in \mathscr{L}(H_n)$, by symmetry the same estimate is true for the resolvent of the generator A_n^* of the adjoint semigroup $T_n^*(t)$, *i.e.*,

$$\sup_{\substack{\xi>0 \\ n\in\mathbb{N}, \|x_n\|\leq 1}} \xi \int_{\mathbb{R}} \|R(\xi + i\eta, A_n^*)x_n\|^2 \mathrm{d}\eta \leqslant \pi M^2. \tag{3.4}$$

Thus the proof of the "only if" part is complete. For the converse implication, we use the inversion formula in ([7], Cor. 5.16, p. 234)

$$T_n(t)x_n = \frac{1}{2\pi i t} \lim_{\omega\to\infty} \int_{\xi - i\omega}^{\xi + i\omega} \mathrm{e}^{\lambda t} R^2(\lambda, A_n)x_n \mathrm{d}\lambda,\ t > 0.$$

Hence, for $\xi = \frac{1}{t}, t > 0$, in (3.2) it follows that

$$\begin{aligned} \left|\left\langle T_n(t)x_n, x_n^*\right\rangle\right| &\leqslant \frac{e}{2\pi t} \lim_{\omega\to\infty} \int_{\frac{1}{t} - i\omega}^{\frac{1}{t} + i\omega} \left|\left\langle R\left(\frac{1}{t} + i\eta, A_n\right)x_n, R\left(\frac{1}{t} - i\eta, A_n^*\right)x_n^*\right\rangle\right| \mathrm{d}\eta \\ &= \frac{e}{2\pi t}\left(\int_{-\infty}^{+\infty} \left\|R\left(\frac{1}{t} + i\eta, A_n\right)x_n\right\|^2 \mathrm{d}\eta\right)^{\frac{1}{2}} \left(\int_{-\infty}^{+\infty} \left\|R\left(\frac{1}{t} + i\eta, A_n^*\right)x_n^*\right\|^2 \mathrm{d}\eta\right)^{\frac{1}{2}}. \end{aligned}$$

Thus, the integral converges absolutely by the Hölder inequality and our assumptions.

Furthermore,

$$\begin{aligned} \left|\left(T_n(t)x_n, x_n^*\right)\right| &\leqslant \frac{e}{4\pi}\left(\frac{1}{t}\int_{-\infty}^{+\infty} \|R(\lambda, A_n)x_n\|^2 \mathrm{d}\beta + \frac{1}{t}\int_{-\infty}^{+\infty} \|R(\lambda, A_n^*)x_n^*\|^2 \mathrm{d}\beta\right) \\ &\leqslant \frac{eC}{4\pi}\|x_n\|.\|x_n^*\|, \end{aligned}$$

We conclude that $\|T_n(t)\| \leqslant \frac{eC}{4\pi}, \quad t \geq 0,\ n \in \mathbb{N}.$ □

To prove Theorem 3.2, we use the following lemma relating the order of growth of the resolvent operators of A_n on the imaginary axis to its behavior in the right half-plane of $\mathbb{C}$. It is worth noting that for a single operator A, this lemma has been shown in [15,18] by more complicated arguments. Inspired by the proof of Proposition 2.2 in Weiss [26], we will consider a proof which is much easier.

Lemma 3.5. *Let $T_n(t), n \in \mathbb{N}$, be a uniformly bounded sequence of C_0-semigroups on the Hilbert spaces H_n and let A_n be the corresponding infinitesimal generators, such that for all $n \in \mathbb{N}$, $i\mathbb{R} \subset \rho(A_n)$. Then for a fixed $\alpha > 0$ the following conditions are equivalent.*

(i) $$\sup_{\operatorname{Re}\lambda>0,\ n\in\mathbb{N}} \tfrac{\|R(\lambda,A_n)\|}{1+|\lambda|^\alpha} < \infty.$$

(ii) $$\sup_{\operatorname{Re}\lambda>0,\ n\in\mathbb{N}} \|R(\lambda, A_n)A_n^{-\alpha}\| < \infty.$$

(iii) $$\sup_{|s|\geq 1,\ n\in\mathbb{N}} \tfrac{1}{|s|^\alpha}\|R(is, A_n)\| < \infty.$$

Proof. To show the equivalence of (i) and (ii), let S be any subset of $\rho(A_n)$. There exists an $\varepsilon > 0$ such that $R(\lambda, A_n)$ is bounded on $D = \{\lambda \in \mathbb{C}/|\lambda| < \varepsilon\}$. From now on, we suppose that $D \cap S = \emptyset$. By induction we have

$$R(\lambda, A_n)A_n^{-p} = \frac{R(\lambda, A_n)}{\lambda^p} + \sum_{k=0}^{p-1}(-1)^k \frac{A_n^{-(p-k)}}{\lambda^{k+1}}$$

for all $p \in \mathbb{N}$. Here the operators $\displaystyle\sum_{k=0}^{p-1}(-1)^k \frac{A_n^{-(p-k)}}{\lambda^{k+1}}$ are uniformly bounded with respect to $\lambda \in S$ and $n \in \mathbb{N}$. Further, for all $\alpha > 0$, we have

$$\frac{R(\lambda, A_n)}{|\lambda|^\alpha} = \frac{1+|\lambda|^\alpha}{|\lambda|^\alpha}\frac{R(\lambda, A_n)}{1+|\lambda|^\alpha}, 1 \leq \frac{1+|\lambda|^\alpha}{|\lambda|^\alpha} \leq d_\alpha$$

for all $\lambda \in S$ and a positive constant d_α. Hence we have proved that

$$\|R(\lambda, A_n)A_n^{-\alpha}\| \leq d_\alpha \frac{\|R(\lambda, A_n)\|}{1+|\lambda|^\alpha} + c \tag{3.5}$$

and

$$\frac{\|R(\lambda, A_n)\|}{1+|\lambda|^\alpha} \leq \|R(\lambda, A_n)A_n^{-\alpha}\| + c \tag{3.6}$$

for $\alpha = p$ and positive constants c and d_α. If α is any non-integer positive number, take $p \in \mathbb{N}$ such that $p < \alpha < p+1$. Since

$$\begin{aligned} R(\lambda, A_n)A_n^{-\alpha} &= R(\lambda, A_n)A_n^{-p}A_n^{-(\alpha-p)} \\ &= \frac{R(\lambda, A_n)A_n^{-(\alpha-p)}}{\lambda^p} + \sum_{k=0}^{p-1}(-1)^k \frac{A_n^{k-\alpha}}{\lambda^{k+1}}, \end{aligned}$$

we see that it is sufficient to prove the relations (3.5) and (3.6) for $0 < \alpha < 1$. To this end we apply Lemma 2.4 with exponents $0 < \alpha < 1$ and we get

$$\begin{aligned} \|R(\lambda, A_n)x\| &= \|A_n^\alpha R(\lambda, A_n)A_n^{-\alpha}x\| \\ &\leq L(1, 1-\alpha)\|R(\lambda, A_n)A_n^{-\alpha}x\|^{1-\alpha}\|A_nR(\lambda, A_n)A_n^{-\alpha}x\|^\alpha \\ &\leq L(1, 1-\alpha)\|R(\lambda, A_n)A_n^{-\alpha}\|^{1-\alpha}\|(I - \lambda R(\lambda, A_n)A_n^{-\alpha}\|^\alpha\|x\| \\ &\leq L(1, 1-\alpha)(1+|\lambda|^\alpha)\left(\|R(\lambda, A_n)A_n^{-\alpha}\| + d_\alpha\right)\|x\|. \end{aligned}$$

In order to obtain the converse inequality, we use Lemma 2.4 again with exponents $-1 < -\alpha < 0$,

$$\begin{aligned}\|R(\lambda, A_n)A_n^{-\alpha}x\| &\le L(1,\alpha)\|A_n^{-1}R(\lambda, A_n)x\|^{\alpha}\|R(\lambda, A_n)x\|^{1-\alpha}\\ &\le L(1,\alpha)\|\frac{1}{\lambda}A_n^{-1} - R(\lambda, A_n))\|^{\alpha}\|R(\lambda, A_n)\|^{1-\alpha}\|x\|\\ &\le L(1,\alpha)\Big(d_\alpha\frac{\|R(\lambda, A_n)\|}{1+|\lambda^\alpha|} + c\Big)\|x\|.\end{aligned}$$

Then, the equivalence is proved. It remains to prove that (i) and (iii) are equivalent. For this purpose, suppose that $\|R(\lambda, A_n)\| \le C_1(1+|\lambda|^\alpha)$ for all $n \in \mathbb{N}$. Hence,

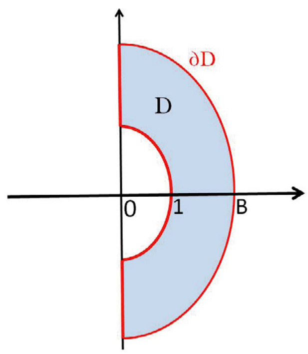
$$\begin{aligned}\|R(is, A_n)\| = \lim_{\beta\to 0^+}\|R(\beta+is, A_n)\| &\le C_1 \lim_{\beta\to 0^+}(1+|\beta+is|^\alpha)\\ &\le C_1(1+|is|^\alpha)\\ &\le C_1(1+|s|^\alpha)\\ &\le C_2|s|^\alpha\end{aligned}$$

for all $n \in \mathbb{N}$ and $|s| > C_0$ for a positive constant C_0. We get that for any $n \in \mathbb{N}$ and s too large

$$\|R(is, A_n)\| < C_2|s|^\alpha,$$

where $C_2 = \frac{C_1(1+C_0^\alpha)}{C_0^\alpha}$. For the converse, let $B \geqslant 1$ and $n \in \mathbb{N}$. Consider the holomorphic function

$$F:\ D := \{\lambda \in \mathbb{C}:\ \mathrm{Re}(\lambda) \geqslant 0, 1 \leqslant |\lambda| \leqslant B\} \longrightarrow H_n, \quad \lambda \longmapsto R(\lambda, A_n)\lambda^{-\alpha}\left(1+\frac{\lambda^2}{B^2}\right).$$



Let $\lambda = Be^{i\varphi}$ for $\varphi \in [\frac{-\pi}{2}, \frac{\pi}{2}]$. If $\varphi \neq \pm\frac{\pi}{2}$, using the estimate $\|R(\lambda, A_n)\| \le \frac{M}{\mathrm{Re}(\lambda)}$, we obtain

$$\begin{aligned}\|F(\lambda)\| &= \|R(\lambda, A_n)\||\lambda|^{-\alpha}|1+e^{i2\varphi}|\\ &\leqslant 2\cos\varphi\frac{M}{B\cos\varphi}B^{-\alpha}\\ &= \frac{2C}{B^{\alpha+1}} \leqslant 2M.\end{aligned}$$

Moreover, $\|F(\pm iB)\| = 0$. For $\lambda = e^{i\varphi}$, we get

$$\begin{aligned}\|F(\lambda)\| &= \|R(\lambda, A_n)\|\left|1+\frac{e^{i2\varphi}}{B^2}\right|\\ &\leqslant 2C,\end{aligned}$$

where we used that $R(\cdot, A_n)$ is continuous on a compact and $|\lambda| < B$. Using (iii), we obtain

$$\begin{aligned}\|F(is)\| &= \|R(is, A_n)\| |s|^{-\alpha} \left|1 - \frac{s^2}{B^2}\right| \\ &\leqslant 2C'\end{aligned}$$

for $1 < |s| < B$. The maximum principle implies that $\|F(\lambda)\| \leqslant 2\max\{M, C, C'\}$ for all $\lambda \in \overline{D}$. Letting $B \to \infty$, we deduce that

$$\|R(\lambda, A_n)\| \leq C^*(1 + |\lambda|^\alpha)$$

for $n \in \mathbb{N}$ and $C^* = 2\max\{M, C, C'\}$. This completes the proof. □

In our situation, estimate (i) allows to control the rate of approach of $\sigma(A_n)$ to the imaginary axis at $\pm i\infty$. In what follows we give the proof of Theorem 3.2.

Proof. We start by proving the equivalence (2) ⇔ (3): assuming (2), we have

$$\begin{aligned}\|T_n(t)A_n^{-p\alpha}\| &= \|[T_n\left(\frac{t}{p}\right) A_n^{-\alpha}]^p\| \\ &\leq (C_2 p)^p t^{-p}\end{aligned}$$

for $t > 0$, $p \in \mathbb{N}$ and $C_2 := \sup\limits_{t \geqslant 0,\ n \in \mathbb{N}} \|tT_n(t)A_n^{-\alpha}\|$. Lemma 2.4 now yields

$$\begin{aligned}\|T_n(t)A_n^{-p\alpha\nu}\| &= \|A_n^{p\alpha(1-\nu)} T_n(t) A_n^{-p\alpha}\| \\ &\leq L(p\alpha, p\alpha\nu)\|A_n^{p\alpha} T_n(t) A_n^{-p\alpha}\|^{1-\nu}.\|T_n(t)A_n^{-p\alpha}\|^\nu \\ &\leq L(p\alpha, p\alpha\nu)\|T_n(t)\|^{1-\nu}.\|T_n(t)A_n^{-p\alpha}\|^\nu \\ &\leq \|L(p\alpha, p\alpha\nu).M^{1-\nu}(C_2 p)^{p\nu} t^{-p\nu}\end{aligned}$$

for $\nu \in (0, 1)$. Choosing $\nu = \frac{1}{p\alpha}$ and $p > \frac{1}{\alpha}$, we obtain

$$\|T_n(t)A_n^{-1}\| \leq C_3 t^{-\frac{1}{\alpha}},$$

where $C_3 = L(\nu, 1)M^{1-\nu}(C_2 p)^{\frac{1}{\alpha}}$ is a constant independent of n. Assume now that (3) is satisfied, *i.e.*, $\exists C_3 > 0$: $\|T_n(t)A_n^{-1}\| < C_3 t^{-\frac{1}{\alpha}}$ for all $t > 0$ and $n \in \mathbb{N}$. Hence,

$$\|T_n(t)A_n^{-p}\| = \|[T_n\left(\frac{t}{p}\right) A_n^{-1}]^p\| \leq C_3^p t^{-\frac{p}{\alpha}}$$

for $t > 0$ and $p \in \mathbb{N}$. We deduce from Lemma 2.4 that

$$\begin{aligned}\|T_n(t)A_n^{-p\nu}\| &= \|A_n^{p(1-\nu)} T_n(t) A_n^{-p}\| \\ &\leq L(p, p\nu)\|A_n^p T_n(t) A_n^{-p}\|^{1-\nu}.\|T_n(t)A_n^{-p}\|^\nu \\ &\leq L(p, p\nu)M^{1-\nu} C_3^{p\nu} t^{-\frac{p\nu}{\alpha}}\end{aligned}$$

for $\nu \in (0, 1)$: $\nu = \frac{\alpha}{p}$ and $\alpha < p$, we deduce that $\|T_n(t)A_n^{-\alpha}\| \leq C_2 t^{-1}$, where $C_2 = L(p, \alpha)M^{1-\nu}C_3^\alpha$ is independent of n. We assume now (1) and show (2). Let $\mathcal{H}_n = H_n \times H_n$. Consider the sequence of operators $\mathcal{A}_n$ on $\mathcal{H}_n$ given by the sequence of matrix

$$\mathcal{A}_n = \begin{pmatrix} A_n & A_n^{-\alpha} \\ 0 & A_n \end{pmatrix} \tag{3.7}$$

with the diagonal domain $D(\mathcal{A}_n) = D(A_n) \times D(A_n)$. Then $\sigma(\mathcal{A}_n) = \sigma(A_n)$, and the sequence of resolvents $R(\lambda, \mathcal{A}_n)$ of $\mathcal{A}_n$ are given by

$$R(\lambda, \mathcal{A}_n) = \begin{pmatrix} R(\lambda, A_n) & R^2(\lambda, A_n)A_n^{-\alpha} \\ 0 & R(\lambda, A_n) \end{pmatrix}, \qquad \lambda \in \rho(A_n). \tag{3.8}$$

The sequence of operators $\mathcal{A}_n$ are the generators of the sequence of C_0-semigroups $(\mathcal{T}_n(t))_{t\geqslant 0}$ on $\mathcal{H}_n$ defined by

$$\mathcal{T}_n(t) = \begin{pmatrix} T_n(t) & tT_n(t)A_n^{-\alpha} \\ 0 & T_n(t) \end{pmatrix}, \tag{3.9}$$

because the resolvents of $\mathcal{A}_n$ and of the generator of $\mathcal{T}_n(t)$ coincide. By the assumption (1) and Lemma 3.4, one has

$$\sup_{\mathrm{Re}\ \lambda>0,\ n\in\mathbb{N}} \|R(\lambda, A_n)A_n^{-\alpha}\| < \infty.$$

Hence, for every $x_n = (x_{n1}, x_{n2}) \in \mathcal{H}_n$ and $\lambda \in \mathbb{C}_+$,

$$\|R(\lambda, \mathcal{A}_n)x_n\|^2 \leq C\left(\|R(\lambda, A_n)x_{n1}\|^2 + \|R(\lambda, A_n)x_{n2}\|^2\right),$$

and similarly

$$\|R(\lambda, \mathcal{A}_n^*)x_n\|^2 \leq C'\left(\|R(\lambda, A_n^*)x_{n1}\|^2 + \|R(\lambda, A_n^*)x_{n2}\|^2\right),$$

where C and C' are constants independent of n. By Lemma 3.4, we have

$$\sup_{\substack{\xi>0 \\ n\in\mathbb{N}, \|x_n\|\leq 1}} \xi \int_{\mathbb{R}} \left(\|R(\xi + i\eta, A_n)x_n\|^2 + \left\|R(\xi + i\eta, A_n^*)x_n\right\|^2\right) \mathrm{d}\eta < \infty,$$

and thus

$$\sup_{\substack{\xi>0 \\ n\in\mathbb{N}, \|x_n\|\leq 1}} \xi \int_{\mathbb{R}} \left(\|R(\xi + i\eta, \mathcal{A}_n)x_n\|^2 + \|R(\xi + i\eta, \mathcal{A}_n^*)x_n\|^2\right)\mathrm{d}\eta < \infty.$$

Therefore, again by Lemma 3.4, $\mathcal{T}_n(\cdot)$ is uniformly bounded. Since $T_n(\cdot)$ is uniformly bounded, the definition of $\mathcal{T}_n(t)$ implies that

$$\sup_{t\geqslant 0,\ n\in\mathbb{N}} \|tT_n(t)A_n^{-\alpha}\| < \infty.$$

For the converse, (it was proved in [5] for a single semigroup) we set

$$m_1(t) := \sup_{s\geqslant t,\ n\in\mathbb{N}} \|T_n(s)A_n^{-1}\| \leq \frac{C}{t^{\frac{1}{\alpha}}}$$

for $\alpha > 0$ and a positive constant C independent of n. Let $u_{0n} \in D(A_n)$, $\tau \in \mathbb{R}$ and $f_{0n} = (i\tau - A_n)u_{0n}$. Let $v_n(t) = \mathrm{e}^{it\tau}u_{0n}$. Then

$$\partial_t v_n = A_n v_n + \mathrm{e}^{it\tau}(i\tau - A_n)u_{0n} = \mathrm{e}^{it\tau}f_{0n}, \quad v(0) = u_{0n}.$$

By Duhamel formula

$$v_n(t) = T_n(t)u_{0n} + \int_0^t T_n(t-s)\mathrm{e}^{is\tau}f_{0n}\mathrm{d}s.$$

Thus, by the boundedness of the sequence $T_n(t)$, we obtain that

$$\begin{aligned} \|u_{0n}\| = \|v_n(t)\| &\leq m_1(t)\|A_n u_{0n}\| + Mt\|f_{0n}\| \\ &\leq m_1(t)\left(|\tau|\|u_{0n}\| + \|f_{0n}\|\right) + Mt\|f_{0n}\| \\ &\leq \frac{C}{t^{\frac{1}{\alpha}}}\|f_{0n}\| + \frac{C|\tau|}{t^{\frac{1}{\alpha}}}\|u_{0n}\| + Mt\|f_{0n}\|. \end{aligned}$$

If we choose $C|\tau| \le \frac{t^{\frac{1}{\alpha}}}{2}$, then

$$\|u_{0n}\| \le \left(\frac{C}{t^{\frac{1}{\alpha}}} + Mt\right)\|f_{0n}\| + \frac{1}{2}\|u_{0n}\|,$$
$$\|u_{0n}\| \le \left(\frac{2C}{t^{\frac{1}{\alpha}}} + 2Mt\right)\|f_{0n}\|.$$

Taking $t = 2^\alpha C^\alpha |\tau|^\alpha$ yields that

$$\|R(i\tau, A_n)\| \le C'|\tau|^\alpha$$

for all $n \in \mathbb{N}$ and $|\tau| > 1$, which completes the proof of Theorem 3.2. □

Proposition 3.6. *Let $T_n(t), n \in \mathbb{N}$, be a uniformly bounded sequence of C_0-semigroups on Hilbert spaces H_n and let A_n be the corresponding infinitesimal generators, such that*

$$i\mathbb{R} \subset \rho(A_n),\ n \in \mathbb{N},$$
$$\sup_{n\in\mathbb{N}} \|A_n^{-1}\| < \infty,$$

and

$$\sup_{\rho\in\mathbb{R},\ n\in\mathbb{N}} \|R(i\rho, A_n)A_n^{-\alpha}\| < \infty,$$

for a constant $\alpha > 0$. Then there exist a positive constant δ independent of n such that $[0, \delta] \subset \rho(A_n)$ and we have

$$\sup_{\mathrm{Re}\lambda\ge -\delta,\ n\in\mathbb{N}} \left\{ \frac{|\mathrm{Im}\lambda|^{-\alpha}}{|\mathrm{Re}\lambda|},\ \lambda \in \sigma_p(A_n) \right\} < \infty. \tag{3.10}$$

Proof. Due to the uniform boundedness of the sequence $T_n(t)$, Lemma 3.4 yields that for all $n \in \mathbb{N}$, $\sigma(A_n) \subset \mathbb{C}_-$. In particular for all $n \in \mathbb{N}$ and $\lambda \in \sigma_p(A_n)$: $\mathrm{Re}(\lambda) \leqslant 0$. From the uniform boundedness of $\|A_n^{-1}\|$, we get that $\sup_{n\in\mathbb{N}}\mathrm{Re}\lambda < 0$, then there exists $\delta > 0$ independent of n such that for all $\lambda \in \sigma_p(A_n)$: $\mathrm{Re}\lambda \geqslant -\delta$.

As in the proof of Theorem 3.2, we see that $(\lambda - A_n)^{-1}A_n^{-\alpha}$ is uniformly bounded for $n \in \mathbb{N}$ and $\mathrm{Re}\lambda > 0$. Lemma 3.5 and the continuity of the resolvent operators then yield

$$\frac{1}{|\mathrm{Re}\lambda|} \le \frac{1}{\mathrm{dist}(i\mathrm{Im}\lambda, \sigma(A_n))} \le \|(i\mathrm{Im}\lambda - A_n)^{-1}\| \le C_1(1 + |\mathrm{Im}\lambda|^\alpha) \le C_2|\mathrm{Im}\lambda|^\alpha$$

for all $n \in \mathbb{N}$, for all $\lambda \in \sigma(A_n)$ with $-\delta \le \mathrm{Re}\lambda < 0$ and constants C_k not depending on n and λ. □

Remark 3.7. To show that δ is independent of n, we have restricted the statement of Proposition 3.7 found in [4] for a single operator, to the point spectrum. Otherwise, we could not show the independence of δ with respect to n, for a family of operators A_n. Proposition 3.6 claims that in general one cannot deduce uniform polynomial stability from the pure spectral criterion (3.10). In [4], the authors showed that for normal single C_0-semigroups, the spectral estimate (3.10) is necessary and sufficient to ensure polynomial stability. Their result can be adapted for a sequence of normal C_0-semigroups.

4. Applications

4.1. Application 1: Abstract thermoelastic models

In order to illustrate our result of uniform polynomial stability, we consider the family of abstract hyperbolic-parabolic systems related to thermoelastic models

$$\begin{cases} \ddot{u}_n + \rho B_n u_n - \mu B_n^\tau \theta_n = 0, \\ \dot{\theta}_n + \kappa B_n \theta_n + \sigma B_n^\tau \dot{u}_n = 0, \\ u_n(0) = u_{0n},\ \dot{u}_n(0) = u_{1n},\ \theta_n(0) = \theta_{0n}, \end{cases} \tag{4.1}$$

where ρ, μ, κ and σ are positive constants, $0 \le \tau < \frac{1}{2}$. The superscript $\dot{}$ denotes partial differentiation with respect to time, *i.e.* $\dot{} = \partial \cdot /\partial t$. We assume that H_n is a family of separable Hilbert spaces equipped with the inner product $\langle .,. \rangle$ and norm $||.||$. We suppose that the sequence of operators $B_n : D(B_n) \subset H_n \to H_n$ is self-adjoint positive definite with the power B_n^{-s} of B_n is a compact linear operator in H_n for a positive real number s, that $0 \in \rho(B_n)$, the resolvent set of B_n, and that $\sup_{n\in\mathbb{N}} ||B_n^{\frac{-1}{2}}|| < \infty$. Let $D(B_n^{\frac{1}{2}})$ be the completion of $D(B_n)$, we introduce the state spaces $\mathcal{H}_n = D(B_n^{\frac{1}{2}}) \times H_n \times H_n$ with the inner product

$$\Big\langle (u_n, v_n, \theta_n), (\tilde{u}_n, \tilde{v}_n, \tilde{\theta}_n) \Big\rangle_{\mathcal{H}_n} = \Big\langle B_n^{\frac{1}{2}} u_n, B_n^{\frac{1}{2}} \tilde{u}_n \Big\rangle_{H_n} + \frac{1}{\rho} \langle v_n, \tilde{v}_n \rangle_{H_n} + \frac{\mu}{\rho\sigma} \langle \theta_n, \tilde{\theta}_n \rangle_{H_n}$$

and the induced norm

$$||(u_n, v_n, \theta_n)||^2_{\mathcal{H}_n} = \Big|\Big| B_n^{\frac{1}{2}} u_n \Big|\Big|^2_{H_n} + \frac{1}{\rho} ||v_n||^2_{H_n} + \frac{\mu}{\rho\sigma} ||\theta_n||^2_{H_n}.$$

We define the energy of system (4.1) by

$$E_n(t) = \frac{1}{2} \left\{ \Big|\Big| B_n^{\frac{1}{2}} u_n \Big|\Big|^2_{H_n} + \frac{1}{\rho} ||v_n||^2_{H_n} + \frac{\mu}{\rho\sigma} ||\theta_n||^2_{H_n} \right\}.$$

We also consider the sequence of linear operators $\mathcal{A}_{\tau,n} : D(\mathcal{A}_{\tau,n}) \subset \mathcal{H}_n \to \mathcal{H}_n$

$$\mathcal{A}_{\tau,n} = \begin{pmatrix} 0 & I_n & 0 \\ -\rho B_n & 0 & \mu B_n^\tau \\ 0 & -\sigma B_n^\tau & -\kappa B_n \end{pmatrix}, \tag{4.2}$$

whose domain is given by $D(\mathcal{A}_{\tau,n}) = D(B_n) \times D(B_n^{\frac{1}{2}}) \times D(B_n)$. So that the family of systems (4.1) can be rewritten as the following initial value problem

$$\frac{\mathrm{d}U_n}{\mathrm{d}t} = \mathcal{A}_{\tau,n} U_n, \quad U_n(0) = U_{0n}, \tag{4.3}$$

with $U_n = (u_n, \dot{u}_n, \theta_n)$ and $U_{0n} = (u_{0n}, u_{1n}, \theta_{0n})$.

Proposition 4.1. *The sequence of operators $\mathcal{A}_{\tau,n}$ generates a family of C_0-semigroups of contractions denoted by $S_{\tau,n}(t)$.*

Proof. We will show that the sequence of operators $\mathcal{A}_{\tau,n}(n = 0, 1, \ldots)$ is dissipative and $0 \in \rho(\mathcal{A}_{\tau,n})$, then our conclusion will follow using the well known Lumer−Phillips theorem (see [7,23]). For the density of $D(\mathcal{A}_{\tau,n})$ in $\mathcal{H}_n$ we use also ([7], Cor. 3.20, p. 86) (see also [23], Thm. 4.6, p. 16). Let $U_n = (u_n, v_n, \theta_n) \in D(\mathcal{A}_{\tau,n})$, then

$$\begin{aligned} \langle \mathcal{A}_{\tau,n} U_n, U_n \rangle_{\mathcal{H}_n} &= \langle (v_n, -\rho B_n u_n + \mu B_n^\tau \theta_n, -\kappa B_n \theta_n - \sigma B_n^\tau v_n), (u_n, v_n, \theta_n) \rangle_{\mathcal{H}_n} \\ &= \Big\langle B_n^{\frac{1}{2}} v_n, B_n^{\frac{1}{2}} u_n \Big\rangle + \frac{1}{\rho} \langle -\rho B_n u_n + \mu B_n^\tau \theta_n, v_n \rangle - \frac{\mu}{\rho\sigma} \langle \kappa B_n \theta_n + \sigma B_n^\tau v_n, \theta_n \rangle \\ &= \Big\langle B_n^{\frac{1}{2}} v_n, B_n^{\frac{1}{2}} u_n \Big\rangle - \Big\langle B_n^{\frac{1}{2}} u_n, B_n^{\frac{1}{2}} v_n \Big\rangle + \frac{\mu}{\rho} \langle B_n^\tau \theta_n, v_n \rangle - \frac{\mu\kappa}{\rho\sigma} \Big\| B_n^{\frac{1}{2}} \theta_n \Big\|^2 - \frac{\mu}{\rho} \langle v_n, B_n^\tau \theta_n \rangle. \end{aligned}$$

Therefore, taking the real part in the equation above, we conclude that

$$\mathrm{Re}\Big\langle \mathcal{A}_{\tau,n} U_n, U_n \Big\rangle_{\mathcal{H}_n} = -\frac{\mu\kappa}{\rho\sigma} \Big\| B_n^{\frac{1}{2}} \theta_n \Big\|^2 \le 0, \tag{4.4}$$

and then the sequence of operators $\mathcal{A}_{\tau,n}$ is dissipative. On the other hand, given $F_n = (f_n, g_n, h_n) \in \mathcal{H}_n$, there exists only one vector $U_n = (u_n, v_n, \theta_n) \in D(\mathcal{A}_{\tau,n})$ such that $\mathcal{A}_{\tau,n} U_n = F_n$ or

$$v_n = f_n \quad \text{in} \quad D(B_n^{\frac{1}{2}}), \tag{4.5}$$

$$-\rho B_n u_n + \mu B_n^\tau \theta_n = g_n \quad \text{in} \quad H_n, \tag{4.6}$$

$$-\kappa B_n \theta_n - \sigma B_n^\tau v_n = h_n \quad \text{in} \quad H_n. \tag{4.7}$$

In fact, taking $v_n = f_n$ in (4.7), we obtain the equation

$$B_n \theta_n = \frac{-\sigma}{\kappa} B_n^\tau f_n - \frac{1}{\kappa} h_n. \tag{4.8}$$

Since $\frac{\sigma}{\kappa} B_n^\tau f_n + \frac{1}{\kappa} h_n \in H_n$ and $0 \in \rho(B_n)$ there exists only one function $\theta_n \in D(B_n)$ that satisfies (4.8). From (4.6), we find that

$$B_n u_n = \frac{\mu}{\rho} B_n^\tau \theta_n - \frac{1}{\rho} g_n, \tag{4.9}$$

and since that $\frac{\mu}{\rho} B_n^\tau \theta_n - \frac{1}{\rho} g_n \in H_n$, there exists only one solution u_n for (4.9) with $u_n \in D(B_n)$. Therefore $U_n = (u_n, v_n, \theta_n) \in D(\mathcal{A}_{\tau,n})$, which achieves the proof. □

Lemma 4.2. *The family of semigroups $S_{\tau,n}(t)$ verifies $i\mathbb{R} \subset \rho(\mathcal{A}_{\tau,n}), \quad n \in \mathbb{N}$.*

Proof. We show this result by a contradiction argument. That is, let us suppose that there exist $n_0 \in \mathbb{N}$ and $0 \neq \beta \in \mathbb{R}$, such that $i\beta$ is in the spectrum of $\mathcal{A}_{\tau,n_0}$. Since $0 \in \rho(B_n)$ and the power B_n^{-s} of B_n compact in H_n for any positive real number s, then the sequence of operators $\mathcal{A}_{\tau,n_0}^{-1}$ is compact and $i\beta$ must be an eigenvalue of $\mathcal{A}_{\tau,n_0}$. Thus there is a sequence of vector function $U_{n_0} = (u_{n_0}, v_{n_0}, \theta_{n_0}) \in D(\mathcal{A}_{\tau,n_0})$, $\|U_{n_0}\|_{\mathcal{H}_{n_0}} = 1$, such that $i\beta U_{n_0} - \mathcal{A}_{\tau,n} U_{n_0} = 0$ or equivalently

$$\begin{aligned} i\beta u_{n_0} - v_{n_0} &= 0, \\ i\beta v_{n_0} + \rho B_{n_0} u_{n_0} - \mu B_{n_0}^\tau \theta_{n_0} &= 0, \\ i\beta \theta_{n_0} + \kappa B_{n_0} \theta_{n_0} + \sigma B_{n_0}^\tau v_{n_0} &= 0. \end{aligned}$$

Taking the real part of the inner product of $i\beta U_{n_0} - \mathcal{A}_{\tau,n_0} U_{n_0} = 0$ with U_{n_0}, we obtain

$$\operatorname{Re} \langle \mathcal{A}_{\tau,n} U_{n_0}, U_{n_0} \rangle_{\mathcal{H}_{n_0}} = -\frac{\mu\kappa}{\rho\sigma} \left\| B_{n_0}^{\frac{1}{2}} \theta_{n_0} \right\|^2 = 0.$$

Thus $\theta_{n_0} = 0$ and then $u_{n_0} = v_{n_0} = 0$, which give the contradiction. □

Remark 4.3. The following theorem shows that the family of operators $S_{\tau,n}(t)$ associated to system (4.1) is uniformly polynomially stable with rate $t^{-1/2(1-2\tau)}$. Moreover, this rate is optimal. The proof is based on the ideas in [19, 21] and on a recent paper [13] in which the authors gave a detailed review about the region of stability and optimality of such kind of systems.

Theorem 4.4. *Assume now that $0 \le \tau < \frac{1}{2}$. Then, the semigroups generated by $\mathcal{A}_{\tau,n}$ defined in (4.2) is uniformly polynomially stable with order no less than $\alpha = 2(1 - 2\tau)$.*

Proof. First of all, we show that $\sup_{n\in\mathbb{N}} ||\mathcal{A}_{\tau,n}^{-1}|| < \infty$. Simple calculation shows that

$$\mathcal{A}_{\tau,n}^{-1} = \begin{pmatrix} -\frac{\mu\sigma}{\rho\kappa} B_n^{2\tau-2} & -\frac{1}{\rho} B_n^{-1} & -\frac{\mu}{\rho\kappa} B_n^{\tau-2} \\ I_n & 0 & 0 \\ -\frac{\sigma}{\kappa} B_n^{\tau-1} & 0 & -\frac{1}{\kappa} B_n^{-1} \end{pmatrix}. \tag{4.10}$$

By hypothesis on τ, $0 \in \rho(B_n)$ and since $\sup_{n\in\mathbb{N}} ||B_n^{\frac{-1}{2}}|| < \infty$, it follows that $\sup_{n\in\mathbb{N}} ||\mathcal{A}_{\tau,n}^{-1}|| < \infty$.

Since $\mathcal{A}_{\tau,n}$ verifies the hypothesis of Theorem 3.2, it is sufficient to show that

$$\sup_{|\beta|\geq 1,\ n\in\mathbb{N}} \frac{1}{|\beta|^{2(1-2\tau)}} \|(i\beta I_{3n} - \mathcal{A}_{\tau,n})^{-1}\| < \infty. \tag{4.11}$$

This will be done by contradiction, so suppose (4.11) is not true, then there exists a sequence $\beta_m \in \mathbb{R}$, $\beta_m \to \infty$ (as $m \to \infty$), a sequence $U_m \in D(\mathcal{A}_{\tau,n})$, $m = 0, 1, \dots,$ with

$$\|U_m\|^2_{\mathcal{H}_n} = \|B_n^{\frac{1}{2}} u_m\|^2 + \|v_m\|^2 + \|\theta_m\|^2 = 1, \tag{4.12}$$

and a subsequence of $\mathcal{A}_{\tau,n}$, still denoted by $\mathcal{A}_{\tau,n}$ such that

$$\lim_{m\to\infty} \|\beta_m^{2(1-2\tau)}(i\beta_m I_{3n} - \mathcal{A}_{\tau,n})U_m\|_{\mathcal{H}_n} = 0, \tag{4.13}$$

that is,

$$\beta_m^{2(1-2\tau)}\left(i\beta_m B_n^{\frac{1}{2}} u_m - B_n^{\frac{1}{2}} v_m\right) \to 0, \quad \text{in } H_n \tag{4.14}$$

$$\beta_m^{2(1-2\tau)}\left(i\beta_m v_m + \rho B_n u_m - \mu B_n^{\tau}\theta_m\right) \to 0, \quad \text{in } H_n \tag{4.15}$$

$$\beta_m^{2(1-2\tau)}\left(i\beta_m \theta_m + \kappa B_n \theta_m + \sigma B_n^{\tau} v_m\right) \to 0 \quad \text{in } H_n. \tag{4.16}$$

Our goal is to obtain $\|U_m\|_{\mathcal{H}_n} \to 0$ as $m \to \infty$, thus a contradiction.

By the dissipativeness of the operator $\mathcal{A}_{\tau,n}$

$$\operatorname{Re}\langle \beta_m^{2(1-2\tau)}(i\beta_m I - \mathcal{A}_{\tau,n})U_m, U_m\rangle_{\mathcal{H}_n} = \frac{\mu\kappa}{\rho\sigma}\left\|\beta_m^{1-2\tau} B_n^{\frac{1}{2}}\theta_m\right\|^2_{H_n}, \tag{4.17}$$

then it follows

$$\lim_{m\to\infty}\left\|\beta_m^{1-2\tau} B_n^{\frac{1}{2}}\theta_m\right\|_{H_n} = 0, \tag{4.18}$$

which further leads to

$$\left\|B_n^{\frac{1}{2}}\theta_m\right\| \to 0, \quad \|\theta_m\| \to 0 \tag{4.19}$$

since $0 \leqslant \tau < \frac{1}{2}$ and $\sup\limits_{n\in\mathbb{N}}\|B_n^{\frac{-1}{2}}\| < \infty$.

Next, we show that v_m also converges to zero. Acting a bounded operator $\lambda_m^{-(1-2\tau)} B_n^{-\frac{1}{2}}$ on (4.16) and applying (4.18), we have

$$i\beta_m^{2-2\tau} B_n^{-\frac{1}{2}}\theta_m + \sigma\beta_m^{1-2\tau} B_n^{\tau-\frac{1}{2}} v_m \to 0, \quad \text{in } H_n. \tag{4.20}$$

Since $0 < 1 - 2\tau \leqslant 1$, by Lemma 2.4, we have

$$\left\|\beta_m^{-(1-2\tau)} B_n^{\frac{1}{2}-\tau} v_m\right\| = \left\|\left(\beta_m^{-1} B_n^{\frac{1}{2}}\right)^{1-2\tau} v_m\right\| \leqslant L\left\|\beta_m^{-1} B_n^{\frac{1}{2}} v_m\right\|^{1-2\tau}\|v_m\|^{2\tau} \leqslant L. \tag{4.21}$$

Here, we have used the (uniform) boundedness of $\|\beta_m^{-1} B_n^{\frac{1}{2}} v_m\|$ which can be obtained from (4.14) and the (uniform) boundedness of $\|B_n^{\frac{1}{2}} u_m\|$. Taking inner product of (4.20) with $\beta_m^{-(1-2\tau)} B_n^{\frac{1}{2}-\tau} v_m$ in H_n yields

$$\langle i\beta_m B_n^{-\tau}\theta_m, v_m\rangle + \|v_m\|^2 \to 0. \tag{4.22}$$

Therefore, we only need to show

$$\|\beta_m B_n^{-\tau}\theta_m\| \to 0. \tag{4.23}$$

By the facts that $-\frac{1}{2} < -\tau \leqslant 0$, we apply Lemma 2.4 again

$$\|\beta_m B_n^{-\tau}\theta_m\| \leqslant L\|\beta_m\theta_m\|^{1-2\tau}\|\beta_m B_n^{-\frac{1}{2}}\theta_m\|^{2\tau} \leqslant C\|\beta_m\theta_m\|, \tag{4.24}$$

where $C = L\sup\limits_{n\in\mathbb{N}}\|B_n^{\frac{-1}{2}}\|^{2\tau} < \infty$. Thus, (4.23) follows from (4.18) and the (uniform) boundedness of $\|B_n^{\frac{-1}{2}}\|$. We have obtained

$$\|v_m\| \to 0. \tag{4.25}$$

Finally, we take inner product of (4.14) with $\rho\beta^{-2(1-2\tau)-1}B_n^{\frac{1}{2}}u_m$ in H_n and take inner product of (4.15) with $\beta^{-2(1-2\tau)-1}v_m$ in H_n, respectively. Thus, the imaginary part of their difference yields

$$\left(\left\|v_m\right\|^2 - \rho\left\|B_n^{\frac{1}{2}}u_m\right\|^2\right) - \mu\mathrm{Im}\langle\beta_m^{-1}B_n^{\tau}\theta_m, v_m\rangle \to 0. \tag{4.26}$$

From (4.18) and the boundedness of $\|v_m\|$, we have

$$\langle\beta_m^{-1}B_n^{\tau}\theta_m, v_m\rangle \to 0. \tag{4.27}$$

Then, it follows from (4.26) and (4.27) that

$$\|v_m\|^2 - \rho\|B_n^{\frac{1}{2}}u_m\|^2 \to 0. \tag{4.28}$$

Therefore, together with (4.25) and (4.28), we get

$$\left\|B_n^{\frac{1}{2}}u_m\right\|_{H_n}^2 \to 0. \tag{4.29}$$

Combining (4.19), (4.25) and (4.29), we have the promised contradiction. Theorem 3.2 allows us to conclude that the family of semigroups $S_{\tau,n}(t)$ is uniformly polynomially stable of order $\alpha = 2(1-2\tau)$. □

Remark 4.5. Throughout the proof of Theorem 4.4, we have used the uniform boundedness of the family of operators $B_n^{\frac{-1}{2}}$. However, due to Lemma 2.4, we can show that this hypothesis is equivalent to the fact that B_n^{-1} is uniformly bounded. This observation will be useful in the following.

4.2. Application 2: Semi-discrete hyperbolic/parabolic systems

We consider a weakly coupled hyperbolic/parabolic $1-d$ system on the bounded domain $\Omega = (0,\pi)$

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + \gamma\theta(x,t) = 0 & \text{in } (0,\pi)\times(0,\infty), \\ \theta_t(x,t) - k\theta_{xx}(x,t) - \gamma u_t(x,t) = 0 & \text{in } (0,\pi)\times(0,\infty), \\ u(x,t)\mid_{x=0,\pi} = 0 = \theta(x,t)\mid_{x=0,\pi} & \text{on } (0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \theta(x,0) = \theta_0(x) & \text{on } (0,\pi), \end{cases} \tag{4.30}$$

where u is proportional to the displacement and θ is the relative temperature. The constants $\gamma > 0$ and $k > 0$ represent, respectively, the coupling parameter and thermal diffusivity. For simplicity, we take $k = 1$, without affecting the proof of our result. By introducing a new variable $v = u_t$, (4.30) can be reduced to the following abstract first order evolution equation

$$\begin{cases} \frac{\mathrm{d}z}{\mathrm{d}t} = \mathcal{A}z \\ z(0) = z_0, \end{cases} \tag{4.31}$$

with

$$z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} \tag{4.32}$$

and

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 \\ D^2 & 0 & -\gamma I \\ 0 & \gamma I & D^2 \end{pmatrix}. \tag{4.33}$$

Let the state space

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \tag{4.34}$$

equipped with the norm

$$\|z\|_{\mathcal{H}} = \left(\|Dz_1\|_{L^2}^2 + \|z_2\|_{L^2}^2 + \|z_3\|_{L^2}^2 \right)^{\frac{1}{2}}, \tag{4.35}$$

and $\mathcal{D}(\mathcal{A}) = H^2 \cap H_0^1 \times H_0^1 \times H^2 \cap H_0^1$. Here we have used the notation $D = \partial/\partial x,\ D^2 = \partial^2/\partial x^2$. It is well known (see [2]) that the semigroup operator $\mathrm{e}^{t\mathcal{A}}$ associated to (4.31) is not exponentially stable. However, it has been shown in [19] that the C_0-semigroups generated by $\mathcal{A}$ is polynomially stable.

Remark 4.6. System (4.30) turns out to be a concrete realization of (4.1) (without the index n), corresponding to the choice $\rho = \kappa = 1$, $\mu = \sigma = -\gamma$, $\tau = 0$, $H = L^2(\Omega)$ and

$$B = -D^2 \quad \text{with domain} \quad D(B) = H^2 \cap H_0^1(\Omega).$$

According to Theorem 4.4, we can show that system (4.30) is polynomially stable of optimal order $\alpha = 2$.

Inspired from the work done in [19,21], and by means of a particular approximate scheme that is often referred to as the Galerkin method, we formulate (4.30) as an abstract approximate problem (1.1). Afterwards, we will show that the approximate scheme of (4.30) is uniformly polynomially stable. We also provide a convergence proof of this scheme. Let $H_n^1(\Omega)$, $H_n^2(\Omega)$ and $H_n^3(\Omega)$ be the n-dimensional subspace of $H_0^1(\Omega)$, $L^2(\Omega)$ and $L^2(\Omega)$ with basis $\{\phi_1, \ldots, \phi_n\}$, $\{\psi_1, \ldots, \psi_n\}$ and $\{\xi_1, \ldots, \xi_n\}$, respectively (see [22]). Since $H^2 \cap H_0^1$ is dense in L^2, we can choose $\phi_i \in H^2 \cap H_0^1$, $\psi_i \in H_0^1$ and $\xi_i \in H_0^1$. Let $\mathcal{H}_n = H_1^n(\Omega) \times H_2^n(\Omega) \times H_3^n(\Omega)$ with a basis

$$E_j = \begin{pmatrix} \phi_j \\ 0 \\ 0 \end{pmatrix}, \quad E_{n+j} = \begin{pmatrix} 0 \\ \psi_j \\ 0 \end{pmatrix}, \quad E_{2n+j} = \begin{pmatrix} 0 \\ 0 \\ \xi_j \end{pmatrix}, \qquad j = 1, \ldots, n. \tag{4.36}$$

The inner product on $\mathcal{H}_n$ is the one induced by the $\mathcal{H}$ product. We consider the approximation to the solution of (4.31) in the form

$$z_n = \sum_{j=1}^{3n} \tilde{z}_j(t) E_j(x), \tag{4.37}$$

which is required to satisfy the following variational system

$$\begin{cases} (\dot{z}_n, E_j)_{\mathcal{H}} = (\mathcal{A} z_n, E_j)_{\mathcal{H}}, \qquad j = 1, \ldots, 3n \\ z_n(0) = z_n^0, \end{cases} \tag{4.38}$$

where z_n^0 is the approximation of z_0 with respect to the basis $\{E_j\}_{j=1}^{3n}$. Then, we have

$$\begin{aligned} M_n \dot{\tilde{y}}_n &= \begin{bmatrix} M_n^{(1)} & & \\ & M_n^{(2)} & \\ & & M_n^{(3)} \end{bmatrix} \begin{bmatrix} \dot{\tilde{z}}_n^{(1)} \\ \dot{\tilde{z}}_n^{(2)} \\ \dot{\tilde{z}}_n^{(3)} \end{bmatrix} \\ &= \begin{bmatrix} 0 & \tilde{D}_n^T & 0 \\ -\tilde{D}_n & 0 & -\gamma \tilde{F}_n \\ 0 & \gamma \tilde{F}_n^T & -G_n \end{bmatrix} \begin{bmatrix} \tilde{z}_n^{(1)} \\ \tilde{z}_n^{(2)} \\ \tilde{z}_n^{(3)} \end{bmatrix} = \tilde{A}_n \tilde{y}_n \end{aligned} \tag{4.39}$$

with

$$\begin{aligned}(M_n^{(1)})_{ij} &= (D\phi_i, D\phi_j)_{L^2}, \quad (M_n^{(2)})_{ij} = (\psi_i, \psi_j)_{L^2}, \quad (M_n^{(3)})_{ij} = (\xi_i, \xi_j)_{L^2},\\ (\tilde{D}_n)_{ij} &= (D\phi_i, D\psi_j)_{L^2}, \quad (\tilde{F}_n)_{ij} = (\xi_i, \psi_j)_{L^2}, \quad (G_n)_{ij} = (D\xi_i, D\xi_j)_{L^2}\end{aligned} \tag{4.40}$$

and

$$\tilde{z}_n^{(i)} = (\tilde{z}_{(i-1)n+1}(t), \ldots, \tilde{z}_{in}(t))^T, \qquad i = 1, 2, 3. \tag{4.41}$$

By construction, the matrix $M_n^{(i)}$ is symmetric and positive definite. Therefore, there exists a lower triangle matrix $L_n^{(i)}$ such that $M_n^{(i)} = (L_n^{(i)})(L_n^{(i)})^T$. Let $L_n = \operatorname{diag}(L_n^{(1)}, L_n^{(2)}, L_n^{(3)})$ and denote $L_n^T \tilde{y}_n$ by $\bar{y}_n$. Then, to obtain approximate solution z_n, we are led to solve ordinary differential equations

$$\begin{cases} \dot{\bar{y}}_n = A_n \bar{y}_n \\ \bar{y}_n(0) = \bar{y}_n^0 := L_n^T z_n^0, \end{cases} \tag{4.42}$$

with

$$A_n = \begin{bmatrix} 0 & (L_1^T)^{-1}\tilde{D}_n^T L_2^{-1} & 0 \\ -(L_2^T)^{-1}\tilde{D}_n L_1^{-1} & 0 & -\gamma(L_2^T)^{-1}\tilde{F}_n L_3^{-1} \\ 0 & \gamma(L_3^T)^{-1}\tilde{F}_n^T L_2^{-1} & -(L_3^T)^{-1} G_n L_3^{-1} \end{bmatrix}. \tag{4.43}$$

It is easy to see that

$$\operatorname{Re}(A_n \bar{y}_n, \bar{y}_n)_{\mathbb{C}^{3n}} = -(G_n L_3^{-1} \bar{z}_n^{(3)}, L_3^{-1} \bar{z}_n^{(3)})_{\mathbb{C}^n} \le 0, \tag{4.44}$$

provided that G_n is semipositive definite. In that case, A_n generates a C_0-semigroup $T_n(t)$ of contraction on $\mathcal{H}_n$.

4.2.1. Finite difference method

Given $n \in \mathbb{N}$, we set $\Delta = \dfrac{\pi}{n}$ and introduce the net

$$x_0 = 0 < x_1 = \Delta < \ldots < x_{n-1} = (n-1)\Delta < x_n = \pi$$

with $x_j = j\Delta$, $j = 0, \ldots, n$. We then introduce the following finite-difference semi-discretization of system (4.30)

$$\begin{cases} \ddot{u}_j(t) + \frac{1}{\Delta^2}[u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)] + \gamma\theta_j(t) = 0, & t > 0,\ j = 1, \ldots, n-1 \\ \dot{\theta}_j(t) + \frac{1}{\Delta^2}[\theta_{j+1}(t) - 2\theta_j(t) + \theta_{j-1}(t)] - \gamma\dot{u}_j(t) = 0, & t > 0,\ j = 1, \ldots, n-1 \\ u_0(t) = u_n(t) = \theta_0(t) = \theta_n(t) = 0, & t > 0 \\ u_j(0) = u_{0j}, \quad \dot{u}_j(0) = u_{1j}, \quad \theta_j(0) = \theta_{0j},\ j = 0, \ldots, n \end{cases} \tag{4.45}$$

which is equivalent to the following system

$$\begin{cases} \ddot{\mathbf{u}}_n + B_n \mathbf{u}_n + \gamma \boldsymbol{\theta}_n = 0, \\ \dot{\boldsymbol{\theta}}_n + B_n \boldsymbol{\theta}_n - \gamma \dot{\mathbf{u}}_n = 0, \\ \mathbf{u}_n(0) = \mathbf{u}_{0n}, \quad \dot{\mathbf{u}}_n(0) = \mathbf{u}_{1n}, \quad \boldsymbol{\theta}_n(0) = \boldsymbol{\theta}_{0n}, \end{cases} \tag{4.46}$$

with $\mathbf{u}_n = (u_1(t), u_2(t), \ldots, u_{n-1}(t))^T$, $\dot{\mathbf{u}}_n = (\dot{u}_1(t), \dot{u}_2(t), \ldots, \dot{u}_{n-1}(t))^T$, $\boldsymbol{\theta}_n = (\theta_1(t), \theta_2(t), \ldots, \theta_{n-1}(t))^T$, $\mathbf{u}_{0n}$, $\mathbf{u}_{1n}$, $\boldsymbol{\theta}_{0n}$ are an approximation of the initial data in (4.30), and

$$B_n = \frac{1}{\Delta^2} \begin{bmatrix} 2 & -1 & & & \mathbf{0} \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ \mathbf{0} & & & -1 & 2 \end{bmatrix} \tag{4.47}$$

is symmetric, positive definite. Moreover, the eigenvalues associated to B_n can be computed explicitly. We have (see [17], p. 456)

$$\lambda_k^{B_n}(\Delta) = \frac{4}{\Delta^2}\sin^2\left(\frac{k\Delta}{2}\right), \quad k = 1, \ldots, n-1,$$

and verify

$$0 < \lambda_1^{B_n} < \lambda_2^{B_n} < \ldots < \lambda_{n-1}^{B_n}.$$

From the expression above, we have

$$\lambda_{n-1}^{B_n}\Delta^2 \to 4, \text{ as } \Delta \to 0.$$

Indeed,

$$\begin{aligned}\lambda_{n-1}^{B_n}\Delta^2 &= 4\sin^2\left(\frac{k\Delta}{2}\right)\\ &= 4\sin^2\left(\frac{\pi}{2} - \frac{\Delta}{2}\right)\\ &= 4\cos^2\left(\frac{\Delta}{2}\right) \to 4, \text{ as } \Delta \to 0.\end{aligned}$$

It is also easy to see that, for j fixed,

$$\lambda_j^{B_n} \to j^2, \text{ as } \Delta \to 0.$$

As we deal with finite dimensional space, due to Tikhonov Theorem, see [25], it is sufficient to show that B_n^{-1} is uniformly bounded for one norm, and hence for every norm on $\mathbb{R}^{(n-1)\times(n-1)}$. We have

$$\begin{aligned}\| B_n^{-1} \|_2 &= \max_{1\le k\le n-1}\left(\frac{1}{\left|\lambda_k^{B_n}\right|}\right)\\ &= \frac{1}{\min\limits_{1\le k\le n-1}\left|\lambda_k^{B_n}\right|}\\ &\leqslant 1.\end{aligned}$$

It follows that $\sup\limits_{n\in\mathbb{N}}||B_n^{-1}|| < \infty$. Therefore, Theorem 4.4 states that the approximate system (4.70) decays uniformly polynomially to zero, with $\alpha = 2$.

4.2.2. Finite element method

The classical finite element method is to divide the domain $\Omega = [0,\pi]$ into subintervals, usually in equal length, and use spline functions for the approximation. Here, we choose ϕ_j, ψ_j, and ξ_j to be the linear spline functions

$$h_j(x) = \begin{cases} 1 - \dfrac{1}{\Delta} \mid x - j\Delta \mid, & x \in [(j-1)\Delta, (j+1)\Delta],\\ 0, & \text{otherwise}\end{cases}$$

for all $j = 1, \dots, n-1$, with $\Delta = \dfrac{\pi}{n}$. A straightforward calculation following (4.40) yields

$$M_n^{(1)} = \frac{1}{\Delta}\begin{bmatrix} 2 & -1 & & & \mathbf{0} \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ \mathbf{0} & & & -1 & 2 \end{bmatrix}, \quad M_n^{(2)} = M_n^{(3)} = \Delta \begin{bmatrix} \frac{2}{3} & \frac{1}{6} & & & \mathbf{0} \\ \frac{1}{6} & \frac{2}{3} & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \frac{2}{3} & \frac{1}{6} \\ \mathbf{0} & & & \frac{1}{6} & \frac{2}{3} \end{bmatrix} \tag{4.48}$$

$$\tilde{D}_n = M_n^{(1)},\ \tilde{F}_n = M_n^{(2)},\ G_n = M_n^{(1)}. \tag{4.49}$$

Both $M_n^{(1)}$ and $M_n^{(2)}$ are symmetric and positive definite. Since $M_n^{(i)}$, $i = 1, 2$ is invertible and by introducing a new matrix $B_n = (M_n^{(2)})^{-1} M_n^{(1)}$, (4.39) can be written in the following form

$$\begin{cases} \ddot{\mathbf{u}}_n + B_n \mathbf{u}_n + \gamma \boldsymbol{\theta}_n = 0, \\ \dot{\boldsymbol{\theta}}_n + B_n \boldsymbol{\theta}_n - \gamma \dot{\mathbf{u}}_n = 0, \\ \mathbf{u}_n(0) = \mathbf{u}_{0n}, \quad \dot{\mathbf{u}}_n(0) = \mathbf{u}_{1n}, \quad \boldsymbol{\theta}_n(0) = \boldsymbol{\theta}_{0n}, \end{cases} \tag{4.50}$$

where $\mathbf{u}_n = \tilde{z}_n^{(1)}$, $\dot{\mathbf{u}}_n = \tilde{z}_n^{(2)}$ and $\boldsymbol{\theta}_n = \tilde{z}_n^{(3)}$, defined as in (4.41), provided the initial data $\mathbf{u}_{0n}$, $\mathbf{u}_{1n}$, $\boldsymbol{\theta}_{0n}$ are an approximation of the initial data in (4.30).

To show that system (4.50) fits in the abstract setting of Theorem 4.4, we need to establish some properties of the sequence B_n. Let us check that B_n is symmetric, positive definite and that $\sup\limits_{n\in\mathbb{N}} ||B_n^{-1}|| < \infty$.

B_n *is symmetric:* according to the invertibility of $M_n^{(2)}$, we only have to check that $M_n^{(1)}$ and $M_n^{(2)}$ commute. Taking

$$J = \begin{bmatrix} 0 & 1 & & & \mathbf{0} \\ 1 & 0 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 0 & 1 \\ \mathbf{0} & & & 1 & 0 \end{bmatrix}. \tag{4.51}$$

We have $M_n^{(1)} = \frac{1}{\Delta}(2I - J)$ and $M_n^{(2)} = \frac{\Delta}{3}(2I + \frac{1}{2}J)$. Thus matrix $M_n^{(1)}$ and $M_n^{(2)}$ commute.
B_n *is positive definite:* we consider a nonsingular matrix P such that $P^{-1} M_n^{(1)} P = D$, D being a diagonal matrix. From the expressions above, we have

$$P^{-1} J P = (2I - \Delta D) \text{ and } P^{-1} M_n^{(2)} P = \Delta\left(I - \frac{\Delta}{6} D\right)$$

which are diagonal matrix. Thus, matrix B_n and $M_n^{(1)}$ have the same eigenvectors. Moreover, eigenvalues of matrix $M_n^{(1)}$ can be computed explicitly

$$\lambda_k^{M_n^{(1)}} = \frac{4}{\Delta}\sin^2\left(\frac{k\Delta}{2}\right), \quad k = 1, \dots, n-1 \tag{4.52}$$

which implies that

$$\lambda_k^{B_n} = \frac{\lambda_k^{M_n^{(1)}}}{\Delta - \frac{\Delta^2}{6}\lambda_k^{M_n^{(1)}}} = \frac{6}{\Delta^2}\left[\frac{\sin^2(\frac{k\Delta}{2})}{\frac{3}{2} - \sin^2(\frac{k\Delta}{2})}\right] = \frac{6}{\Delta^2}\left[\frac{1 - \cos(k\Delta)}{2 + \cos(k\Delta)}\right]. \tag{4.53}$$

It is clear that $\lambda_k^{B_n} > 0$ for all $k = 1, \dots, n-1$. Thus, B_n is positive definite.

B_n^{-1} *is uniformly bounded:* From the expressions above, and especially (4.53), we have

$$\lambda_{n-1}^{B_n}\Delta^2 \to 12, \text{ as } \Delta \to 0.$$

Indeed,

$$\begin{aligned}\lambda_{n-1}^{B_n}\Delta^2 &= 6\left[\frac{1-\cos((n-1)\Delta)}{2+\cos((n-1)\Delta)}\right]\\ &= 6\left[\frac{1-\cos(\pi-\Delta)}{2+\cos(\pi-\Delta)}\right]\\ &= 6\left[\frac{1+\cos(\Delta)}{2-\cos(\Delta)}\right] \to 12, \text{ as } \Delta \to 0.\end{aligned}$$

It is also easy to see that, for j fixed,

$$\lambda_j^{B_n} \to j^2, \text{ as } \Delta \to 0.$$

Indeed,

$$\lim_{\Delta\to 0}\lambda_j^{B_n} = 2\lim_{\Delta\to 0}\frac{1-\cos(j\Delta)}{\Delta^2} = j\lim_{\Delta\to 0}\frac{\sin(j\Delta)}{\Delta} = j^2.$$

Thus, we have

$$\begin{aligned}\| B_n^{-1} \|_2 &= \max_{1\leq k\leq n-1}\left(\frac{1}{\left|\lambda_k^{B_n}\right|}\right)\\ &= \frac{1}{\min\limits_{1\leq k\leq n-1}\left|\lambda_k^{B_n}\right|}\\ &\leqslant 1.\end{aligned}$$

It follows that $\sup\limits_{n\in\mathbb{N}}||B_n^{-1}|| < \infty$. Therefore, Theorem 4.4 allows to conclude that system (4.50) decays uniformly polynomially to zero, with $\alpha = 2$.

Remark 4.7. The spectral analysis used above is inspired from the one in [16], where the authors used a similar analysis to investigate the observability property for the space semi-discretizations of the $1-d$ wave equations.

Remark 4.8. When the coupling terms ($\gamma\theta$ and γu_t) are replaced by ($\gamma\theta_x$ and γu_{tx}), system (4.30) becomes exponentially stable (see [20]). However, the question of showing the uniform exponential decay of solutions associated to the finite element scheme still remains open as has been mentioned in [21].

4.2.3. Spectral element method

Spectral element method is to choose the eigenvectors of the system as the basis vectors. Here, we will use the eigenvectors of the uncoupled hyperbolic parabolic system, *i.e.*, $\gamma = 0$ in (4.33). Let

$$\phi_j = \psi_j = \xi_j = \sqrt{\frac{2}{\pi}}\sin jx, \qquad j = 1, \ldots, n \tag{4.54}$$

we can choose the finite dimensional subspaces $H_j^n(\Omega) \subset H_0^1(\Omega)$, $j = 1, 2, 3$, as $\text{lin}\{\phi_i : i = 1, \ldots, n\}$. A straightforward calculation following (4.40) and (4.43) yields

$$A_n = \begin{bmatrix} 0 & D_n & 0\\ -D_n & 0 & -\gamma I_n\\ 0 & \gamma I_n & -D_n^2 \end{bmatrix} \tag{4.55}$$

with

$$D_n = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & n \end{bmatrix}. \tag{4.56}$$

Notice that with the previous choice of basis in $\mathcal{H}_n$, (4.37) defines an isomorphism between $\mathcal{H}_n$ and $\mathbb{R}^{3n}$ which is equipped with the usual inner product. Let

$$\mathcal{A}_n = P_n \mathcal{A} P_n, \tag{4.57}$$

where P_n is the orthogonal projection from $\mathcal{H}$ to $\mathcal{H}_n$. Then (4.38) can be considered as an evolution equation in $\mathcal{H}_n$

$$\begin{cases} \frac{\mathrm{d}z_n}{\mathrm{d}t} = \mathcal{A}_n z_n, \\ z_n(0) = z_n^0. \end{cases}$$

Notice that for $z_n \in \mathcal{H}_n$, we have $(\mathcal{A}_n z_n, z_n)_{\mathcal{H}_n} = (A_n \bar{y}_n, \bar{y}_n)_{\mathbb{R}^{3n}}$.

Lemma 4.9. *The family of generators $\mathcal{A}_n$ satisfies $i\mathbb{R} \subset \rho(\mathcal{A}_n), n \in \mathbb{N}$, and $\sup_{n\in\mathbb{N}} ||\mathcal{A}_n^{-1}|| < \infty$.*

Proof. We show this result by a contradiction argument. That is, let us suppose that there exist a fixed $m \in \mathbb{N}$ and $0 \neq \beta \in \mathbb{R}$, such that $i\beta$ is in the spectrum of $\mathcal{A}_m$. Since the operator $\mathcal{A}_m$ is of finite rank, it is compact, thus $i\beta$ is an eigenvalue of $\mathcal{A}_m$. Therefore, there exists a sequence of vector function $z_m \in \mathcal{H}_m$, $\|z_m\|_{\mathcal{H}_m} = 1$ and by (4.37) there exists $y_m = (u_m, v_m, \theta_m) \in \mathbb{R}^{3m}$, $\|y_m\|_{\mathbb{R}^{3m}} = 1$, accordingly. It follows from the definition of A_n that $i\beta z_m - \mathcal{A}_m z_m = 0$ is equivalent to $i\beta y_m - A_m y_m = 0$, *i.e.*

$$i\beta u_m - D_m v_m = 0,$$

$$i\beta v_m + D_m u_m + \gamma \theta_m = 0,$$

$$i\beta \theta_m - \gamma v_m + D_m^2 \theta_m = 0.$$

Taking the inner product of $i\beta z_m - \mathcal{A}_m z_m = 0$ with z_m, we obtain

$$(\mathcal{A}_m z_m, z_m)_{\mathcal{H}_m} = (A_m \bar{y}_m, \bar{y}_m)_{\mathbb{R}^{3m}} = -\|D_m \theta_m\|^2 = 0.$$

Hereafter we also denote by $\|\cdot\|$ the l^2 norm in $\mathbb{R}^m$ or $\mathbb{C}^m$ when no confusion occurs. Since D_m is invertible, $\theta_m = 0$ and then $u_m = v_m = 0$, which gives the contradiction. Therefore, $i\mathbb{R} \subset \rho(\mathcal{A}_n)$ for all $n \in \mathbb{N}$. As A_m is the matrix representation of the operator $\mathcal{A}_m$, it is sufficient to show that $\sup_{n\in\mathbb{N}} ||A_n^{-1}|| < \infty$. Simple calculations show that

$$A_n^{-1} = \begin{bmatrix} -\gamma^2 D_n^{-4} & -D_n^{-1} & \gamma D_n^{-3} \\ D_n^{-1} & 0 & 0 \\ \gamma D_n^{-3} & 0 & -D_n^{-2} \end{bmatrix}. \tag{4.58}$$

Therefore,

$$\|A_n^{-1}\|_\infty = \gamma^2 + \gamma + 1,$$

and this ends the proof. $\square$

The following theorem claims that the approximate system still decays to zero with an order no less than $\alpha = 2$.

Theorem 4.10. *The semigroups generated by $\mathcal{A}_n$, $n \in \mathbb{N}$, defined in (4.57) are uniformly polynomially stable with $\alpha = 2$.*

Proof. The family $\mathcal{A}_n, n \in \mathbb{N}$, satisfies the hypothesis of our abstract result. Let us verify that

$$\sup_{|\beta|\geq 1,\, n\in\mathbb{N}} \frac{1}{|\beta|^2} \|(i\beta I_{3n} - \mathcal{A}_n)^{-1}\| < \infty. \tag{4.59}$$

This will be done by a contradiction argument. If (4.59) is not true, then there must exist a subsequence of $\mathcal{A}_n$, still denoted by $\mathcal{A}_n$, a sequence $\beta_m \in \mathbb{R}^+$, $\beta_m \to \infty$ (as $m \to \infty$), and a sequence $z_m \in \mathcal{H}_n$ with $\|z_m\|_{\mathcal{H}_n} = 1$ such that as $m \to \infty$

$$\lim_{m\to\infty} \|\beta_m^2 (i\beta_m I_{3n} - \mathcal{A}_n) z_m\| = 0. \tag{4.60}$$

Let $y_m = (u_m, v_m, \theta_m) \in \mathbb{R}^{3n}$ be the corresponding coordinate vector to z_m. Then (4.60) is equivalent to

$$\lim_{m\to\infty} \|\beta_m^2 (i\beta_m I_{3n} - A_n) y_m\| = 0,$$

that is,

$$\|\beta_m^2 (i\beta_m u_m - D_n v_m)\| \to 0, \tag{4.61}$$

$$\|\beta_m^2 (i\beta_m v_m + \gamma\theta_m + D_n u_m)\| \to 0, \tag{4.62}$$

$$\|\beta_m^2 ((i\beta_m I_n + D_n^2)\theta_m - \gamma v_m)\| \to 0. \tag{4.63}$$

Our goal is to obtain $\|y_m\|_{\mathbb{R}^{3n}} \to 0$ as $m \to \infty$, thus a contradiction. Since $\|y_m\|_{\mathbb{R}^{3n}} = 1$, one has

$$|\mathrm{Re}(\beta_m^2 (i\beta_m I_{3n} - A_n) y_m, y_m)| \leq \|\beta_m^2 (i\beta_m I_{3n} - A_n) y_m\|. \tag{4.64}$$

Hence

$$\|\beta_m D_n \theta_m\|^2 = \mathrm{Re}(\beta_m^2 (i\beta_m I_{3n} - A_n) y_m, y_m) \to 0, \tag{4.65}$$

then, it follows

$$\|\beta_m \theta_m\| \leq \|\beta_m D_n \theta_m\| \to 0, \tag{4.66}$$

which further leads to

$$\|\theta_m\| \to 0. \tag{4.67}$$

It follows from $\|y_m\|_{\mathbb{R}^{3n}} = 1$ and (4.67) that

$$\left\| \begin{pmatrix} u_m \\ v_m \end{pmatrix} \right\|_{\mathbb{R}^{2n}}^2 \to 1. \tag{4.68}$$

Next, we show that $\|v_m\|$ also converges to zero. Taking the inner product of (4.63) with v_m yields

$$i(\beta_m \theta_m, v_m) + (D_n \theta_m, D_n v_m) - \gamma \|v_m\|^2 \to 0. \tag{4.69}$$

It follows from (4.66) and (4.68) that

$$|(\beta_m \theta_m, v_m)| \to 0.$$

The inner product of (4.61) by $D_n \theta_m$, (4.66) and (4.68) yields

$$(D_n \theta_m, D_n v_m) \to 0,$$

and thus

$$\|v_m\| \to 0. \tag{4.70}$$

On the other hand, using the difference between the inner product of (4.61) with $D_n^{-1} v_m$ and the inner product of (4.62) with $D_n^{-1} u_m$, we obtain

$$\|u_m\|^2 + \|v_m\|^2 + \gamma(\theta_m, D_n^{-1} u_m) \to 0,$$

since $(\theta_m, D_n^{-1}u_m) \to 0$ due to (4.67) and (4.68). We deduce from (4.70) that

$$\|u_m\| \to 0. \tag{4.71}$$

Thus, we have the promised contradiction, and the proof of Theorem 4.10 is complete. □

Remark 4.11. Due to Theorem 4.4, we can still show that the spectral element scheme still decays uniformly polynomially to zero. Moreover, as we deal with the Laplace operator basis, system (4.30) can be diagonalized, and $T_n(t) = P_n T(t) P_n$, where P_n is the projection on the space of the n first eigenfunctions of the Laplace operator. This makes the polynomial stability of the semigroup $T_n(t)$ a straightforward consequence of the polynomial stability of $T(t)$.

Now we will prove the strong convergence of the approximating semigroups $T_n(t)$ of (4.42) to $T(t) := \mathrm{e}^{t\mathcal{A}}$ and $T_n^*(t)$ to $T^*(t)$. This is obtained with the help of a general version of the Trotter−Kato Theorem proved in ([23], Thm. 4.5, p. 88) that is appropriated when the approximated semigroups are defined in proper subspaces. The basic idea is that the convergence of the approximated semigroups is equivalent to the convergence of their generators, hence we prove such a convergence result for both examples of the previous section.

As mentioned before, the matrix A_n in (4.55) is the matrix representation of the operator $\mathcal{A}_n$. It is easy to see that $\mathcal{D} := \mathcal{D}(\mathcal{A}) \cap (H^4 \times H^3 \times H^4)$ is dense in $\mathcal{H}$. Since $(I - \mathcal{A})\mathcal{D}(\mathcal{A}) = \mathcal{H}$, we also know that $(I - \mathcal{A})\mathcal{D}$ is dense in $\mathcal{H}$. With the dissipativeness of $\mathcal{A}$ and $\mathcal{A}_n$, by the Trotter–Kato theorem, we only need to show $\mathcal{A}_n z \to \mathcal{A}z$ in $\mathcal{H}$ for all $z \in \mathcal{D}$ for the strong convergence of the approximation semigroups $T_n(t)$ to $T(t)$.

Theorem 4.12. *$T_n(t)$, $T_n^*(t) \underset{\longrightarrow}{s} T(t)$, $T^*(t)$ in $\mathcal{H}$, respectively. Moreover, the convergence is uniform in bounded t-intervals.*

Proof. Let $z \in \mathcal{D}$. Then

$$z = \sum_{j=1}^{\infty} \left[a_j \begin{pmatrix} \frac{1}{j}\sin jx \\ 0 \\ 0 \end{pmatrix} + b_j \begin{pmatrix} 0 \\ \sin jx \\ 0 \end{pmatrix} + c_j \begin{pmatrix} 0 \\ 0 \\ \sin jx \end{pmatrix} \right]$$

with $\{a_j j^3, b_j j^3, c_j j^4\}_{j\geq 1}$ being l^2 sequences. Furthermore, we have

$$\mathcal{A}z = \begin{pmatrix} \sum_{i=1}^{\infty} b_i \sin ix \\ \sum_{i=1}^{\infty} (-a_i i - \gamma c_i) \sin ix \\ \sum_{i=1}^{\infty} (\gamma b_i - c_i i^2) \sin ix \end{pmatrix}$$

and

$$\mathcal{A}_n z = \begin{pmatrix} \sum_{i=1}^{n} b_i \sin ix \\ \sum_{i=1}^{n} (-a_i i - \gamma c_i) \sin ix \\ \sum_{i=1}^{n} (\gamma b_i - c_i i^2) \sin ix \end{pmatrix}.$$

Now, computing $\mathcal{A}z - \mathcal{A}_n z$, we obtain

$$\begin{pmatrix} \sum_{i=n+1}^{\infty} b_i \sin ix \\ \sum_{i=n+1}^{\infty} (-a_i i - \gamma c_i) \sin ix \\ \sum_{i=n+1}^{\infty} (\gamma b_i - c_i i^2) \sin ix \end{pmatrix} := R_n.$$

It follows from $\mathcal{A}z \in \mathcal{H}$ that $\|R_n\| \to 0$ as $n \to \infty$. Thus, we have proved

$$\lim_{n\to+\infty} \|\mathcal{A}z - \mathcal{A}_n z\|_{\mathcal{H}} = 0, \qquad \forall z \in \mathcal{D}.$$

The convergence of approximate adjoint semigroups can be verified in a similar way, since $\mathcal{A}$ and $\mathcal{A}^*$ only differ by the sign in front of the coupling coefficient γ. □

5. Numerical experiments

As has been demonstrated in Sections 3 and 4, preserving polynomial stability for the general approximation schemes of thermoelastic system (4.1) can be a complicated problem, due to the structure of the matrix A_n defined in (4.43). On the other hand, for any given approximation scheme, we can compute the eigenvalues of $\mathcal{A}_n$ to observe the trends in their location, as well as the stability behavior. For systems with polynomial decay, it is well known that the eigenvalues are approaching the imaginary axis at $\pm\infty$.

In this section, we consider the three approximation schemes presented in Section 4.2. For each of them, the matrix $\mathcal{A}_n$ is constructed and its eigenvalues are computed. Due to Proposition 3.6 we will show that for the three approximation schemes, we have a uniform spectral estimate (3.10). Finally, we show numerically the effect of the smoothness of the initial data on the rate of decay of energy associated to the finite element scheme (4.50), a fact which has been shown in Theorem 3.2. In all the following examples, we take $\gamma = 0.1$. Since the real eigenvalues of the matrix $\mathcal{A}_n$ are much smaller than the imaginary part of the complex eigenvalues, it is enough to observe the complex ones only. Otherwise, due to scaling, there are several negative real eigenvalues which are not plotted.

5.1. Uniform spectral estimate

It has been shown in Section 4 that the approximate schemes of thermoelastic system (4.30), either by finite difference, finite element or spectral element method are uniformly polynomially stable with $\alpha = 2$. In what follows, we set d to be the spectral distance (3.10) defined in Proposition 3.6,

$$d = \sup_{\mathrm{Re}\lambda \geq -\delta,\ n\in\mathbb{N}} \left\{ \frac{|\mathrm{Im}\lambda|^{-2}}{|\mathrm{Re}\lambda|},\ \lambda \in \sigma(\mathcal{A}_n) \right\},$$

with $\delta = -\min(\mathrm{Re}(\lambda)) > 0$.

Throughout Table 1, we notice that for the three approximate schemes, these distances are uniform. The location of the eigenvalues derived from these approximation schemes are also plotted for $n = 8, 16, 24, 32$. We observe from Figures 1–3, that for the three approximate schemes, for fixed n, the eigenvalues of higher frequency modes are closer to the imaginary axis, which is in perfect agreement with our theory.

5.2. Role of smoothness of the initial data on the rate of decay of energy

It has been demonstrated in Theorem 3.2 ((2) $\Leftrightarrow$ (3)) that the rate of decay of energy can be improved according to the smoothness of the initial data. To show this fact, we use a uniform mesh with $n = 8$ elements, fix the final time in $T = 100$, use $\Delta t = 10^{-2}$ and consider the following initial conditions for u and θ

$$u(x,0) = 0, \qquad \theta(x,0) = 0, \qquad u_t(x,0) = \sqrt{\frac{2}{\pi}}\sin(jx),\ j = 1,2,3.$$

In the following, we present the graphics of energy of the thermoelastic model (4.30), $E_n^j(t)$, $j = 1,2,3$, associated to the finite element method studied in Section 4.

Table 1. Uniform spectral estimate for the finite difference method (fdm), finite element method (fem) and spectral element method (sem) in the case of Dirichlet−Dirichlet boundary conditions.

n	d (fdm)	d (fem)	d (sem)
8	2.062779×10^2	2.022197×10^2	2.031240×10^2
16	2.015359×10^2	2.005207×10^2	2.007812×10^2
24	2.006796×10^2	2.002283×10^2	2.003472×10^2
32	2.003816×10^2	2.001278×10^2	2.001953×10^2

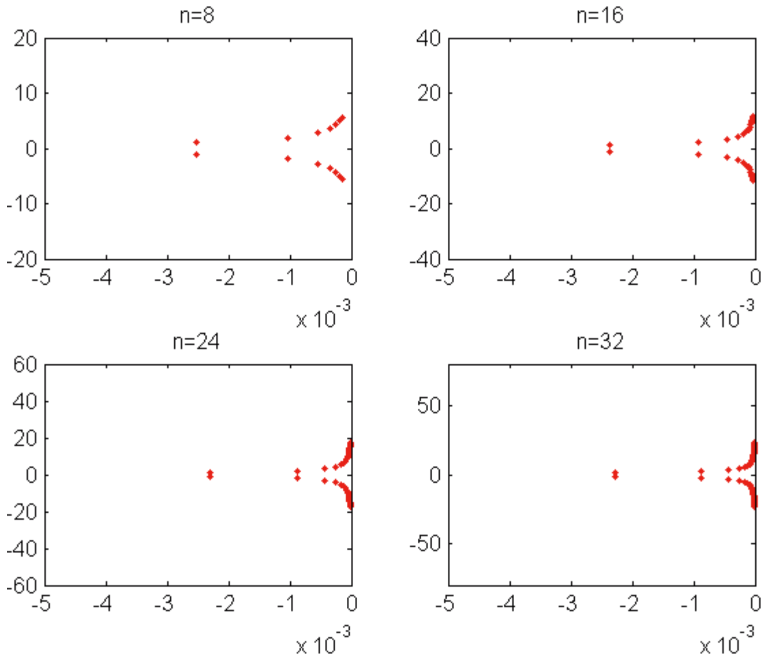


FIGURE 1. Location of the complex eigenvalues of the matrix $\mathcal{A}_n$ for the finite difference method in the case of Dirichlet−Dirichlet boundary conditions.

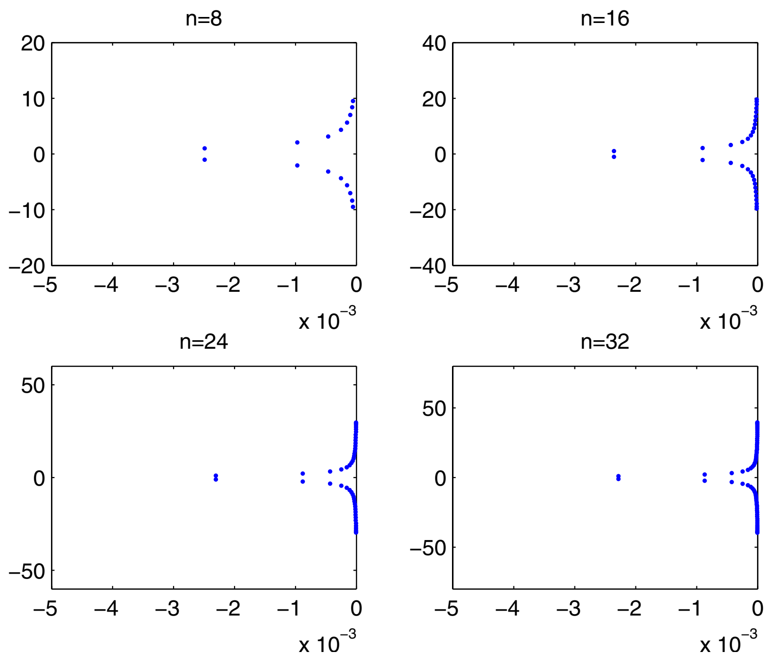


FIGURE 2. Location of the complex eigenvalues of the matrix $\mathcal{A}_n$ for the finite element method in the case of Dirichlet−Dirichlet boundary conditions.

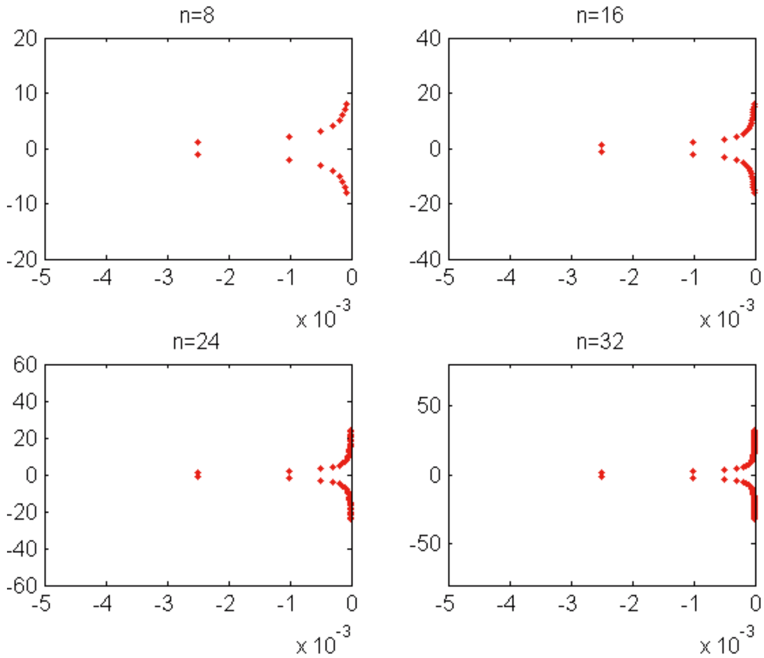


FIGURE 3. Location of the complex eigenvalues of the matrix $\mathcal{A}_n$ for the spectral element method in the case of Dirichlet−Dirichlet boundary conditions.

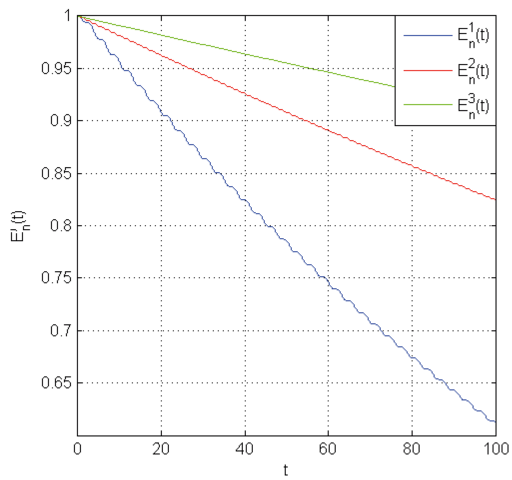


FIGURE 4. Role of smoothness of the initial data on the rate of decay of energy of the linear thermoelastic system (4.30).

Through Figure 4, we notice that for $j = 1$, the approximate energy $E_n^1(t)$ decays to zero as the time t increases. Moreover, we can observe that the decay rate of energy decreases as j increases, that is, the initial data is very oscillating. This means that the rate of decay of the approximated energy $E_n(t)$ is very sensitive to the choice of the initial data. This fact is consistent with the main result of this paper.

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