ESAIM: COCV 22 (2016) 64–87
DOI: 10.1051/cocv/2014066

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

A NOTION OF COMPLIANCE ROBUSTNESS IN TOPOLOGY OPTIMIZATION

Samuel Amstutz[1] and Marc Ciligot-Travain[1]

Abstract. The goal of this paper is twofold. On one hand, our work revisits the minimization of the robust compliance in shape optimization, with a more natural and more general approach than what has been done before. On the other hand, following a more recent viewpoint on robust optimization, we study the maximization of the so-called stability radius for a fixed maximal compliance. We provide theorical as well as numerical results.

Mathematics Subject Classification. 49Q10, 49M29, 74P05, 74P10, 74P15, 90C20.

Received March 26, 2014. Revised September 26, 2014.
Published online September 2, 2015.

1. Introduction

The compliance $C(\Omega, f_\Omega)$ of a linear elastic structure occupying a domain Ω and submitted to a load f_Ω is defined as the work done by the load, or equivalently as the stored elastic energy. Minimizing the compliance for a fixed load is a very standard shape optimization problem, for which a wide range of methods have been developed, see *e.g.* [1,11] and the references therein. However, it often occurs that the load is not known exactly. In this work we suppose that it takes the form $f_\Omega^N + B_\Omega \xi$, $\xi \in r\mathbb{B}$, with $r > 0$, $\mathbb{B}$ the closed unit ball of a Hilbert space, f_Ω^N a nominal load and B_Ω a linear operator. The robust compliance (also called principal compliance) is then defined by

$$j_{wc}(\Omega) = \sup_{\xi \in r\mathbb{B}} C(\Omega, f_\Omega^N + B_\Omega \xi).$$

The robust compliance may replace the compliance when the load is uncertain, so that minimizing the robust compliance is just minimizing the compliance 'in the worst case'. The way from compliance to robust compliance is just an illustration of the transition from optimization to robust optimization. The robust compliance has been first studied in [13], see also [14]. The worst case point of view has been applied to other criteria in [2,8,23].

The goal of this paper is twofold. On one hand, our work revisits the paper by De Gournay *et al.* [15] about the minimization of the robust compliance, with a more natural (and more general) approach. By definition, computing the robust compliance amounts to solving a quadratic optimization problem with a norm constraint in infinite dimension. In [15], the authors clearly announced that they renounced to follow this direct way because they did not see how to proceed, and used instead a formulation where the displacement field is chosen as main unknown. We show in the present article how to deal with the direct formulation.

Keywords and phrases. Robustness, stability radius, compliance, topological derivative, topology optimization.

[1] Laboratoire de Mathématiques d'Avignon, Faculté des Sciences, 33 rue Pasteur, 84000 Avignon, France.
marc.ciligot@univ-avignon.fr

On the other hand, following another and more recent viewpoint on robust optimization, we study the maximization of the so-called stability radius for a fixed maximal compliance. This is actually the main purpose of this paper, and we shall enter a little more into details. Suppose that the compliance should not exceed α. Then the stability radius relatively to the level-set $[C(\Omega, f_\Omega^N + B_\Omega.) \leq \alpha]$ is defined as

$$\sqrt{2j_{sr}(\Omega)} = \sup\left\{r > 0 \,\middle|\, r\mathbb{B} \subset [C(\Omega, f_\Omega^N + B_\Omega.) \leq \alpha]\right\} = \operatorname{dist}\left(0, [C(\Omega, f_\Omega^N + B_\Omega.) > \alpha]\right).$$

Actually we shall rather consider the squared quantity $j_{sr}(\Omega)$, as it will be identified with the value of a quadratic programming problem. Then it is natural to look for the domain Ω which maximizes $j_{sr}(\Omega)$. In other words, we seek the shape which tolerates the greatest deviation from the nominal load, in the sense that the compliance remains below α. This is a kind of robustness optimization. Actually, we will not necessarily find globally optimal shapes. It is more exact to say that we show how to improve the stability radius of a given shape until reaching, with respect to a certain class of perturbations, a locally optimal design.

The notion of stability radius appeared in robust control (see, *e.g.*, [24, 25]), and has been developed in its full generality, but not from a very mathematical point of view, at the end of the 90's by Ben–Haim (see [9] and the references therein). Compared to the worst case approach, this one avoids fixing r *a priori*, which is not necessarily easy and natural in some circumstances. We think that, in many situations, it is more natural to fix an upper bound α for the objective function, here the compliance. This amounts somehow to fixing some specifications.

The paper is organized as follows. In Section 2, we describe the general mathematical setting of our problems. In Section 3, we first show that the two robust criteria, namely $j_{wc}(\Omega)$ and $j_{sr}(\Omega)$, are the value functions of some quadratic programs with equality constraints. More precisely, the objective function of each subproblem is a quadratic functional and the equality constraint is associated with another quadratic functional. Such problems are known in the literature as trust-region subproblems, and have been extensively studied in the finite dimensional setting, see *e.g.* [16–18, 22, 29, 32, 35–37]. Here we prove the existence of critical loads (*i.e.* solutions of the subproblems) in arbitrary dimension, for both problems. Finally, using a strong duality argument for Lagrangian duality extending known results in finite dimension, we show the existence of a unique solution of the dual problem and give a complete description of the critical loads based on this solution. In Section 4, for both problems again, we give an expression of the Hadamard semiderivative of the two criteria relatively to the quadratic functionals depending on Ω, based on the Lagrangian and the solution of the dual problem. Sections 5 through 7 specifically deal with the optimization problem with respect the shape Ω. To keep concise and avoid repetitions, we concentrate on the maximization of the stability radius. Our procedure relies on the concept of topological derivative [21, 30, 34], which evaluates the variation of the objective functional with respect to small topological perturbations. In Section 5, we deduce from Section 4 the expression of the topological derivative of the stability radius. The optimization algorithm is described in Section 6, while Section 7 reports on some numerical computations.

2. General setting

We denote by Ω the domain to be optimized, and by $\mathscr{E}$ the set of admissible domains. For each $\Omega \in \mathscr{E}$ we are given a reflexive Banach space $\mathcal{V}_\Omega$. We denote by $\|.\|$ the norm on $\mathcal{V}_\Omega$ and by $\langle ., . \rangle$ the duality pairing between $\mathcal{V}_\Omega'$ and $\mathcal{V}_\Omega$, where $\mathcal{V}_\Omega'$ stands for the continuous dual space of $\mathcal{V}_\Omega$. We also consider a continuous and self-adjoint positive definite isomorphism A_Ω from $\mathcal{V}_\Omega$ into $\mathcal{V}_\Omega'$. We associate to each $f_\Omega \in \mathcal{V}_\Omega'$ the vector $u_{\Omega,f_\Omega} = A_\Omega^{-1} f_\Omega \in \mathcal{V}_\Omega$ and the scalar

$$C(\Omega, f_\Omega) = \frac{1}{2}\langle f_\Omega, u_{\Omega,f_\Omega}\rangle = \frac{1}{2}\langle f_\Omega, A_\Omega^{-1} f_\Omega\rangle.$$

Referring to the context of structural mechanics, we will subsequently call f_Ω the load, u_{Ω,f_Ω} the displacement field, and $C(\Omega, f_\Omega)$ the compliance (actually the half compliance). We will consider a parameterized family of

loads of the form

$$f_\Omega = f_\Omega^N + B_\Omega \xi, \qquad \xi \in \mathcal{H},$$

where $f_\Omega^N \in \mathcal{V}_\Omega'$ is a given nominal load, $\mathcal{H}$ is a fixed (*i.e.*, independent of Ω) separable Hilbert space and $B_\Omega : \mathcal{H} \to \mathcal{V}_\Omega'$ is a linear, compact and injective operator. We set

$$q(\Omega, \xi) = C(\Omega, f_\Omega^N + B_\Omega \xi).$$

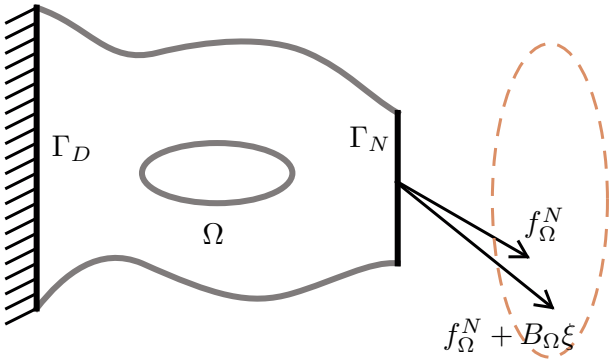
Hence we have, for $\xi \in \mathcal{H}$,

$$q(\Omega, \xi) = \frac{1}{2}\langle Q_\Omega \xi, \xi\rangle + \langle b_\Omega, \xi\rangle + c_\Omega$$

with

$$Q_\Omega = B_\Omega^* A_\Omega^{-1} B_\Omega, \qquad b_\Omega = B_\Omega^* A_\Omega^{-1} f_\Omega^N, \qquad c_\Omega = \frac{1}{2}\left\langle f_\Omega^N, A_\Omega^{-1} f_\Omega^N \right\rangle. \tag{2.1}$$

Note that $Q_\Omega : \mathcal{H} \to \mathcal{H}$ is a compact self-adjoint operator satisfying $Q_\Omega \geq 0$ and $Q_\Omega \neq 0$ (one supposes $\mathcal{H} \neq \{0\}$).



Before continuing with the abstract framework, let us give a typical concrete example. We consider the problem of compliance minimization for a structure submitted to an uncertain load. The structure is represented by a domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or $d = 3$), whose boundary is split into three disjoint subsets Γ, Γ_D and Γ_N with $\mathrm{meas}(\Gamma_D) > 0$. Homogeneous Dirichlet boundary conditions are prescribed on Γ_D, and Neumann boundary conditions are prescribed on $\Gamma \cup \Gamma_N$, with zero force on Γ. We denote by $\mathrm{H}^1_D(\Omega)^d$ the space of vector fields belonging to $\mathrm{H}^1(\Omega)^d$ with vanishing trace on Γ_D. For a given load $f_\Omega \in (\mathrm{H}^1_D(\Omega)^d)'$ and a given displacement $u \in \mathrm{H}^1_D(\Omega)^d$, the elastic energy is the quadratic functional in u defined by

$$E_\Omega(u, f_\Omega) = -\frac{1}{2}\int_\Omega He(u) : e(u) + \langle f_\Omega, u\rangle$$

where $e(u) = (\nabla u + \nabla u^T)/2$ is the strain tensor and H is the fourth-order elasticity tensor (Hooke's tensor) such that $He(u)$ is the stress tensor. The compliance is defined as

$$C(\Omega, f_\Omega) = \max_{u \in \mathrm{H}^1_D(\Omega)^d} E(u, f_\Omega).$$

Therefore, setting $\mathcal{V}_\Omega = \mathrm{H}^1_D(\Omega)^d$, the operator $A_\Omega : \mathcal{V}_\Omega \to \mathcal{V}_\Omega'$ is defined by

$$\langle A_\Omega u, v\rangle = \int_\Omega He(u) : e(v)$$

and the compliance admits the expression

$$C(\Omega, f_\Omega) = \frac{1}{2}\langle f_\Omega, A_\Omega^{-1} f_\Omega\rangle.$$

We choose $\mathcal{H}$ as a closed subspace of $L^2(\Gamma_N)^d$, and we define $B_\Omega : \mathcal{H} \to \mathcal{V}'_\Omega$ by

$$B_\Omega \xi : u \in \mathcal{V}_\Omega \mapsto \int_{\Gamma_N} \xi.u|_{\Gamma_N}.$$

The compactness of B_Ω is due to the compactness of the Sobolev embedding $L^2(\Gamma_N) \to H^{-1/2}(\Gamma_N)$. A very standard problem in optimal design consists in minimizing the compliance with a fixed load f_Ω^N, *i.e.*,

$$\text{Minimize} \quad q(\Omega, 0) = C(\Omega, f_\Omega^N), \quad \Omega \in \mathscr{E},$$

where the set $\mathscr{E}$ can include constraints. One speaks of robust compliance minimization when, at the same time, perturbations of the load are considered.

Let us now come back to the general case. In this paper we will investigate two notions of robustness and the associated optimization problems.

(1) **Stability radius as compliance robustness.** Given a threshold $\alpha > C(\Omega, f_\Omega^N) = c_\Omega$, the functional j_{sr} is defined by

$$j_{sr}(\Omega) = \frac{1}{2}\operatorname{dist}(0, [q(\Omega, .) > \alpha])^2 = \inf_{q(\Omega,\xi)>\alpha} \frac{1}{2}\|\xi\|^2 = \inf_{C(\Omega, f_\Omega^N + B_\Omega\xi)>\alpha} \frac{1}{2}\|\xi\|^2. \tag{2.2}$$

Increasing this value amounts to increasing the distance to unfeasability, where unfeasability means that the compliance is greater than α. This leads to considering the optimization problem:

$$\text{Maximize } j_{sr}(\Omega), \quad \Omega \in \mathscr{E},$$

hence one speaks of robustness maximization.

(2) **Robust compliance in the worst case sense.** Given a radius $r > 0$, the worst case compliance is defined by

$$j_{wc}(\Omega) = \sup_{\|\xi\|\le r} q(\Omega, \xi) = \sup_{\|\xi\|\le r} C(\Omega, f_\Omega^N + B_\Omega\xi). \tag{2.3}$$

This is the maximal compliance obtained for a given family of loads. One naturally wants to minimize this quantity, leading to the so-called worst case compliance minimization problem:

$$\text{Minimize } j_{wc}(\Omega), \quad \Omega \in \mathscr{E}.$$

In fact it is easily checked that the inequalities in (2.2) and (2.3) can be replaced by equalities, *i.e.* we have

$$j_{sr}(\Omega) = \frac{1}{2}\operatorname{dist}(0, [q(\Omega, .) = \alpha])^2 = \inf_{q(\Omega,\xi)=\alpha} \frac{1}{2}\|\xi\|^2 = \inf_{C(\Omega, f_\Omega^N + B_\Omega\xi)=\alpha} \frac{1}{2}\|\xi\|^2. \tag{2.4}$$

$$j_{wc}(\Omega) = \sup_{\|\xi\|=r} q(\Omega, \xi) = \sup_{\|\xi\|=r} C(\Omega, f_\Omega^N + B_\Omega\xi). \tag{2.5}$$

3. Expression of the robust criteria

In all this section the domain Ω is fixed. Hence, for notational simplicity, we drop the subscript Ω, denoting Q_Ω, b_Ω, c_Ω by Q, b, c.

Actually, the framework of this section does not require that Q, b, c be necessarily defined by (2.1). It is sufficient to assume that $Q : \mathcal{H} \to \mathcal{H}$ is a compact, self-adjoint operator with $Q \geq 0$, $Q \neq 0$, $b \in \mathcal{H}$, and $\alpha, r, c \in \mathbb{R}$ satisfy $c < \alpha$ and $r > 0$.

We shall give practical procedures to compute the values of $j_{sr}(\Omega)$ and $j_{wc}(\Omega)$.

The two optimization problems appearing in (2.4) and (2.5) can be formulated in the form

$$\text{Minimize } q_1(\xi), \quad \text{subject to } q_2(\xi) = 0, \quad \xi \in \mathcal{H}, \tag{$\mathscr{P}$}$$

with two quadratic functionals q_1 and q_2 written as

$$q_1(\xi) = \frac{1}{2}\langle Q_1\xi, \xi\rangle + \langle b_1, \xi\rangle + c_1, \quad q_2(\xi) = \frac{1}{2}\langle Q_2\xi, \xi\rangle + \langle b_2, \xi\rangle + c_2.$$

In all this section, these quantities are defined as follows, where I is the identity of $\mathcal{H}$.

(1) Compliance robustness :

$$q_1(\xi) = \frac{1}{2}\|\xi\|^2, \quad q_2(\xi) = q(\Omega, \xi) - \alpha, \tag{3.1}$$

i.e.,

$$\begin{aligned} &Q_1 = I, && b_1 = 0, && c_1 = 0,\\ &Q_2 = Q, && b_2 = b, && c_2 = c - \alpha. \end{aligned}$$

(2) Worst case robust compliance :

$$q_1(\xi) = -q(\Omega, \xi), \quad q_2(\xi) = \frac{1}{2}\|\xi\|^2 - \frac{1}{2}r^2, \tag{3.2}$$

i.e.,

$$\begin{aligned} &Q_1 = -Q, && b_1 = -b, && c_1 = -c,\\ &Q_2 = I, && b_2 = 0, && c_2 = -r^2/2. \end{aligned}$$

Problems of form ($\mathscr{P}$) are known in the literature as trust region problems (or subproblems). They have been extensively studied, but almost always in the case of a finite dimensional space $\mathcal{H}$, see *e.g.* [16–18,22,29,32,35–37].

3.1. Existence of critical loads

Theorem 3.1. *Problem ($\mathscr{P}$) admits at least a solution*

Proof. Consider first the case (3.1). In this case, Q_2 is compact with $Q_2 = Q \geq 0$, $Q_2 \neq 0$ and $c_2 = c - \alpha < 0$. Hence $q_2(0) < 0$ and there exists ξ_0 such that $q_2(\xi_0) = 0$, so $[q_2 = 0] \neq \emptyset$. Let (ξ_n) be a minimizing sequence. For any $\beta > 0$, $[q_1 \leq \beta]$ is a closed ball. Thus, up to a subsequence, there exists $\xi \in \mathcal{H}$ such that $\xi_n \rightharpoonup \xi$. By compactness of Q_2, we have $q_2(\xi) = \lim q_2(\xi_n) = 0$. Since q_1 is convex and continuous, it is weakly lower-semicontinuous, thus $q_1(\xi) \leq \liminf q_1(\xi_n) = \inf_{[q_2=0]} q_1$.

Let us turn to the second case (3.2). Recall that, in this case, Q_1 is compact, negative semi-definite. Let (ξ_n) be a minimizing sequence of the problem $\inf_{[q_2\leq 0]} q_1$. Since $[q_2 \leq 0]$ is a closed ball, up to a subsequence, there exists $\xi \in \mathcal{H}$ such that $\xi_n \rightharpoonup \xi$. By compactness of Q_1, we have $q_1(\xi) = \lim q_1(\xi_n) = \inf_{[q_2\leq 0]} q_1$. As q_2 is convex and continuous, it is weakly lower-semicontinuous, thus $q_2(\xi) \leq \liminf q_2(\xi_n) \leq 0$. Therefore, the infimum $\inf_{[q_2\leq 0]} q_1$ is attained at some points of $[q_2 \leq 0]$. Finally, as q_1 is concave and $[q_2 \leq 0]$ is convex, at least one of these points can be found in $[q_2 = 0]$. □

Remark 3.2. From the proof it appears that Problem ($\mathscr{P}$) still admits solutions if (Q, b, c), with $Q : \mathcal{H} \to \mathcal{H}$ self-adjoint, compact, is sufficiently close to $(\bar{Q}, \bar{b}, \bar{c})$ with $\bar{Q} : H \to H$ self-adjoint, compact, $\bar{Q} \geq 0$, $\bar{Q} \neq 0$ and $\bar{c} < \alpha$.

3.2. Dual formulation: general framework

For all $(\xi, \mu) \in \mathcal{H} \times \mathbb{R}$ we associate to Problem ($\mathscr{P}$) the Lagrangian

$$L(\xi, \mu) = q_1(\xi) + \mu q_2(\xi) = \frac{1}{2}\langle (Q_1 + \mu Q_2)\xi, \xi\rangle + \langle b_1 + \mu b_2, \xi\rangle + c_1 + \mu c_2.$$

The dual criterion is

$$g(\mu) = \inf_{\xi \in \mathcal{H}} L(\xi, \mu).$$

For any self-adjoint linear continuous operator $T : \mathcal{H} \to \mathcal{H}$, we have

$$\mathcal{H} = \ker T \overset{\perp}{\oplus} \mathrm{cl}(\mathrm{im}\ T),$$

hence the restriction $T|_{\mathrm{cl}(\mathrm{im}\ T)} : \mathrm{cl}(\mathrm{im}\ T) \to \mathrm{im}\ T$ is a bijection. We denote by $T^\dagger := T|^{-1}_{\mathrm{cl}(\mathrm{im}\ T)} : \mathrm{im}\ T \to \mathrm{cl}(\mathrm{im}\ T)$ the inverse operator, which, by virtue of the open mapping theorem, is continuous as soon as $\mathrm{im}\ T$ is closed. For all $\mu \in \mathbb{R}$ such that $b_1 + \mu b_2 \in \mathrm{im}\,(Q_1 + \mu Q_2)$ we set

$$\psi(\mu) = -\frac{1}{2}\langle (Q_1 + \mu Q_2)^\dagger (b_1 + \mu b_2), b_1 + \mu b_2\rangle + c_1 + \mu c_2.$$

Lemma 3.3. *The dual criterion is expressed by*

$$g(\mu) = \begin{cases} -\infty & \text{if } Q_1 + \mu Q_2 \not\geq 0, \\ -\infty & \text{if } Q_1 + \mu Q_2 \geq 0 \text{ and } b_1 + \mu b_2 \notin \mathrm{im}\,(Q_1 + \mu Q_2), \\ \psi(\mu) & \text{if } Q_1 + \mu Q_2 \geq 0 \text{ and } b_1 + \mu b_2 \in \mathrm{im}\,(Q_1 + \mu Q_2). \end{cases}$$

Proof. We first note that the condition

$$\forall \mu \in \mathbb{R}, \qquad Q_1 + \mu Q_2 \geq 0 \Longrightarrow \mathrm{im}\,(Q_1 + \mu Q_2) \text{ is closed} \tag{3.3}$$

is fulfilled in the two cases under study. Indeed, in the first case, we have $Q_1 + \mu Q_2 = I + \mu Q$, whose image is always closed since Q is compact. In the second case we have $Q_1 + \mu Q_2 = -Q + \mu I$, and $Q_1 + \mu Q_2 \geq 0$ implies $\mu > 0$, whereby we conclude as before. For simplicity, we set $Q_\mu = Q_1 + \mu Q_2$, $b_\mu = b_1 + \mu b_2$, $c_\mu = c_1 + \mu c_2$, so that

$$g(\mu) = \inf_{\xi \in \mathcal{H}} q_\mu(\xi) := \frac{1}{2}\langle Q_\mu \xi, \xi\rangle + \langle b_\mu, \xi\rangle + c_\mu.$$

If $Q_\mu \not\geq 0$, it is clear that $g(\mu) = -\infty$. Therefore we assume now that $Q_\mu \geq 0$. By (3.3), $\mathrm{im}\ Q_\mu$ is closed, thus

$$\mathcal{H} = \ker Q_\mu \overset{\perp}{\oplus} \mathrm{im}\ Q_\mu.$$

For all $\xi \in \mathcal{H}$, we make the decomposition $\xi = \xi_1 + \xi_2$, with $\xi_1 \in \ker Q_\mu$ and $\xi_2 \in \mathrm{im}\ Q_\mu$. We get

$$q_\mu(\xi) = q_\mu(\xi_2) + \langle b_\mu, \xi_1\rangle.$$

Two cases can arise.

(1) If $b_\mu \notin \mathrm{im}\ Q_\mu$, choosing $\xi = t\tilde{b}_\mu$ with $t \in \mathbb{R}$ and $\tilde{b}_\mu$ the orthogonal projection of b_μ onto $\ker Q_\mu$, we obtain

$$q_\mu(\xi) = t\|b_\mu\|^2.$$

Letting t go to $-\infty$ yields $g(\mu) = -\infty$.

(2) If $b_\mu \in \operatorname{im} Q_\mu$, we have $q_\mu(\xi) = q_\mu(\xi_2)$ for all $\xi \in \mathcal{H}$, and

$$g(\mu) = \inf_{\xi_2 \in \operatorname{im} Q_\mu} q_\mu(\xi_2) := \frac{1}{2}\langle Q_\mu \xi_2, \xi_2\rangle + \langle b_\mu, \xi_2\rangle + c_\mu.$$

The unique minimizer of this quadratic problem is $\xi_2 = -Q_\mu^\dagger b_\mu$, and the value of the minimum is $-\frac{1}{2}\langle Q_\mu^\dagger b_\mu, b_\mu\rangle + c_\mu$, *i.e.*, $\psi(\mu)$. □

The dual problem is

$$\text{Maximize } \psi(\mu) \quad \text{subject to } \begin{cases} Q_1 + \mu Q_2 \geq 0, \\ b_1 + \mu b_2 \in \operatorname{im}(Q_1 + \mu Q_2), \end{cases} \quad \mu \in \mathbb{R}. \tag{$\mathscr{D}$}$$

The following result is an adaptation of (Thm. 2.1 of [37]). Due to its importance in the sequel, we nevertheless give a proof.

Theorem 3.4. *Let $\bar\xi \in \mathcal{H}$. The following statements are equivalent:*

(1) *$\bar\xi$ is a (global) minimizer of ($\mathscr{P}$);*
(2) *$q_2(\bar\xi) = 0$ and there exists $\bar\mu \in \mathbb{R}$ such that*

$$\partial_\xi L(\bar\xi, \bar\mu) = (Q_1 + \bar\mu Q_2)\bar\xi + (b_1 + \bar\mu b_2) = 0, \tag{3.4}$$
$$\partial^2_{\xi\xi} L(\bar\xi, \bar\mu) = Q_1 + \bar\mu Q_2 \geq 0; \tag{3.5}$$

(3) *there exists $\bar\mu \in \mathbb{R}$ such that*

$$\bar\xi \in \operatorname{argmin} L(., \bar\mu) \cap [q_2 = 0]. \tag{3.6}$$

Proof. We shall prove the implications (1) ⇒ (2) ⇒ (3) ⇒ (1).
First step. Let us assume that $\bar\xi$ is a minimizer of ($\mathscr{P}$). As the constraint is scalar, the classical constraint qualification reduces to

$$\nabla q_2(\bar\xi) = Q_2\bar\xi + b_2 \neq 0. \tag{3.7}$$

In case (3.1), $Q_2\bar\xi + b_2 = Q\bar\xi + b \neq 0$ since otherwise the constraint $q_2(\xi) = 0$ would yield $\langle Q\bar\xi, \bar\xi\rangle = 2(c-\alpha) < 0$. In case (3.2), $Q_2\bar\xi + b_2 = \bar\xi \neq 0$ since $\|\bar\xi\| = r > 0$.

Therefore, by the first order necessary optimality condition (see *e.g.*, [27] 1.1.1, Thm. 1), there exists $\bar\mu \in \mathbb{R}$ such that

$$\partial_\xi L(\bar\xi, \bar\mu) = 0.$$

The second order necessary optimality condition reads (see *e.g.*, [27] 7.2.1, Cor. of Thm. 1)

$$\langle \partial^2_{\xi\xi} L(\bar\xi, \bar\mu)\zeta, \zeta\rangle \geq 0 \qquad \forall \zeta \in T_{\bar\xi}, \tag{3.8}$$
$$T_{\bar\xi} = \{\zeta \in \mathcal{H}, \langle Q_2\bar\xi + b_2, \zeta\rangle = 0\}.$$

Now, suppose that $\zeta \notin T_{\bar\xi}$. We assume first that $Q_2\zeta \neq 0$. Since $Q_2 \geq 0$, this entails $\langle Q_2\zeta, \zeta\rangle > 0$. Set

$$\bar t = -2\frac{\langle Q_2\bar\xi + b_2, \zeta\rangle}{\langle Q_2\zeta, \zeta\rangle} \neq 0, \qquad \xi = \bar\xi + \bar t\zeta.$$

After calculation we find that $q_2(\xi) = q_2(\bar\xi) = 0$. This implies

$$L(\xi, \bar\mu) - L(\bar\xi, \bar\mu) = q_1(\xi) - q_1(\bar\xi) \geq 0. \tag{3.9}$$

Yet we have

$$L(\xi, \bar\mu) - L(\bar\xi, \bar\mu) = \bar t\langle (Q_1 + \bar\mu Q_2)\bar\xi + b_1 + \bar\mu b_2, \zeta\rangle + \frac{\bar t^2}{2}\langle (Q_1 + \bar\mu Q_2)\zeta, \zeta\rangle. \tag{3.10}$$

Combining (3.4), (3.9), (3.10) and $\bar{t} \neq 0$, we derive that $\langle (Q_1 + \bar{\mu} Q_2)\zeta, \zeta\rangle \geq 0$. Assume now that $Q_2\zeta = 0$. As $Q_2 \neq 0$, we choose some $\xi_0 \in \mathcal{H}$ such that $Q_2\xi_0 \neq 0$. Let (t_n) be an arbitrary sequence of positive real numbers such that $t_n \to 0$ as $n \to \infty$. We set $\zeta_n = \zeta + t_n\xi_0$. For all n we have $Q_2\zeta_n = t_n Q_2\xi_0 \neq 0$, hence $\langle (Q_1 + \bar{\mu} Q_2)\zeta_n, \zeta_n\rangle \geq 0$. Passing to the limit yields $\langle (Q_1 + \bar{\mu} Q_2)\zeta, \zeta\rangle \geq 0$. We have proved that $Q_1 + \bar{\mu} Q_2 \geq 0$. In other words, the second order optimality condition (3.8) holds for any ζ, which is a typical fact in quadratic programming.

Second step. Since the function $\xi \mapsto L(\xi, \bar{\mu})$ is quadratic, a Taylor expansion immediately shows that the conditions (3.4) and (3.5) imply $\bar{\xi} \in \operatorname{argmin} L(., \bar{\mu})$.

Third step. We remark that, for all $\mu \in \mathbb{R}$, it holds

$$[L(., \mu) = q_1 + \delta_{[q_2=0]}] = [q_2 = 0], \tag{3.11}$$

with δ_C the indicator function of the set C, *i.e.*, $\delta_C(x) = 0$ if $x \in C$ and $\delta_C(x) = +\infty$ if $x \notin C$. By (A.2) in the appendix, we have $\bar{\xi} \in \operatorname{argmin} L(., \bar{\mu}) \cap [q_2 = 0] \subset \operatorname{argmin} q_1 + \delta_{[q_2=0]}$. □

Remark 3.5. One can also deduce the implication (1) ⇒ (3) of Theorem 3.4 from the following version of the S-lemma (see [19], for the original version in Russian).

Theorem 3.6 (S-lemma, [20], Thm. 2.1)**.** *Let $s_i : \mathcal{H} \to \mathbb{R}$, $i = 1, 2$, be two quadratic functionals of the form $s_i : \xi \to \frac{1}{2}\langle S_i\xi, \xi\rangle + \langle d_i, \xi\rangle + e_i$, $S_i : \mathcal{H} \to \mathcal{H}$ continuous and self-adjoint, $d_i \in \mathcal{H}$ and $e_i \in \mathbb{R}$. Suppose that s_2 is nonlinear, that there exist $\xi_+, \xi_- \in \mathcal{H}$ such that $s_2(\xi_+) > 0$, $s_2(\xi_-) < 0$ and*

$$\forall \xi \in \mathcal{H}, \quad s_2(\xi) = 0 \Longrightarrow s_1(\xi) \geq 0.$$

Then there exists $\lambda \in \mathbb{R}$ such that

$$\forall \xi \in \mathcal{H}, \quad s_1(\xi) - \lambda s_2(\xi) \geq 0.$$

In fact, one sets $s_1 = q_1 - q_1(\bar{\xi})$, $s_2 = q_2$, s_2 is nonlinear because $Q_2 \neq 0$, and there exist $\xi_+, \xi_- \in \mathcal{H}$ such that $s_2(\xi_+) > 0$, $s_2(\xi_-) < 0$ due to (3.7) and $q_2(\bar{\xi}) = 0$. Then (1) ⇒ (3) of Theorem 3.4 results from the S-lemma 3.6.

The following Theorem is a consequence of Theorem 3.4 together with general results on duality (*cf.* Appendix A).

Theorem 3.7. *The primal problem ($\mathscr{P}$) and the dual problem ($\mathscr{D}$) have the same optimal values. In addition, ($\mathscr{D}$) admits solutions, and, if $\bar{\mu}$ is one of these solutions, we have*

$$\operatorname{argmin}(\mathscr{P}) = \operatorname{argmin} L(., \bar{\mu}) \cap [q_2 = 0].$$

Proof. In view of Theorem 3.1, Theorem 3.4 and (3.11), one uses successively (A.5), (A.3) and (A.6). □

3.3. Expression of the critical loads for the compliance robustness

In this case, with the notations (2.1), the primal problem ($\mathscr{P}$) and the dual problem ($\mathscr{D}$) read, respectively,

$$\text{Minimize } \frac{1}{2}\|\xi\|^2 \quad \text{subject to } q(\xi) := \frac{1}{2}\langle Q\xi, \xi\rangle + \langle b, \xi\rangle + c = \alpha, \quad \xi \in \mathcal{H}, \tag{3.12}$$

$$\text{Maximize } \psi_{sr}(\mu) = -\frac{1}{2}\langle (I + \mu Q)^{\dagger}(\mu b), \mu b\rangle + \mu(c - \alpha) \text{ subject to } \begin{cases} I + \mu Q \geq 0, \\ \mu b \in \operatorname{im}(I + \mu Q), \end{cases} \quad \mu \in \mathbb{R}. \tag{3.13}$$

We denote by $\lambda_{\max}$ the largest eigenvalue of Q. The following theorem refines Theorem 3.7 and uses the same terminology as [37].

Theorem 3.8. *The primal problem (3.12) and the dual problem (3.13) have the same optimal values. The dual problem (3.13) admits a unique solution $\bar{\mu} \in \mathbb{R}$, which can be computed in the following way.*

- *Easy case: $b \notin \ker(Q - \lambda_{\max} I)^{\perp}$. Then $\bar{\mu}$ is the unique solution in $]-1/\lambda_{\max}, 0[$ of the equation*
$$q(-\mu(I + \mu Q)^{-1} b) = \alpha. \tag{3.14}$$
- *Hard case I: $b \in \ker(Q - \lambda_{\max} I)^{\perp}$ and*
$$q((\lambda_{\max} I - Q)^{\dagger} b) > \alpha. \tag{3.15}$$
Then $\bar{\mu}$ is also the unique solution in $]-1/\lambda_{\max}, 0[$ of (3.14).
- *Hard case II: $b \in \ker(Q - \lambda_{\max} I)^{\perp}$ and $q((\lambda_{\max} I - Q)^{\dagger} b) \leq \alpha$. Then $\bar{\mu} = -1/\lambda_{\max}$.*

The set Ξ of solutions of the primal problem (3.12) is given by the following expressions.

- *Easy case and Hard case I. There is a unique critical load given by*
$$\Xi = \{-\bar{\mu}(I + \bar{\mu} Q)^{-1} b\}.$$
- *Hard case II. The set of critical loads is*
$$\Xi = \Big[\{(\lambda_{\max} I - Q)^{\dagger} b\} + \ker(Q - \lambda_{\max} I)\Big] \cap [q = \alpha].$$

Proof. Let us reformulate the constraints of the dual problem (3.13). The first one is equivalent to $\mu \geq -1/\lambda_{\max}$, and we have by the classical theory of compact perturbations of the identity $\operatorname{im}(I + \mu Q) = \ker(I + \mu Q)^{\perp} = \mathcal{H}$ as soon as $\mu > -1/\lambda_{\max}$. Therefore we have

$$\left\{ \begin{array}{l} I + \mu Q \geq 0 \\ \mu b \in \operatorname{im}(I + \mu Q) \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{l} \mu > -1/\lambda_{\max} \text{ if } b \notin \ker(Q - \lambda_{\max} I)^{\perp} \\ \mu \geq -1/\lambda_{\max} \text{ if } b \in \ker(Q - \lambda_{\max} I)^{\perp} \end{array} \right\}.$$

If $-1/\lambda_{\max} < \mu$ we have

$$\psi_{sr}(\mu) = -\frac{\mu^2}{2} \langle (I + \mu Q)^{-1} b, b \rangle + \mu(c - \alpha). \tag{3.16}$$

Differentiating entails

$$\psi'_{sr}(\mu) = -\mu \langle (I + \mu Q)^{-1} b, b \rangle + \frac{\mu^2}{2} \langle (I + \mu Q)^{-2} Q b, b \rangle + (c - \alpha) \tag{3.17}$$
$$= q(-\mu(I + \mu Q)^{-1} b) - \alpha. \tag{3.18}$$

Replacing μQ by $(I + \mu Q) - I$ in the second term of the right hand side of (3.17) leads to the alternative expressions

$$\psi'_{sr}(\mu) = -\frac{\mu}{2} \langle \left((I + \mu Q)^{-1} + (I + \mu Q)^{-2} \right) b, b \rangle + (c - \alpha) \tag{3.19}$$
$$= -\frac{\mu}{2} \langle (I + \mu Q)^{-2} (2I + \mu Q) b, b \rangle + (c - \alpha). \tag{3.20}$$

Differentiating another time from (3.19) yields

$$\psi''_{sr}(\mu) = -\frac{1}{2} \langle \left((I + \mu Q)^{-1} + (I + \mu Q)^{-2} \right) b, b \rangle + \frac{\mu}{2} \langle \left((I + \mu Q)^{-2} + 2(I + \mu Q)^{-3} \right) Q b, b \rangle$$
$$= -\langle (I + \mu Q)^{-3} b, b \rangle. \tag{3.21}$$

Let Λ be the set of eigenvalues of Q, $\Lambda^\star := \Lambda \setminus \{0\}$ and, for $\lambda \in \mathbb{R}$, denote by b_λ the orthogonal projection of b on $\ker(Q-\lambda I)$. From (3.20) we get for all $\mu \in]-1/\lambda_{\max}, +\infty[$

$$\begin{aligned}\psi'_{sr}(\mu) &= -\frac{\mu}{2}\sum_{\lambda\in\Lambda}\frac{2+\mu\lambda}{(1+\mu\lambda)^2}\|b_\lambda\|^2 + c - \alpha \\ &= -\mu\|b_0\|^2 + \frac{1}{2}\sum_{\lambda\in\Lambda^\star}\left(\frac{1}{\lambda(1+\mu\lambda)^2} - \frac{1}{\lambda}\right)\|b_\lambda\|^2 + c - \alpha. \end{aligned} \tag{3.22}$$

Note that $\psi'_{sr}(0) = c - \alpha < 0$.

- *Easy case:* $b \notin \ker(Q-\lambda_{\max} I)^\perp$. Then $b_{\lambda_{\max}} \neq 0$ and, due to (3.22), $\lim_{\mu\to(-1/\lambda_{\max})^+}\psi'_{sr}(\mu) = +\infty$. Due to (3.21), it follows that ψ'_{sr} is decreasing and ψ_{sr} admits a unique maximizer $\bar\mu$ on $]-1/\lambda_{\max}, +\infty[$. It is characterized by $\psi'_{sr}(\bar\mu) = 0$ and obviously belongs to $]-1/\lambda_{\max}, 0[$.
- *Hard case:* $b \in \ker(Q-\lambda_{\max} I)^\perp$. Then the expressions (3.16)–(3.18) remain true for $\mu = -1/\lambda_{\max}$, provided that the restriction of $I+\mu Q$ to $\ker(Q-\lambda_{\max} I)^\perp$ is considered. In particular,

$$\psi'_{sr}(-1/\lambda_{\max}) = q\left((\lambda_{\max} I - Q)^\dagger b\right) - \alpha$$

  is finite.
  If $\psi'_{sr}(-1/\lambda_{\max}) > 0$ (hard case I), then $b \neq 0$. Due to (3.21), it follows that ψ'_{sr} is decreasing and ψ_{sr} admits a unique maximizer $\bar\mu$ in $]-1/\lambda_{\max}, +\infty[$, characterized by $\psi'_{sr}(\bar\mu) = 0$ and clearly belonging to $]-1/\lambda_{\max}, 0[$.
  If $\psi'_{sr}(-1/\lambda_{\max}) \leq 0$ (hard case II), in view of (3.21) and $\psi'_{sr}(0) = c - \alpha < 0$, ψ_{sr} is decreasing and $-1/\lambda_{\max}$ is the unique maximizer of ψ_{sr} in $[-1/\lambda_{\max}, +\infty[$.

By Theorem 3.7, the set of solutions of the primal problem is given by

$$\Xi = \operatorname{argmin} L(.,\bar\mu) \cap [q = \alpha].$$

Here the Lagrangian is

$$L(\xi,\bar\mu) = \frac{1}{2}\langle (I+\bar\mu Q)\xi, \xi\rangle + \bar\mu\langle b, \xi\rangle + \bar\mu(c-\alpha).$$

In the easy case and the hard case I, $L(.,\bar\mu)$ is strictly convex since $\bar\mu > -1/\lambda_{\max}$. It admits as unique minimizer

$$\bar\xi = -\bar\mu(I+\bar\mu Q)^{-1} b.$$

From (3.18) and $\psi'_{sr}(\bar\mu) = 0$ we derive that $q(\bar\xi) = \frac{1}{2}\langle Q\bar\xi, \bar\xi\rangle + \langle b, \bar\xi\rangle + c = \alpha$.

In the hard case II, we have $\bar\mu = -1/\lambda_{\max}$ and

$$\operatorname{argmin} L(.,\bar\mu) = \{\xi \in \mathcal{H}, (Q-\lambda_{\max} I)\xi = -b\}.$$

Using that $b \in \ker(Q-\lambda_{\max} I)^\perp = \operatorname{im}(Q-\lambda_{\max} I)$ we obtain

$$\operatorname{argmin} L(.,\bar\mu) = \{(\lambda_{\max} I - Q)^\dagger b\} + \ker(Q-\lambda_{\max} I).$$

Observe that $q(-(Q-\lambda_{\max} I)^\dagger b) \leq \alpha$ confirms that $\Xi \neq \emptyset$. □

Remark 3.9. If $f_\Omega^N \in \operatorname{im} B_\Omega$, then the expressions (3.14) and (3.15) can be conveniently rewritten. Indeed, writing $f_\Omega^N = B_\Omega \xi^N$ yields $b = Q\xi^N$ as well as $c = \frac{1}{2}\langle Q\xi^N, \xi^N\rangle$. Plugging these expressions into (3.14) entails after simplification

$$\psi'_{sr}(\mu) = q\left((I+\mu Q)^{-1}\xi^N - \xi^N\right) - \alpha = C\left(\Omega, B_\Omega(I+\mu Q)^{-1}\xi^N\right) - \alpha.$$

To conclude this section, we present an interesting connection between the dual problem (3.13) and a semidefinite programming problem inspired by [10].

We will need the following version of the S-lemma in Hilbert spaces . It relies on slightly different assumptions from Theorem 3.6, but its statement is contained in the same reference (see again [19] for the original version in Russian).

Theorem 3.10 (S-lemma [20], Thm. 2.1)**.** *Let $s_i : \mathcal{H} \to \mathbb{R}$, $i = 1, 2$, be two quadratic functionals of the form $s_i : \xi \to \frac{1}{2}\langle S_i\xi, \xi\rangle + \langle d_i, \xi\rangle + e_i$, $S_i : \mathcal{H} \to \mathcal{H}$ continuous and self-adjoint, $d_i \in \mathcal{H}$ and $e_i \in \mathbb{R}$, $i = 1, 2$. Suppose that there exists $\xi_+ \in \mathcal{H}$ such that $s_2(\xi_+) > 0$ and*

$$\forall \xi \in \mathcal{H}, \quad s_2(\xi) \geq 0 \Longrightarrow s_1(\xi) \geq 0.$$

Then there exists $\lambda \geq 0$ such that

$$\forall \xi \in \mathcal{H}, \quad s_1(\xi) - \lambda s_2(\xi) \geq 0.$$

Moreover, if there exists ξ_- such that $s_1(\xi_-) < 0$, one can suppose that $\lambda > 0$.

Let us consider the following problem:

$$\text{Maximize } \rho \quad \text{subject to } \rho\mathbb{B} \subset [q \leq \alpha], \quad \rho \in \mathbb{R}_+^\star, \tag{3.23}$$

whose (unique) solution is $\rho = \text{dist}(0, [q = \alpha]) = \text{dist}(0, [q > \alpha]) \in \mathbb{R}_+^\star$ since $c = q(0) < \alpha$ and $Q \geq 0$, $Q \neq 0$ so there exists ξ_0 such that $q(\xi_0) \geq \alpha$. Yet we have for all $\rho > 0$

$$\begin{aligned}
\rho\mathbb{B} \subset [q \leq \alpha] &\iff \forall \xi \in \mathcal{H},\ \|\xi\| \leq \rho \Rightarrow q(\xi) \leq \alpha, \\
&\iff \forall \xi \in \mathcal{H},\ \|\xi\| \leq 1 \Rightarrow q(\rho\xi) \leq \alpha, \\
&\iff \forall \xi \in \mathcal{H},\ 1 - \|\xi\|^2 \geq 0 \Rightarrow \alpha - q(\rho\xi) \geq 0.
\end{aligned}$$

As $1 - \|\xi\|^2 = 1 > 0$ for $\xi = 0$ and $\alpha - q(\rho\xi) < 0$ for some ξ, using Theorem 3.10, one obtains

$$\rho\mathbb{B} \subset [q \leq \alpha] \Longleftrightarrow \exists \lambda > 0, \forall \xi \in \mathcal{H},\ \alpha - q(\rho\xi) - \lambda[1 - \|\xi\|^2] \geq 0.$$

But $q(\xi)$ can be expressed as

$$q(\xi) = \max_{v \in \mathcal{V}_\Omega} -\frac{1}{2}\langle A_\Omega v, v\rangle + \langle f_\Omega^N + B_\Omega\xi, v\rangle,$$

whereby

$$\alpha - q(\rho\xi) - \lambda[1 - \|\xi\|^2] \geq 0 \Longleftrightarrow \forall v \in \mathcal{V}_\Omega,\ \frac{1}{2}\langle A_\Omega v, v\rangle + \lambda\|\xi\|^2 - \rho\langle B_\Omega\xi, v\rangle - \langle f_\Omega^N, v\rangle + \alpha - \lambda \geq 0.$$

Changing (ξ, v) into $(s^{-1}\xi, s^{-1}v)$ for any $s \neq 0$, one also has

$$\begin{aligned}
\forall \xi \in \mathcal{H},\ \alpha - q(\rho\xi) - \lambda[1 - \|\xi\|^2] \geq 0 &\Longleftrightarrow \forall (v, \xi, s) \in \mathcal{V}_\Omega \times \mathcal{H} \times \mathbb{R}, \\
&\quad \frac{1}{2}\langle A_\Omega v, v\rangle + \lambda\|\xi\|^2 - \rho\langle B_\Omega\xi, v\rangle - s\langle f_\Omega^N, v\rangle + (\alpha - \lambda)s^2 \geq 0, \\
&\Longleftrightarrow \begin{bmatrix} A_\Omega & \rho B_\Omega & (f_\Omega^N)^* \\ \rho B_\Omega^* & 2\lambda I & 0 \\ f_\Omega^N & 0 & 2(\alpha - \lambda) \end{bmatrix} \geq 0,
\end{aligned}$$

where $(f_\Omega^N)^* : \mathbb{R} \to \mathcal{V}_\Omega'$ is defined as $s \to sf_\Omega^N$. Finally

$$\rho\mathbb{B} \subset [q \leq \alpha] \Longleftrightarrow \exists \lambda > 0, \begin{bmatrix} A_\Omega & \rho B_\Omega & (f_\Omega^N)^* \\ \rho B_\Omega^* & 2\lambda I & 0 \\ f_\Omega^N & 0 & 2(\alpha - \lambda) \end{bmatrix} \geq 0. \tag{3.24}$$

Therefore, $\bar{\rho} > 0$ is the solution of (3.23) if and only if there exists $\bar{\lambda} > 0$ such that $(\bar{\rho}, \bar{\lambda})$ is solution of:

$$\text{Maximize } \rho \quad \text{subject to } \begin{bmatrix} A_\Omega & \rho B_\Omega & (f_\Omega^N)^* \\ \rho B_\Omega^* & \lambda I & 0 \\ f_\Omega^N & 0 & 2\alpha - \lambda \end{bmatrix} \geq 0, \quad \rho, \lambda \in \mathbb{R}_+^\star. \tag{3.25}$$

Note that the above matrix is affine in (ρ, λ) and that, as Problem (3.23) admits a solution, it is also the case for Problem (3.25).

Proposition 3.11. *Problem (3.25) admits a unique solution $(\bar{\rho}, \bar{\lambda})$. This solution is such that $\bar{\rho} = \operatorname{dist}(0, [q > \alpha])$ and $-\bar{\rho}^2/\bar{\lambda}$ is the solution of the dual problem (3.13).*

Proof. Let $(\bar{\rho}, \bar{\lambda})$ be a solution of (3.25). The constraint implies

$$\forall \xi \in \mathcal{H}, \quad \alpha - q(\bar{\rho}\xi) - \frac{\bar{\lambda}}{2}[1 - \|\xi\|^2] \geq 0. \tag{3.26}$$

Let $\bar{\xi}$ be a solution of the primal problem ($\mathscr{P}$). Substituting ξ for $\bar{\rho}\xi$ in (3.26) and multiplying by $\bar{\rho}^2/\bar{\lambda}$ we arrive at

$$\forall \xi \in \mathcal{H}, \quad \frac{1}{2}\|\xi\|^2 - \frac{\bar{\rho}^2}{\bar{\lambda}}[q(\xi) - \alpha] \geq \frac{1}{2}\bar{\rho}^2 = \frac{1}{2}\|\bar{\xi}\|^2.$$

As $q(\bar{\xi}) = \alpha$ we can write

$$\forall \xi \in \mathcal{H}, \quad \frac{1}{2}\|\xi\|^2 - \frac{\bar{\rho}^2}{\bar{\lambda}}[q(\xi) - \alpha] \geq \frac{1}{2}\|\bar{\xi}^2\| - \frac{\bar{\rho}^2}{\bar{\lambda}}[q(\bar{\xi}) - \alpha].$$

It follows that

$$\bar{\xi} \in \operatorname{argmin} L(., -\bar{\rho}^2/\bar{\lambda}) \cap [L(., -\bar{\rho}^2/\bar{\lambda}) = \frac{1}{2}\|.\|^2]$$

thus, by virtue of (A.7), $-\bar{\rho}^2/\bar{\lambda}$ is the solution of the dual problem (3.13). The uniqueness of the solution of the dual problem and the implication just proved imply the uniqueness of the solution of problem (3.25). □

3.4. Expression of the critical loads for the robust compliance in the worst case sense

Here, using again the notation (2.1), the primal problem ($\mathscr{P}$) and the dual problem ($\mathscr{D}$) are equivalent, respectively, to:

$$\text{Minimize } -q(\xi) = -\frac{1}{2}\langle Q\xi, \xi\rangle - \langle b, \xi\rangle - c \quad \text{subject to } \|\xi\| = r, \quad \xi \in \mathcal{H}, \tag{3.27}$$

$$\text{Maximize } \frac{1}{2}\langle (Q - \mu I)^\dagger b, b\rangle - c - \mu\frac{r^2}{2} \quad \text{subject to } \begin{cases} -Q + \mu I \geq 0, \\ b \in \operatorname{im}(Q - \mu I), \end{cases} \quad \mu \in \mathbb{R}. \tag{3.28}$$

The largest eigenvalue of Q is still denoted by $\lambda_{\max}$. Applying Theorem 3.7 similarly to Theorem 3.8 provides the following result.

Theorem 3.12. *The primal problem (3.27) and the dual problem (3.28) have the same optimal values. The dual problem (3.28) admits a unique solution $\bar{\mu} \in \mathbb{R}$, which can be computed in the following way.*

- *Easy case: $b \notin \ker(Q - \lambda_{\max} I)^\perp$. Then $\bar{\mu}$ is the unique solution in $]\lambda_{\max}, +\infty[$ of the equation*

$$\|(-Q + \mu I)^{-1} b\| = r. \tag{3.29}$$

- *Hard case I: $b \in \ker(Q - \lambda_{\max} I)^\perp$ and $\|(-Q + \lambda_{\max} I)^\dagger b\| > r$. Then $\bar{\mu}$ is also the unique solution in $]\lambda_{\max}, +\infty[$ of (3.29).*

- *Hard case II:* $b \in \ker(Q - \lambda_{\max} I)^{\perp}$ *and* $\|(-Q + \lambda_{\max} I)^{\dagger} b\| \leq r$. *Then* $\bar{\mu} = \lambda_{\max}$.

The set Ξ *of solutions of the primal problem* (3.27) *is given by the following expressions.*

- *Easy case and Hard case I. There is a unique critical load given by*

$$\Xi = \{(-Q + \bar{\mu} I)^{-1} b\}.$$

- *Hard case II. The set of critical loads is*

$$\Xi = \left[\{(-Q + \lambda_{\max} I)^{\dagger} b\} + \ker(Q - \lambda_{\max} I)\right] \cap [\,\|\,.\,\| = r].$$

As in the preceding subsection, we present a connection between the dual problem (3.28) and a semidefinite programming problem. Let us consider the following problem:

$$\text{Minimize } \alpha \quad \text{subject to } r\mathbb{B} \subset [q \leq \alpha], \quad \alpha \in \mathbb{R}, \tag{3.30}$$

where $r > 0$ and whose (unique) solution is $\alpha = \sup_{\|\xi\| \leq r} q(\xi)$.

Therefore, using (3.24), $\bar{\alpha}$ is the solution of (3.30) if and only if there exists $\bar{\lambda} > 0$ such that $(\bar{\alpha}, \bar{\lambda})$ is solution of:

$$\text{Minimize } \alpha \quad \text{subject to } \begin{bmatrix} A_\Omega & rB_\Omega & (f_\Omega^N)^* \\ rB_\Omega^* & \lambda I & 0 \\ f_\Omega^N & 0 & 2\alpha - \lambda \end{bmatrix} \geq 0, \quad \alpha \in \mathbb{R}, \lambda \in \mathbb{R}_+^\star. \tag{3.31}$$

Note that the above matrix is affine in (α, λ) and that, as problem (3.30) admits a solution, the same holds for problem (3.31). Similarly to Proposition 3.11 we obtain the following.

Proposition 3.13. *Problem* (3.31) *admits a unique solution* $(\bar{\alpha}, \bar{\lambda})$. *This solution is such that* $\bar{\alpha} = \sup_{\|\xi\| \leq r} q(\xi)$ *and* $\bar{\lambda}/r^2$ *is the solution of the dual problem* (3.28).

4. Hadamard semiderivative of the robust criteria

The functionals $j_{sr}(\Omega)$ and $j_{wc}(\Omega)$ can be written as

$$j_{sr}(\Omega) = J_{sr}(w_\Omega), \quad j_{wc}(\Omega) = J_{wc}(w_\Omega), \tag{4.1}$$

with $w_\Omega = (Q_\Omega, b_\Omega, c_\Omega)$ defined by (2.1), $w_\Omega \in W := \mathcal{K}_s(\mathcal{H}) \times \mathcal{H} \times \mathbb{R}$, where $\mathcal{K}_s(\mathcal{H})$ stands for the space of self-adjoint compact linear operators from $\mathcal{H}$ into itself. Moreover $J_{sr}, J_{wc} : W \to \mathbb{R} \cup \{+\infty\}$ are defined, for $w \in W$, by

$$J_{sr}(w) = \inf_{q(w,\xi) = \alpha} \frac{1}{2}\|\xi\|^2, \tag{4.2}$$

and

$$J_{wc}(w) = \sup_{\|\xi\| = r} q(w, \xi) \tag{4.3}$$

with, for $w = (Q, b, c) \in W$,

$$q(w, \xi) = \frac{1}{2}\langle Q\xi, \xi\rangle + \langle b, \xi\rangle + c.$$

In view of (4.1), calculating the shape derivative or the topological derivative of j_{sr} or j_{wc} requires to calculate the Hadamard semiderivative of J_{sr} or J_{wc}. This is the aim of this section. The sensitivity of w with respect to Ω, specifically its topological derivative, will be studied in Section 5.

Our approach, on one hand, adapts the proof of Theorem 4.24 in [12] to a slightly different context, while, on the other hand, it shows that the modified assumptions of this Theorem, in particular the strong assumption (iii), are fulfilled.

In all this section, we consider some $\bar{w} = (\bar{Q}, \bar{b}, \bar{c}) \in W$ with $\bar{Q} : \mathcal{H} \to \mathcal{H}$ a self–ajoint compact operator such that $\bar{Q} \geq 0$, $\bar{Q} \neq 0$ and some $\alpha, r \in \mathbb{R}$ with $\bar{c} < \alpha$, $r > 0$.

4.1. Hadamard semiderivatives

Let X be a Banach space and consider a function $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$. If $z, d \in X$ and $f(z) \in \mathbb{R}$ the upper and lower Hadamard semiderivatives of f at point z in direction d are defined, respectively, as

$$f'_+(z,d) = \limsup_{\substack{t\downarrow 0\\ v\to d}} \frac{f(z+tv)-f(z)}{t}, \quad f'_-(z,d) = \liminf_{\substack{t\downarrow 0\\ v\to d}} \frac{f(z+tv)-f(z)}{t}.$$

If $f'_+(z,d) = f'_-(z,d)$, their common value is called the Hadamard semiderivative of f at point z in direction d, denoted by $f'(z,d)$.

4.2. Hadamard semiderivative of the compliance robustness

Let $w \in W$ and denote by $\Xi(w)$ the minimizing set of (4.2). Recall that the Lagrangian of this problem is

$$L(w,\xi,\mu) = \frac{1}{2}\|\xi\|^2 + \mu(q(w,\xi)-\alpha),$$

and that the dual criterion is

$$g(w,\mu) = \inf_{\xi\in\mathcal{H}} L(w,\xi,\mu).$$

We denote by $\bar\mu$ the unique solution of the dual problem for $\bar w$, *i.e.* the unique element of $\operatorname{argmax} g(\bar w, .)$.

Lemma 4.1. *Let $\bar\xi \in \mathcal{H}$ be such that $q(\bar w, \bar\xi) = \alpha$. There exists a neighborhood $\mathcal{W}$ of $\bar w$ and a function $S_{\bar w,\bar\xi} : \mathcal{W} \to \mathcal{H}$ of class $\mathscr{C}^\infty$ such that $S_{\bar w,\bar\xi}(\bar w) = \bar\xi$ and*

$$q(w, S_{\bar w,\bar\xi}(w)) = \alpha \qquad \forall w \in \mathcal{W}.$$

Proof. Consider the function $F : (w,s) \in W \times \mathbb{R} \mapsto q(w, s\bar\xi) - \alpha$, which is clearly of class $\mathscr{C}^\infty$. We have

$$\partial_s F(\bar w, 1) = \langle \bar Q\bar\xi, \bar\xi\rangle + \langle \bar b, \bar\xi\rangle = \frac{1}{2}\langle \bar Q\bar\xi, \bar\xi\rangle - (\bar c - \alpha),$$

due to $q(\bar w, \bar\xi) = \alpha$. Using that $\bar Q \geq 0$ and $\bar c < \alpha$ we infer $\partial_s F(\bar w, 1) > 0$. The implicit function theorem leads to the result. □

Lemma 4.2. *We have, for any $\bar h \in W$,*

$$(J_{sr})'_+(\bar w, \bar h) \leq \inf_{\bar\xi\in\Xi(\bar w)} \partial_w L(\bar w, \bar\xi, \bar\mu)\bar h.$$

Proof. Choose an arbitrary $\bar\xi \in \Xi(\bar w)$. Let $(t_n, h_n) \in \mathbb{R}^*_+ \times W$ be such that $t_n \to 0$, $h_n \to \bar h$, and

$$(J_{sr})'_+(\bar w, \bar h) = \lim_{n\to+\infty} \frac{J_{sr}(\bar w + t_n h_n) - J_{sr}(\bar w)}{t_n}.$$

We assume that n is large enough so that $w_n := \bar w + t_n h_n \in \mathcal{W}$. Denoting $\xi_n := S_{\bar w,\bar\xi}(w_n)$, we have $q(w_n, \xi_n) = \alpha$, hence (4.2) entails

$$J_{sr}(w_n) \leq \frac{1}{2}\|\xi_n\|^2 = L(w_n, \xi_n, \mu)$$

for any $\mu \in \mathbb{R}$. As $\bar\xi \in \Xi(\bar w)$, (4.2) also yields

$$J_{sr}(\bar w) = \frac{1}{2}\|\bar\xi\|^2 = L(\bar w, \bar\xi, \mu).$$

Therefore we have

$$\frac{J_{sr}(w_n) - J_{sr}(\bar{w})}{t_n} \leq \frac{L(w_n, \xi_n, \mu) - L(\bar{w}, \bar{\xi}, \mu)}{t_n}.$$

For all w in a neighborhood of $\bar{w}$ we set

$$\Phi(w) = L(w, S_{\bar{w},\bar{\xi}}(w), \mu).$$

We have

$$L(w_n, \xi_n, \mu) - L(\bar{w}, \bar{\xi}, \mu) = \Phi(w_n) - \Phi(\bar{w}) = d\Phi(\bar{w})(t_n h_n) + o_{n\to+\infty}(t_n h_n),$$

since Φ is Fréchet differentiable by composition. The chain rule gives, for $v \in \mathcal{H}$,

$$d\Phi(\bar{w})(v) = \partial_w L(\bar{w}, \bar{\xi}, \mu)v + \partial_\xi L(\bar{w}, \bar{\xi}, \mu)(dS_{\bar{w},\bar{\xi}}(\bar{w})v).$$

We can choose $\mu = \bar{\mu}$, for which it holds $\partial_\xi L(\bar{w}, \bar{\xi}, \bar{\mu}) = 0$. We arrive at $d\Phi(\bar{w})(v) = \partial_w L(\bar{w}, \bar{\xi}, \bar{\mu})v$, and

$$\frac{J_{sr}(\bar{w} + t_n h_n) - J_{sr}(\bar{w})}{t_n} \leq \partial_w L(\bar{w}, \bar{\xi}, \bar{\mu})h_n + o(1).$$

Passing to the limit yields

$$(J_{sr})'_+(\bar{w}, \bar{h}) \leq \partial_w L(\bar{w}, \bar{\xi}, \bar{\mu})\bar{h}.$$

This being true for any $\bar{\xi} \in \Xi(\bar{w})$, we arrive at the desired result. □

Lemma 4.3. *For any $\bar{h} \in W$ we have*

$$(J_{sr})'_-(\bar{w}, \bar{h}) \geq \inf_{\bar{\xi}\in\Xi(\bar{w})} \partial_w L(\bar{w}, \bar{\xi}, \bar{\mu})\bar{h}.$$

Proof. Let $(t_n, h_n) \in \mathbb{R}^*_+ \times W$ be such that $t_n \to 0$, $h_n \to \bar{h}$, and

$$(J_{sr})'_-(\bar{w}, \bar{h}) = \lim_{n\to+\infty} \frac{J_{sr}(\bar{w} + t_n h_n) - J_{sr}(\bar{w})}{t_n}.$$

For all n we set $w_n = \bar{w} + t_n h_n$ and choose some $\xi_n \in \Xi(w_n)$ (*cf.* Rem. 3.2).

Step 1. We choose an arbitrary $\xi \in \mathcal{H}$ such that $q(\bar{w}, \xi) = \alpha$. By Lemma 4.1, there exists a neighborhood $\mathcal{W}$ of $\bar{w}$ and a function $S_{\bar{w},\xi} : \mathcal{W} \to \mathcal{H}$ of class $\mathscr{C}^\infty$ such that $S_{\bar{w},\xi}(\bar{w}) = \xi$ and

$$q(w, S_{\bar{w},\xi}(w)) = \alpha \qquad \forall w \in \mathcal{W}.$$

Since $w_n \to \bar{w}$, it holds for n large enough $q(w_n, S_{\bar{w},\xi}(w_n)) = \alpha$, hence

$$\frac{1}{2}\|\xi_n\|^2 \leq \frac{1}{2}\|S_{\bar{w},\xi}(w_n)\|^2. \tag{4.4}$$

This shows that the sequence (ξ_n) is bounded. Therefore there exists $\bar{\xi} \in \mathcal{H}$ such that $\xi_n \rightharpoonup \bar{\xi}$ weakly for some non-relabeled subsequence.

Step 2. We shall show that $\bar{\xi} \in \Xi(\bar{w})$. From $w_n = \bar{w} + t_n h_n$ and $q(w_n, \xi_n) = \alpha$, denoting $h_n = (Q_n, b_n, c_n)$, we obtain

$$q(w_n, \xi_n) = \frac{1}{2}\langle(\bar{Q} + t_n Q_n)\xi_n, \xi_n\rangle + \langle\bar{b} + t_n b_n, \xi_n\rangle + \bar{c} + t_n c_n = \alpha. \tag{4.5}$$

We have $\bar{Q} + t_n Q_n \to \bar{Q}$, $\bar{Q}$ compact, $\bar{b} + t_n b_n \to \bar{b}$, $\bar{c} + t_n c_n \to \bar{c}$ and $\xi_n \rightharpoonup \bar{\xi}$. Hence $(\bar{Q} + t_n Q_n)\xi_n \to \bar{Q}\bar{\xi}$ strongly and $\frac{1}{2}\langle(\bar{Q} + t_n Q_n)\xi_n, \xi_n\rangle \to \frac{1}{2}\langle\bar{Q}\bar{\xi}, \bar{\xi}\rangle$. Passing to the limit in (4.5) yields

$$\frac{1}{2}\langle\bar{Q}\bar{\xi}, \bar{\xi}\rangle + \langle\bar{b}, \bar{\xi}\rangle + \bar{c} = q(\bar{w}, \bar{\xi}) = \alpha.$$

Moreover, since $\xi_n \rightharpoonup \bar{\xi}$ and $S_{\bar{w},\xi}(w_n) \to S_{\bar{w},\xi}(\bar{w}) = \xi$, (4.4) entails

$$\frac{1}{2}\|\bar{\xi}\|^2 \leq \frac{1}{2}\liminf_{n\to+\infty}\|\xi_n\|^2 \leq \frac{1}{2}\liminf_{n\to+\infty}\|S_{\bar{w},\xi}(w_n)\|^2 = \frac{1}{2}\|\xi\|^2.$$

This proves that $\bar{\xi} \in \Xi(\bar{w})$.

Step 3. From Theorem 3.7, we have $J_{sr}(\bar{w}) = g(\bar{w},\bar{\mu}) = \inf_{\xi\in\mathcal{H}} L(\bar{w},\xi,\bar{\mu})$. It follows that $J_{sr}(\bar{w}) \leq L(\bar{w},\xi_n,\bar{\mu})$. As $\xi_n \in \Xi(w_n)$, we have $J_{sr}(w_n) = \frac{1}{2}\|\xi_n\|^2 = L(w_n,\xi_n,\bar{\mu})$. We arrive at

$$\frac{J_{sr}(w_n) - J_{sr}(\bar{w})}{t_n} \geq \frac{L(w_n,\xi_n,\bar{\mu}) - L(\bar{w},\xi_n,\bar{\mu})}{t_n}.$$

Yet we have

$$\frac{L(w_n,\xi_n,\bar{\mu}) - L(\bar{w},\xi_n,\bar{\mu})}{t_n} = \bar{\mu}\left[\frac{1}{2}\langle Q_n\xi_n,\xi_n\rangle + \langle b_n,\xi_n\rangle + c_n\right].$$

Denoting $\bar{h} = (Q,b,c)$ and using that $Q_n \to Q$ compact, $b_n \to b$, $c_n \to c$ and $\xi_n \rightharpoonup \bar{\xi}$ weakly, one obtains

$$\frac{L(w_n,\xi_n,\bar{\mu}) - L(\bar{w},\xi_n,\bar{\mu})}{t_n} \longrightarrow \bar{\mu}\left[\frac{1}{2}\langle Q\bar{\xi},\bar{\xi}\rangle + \langle b,\xi\rangle + c\right] = \partial_w L(\bar{w},\bar{\xi},\bar{\mu})\bar{h}.$$

It follows that

$$(J_{sr})'_-(\bar{w},\bar{h}) = \lim_{n\to+\infty}\frac{J_{sr}(\bar{w}+t_nh_n) - J_{sr}(\bar{w})}{t_n} \geq \partial_w L(\bar{w},\bar{\xi},\bar{\mu})\bar{h}.$$

The proof is therefore complete. □

From Lemmas 4.2 and 4.3, we derive the following.

Theorem 4.4. *For any $\bar{h} \in W$, J_{sr} admits a Hadamard semiderivative at point $\bar{w}$ in the direction $\bar{h}$ given by*

$$J'_{sr}(\bar{w},\bar{h}) = \inf_{\bar{\xi}\in\Xi(\bar{w})}\partial_w L(\bar{w},\bar{\xi},\bar{\mu})\bar{h} = \inf_{\bar{\xi}\in\Xi(\bar{w})}\bar{\mu}q(\bar{h},\bar{\xi}),$$

where $\bar{\mu}$ is the unique solution of the dual problem (3.13).

4.3. Hadamard semiderivative of the robust compliance in the worst case sense

We denote by $\Xi(\bar{w})$ the maximizing set of (4.3) with $w = \bar{w}$, which has been obtained in Theorem 3.12. Slightly adapting the proof of Theorem 4.13 in [12], one obtains the following result.

Theorem 4.5. *For any $\bar{h} \in W$, the worst case functional J_{wc} admits a Hadamard semiderivative at point $\bar{w}$ in the direction $\bar{h}$ given by*

$$J'_{wc}(\bar{w},\bar{h}) = \sup_{\bar{\xi}\in\Xi(\bar{w})}\partial_w q(\bar{w},\bar{\xi})\bar{h} = \sup_{\bar{\xi}\in\Xi(\bar{w})} q(\bar{h},\bar{\xi}).$$

5. Topological derivative of the robust criteria

Consider a reference domain $\Omega = \Omega_0 \in \mathcal{E}$ and a family of perturbed domains $(\Omega_t)_{t>0}$ such that, for all t small enough, $\Omega_t \in \mathcal{E}$. We choose for simplicity a nominal load of the form $f^N_\Omega = B_\Omega\xi^N \in \mathcal{V}'_\Omega$ with $\xi^N \in \mathcal{H}$. As in the previous sections, set $w_\Omega = (Q_\Omega, b_\Omega, c_\Omega)$ with

$$Q_\Omega = B^*_\Omega A^{-1}_\Omega B_\Omega, \quad b_\Omega = B^*_\Omega A^{-1}_\Omega B_\Omega\xi^N = Q_\Omega\xi^N, \quad c_\Omega = \frac{1}{2}\langle B^*_\Omega A^{-1}_\Omega B_\Omega\xi^N,\xi^N\rangle = \frac{1}{2}\langle Q_\Omega\xi^N,\xi^N\rangle.$$

We make the following assumption, which will be verified for specific problems in Section 6.

Assumption 5.1. There exists $\delta > 0$ and a self-adjoint linear operator $G_\Omega : \mathcal{H} \to \mathcal{H}$ such that

$$\langle Q_{\Omega_t}\xi, \xi\rangle - \langle Q_\Omega \xi, \xi\rangle = t\langle G_\Omega \xi, \xi\rangle + O(t^{1+\delta}) \qquad \forall \xi \in \mathcal{H}.$$

Lemma 5.2. *The function $t \in \mathbb{R}_+ \mapsto Q_{\Omega_t} \in \mathcal{L}(\mathcal{H})$ admits a right derivative at 0 given by*

$$\frac{\mathrm{d}}{\mathrm{d}t}[Q_{\Omega_t}]_{|t=0} = G_\Omega.$$

Proof. Assumption 5.1 and the polarization identity entail

$$\langle Q_{\Omega_t}\xi, \zeta\rangle - \langle Q_\Omega, \xi, \zeta\rangle = t\langle G_\Omega \xi, \zeta\rangle + O(t^{1+\delta}) \qquad \forall \xi, \zeta \in \mathcal{H},$$

that is,

$$\forall \xi, \zeta \in \mathcal{H}, \qquad \sup_{t>0} t^{-\delta} \left| \left\langle \left(\frac{Q_{\Omega_t} - Q_\Omega}{t} - G_\Omega \right) \xi, \zeta \right\rangle \right| < +\infty.$$

By the Banach–Steinhaus's theorem we obtain

$$\forall \xi \in \mathcal{H}, \qquad \sup_{t>0} t^{-\delta} \left\| \left(\frac{Q_{\Omega_t} - Q_\Omega}{t} - G_\Omega \right) \xi \right\|_{\mathcal{H}} < +\infty.$$

Another application of the Banach-Steinhaus theorem yields

$$\sup_{t>0} t^{-\delta} \left\| \frac{Q_{\Omega_t} - Q_\Omega}{t} - G_\Omega \right\|_{\mathcal{L}(\mathcal{H})} < +\infty.$$

In particular we have

$$\lim_{t\downarrow 0} \left\| \frac{Q_{\Omega_t} - Q_\Omega}{t} - G_\Omega \right\|_{\mathcal{L}(\mathcal{H})} = 0,$$

and the proof is achieved. □

The following theorem states the right derivative of $t \to j_{sr}(\Omega_t)$.

Theorem 5.3. *The function $t \mapsto j_{sr}(\Omega_t)$ admits a right derivative at 0 given by*

$$\frac{\mathrm{d}}{\mathrm{d}t}[j_{sr}(\Omega_t)]_{|t=0} = \inf_{\bar\xi \in \Xi} \frac{1}{2}\bar\mu\langle \xi^N + \bar\xi, G_\Omega(\xi^N + \bar\xi)\rangle, \tag{5.1}$$

where Ξ is the set of solutions of the primal problem (3.12) and $\bar\mu$ is the solution of the dual problem (3.13).

Proof. By Lemma 5.2, we get that the map $t \mapsto w_{\Omega_t}$ admits a right derivative at 0 given by

$$\frac{\mathrm{d}}{\mathrm{d}t}[Q_{\Omega_t}]_{|t=0} = G_\Omega, \qquad \frac{\mathrm{d}}{\mathrm{d}t}[b_{\Omega_t}]_{|t=0} = G_\Omega \xi^N, \qquad \frac{\mathrm{d}}{\mathrm{d}t}[c_{\Omega_t}]_{|t=0} = \frac{1}{2}\langle G_\Omega \xi^N, \xi^N\rangle. \tag{5.2}$$

Next, with the notation of Section 4, we have

$$j_{sr}(\Omega_t) = J_{sr}(w_{\Omega_t}).$$

By composition (see, *e.g.*, [12], Prop. 2.47), the function $t \mapsto j_{sr}(\Omega_t)$ admits a right derivative at 0 given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[j_{sr}(\Omega_t)\right]_{|t=0} = J'_{sr}\left(w_\Omega, \frac{\mathrm{d}}{\mathrm{d}t}[w_{\Omega_t}]_{|t=0}\right).$$

Theorem 4.4 yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[j_{sr}(\Omega_t)\right]_{|t=0} = \inf_{\bar{\xi}\in\Xi} \bar{\mu}\left[\frac{1}{2}\left\langle \frac{\mathrm{d}}{\mathrm{d}t}[Q_{\Omega_t}]_{|t=0}\bar{\xi},\bar{\xi}\right\rangle + \left\langle \frac{\mathrm{d}}{\mathrm{d}t}[b_{\Omega_t}]_{|t=0},\bar{\xi}\right\rangle + \frac{\mathrm{d}}{\mathrm{d}t}[c_{\Omega_t}]_{|t=0}\right].$$

Using (5.2) we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[j_{sr}(\Omega_t)\right]_{|t=0} = \inf_{\bar{\xi}\in\Xi}\ \bar{\mu}\left[\frac{1}{2}\left\langle G_\Omega\bar{\xi},\bar{\xi}\right\rangle + \left\langle G_\Omega\xi^N,\bar{\xi}\right\rangle + \frac{1}{2}\left\langle G_\Omega\xi^N,\xi^N\right\rangle\right].$$

A rearrangement completes the proof. □

6. Algorithm

6.1. Problem setting

In the examples we will present we want to minimize

$$\mathcal{J}(\Omega) := \Phi(j_{sr}(\Omega)) + \ell|\Omega|,$$

where $\Phi : \mathbb{R}_+^* \to \mathbb{R}$ is a smooth and decreasing function, ℓ is a user-given Lagrange multiplier, and $|\Omega|$ is the Lebesgue measure of Ω. In order to maintain $j_{sr}(\Omega)$ positive during the iterations, we further assume that $\lim_{t\to 0}\Phi(t) = +\infty$. In our computations, for numerical purposes, we have used the function with moderate growth $\Phi(t) = -\log t$.

We choose $\mathcal{E}$ as the set of all subdomains of a fixed "hold-all" domain $D \subset \mathbb{R}^N$. Our model problem is that of linear elasticity, with the following standard framework. The domain Ω is occupied by an elastic material of unitary Young modulus, and its complement $D\backslash\Omega$ is filled with a weak phase, *i.e.*, a fictitious material with small Young modulus ε. This permits to formulate the equilibrium equations, represented by the operator A_Ω, in the fixed domain D. Therefore the function space $\mathcal{V}_\Omega$ is the subspace $H^1_D(D)^N$ including the Dirichlet boundary condition on the appropriate part Γ_D of $\partial\Omega$. The Hilbert space $\mathcal{H}$ and the operator $B_\Omega : \mathcal{H} \to \mathcal{V}'_\Omega$ may be arbitrary but B_Ω is assumed to be independent of Ω.

For some $\hat{x} \in D \setminus \partial\Omega$, we consider the *topological* perturbation

$$\Omega_t = \begin{cases} \Omega \setminus \overline{\mathbb{B}(\hat{x},\rho(t))} & \text{if } \hat{x} \in \Omega, \\ (\Omega \cup \mathbb{B}(\hat{x},\rho(t))) \cap D & \text{if } \hat{x} \in D \setminus \overline{\Omega}, \end{cases}$$

with $\rho(t) = t^{1/N}$.

6.2. Optimality condition

The derivative of $\mathcal{J}(\Omega_t)$ with respect to t is the so called topological derivative. It is given by the chain rule:

$$g_\Omega(\hat{x}) := \frac{\mathrm{d}}{\mathrm{d}t}[\mathcal{J}(\Omega_t)]_{|t=0} = \Phi'(j_{sr}(\Omega))\frac{\mathrm{d}}{\mathrm{d}t}[j_{sr}(\Omega_t)]_{|t=0} + \ell\frac{\mathrm{d}}{\mathrm{d}t}[|\Omega_t|]_{|t=0}.$$

Of course, this is only valid if the topological derivatives $\frac{\mathrm{d}}{\mathrm{d}t}[j_{sr}(\Omega_t)]_{|t=0}$ and $\frac{\mathrm{d}}{\mathrm{d}t}[|\Omega_t|]_{|t=0}$ exist. For this latter one this is obviously true. In the two dimensional case $N = 2$ in which we henceforth place ourselves we have

$$\frac{\mathrm{d}}{\mathrm{d}t}[|\Omega_t|]_{|t=0} = \begin{cases} -\pi & \text{if } \hat{x} \in \Omega, \\ \pi & \text{if } \hat{x} \in D \setminus \overline{\Omega}. \end{cases}$$

The expression of $\frac{\mathrm{d}}{\mathrm{d}t}[j_{sr}(\Omega_t)]_{|t=0}$ has been obtained in Theorem 5.3 upon Assumption 5.1. The operator G_Ω that satisfies Assumption 5.1 is associated with the topological derivative of the classical compliance. Its expression is known as (see [4,6]):

$$\langle G_\Omega\xi,\xi\rangle = -\pi\frac{r-1}{\kappa r+1}\frac{\kappa+1}{2}\left[2\sigma : e + \frac{(\varepsilon-1)(\kappa-2)}{\kappa+2\varepsilon-1}\operatorname{tr}\sigma\operatorname{tr}e\right],$$

with

$$r = \begin{cases} \varepsilon & \text{if } \hat{x} \in \Omega, \\ \varepsilon^{-1} & \text{if } \hat{x} \in D \setminus \overline{\Omega}, \end{cases}$$

$\kappa = (\lambda_L + 3\mu_L)/(\lambda_L + \mu_L)$, λ_L, μ_L the Lamé coefficients of the material, and (σ, e) the stress and strain tensors respectively at point $\hat{x}$ for the load $B_\Omega \xi$.

A necessary optimality condition for this class of perturbations is clearly

$$g_\Omega(\hat{x}) \geq 0 \qquad \forall \hat{x} \in D, \tag{6.1}$$

which is the starting point of our algorithm.

6.3. Description of the algorithm

In order to solve (6.1) we use the algorithm introduced in [6] and further analyzed in [5]. We recall its main features. Each domain Ω is represented by a smooth function $\psi_\Omega : D \to \mathbb{R}$ such that

$$\Omega = \{x \in D, \psi_\Omega(x) < 0\}.$$

Such a level-set representation is very standard in shape optimization. It is usually combined with a Hamilton–Jacobi evolution equation, see [3,31] to cite only the seminal works. However the Hamilton–Jacobi's formulation does not allow to easily nucleate holes, which can be a serious limitation in topology optimization. We proceed differently, defining first the signed topological derivative as

$$\tilde{g}_\Omega(\hat{x}) = \begin{cases} -g_\Omega(\hat{x}) & \text{if } \hat{x} \in \Omega, \\ g_\Omega(\hat{x}) & \text{if } \hat{x} \notin \Omega. \end{cases}$$

It appears that (6.1) will be solved as soon as

$$\tilde{g}_\Omega \sim \psi_\Omega, \tag{6.2}$$

with the equivalence relation $\sim$ defined by

$$\psi_1 \sim \psi_2 \iff \exists \alpha > 0, \psi_1 \sim \alpha \psi_2.$$

We apply to (6.2) the fixed point iteration with relaxation, *i.e.*, the update of the function ψ_Ω at iteration k is

$$\psi_{\Omega_{k+1}} \sim (1 - \omega_k)\psi_{\Omega_k} + \omega_k \tilde{g}_{\Omega_k}.$$

The parameter $\omega_k \in]0, 1]$ acts as step size and is fixed at every iteration by a line search.

Remark 6.1. Consider a combination of disjoint topological perturbations, such as for instance

$$\Omega_t = \Omega \setminus \bigcup_{i=1}^{n} \overline{B}(\hat{x}_i, \rho(t)).$$

The topological derivative of the compliance is additive with respect to the perturbation (see [7]), *i.e.*,

$$G_\Omega = \sum_{i=1}^{n} G_{\Omega,i},$$

where $G_{\Omega,i}$ is the topological derivative for a single perturbation. The topological derivative of $\Omega \to j_{sr}(\Omega)$ for the combination of perturbations is then

$$\frac{\mathrm{d}}{\mathrm{d}t}[j_{sr}(\Omega_t)]_{|t=0} = \inf_{\bar{\xi} \in \Xi} \bar{\mu} \sum_{i=1}^{n} \langle G_{\Omega,i}(\xi^N + \bar{\xi}), \xi^N + \bar{\xi} \rangle \geq \sum_{i=1}^{n} \inf_{\bar{\xi} \in \Xi} \bar{\mu} \langle G_{\Omega,i}(\xi^N + \bar{\xi}), \xi^N + \bar{\xi} \rangle.$$

This means that j_{sr} is superadditive with respect to the perturbation, hence a combination of descent directions for the functional $\mathcal{J}$ still provides a descent direction (recall that Φ is decreasing).

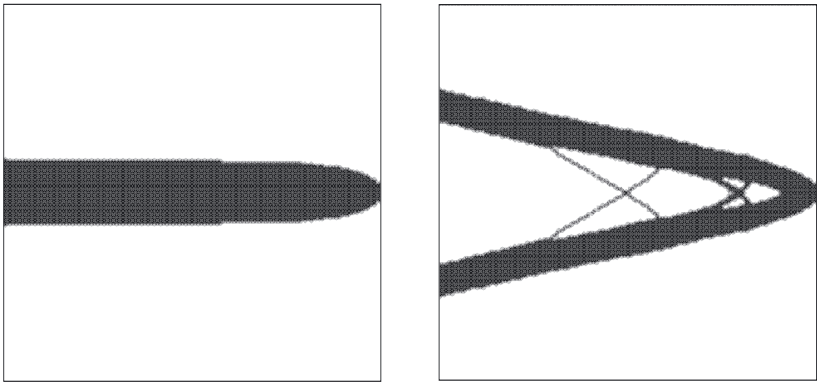


FIGURE 1. Beam: obtained domains for one load (*left*) and two loads (*right*).

For solving the dual problem (3.14) in the easy case and the hard case I, we have contented ourselves with the bisection method, as in our examples the space $\mathcal{H}$ was of small dimension.

Also, we have always encountered simple eigenvalues $\lambda_{\max}$. In such cases the set of critical loads Ξ consists of at most two points, therefore solving the minimization problem in (5.1) is trivial. Further investigation would be needed to design a numerical procedure able to deal with the general case.

7. Numerical examples

In the subsequent computations, the linear elasticity equations are solved by means of $P1$ finite elements (recall that the space dimension is $N = 2$), leading to a stiffness matrix denoted by K_Ω. We consider a finite-dimensional space of perturbations $\mathcal{H}$, say $\mathcal{H} = \mathbb{R}^m$ equipped with its canonical inner product and its canonical basis $(e_i)_{i=1}^m$. Each e_i corresponds to an applied load ϕ_i and a force vector F_i in the finite element framework. Then the matrix Q_Ω admits the entries:

$$[Q_\Omega]_{ij} = \frac{1}{2} F_i^T K_\Omega^{-1} F_j.$$

7.1. Beam

The hold all domain D is the unit square $]0,1[\times]0,1[$, with a Dirichlet boundary condition on the left side. We denote by p the middle of the right side and by ϕ_1 the unit horizontal force applied at p. The nominal load f_Ω^N corresponds to ϕ_1.

At first we consider a one-dimensional space $\mathcal{H}$, for which the value $\xi = 1$ of the parameter corresponds to the force ϕ_1. The threshold α is chosen as 10 times the compliance of the initial domain, which is the band $]0,1[\times]0.4,0.6[$, under the nominal load. The Lagrange multiplier for the area is fixed to $\ell = 10$. In this situation, only the easy case occurs. The optimized domain is represented in Figure 1, *left*. The convergence history of the criterion $\mathcal{J}$ is shown in Figure 2, *left*.

Next we add a unit vertical force ϕ_2, still applied at point p, and represented by the parameter $\xi = (0,1)$ in the space $\mathcal{H} = \mathbb{R}^2$. The force ϕ_1, represented by the parameter $\xi^N = (1,0)$, remains the nominal load. The other data are unchanged. The optimized domain is given in Figure 1, *right*, with the convergence history in Figure 2, *right*. For comparison, note that the area of this domain (0.168) is close to the area obtained in the previous case (0.160). The configuration at convergence is the hard case II, with the critical loads (including the nominal load) corresponding to the forces $1.09\phi_1 \pm 0.76\phi_2$.

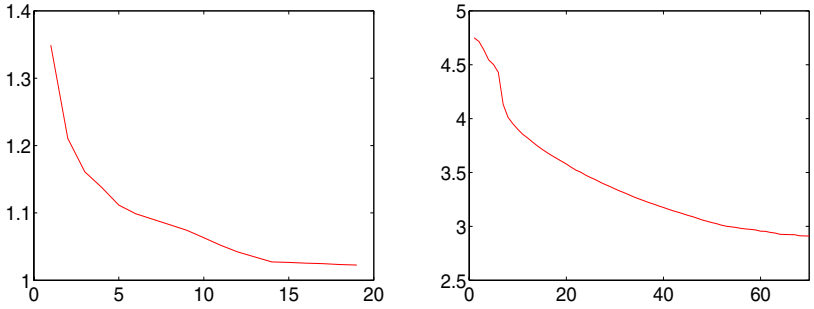


FIGURE 2. Beam: convergence histories for one load (*left*) and two loads (*right*).

7.2. Mast

For this problem the hold-all domain D is the union of the rectangles $]-1,1[\times]0,4[$ and $]-2,2[\times]4,6[$. A Dirichlet boundary condition is applied at the bottom side. We consider 4 forces:

- ϕ_1 is the force of components $(0,-1)$ applied at point $p_1=(-1,4)$;
- ϕ_2 is the force of components $(0,-1)$ applied at point $p_2=(1,4)$;
- ϕ_3 is the force of components $(1,0)$ applied at point p_1;
- ϕ_4 is the force of components $(1,0)$ applied at point p_2.

The nominal load corresponds to the forces ϕ_1 and ϕ_2 applied simultaneously.

As first case we again consider a one-dimensional space of perturbations, spanned by the nominal load. The forces ϕ_3 and ϕ_4 are not taken into account. The initialization is the full domain D, and α is chosen as 10 times the compliance of this domain under the nominal load. The Lagrange multiplier ℓ is fixed to 0.8. The optimized domain is represented in Figure 3, *left*.

Then we consider a two-dimensional space of perturbations, spanned by the forces ϕ_1 and ϕ_2 applied independently. All the other data are unchanged. The optimized domain is represented in Figure 3, *middle*. Due to the symmetry of the problem, this is a hard case II. The critical loads are $1.98\phi_1+1.01\phi_2$ and $1.01\phi_1+1.98\phi_2$.

Finally we consider four independent perturbations, given by the forces ϕ_1, ϕ_2, ϕ_3 and ϕ_4. We obtain the domain represented in Figure 3, *right*. This is again a hard case II, with the critical loads given by $1.28\phi_1+1.09\phi_2-0.30\phi_3-0.39\phi_4$ and its symmetric $1.09\phi_1+1.28\phi_2+0.39\phi_3+0.30\phi_4$.

From the first case to the last one, we clearly observe, first, a stiffening under non-symmetric vertical load, then, a stiffening under horizontal load.

APPENDIX A. LAGRANGIAN DUALITY

Here we gather useful results on general Lagrangian duality theory, which are essentially reformulations of classical results found in [26, 28, 33]. We nevertheless provide concise proofs for completeness.

Let X, Y be two sets and $L : X \times Y \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ be an application, called the *Lagrangian*. We define

$$f(x) = \sup_{y\in Y} L(x,y), \qquad x \in X,$$

$$g(y) = \inf_{x\in X} L(x,y), \qquad y \in Y.$$

The duality theory aims at finding relations between the *primal problem*

$$\text{Minimize } f(x), \quad x \in X, \tag{$\mathscr{P}$}$$

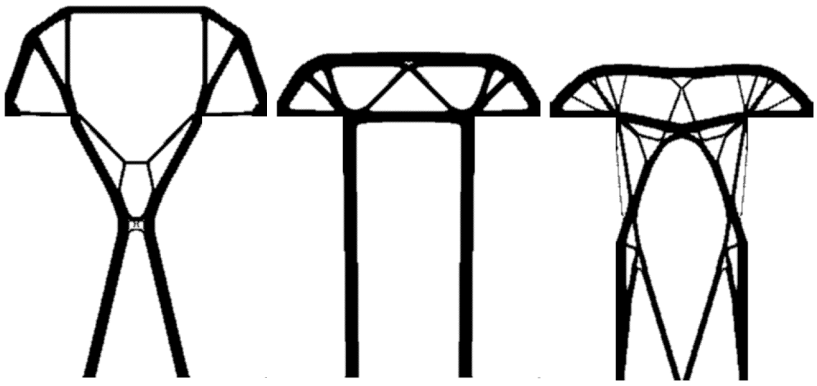


FIGURE 3. Mast: obtained domains for one load (*left*), two loads (*middle*) and four loads (*right*).

and the so-called *dual problem*

$$\text{Maximize } g(y), \quad y \in Y. \tag{$\mathscr{D}$}$$

We denote by $v(\mathscr{P}) = \inf_{x\in X} f(x)$ and $v(\mathscr{D}) = \sup_{y\in Y} g(y)$ the values of the primal and the dual problems, respectively.

Moreover, for any $y \in Y$, we consider the problem

$$\text{Minimize } L(x, y), \quad x \in X, \tag{$\mathscr{L}_y$}$$

with value $v(\mathscr{L}_y) = \inf_{x\in X} L(x, y) = g(y)$. We always have (weak duality):

$$\forall y \in Y, \qquad v(\mathscr{L}_y) \le v(\mathscr{D}) \le v(\mathscr{P}). \tag{A.1}$$

Theorem A.1.

(1) *For all $y \in Y$ it holds*

$$\operatorname{argmin}(\mathscr{L}_y) \cap [L(.,y) = f] \subset \operatorname{argmin}(\mathscr{P}). \tag{A.2}$$

(2) *For all $y \in Y$ it holds*

$$v(\mathscr{P}) = v(\mathscr{L}_y) \Longleftrightarrow \begin{cases} y \in \operatorname{argmax}(\mathscr{D}), \\ v(\mathscr{D}) = v(\mathscr{P}). \end{cases} \tag{A.3}$$

(3) *We have the relations:*

$$v(\mathscr{P}) = v(\mathscr{L}_y) \Longrightarrow \operatorname{argmin}(\mathscr{L}_y) \cap [L(.,y) = f] = \operatorname{argmin}(\mathscr{P}). \tag{A.4}$$

(4) *For all $y \in Y$ it holds*

$$\operatorname{argmin}(\mathscr{L}_y) \cap [L(.,y) = f] \neq \emptyset \Longrightarrow v(\mathscr{L}_y) = v(\mathscr{P}), \tag{A.5}$$

hence

$$\operatorname{argmin}(\mathscr{L}_y) \cap [L(.,y) = f] \neq \emptyset \Longrightarrow \operatorname{argmin}(\mathscr{L}_y) \cap [L(.,y) = f] = \operatorname{argmin}(\mathscr{P}). \tag{A.6}$$

(5) *We always have*

$$\Big\{ y \in Y \;\Big|\; \operatorname{argmin}(\mathscr{L}_y) \cap [L(.,y) = f] \neq \emptyset \Big\} \subset \operatorname{argmax}(\mathscr{D}). \tag{A.7}$$

Proof.

(1) If $\bar{x} \in \operatorname{argmin}(\mathscr{L}_y) \cap [L(.,y) = f]$ then, for all $x \in X$, $f(\bar{x}) = L(\bar{x},y) \leq L(x,y) \leq f(x)$.
(2) If $v(\mathscr{L}_y) = v(\mathscr{P})$, using (A.1), one obtains $v(\mathscr{D}) = v(\mathscr{P}) = v(\mathscr{L}_y) = g(y)$ and $y \in \operatorname{argmax}(\mathscr{D})$. If $y \in \operatorname{argmax}(\mathscr{D})$ and $v(\mathscr{D}) = v(\mathscr{P})$ then $v(\mathscr{L}_y) = g(y) = v(\mathscr{D}) = v(\mathscr{P})$.
(3) If $\bar{x} \in \operatorname{argmin}(\mathscr{P})$ then $L(\bar{x},y) \leq f(\bar{x}) = v(\mathscr{P}) = v(\mathscr{L}_y) \leq L(x,y)$ for all $x \in X$. In particular $L(\bar{x},y) = f(\bar{x})$ and $\bar{x} \in \operatorname{argmin}(\mathscr{L}_y)$. The reverse inclusion is given by (A.2).
(4) If $\bar{x} \in \operatorname{argmin}(\mathscr{L}_y) \cap [L(.,y) = f]$ then

$$v(\mathscr{P}) = \inf_{x \in X} f(x) \leq f(\bar{x}) = L(\bar{x},y) = v(\mathscr{L}_y) = \inf_{x \in X} L(x,y) \leq \inf_{x \in X} f(x) = v(\mathscr{P}).$$

(5) If $\operatorname{argmin}(\mathscr{L}_y) \cap [L(.,y) = f] \neq \emptyset$, then using (A.5), one infers $v(\mathscr{L}_y) = v(\mathscr{P})$. We conclude using (A.3). □

Acknowledgements. The authors thank the anonymous referees for their careful reading, useful comments and suggestions.

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