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NONSMOOTH PROBLEMS OF CALCULUS OF VARIATIONS *VIA* CODIFFERENTIATION*

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Abstract. In this paper multidimensional nonsmooth, nonconvex problems of the calculus of variations with codifferentiable integrand are studied. Special classes of codifferentiable functions, that play an important role in the calculus of variations, are introduced and studied. The codifferentiability of the main functional of the calculus of variations is derived. Necessary conditions for the extremum of a codifferentiable function on a closed convex set and its applications to the nonsmooth problems of the calculus of variations are described. Necessary optimality conditions in the main problem of the calculus of variations and in the problem of Bolza in the nonsmooth case are derived. Examples comparing presented results with other approaches to nonsmooth problems of the calculus of variations are given.

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1. Introduction

In this paper nonsmooth problems of the calculus of variations are studied. These problems were first studied in the works of Rockafellar [21–23]. After these works many different approaches were suggested to studying nonsmooth problems of the calculus of variations (*cf.*, for details, [5–7, 14, 15, 17–19, 25, 26]); however, all existing approaches have some disadvantages, that make their practical applications quite difficult.

As a rule, nonsmooth problems are studied by different homogeneous approximations of the increment of a function, such as the Clarke subdifferential [7] or the proximal subgradient [15]. But all these approximations are not continuous functions of points in a nonsmooth case. A lack of continuity makes the construction of effective numerical methods based on homogeneous approximations a very difficult task. The other disadvantage of a "homogeneous" approach is that computing an approximation is very complicated because there does not exist a convenient calculus of these approximations (*cf.*, for instance, formulae for computing the Clarke subdifferential [7] and "fuzzy calculus" of the proximal subgradients [15]).

In this paper nonsmooth problems of the calculus of variations are studied by the notion of codifferentiability. The concept of codifferentiable function was introduced by Demyanov [9, 10] (*cf.*, also, [11, 13]). Although an

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approach based on codifferentials does not allow to study nonsmooth optimization problems under such general assumptions as some other nonsmooth methods, it has its own benefits. An approximation of the increment of a function based on the codifferential is nonhomogeneous and is usually a continuous function of points. That is why it is easy to construct effective numerical methods based on the concept of codifferential (*cf.* method of codifferential descent in [11] and method of truncated codifferential descent and its applications in [12]). Also, there exists a well-developed codifferential calculus [11, 13] and formulae for computing codifferentials are very simple. These advantages make an approach based on the notion of codifferentiability more appealing for many practical applications than other existing approaches.

However, one should mention some limitations of the approach based on codifferentiability (and the closely related notion of quasidifferentiability [11]). Unlike the Clarke subdifferential and some other types of subdifferentials, a codifferential of a Lipschitz continuous function can be empty, but it is worth mentioning that the difference of two continuous convex functions defined on a normed space is always codifferentiable. Also, no characterization of the class of codifferentiable (or quasidifferentiable) functions is known. Moreover, despite the fact that there exists a convenient and elaborate calculus of codifferentiable and quasidifferentiable functions, there are no algorithms, in general, for the construction of elements in codifferential or quasidifferential in the case when they are not empty. However, codifferential was proved to be a very efficient tool for solving nonsmooth optimization problems in the case when codifferentials of the functions under consideration can be effectively computed [3, 4].

The main goal of our study is to prove that the main functional of the calculus of variations with codifferentiable integrand is codifferentiable. In order to do that we introduce and study several special classes of codifferentiable functions. Also, we derive necessary conditions for the extremum of a codifferentiable function on a closed convex set and apply them to studying the main problem of the calculus of variations and the problem of Bolza in the nonsmooth case. In the end, we provide two examples demonstrating that the necessary optimality conditions derived in this paper are better than the existing necessary optimality conditions for nonsmooth problems of the calculus of variations.

2. Necessary conditions for the extremum of a codifferentiable function

In this section we discuss necessary conditions for the extremum of a codifferentiable function on a closed convex set. We will apply these conditions to the study of nonsmooth problems of the calculus of variations.

We introduce the notation first. We denote by $(E, \|\cdot\|)$ a real normed space. As usual, its topological dual space is denoted by E^* and the weak* topology on E^* is denoted by w^* or $\sigma(E^*, E)$. The standard topology on the real line $\mathbb{R}$ is denoted by τ, the inner product in $\mathbb{R}^d$ is denoted by $\langle\cdot,\cdot\rangle$. Denote by $\operatorname{co} A$ the convex hull of a set $A \subset E$.

Let us recall the definition of codifferentiable function and related notions. Let $S \subset \mathbb{R}^d$ be an open set.

Definition 2.1. A function $f\colon S \to \mathbb{R}$ is said to be codifferentiable at a point $x \in S$ if there exists a pair of nonvoid compact convex sets $\underline{d}f(x), \overline{d}f(x) \subset \mathbb{R}^{d+1}$ such that for any admissible argument increment Δx (*i.e.* $\operatorname{co}\{x, x+\Delta x\} \subset S$) the corresponding function increment is represented as

$$f(x+\Delta x) - f(x) = \max_{[a,v]\in \underline{d}f(x)} (a + \langle v, \Delta x\rangle) + \min_{[b,w]\in \overline{d}f(x)} (b + \langle w, \Delta x\rangle) + o(\Delta x, x),$$

where

$$\max_{[a,v]\in \underline{d}f(x)} a + \min_{[b,w]\in \overline{d}f(x)} b = 0, \quad \frac{o(\alpha\Delta x, x)}{\alpha} \to 0 \text{ as } \alpha \downarrow 0.$$

Remark 2.2. We write $\alpha \downarrow 0$ instead of $\alpha \to +0$.

The following definition is a natural generalization of the notion of codifferentiation to the infinite-dimensional case.

Definition 2.3. Let $S \subset E$ be an open set. A function $f \colon S \to \mathbb{R}$ is said to be codifferentiable at a point $x \in S$ if there exists a pair of nonvoid convex sets $\underline{d} f(x), \overline{d} f(x) \subset \mathbb{R} \times E^*$ that are compact in the topological product $(\mathbb{R}, \tau) \times (E^*, w^*)$ and such that for any admissible argument increment $\Delta x \in E$

$$f(x + \Delta x) - f(x) = \max_{[a,\varphi] \in \underline{d} f(x)} (a + \varphi(\Delta x)) + \min_{[b,\psi] \in \overline{d} f(x)} (b + \psi(\Delta x)) + o(\Delta x, x),$$

where

$$\max_{[a,\varphi] \in \underline{d} f(x)} a + \min_{[b,\psi] \in \overline{d} f(x)} b = 0, \quad \frac{o(\alpha \Delta x, x)}{\alpha} \to 0 \text{ as } \alpha \downarrow 0.$$

A pair of sets $Df(x) = [\underline{d} f(x), \overline{d} f(x)]$ is called a codifferential of f at a point x, the set $\underline{d} f(x)$ is called a hypodifferential, and the set $\overline{d} f(x)$ is referred to as a hyperdifferential. Note that a codifferential is not unique. A function f is said to be hypodifferentiable at a point x if there exists a codifferential of the form $Df(x) = [\underline{d} f(x), \{0\}]$ and hyperdifferentiable at a point x if there exists a codifferential of the form $Df(x) = [\{0\}, \overline{d} f(x)]$.

Remark 2.4. It is easy to see that for any convex and compact in the topology $\tau \times w^*$ set $S \subset \mathbb{R} \times E^*$ the pair $[\underline{d} f(x) + S, \overline{d} f(x) - S]$ is a codifferential of f at x. Therefore there is an interesting and unresolved question concerning how to find a codifferential $Df(x)$ which is minimal, in some sense. For some results on the closely related problem of finding a minimal, in some sense, quasidifferential see [20, 24].

Remark 2.5. The direct product $\mathbb{R} \times E^*$ can be equipped with the norm $\|[a, \varphi]\|_p = (|a|^p + \|\varphi\|^p)^{\frac{1}{p}}$, where $1 \le p < \infty$, or $\|[a, \varphi]\|_\infty = \max\{|a|, \|\varphi\|\}$. It is clear that all norms $\| \cdot \|_p$, $1 \le p \le \infty$ are equivalent. Every norm $\| \cdot \|_p$ induces the Hausdorff metric on the set of all closed bounded subsets of the space $(\mathbb{R} \times E^*, \| \cdot \|_p)$. Since all norms $\| \cdot \|_p$, $1 \le p \le \infty$, are equivalent, then all corresponding Hausdorff metrics are also equivalent. Therefore, hereafter we will refer to Hausdorff metric without specifying a norm on $\mathbb{R} \times E^*$ that induces given metric.

A function f is said to be continuously codifferentiable at a point x if it is codifferentiable in a neighbourhood of x and there exists a mapping $y \to Df(y) = [\underline{d} f(y), \overline{d} f(y)]$ such that the mappings $y \to \underline{d} f(y)$ and $y \to \overline{d} f(y)$ are Hausdorff continuous at this point. One can also define continuously hypodifferentiable functions and continuously hyperdifferentiable functions.

Remark 2.6. The class of continuously codifferentiable functions is quite large. This class forms a linear space closed under the main algebraic operations (such as multiplication), the pointwise maximum and the pointwise minimum of a finite family of its elements (see [11, 13] for a codifferential calculus).

Let us give several examples of important classes of functions that are contained in the class of continuously codifferentiable functions. Any convex function $f \colon \mathbb{R}^d \to \mathbb{R}$ is hypodifferentiable on $\mathbb{R}^d$ and continuously hypodifferentiable on any bounded subset of $\mathbb{R}^d$ (*cf.* [11], Sect. 4.1), and any norm is continuously hypodifferentiable on the whole space ([13], example 3.2). Let $S \subset E$ be an open set, functions $f_i, g_j \colon S \to \mathbb{R}$ be continuously Gâteaux differentiable at a point $x \in S$, $i \in I = \{1, \dots, k\}$, $j \in J = \{1, \dots, l\}$. Let us show that the function

$$f(\cdot) = \max_{i \in I} f_i(\cdot) + \min_{j \in J} g_j(\cdot)$$

is continuously codifferentiable at x. Indeed, for any admissible $\Delta x \in E$ one has

$$f(x + \Delta x) - f(x) = \max_{i \in I} \left(f_i(x + \Delta x) - \max_{i \in I} f_i(x) \right) + \min_{j \in J} \left(g_j(x + \Delta x) - \min_{j \in J} g_j(x) \right).$$

Since the functions f_i and g_j are condinuously Gâteaux differentiable at x then

$$\begin{aligned} f(x + \Delta x) - f(x) = &\max_{i \in I} \left(f_i(x) - \max_{i \in I} f_i(x) + f_i'[x](\Delta x) + o_{f_i}(\Delta x, x) \right) \\ &+ \min_{j \in J} \left(g_j(x) - \min_{j \in J} g_j(x) + g_j'[x](\Delta x) + o_{g_j}(\Delta x, x) \right), \end{aligned}$$

where $o_{f_i}(\alpha\Delta x, x)/\alpha \to 0$ and $o_{g_j}(\alpha\Delta x, x)/\alpha \to 0$ as $\alpha \downarrow 0$, $f_i'[x]$ and $g_j'[x]$ are the Gâteaux derivatives of the functions f_i and g_j, respectively, $i \in I$, $j \in J$. Therefore one gets

$$f(x+\Delta x) - f(x) = \max_{i \in I}\Big(f_i(x) - \max_{i\in I} f_i(x) + f_i'[x](\Delta x)\Big) + \min_{j\in J}\Big(g_j(x) - \min_{j\in J} g_j(x) + g_j'[x](\Delta x)\Big) + o(\Delta x, x),$$

where $o(\alpha\Delta x, x)/\alpha \to 0$ as $\alpha \downarrow 0$. Thus the function f is continuously codifferentiable at x, and there is a codifferential of f at x of the form

$$Df(x) = \Big[\operatorname{co}\big\{[f_i(x) - \max_{i\in I} f_i(x), f_i'[x]] \mid i \in I\big\},\ \operatorname{co}\big\{[g_j(x) - \min_{j\in J} g_j(x), g_j'[x]] \mid j\in J\big\}\Big].$$

In particular, if $E = \mathbb{R}^d$ then

$$Df(x) = \left[\operatorname{co}\left\{\left[f_i(x) - \max_{i\in I} f_i(x), \frac{\partial f_i}{\partial x_1}(x), \ldots, \frac{\partial f_i}{\partial x_d}(x)\right] \,\middle|\, i\in I\right\},\right.$$
$$\left.\operatorname{co}\left\{\left[g_j(x) - \min_{j\in J} g_j(x), \frac{\partial g_j}{\partial x_1}(x), \ldots, \frac{\partial g_j}{\partial x_d}(x)\right] \,\middle|\, j\in J\right\}\right].$$

Remark 2.7. Let a function $f\colon S \to \mathbb{R}$ be codifferentiable at a point x. One can always suppose that the following equalities hold true

$$\max_{[a,\varphi]\in \underline{d}f(x)} a = \min_{[b,\psi]\in \overline{d}f(x)} b = 0. \tag{2.1}$$

In fact, if equalities (2.1) do not hold true, then one can consider the pair $[A, B] = [\underline{d}f(x) - \{[\overline{a}, 0]\}, \overline{d}f(x) + \{[\overline{a}, 0]\}]$, where

$$\overline{a} = \max_{[a,\varphi]\in\underline{d}f(x)} a = -\min_{[b,\psi]\in\overline{d}f(x)} b,$$

that, as it is easy to check, satisfies the definition of codifferential and equalities (2.1).

Note that if one uses standard formulae for computing codifferentials (*cf.* [11,13]), then equalities (2.1) always hold true.

Let a function $f\colon S\to\mathbb{R}$ be codifferentiable on an open set $S \subset E$, *i.e.* f is codifferentiable at any point $x\in S$, and let $A \subset S$ be a nonvoid closed convex set. Consider the problem of minimizing the function f on the set A.

Theorem 2.8. *Suppose that the function f has a local minimum on the set A at a point $x^* \in A$. Then for any $[0,\psi] \in \overline{d}f(x^*)$ the function*

$$g(x) = \max_{[a,\varphi]\in\underline{d}f(x^*)} (a + \varphi(x)) + \psi(x), x\in E,$$

has a global minimum on the set $A - x^*$ at the origin. Moreover, if $(E, \|\cdot\|)$ is a Banach space, then

$$(\underline{d}f(x^*) + \{[0,\psi]\}) \cap (\{0\} \times (-N(A, x^*))) \neq \emptyset \quad \forall [0,\psi]\in\overline{d}f(x^*), \tag{2.2}$$

where $N(A, x^*) = \{p \in E^* \mid p(a - x^*) \le 0 \quad \forall a \in A\}$ is the normal cone to the set A at the point x^*.

Proof. Ab absurdo, suppose that there exists $[0,\psi]\in\overline{d}f(x^*)$ for which the function

$$g(x) = \max_{[a,\varphi]\in\underline{d}f(x^*)} (a + \varphi(x)) + \psi(x), x\in E,$$

does not attain a global minimum on the set $A - x^*$ at the origin. Then there exists a point $y \in A - x^*$, $y \neq 0$, such that $g(y) = -c < 0 = g(0)$ (the last equality holds true by virtue of Rem. 2.7). Since A is convex, then $\text{co}\{x^*, x^* + y\} \subset A$. It is clear that the function g is convex, therefore for all $\alpha \in [0,1]$

$$g(\alpha y) = g(\alpha y + (1-\alpha)0) \leq \alpha g(y) + (1-\alpha)g(0) = -c\alpha. \tag{2.3}$$

By the definition of codifferentiable function, there exists $\alpha_0 \in (0,1)$ such that for all $\alpha \in (0, \alpha_0)$

$$f(x^* + \alpha y) - f(x^*) \leq \max_{[a,\varphi] \in \underline{d} f(x^*)} (a + \varphi(\alpha y)) + \min_{[b,\psi] \in \overline{d} f(x^*)} (b + \psi(\alpha y)) + \frac{c}{2}\alpha \leq g(\alpha y) + \frac{c}{2}\alpha.$$

Taking into account (2.3), we find that for any $\alpha \in (0, \alpha_0)$

$$f(x^* + \alpha y) - f(x^*) \leq -\frac{c}{2}\alpha < 0,$$

which contradicts the definition of the point x^*.

It remains to prove that if $(E, \|\cdot\|)$ is a Banach space, then (2.2) holds true. Indeed, since g attains a global minimum on the set $A - x^*$ at the origin, then by virtue of the necessary and sufficient condition for the minimum of a convex function on a convex set ([16], Thm. 1.1.2) one has

$$\partial g(0) \cap (-N(A - x^*, 0)) \neq \emptyset.$$

Applying the theorem about the subdifferential of the supremum ([16], Thm. 4.2.3), one gets

$$\partial g(0) = \{p \in E^* \mid p = \varphi + \psi, [0, \varphi] \in \underline{d} f(x^*)\}$$

(it is easy to verify that the set on the right-hand side is convex and closed in the weak* topology). Hence the desired result immediately follows from the obvious inclusion $\{0\} \times \partial g(0) \subset \underline{d} f(x^*) + \{[0, \psi]\}$ and the equality $N(A - x^*, 0) = N(A, x^*)$. □

Corollary 2.9. *Suppose that the function f has a local maximum on the set A at a point $x^* \in A$. Then for any $[0, \varphi] \in \underline{d} f(x^*)$ the function*

$$h(x) = \min_{[b,\psi] \in \overline{d} f(x^*)} (b + \psi(x)) + \varphi(x), x \in E,$$

has a maximum on the set $A - x^*$ at the point 0. Moreover, if $(E, \|\cdot\|)$ is a Banach space, then

$$(\overline{d} f(x^*) + \{[0, \varphi]\}) \cap (\{0\} \times N(A, x^*)) \neq \emptyset \quad \forall [0, \varphi] \in \underline{d} f(x^*).$$

3. Two problems of the calculus of variations

In this section we will describe two problems of the calculus of variations that are the main subject of our study. Both of these problems have its own difficult points. However, the ideas and results that were obtained during the study of one problem helped us to understand better the other one and vice versa. This is the main reason to consider both of these problems together.

Let us introduce the additional notation. In the subsequent sections $\Omega \subset \mathbb{R}^d$ will be an open bounded set and $|\cdot|$ will be the Euclidean norm on $\mathbb{R}^d$.

We denote by $C^1(\overline{\Omega})$ a vector space of all those $f \in C^1(\Omega)$ (*i.e.* f is continuously differentiable on Ω) for which functions f and $\frac{\partial f}{\partial x_i}$, $i \in \{1, \ldots, d\}$, are bounded and uniformly continuous on Ω (then there exist unique, bounded, continuous extensions of the function f and all its first order partial derivatives to the closure $\overline{\Omega}$ of Ω). $C^1(\overline{\Omega})$ is a Banach space with the norm given by

$$\|f\|_{C^1} = \max\left\{\sup_{x \in \Omega} |f(x)|, \sup_{x \in \Omega} \left|\frac{\partial f}{\partial x_1}(x)\right|, \ldots, \sup_{x \in \Omega} \left|\frac{\partial f}{\partial x_d}(x)\right|\right\}.$$

The vector space of all functions $f \in C^1(\overline{\Omega})$ vanishing on the boundary of Ω is denoted by $C_0^1(\overline{\Omega})$. $C^1(\overline{\Omega}, \mathbb{R}^m)$ is a Banach space of all functions $f = (f_1, \ldots, f_m)$ mapping $\overline{\Omega}$ to $\mathbb{R}^m$ such that $f_i \in C^1(\overline{\Omega})$, $i \in \{1, \ldots, m\}$, endowed with the norm

$$\|f\|_{C^1} = \max\left\{\sup_{x\in\Omega}|f(x)|, \sup_{x\in\Omega}|\nabla f(x)|\right\},$$

where

$$\nabla f(x) = \left(\frac{\partial f_j}{\partial x_i}(x)\right)_{1\le i\le d}^{1\le j\le m} \in \mathbb{R}^{m\times d}, \quad x \in \Omega.$$

The space $C_0^1(\overline{\Omega}, \mathbb{R}^m)$ is defined in the same way as $C_0^1(\overline{\Omega})$.

Consider the space $L_p(\Omega, \mathbb{R}^m)$. This is a Banach space equipped with the norm

$$\|u\|_p = \left(\int_\Omega |u(x)|^p \mathrm{d}x\right)^{\frac{1}{p}}$$

in the case $1 \le p < \infty$ and $\|u\|_\infty = \operatorname{esssup}_{x\in\Omega}|u(x)|$ in the case $p = \infty$, where $u = (u_1, \ldots, u_m) \in L_p(\Omega, \mathbb{R}^m)$. The Sobolev space $W^{1,p}(\Omega, \mathbb{R}^m)$ is endowed with the norm $\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p$. The closure of the space $C_0^\infty(\Omega)$ in the Sobolev space $W^{1,p}(\Omega)$ is referred to as $W_0^{1,p}(\Omega)$, $1 \le p \le \infty$. Here, as usual, $C_0^\infty(\Omega)$ is a subspace of $C^\infty(\Omega)$ consisting of all functions with compact support. If $1 \le p \le \infty$, then we denote by q the conjugate index to p, *i.e.* $1 \le q \le \infty$ and $1/p + 1/q = 1$.

Let us consider the following functional

$$\mathcal{I}_C(u) = \int_\Omega f(x, u(x), \nabla u(x))\,\mathrm{d}x$$

defined on the space $C^1(\overline{\Omega}, \mathbb{R}^m)$. Here $f\colon \overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}$, $f = f(x, u, \xi)$, is a given continuous function, $u = (u_1, \ldots, u_m) \in C^1(\overline{\Omega}, \mathbb{R}^m)$. Fix an arbitrary $u_0 \in C^1(\overline{\Omega}, \mathbb{R}^m)$ and denote by $A_C = \{u_0 + u \mid u \in C_0^1(\overline{\Omega}, \mathbb{R}^m)\}$ a closed convex subset of $C^1(\overline{\Omega}, \mathbb{R}^m)$. We will consider the following problem of the calculus of variations

$$\mathcal{I}_C(u) \to \text{extr}, \quad u \in A_C. \tag{3.4}$$

We will also consider the following functional

$$\mathcal{I}_W(u) = \int_\Omega g(x, u(x), \nabla u(x))\,\mathrm{d}x,$$

defined on the space $W^{1,p}(\Omega, \mathbb{R}^m)$, $1 \le p \le \infty$. Here $u = (u_1, \ldots, u_m) \in W^{1,p}(\Omega, \mathbb{R}^m)$ and $g\colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}$, $g = g(x, u, \xi)$, is a given function satisfying the Caratheodory condition (*i.e.* the function $(u, \xi) \to g(x, u, \xi)$ is continuous for almost every $x \in \Omega$ and the function $x \to g(x, u, \xi)$ is measurable for all $u \in \mathbb{R}^m$, $\xi \in \mathbb{R}^{m\times d}$) and the growth condition: there exist $C \ge 0$ and $\beta \in L_1(\Omega)$, $\beta \ge 0$, such that for a.e. $x \in \Omega$

$$|g(x, u, \xi)| \le \beta(x) + C(|u|^p + |\xi|^p) \quad \forall (u, \xi) \in \mathbb{R}^m \times \mathbb{R}^{m\times d}, \tag{3.5}$$

in the case $1 \le p < \infty$, and for any $N \in \mathbb{N}$ there exists a function $\beta_N \in L_1(\Omega)$, $\beta_N \ge 0$ such that for a.e. $x \in \Omega$

$$|g(x, u, \xi)| \le \beta_N(x) \quad \forall (u, \xi) \in B_N,$$

in the case $p = \infty$. Here $B_N = \{(u, \xi) \in \mathbb{R}^m \times \mathbb{R}^{m\times d} \mid |u| + |\xi| \le N\}$.

Remark 3.1. With the use of the Sobolev embedding theorem, the inequality (3.5) can be improved under some additional assumptions on the set Ω (*cf.*, for instance, [8], Sect. 3.4.2).

Fix an arbitrary $v_0 \in W^{1,p}(\Omega, \mathbb{R}^m)$ and denote by $A_W = \{v_0 + u \in W^{1,p}(\Omega, \mathbb{R}^m) \mid u \in W_0^{1,p}(\Omega, \mathbb{R}^m)\}$ a closed convex subset of $W^{1,p}(\Omega, \mathbb{R}^m)$. We will consider the following problem of the calculus of variations

$$\mathcal{I}_W(u) \to \text{extr}, \quad u \in A_W \tag{3.6}$$

along with the problem (3.4).

In the subsequent sections we will show that the functionals $\mathcal{I}_C$ and $\mathcal{I}_W$ are codifferentiable under some assumptions on the functions f and g. With the use of Theorem 2.8, we will derive necessary optimality conditions for problems (3.4) and (3.6).

4. Special codifferentiable functions

In this section we will introduce several special classes of codifferentiable functions that will play an important role in studying the functionals $\mathcal{I}_C$ and $\mathcal{I}_W$. Let X be an arbitrary nonvoid set, $S \subset E$ be an open set.

Definition 4.1. A function $f\colon X \times S \to \mathbb{R}$, $f = f(x,y)$, is said to be codifferentiable with respect to y at a point $(x_0, y_0) \in X \times S$ if the function $g(y) = f(x_0, y)$, $y \in S$, is codifferentiable at the point y_0. It obviously means that there exists a pair of nonvoid convex sets $\underline{d}_y f(x_0, y_0)$, $\overline{d}_y f(x_0, y_0) \subset \mathbb{R} \times E^*$ that are compact in the topological product $(\mathbb{R}, \tau) \times (E^*, w^*)$ and such that for any admissible argument increment Δy (*i.e.* $\text{co}\{y, y + \Delta y\} \subset S$)

$$f(x_0, y_0 + \Delta y) - f(x_0, y_0) = \Phi_f(x_0, y_0; \Delta y) + \Psi_f(x_0, y_0; \Delta y) + o(\Delta y; x_0, y_0),$$

where

$$\Phi_f(x_0, y_0; \Delta y) = \max_{[a,\varphi] \in \underline{d}_y f(x_0,y_0)} (a + \varphi(\Delta y)), \quad \Psi_f(x_0, y_0; \Delta y) = \min_{[b,\psi] \in \overline{d}_y f(x_0,y_0)} (b + \psi(\Delta y)),$$
$$\Phi_f(x_0, y_0; 0) + \Psi_f(x_0, y_0, 0) = 0, \quad \frac{o(\alpha \Delta y; x_0, y_0)}{\alpha} \to 0 \text{ as } \alpha \downarrow 0.$$

A pair of sets $D_y f(x,y) = [\underline{d}_y f(x,y), \overline{d}_y f(x,y)]$ is called a codifferential of f with respect to y at a point (x,y), the set $\underline{d}_y f(x,y)$ is referred to as a hypodifferential with respect to y, and the set $\overline{d}_y f(x,y)$ is called a hyperdifferential with respect to y. A function f is said to be codifferentiable with respect to y on $X \times S$ if f is codifferentiable at every point $(x,y) \in X \times S$. As in the case of an ordinary codifferential, a codifferential with respect to y is not unique.

Let (X, σ) be an aribitrary topological space. A function $f\colon X \times S \to \mathbb{R}$ is said to be continuously codifferentiable with respect to y at a point $(x_0, y_0) \in X \times S$ if it is codifferentiable with respect to y in a neighbourhood of (x_0, y_0) and there exists a mapping $(x,y) \to D_y f(x,y)$ such that the mappings $(x,y) \to \underline{d}_y f(x,y)$ and $(x,y) \to \overline{d}_y f(x,y)$ are Hausdorff continuous at this point.

Remark 4.2. It is clear that a codifferential with respect to y has the same properties as an ordinary codifferential. In particular, it is easy to derive formulae for computing a codifferential with respect to y and assertions about its continuity.

Let us obtain some useful properties of a function that is continuously codifferentiable with respect to y.

Proposition 4.3. *Let (X, σ) be a topological space, and suppose that a function $f\colon X \times S \to \mathbb{R}$, $f = f(x,y)$, is continuously codifferentiable with respect to y at a point $(x_0, y_0) \in X \times S$. Then for any $\Delta y_0 \in E$ the functions $(x, y, \Delta y) \to \Phi_f(x, y; \Delta y)$ and $(x, y, \Delta y) \to \Psi_f(x, y; \Delta y)$ are continuous at the point $(x_0, y_0, \Delta y_0)$*

Proof. We consider only the function $\Phi_f(x, y; \Delta y)$ since the assertion for the function $\Psi_f(x, y; \Delta y)$ is proved in a similar way. Fix arbitrary $\varepsilon > 0$ and $\Delta y_0 \in E$. Since the function f is continuously codifferentiable with respect to y at (x_0, y_0), then there exist a neighbourhood $V_{x_0} \in \sigma$ of the point x_0 and $\delta_1 > 0$ such that for all $x \in V_{x_0}$ and $y \in E$, $\|y - y_0\| < \delta_1$

$$\rho_H(\underline{d}_y f(x_0, y_0), \underline{d}_y f(x, y)) = \sup_{[a_2,\varphi_2]\in\underline{d}_y f(x,y)} \inf_{[a_1,\varphi_1]\in\underline{d}_y f(x_0,y_0)} (|a_1 - a_2| + \|\varphi_1 - \varphi_2\|) + \sup_{[a_1,\varphi_1]\in\underline{d}_y f(x_0,y_0)} \inf_{[a_2,\varphi_2]\in\underline{d}_y f(x,y)} (|a_1 - a_2| + \|\varphi_1 - \varphi_2\|) < \frac{\varepsilon}{6(\|\Delta y_0\| + 1)}, \quad (4.7)$$

where ρ_H is a Hausdorff metric. The set $\underline{d}_y f(x, y)$ is compact in $(\mathbb{R}, \tau) \times (E^*, w^*)$, then it is bounded (*cf.* [13], Thm. 2.1). Hence by (4.7) one gets that

$$C = \sup\{|a| + \|\varphi\| \mid [a, \varphi] \in \underline{d}_y f(x, y), x \in V_{x_0}, y \in E, \|y - y_0\| < \delta_1\} < +\infty.$$

Denote $\delta_2 = \varepsilon/3C$ and fix an arbitrary $x \in V_{x_0}$, $y \in E$, $\Delta y \in E$ such that $\|y - y_0\| < \delta_1$, $\|\Delta y - \Delta y_0\| < \delta_2$. By definition of Φ_f, there exists $[a_1, \varphi_1] \in \underline{d}_y f(x_0, y_0)$ for which $\Phi_f(x_0, y_0; \Delta y_0) = a_1 + \varphi_1(\Delta y_0)$. It follows from (4.7), that there exists $[a_2, \varphi_2] \in \underline{d}_y f(x, y)$ such that

$$|a_1 - a_2| + \|\varphi_1 - \varphi_2\| < \frac{\varepsilon}{3(\|\Delta y_0\| + 1)} < \frac{\varepsilon}{3}.$$

We have

$$|\varphi_1(\Delta y_0) - \varphi_2(\Delta y)| \leq |\varphi_1(\Delta y_0) - \varphi_2(\Delta y_0)| + |\varphi_2(\Delta y_0) - \varphi_2(\Delta y)| \leq \leq \|\varphi_1 - \varphi_2\|\|\Delta y_0\| + \|\varphi_2\|\|\Delta y_0 - \Delta y\| \leq \frac{\varepsilon}{3(\|\Delta y_0\| + 1)}\|\Delta y_0\| + C\frac{\varepsilon}{3C} < \frac{2\varepsilon}{3}.$$

Hence

$$\Phi_f(x_0, y_0; \Delta y_0) = a_1 + \varphi_1(\Delta y_0) \leq a_2 + \varphi_2(\Delta y) + \varepsilon \leq \Phi_f(x, y; \Delta y) + \varepsilon.$$

Arguing in the same way we get the inverse inequality

$$\Phi_f(x, y; \Delta y) \leq \Phi_f(x_0, y_0; \Delta y_0) + \varepsilon.$$

Therefore for any $x \in V_{x_0}$, $y \in E$ and $\Delta y \in E$ such that $\|y - y_0\| < \delta_1$, $\|\Delta y - \Delta y_0\| < \delta_2$ the following inequality holds true

$$|\Phi_f(x_0, y_0; \Delta y_0) - \Phi_f(x, y; \Delta y)| < \varepsilon.$$

Thus, the proof is complete. □

Proposition 4.4. *Let (X, σ) be a topological space, $E = \mathbb{R}^k$, $S \subset E$ be an open set. Suppose that a function $f : X \times S \to \mathbb{R}$, $f = f(x, y)$, is continuously codifferentiable with respect to y on $X \times S$. Then for any compact set $K \subset X \times S$ and bounded set $Q \subset \mathbb{R}^k$ there exists $L > 0$ such that for all $(x, y) \in K$ and $\Delta y_1, \Delta y_2 \in Q$*

$$|\Phi_f(x, y; \Delta y_1) - \Phi_f(x, y; \Delta y_2)| \leq L|\Delta y_1 - \Delta y_2|, \quad |\Psi_f(x, y; \Delta y_1) - \Psi_f(x, y; \Delta y_2)| \leq L|\Delta y_1 - \Delta y_2|.$$

Proof. Let us prove the assertion for Φ_f. Denote $r = \sup_{\Delta y\in Q} |\Delta y|$. By the previous proposition, the function Φ_f is continuous. Since by the Tichonoff theorem the set $K \times \{\Delta y \in \mathbb{R}^k \mid |\Delta y| \leq 2r\}$ is compact, then

$$c = \max_{(x,y)\in K, |\Delta y|\leq 2r} |\Phi_f(x, y; \Delta y)| < +\infty.$$

Fix an arbitrary $(x, y) \in K$ and $\Delta y_1, \Delta y_2 \in Q$, $\Delta y_1 \neq \Delta y_2$. Denote $\theta = \frac{|\Delta y_1 - \Delta y_2|}{r + |\Delta y_1 - \Delta y_2|} \in (0, 1)$ and $\Delta y = \frac{1}{\theta}(\Delta y_1 - (1 - \theta)\Delta y_2)$. We have

$$|\Delta y - \Delta y_1| = \frac{1 - \theta}{\theta}|\Delta y_1 - \Delta y_2| = \frac{r}{r + |\Delta y_1 - \Delta y_2|}\frac{r + |\Delta y_1 - \Delta y_2|}{|\Delta y_1 - \Delta y_2|}|\Delta y_1 - \Delta y_2| = r,$$

thus $|\Delta y| \leq 2r$ and $|\Phi_f(x, y; \Delta y)| \leq c$.

By definition of the function Φ_f, one has that the mapping $\Delta y \to \Phi_f(x, y; \Delta y)$ is convex for any $(x, y) \in X \times E$. Hence

$$(\theta + (1-\theta))\Phi_f(x, y; \Delta y_1) = \Phi_f(x, y; \Delta y_1) = \Phi_f(x, y; \theta \Delta y + (1-\theta)\Delta y_2) \leq \theta c + (1-\theta)\Phi_f(x, y; \Delta y_2),$$

therefore

$$\Phi_f(x, y; \Delta y_1) - \Phi_f(x, y; \Delta y_2) \leq \frac{\theta}{1-\theta}(c - \Phi_f(x, y; \Delta y_1)) \leq \frac{2c}{r}|\Delta y_1 - \Delta y_2|.$$

Since $\Delta y_1, \Delta y_2 \in Q$ are arbitrary, then

$$\Phi_f(x, y; \Delta y_2) - \Phi_f(x, y; \Delta y_1) \leq \frac{2c}{r}|\Delta y_2 - \Delta y_1|.$$

It remains to denote $L = 2c/r$. □

Remark 4.5. The proof of Proposition 4.4 is based on the well-known proof of the Lipschitz continuity of a convex function.

Let us consider a particular case of Definition 4.1, that is the most important for the study of nonsmooth problems of the calculus of variations.

Definition 4.6. Let $X \subset \mathbb{R}^d$ be an arbitrary nonvoid set, a function $f\colon X \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}$, $f = f(x, u, \xi)$. Denote $E = \mathbb{R}^m \times \mathbb{R}^{m\times d}$. The function f is said to be codifferentiable with respect to u and ξ at a point $(x_0, u_0, \xi_0) \in X \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$ if the function $g(x, y) = f(x, u, \xi)$, $y = (u, \xi)$, is codifferentiable with respect to y at the point $(x_0, y_0) \in X \times E$, $y_0 = (u_0, \xi_0)$. It means that there exist nonvoid compact convex sets $\underline{d}_{u,\xi} f(x_0, u_0, \xi_0)$, $\overline{d}_{u,\xi} f(x_0, u_0, \xi_0) \subset \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$ such that for all $\Delta u \in \mathbb{R}^m$ and $\Delta \xi \in \mathbb{R}^{m\times d}$

$$f(x_0, u_0 + \Delta u, \xi_0 + \Delta \xi) - f(x_0, u_0, \xi_0) = \Phi_f(x_0, u_0, \xi_0; \Delta u, \Delta \xi) + \Psi_f(x_0, u_0, \xi_0; \Delta u, \Delta \xi) + o(\Delta u, \Delta \xi; x_0, u_0, \xi_0),$$

where

$$\Phi_f(x_0, u_0, \xi_0; \Delta u, \Delta \xi) = \max_{[a, v_1, v_2] \in \underline{d}_{u,\xi} f(x_0, u_0, \xi_0)} (a + \langle v_1, \Delta u\rangle + \langle v_2, \Delta \xi\rangle),$$

$$\Psi_f(x_0, u_0, \xi_0; \Delta u, \Delta \xi) = \min_{[b, w_1, w_2] \in \overline{d}_{u,\xi} f(x_0, u_0, \xi_0)} (b + \langle w_1, \Delta u\rangle + \langle w_2, \Delta \xi\rangle),$$

$$\Phi_f(x_0, u_0, \xi_0; 0, 0) + \Psi_f(x_0, u_0, \xi_0; 0, 0) = 0, \qquad \frac{o(\alpha \Delta u, \alpha \Delta \xi; x_0, u_0, \xi_0)}{\alpha} \to 0 \text{ as } \alpha \downarrow 0.$$

Remark 4.7. All notions and assertions connected with a codifferentiation (such as a continuous codifferentiation) are easily transferred to the case of a codifferentiability with respect to u and ξ. In particular, one can always suppose that $\Phi_f(x, u, \xi; 0, 0) = \Psi_f(x, u, \xi; 0, 0) = 0$.

We introduce several auxiliary definitions that will allow us to use "a codifferentiation of an integral with respect to a parameter". These definitions will be vital for the study of nonsmooth problems of the calculus of variations. As previously mentioned, $\Omega \subset \mathbb{R}^d$ is an open bounded set.

Definition 4.8. A function $f\colon \overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}$, $f = f(x, u, \xi)$, is said to be codifferentiable with respect to u and ξ on $\overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$ uniformly with respect to $C^1(\overline{\Omega}, \mathbb{R}^m)$ if f is codifferentiable with respect to u and ξ on $\overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$ and for all $u, h \in C^1(\overline{\Omega}, \mathbb{R}^m)$, $x \in \overline{\Omega}$ and $\alpha \geq 0$

$$\begin{aligned} &f(x, u(x) + \alpha h(x), \nabla u(x) + \alpha \nabla h(x)) - f(x, u(x), \nabla u(x)) \\ &\quad - \Phi_f(x, u(x), \nabla u(x); \alpha h(x), \alpha \nabla h(x)) - \Psi_f(x, u(x), \nabla u(x); \alpha h(x), \alpha \nabla h(x)) = \alpha \varepsilon_f(x, \alpha), \end{aligned}$$

where $\varepsilon_f(x, \alpha) \to 0$ as $\alpha \downarrow 0$ uniformly with respect to $x \in \overline{\Omega}$.

Let us adduce some propositions that help to check whether a function is codifferentiable with respect to u and ξ uniformly with respect to $C^1(\overline{\Omega}, \mathbb{R}^m)$. The following proposition immediately follows from the well-known properties of a continuously differentiable function.

Proposition 4.9. *Let a function* $f\colon \overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}$, $f = f(x,u,\xi)$, *be continuous and continuously differentiable with respect to* u_i *and* ξ_{ij} *on* $\overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$, $i \in \{1,\dots,m\}$, $j \in \{1,\dots,d\}$. *Then the function* f *is continuously codifferentiable with respect to* u *and* ξ *on* $\overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$ *uniformly with respect to* $C^1(\overline{\Omega}, \mathbb{R}^m)$.

Applying assertions about computing a codifferential (*cf.* [11], Lems. 4.2.1–4.2.4) and Proposition 4.4, it is easy to prove the following propositions.

Proposition 4.10. *Let functions* $f_1, f_2\colon \overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}$, $f_i = f_i(x,u,\xi)$, *be codifferentiable with respect to* u *and* ξ *on* $\overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$ *uniformly with respect to* $C^1(\overline{\Omega}, \mathbb{R}^m)$, *and let* $c_1, c_2 \in \mathbb{R}$ *be given real numbers. Then the function* $c_1 f_1 + c_2 f_2$ *is codifferentiable with respect to* u *and* ξ *on* $\overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$ *uniformly with respect to* $C^1(\overline{\Omega}, \mathbb{R}^m)$.

Proposition 4.11. *Let functions* $f_1, f_2\colon \overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}$, $f_i = f_i(x,u,\xi)$, *be codifferentiable with respect to* u *and* ξ *on* $\overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$ *uniformly with respect to* $C^1(\overline{\Omega}, \mathbb{R}^m)$. *Then the function* $f_1 \cdot f_2$ *is codifferentiable with respect to* u *and* ξ *on* $\overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$ *uniformly with respect to* $C^1(\overline{\Omega}, \mathbb{R}^m)$.

Proposition 4.12. *Let functions* $f_i\colon \overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}$, $f_i = f_i(x,u,\xi)$, $i \in I = \{1,\dots,k\}$, *be codifferentiable with respect to* u *and* ξ *on* $\overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$ *uniformly with respect to* $C^1(\overline{\Omega}, \mathbb{R}^m)$. *Then the functions* $g_1 = \max_{i\in I} f_i$ *and* $g_2 = \min_{i\in I} f_i$ *are codifferentiable with respect to* u *and* ξ *on* $\overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$ *uniformly with respect to* $C^1(\overline{\Omega}, \mathbb{R}^m)$.

Proposition 4.13. *Let function* $f\colon \overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}$, $f = f(x,u,\xi)$, *be continuous and codifferentiable with respect to* u *and* ξ *on* $\overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$ *uniformly with respect to* $C^1(\overline{\Omega}, \mathbb{R}^m)$, *and suppose that* $f(x,u,\xi) \neq 0$ *for all* $x \in \overline{\Omega}$, $u \in \mathbb{R}^m$ *and* $\xi \in \mathbb{R}^{m\times d}$. *Then the function* $g = 1/f$ *is codifferentiable with respect to* u *and* ξ *on* $\overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$ *uniformly with respect to* $C^1(\overline{\Omega}, \mathbb{R}^m)$.

For instance, let us prove Proposition 4.12.

Proof. We consider only the function g_1. Fix arbitrary $u, h \in C^1(\overline{\Omega}, \mathbb{R}^m)$. For all $\alpha \geq 0$ and $x \in \overline{\Omega}$ one has

$$
\begin{aligned}
&g_1(x, u(x) + \alpha h(x), \nabla u(x) + \alpha \nabla h(x)) - g_1(x, u(x), \nabla u(x)) \\
&= \max_{i\in I} \Big(f_i(x, u(x), \nabla u(x)) + \Phi_{f_i}(x, u(x), \nabla u(x); \alpha h(x), \alpha \nabla h(x)) \\
&\quad + \Psi_{f_i}(x, u(x), \nabla u(x); \alpha h(x), \alpha \nabla h(x)) + \alpha \varepsilon_{f_i}(x,\alpha) \Big) - g_1(x, u(x), \nabla u(x)) \\
&= \max_{i\in I} \Big(f_i(x, u(x), \nabla u(x)) - g_1(x, u(x), \nabla u(x)) + \Phi_{f_i}(x, u(x), \nabla u(x); \alpha h(x), \alpha \nabla h(x)) \\
&\quad + \Psi_{f_i}(x, u(x), \nabla u(x); \alpha h(x), \alpha \nabla h(x)) \Big) + \alpha \varepsilon_{g_1}(x,\alpha),
\end{aligned}
$$

where

$$
\min_{i\in I} \varepsilon_{f_i}(x,\alpha) \leq \varepsilon_{g_1}(x,\alpha) \leq \max_{i\in I} \varepsilon_{f_i}(x,\alpha).
$$

It is clear that $\varepsilon_{g_1}(x,\alpha) \to 0$ as $\alpha \downarrow 0$ uniformly with respect to $x \in \overline{\Omega}$. It remains to note that

$$
\begin{aligned}
&\max_{i\in I} \Big(f_i(x, u(x), \nabla u(x)) - g_1(x, u(x), \nabla u(x)) \\
&\quad + \Phi_{f_i}(x, u(x), \nabla u(x); \alpha h(x), \alpha \nabla h(x)) + \Psi_{f_i}(x, u(x), \nabla u(x); \alpha h(x), \alpha \nabla h(x)) \Big) \\
&= \Phi_{g_1}(x, u(x), \nabla u(x); \alpha h(x), \alpha \nabla h(x)) + \Psi_{g_1}(x, u(x), \nabla u(x); \alpha h(x), \alpha \nabla h(x))
\end{aligned}
$$

(*cf.* for details [11], the proof of Lem. 4.2.4). □

Fix an arbitrary $1 \leq p \leq \infty$. In order to give a correct definition of codifferentiation with respect to u and ξ uniformly with respect to the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^m)$, we need the following definition.

Definition 4.14. Let a function $f\colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \to \mathbb{R}$, $f = f(x, u, \xi)$, be codifferentiable with respect to u and ξ on $\Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$. A codifferential $D_{u,\xi} f(x, u, \xi)$ of the function f with respect to u and ξ is said to satisfy the growth condition if there exists a codifferential mapping $(x, u, \xi) \to D_{u,\xi} f(x, u, \xi)$, $(x, u, \xi) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$, such that there exist almost everywhere nonnegative functions $\beta, \gamma \in L_1(\Omega)$, $\beta_1, \beta_2, \gamma_1, \gamma_2 \in L_q(\Omega)$ and nonnegative real numbers $C, C_1, C_2, D, D_1, D_2 \in \mathbb{R}$ such that for almost every $x \in \Omega$ and for all $u \in \mathbb{R}^m$, $\xi \in \mathbb{R}^{m \times d}$

$$|a| \leq \beta(x) + C(|u|^p + |\xi|^p), \quad |v_1| \leq \beta_1(x) + C_1(|u|^{p-1} + |\xi|^{p-1}), \quad |v_2| \leq \beta_2(x) + C_2 \left(|u|^{p-1} + |\xi|^{p-1}\right)$$

for all $[a, v_1, v_2] \in \underline{d}_{u,\xi} f(x, u, \xi)$ and

$$|b| \leq \gamma(x) + D(|u|^p + |\xi|^p), \quad |w_1| \leq \gamma_1(x) + D_1(|u|^{p-1} + |\xi|^{p-1}), \quad |w_2| \leq \gamma_2(x) + D_2 \left(|u|^{p-1} + |\xi|^{p-1}\right)$$

for all $[b, w_1, w_2] \in \overline{d}_{u,\xi} f(x, u, \xi)$ in the case $1 \leq p < \infty$, and for all $N \in \mathbb{N}$ there exist almost everywhere nonnegative functions $\beta^{(N)}$, $\beta_1^{(N)}$, $\beta_2^{(N)}$, $\gamma^{(N)}$, $\gamma_1^{(N)}$, $\gamma_2^{(N)} \in L_1(\Omega)$ such that for almost every $x \in \Omega$ and for all $(u, \xi) \in B_N$

$$|a| \leq \beta^{(N)}(x), \quad |v_1| \leq \beta_1^{(N)}(x), \quad |v_2| \leq \beta_2^{(N)}(x) \quad \forall [a, v_1, v_2] \in \underline{d}_{u,\xi} f(x, u, \xi),$$
$$|b| \leq \gamma^{(N)}(x), \quad |w_1| \leq \gamma_1^{(N)}(x), \quad |w_2| \leq \gamma_2^{(N)}(x) \quad \forall [b, w_1, w_2] \in \overline{d}_{u,\xi} f(x, u, \xi),$$

in the case $p = \infty$.

Remark 4.15. With the use of the Sobolev embedding theorem, the inequalities in the previous definition can be improved under some additional assumptions on the set Ω (*cf.*, for instance, [8], Sect. 3.4.2).

The next proposition follows directly from the formulae for computing a codifferential.

Proposition 4.16. *Let $1 \leq p \leq \infty$, functions $f_i\colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \to \mathbb{R}$, $f_i = f_i(x, u, \xi)$, be codifferentiable with respect to u and ξ on $\Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$ and satisfy the growth condition, $i \in I = \{1, \ldots, k\}$. Suppose that codifferentials of the functions f_i with respect to u and ξ satisfy the growth condition, $i \in I$. Then for any real numbers $c_i \in \mathbb{R}$, $i \in I$, the functions $\sum_{i=1}^k c_i f_i$, $\max_{i \in I} f_i$ and $\min_{i \in I} f_i$ are codifferentiable with respect to u and ξ on $\Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$, satisfy the growth condition and their codifferentials with respect to u and ξ also satisfy the growth condition.*

Note an obvious property of a codifferentiable function, that has a codifferential satisfying the growth condition.

Proposition 4.17. *Let $1 \leq p \leq \infty$, a function $f\colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \to \mathbb{R}$, $f = f(x, u, \xi)$, be codifferentiable with respect to u and ξ on $\Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$, and suppose that a codifferential of f with respect to u and ξ satisfy the growth condition. Then for any $u \in W^{1,p}(\Omega, \mathbb{R}^m)$ all measurable selections of the set–valued mapping $x \to \underline{d}_{u,\xi} f(x, u(x), \nabla u(x))$ (or $x \to \overline{d}_{u,\xi} f(x, u(x), \nabla u(x))$) belong to the space $L_1(\Omega) \times L_q(\Omega, \mathbb{R}^m) \times L_q(\Omega, \mathbb{R}^{m \times d})$.*

Definition 4.18. Let a function $f\colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \to \mathbb{R}$, $f = f(x, u, \xi)$, satisfy the Caratheodory condition and the growth condition. The function f is said to be codifferentiable with respect to u and ξ on $\Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$ uniformly with respect to $W^{1,p}(\Omega, \mathbb{R}^m)$, if

1. f is codifferentiable with respect to u and ξ on $\Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$,
2. the set–valued maps $(x, u, \xi) \to \underline{d}_{u,\xi} f(x, u, \xi)$ and $(x, u, \xi) \to \overline{d}_{u,\xi} f(x, u, \xi)$, $x \in \Omega$, $u \in \mathbb{R}^m$, $\xi \in \mathbb{R}^{m \times d}$, satisfy the Caratheodory condition,

3. a codifferential of the function f with respect to u and ξ satisfies the growth condition,
4. for all $u, h \in W^{1,p}(\Omega, \mathbb{R}^m)$, $\alpha \geq 0$ and for a.e. $x \in \Omega$

$$\begin{aligned} &f(x, u(x) + \alpha h(x), \nabla u(x) + \alpha \nabla h(x)) - f(x, u(x), \nabla u(x)) \\ &- \Phi_f(x, u(x), \nabla u(x); \alpha h(x), \alpha \nabla h(x)) \\ &- \Psi_f(x, u(x), \nabla u(x); \alpha h(x), \alpha \nabla h(x)) = \alpha \varepsilon_f(x, \alpha), \end{aligned}$$

where $\int_\Omega |\varepsilon_f(x, \alpha)| \, \mathrm{d}x \to 0$ as $\alpha \downarrow 0$.

Remark 4.19. Definition 4.18 is correct in the sense that under our assumptions $\varepsilon(\cdot, \alpha) \in L_1(\Omega)$ for all $\alpha \geq 0$. Indeed, since the function f satisfies the Caratheodory condition, the growth condition and $u, h \in W^{1,p}(\Omega, \mathbb{R}^m)$, then $f(\cdot, u(\cdot) + \alpha h(\cdot), \nabla u(\cdot) + \alpha \nabla h(\cdot)) \in L_1(\Omega)$ for all $\alpha \geq 0$.

The set–valued mapping $(x, u, \xi) \to \underline{d}_{u,\xi} f(x, u, \xi)$ satisfies the Caratheodory condition, hence the set–valued mapping $x \to \underline{d}_{u,\xi} f(x, u(x), \nabla u(x))$ is measurable (*cf.* [2], Thm. 8.2.8). It is clear, that the mapping $(x, a, v_1, v_2) \to a + \langle v_1, \alpha h(x) \rangle + \langle v_2, \alpha \nabla h(x) \rangle$, $x \in \Omega$, $[a, v_1, v_2] \in \mathbb{R}^{1+m+m\times d}$, satisfies the Caratheodory condition, therefore the set–valued mapping

$$x \to \{a + \langle v_1, \alpha h(x) \rangle + \langle v_2, \alpha \nabla h(x) \rangle \mid [a, v_1, v_2] \in \underline{d}_{u,\xi} f(x, u(x), \nabla u(x))\}$$

is measurable (*cf.* [2], Thm. 8.2.8). Then it is easy to check, that the mapping

$$x \to \Phi_f(x, u(x), \nabla u(x); \alpha h(x), \alpha \nabla h(x)) = \max_{[a, v_1, v_2] \in \underline{d}_{u,\xi} f(x, u(x), \nabla u(x))} (a + \langle v_1, \alpha h(x) \rangle + \langle v_2, \alpha \nabla h(x) \rangle)$$

is measurable. Applying the fact that a codifferential of f satisfies the growth condition, it is easy to verify that $\Phi_f(\cdot, u(\cdot), \nabla u(\cdot); \alpha h(\cdot), \alpha \nabla h(\cdot)) \in L_1(\Omega)$. Arguing in the same way, one can find that the function $\Psi_f(\cdot, u(\cdot), \nabla u(\cdot); \alpha h(\cdot), \alpha \nabla h(\cdot)) \in L_1(\Omega)$. Thus, $\varepsilon(\cdot, \alpha) \in L_1(\Omega)$.

Let us consider several assertions, that help to verify whether a function is codifferentiable with respect to u and ξ uniformly with respect to $W^{1,p}(\Omega, \mathbb{R}^m)$.

Proposition 4.20. *Let $1 \leq p \leq \infty$, a function $f \colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}$ satisfy the Caratheodory condition and the growth condition, and suppose that there exist all partial derivatives $\frac{\partial f}{\partial u_i}$ and $\frac{\partial f}{\partial \xi_{ij}}$ satisfying the Caratheodory condition on $\Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$, $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, d\}$. Suppose also that there exist a.e. nonnegative functions $\beta_i, \gamma_{ij} \in L_q(\Omega)$ and real numbers $C_i \geq 0$ and $D_{ij} \geq 0$, $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, d\}$, such that for a.e. $x \in \Omega$ and for all $u \in \mathbb{R}^m$, $\xi \in \mathbb{R}^{m\times d}$*

$$\left| \frac{\partial f}{\partial u_i}(x, u, \xi) \right| \leq \beta_i(x) + C_i(|u|^{p-1} + |\xi|^{p-1}) \quad \forall i \in \{1, \ldots, m\},$$

$$\left| \frac{\partial f}{\partial \xi_{ij}}(x, u, \xi) \right| \leq \gamma_{ij}(x) + D_{ij}(|u|^{p-1} + |\xi|^{p-1}) \quad \forall i \in \{1, \ldots, m\}, \;\; j \in \{1, \ldots, d\}$$

in the case $1 \leq p < \infty$, and for any $N \in \mathbb{N}$ there exist a.e. nonnegative functions $\beta_i^{(N)}, \gamma_{ij}^{(N)} \in L_1(\Omega)$, $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, d\}$, such that for a.e. $x \in \Omega$ and for all $(u, \xi) \in B_N$, $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, d\}$

$$\left| \frac{\partial f}{\partial u_i}(x, u, \xi) \right| \leq \beta_i^{(N)}(x), \quad \left| \frac{\partial f}{\partial \xi_{ij}}(x, u, \xi) \right| \leq \gamma_{ij}^{(N)}(x)$$

in the case $p = \infty$. Then the function f is codifferentiable with respect to u and ξ on $\Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$ uniformly with respect to $W^{1,p}(\Omega, \mathbb{R}^m)$.

Proof. *cf.* the proof of Theorem 3.37 in [8]. □

It is easy to check that the following proposition holds true.

Proposition 4.21. *Let $1 \le p \le \infty$, functions $f_i \colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}$, $f_i = f_i(x,u,\xi)$, $i \in I = \{1,\dots,k\}$, satisfy the Caratheodory condition and the growth condition, and suppose that the functions f_i are codifferentiable with respect to u and ξ on $\Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$ uniformly with respect to $W^{1,p}(\Omega,\mathbb{R}^m)$. Then for any numbers $c_i \in \mathbb{R}$, $i \in I$, the functions $\sum_{i=1}^k c_i f_i$, $\max_{i\in I} f_i$ and $\min_{i\in I} f_i$ are codifferentiable with respect to u and ξ on $\Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$ uniformly with respect to $W^{1,p}(\Omega,\mathbb{R}^m)$.*

Remark 4.22. There are two interesting open questions. Let a function $f \colon \overline{\Omega}\times\mathbb{R}^m\times\mathbb{R}^{m\times d} \to \mathbb{R}$ be continuously codifferentiable with respect to u and ξ on its domain. Is f codifferentiable with respect to u and ξ uniformly with respect to $C^1(\overline{\Omega},\mathbb{R}^m)$? Let a function $g \colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}$ satisfy all conditions of Definition 4.18 except the last one. Is g codifferentiable with respect to u and ξ uniformly with respect to $W^{1,p}(\Omega,\mathbb{R}^m)$?

5. Codifferentiability of the Main functionals

Let us study the functionals $\mathcal{I}_C$ and $\mathcal{I}_W$ in the case, when the integrands f and g are codifferentiable with respect to u and ξ on their domain.

For the proof of a codifferentiability of the functionals $\mathcal{I}_C$ and $\mathcal{I}_W$, we need the theorem that the space $C^1(\overline{\Omega},\mathbb{R}^m)$ is dense in the Sobolev space. This theorem holds true only if the set Ω has the segment property (*cf.* [1], Sect. 3.17), *i.e.* if for every $x \in \operatorname{bd}\Omega$ there exist an open set $U_x \subset \mathbb{R}^d$ and a nonzero vector $y_x \in \mathbb{R}^d$ such that $x \in U_x$ and if $z \in \overline{\Omega} \cap U_x$, then $z + ty_x \in \Omega$ for $t \in (0,1)$. If the set Ω has this property then it must have $(n-1)$-dimensional boundary and cannot simultaneously lie on both sides of any given part of its boundary.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^d$ be an open bounded set having the segment property, $1 \le p \le \infty$, and let a function $g \colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}$, $g = g(x,u,\xi)$, satisfy the Caratheodory condition and the growth condition. Suppose that g is codifferentiable with respect to u and ξ on $\Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$ uniformly with respect to $W^{1,p}(\Omega,\mathbb{R}^m)$. Suppose also that in the case $p = \infty$ for any $N \in \mathbb{N}$ there exists $C^{(N)} > 0$ such that for a.e. $x \in \Omega$ and for all $(u,\xi) \in B_N$*

$$|a| \le C^{(N)}, |v_1| \le C^{(N)}, |v_2| \le C^{(N)} \quad \forall [a,v_1,v_2] \in \underline{d}_{u,\xi} g(x,u,\xi),$$
$$|b| \le C^{(N)}, |w_1| \le C^{(N)}, |w_2| \le C^{(N)} \quad \forall [b,w_1,w_2] \in \overline{d}_{u,\xi} g(x,u,\xi)$$

(*in particular, one can suppose that the function g is continuously codifferentiable on its domain*). *Then the functional*

$$\mathcal{I}_W(u) = \int_\Omega g(x,u(x),\nabla u(x))\,\mathrm{d}x,$$

defined on the space $W^{1,p}(\Omega,\mathbb{R}^m)$, is codifferentiable on its domain, and there is a codifferential of the functional $\mathcal{I}_W$ at a point $u \in W^{1,p}(\Omega,\mathbb{R}^m)$ of the form

$$\underline{d}\mathcal{I}_W(u) = \Big\{ [A,\varphi] \in \mathbb{R} \times (W^{1,p}(\Omega,\mathbb{R}^m))^* \;\Big|\; A = \int_\Omega a(x)\,\mathrm{d}x,$$
$$\varphi(h) = \int_\Omega (\langle v_1(x), h(x)\rangle + \langle v_2(x), \nabla h(x)\rangle)\,\mathrm{d}x \quad \forall h \in W^{1,p}(\Omega,\mathbb{R}^m),$$
$$[a(\cdot),v_1(\cdot),v_2(\cdot)] \textit{ is a measurable selection of the map } x \to \underline{d}_{u,\xi} g(x,u(x),\nabla u(x)) \Big\}$$

and

$$\overline{d}\mathcal{I}_W(u) = \Big\{ [B,\psi] \in \mathbb{R} \times (W^{1,p}(\Omega,\mathbb{R}^m))^* \;\Big|\; B = \int_\Omega b(x)\,\mathrm{d}x,$$
$$\psi(h) = \int_\Omega (\langle w_1(x), h(x)\rangle + \langle w_2(x), \nabla h(x)\rangle)\,\mathrm{d}x \quad \forall h \in W^{1,p}(\Omega,\mathbb{R}^m),$$
$$[b(\cdot),w_1(\cdot),w_2(\cdot)] \textit{ is a measurable selection of the map } x \to \overline{d}_{u,\xi} g(x,u(x),\nabla u(x)) \Big\}$$

We divide the proof of Theorem 5.1 into several lemmas.

Lemma 5.2. *Let $\Omega \subset \mathbb{R}^d$ be an open bounded set, $1 \le p \le \infty$, and let a function $g \colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \to \mathbb{R}$, $g = g(x, u, \xi)$ satisfy the assumptions of Theorem 5.1. Then for all $u, h, \in W^{1,p}(\Omega, \mathbb{R}^m)$ and $\alpha \ge 0$*

$$\mathcal{I}_W(u + \alpha h) - \mathcal{I}_W(u) = \max_{[A,\varphi] \in \underline{d}\mathcal{I}_W(u)} (A + \varphi(\alpha h)) + \min_{[B,\psi] \in \overline{d}\mathcal{I}_W(u)} (B + \psi(\alpha h)) + o(\alpha),$$

where $o(\alpha)/\alpha \to 0$ as $\alpha \downarrow 0$ and the sets $\underline{d}\mathcal{I}_W(u)$ and $\underline{d}\mathcal{I}_W(u)$ are defined in Theorem 5.1.

Proof. Since the function g is codifferentiable with respect to u and ξ uniformly with respect to $W^{1,p}(\Omega, \mathbb{R}^m)$, then for all $u, h \in W^{1,p}(\Omega, \mathbb{R}^m)$ and $\alpha \ge 0$

$$\mathcal{I}_W(u + \alpha h) - \mathcal{I}_W(u) = \int_\Omega \Phi_g(x, u(x), \nabla u(x); \alpha h(x), \alpha \nabla h(x)) \, \mathrm{d}x + \int_\Omega \Psi_g(x, u(x), \nabla u(x); \alpha h(x), \alpha \nabla h(x)) \, \mathrm{d}x + o(\alpha),$$

where $o(\alpha)/\alpha \to 0$ as $\alpha \downarrow 0$. Note that by virtue of Remark 4.19 one has that $\Phi_g(\cdot, u(\cdot), \nabla u(\cdot); \alpha h(\cdot), \alpha \nabla h(\cdot)) \in L_1(\Omega)$ and $\Psi_g(\cdot, u(\cdot), \nabla u(\cdot); \alpha h(\cdot), \alpha \nabla h(\cdot)) \in L_1(\Omega)$.

Let $[a(\cdot), v_1(\cdot), v_2(\cdot)]$ be an arbitrary measurable selection of the set-valued mapping $x \to \underline{d}_{u,\xi} g(x, u(x), \nabla u(x))$. Since a codifferential of the function g with respect to u and ξ satisfies the growth condition with index p, then according to Proposition 4.17 one has that $a(\cdot) \in L_1(\Omega)$, $v_1(\cdot) \in L_q(\Omega, \mathbb{R}^m)$ and $v_2(\cdot) \in L_q(\Omega, \mathbb{R}^{m \times d})$. It is clear that for all $\alpha \ge 0$ and for a.e. $x \in \Omega$

$$\Phi_g(x, u(x), \nabla u(x); \alpha h(x), \alpha \nabla h(x)) \ge a(x) + \langle v_1(x), \alpha h(x) \rangle + \langle v_2(x), \alpha \nabla h(x) \rangle$$

Since for a.e. $x \in \Omega$

$$\Phi_g(x, u(x), \nabla u(x); \alpha h(x), \alpha \nabla h(x)) \in \{ a + \langle v_1, \alpha h(x) \rangle + \langle v_2, \alpha \nabla h(x) \rangle \mid [a, v_1, v_2] \in \underline{d}_{u,\xi} g(x, u(x), \nabla u(x)) \},$$

then by the well-known Filippov theorem (*cf.*, for example, [2], Thm. 8.2.10) there exists a measurable selection $[a(\cdot), v_1(\cdot), v_2(\cdot)]$ of the set-valued mapping $x \to \underline{d}_{u,\xi} g(x, u(x), \nabla u(x))$ such that for a.e. $x \in \Omega$

$$\Phi_g(x, u(x), \nabla u(x); \alpha h(x), \alpha \nabla h(x)) = a(x) + \langle v_1(x), \alpha h(x) \rangle + \langle v_2(x), \alpha \nabla h(x) \rangle$$

Thus

$$\int_\Omega \Phi_g(x, u(x), \nabla u(x); \alpha h(x), \alpha \nabla h(x)) \, \mathrm{d}x = \max \int_\Omega (a(x) + \langle v_1(x), \alpha h(x) \rangle + \langle v_2(x), \alpha \nabla h(x) \rangle) \, \mathrm{d}x.$$

Here, the maximum on the right-hand side is taken over all measurable selections $[a(\cdot), v_1(\cdot), v_2(\cdot)]$ of the set-valued mapping $x \to \underline{d}_{u,\xi} g(x, u(x), \nabla u(x))$. Taking into account the form of the set $\underline{d}\mathcal{I}_W(u)$ one has that

$$\int_\Omega \Phi_g(x, u(x), \nabla u(x); \alpha h(x), \alpha \nabla h(x)) \, \mathrm{d}x = \max_{[A,\varphi] \in \underline{d}\mathcal{I}_W(u)} (A + \varphi(\alpha h)).$$

The rest of the proof is obvious. □

Remark 5.3. It is easy to see that in the previous lemma

$$\max_{[A,\varphi] \in \underline{d}\mathcal{I}_W(u)} A = \int_\Omega \Phi_g(x, u(x), \nabla u(x); 0, 0) \, \mathrm{d}x = \int_\Omega \Psi_g(x, u(x), \nabla u(x); 0, 0) \, \mathrm{d}x = \min_{[B,\psi] \in \overline{d}\mathcal{I}_W(u)} B.$$

Therefore $\max_{[A,\varphi] \in \underline{d}\mathcal{I}_W(u)} A = \min_{[B,\psi] \in \overline{d}\mathcal{I}_W(u)} B = 0$.

Lemma 5.4. *Let $\Omega \subset \mathbb{R}^d$ be an open bounded set, $1 \le p \le \infty$, and let a function g satisfy the assumptions of Theorem 5.1. Then the sets $\underline{d}\mathcal{I}_W(u)$ and $\overline{d}\mathcal{I}_W(u)$ are convex and bounded.*

Proof. We will consider only the set $\underline{d}\mathcal{I}_W(u)$. The convexity of the set $\underline{d}\mathcal{I}_W(u)$ follows directly from the convexity of a hypodifferential with respect to u and ξ.

Suppose that $1 \le p < \infty$. With the use of the Hölder inequality, it is easy to show that for all $[A, \varphi] \in \underline{d}\mathcal{I}_W(u)$

$$|A| \le \|\beta\|_1 + C\left((\|u\|_p)^p + (\|\nabla u\|_p)^p\right),$$
$$|\varphi(h)| \le \left(\|\beta_1\|_q + C_1(\|u\|_p)^{\frac{p}{q}} + C_1(\|\nabla u\|_p)^{\frac{p}{q}}\right)\|h\|_p + \left(\|\beta_2\|_q + C_2(\|u\|_p)^{\frac{p}{q}} + C_2(\|\nabla u\|_p)^{\frac{p}{q}}\right)\|\nabla h\|_p,$$

where $\beta, \beta_1, \beta_2, C, C_1$ and C_2 are from the definition of the codifferential's growth condition and $h \in W^{1,p}(\Omega, \mathbb{R}^m)$ is arbitrary. Thus, the set $\underline{d}\mathcal{I}_W(u)$ is bounded.

Consider the case $p = \infty$. Fix an arbitrary $u \in W^{1,\infty}(\Omega, \mathbb{R}^m)$. It is clear that there exists $N \in \mathbb{N}$ such that $|u(x)| + |\nabla u(x)| \le N$ for a.e. $x \in \Omega$. Therefore, applying the fact that a codifferential of the function g with respect to u and ξ satisfies the growth condition one has that there exist $\beta^{(N)}, \beta_1^{(N)}, \beta_2^{(N)} \in L_1(\Omega)$ such that for a.e. $x \in \Omega$

$$|a| \le \beta^{(N)}(x), |v_1| \le \beta_1^{(N)}(x), |v_2| \le \beta_2^{(N)}(x) \quad \forall [a, v_1, v_2] \in \underline{d}_{u,\xi} g(x, u(x), \nabla u(x)).$$

Thus for any $[A, \varphi] \in \underline{d}\mathcal{I}_W(u)$ and for all $h \in W^{1,\infty}(\Omega, \mathbb{R}^m)$

$$|A| \le \|\beta^{(N)}\|_1, |\varphi(h)| \le \|\beta_1^{(N)}\|_1\|h\|_\infty + \|\beta_2^{(N)}\|_1\|\nabla h\|_\infty.$$

Hence, the set $\underline{d}\mathcal{I}_W(u)$ is bounded in the case $p = \infty$. □

Since an arbitrary set $A \subset \mathbb{R} \times E^*$ is compact in the topology $\tau \times w^*$ iff it is bounded and closed in the topology $\tau \times w^*$ ([13], Thm. 2.1), then in order to prove Theorem 5.1 it remains to show that the sets $\underline{d}\mathcal{I}_W(u)$ and $\underline{d}\mathcal{I}_W(u)$ are closed in the topology $\tau \times w^*$.

For the proof of the closedness of the sets $\underline{d}\mathcal{I}_W(u)$ and $\overline{d}\mathcal{I}_W(u)$ we need a simple auxiliary assertion about Bochner integral. For the sake of completeness we will give a brief proof of it (see [27], Chap. 5 for the definition and detailed study of Bochner integral).

Lemma 5.5. *Let $(X, \mathcal{A}, \mu)$ be a complete σ–finite measure space, and let a mapping $\eta = (\eta_1, \eta_2, \ldots, \eta_n, \ldots)\colon X \to \ell_2$ be measurable. Suppose that $\|\eta\|_{\ell_2} \in L_1(X, \mathcal{A}, \mu)$ and $\zeta = (\int_X \eta_1\, d\mu, \int_X \eta_2\, \mathrm{d}\mu, \ldots, \int_X \eta_n\, \mathrm{d}\mu, \ldots) \in \ell_2$. Then the function η is Bochner integrable and $\int_X \eta\, \mathrm{d}\mu = \zeta$.*

Proof. If the function η is simple, then the proof is obvious. Let η be an arbitrary function satisfying the assumptions of the lemma. Since η is measurable and $\|\eta\|_{\ell_2} \in L_1(X, \mathcal{A}, \mu)$, then function η is Bochner integrable and for any $n \in \mathbb{N}$ the function $\eta_n \in L_1(X, \mathcal{A}, \mu)$ (*cf.*, for instance, [27], Chap. 5, Sect. 5). It remains to prove that $\int_X \eta\, \mathrm{d}\mu = \zeta$. Fix an arbitrary sequence of simple functions $\eta^{(k)}\colon X \to \ell_2$ such that $\eta^{(k)}$ converges to η almost everywhere and

$$\lim_{k\to\infty} \int_X \|\eta^{(k)} - \eta\|_{\ell_2} \mathrm{d}\mu = 0. \tag{5.8}$$

Then, by definition $\int_X \eta\, \mathrm{d}\mu = \lim_{k\to\infty} \int_\Omega \eta^{(k)}\, \mathrm{d}\mu$. Therefore it is sufficient to prove that $\lim_{k\to\infty} \int_X \eta^{(k)}\, \mathrm{d}\mu = \zeta$.

Fix an arbitrary $\varepsilon > 0$. Since $\zeta \in \ell_2$ and $\int_X \eta\, \mathrm{d}\mu \in \ell_2$, then there exists $m \in \mathbb{N}$ for which

$$\left(\sum_{n=m+1}^{\infty} |\zeta_n|^2\right)^{\frac{1}{2}} = \left(\sum_{n=m+1}^{\infty} \left|\int_X \eta_n\, \mathrm{d}\mu\right|^2\right)^{\frac{1}{2}} < \varepsilon, \quad \left(\sum_{n=m+1}^{\infty} \left|\left(\int_X \eta\, \mathrm{d}\mu\right)_n\right|^2\right)^{\frac{1}{2}} < \frac{\varepsilon}{2}\cdot \tag{5.9}$$

Here $\left(\int_X \eta\, \mathrm{d}\mu\right)_n$ is the n–th term of the sequence $\int_X \eta\, \mathrm{d}\mu$.

From (5.8) it follows, that there exists $k_1 \in \mathbb{N}$ such that for all $k \geq k_1$

$$\left| \left(\sum_{n=m+1}^{\infty} \left| \int_X \eta_n^{(k)} \, \mathrm{d}\mu \right|^2 \right)^{\frac{1}{2}} - \left(\sum_{n=m+1}^{\infty} \left| \left(\int_X \eta \, \mathrm{d}\mu \right)_n \right|^2 \right)^{\frac{1}{2}} \right|$$
$$\leq \left(\sum_{n=m+1}^{\infty} \left| \int_X \eta_n^{(k)} \, \mathrm{d}\mu - \left(\int_X \eta \, \mathrm{d}\mu \right)_n \right|^2 \right)^{\frac{1}{2}} \leq \left\| \int_X \eta^{(k)} \, \mathrm{d}\mu - \int_X \eta \mathrm{d}\mu \right\|_{\ell_2} < \frac{\varepsilon}{2}. \quad (5.10)$$

Here, we used the equality $\left(\int_X \eta^{(k)} \, d\mu\right)_n = \int_X \eta_n^{(k)} \, \mathrm{d}\mu$ that follows from the fact that the lemma holds true for simple functions. Hence (5.9) and (5.10) imply that for all $k \geq k_1$

$$\left(\sum_{n=m+1}^{\infty} \left| \int_X \eta_n^{(k)} \, \mathrm{d}\mu \right|^2 \right)^{\frac{1}{2}} < \varepsilon. \quad (5.11)$$

It is clear, that for all $n \in \mathbb{N}$ the sequence $\eta_n^{(k)}$ converges to η_n almost everywhere and

$$\int_X |\eta_n^{(k)} - \eta_n| \, \mathrm{d}\mu \leq \int_X \|\eta^{(k)} - \eta\|_{\ell_2} \mathrm{d}\mu \to 0 \text{ as } k \to \infty.$$

Therefore there exists $k_2 \in \mathbb{N}$ such that for all $k \geq k_2$

$$\left(\sum_{n=1}^{m} \left| \int_X |\eta_n^{(k)} - \eta_n| \, \mathrm{d}\mu \right|^2 \right)^{\frac{1}{2}} < \varepsilon.$$

Hence applying (5.9) and (5.11) one has that for all $k \geq \max\{k_1, k_2\}$

$$\left\| \int_X \eta^{(k)} \, \mathrm{d}\mu - \zeta \right\|_{\ell_2} = \left(\sum_{n=1}^{\infty} \left| \int_X \eta_n^{(k)} \, \mathrm{d}\mu - \int_X \eta_n \, \mathrm{d}\mu \right|^2 \right)^{\frac{1}{2}}$$
$$\leq \left(\sum_{n=1}^{m} \left| \int_X |\eta_n^{(k)} - \eta_n| \, \mathrm{d}\mu \right|^2 \right)^{\frac{1}{2}} + \left(\sum_{n=m+1}^{\infty} \left| \int_X \eta_n^{(k)} \, \mathrm{d}\mu \right|^2 \right)^{\frac{1}{2}} + \left(\sum_{n=m+1}^{\infty} \left| \int_X \eta_n \, \mathrm{d}\mu \right|^2 \right)^{\frac{1}{2}} < 3\varepsilon,$$

that completes the proof. □

Lemma 5.6. *Let $1 \leq p \leq \infty$, a set Ω and a function g satisfy the assumptions of Theorem 5.1. Then for all $u \in W^{1,p}(\Omega, \mathbb{R}^m)$ the sets $\underline{d}\mathcal{I}_W(u)$ and $\overline{d}\mathcal{I}_W(u)$ are closed in the topology $\tau \times w^*$.*

Proof. We will prove the assertion for $\underline{d}\mathcal{I}_W(u)$ since the assertion for $\overline{d}\mathcal{I}_W(u)$ is proved in a similar way.

Consider the case $1 \leq p < \infty$. Our aim is to construct a set-valued mapping such that the limit points of the set $\underline{d}\mathcal{I}_W(u)$ are closely related, in some sense, to the limit points of the Aumann integral of this set-valued mapping. Then applying the closedness of the Aumann integral we will get the required result (see [2], Sect. 8.6 for the definition and basic properties of the Aumann integral of a set-valued mapping).

The space $W^{1,p}(\Omega, \mathbb{R}^m)$ is separable ([1], Thm. 3.5). Let $\{y_n\}_{n=1}^{\infty} \subset W^{1,p}(\Omega, \mathbb{R}^m)$ be a countable dense subset. The space $C^1(\overline{\Omega}, \mathbb{R}^m)$ is dense in $W^{1,p}(\Omega, \mathbb{R}^m)$ ([1], Thm. 3.18), therefore for all $n, k \in \mathbb{N}$ there exists $z_{nk} \in C^1(\overline{\Omega}, \mathbb{R}^m)$ such that $\|z_{nk} - y_n\|_{1,p} < \frac{1}{k}$. It is clear that the set $\{z_{nk}\}_{n,k=1}^{\infty} \subset C^1(\overline{\Omega}, \mathbb{R}^m)$ is countable and dense in $W^{1,p}(\Omega, \mathbb{R}^m)$. For convenience sake we denote this set by $\{z_n\}_{n=1}^{\infty}$. Denote $h_n = \frac{1}{n^2} \frac{z_n}{\|z_n\|_{C^1}}$, then

$$|h_n(x)| \leq \frac{1}{n^2}, \quad |\nabla h_n(x)| \leq \frac{1}{n^2} \quad \forall x \in \Omega.$$

Introduce the mapping $F\colon \Omega \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \ell_2$,

$$F(x, a, v_1, v_2) = (a, \langle v_1, h_1(x)\rangle + \langle v_2, \nabla h_1(x)\rangle, \dots, \langle v_1, h_n(x)\rangle + \langle v_2, \nabla h_n(x)\rangle, \dots).$$

It is easy to see that for all $x \in \Omega$ and $[a, v_1, v_2] \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$, one has that $F(x, a, v_1, v_2) \in \ell_2$ and the mapping F is continuous, hence it satisfies the Caratheodory condition.

Fix an arbitrary $u \in W^{1,p}(\Omega, \mathbb{R}^m)$ and consider the set-valued mapping $x \to F(x, \underline{d}_{u,\xi} g(x, u(x), \nabla u(x))$. By virtue of Theorem 8.2.8 from [2], one has that the mapping $x \to F(x, \underline{d}_{u,\xi} g(x, u(x), \nabla u(x)))$ is measurable.

Note that applying the Filippov theorem, one can show that the mapping $\eta(\cdot) = (\eta_0(\cdot), \eta_1(\cdot), \dots, \eta_n(\cdot), \dots)$, $\eta\colon \Omega \to \ell_2$ is a measurable selection of the set-valued mapping $x \to F(x, \underline{d}_{u,\xi} g(x, u(x), \nabla u(x)))$ if and only if there exists a measurable selection $[a(\cdot), v_1(\cdot), v_2(\cdot)]$ of the mapping $x \to \underline{d}_{u,\xi} g(x, u(x), \nabla u(x))$ such that for all $n \in \mathbb{N}$ and for a.e. $x \in \Omega$

$$\eta_0(x) = a(x), \quad \eta_n(x) = \langle v_1(x), h_n(x)\rangle + \langle v_2(x), \nabla h_n(x)\rangle.$$

Consider the Aumann integral $\int_\Omega F(x, \underline{d}_{u,\xi} g(x, u(x), \nabla u(x)))\mathrm{d}x$. Let a map $\eta(\cdot) = (\eta_0(\cdot), \eta_1(\cdot), \dots, \eta_n(\cdot), \dots)$, $\eta\colon \Omega \to \ell_2$ be a measurable selection of the set-valued mapping $x \to F(x, \underline{d}_{u,\xi} g(x, u(x), \nabla u(x)))$. Let us show that η satisfies the assumptions of Lemma 5.5. As earlier mentioned, there exists a measurable selection $[a(\cdot), v_1(\cdot), v_2(\cdot)]$ of the set-valued mapping $x \to \underline{d}_{u,\xi} g(x, u(x), \nabla u(x))$ such that for all $n \in \mathbb{N}$ and for a.e. $x \in \Omega$

$$\eta_0(x) = a(x), \quad \eta_n(x) = \langle v_1(x), h_n(x)\rangle + \langle v_2(x), \nabla h_n(x)\rangle.$$

Since a codifferential of the function g with respect to u and ξ satisfies the growth condition, then applying the Hölder inequality one has that for all $n \in \mathbb{N}$

$$\int_\Omega |\eta_0(x)|\,\mathrm{d}x \le \|\beta\|_1 + C\left((\|u\|_p)^p + (\|\nabla u\|_p)^p\right) = \theta_0,$$

$$\begin{aligned}\int_\Omega |\eta_n(x)|\,\mathrm{d}x &\le \int_\Omega |v_1(x)||h_n(x)|\,\mathrm{d}x + \int_\Omega |v_2(x)||\nabla h_n(x)|\,\mathrm{d}x \\ &\le \frac{(\mu(\Omega))^{\frac{1}{p}}}{n^2}\left[\|\beta_1\|_q + \|\beta_2\|_q + (C_1 + C_2)\left((\|u\|_p)^{\frac{p}{q}} + (\|\nabla u\|_p)^{\frac{p}{q}}\right)\right] = \frac{\theta}{n^2},\end{aligned}$$

where μ is the Lebesgue measure on $\mathbb{R}^d$, $\beta, \beta_1, \beta_2, C, C_1, C_2$ are from the definition of the growth condition of a codifferential and

$$\theta = (\mu(\Omega))^{\frac{1}{p}}\left[\|\beta_1\|_q + \|\beta_2\|_q + (C_1 + C_2)\left((\|u\|_p)^{\frac{p}{q}} + (\|\nabla u\|_p)^{\frac{p}{q}}\right)\right].$$

Hence one gets, that $\zeta = (\int_\Omega \eta_0(x)\,\mathrm{d}x, \dots, \int_\Omega \eta_n(x)\,\mathrm{d}x, \dots) \in \ell_2$ and

$$\int_\Omega \|\eta(x)\|_{\ell_2}\,\mathrm{d}x = \int_\Omega \left(\sum_{n=0}^\infty |\eta_n(x)|^2\right)^{\frac{1}{2}}\mathrm{d}x \le \int_\Omega \sum_{n=0}^\infty |\eta_n(x)|\,\mathrm{d}x = \sum_{n=0}^\infty \int_\Omega |\eta_n(x)|\,\mathrm{d}x \le \theta_0 + \theta\sum_{n=1}^\infty \frac{1}{n^2} < \infty.$$

Thus, the function η satisfies the assumptions of Lemma 5.5. Therefore η is Bochner integrable and $\int_\Omega \eta(x)\,\mathrm{d}x = \zeta$. As a result, one has

$$\begin{aligned}\int_\Omega F(x, \underline{d}_{u,\xi} g(x, u(x), \nabla u(x)))\,\mathrm{d}x = \Bigg\{ \left(\int_\Omega \eta_0(x)\,\mathrm{d}x, \dots, \int_\Omega \eta_n(x)\,\mathrm{d}x, \dots\right) \Bigg| \\ \eta_0(x) = a(x), \eta_n(x) = \langle v_1(x), h_n(x)\rangle + \langle v_2(x), \nabla h_n(x)\rangle, n \in \mathbb{N}, \\ [a(\cdot), v_1(\cdot), v_2(\cdot)]\text{is a measurable selection of the map } x \to \underline{d}_{u,\xi} g(x, u(x), \nabla u(x))\Bigg\}.\end{aligned}$$

Furthermore, one has that for any measurable selection η of the map $x \to F(x, \underline{d}_{u,\xi} g(x, u(x), \nabla u(x)))$ the following inequality holds true

$$\left(\sum_{n=m+1}^{\infty} \left| \int_{\Omega} \eta_n(x) \, \mathrm{d}x \right|^2 \right)^{\frac{1}{2}} \le \theta \sum_{n=m+1}^{\infty} \frac{1}{n^2} \to 0 \text{ as } m \to \infty, \tag{5.12}$$

i.e. the series remainder converges to zero uniformly with respect to all measurable selections η.

Let $[A_0, \varphi_0]$ be a limit point of the set $\underline{d}\mathcal{I}_W(u)$ in the topology $\tau \times w^*$. Let us show, that the point $(A_0, \varphi_0(h_1), \ldots, \varphi_0(h_n), \ldots)$ is a limit point of the Aumann integral $\int_{\Omega} F(x, \underline{d}_{u,\xi} g(x, u(x), \nabla u(x))) \, \mathrm{d}x$ in the norm topology in ℓ_2.

By definition one has that $\varphi_0 \in (W^{1,p}(\Omega, \mathbb{R}^m))^*$, then (*cf.* [1], Thm. 3.8) there exist $r_1 \in L_q(\Omega, \mathbb{R}^m)$ and $r_2 \in L_q(\Omega, \mathbb{R}^{m \times d})$ such that

$$\varphi_0(h) = \int_{\Omega} (\langle r_1(x), h(x) \rangle + \langle r_2(x), \nabla h(x) \rangle) \, \mathrm{d}x \quad \forall h \in W^{1,p}(\Omega, \mathbb{R}^m).$$

Therefore for all $n \in \mathbb{N}$

$$|\varphi_0(h_n)| \le \int_{\Omega} |\langle r_1(x), h_n(x) \rangle + \langle r_2(x), \nabla h_n(x) \rangle| \, \mathrm{d}x \le (\|r_1\|_q + \|r_2\|_q) \mu(\Omega)^{\frac{1}{p}} \frac{1}{n^2}.$$

Hence one gets that $(A_0, \varphi_0(h_1), \ldots, \varphi_0(h_n), \ldots) \in \ell_2$. Fix an arbitrary $\varepsilon > 0$. It is clear that there exists $N_0 \in \mathbb{N}$ for which

$$\theta \sum_{n=N_0+1}^{\infty} \frac{1}{n^2} < \frac{\varepsilon}{3}, \quad \left(\sum_{n=N_0+1}^{\infty} |\varphi_0(h_n)|^2 \right)^{\frac{1}{2}} < \frac{\varepsilon}{3}.$$

Since $[A_0, \varphi_0]$ is a limit point of the set $\underline{d}\mathcal{I}_W(u)$ in the topology $\tau \times w^*$, then there exists a measurable selection $[a(\cdot), v_1(\cdot), v_2(\cdot)]$ of the set-valued mapping $x \to \underline{d}_{u,\xi} g(x, u(x), \nabla u(x))$ such that

$$\left| A_0 - \int_{\Omega} a(x) \, \mathrm{d}x \right| + \sum_{n=1}^{N_0} \left| \varphi_0(h_n) - \int_{\Omega} (\langle v_1(x), h_n(x) \rangle + \langle v_2(x), \nabla h_n(x) \rangle) \, \mathrm{d}x \right| < \frac{\varepsilon}{3}.$$

Denote

$$\overline{\eta}(x) = (a(x), \langle v_1(x), h_1(x) \rangle + \langle v_2(x), \nabla h_1(x) \rangle, \ldots, \langle v_1(x), h_n(x) \rangle + \langle v_2(x), \nabla h_n(x) \rangle, \ldots), \quad x \in \Omega.$$

Then $\overline{\eta}(x) \in F(x, \underline{d}_{u,\xi} g(x, u(x), \nabla u(x)))$ for a.e. $x \in \Omega$ and $\int_{\Omega} \overline{\eta}(x) \, \mathrm{d}x \in \int_{\Omega} F(x, \underline{d}_{u,\xi} g(x, u(x), \nabla u(x))) \, \mathrm{d}x$. Hence, by (5.12), one has

$$\begin{aligned}
&\left\| \int_{\Omega} \overline{\eta}(x) \, \mathrm{d}x - (A_0, \varphi_0(h_1), \ldots, \varphi_0(h_n), \ldots) \right\|_{\ell_2} \\
&= \left(\left| \int_{\Omega} a(x) \, \mathrm{d}x - A_0 \right|^2 + \sum_{n=1}^{\infty} \left| \int_{\Omega} (\langle v_1(x), h_n(x) \rangle + \langle v_2(x), \nabla h_n(x) \rangle) \, \mathrm{d}x - \varphi_0(h_n) \right|^2 \right)^{\frac{1}{2}} \\
&\le \left| A_0 - \int_{\Omega} a(x) \, \mathrm{d}x \right| + \sum_{n=1}^{N_0} \left| \varphi_0(h_n) - \int_{\Omega} (\langle v_1(x), h_n(x) \rangle + \langle v_2(x), \nabla h_n(x) \rangle) \, \mathrm{d}x \right| \\
&\quad + \left(\sum_{n=N_0+1}^{\infty} |\varphi_0(h_n)| \right)^{\frac{1}{2}} + \left(\sum_{n=N_0+1}^{\infty} \left| \int_{\Omega} \overline{\eta}_n(x) \, \mathrm{d}x \right|^2 \right)^{\frac{1}{2}} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, then the point $(A_0, \varphi_0(h_1), \ldots, \varphi_0(h_n), \ldots)$ is a limit point of the Aumann integral $\int_\Omega F(x, \underline{d}_{u,\xi} g(x, u(x), \nabla u(x)))\, \mathrm{d}x$.

The space ℓ_2 is separable and reflexive. It is easy to verify, that the mapping $x \to F(x, \underline{d}_{u,\xi} g(x, u(x), \nabla u(x))$ is integrably bounded (*cf.* [2], Sect. 8.6). Hence, the set $\int_\Omega F(x, \underline{d}_{u,\xi} g(x, u(x), \nabla u(x)))\, \mathrm{d}x$ is closed (*cf.* [2], Thm. 8.6.4). Thus, there exists a measurable selection $[a(\cdot), v_1(\cdot), v_2(\cdot)]$ of the mapping $x \to \underline{d}_{u,\xi} g(x, u(x), \nabla u(x))$ such that for all $n \in \mathbb{N}$

$$A_0 = \int_\Omega a(x)\, \mathrm{d}x, \quad \varphi_0(h_n) = \int_\Omega (\langle v_1(x), h_n(x) \rangle + \langle v_2(x), \nabla h_n(x) \rangle)\, \mathrm{d}x.$$

Since φ_0 is a linear functional, then

$$\varphi_0(z_n) = \int_\Omega (\langle v_1(x), z_n(x) \rangle + \langle v_2(x), \nabla z_n(x) \rangle)\, \mathrm{d}x \quad \forall n \in \mathbb{N}.$$

The set $\{z_n\}_{n=1}^\infty$ is dense in $W^{1,p}(\Omega, \mathbb{R}^m)$ by definition, and the linear functional φ_0 is continuous, hence

$$\varphi_0(z) = \int_\Omega (\langle v_1(x), z(x) \rangle + \langle v_2(x), \nabla z(x) \rangle)\, \mathrm{d}x \quad \forall z \in W^{1,p}(\Omega, \mathbb{R}^m),$$

therefore $[A_0, \varphi_0] \in \underline{d}\mathcal{I}_W(u)$ and the set $\underline{d}\mathcal{I}_W(u)$ is closed.

Suppose now that $p = \infty$. Fix an arbitrary $u \in W^{1,\infty}(\Omega, \mathbb{R}^m)$. It is obvious that there exists $N \in \mathbb{N}$ such that $|u(x)| + |\nabla u(x)| \le N$. Hence, under our assumptions there exists $C^{(N)} > 0$ such that

$$|a| \le C^{(N)}, \ |v_1| \le C^{(N)}, \ |v_2| \le C^{(N)} \quad \forall [a, v_1, v_2] \in \underline{d}_{u,\xi} g(x, u(x), \nabla u(x)). \tag{5.13}$$

Let $[A, \varphi] \in \underline{d}\mathcal{I}_W(u)$ then, by definition, there exists a measurable selection $[a(\cdot), v_1(\cdot), v_2(\cdot)]$ of the mapping $x \to \underline{d}_{u,\xi} g(x, u(x), \nabla u(x))$ such that for all $h \in W^{1,\infty}(\Omega, \mathbb{R}^m)$

$$A = \int_\Omega a(x)\, \mathrm{d}x, \quad \varphi(h) = \int_\Omega (\langle v_1(x), h(x) \rangle + \langle v_2(x), \nabla h(x) \rangle)\, \mathrm{d}x.$$

Taking into account (5.13), one has

$$|\varphi(h)| \le C^{(N)} \|h\|_{1,1} \quad \forall h \in W^{1,\infty}(\Omega, \mathbb{R}^m). \tag{5.14}$$

Since the space $C^1(\overline{\Omega}, \mathbb{R}^m)$ is dense in $W^{1,1}(\Omega, \mathbb{R}^m)$ ([1], Thm. 3.18), then it is clear that $W^{1,\infty}(\Omega, \mathbb{R}^m)$ is dense in $W^{1,1}(\Omega, \mathbb{R}^m)$. Therefore there exists a unique extension $\widehat{\varphi} \in (W^{1,1}(\Omega, \mathbb{R}^m))^*$ of the functional φ on the space $W^{1,1}(\Omega, \mathbb{R}^m)$. Denote by $\underline{d}\widehat{\mathcal{I}}_W(u)$ the set of all pairs $[A, \widehat{\varphi}]$, where $[A, \varphi] \in \underline{d}\mathcal{I}_W(u)$ and $\widehat{\varphi}$ is a unique continuous extension of φ on the space $W^{1,1}(\Omega, \mathbb{R}^m)$. It is clear that

$$\underline{d}\widehat{\mathcal{I}}_W(u) = \Bigg\{ [A, \widehat{\varphi}] \in \mathbb{R} \times (W^{1,1}(\Omega, \mathbb{R}^m))^* \ \Bigg| \ A = \int_\Omega a(x)\, \mathrm{d}x,$$
$$\widehat{\varphi}(h) = \int_\Omega (\langle v_1(x), h(x) \rangle + \langle v_2(x), \nabla h(x) \rangle)\, \mathrm{d}x \quad \forall h \in W^{1,1}(\Omega, \mathbb{R}^m),$$
$$[a(\cdot), v_1(\cdot), v_2(\cdot)] \text{ is a measurable selection of the map } x \to \underline{d}_{u,\xi} g(x, u(x), \nabla u(x)) \Bigg\}.$$

Let the pair $[A_0, \varphi_0]$ be a limit point of the set $\underline{d}\mathcal{I}_W(u)$ in the topology $\tau \times \sigma((W^{1,\infty}(\Omega, \mathbb{R}^m))^*, W^{1,\infty}(\Omega, \mathbb{R}^m))$. Then for any $\varepsilon > 0$ and $h \in W^{1,\infty}(\Omega, \mathbb{R}^m)$ there exists $[A, \varphi] \in \underline{d}\mathcal{I}_W(u)$ such that

$$|A - A_0| < \varepsilon, \quad |\varphi_0(h) - \varphi(h)| < \varepsilon$$

Applying (5.14), it is easy to show that

$$|\varphi_0(h)| \le C^{(N)} \|h\|_{1,1} \quad \forall h \in W^{1,\infty}(\Omega, \mathbb{R}^m).$$

Thus, there exists a unique extension $\widehat{\varphi}_0 \in (W^{1,1}(\Omega, \mathbb{R}^m))^*$ of the functional φ_0 on the space $W^{1,1}(\Omega, \mathbb{R}^m)$. Moreover, it is easy to check that the pair $[A_0, \widehat{\varphi}_0]$ is a limit point of the set $\underline{d}\widehat{\mathcal{I}}_W(u)$ in the topology $\tau \times \sigma((W^{1,1}(\Omega, \mathbb{R}^m))^*, W^{1,1}(\Omega, \mathbb{R}^m))$.

Arguing in the same way as in the proof of the case $1 \le p < \infty$, one can find that $[A, \widehat{\varphi}_0] \in \underline{d}\widehat{\mathcal{I}}_W(u)$, therefore $[A_0, \varphi_0] \in \underline{d}\mathcal{I}_W(u)$. Thus, the proof is complete. □

An analogous result for the functional $\mathcal{I}_C$ also holds true.

Theorem 5.7. *Let $\Omega \subset \mathbb{R}^d$ be an open bounded set having the segment property, and let a function $f \colon \overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \to \mathbb{R}$, $f = f(x, u, \xi)$, be continuous and continuously codifferentiable with respect to u and ξ on $\overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$ uniformly with respect to $C^1(\overline{\Omega}, \mathbb{R}^m)$. Then the functional*

$$\mathcal{I}_C(u) = \int_\Omega f(x, u(x), \nabla u(x))\, \mathrm{d}x,$$

defined on the space $C^1(\overline{\Omega}, \mathbb{R}^m)$, is continuously codifferentiable at every point $u \in C^1(\overline{\Omega}, \mathbb{R}^m)$ and there is a codifferential of the functional $\mathcal{I}_C$ at a point u of the form

$$\underline{d}\mathcal{I}_C(u) = \Big\{ [A, \varphi] \in \mathbb{R} \times (C^1(\overline{\Omega}, \mathbb{R}^m))^* \ \Big|\ A = \int_\Omega a(x)\, \mathrm{d}x,$$
$$\varphi(h) = \int_\Omega (\langle v_1(x), h(x)\rangle + \langle v_2(x), \nabla h(x)\rangle)\, \mathrm{d}x \quad \forall h \in C^1(\overline{\Omega}, \mathbb{R}^m),$$
$$[a(\cdot), v_1(\cdot), v_2(\cdot)] \text{ is a measurable selection of the map } x \to \underline{d}_{u,\xi} f(x, u(x), \nabla u(x)) \Big\}$$

and

$$\overline{d}\mathcal{I}_C(u) = \Big\{ [B, \psi] \in \mathbb{R} \times (C^1(\overline{\Omega}, \mathbb{R}^m))^* \ \Big|\ B = \int_\Omega b(x)\, \mathrm{d}x,$$
$$\psi(h) = \int_\Omega (\langle w_1(x), h(x)\rangle + \langle w_2(x), \nabla h(x)\rangle)\, \mathrm{d}x \quad \forall h \in C^1(\overline{\Omega}, \mathbb{R}^m),$$
$$[b(\cdot), w_1(\cdot), w_2(\cdot)] \text{ is a measurable selection of the map } x \to \overline{d}_{u,\xi} f(x, u(x), \nabla u(x)) \Big\}$$

Proof. Fix an arbitrary $u \in C^1(\overline{\Omega}, \mathbb{R}^m)$. Since f is continuously codifferentiable with respect to u and ξ on its domain, then there exists $C > 0$ such that

$$|a| \le C, \quad |v_1| \le C, \quad |v_2| \le C \quad \forall [a, v_1, v_2] \in \underline{d}_{u,\xi} f(x, u(x), \nabla u(x)),$$
$$|b| \le C, \quad |w_1| \le C, \quad |w_2| \le C \quad \forall [b, w_1, w_2] \in \overline{d}_{u,\xi} f(x, u(x), \nabla u(x))$$

and, by virtue of Proposition 4.3, the functions $\Phi_f(\cdot, u(\cdot), \nabla u(\cdot); \alpha h(\cdot), \alpha \nabla h(\cdot))$, $\Psi_f(\cdot, u(\cdot), \nabla u(\cdot); \alpha h(\cdot), \alpha \nabla h(\cdot))$ are continuous. Applying these facts and arguing in a similar way to the proof of Theorem 5.1 in the case $p = \infty$ one can show that the functional $\mathcal{I}_C$ is codifferentiable at the point u and it has a codifferential of the form stated in the theorem. It remains to prove that the functional $\mathcal{I}_C$ is continuously codifferentiable.

Let us prove the Hausdorff continuity of the maps $u \to \underline{d}\mathcal{I}_C(u)$ and $u \to \overline{d}\mathcal{I}_C(u)$. We will consider only the hypodifferential of the functional $\mathcal{I}_C$, since the continuity of the hyperdifferential is proved in a similar way.

Fix an arbitrary $\varepsilon > 0$ and denote

$$K = \{(x, u_0, \xi_0) \mid x \in \overline{\Omega}, |u_0| \le \|u\|_{C^1} + 1, |\xi_0| \le \|u\|_{C_1} + 1\}.$$

It is clear that K is compact. Since f is continuously codifferentiable with respect to u and ξ on its domain, then the mapping $(x, u, \xi) \to \underline{d}_{u,\xi} f(x, u, \xi)$ is uniformly Hausdorff continuous on K. Hence, there exists $\delta \in (0, 1)$ such that for all $(x, u^{(1)}, \xi^{(1)}), (x, u^{(2)}, \xi^{(2)}) \in K$ such that $|u^{(1)} - u^{(2)}| < \delta$ and $|\xi^{(1)} - \xi^{(2)}| < \delta$

$$\rho_H(\underline{d}_{u,\xi} f(x, u^{(1)}, \xi^{(1)}), \underline{d}_{u,\xi} f(x, u^{(2)}, \xi^{(2)})) < \varepsilon. \tag{5.15}$$

Thus, for any $h \in C^1(\overline{\Omega}, \mathbb{R}^m)$ such that $\|h\|_{C^1} < \delta$ one has that for all $x \in \Omega$

$$\rho_H(\underline{d}_{u,\xi} f(x, u(x) + h(x), \nabla u(x) + \nabla h(x)), \underline{d}_{u,\xi} f(x, u(x), \nabla u(x))) < \varepsilon. \tag{5.16}$$

Fix arbitrary $h \in C^1(\overline{\Omega}, \mathbb{R}^m)$, $\|h\|_{C^1} < \delta$ and $[A, \varphi] \in \underline{d}\mathcal{I}_C(u + h)$. Then there exists a measurable selection $[a(\cdot), v_1(\cdot), v_2(\cdot)]$ of the mapping $x \to \underline{d}_{u,\xi} f(x, u(x) + h(x), \nabla u(x) + \nabla h(x))$ such that for all $z \in C^1(\overline{\Omega}, \mathbb{R}^m)$

$$A = \int_\Omega a(x)\,\mathrm{d}x, \quad \varphi(z) = \int_\Omega (\langle v_1(x), z(x)\rangle + \langle v_2(x), \nabla z(x)\rangle)\,\mathrm{d}x.$$

It follows from (5.16) that for a.e. $x \in \Omega$

$$\rho(\underline{d}_{u,\xi} f(x, u(x), \nabla u(x)), \{[a(x), v_1(x), v_2(x)]\})$$
$$= \inf \Big\{ \max\{|\overline{a}(x) - a(x)|, |\overline{v}_1(x) - v_1(x)|, |\overline{v}_2(x) - v_2(x)\} \;\Big|\; [\overline{a}(x), \overline{v}_1(x), \overline{v}_2(x)] \in \underline{d}_{u,\xi} f(x, u(x), \nabla u(x)) \Big\} < \varepsilon.$$

Therefore, by virtue of Corollary 8.2.13 from [2], there exists a measurable selection $[\overline{a}(\cdot), \overline{v}_1(\cdot), \overline{v}_2(\cdot)]$ of the mapping $x \to \underline{d}_{u,\xi} f(x, u(x), \nabla u(x))$ such that for a.e. $x \in \Omega$

$$|a(x) - \overline{a}(x)| < \varepsilon, \quad |v_1(x) - \overline{v}_1(x)| < \varepsilon, \quad |v_2(x) - \overline{v}_2(x)| < \varepsilon.$$

Denote $\overline{A} = \int_\Omega \overline{a}(x)\,\mathrm{d}x$ and

$$\overline{\varphi}(z) = \int_\Omega (\langle \overline{v}_1(x), z(x)\rangle + \langle \overline{v}_2(x), \nabla z(x)\rangle)\,\mathrm{d}x \quad \forall z \in C^1(\overline{\Omega}, \mathbb{R}^m).$$

It is clear that $[\overline{A}, \overline{\varphi}] \in \underline{d}\mathcal{I}_C(u)$ and there exists $M > 0$, depending only on the measure of Ω, d and m, such that $\|[A, \varphi] - [\overline{A}, \overline{\varphi}]\| < M\varepsilon$. Arguing in the same way one can prove that for any $[\overline{A}, \overline{\varphi}] \in \underline{d}\mathcal{I}_C(u)$ there exists $[A, \varphi] \in \underline{d}\mathcal{I}_C(u + h)$ such that $\|[A, \varphi] - [\overline{A}, \overline{\varphi}]\| < M\varepsilon$. Thus

$$\rho_H(\underline{d}\mathcal{I}_C(u), \underline{d}\mathcal{I}_C(u + h)) < M\varepsilon \quad \forall h \in C^1(\overline{\Omega}, \mathbb{R}^m), \|h\| < \delta,$$

that completes the proof. □

Remark 5.8. Arguing in a similar way to the proof of the previous theorem, one can show that if in Theorem 5.1 the function g is continuously codifferentiable with respect to u and ξ on its domain, then the functional $\mathcal{I}_W$ is continuously codifferentiable in the case $p = \infty$.

6. Necessary optimality conditions in nonsmooth problems of the calculus of variations

Let us derive necessary conditions for the extremum of the functional $\mathcal{I}_W$ on the set A_W. We will consider only necessary conditions for a minimum, since necessary conditions for a maximum are derived in a similar way.

Theorem 6.1. *Let a set Ω and a function g satisfy the assumptions of Theorem 5.1, and suppose that the functional $\mathcal{I}_W$ has a local minimum on the set A_W at a point $u^* \in W^{1,p}(\Omega, \mathbb{R}^m)$. Then for any measurable selection $[b(\cdot), w_1(\cdot), w_2(\cdot)]$ of the mapping $x \to \overline{d}_{u,\xi} g(x, u^*(x), \nabla u^*(x))$ such that $b(x) = 0$ for a.e. $x \in \Omega$, there exists a function $\zeta \in L_q(\Omega, \mathbb{R}^{m \times d})$ such that there exists* $\operatorname{div} \zeta = (\operatorname{div}(\zeta_{11}, \ldots, \zeta_{1d}), \ldots, \operatorname{div}(\zeta_{m1}, \ldots, \zeta_{md})) \in L_q(\Omega, \mathbb{R}^m)$ *and for a.e. $x \in \Omega$*

$$[0, \operatorname{div}(\zeta)(x), \zeta(x)] \in \underline{d}_{u,\xi} g(x, u^*(x), \nabla u^*(x)) + \{[0, w_1(x), w_2(x)]\}.$$

Proof. It is easy to see, that

$$N(A_W, u^*) = \{\varphi \in (W^{1,p}(\Omega, \mathbb{R}^m))^* \mid \varphi(h) = 0 \quad \forall h \in W_0^{1,p}(\Omega, \mathbb{R}^m)\}.$$

Applying Theorems 2.8 and 5.1 one easily get that for any measurable selection $[b(\cdot), w_1(\cdot), w_2(\cdot)]$ of the mapping $x \to \overline{d}_{u,\xi} g(x, u^*(x), \nabla u^*(x))$ such that $b(x) = 0$ for a.e. $x \in \Omega$, there exists a measurable selection $[a(\cdot), v_1(\cdot), v_2(\cdot)]$ of the mapping $x \to \underline{d}_{u,\xi} g(x, u^*(x), \nabla u^*(x))$ such that $a(x) = 0$ for a.e. $x \in \Omega$ and for all $h \in W_0^{1,p}(\Omega, \mathbb{R}^m)$

$$\int_\Omega (\langle v_1(x) + w_1(x), h(x) \rangle + \langle v_2(x) + w_2(x), \nabla h(x) \rangle) \, \mathrm{d}x = 0. \tag{6.17}$$

Therefore, by definition, $v_1 + w_1 = \operatorname{div}(v_2 + w_2)$. It remains to denote $\zeta = v_2 + w_2$. ☐

An analogous theorem for the functional $\mathcal{I}_C$ also holds true.

Theorem 6.2. *Let a set Ω and a function f satisfy the assumptions of Theorem 5.7, and suppose that the functional $\mathcal{I}_C$ has a local minimum on the set A_C at a point $u^* \in C^1(\overline{\Omega}, \mathbb{R}^m)$. Then for any measurable selection $[b(\cdot), w_1(\cdot), w_2(\cdot)]$ of the mapping $x \to \overline{d}_{u,\xi} f(x, u^*(x), \nabla u^*(x))$ such that $b(x) = 0$ for a.e. $x \in \Omega$, there exist a function $\zeta \in L_\infty(\Omega, \mathbb{R}^{m \times d})$ such that there exists* $\operatorname{div} \zeta \in L_\infty(\Omega, \mathbb{R}^m)$ *and for a.e. $x \in \Omega$*

$$[0, \operatorname{div}(\zeta)(x), \zeta(x)] \in \underline{d}_{u,\xi} f(x, u^*(x), \nabla u^*(x)) + \{[0, w_1(x), w_2(x)]\}.$$

In the case $d = 1$ necessary conditions for a minimum can be formulated in a different way.

Corollary 6.3. *Let $d = 1$, $\Omega = (a, b)$, and let a function g satisfy the assumptions of Theorem 5.1. Suppose that the functional $\mathcal{I}_W$ has a local minimum on the set A_W at a point $u^* \in W^{1,p}((a, b), \mathbb{R}^m)$. Then for any measurable selection $[b(\cdot), w_1(\cdot), w_2(\cdot)]$ of the mapping $x \to \overline{d}_{u,\xi} g(x, u^*(x), (u^*)'(x))$ such that $b(x) = 0$ for a.e. $x \in (a, b)$, there exist a vector $c \in \mathbb{R}^m$ and a measurable selection $[a(\cdot), v_1(\cdot), v_2(\cdot)]$ of the mapping $x \to \underline{d}_{u,\xi} g(x, u^*(x), (u^*)'(x))$ such that $a(x) = 0$ for a.e. $x \in (a, b)$ and*

$$\int_x^b (v_1(y) + w_1(y)) \, dy + v_2(x) + w_2(x) = c \text{ for a.e. } x \in (a, b). \tag{6.18}$$

or, equivalently, for any measurable selection $[b(\cdot), w_1(\cdot), w_2(\cdot)]$ of the mapping $x \to \overline{d}_{u,\xi} g(x, u^*(x), (u^*)'(x))$ such that $b(x) = 0$ for a.e. $x \in (a, b)$, there exist an absolutely continuous function $\zeta \colon (a, b) \to \mathbb{R}^m$ such that for a.e. $x \in (a, b)$

$$[0, \zeta'(x), \zeta(x)] \in \underline{d}_{u,\xi} g(x, u^*(x), (u^*)'(x)) + \{[0, w_1(x), w_2(x)]\}.$$

Moreover, $\zeta \in W^{1,q}((a, b), \mathbb{R}^m)$.

Proof. Integrating the first term on the left-hand side of (6.17) by parts one find that for all $h \in C_0^\infty((a, b), \mathbb{R}^m)$

$$\int_a^b \Big\langle \int_x^b (v_1(y) + w_1(y)) \, dy + v_2(x) + w_2(x), h'(x) \Big\rangle \mathrm{d}x = 0.$$

Applying the du Bois–Reymond lemma (*cf.*, for instance, [16], Sect. 2.2), one gets that there exists $c \in \mathbb{R}^m$ such that the equation (6.18) holds true. In order to get the equivalent result it remains to denote $\zeta(x) = -\int_x^b (v_1(y) + w_1(y)) \, dy + c$. ☐

Necessary optimality conditions can be transformed into a more convenient form for different particular functionals. Let us give an example of how one can transform the condition for a minimum stated in Corollary 6.3.

Proposition 6.4. *Let $d = 1$, $\Omega = (a, b)$, $f = \max_{i\in I} f_i + \min_{j\in J} g_j$, where $I = \{1, \ldots, k\}$, $J = \{1, \ldots, l\}$, functions $f_i, g_j \colon [a, b] \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$, $f_i = f_i(x, u, \xi)$, $g_j = g_j(x, u, \xi)$, are continuous and continuously differentiable with respect to u_s and ξ_r on $[a, b] \times \mathbb{R}^m \times \mathbb{R}^m$, $i \in I$, $j \in J$, $s, r \in \{1, \ldots, m\}$. Suppose that the functional $\mathcal{I}_C$ has a local minimum on the set $A_C = \{u \in C^1([a, b], \mathbb{R}^m) \mid u(a) = u_1, u(b) = u_2\}$ where $u_1, u_2 \in \mathbb{R}^m$ are given vectors, at a point u^*. Then for any measurable functions $\alpha_j \colon (a, b) \to [0, 1]$, $j \in J$, such that $\alpha_1 + \ldots + \alpha_l = 1$ a.e. and for any $j \in J$*

$$\alpha_j(x)(g_j(x, u^*(x), (u^*)'(x)) - \min_{j\in J} g_j(x, u^*(x), (u^*)'(x))) = 0 \text{ for a.e. } x \in (a, b),$$

there exist a vector $c = (c_1, \ldots, c_m) \in \mathbb{R}^m$ and measurable functions $\lambda_i \colon (a, b) \to [0, 1]$ such that $\lambda_1 + \ldots + \lambda_k = 1$ a.e., for any $i \in I$

$$\lambda_i(x)(\max_{i\in I} f_i(x, u^*(x), (u^*)'(x)) - f_i(x, u^*(x), (u^*)'(x))) = 0 \text{ for a.e. } x \in (a, b)$$

and for all $s \in \{1, \ldots, m\}$ and for a.e. $x \in (a, b)$

$$\int_x^b \left(\sum_{i\in I} \lambda_i(t) \frac{\partial f_i}{\partial u_s}(t, u^*(t), (u^*)'(t)) + \sum_{j\in J} \alpha_j(t) \frac{\partial g_j}{\partial u_s}(t, u^*(t), (u^*)'(t)) \right) \mathrm{d}t + \sum_{i\in I} \lambda_i(x) \frac{\partial f_i}{\partial \xi_s}(x, u^*(x), (u^*)'(x)) + \sum_{j\in J} \alpha_j(x) \frac{\partial g_j}{\partial \xi_s}(x, u^*(x), (u^*)'(x)) = c_s.$$

Proof. By virtue of Propositions 4.9, 4.10 and 4.12, one has that the function f is continuous and continuously codifferentiable with respect to u and ξ on $[a, b] \times \mathbb{R}^m \times \mathbb{R}^m$ uniformly with respect to $C^1([a, b], \mathbb{R}^m)$ and there is a codifferential of the function f with respect to u and ξ of the form $D_{u,\xi} f(x, u, \xi) = [\underline{d}_{u,\xi} f(x, u, \xi), \overline{d}_{u,\xi} f(x, u, \xi)]$, where (*cf.* Rem. 2.6)

$$\underline{d}_{u,\xi} f(x, u, \xi) = \operatorname{co} \Big\{ \Big(f_i(x, u, \xi) - \max_{i\in I} f_i(x, u, \xi), \frac{\partial f_i}{\partial u_1}(x, u, \xi), \ldots, \frac{\partial f_i}{\partial u_m}(x, u, \xi), \frac{\partial f_i}{\partial \xi_1}(x, u, \xi), \ldots, \frac{\partial f_i}{\partial \xi_m}(x, u, \xi) \Big) \Bigm| i \in I \Big\}$$

and

$$\overline{d}_{u,\xi} f(x, u, \xi) = \operatorname{co} \Big\{ \Big(g_j(x, u, \xi) - \min_{i\in I} g_j(x, u, \xi), \frac{\partial g_j}{\partial u_1}(x, u, \xi), \ldots, \frac{\partial g_j}{\partial u_m}(x, u, \xi), \frac{\partial g_j}{\partial \xi_1}(x, u, \xi), \ldots, \frac{\partial g_j}{\partial \xi_m}(x, u, \xi) \Big) \Bigm| j \in J \Big\}.$$

Let us describe all measurable selections of the map $x \to \underline{d}_{u,\xi} f(x, u(x), u'(x))$, where $u \in C^1([a, b], \mathbb{R}^m)$ is arbitrary. Denote

$$\Lambda = \{(\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k \mid \lambda_i \geq 0, i \in I, \lambda_1 + \ldots + \lambda_k = 1\}$$

and introduce the mapping $F \colon [a, b] \times \mathbb{R}^k \to \mathbb{R}^{1+2m}$,

$$F(x, \lambda_1, \ldots, \lambda_k) = \Bigg(\sum_{i\in I} \lambda_i (f_i(x, u(x), u'(x)) - \max_{i\in I} f_i(x, u(x), u'(x))), \sum_{i\in I} \lambda_i \frac{\partial f_i}{\partial u_1}(x, u(x), u'(x)), \ldots, \sum_{i\in I} \lambda_i \frac{\partial f_i}{\partial u_m}(x, u(x), u'(x)), \sum_{i\in I} \lambda_i \frac{\partial f_i}{\partial \xi_1}(x, u(x), u'(x)), \ldots, \sum_{i\in I} \lambda_i \frac{\partial f_i}{\partial \xi_m}(x, u(x), u'(x)) \Bigg).$$

It is easy to verify that the mapping F satisfies the Caratheodory condition. Let $[a(\cdot), v_1(\cdot), v_2(\cdot)]$ be a measurable selection of the mapping $x \to \underline{d}_{u,\xi} f(x, u^*(x), (u^*)'(x))$, then for a.e. $x \in (a,b)$

$$[a(x), v_1(x), v_2(x)] \in F(x, \Lambda).$$

By virtue of the Filippov theorem one has that there exists a measurable selection $(\lambda_1(\cdot), \ldots, \lambda_k(\cdot))$ of the constant mapping $x \to \Lambda$ such that for a.e. $x \in (a,b)$

$$\begin{aligned} a(x) &= \sum_{i \in I} \lambda_i(x)(f_i(x, u(x), u'(x)) - \max_{i \in I} f_i(x, u(x), u'(x))), \\ v_1(x) &= \left(\sum_{i \in I} \lambda_i(x) \frac{\partial f_i}{\partial u_1}(x, u(x), u'(x)), \ldots, \sum_{i \in I} \lambda_i(x) \frac{\partial f_i}{\partial u_m}(x, u(x), u'(x)) \right), \\ v_2(x) &= \left(\sum_{i \in I} \lambda_i(x) \frac{\partial f_i}{\partial \xi_1}(x, u(x), u'(x)), \ldots, \sum_{i \in I} \lambda_i(x) \frac{\partial f_i}{\partial \xi_m}(x, u(x), u'(x)) \right). \end{aligned}$$

Furthermore, $\lambda_i \geq 0$ and $\lambda_1 + \ldots + \lambda_k = 1$ almost everywhere. Hence, applying Corollary 6.3 it is easy to get the required result. □

7. Problem of Bolza

In this section we will briefly discuss necessary optimality conditions in the problem of Bolza. Let $1 \leq p \leq \infty$, $d = 1$, $\Omega = (a,b)$, and consider the following problem of Bolza

$$\mathcal{I}(u) = g_0(u(a), u(b)) + \int_a^b g(x, u(x), u'(x))\, \mathrm{d}x \to \text{extr}, \tag{7.19}$$

where $u \in W^{1,p}((a,b), \mathbb{R}^m)$. Here, as usual, we identify the Sobolev space $W^{1,p}((a,b), \mathbb{R}^m)$ with the space consisting of all absolutely continuous functions $u \colon [a,b] \to \mathbb{R}^m$, such that $u' \in L_p((a,b), \mathbb{R}^m)$. We will assume that the function $g \colon (a,b) \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$, $g = g(x,u,\xi)$ satisfies the assumptions of Theorem 5.1, and the function $g_0 \colon \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is codifferentiable on its domain.

Theorem 7.1. *Let the functional $\mathcal{I}$ have a local minimum at a point $u^* \in W^{1,p}((a,b), \mathbb{R}^m)$. Then for any $[0, r_1, r_2] \in \overline{d} g_0(u^*(a), u^*(b))$ and for any measurable selection $[b(\cdot), w_1(\cdot), w_2(\cdot)]$ of the mapping $x \to \overline{d}_{u,\xi} g(x, u^*(x), (u^*)'(x))$ such that $b(x) = 0$ for a.e. $x \in (a,b)$ there exists an absolutely continuous function $\zeta \colon [a,b] \to \mathbb{R}^d$ such that $\zeta' \in L_q((a,b), \mathbb{R}^d)$, for a.e. $x \in (a,b)$*

$$[0, \zeta'(x), \zeta(x)] \in \underline{d}_{u,\xi} g(x, u^*(x), (u^*)'(x)) + \{[0, w_1(x), w_2(x)]\}$$

and the following transversality condition holds true

$$[0, \zeta(a), -\zeta(b)] \in \underline{d} g_0(u^*(a), u^*(b)) + \{[0, r_1, r_2]\}.$$

Proof. Define the functional $\mathcal{J} \colon W^{1,p}((a,b), \mathbb{R}^m) \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$,

$$\mathcal{J}(u, y, z) = g_0(y, z) + \int_a^b g(x, u(x), u'(x))\, \mathrm{d}x$$

and fix arbitrary $h \in W^{1,p}((a,b),\mathbb{R}^m)$, $y, \Delta y, z, \Delta z \in \mathbb{R}^m$. Since the function g satisfies the assumptions of Theorem 5.1, then it is easy to verify that for any $\alpha \geq 0$

$$\begin{aligned} &\mathcal{J}(u^* + \alpha h, y + \alpha \Delta y, z + \alpha \Delta z) - \mathcal{J}(u^*, y, z) \\ &= \max\left(\delta + \int_a^b a(x)\,\mathrm{d}x + \alpha\langle s_1, \Delta y\rangle + \alpha\langle s_2, \Delta z\rangle + \int_a^b \langle v_1(x), \alpha h(x)\rangle\,\mathrm{d}x + \int_a^b \langle v_2(x), \alpha h'(x)\rangle\,\mathrm{d}x\right) \\ &\quad + \min\left(\gamma + \int_a^b b(x)\,\mathrm{d}x + \alpha\langle r_1, \Delta y\rangle + \alpha\langle r_2, \Delta z\rangle + \int_a^b \langle w_1(x), \alpha h(x)\rangle\,\mathrm{d}x + \int_a^b \langle w_2(x), \alpha h'(x)\rangle\,\mathrm{d}x\right) + o(\alpha), \end{aligned} \tag{7.20}$$

where $o(\alpha)/\alpha \to 0$ as $\alpha \downarrow 0$. Here the maximum on the right-hand side is taken over all $[\delta, s_1, s_2] \in \underline{d}g_0(y,z)$ and measurable selections $[a(\cdot), v_1(\cdot), v_2(\cdot)]$ of the mapping $x \to \underline{d}_{u,\xi} g(x, u^*(x), (u^*)'(x))$, and the minimum on the right-hand side is taken over all $[\gamma, r_1, r_2] \in \overline{d}g_0(y,z)$ and measurable selections $[b(\cdot), w_1(\cdot), w_2(\cdot)]$ of the mapping $x \to \overline{d}_{u,\xi} g(x, u^*(x), (u^*)'(x))$. Hence, by (7.20), the functional $\mathcal{J}$ is codifferentiable.

Since the functional $\mathcal{I}$ has a local minimum at the point u^*, then the functional $\mathcal{J}$ has a local minimum on the closed convex set

$$A = \{(u, y, z) \in W^{1,p}((a,b),\mathbb{R}^m) \times \mathbb{R}^m \times \mathbb{R}^m \mid y = u(a), z = u(b)\}$$

at the point $(u^*(\cdot), u^*(a), u^*(b))$. It is easy to show that

$$\begin{aligned} N(A, (u^*, u^*(a), u^*(b))) = \{&(\varphi, y, z) \in (W^{1,p}((a,b),\mathbb{R}^m))^* \times \mathbb{R}^m \times \mathbb{R}^m \mid \\ &\varphi(h) + \langle y, h(a)\rangle + \langle z, h(b)\rangle = 0 \quad \forall h \in W^{1,p}((a,b),\mathbb{R}^m)\}. \end{aligned}$$

Applying the necessary condition for the minimum of a codifferentiable function (Thm. 2.8), one has that for any $[0, r_1, r_2] \in \overline{d}g_0(u^*(a), u^*(b))$ and for any measurable selection $[b(\cdot), w_1(\cdot), w_2(\cdot)]$ of the mapping $x \to \overline{d}_{u,\xi} g(x, u^*(x), (u^*)'(x))$ such that $b(x) = 0$ for a.e. $x \in (a,b)$, there exist $[0, s_1, s_2] \in \underline{d}g_0(u^*(a), u^*(b))$ and a measurable selection $[a(\cdot), v_1(\cdot), v_2(\cdot)]$ of the set-valued mapping $x \to \underline{d}_{u,\xi} g(x, u^*(x), (u^*)'(x))$ such that $a(x) = 0$ for a.e. $x \in (a,b)$ and for all $h \in W^{1,p}((a,b),\mathbb{R}^m)$

$$\int_a^b \left(\langle v_1(x) + w_1(x), h(x)\rangle + \langle v_2(x) + w_2(x), h'(x)\rangle\right)\,\mathrm{d}x + \langle s_1 + r_1, h(a)\rangle + \langle s_2 + r_2, h(b)\rangle = 0. \tag{7.21}$$

Therefore for any $h \in C_0^\infty([a,b],\mathbb{R}^d)$, one has

$$\int_a^b \left\langle \left(\int_x^b (v_1(t) + w_1(t))\,\mathrm{d}t + v_2(x) + w_2(x)\right), h'(x) \right\rangle\,\mathrm{d}x = 0,$$

then applying the du Bois–Reymond lemma one gets that there exists $c \in \mathbb{R}^m$ such that

$$\int_x^b (v_1(t) + w_1(t))\,\mathrm{d}t + v_2(x) + w_2(x) = c \text{ for a.e. } x \in (a,b).$$

Define $\zeta(x) = -\int_x^b (v_1(\tau) + w_1(\tau))\,d\tau + c$, $x \in [a,b]$. It is clear that ζ is absolutely continuous, $\zeta' = v_1 + w_1 \in L_q((a,b),\mathbb{R}^m)$, $\zeta = v_2 + w_2$ a.e. and for a.e. $x \in (a,b)$

$$[0, \zeta'(x), \zeta(x)] \in \underline{d}_{u,\xi} g(x, u^*(x), (u^*)'(x)) + \{[0, w_1(x), w_2(x)\}.$$

Substituting $v_2 + w_2$ for ζ and $v_1 + w_1$ for ζ' in (7.21) and integrating by parts one gets that for any $h \in W^{1,p}((a,b),\mathbb{R}^m)$

$$\langle s_1 + r_1 - \zeta(a), h(a)\rangle + \langle s_2 + r_2 + \zeta(b), h(b)\rangle = 0,$$

therfore $\zeta(a) = s_1 + r_1$ and $\zeta(b) = -s_2 - r_2$ or, equivalently,

$$[0, \zeta(a), -\zeta(b)] \in \underline{d} g_0(u^*(a), u^*(b)) + [0, r_1, r_2].$$

Thus, the proof is complete. □

Remark 7.2. It is easy to derive necessary conditions for a maximum in the problem of Bolza (7.19). One can also consider the problem of Bolza for a functional defined on $C^1([a,b], \mathbb{R}^d)$ and get similar results.

Remark 7.3. In this paper we studied only the main problem of the calculus of variations and the problem of Bolza. However, the techniques developed in the article can be applied to the study of more difficult problems. In particular, one can consider the problem

$$\mathcal{I}(u) = \int_a^b f(x, u(x), u'(x))\,\mathrm{d}x \to \inf, \quad g(x, u(x), u'(x)) \le 0, \quad u(a) = u_1, \quad u(b) = u_2.$$

It is clear that u^* is a point of local minimum in this problem if and only if u^* is a point of local minimum in the problem

$$\max\left\{\mathcal{I}(u) - \mathcal{I}(u^*), \int_a^b \max\{g(x, u(x), u'(x)), 0\}\,\mathrm{d}x\right\} \to \inf, \quad u(a) = u_1, \quad u(b) = u_2.$$

Applying the necessary conditions for a minimum of a codifferentiable function on a convex set one can easily obtain necessary optimality conditions in the latter problems and, as a result, in the initial one. Also, the author supposes that the approach based on codifferentiation can be applied to the study of various optimal control problems.

8. Examples

We consider two examples which demonstrate benefits of the necessary optimality conditions stated in Theorems 6.1 and 7.1. In the first example we compare the necessary conditions for a minimum (Thm. 6.1) in the case $d > 1$ with Clarke's optimality conditions [7]. In the second one we compare the necessary optimality conditions in the problem of Bolza stated in Theorem 7.1 with Clarke's separated Euler condition [5], Clarke's Euler–Lagrange condition [6, 7] and necessary optimality conditions derived by Ioffe and Rockafellar in [15].

Example 1. Let $d = 2$, $\Omega = (-1,1) \times (-1,1)$, $1 \le p < \infty$, $g(u, \xi) = \max\{u, 0\} + \min\{\xi_1 - \xi_2, 0\}$, *i.e.*

$$\mathcal{I}_W(u) = \int_\Omega \left(\max\{u(x), 0\} + \min\left\{\frac{\partial u}{\partial x_1}(x) - \frac{\partial u}{\partial x_2}(x), 0\right\}\right)\mathrm{d}x,$$

and $v_0 = 0$, *i.e.* $A_W = W_0^{1,p}(\Omega)$. We want to check non-optimality of the function $u_0(x) \equiv 0$. It is easy to verify that

$$\partial_{Cl} g(u_0(x), \nabla u_0(x)) = \partial_{Cl} g(0,0) = \operatorname{co}\{(1,1,-1), (1,0,0), (0,1,-1), (0,0,0)\}.$$

Here $\partial_{Cl} g(0,0)$ is the Clarke subdifferential of the function g at the point $(0,0)$. Since $(0,0) \in \partial_{Cl} g(u_0(x), \nabla u_0(x))$ for any $x \in \Omega$, then Clarke's necessary optimality condition ([7], Thm. 4.6.1) is satisfied.

It is obvious that the function g satisfies the assumptions of Theorem 5.1. Thus we can use Theorem 6.1 to check non-optimality of the function u_0. Applying formulae for computing a codifferential [11, 13], one has

$$\underline{d}_{u,\xi} g(u_0(x), \nabla u_0(x)) = \operatorname{co}\{(0,1,0,0), 0\}, \quad \overline{d}_{u,\xi} g(u_0(x), \nabla u_0(x)) = \operatorname{co}\{(0,0,1,-1), 0\}.$$

The function $[0, 0, x_1^2, -x_1^2]$ is a measurable selection of the set-valued mapping $x \to \overline{d}_{u,\xi} g(u_0(x), \nabla u_0(x))$. Therefore, if the necessary condition for a minimum (Thm. 6.1) is satisfied, then there exists a function $\zeta \in L_q(\Omega, \mathbb{R}^2)$ such that there exists $\operatorname{div}\zeta \in L_q(\Omega)$ and for a.e. $x \in \Omega$

$$[0, \operatorname{div}(\zeta)(x), \zeta(x)] \in \underline{d}_{u,\xi} g(u_0(x), \nabla u_0(x)) + \{[0, 0, x_1^2, -x_1^2]\} = \operatorname{co}\{[0, 1, x_1^2, -x_1^2], [0, 0, x_1^2, -x_1^2]\}.$$

Hence $\zeta(x) = (x_1^2, -x_1^2)$ and $\operatorname{div}(\zeta)(x) = 2x_1$ a.e., but for any $(x_1, x_2) \in \Omega$, $x_1 \notin (0, 1/2)$

$$[0, 2x_1, x_1^2, -x_1^2] \notin \operatorname{co}\{[0, 1, x_1^2, -x_1^2], [0, 0, x_1^2, -x_1^2]\}.$$

Thus, the function $u_0(x) \equiv 0$ is non-optimal.

Example 2. Let $d = 1$, $\Omega = (0, 1)$, $p = 1$. Consider the following problem of Bolza

$$\mathcal{I}(u) = u(0) - \gamma u(1) + \int_0^1 \max\{|u'(x)| - |u(x)|, 0\}\, \mathrm{d}x.$$

We want to check non-optimality of the function $u_\alpha(x) = \alpha \mathrm{e}^x$ for any $\alpha \geq 0$ and $\gamma \in \mathbb{R}$ (*cf.* [15], example 2). It was shown in [15] that Clarke's separated Euler condition fails to disqualify u_α as non-optimal for any $\alpha > 0$ and $\gamma \in [0, 1]$. Clarke's Euler–Lagrange condition fails to disqualify $u(x) \equiv 0$ ($\alpha = 0$) for $\gamma \in [0, 1]$ and the necessary optimality condition derived in [15] is satisfied, when $u(x) \equiv 0$ and $\gamma \in [\mathrm{e}^{-1}, 1]$. Also, both Clarke's Euler–Lagrange conditions and the necessary optimality condition derived in [15] are satisfied for u_α, when $\gamma = \mathrm{e}^{-1}$ and $\alpha > 0$.

We will show that the necessary condition for a minimum in the problem of Bolza stated in Theorem 7.1 is not satisfied for any $\alpha \geq 0$ and $\gamma \in \mathbb{R}$, except the case $\gamma = \mathrm{e}^{-1}$, when $\alpha > 0$. Indeed, since $g_0(y, z) = y - \gamma z$ and $Dg_0(y, z) = [\{[0, 1, -\gamma]\}, \{0\}]$, then the transversality condition is $\zeta(0) = 1$, $\zeta(1) = \gamma$. One has

$$g(x, u, \xi) = \max\{|\xi|, |u|\} - |u| = \max\{\xi, -\xi, u, -u\} + \min\{u, -u\},$$

therefore the function g satisfies the assumptions of Theorem 5.1 and for any $\alpha \geq 0$

$$\underline{d}_{u,\xi} g(x, u_\alpha(x), u'_\alpha(x)) = \operatorname{co}\{(0, 0, 1), (-2\alpha \mathrm{e}^x, 0, -1), (0, 1, 0), (-2\alpha \mathrm{e}^x, -1, 0)\}, \tag{8.22}$$

$$\overline{d}_{u,\xi} g(x, u_\alpha(x), u'_\alpha(x)) = \operatorname{co}\{(2\alpha \mathrm{e}^x, 1, 0), (0, -1, 0)\}. \tag{8.23}$$

If $\alpha > 0$, then the only measurable selection of the set-valued mapping $x \to \overline{d}_{u,\xi} g(\cdot, u_\alpha(x), u'_\alpha(x))$ such that $b(x) = 0$ for a.e. $x \in (0, 1)$ is $[b(\cdot), w_1(\cdot), w_2(\cdot)] = [0, -1, 0]$. Suppose that the necessary condition for a minimum (Thm. 7.1) is satisfied. Then there exists $\zeta \in W^{1,1}(0, 1)$ such that $\zeta(0) = 1$, $\zeta(1) = \gamma$ and for a.e. $x \in (0, 1)$

$$[0, \zeta'(x), \zeta(x)] \in \underline{d}_{u,\xi} g(x, u_\alpha(x), u'_\alpha(x)) + \{[0, -1, 0]\} = \operatorname{co}\{(0, -1, 1), (-2\alpha \mathrm{e}^x, -1, -1), (0, 0, 0), (-2\alpha \mathrm{e}^x, -2, 0)\}.$$

Hence $\zeta' = -\zeta$ a.e. and $\zeta(x) = c\mathrm{e}^{-x}$. Applying the transversality condition $\zeta(0) = 1$, one has that $\zeta(x) = \mathrm{e}^{-x}$. Thus, u_α is non-optimal for any $\gamma \neq \mathrm{e}^{-1}$.

Consider the case $\alpha = 0$, *i.e.* $u_\alpha(x) \equiv 0$, and suppose that u_α satisfies the necessary condition for a minimum. By Theorem 7.1 there exists $\zeta \in W^{1,1}(0, 1)$ such that $\zeta(0) = 1$, $\zeta(1) = \gamma$ and for a.e. $x \in (0, 1)$

$$[0, \zeta'(x), \zeta(x)] \in \underline{d}_{u,\xi} g(x, u_\alpha(x), u'_\alpha(x)) + \{[0, -1, 0]\} = \operatorname{co}\{(0, -1, 1), (0, -1, -1), (0, 0, 0), (0, -2, 0)\}.$$

Therefore $\zeta'(x) \leq 0$ for a.e. $x \in (0, 1)$ and $\zeta(1) \leq \zeta(0) = 1$. Hence the necessary condition for a minimum is not satisfied for any $\gamma > 1$. Analogously, there exists $\zeta \in W^{1,1}(0, 1)$ such that $\zeta(0) = 1$, $\zeta(1) = \gamma$ and for almost every $x \in (0, 1)$

$$[0, \zeta'(x), \zeta(x)] \in \underline{d}_{u,\xi} g(x, u_\alpha(x), u'_\alpha(x)) + \{[0, 1, 0]\} = \operatorname{co}\{(0, 1, 1), (0, 1, -1), (0, 2, 0), (0, 0, 0)\}.$$

(*cf.* (8.22), and (8.23)). Thus $\zeta'(x) \geq 0$ for a.e. $x \in (0, 1)$ and $\zeta(1) \geq \zeta(0) = 1$. Let us show that $\zeta(1) > 1$. If $\zeta'(x) > 0$ on a set of positive measure, then $\zeta(1) > \zeta(0) = 1$. If $\zeta'(x) = 0$ a.e., then $\zeta(x) = 0$ for a.e. $x \in (0, 1)$, which contradicts the transversality condition $\zeta(0) = 1$. Therefore $\zeta(1) > 1$ and for any $\gamma \leq 1$ the necessary condition for a minimum is not satisfied.

Two previous examples show that the necessary optimality condition stated in Theorem 6.1 and the necessary optimality condition in the problem of Bolza stated in Theorem 7.1 are better then Clarke's optimality conditions and the necessary optimality condition obtained in [15]. We believe that this is a usual situation, because codifferential, as a nonhomogeneous approximation, provides more information about function's behaviour than homogeneous approximations used in Clarke's works and in [15].

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