ESAIM: COCV 20 (2014) 771–802
DOI: 10.1051/cocv/2013083

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

THE VALUE FUNCTION REPRESENTING HAMILTON–JACOBI EQUATION WITH HAMILTONIAN DEPENDING ON VALUE OF SOLUTION

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Abstract. In the paper we investigate the regularity of the value function representing Hamilton–Jacobi equation: $-U_t + H(t,x,U,-U_x) = 0$ with a final condition: $U(T,x) = g(x)$. Hamilton–Jacobi equation, in which the Hamiltonian H depends on the value of solution U, is represented by the value function with more complicated structure than the value function in Bolza problem. This function is described with the use of some class of Mayer problems related to the optimal control theory and the calculus of variation. In the paper we prove that absolutely continuous functions that are solutions of Mayer problem satisfy the Lipschitz condition. Using this fact we show that the value function is a bilateral solution of Hamilton–Jacobi equation. Moreover, we prove that continuity or the local Lipschitz condition of the function of final cost g is inherited by the value function. Our results allow to state the theorem about existence and uniqueness of bilateral solutions in the class of functions that are bounded from below and satisfy the local Lipschitz condition. In proving the main results we use recently derived necessary optimality conditions of Loewen–Rockafellar [P.D. Loewen and R.T. Rockafellar, *SIAM J. Control Optim.* **32** (1994) 442–470; P.D. Loewen and R.T. Rockafellar, *SIAM J. Control Optim.* **35** (1997) 2050–2069].

Mathematics Subject Classification. 49J52, 49L25, 35B37.

Received January 29, 2013. Revised October 16, 2013.
Published online May 21, 2014.

1. Introduction

The classic Cauchy problem for Hamilton–Jacobi equation, that the Hamiltonian depends on the value of the solution, is a partial differential equation of the first order with the final condition:

$$\begin{array}{c} -U_t + H(t,x,U,-U_x) = 0 \text{ in } (0,T)\times\mathbb{R}^n, \\ U(T,x) = g(x) \text{ in } \mathbb{R}^n. \end{array} \tag{1.1}$$

The Hamilton–Jacobi equations, with the Hamiltonian $H(t,x,u,p)$ convex with respect to p, are one of the main objects in the optimal control theory. It was motivated by Bellman's idea connecting solutions of (1.1) with optimization problems described by a dual function to the Hamiltonian. This function, called the Lagrangian and denoted by L, is derived from H using the Legendre–Fenchel transform as follows:

$$L(t,x,u,v) = \sup_{p\in\mathbb{R}^n} \{ \langle v,p\rangle - H(t,x,u,p) \}. \tag{1.2}$$

Keywords and phrases. Hamilton–Jacobi equation, optimal control, nonsmooth analysis, viability theory, viscosity solution.

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In this paper we assume that L can take the value $+\infty$. It influences the generality of considered problems. Throughout the paper Q will denote the multivalued mapping defined by the formula

$$Q(t,x,u)=\{(v,-r)\in\mathbb{R}^n\times\mathbb{R}\,:\,r\geqslant L(t,x,u,v)\}. \tag{1.3}$$

Let $\Psi_K:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ denote *the indicator function* of a set $K\subset\mathbb{R}^n$ that is given by $\Psi_K(v)=0$ for $v\in K$ and $\Psi_K(v)=+\infty$, otherwise. Consider the following Mayer's problem parameterized by (t_0,x_0):

$$\begin{array}{ll}\text{minimize} & u(t_0)+\Psi_{x_0}(x(t_0))+\Psi_{\operatorname{epi} g}(x(T),u(T))\\ \text{subject to} & (\dot x(t),\dot u(t))\in Q(t,x(t),u(t))\;\; a.e.\; t\in[t_0,T],\end{array} \tag{$\mathcal{P}_{t_0,x_0}$}$$

where we minimize ($\mathcal{P}_{t_0,x_0}$) over $\mathcal{A}([t_0,T],\mathbb{R}^n\times\mathbb{R})$ (the space of all absolutely continuous functions (x,u) : $[t_0,T]\to\mathbb{R}^n\times\mathbb{R}$). We will show that Mayer's problem ($\mathcal{P}_{t_0,x_0}$), whose bounds on the dynamics are given by differential inclusions coming from Lagrangians, include some problems from the optimal control theory and the calculus of variation.

The set $\operatorname{dom} f:=\{x\,:\,f(x)\neq\pm\infty\}$ is called *the effective domain* of f. An extended-real-valued function is called *proper* if it never takes the value $-\infty$ and $\operatorname{dom} f\neq\emptyset$. When $H(t,x,u,\cdot)$ is proper, lower semicontinuous (lsc) and convex for each (t,x,u), then $L(t,x,u,\cdot)$ has also these properties. Furthermore, we can retrieve H from L by performing the Legendre–Fenchel transform a second time:

$$H(t,x,u,p)=\sup_{v\in\mathbb{R}^n}\{\,\langle p,v\rangle-L(t,x,u,v)\,\}. \tag{1.4}$$

Thus we have a one-to-one correspondence between Hamiltonians and Lagrangians in this convex case, and every equation of the form (1.1) can be related to a problem of the form ($\mathcal{P}_{t_0,x_0}$).

The function $(x,u)(\cdot)\in\mathcal{A}([a,b],\mathbb{R}^n\times\mathbb{R})$ is called *L-solution* on $[a,b]$, if $(\dot x(t),\dot u(t))\in Q(t,x(t),u(t))$ for a.e. $t\in[a,b]$. We define *the value function* $V:[0,T]\times\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ as the optimal value of ($\mathcal{P}_{t_0,x_0}$) parameterized by (t_0,x_0) *i.e.*

Definition 1.1. For $t_0=T$ we have $V(t_0,x_0):=g(x_0)$ and for $t_0<T$ we have (*we set* $\inf\emptyset:=+\infty$)

$$V(t_0,x_0):=\inf\left\{u_0:\begin{array}{l}\text{there exists } L\text{-solution }\;(x,u)(\cdot)\;\text{ on }\;[t_0,T]\\ \text{such that }\;(x,u)(t_0)=(x_0,u_0)\;\text{ and }\;u(T)\geqslant g(x(T)).\end{array}\right\}$$

When the value function is real-valued and differentiable, it is known that it satisfies (1.1) in the classical sense. However, in many situations the value function is extended-real-valued and merely lsc, therefore one introduces nonsmooth solutions in such a way that the value function V is the unique solution of equation (1.1) in terms of previously proposed.

In 1980 Crandall and Lions [9] introduced *viscosity solutions*, with Crandall, Evans, and Lions giving a simplified approach [10]. Viscosity solutions attracted a lot of attention, and over subsequent years a sizeable literature developed from many authors dealing with, among other issues, existence and uniqueness of solutions. Usually in the viscosity theory we have some assumptions about uniform continuity and boundedness of solutions and uniform continuity of Hamiltonian (see [2,3]). In our paper assumptions are less restrictive.

In [4] Baron–Jensen extended viscosity solutions to semicontinuous functions for Hamiltonians that are convex with respect to the last variable. Moreover, they proved existence and uniqueness results. Frankowska [12] using different methods (the viability theory), obtained similar results to those of Baron–Jensen. Results of [4, 12] are obtained supposing a linear growth of the Hamiltonian H that implies boundedness of the effective domain $L(t,x,u,\cdot)$, where L can be derived from H *via* (1.2). This assumption contains problems of the optimal control theory, but not interesting problems of the calculus of variation.

Results of Dal Maso and Frankowska [11], Galbraith [14] and Plaskacz–Quincampoix [20] show that using methods from the viability theory we can prove existence and uniqueness of solutions if we replace the assumption about a linear growth by much weaker one concerning a mild growth of the Hamiltonian. If the Hamiltonian

H satisfies the mild growth condition, then the effective domain of $L(t,x,u,\cdot)$ can be the unbounded set. This condition contains problems of the calculus of variation that are interested for us, *i.e.* when the Lagrangian is finite and satisfies a coercivity-type condition.

In [20] Plaskacz–Quincampoix considered Cauchy problem (1.1) for Hamiltonians H not only satisfying a mild growth condition but also depending on values of the solution U. Results of papers [4, 11, 12, 14] are obtained for Hamiltonians H independent of values of the solution U. In such a case the representative of Cauchy problem (1.1) is the value function of Bolza problem.

Results of Plaskacz and Quincampoix [20] only partially answer questions concerning existence and uniqueness of solutions of Cauchy problem (1.1). In this paper we solve important problems that occurred in [20], but we also study issues that were not considered in [20]. We were motivated to continue studies from [20] by papers of Loewen–Rockafellar [17–19] and Galbraith [14, 15]. It began at the moment that we solved that the Lipschitz-type condition from [20] is equivalent to the one found in [17], that in the form of subgradients also occurs in [14, 15, 18, 19] (see Sect. 2). It helped us to deal and understand better the problems related to Cauchy problem (1.1), although in papers of Loewen and Rockafellar considered are Hamiltonians $H(t,x,u,p)$ independent of the variable u.

Since we use the nonsmooth analysis we need a notion of a subgradient. We use the notation adopted in Rockafellar and Wets [23]. For a vector $v\in\mathbb{R}^n$ and a function $f:\mathbb{R}^n\to\overline{\mathbb{R}}$,

(i) v is a regular subgradient of f at $x\in\operatorname{dom} f$, denoted $v\in\widehat{\partial} f(x)$, if

$$\liminf_{x'\to x,\, x'\neq x}\frac{f(x')-f(x)-\langle v,x'-x\rangle}{|x'-x|}\geqslant 0;$$

(ii) v is a general subgradient of f at $x\in\operatorname{dom} f$, denoted $v\in\partial f(x)$, if there are sequences $x_n\to x$ with $f(x_n)\to f(x)$ and $v_n\in\widehat{\partial} f(x_n)$ with $v_n\to v$;

(iii) v is a horizon subgradient of f at $x\in\operatorname{dom} f$, denoted $v\in\partial^\infty f(x)$, if there are sequences $x_n\to x$ with $f(x_n)\to f(x)$ and $v_n\in\widehat{\partial} f(x_n)$ with $\tau_n v_n\to v$ for some sequence $\tau_n\to 0+$.

Given a closed set K and a point $x\in K$ we can define the general normal cone $N_K(x)$ and the regular normal cone $\widehat{N}_K(x)$ from the corresponding subgradients by

$$N_K(x)=\partial\Psi_K(x),\qquad\qquad \widehat{N}_K(x)=\widehat{\partial}\Psi_K(x). \tag{1.5}$$

For a function $f:\mathbb{R}^n\to\overline{\mathbb{R}}$ and a point $x\in\operatorname{dom} f$, the subderivative function $df(x):\mathbb{R}^n\to\overline{\mathbb{R}}$ is defined by

$$df(x)(w)=\liminf_{w'\to w,\,\tau\to 0+}\frac{f(x+\tau w')-f(x)}{\tau}\cdot$$

Mayer problems of the form $(\mathcal{P}_{t_0,x_0})$ contain not only Bolza problems but also problems with a discount. In Section 3 we prove that the problem with a discount given by L_0 and λ can be written as Mayer problem $(\mathcal{P}_{t_0,x_0})$ with the Lagrangian $L(t,x,u,v):=L_0(t,x,v)-\lambda(t,x)u$. This fact shows the difference between problems considered in current paper and [1, 4, 8, 9, 11, 14, 15, 18, 19]. Investigating properties of Lagrangians L of the above type is possible by introducing weaker, than in the paper [20], assumption: $L(t,x,u,v)\geqslant -C_1-C_2u_+$, where $u_+:=\max\{0,u\}$ and $C_1,C_2\geqslant 0$. In [20] only nonnegativity of the Lagrangian is used.

Baron–Jensen [4] considered extended viscosity solutions in the class of upper semicontinuous functions and called them "upper semicontinuous solutions". Suitable reformulation replaces the class of upper semicontinuous functions by the class of lower semicontinuous functions. Frankowska [12] considered solutions in the class of lower semicontinuous functions and called them "lower semicontinuous solutions". These solutions are similar to solutions of Baron–Jensen. Our definition of the solution is the same as in Frankowska [12]. We call them "bilateral solutions" because we consider solutions not only in the class of lower semicontinuous functions but also in the class of functions that satisfy locally the Lipschitz condition.

Definition 1.2. We say that a lower semicontinuous function $U : [0,T]\times\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ is a bilateral solution of the equation (1.1), if for every $(t,x)\in \operatorname{dom} U$, $t\in(0,T)$ the following equality holds:

$$-p_t + H(t,x,U(t,x),-p_x) = 0 \quad \textit{for every} \quad (p_t,p_x)\in \widehat{\partial}\, U(t,x). \tag{1.6}$$

The main result of this paper is an answer to the problem of Plaskacz and Quincampoix [20] who asked if there exist minimizers of the value function V, at points of the set $\operatorname{dom} V$, satisfying the Lipschitz condition. The solutions of the problem ($\mathcal{P}_{t_0,x_0}$) are called minimizers of the value functions V at the point (t_0,x_0). The problem stated by Plaskacz and Quincampoix [20] has not been investigated yet. In the literature [1, 5, 7, 8, 21] Lipschitzian minimizers are, in general, considered in the context of Bolza problem, that is a special case of the above problem *i.e.* when the Lagrangian $L(t,x,u,v)$ does not depend on the variable u. Lipschitzian minimizers play significant roles, because they allow to prove, as was noticed by Plaskacz and Quincampoix [20], that the value function V is a solution of Cauchy problem (1.1). Existence of Lipschitzian minimizers is important not only because of the above fact, but also it allows to prove some crucial regularity of the value function that are presented below.

Our method of the solution of the problem of Plaskacz and Quincampoix [20] relies on using necessary conditions of optimality given in [17]. These results can be used because the Lipschitz-type condition from our paper is equivalent to their condition (we have already mentioned that). Basing on results of [17–19] we first prove (see Thm. 5.6), that if a function g is finite and satisfies locally the Lipschitz condition, then for every solution $(\overline{x},\overline{u})(\cdot)$ of the problem ($\mathcal{P}_{t_0,x_0}$) there exists a function $\psi(\cdot)\in\mathcal{A}([t_0,T],\mathbb{R}^n)$ such that

$$\dot{\overline{x}}(t)\in\partial_p\, H(t,\overline{x}(t),\overline{u}(t),\psi(t)) \quad \text{a.e.} \quad t\in[t_0,T]. \tag{1.7}$$

From the inclusion (1.7) we have that a function $\overline{x}(\cdot)$ satisfies the Lipschitz condition on $[t_0,T]$, because the Hamiltonian H(t,x,u,p) is continuous and convex with respect to p. In the case when the function g is extended and lower semicontinuous, we prove the significantly weaker fact than the one above (see Thm. 5.13). Namely, if $d\,V(t_0,x_0)(0,0)=0$ and the function $(\overline{x},\overline{u})(\cdot)$ is a solution of the problem ($\mathcal{P}_{t_0,x_0}$), then there exist a number $\tau\in(t_0,T]$ and a function $\psi(\cdot)\in\mathcal{A}([t_0,\tau],\mathbb{R}^n)$ such that

$$\dot{\overline{x}}(t)\in\partial_p\, H(t,\overline{x}(t),\overline{u}(t),\psi(t)) \quad \text{a.e.} \quad t\in[t_0,\tau]. \tag{1.8}$$

Therefore, in this case the function $\overline{x}(\cdot)$ also satisfies the Lipschitz condition, but only on $[t_0,\tau]$. Next, from the inclusions (1.7), (1.8) one can prove (see Prop. 5.9), that for every solution $(\overline{x},\overline{u})(\cdot)$ of the problem ($\mathcal{P}_{t_0,x_0}$) there exists a function $\overline{\eta}(\cdot)$ that satisfies the Lipschitz condition on $[t_0,T]$, $[t_0,\tau]$, respectively, and the function $(\overline{x},\overline{\eta})(\cdot)$ is a solution of the problem ($\mathcal{P}_{t_0,x_0}$). We know (see Thm. 3.4) that at every point $(t_0,x_0)\in\operatorname{dom} V$ there exists an absolutely continuous function being a minimizer of the value function V. Summarizing our results we can formulate the solution of the problem of Plaskacz and Quincampoix [20] in the following way:

(M1) If the function g is finite and satisfies locally the Lipschitz condition, then at every point $(t_0,x_0)\in\operatorname{dom} V$ there exists a minimizer $(\overline{x},\overline{u})(\cdot)$ satisfying the Lipschitz condition on $[t_0,T]$.

(M2) If the function g is extended and lower semicontinuous, then at every point $(t_0,x_0)\in\operatorname{dom} V$ such that $d\,V(t_0,x_0)(0,0)=0$ there exists a minimizer $(\overline{x},\overline{u})(\cdot)$ satisfying the Lipschitz condition on $[t_0,\tau]$, where τ is some number larger than t_0.

The point (M2) is significantly weaker than Plaskacz and Quincampoix [20] expected, because the Lipschitzian minimizer does not have to exist at every point of $\operatorname{dom} V$. Moreover, at points where it exists, it satisfies the Lipschitz condition only on some neighbourhood.

In Chapter 6 we show that (M2) is sufficient to prove that a lower semicontinuous value function is a bilateral solution of Cauchy problem (1.1). Moreover, in the proof we do not assume the Lipschitz-type condition with respect to the time variable on the Hamiltonian, what is done in [14, 20]. In the literature related to problems of existence and uniqueness one usually assumes continuity of the time variable [12] or only measurability [13].

In the proof of Theorem 4.3 (Existence) we omit this unnatural assumption. Instead, we use the property (Q) from Section 8.5.A in [5] and the epi-continuity property L from [22].

The next result of the paper concerns inheriting regularities, such that continuity and locally Lipschitz continuity, from the function g to the value function V (see Thms. 4.4, 4.5). Such results are known in the literature [2,6] but with significantly more restrictive assumptions than ours. Moreover, such regularities were mainly investigated for value functions in Bolza problem. Value functions appearing here are more general and it requires a new approach to these problems (see Sect. 7). The finiteness of the function g does not imply the finiteness of the value function V corresponding to g and L, because the Lagrangian L is not bounded from above. We obtain continuity and locally Lipschitz continuity of the value function V only on some small neighbourhoods of points from the set $\operatorname{dom} V$. In the proofs, the important role is played by Lemma 3.17 in [20] that concerns some relation between solutions of the inclusion $(\dot{x}, \dot{u}) \in Q(t, x, u)$ and initial conditions (x_0, u_0). Moreover, in the proof of locally Lipschitz continuity of the value function V we need the equi-Lipschitz property of minimizers that can be obtained from (M1) (see Sect. 5.2).

Plaskacz and Quincampoix prove uniqueness in the class of lower semicontinuous functions in the sense of proximal solutions (see [20], Thm. 3.6). In the class of functions, that satisfy locally Lipschitz continuity, proximal solutions are equivalent to bilateral solutions. However, in the class of lower semicontinuous functions proximal solutions are bilateral but not opposite. Our result (M2) is not sufficient to prove that the lower semicontinuous value function V is a proximal solution. The reason of that is the fact that not at every point of the set $\operatorname{dom} V$ there exists a Lipschitzian minimizer. So in the class of lower semicontinuous functions a bilateral solutions seem to be the most suitable. The complicated problem of uniqueness of bilateral solutions in the class of lower semicontinuous functions will be contained in a different paper. In this paper we consider uniqueness in the class of functions that satisfy Lipschitz continuity. The value function belongs to this class, if g satisfies locally Lipschitz continuity. Results about uniqueness in Plaskacz and Quincampoix [20] need Lipschitz-type condition with respect to the time variable, so we cannot use them directly. The fact (M1) allows us to solve this problem (see Thm. 4.6). Consequently, we obtain Theorem 4.6 about uniqueness without Lipschitz-type conditions with respect to the time variable that were needed in Galbraith [14] and Plaskacz–Quincampoix [20].

2. Assumptions and their equivalent forms

Throughout the paper we assume that the Hamiltonian H satisfies, in general, the following conditions

- (H1) $H : [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is upper semicontinuous;
- (H2) $H(t,x,u,p)$ is convex with respect to p for every t,x,u;
- (H3) $H(t,x,u,p)$ is nondecreasing with respect to u for every t,x,p;
- (H4) $H(t,x,u,0) \leqslant C_1 + C_2 u_+$ for all t,x,u and some constants $C_1, C_2 \geqslant 0$;
- (H5) For every $u \in \mathbb{R}$ there exists a convex and finite function $\psi : [0,\infty) \to \mathbb{R}$ and a constant $C > 0$ such that the inequality $H(t,x,u,p) \leqslant \psi(|p|) + C(1+|x|)|p|$ holds for all t,x,p.

Proposition 2.1. *Suppose that L, H are related by (1.2), (1.4), respectively. Then H satisfies (H1)−(H5) if and only if L satisfies (L1)−(L5):*

- (L1) $L : [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *is lower semicontinuous;*
- (L2) $L(t,x,u,v)$ *is convex and proper with respect to v for every t,x,u;*
- (L3) $L(t,x,u,v)$ *is nonincreasing with respect to u for every t,x,v;*
- (L4) $L(t,x,u,v) \geqslant -C_1 - C_2 u_+$ *for all t,x,u,v and some constants $C_1, C_2 \geqslant 0$;*
- (L5) *For every $u \in \mathbb{R}$ there exists a convex, lsc, proper function $\phi : [0,+\infty) \to \overline{\mathbb{R}}$ such that $\lim_{r\to+\infty} \phi(r)/r = +\infty$ and there exists $C > 0$ such that for all t,x,v we have $|v| > C(1+|x|) \;\Rightarrow\; L(t,x,u,v) \geqslant \phi(|v|)$.*

The proof of Proposition 2.1 is omitted, because it is similar to the proof of Proposition 2.1 in [20].

Definition 2.2. The Lagrangian L satisfies the epi-continuity property if for any point (t,x,u,v) where $L(t,x,u,v)$ is finite, and for any sequence $(t_n,x_n,u_n) \to (t,x,u)$, there exists a sequence $v_n \to v$ along which $L(t_n,x_n,u_n,v_n) \to L(t,x,u,v)$.

Now we introduce an idea of lower semicontinuity of a multivalued function F in Kuratowski's sense. We say that a multivalued function F is lower semicontinuous (lsc), if for every (z,y) satisfying $y \in F(z)$ and every sequence $z_n \to z$ there exists a sequence $y_n \in F(z_n)$ such that $y_n \to y$.

Remark 2.3. Let $L(t,x,u,\cdot)$ be proper for all (t,x,u). It is not difficult to show that if L satisfies the epi-continuity property, then multivalued function Q is lsc in the sense of Kuratowski. Moreover, L is lsc and satisfies the epi-continuity property if and only if Q is lsc in the sense of Kuratowski and has a closed graph.

Proposition 2.4. *Suppose that L, H are related by* (1.2), (1.4), *respectively. Let H satisfy* (H1)−(H5) *or, equivalently, L satisfy* (L1)−(L5)*. Then L satisfies the epi-continuity property if and only if H is continuous.*

Proposition 2.4 is a consequence of Proposition 2.1 in [22].

2.1. Lipschitz-type conditions

Throughout the paper, $\mathbb{B}$ we denote by the closed unit ball in the Euclidean space of an appropriate dimension. In order to simplify notation we use z as (x,u).

(L6) For every $R>0$ there exists $C>0$ such that for every $t\in[0,T]$ and $z',z\in R\mathbb{B}$, every $v\in \operatorname{dom} L(t,z,\cdot)$ there exists $v'\in\operatorname{dom} L(t,z',\cdot)$ such that
  (a) $|v'-v| \leqslant C(1+|v|+|L(t,z,v)|)|z'-z|$;
  (b) $L(t,z',v') \leqslant L(t,z,v) + C(1+|v|+|L(t,z,v)|)|z'-z|$.

Proposition 2.5. *Suppose that L, H are related by* (1.2), (1.4)*, respectively. Let H satisfy* (H1)−(H5) *or equivalently L satisfy* (L1)−(L5)*. Then the condition* (L6) *is equivalent to each of the conditions:*

(SL) *For every $R>0$ there exists $C>0$ and $\delta_0>0$ such that for all $t\in[0,T]$, $z',z\in R\mathbb{B}$ and $\delta\geqslant\delta_0$ the following inclusion is satisfied*

$$Q(t,z')\cap\delta(\mathbb{B}\times[-1,1]) \subset Q(t,z) + C\delta|z'-z|(\mathbb{B}\times[-1,1]).$$

(H6) *For every $R>0$ there exists $C>0$ and $\delta_0>0$ such that for all $t\in[0,T]$, $z',z\in R\mathbb{B}$, every $p\in\mathbb{R}^n$ and $\delta\geqslant\delta_0$ the following inequality holds*

$$\inf_{q\in\mathbb{R}^n,\,\theta>0}\{\,\theta H(t,z',q/\theta)+\delta|q-p|+\delta|\theta-1|\,\} \leqslant H(t,z,p)+C\delta(1+|p|)|z'-z|.$$

The assumption (H6), that is used in the main results of this paper, is some type of the Lipschitz condition on the Hamiltonian. In Appendix we prove Proposition 2.5 which states that the condition (H6) is equivalent to conditions (L6) and (SL). The condition (L6) is a restriction on the Lagrangian coming from the paper of Plaskacz and Quincampoix [20]. However, in this paper it does not depend on the variable t. The condition (SL) is a special kind of Aubin continuity of Q and is typical for multivalued functions with unbounded values. It was introduced by Loewen and Rockafellar [17].

In the proofs of the main results we also use the condition:

(GS) For every $R>0$ there exists $C>0$ such that at every point $(t,z,v)\in[0,T]\times R\mathbb{B}\times\mathbb{R}^n$, every $(w,p)\in\partial L(t,z,v)$ satisfies $|w|\leqslant C(1+|v|+|L(t,z,v)|)(1+|p|)$.

For simplicity in dealing with subgradients of $L(t,z,v)$ we use the notation ∂L instead of the more cumbersome (but precise) $\partial_{z,v}L$. If L satisfies the epi-continuity property and (L1)−(L5), then conditions (SL) and (GS) are equivalent. It can be proved in a similar way as the equivalence ([15], Prop. 3.4).

Corollary 2.6. *Suppose that L, H are related by* (1.2), (1.4)*, respectively. Let H be continuous and satisfy* (H1)−(H5) *or equivalently L satisfy the epi-continuity property and* (L1)−(L5)*. Then conditions* (L6), (H6), (SL), (GS) *are equivalent.*

2.2. Examples

We give examples satisfying aforementioned conditions and illustrating differences in assumptions of our paper and that in papers of Plaskacz–Quincampoix [20] and Galbraith [14].

Example 2.7. Assume that $f:[0,T]\to[0,+\infty)$ is a continuous function. Moreover, for $(t,x,p)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n$ and $u\in\mathbb{R}$ define

$$H_f(t,x,u,p)=\frac{\mathrm{e}^u}{1+f(t)+|x|^2}|p|^2.$$

Directly from the construction of the Hamiltonian H_f it is continuous and satisfies conditions (H1)−(H5). Similarly to the paper ([18], s. 1508) it can be proved that the Hamiltonian H_f fulfills (H6). Moreover, H_f does not satisfy conditions ([14], (A2) and [20], (A6)), otherwise it would also fulfill the Lipschitz condition with respect to the variable t, which follows from ([14], Prop. 2.4). But it is impossible, because f is continuous.

Example 2.8. Let the Hamiltonian H_f be the same as in Example 2.7 and $f\equiv 0$. Then the Hamiltonian H_f does not depend on the variable t. If $H(t,x,u,p):=|t-T/2|H_f(x,u,p)$, then the Hamiltonian H is continuous and fulfills (H1)−(H6), but does not satisfy conditions ([14], (A2) and [20], (A6)) in the neighbourhood of $t=T/2$, although it fulfills the locally the Lipschitz condition with respect to the variable t.

3. Value function

In this section we present arguments that allowed us, with weaker assumption (L4) than in [20], to state that the value function still has its properties such that is bounded from below, lower semicontinuous and possesses a minimizer. Besides, we prove that the problem with a discount can be expressed using Mayer's problem ($\mathcal{P}_{t_0,x_0}$).

3.1. Boundedness from below of the value function

The assumption about nonnegativity of the Lagrangian occurring in [20] implies that functions $u(\cdot)$ from Definition 1.1 are nonincreasing. So, with this assumption boundedness of the value function $V(t_0,x_0)$ is a consequence of boundedness of the function $g(\cdot)$, because $u_0\geqslant u(T)\geqslant g(x(T))$. The condition (L4) occurring in this paper does not imply monotonicity of the function $u(\cdot)$, even so the value function is still bounded from below.

Lemma 3.1. *Assume that $[a,b]\subset[0,T]$ and C_1, $C_2\geqslant 0$. If $u:[a,b]\to\mathbb{R}$ is an absolutely continuous function satisfying the inequality*

$$\dot u(t)\leqslant C_1+C_2u(t)_+\quad\text{for a.e. } t\in[a,b], \tag{3.1}$$

then for every $t\in[a,b]$ the following inequalities hold:

(i) $u(t)\leqslant[u(a)_++TC_1]\mathrm{e}^{C_2T}$;
(ii) $\max\{u(b)-u(t),u(t)-u(a)\}\leqslant 2\mathrm{e}^{2T[C_1+C_2]}[u(a)_++1]$;
(iii) $\min\{u(b)-2\mathrm{e}^{2T[C_1+C_2]},\mathrm{e}^{-3-2T[C_1+C_2]}u(b)-2\mathrm{e}^{-3}\}\leqslant u(a)$.

Proof. From the inequality (3.1) and absolute continuity of the function $u(\cdot)$ we have for all $t\in[a,b]$

$$u(t)\leqslant[u(a)_++TC_1]+\int_a^t C_2u(\tau)_+\,\mathrm{d}\tau. \tag{3.2}$$

The right side of the inequality (3.2) is nonnegative, so $u(t)_+\leqslant$ PS (3.2) for every $t\in[a,b]$. From Gronwall's inequality for all $t\in[a,b]$

$$u(t)_+\leqslant[u(a)_++TC_1]\mathrm{e}^{TC_2}.$$

Therefore, the inequality (i) holds, because the inequality $u(t) \leqslant u(t)_+$ is satisfied for every $t \in [a, b]$. We notice that from the inequality (3.1) and (i) for all $t \in [a, b]$

$$\begin{aligned} u(t) - u(a) &\leqslant TC_1 + C_2 \int_a^t u(\tau)_+ \, \mathrm{d}\tau \\ &\leqslant 2\mathrm{e}^{2T[C_1+C_2]}[u(a)_+ + 1]. \end{aligned}$$

Similarly, we show that for all $t \in [a, b]$

$$u(b) - u(t) \leqslant 2\mathrm{e}^{2T[C_1+C_2]}[u(a)_+ + 1].$$

Therefore, the inequality (ii) holds. Finally, notice that the inequality (iii) follows from the (ii). □

Proposition 3.2. *Assume that the Lagrangian L fulfills* (L4) *and the function $g\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is bounded from below. Then, the value function V associated with g and L is also bounded from below.*

Proof. From the definition of the value function $-\dot{u}(t) \geqslant L(t, x(t), u(t), \dot{x}(t))$ for a.e. $t \in [t_0, T]$. Therefore, using the condition (L4) we obtain $\dot{u}(t) \leqslant C_1 + C_2 u(t)_+$ for a.e. $t \in [t_0, T]$. So the inequality (3.1) from Lemma 3.1 holds. The function $g(\cdot)$ is bounded from below, so is the value $u(T)$, because $u(T) \geqslant g(x(T))$. Therefore, using the inequality (iii) from Lemma 3.1 we have the value $u_0 = u(t_0)$ has to be bounded from below despite of the choice of the function $(x, u)(\cdot)$. □

3.2. Minimizers and lower semicontinuity

Existence of minimizers and lower semicontinuity of value functions is proved in [20]. These results are a consequence of [20], Lemma 2.5, which proof is based on criterions of Tonelli–Nagumo and Helly convergences. Similarly to [20] we can prove [20], Lemma 2.5 with weaker assumption (L4), rather than with nonnegativity of the Lagrangian, if we can show that assumptions of aforementioned criterions are satisfied for the sequence of L-solutions fulfilling properties given below.

Let L satisfy (L1)–(L5) and $(t_{n0}, x_{n0}, u_{n0}) \to (t_0, x_0, u_0)$. Assume that the sequence of L-solutions $(x_n, u_n)(\cdot)$ on $[t_{n0}, T]$ fulfills $(x_n, u_n)(t_{n0}) = (x_{n0}, u_{n0})$ and $u_n(T) \geqslant K$ for all $n \in \mathbb{N}$.

Let $M > 0$ be an upper bound of the sequence $\{u_{n0}\}$. From (L4) we have $\dot{u}_n(t) \leqslant C_1 + C_2 u_n(t)_+$ for a.e. $t \in [t_{n0}, T]$. From the inequality (i) in Lemma 3.1 $u_n(t) \leqslant (M + TC_1)\mathrm{e}^{TC_2} =: \widetilde{u}$ for every $t \in [t_{n0}, T]$. For $\widetilde{u}$ there exists a function $\phi(\cdot)$ and a constant $C > 0$ such that the implication from (L5) holds. Let

$$A_n := \{t \in [t_{n0}, T]\,;\; |\dot{x}_n(t)| > C(1 + |x_n(t)|)\}.$$

From the condition (L4) for a.e. $t \in [t_{n0}, T]$

$$-\dot{u}_n(t) \;\;\geqslant\;\; L(t, x_n(t), u_n(t), \dot{x}_n(t)) \;\;\geqslant\;\; -C_1 - C_2\widetilde{u}.$$

If we set $C_3 := C_1 + C_2\widetilde{u}$, then $C_3 \geqslant 0$ and

$$\begin{aligned} \int_{A_n} \phi(|\dot{x}_n(t)|)\, \mathrm{d}t &\leqslant \int_{A_n} L(t, x_n(t), \widetilde{u}, \dot{x}_n(t))\, \mathrm{d}t \\ &\leqslant \int_{t_{n0}}^{T} C_3 + L(t, x_n(t), u_n(t), \dot{x}_n(t))\, \mathrm{d}t \\ &\leqslant TC_3 + M - K. \end{aligned} \tag{3.3}$$

From the inequality (3.3) and boundedness of the sequence $\{|x_n(t_{n0})|\}$ we have that assumptions of Tonelli–Nagumo theorem ([20], Prop. 2.6) are satisfied.

We know that nonnegativity of L implies that functions $u_n(\cdot)$ are nonincreasing, so from properties of the sequence $\{(x_n, u_n)\}(\cdot)$ they are also equi-bounded. This fact allows us to use Helly's theorem ([5], Thm. 15.1). The condition (L4) does not imply monotonicity of functions $u_n(\cdot)$, even so we will show that functions $u_n(\cdot)$ are uniformly bounded and have equi-bounded variation, that allows us to use Helly's theorem. From the inequality (ii) in Lemma 3.1 we obtain

$$u_n(t) \geqslant K - 2\mathrm{e}^{2T[C_1+C_2]}[M+1] \quad \text{ for all } \ t \in [t_{n0}, T].$$

Therefore, using above bounds the sequence $\{u_n(\cdot)\}$ is equi-bounded. We will show that it has equi-bounded variation. Indeed,

$$\begin{aligned} \mathrm{Var}_{[t_{n0},T]} u_n(\cdot) &\leqslant \int_{t_{n0}}^{T} |\dot{u}_n(t)| \,\mathrm{d}t \\ &= \int_{t_{n0}}^{T} 2\dot{u}_n(t)_+ \,\mathrm{d}t - \int_{t_{n0}}^{T} \dot{u}_n(t) \,\mathrm{d}t \\ &\leqslant 2TC_3 + M - K. \end{aligned} \tag{3.4}$$

Definition 3.3. Let V be the value function associated with g and L. We say that at a point $(t_0, x_0) \in \operatorname{dom} V$, $t_0 \in [0, T)$, L-solution $(\overline{x}, \overline{u})(\cdot)$ on $[t_0, T]$ is a minimizer, if it satisfies properties:

$$(\overline{x}, \overline{u})(t_0) = (x_0, V(t_0, x_0)), \qquad \overline{u}(T) \geqslant g(\overline{x}(T)). \tag{3.5}$$

The consequence of the above consideration and ([20], Lem. 2.5) is the following theorem:

Theorem 3.4. *Assume that L satisfies* (L1)−(L5) *and g is lsc and bounded from below. If V is the value function associated with g and L, then at every point $(t_0, x_0) \in \operatorname{dom} V$, $t_0 \in [0, T)$ the value function V has a minimizer. Moreover, V is lsc on $[0, T] \times \mathbb{R}^n$.*

3.3. Problem with a discount

Let the function $L_0(t, x, v)$ and $\lambda(t, x)$ be given. We call the problem of minimization of a functional (3.7) *the problem with a discount*. Consider the Lagrangian $L(t, x, u, v)$ given by

$$L(t, x, u, v) = L_0(t, x, v) - \lambda(t, x)u. \tag{3.6}$$

Notice that the Lagrangian L given by the equality (3.6) takes negative values, so it is not considered in [20].

Notice that if functions L_0 and λ satisfy conditions (L1$^+$)−(L5$^+$) below, then the Lagrangian L given by the equality (3.6) fulfills conditions (L1)−(L5).

(L1$^+$) $L_0 : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous;
(L2$^+$) $L_0(t, x, v)$ is convex, proper with respect to v for every t, x;
(L3$^+$) $\lambda : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function and $0 \leqslant \lambda(t, x) \leqslant C_2$ for all t, x and some constant $C_2 \geqslant 0$;
(L4$^+$) $L_0(t, x, v) \geqslant -C_1$ for all t, x, v and some constant $C_1 \geqslant 0$;
(L5$^+$) There exists a convex, lsc, proper function $\phi : [0, +\infty) \to \overline{\mathbb{R}}$ such that $\lim_{r\to+\infty} \phi(r)/r = +\infty$ and there exists $C > 0$ such that for all t, x, v we have $|v| > C(1 + |x|) \ \Rightarrow \ L_0(t, x, v) \geqslant \phi(|v|)$.

If conditions (L1$^+$)−(L5$^+$) are satisfied, then we can define a functional $J_\lambda(\cdot, t_0, x_0)$ for $x(\cdot) \in \mathcal{A}([t_0, T], \mathbb{R}^n)$ and $x(t_0) = x_0$ by formula:

$$J_\lambda(x(\cdot), t_0, x_0) = g(x(T))\,\mathrm{e}^{-\int_{t_0}^{T} \lambda(\tau, x(\tau))\mathrm{d}\tau} + \int_{t_0}^{T} \mathrm{e}^{-\int_{t_0}^{s} \lambda(\tau, x(\tau))\mathrm{d}\tau} L_0(s, x(s), \dot{x}(s)) \,\mathrm{d}s, \tag{3.7}$$

where g is a function bounded from below.

Theorem 3.5. *Let L and L_0 satisfy the equality (3.6). Assume that* (L1$^+$)-(L5$^+$) *hold. If g is lower semicontinuous and bounded from below, and V is the value function associated with g and L, then the following equality holds*

$$V(t_0,x_0)=\inf_{\substack{x(\cdot)\,\in\,\mathcal{A}([t_0,T],\mathbb{R}^n)\\ x(t_0)\,=\,x_0}} J_\lambda(x(\cdot),t_0,x_0). \tag{3.8}$$

Besides, if the function $(\overline{x},\overline{u})(\cdot)$ is a minimizer, then the function $\overline{x}(\cdot)$ realizes the infimum of the right side of the equality (3.8).

Proof. Notice that the functional $J_\lambda(\cdot,t_0,x_0)$ is bounded from below, so RS (3.8) takes values in $\mathbb{R}\cup\{+\infty\}$.

First, we prove the inequality RS(3.8) $\geqslant V(t_0,x_0)$. Without loss of generality we can assume that RS(3.8) is a real number. Let $\varepsilon>0$, then there exists a function $x(\cdot)\in\mathcal{A}([t_0,T],\mathbb{R}^n)$ such that $x(t_0)=x_0$ and

$$\mathrm{RS}(3.8)+\varepsilon\geqslant J_\lambda(x(\cdot),t_0,x_0). \tag{3.9}$$

Define an absolutely continuous function $u\colon[t_0,T]\to\mathbb{R}$ by

$$\begin{aligned} u(t) &:= J_\lambda(x(\cdot),t,x_0)\\ &= g(x(T))\,\mathrm{e}^{-\int_t^T \lambda(\tau,x(\tau))\mathrm{d}\tau}+\mathrm{e}^{-\int_t^T \lambda(\tau,x(\tau))\mathrm{d}\tau}\int_t^T \mathrm{e}^{\int_s^T \lambda(\tau,x(\tau))\mathrm{d}\tau} L_0(s,x(s),\dot{x}(s))\,\mathrm{d}s. \end{aligned}$$

Notice that $u(T)=g(x(T))$, besides for a.e. $t\in[t_0,T]$

$$\dot{u}(t)\;=\;\lambda(t,x(t))\,u(t)-L_0(t,x(t),\dot{x}(t))\;=\;-L(t,x(t),u(t),\dot{x}(t)).$$

Therefore, from the definition of the value function $u(t_0)\geqslant V(t_0,x_0)$. By (3.9) we have RS(3.8) $+\,\varepsilon\geqslant u(t_0)\geqslant V(t_0,x_0)$ so RS(3.8) $\geqslant V(t_0,x_0)$.

We consider the second inequality $V(t_0,x_0)\geqslant$ PS(3.8). Without loss of generality we can assume that $V(t_0,x_0)$ is bounded. Therefore, at a point (t_0,x_0) there exists a minimizer $(\overline{x},\overline{u})(\cdot)$. So for a.e. $t\in[t_0,T]$,

$$\begin{aligned} \dot{\overline{u}}(t) &\leqslant -L(t,\overline{x}(t),\overline{u}(t),\dot{\overline{x}}(t))\\ &= \lambda(t,\overline{x}(t))\,\overline{u}(t)-L_0(t,\overline{x}(t),\dot{\overline{x}}(t)). \end{aligned} \tag{3.10}$$

From the inequality (3.10) for a.e. $s\in[t_0,T]$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}\left(\overline{u}(s)\mathrm{e}^{-\int_{t_0}^{s}\lambda(\tau,\overline{x}(\tau))\mathrm{d}\tau}\right) &= \mathrm{e}^{-\int_{t_0}^{s}\lambda(\tau,\overline{x}(\tau))\mathrm{d}\tau}\left[\dot{\overline{u}}(s)-\lambda(s,\overline{x}(s))\,\overline{u}(s)\right]\\ &\leqslant -\mathrm{e}^{-\int_{t_0}^{s}\lambda(\tau,\overline{x}(\tau))\mathrm{d}\tau} L_0(s,\overline{x}(s),\dot{\overline{x}}(s)). \end{aligned} \tag{3.11}$$

The function $t\to\overline{u}(t)\mathrm{e}^{-\int_{t_0}^{t}\lambda(\tau,\overline{x}(\tau))\mathrm{d}\tau}$ is absolutely continuous, so from the inequality (3.11),

$$\begin{aligned} \overline{u}(T)\mathrm{e}^{-\int_{t_0}^{T}\lambda(\tau,\overline{x}(\tau))\mathrm{d}\tau}-\overline{u}(t_0) &= \int_{t_0}^{T}\frac{\mathrm{d}}{\mathrm{d}s}\left(\overline{u}(s)\mathrm{e}^{-\int_{t_0}^{s}\lambda(\tau,\overline{x}(\tau))\mathrm{d}\tau}\right)\mathrm{d}s\\ &\leqslant -\int_{t_0}^{T}\mathrm{e}^{-\int_{t_0}^{s}\lambda(\tau,\overline{x}(\tau))\mathrm{d}\tau} L_0(s,\overline{x}(s),\dot{\overline{x}}(s))\,\mathrm{d}s. \end{aligned} \tag{3.12}$$

Putting together inequalities (3.12) and $\overline{u}(T) \geqslant g(\overline{x}(T))$, we obtain

$$g(\overline{x}(T))\mathrm{e}^{-\int_{t_0}^{T} \lambda(\tau,\overline{x}(\tau))\mathrm{d}\tau} + \int_{t_0}^{T} \mathrm{e}^{-\int_{t_0}^{s} \lambda(\tau,\overline{x}(\tau))\mathrm{d}\tau} L_0(s,\overline{x}(s),\dot{\overline{x}}(s))\,\mathrm{d}s \leqslant \overline{u}(t_0). \tag{3.13}$$

The last inequality implies $V(t_0,x_0) \geqslant \mathrm{RS}(3.8)$.

We notice that the second part of the proof of the theorem is implied by the inequality (3.13) and the equality $V(t_0,x_0) = \mathrm{RS}(3.8)$. □

We write the Lipschitz-type condition in the notation of a general subgradient:

(GS$^+$) For every $R > 0$ there exists $C > 0$ such that at every point $(t,x,v) \in [0,T] \times R\mathbb{B} \times \mathbb{R}^n$, every $(w,p) \in \partial L_0(t,x,v)$ satisfies $|w| \leqslant C(1+|v|+|L_0(t,x,v)|)(1+|p|)$.

Proposition 3.6. *Assume that L_0 and λ satisfy* (L1$^+$)–(L5$^+$) *and* (GS$^+$). *If the function $\lambda(t,\cdot)$ fulfills locally the Lipschitz condition uniformly in t, then the Lagrangian L given by* (3.6) *satisfies* (L1)–(L5) *and* (GS).

Proof. We prove the condition (GS). Put $\varphi(t,z) := \lambda(t,x)u$ and $\mathcal{L}_0(t,z,v) := L_0(t,x,v)$, where $z = (x,u)$. The function $\varphi(\cdot,\cdot)$ is bounded on $[0,T] \times R\mathbb{B}$, because it is continuous. Moreover, $\varphi(t,\cdot)$ satisfies locally the Lipschitz condition, so there exists $D > 0$ fulfilling the inequality $|w| \leqslant D$ for every $(t,z) \in [0,T] \times R\mathbb{B}$, $w \in \partial(-\varphi(t,z))$. By ([23], Exercise 10.10) we have $\partial L(t,z,v) = \partial \mathcal{L}_0(t,z,v) + \partial(-\varphi(t,z)) \times \{0\}$. Therefore, it is not difficult to prove that if L_0 satisfies (GS$^+$), then L satisfies (GS). □

4. Main results

The main results of this paper are contained in Theorems 4.1 and 4.2 concerning Lipschitzian minimizers. Using these theorems we prove Theorem 4.3 about existence of bilateral solutions and Theorem 4.5 concerning locally Lipschitz continuity of the value function. Finally, we state the theorem about uniqueness of bilateral solutions in the class of function bounded from below and satisfying locally the Lipschitz condition.

Theorem 4.1. *Suppose that L satisfies the epi-continuity property and* (L1)–(L6). *Let the function g be finite, bounded from below and satisfying the local Lipschitz condition. Then at the point $(t_0,x_0) \in \operatorname{dom} V$ and $t_0 \in [0,T)$ the function $(\overline{x},\overline{u})(\cdot)$, being a minimizer, has the function $\overline{x}(\cdot)$ satisfying the Lipschitz condition. Moreover, there exists the function $\overline{\eta}(\cdot)$ satisfying the Lipschitz condition such that at the point (t_0,x_0) the function $(\overline{x},\overline{\eta})(\cdot)$ is a minimizer.*

Supposing that the function g is finite and satisfies the local Lipschitz condition we obtain from Theorems 3.4 and 4.1 that at every point of the effective domain V there exists a minimizer satisfying the Lipschitz condition. So we solve the problem of Lipschitzian minimizers that is stated in [20].The proofs of Theorems 4.1 and 4.2 are contained in Section 5.

Theorem 4.2. *Suppose that L satisfies the epi-continuity property and* (L1)–(L6). *Let the function g be proper, lsc and bounded from below. If $dV(t_0,x_0)(0,0) = 0$ and $t_0 \in [0,T)$, then at the point (t_0,x_0) the function $(\overline{x},\overline{u})(\cdot)$ being a minimizer, has the function $\overline{x}(\cdot)$ satisfying the Lipschitz condition in the neighbourhood of t_0. Moreover, there exists the function $\overline{\eta}(\cdot)$ satisfying the Lipschitz condition in the neighbourhood of t_0 such that at the point (t_0,x_0) the function $(\overline{x},\overline{\eta})(\cdot)$ is a minimizer.*

Supposing lower semicontinuity of the function g in Theorem 4.2 we obtain much weaker claim than in Theorem 4.1. Namely, Lipschitzian minimizers exist only in suffciently regular points of the effective domain and satisfy the Lipschitz condition only on small neighbourhoods of these points. However, this fact is sufficient to prove that lower semicontinuous value function is a bilateral solution (see theorem below).

Theorem 4.3 (Existence)**.** *Suppose that the Hamiltonian H is continuous and satisfies* (H1)−(H6)*. Let g be proper, lsc and bounded from below, and V be the value function associated with g and L, where L is given by* (1.2)*. Then the value function V is a bilateral solution of* (1.1) *that is bounded from below and satisfies the property:*

$$V(T,x) = \liminf_{t'\to T-,\, x'\to x} V(t',x'). \tag{4.1}$$

The theorem about existence of bilateral solutions is proven in this paper using continuity of the Hamiltonian with respect to the time variable. This result is more general than the one stated in [14, 20], because there are assumptions concerning Lipschitz-type continuity of the Hamiltonian with respect to the time variable. The proof of Theorem 4.3 is contained in Section 6.

Denote $\varDelta(t_0,\tau) := [t_0-\tau, t_0+\tau] \cap [0,T]$ and $O_r(s,y) := \varDelta(s,r) \times \mathbb{B}(y,r)$.

Theorem 4.4. *Suppose that L satisfies the epi-continuity property and* (L1)−(L6) *hold. Let the function g be finite, continuous and bounded from below. If V is the value function associated with g and L, then for every $(s,y) \in \operatorname{dom} V$ there exists $r > 0$ such that the function V is finite and continuous on the set $O_r(s,y)$.*

Theorem 4.5. *Suppose that L satisfies the epi-continuity property and* (L1)−(L6)*. Let g be finite, bounded from below and satisfying the local Lipschitz condition. If V is the value function associated with g and L, then for every $(s_0,y_0) \in \operatorname{dom} V$ there exists $r_0 > 0$ such that V is finite on $O_{r_0}(s_0,y_0)$. Moreover, there exists $k_0 > 0$ such that*

$$|V(s,y) - V(t,x)| \leqslant k_0(|s-t| + |y-x|) \quad \text{for all } (s,y),(t,x) \in O_{r_0}(s_0,y_0).$$

The fact that continuity and local Lipschitz continuity of the function g propagate on the value function V are well-known in the literature. However, in this paper we state them with less restrictive assumptions. Moreover, such regularities were mainly considered for value functions in Bolza problems. Value functions considered in this paper are more general and that requires a new approach to these issues (see Sect. 7). In the proof of the Theorem 4.5 an important role is played by Theorem 4.1. Adding to assumptions in Theorems 4.4 and 4.1 an additional condition that $\exists\, D > 0\ \forall\, t,x\ \exists\, v : L(t,x,0,v) \leqslant D$ we can prove that the value function V is finite on $[0,T] \times \mathbb{R}^n$. This condition can be equivalently formulated as $\exists\, D > 0\ \forall\, t,x : -D \leqslant H(t,x,0,0)$. Now we can easily notice that Hamiltonians from Examples 2.7 and 2.8 satisfy this condition. We want to omit such additional assumptions since they influence the generality of considered problems.

Let $X \subset \mathbb{R}^n$ and $f : X \to \mathbb{R} \cup \{+\infty\}$. We say that *an extended and lsc function $f(\cdot)$ satisfies the local Lipschitz condition*, if for every $x \in \operatorname{dom} f$ there exists $r > 0$ such that on the set $r\mathbb{B} \cap X$ the function f is finite and satisfies the Lipschitz condition.

Theorem 4.6 (Uniqueness)**.** *Suppose that the Hamiltonian H is continuous and satisfies* (H1)−(H6)*. Let g be finite, bounded from below and satisfy the local Lipschitz condition, and V be the value function associated with g and L, where L is given by* (1.2)*. If U is a bilateral solution of equation* (1.1) *that is bounded from below and satisfies locally the Lipschitz condition and $U(T,x) = g(x)$, then $U{=}V$ on $[0,T] \times \mathbb{R}^n$.*

Plaskacz and Quincampoix obtain uniqueness in the class of lower semicontinuous functions in the sense of proximal solutions (see [20], Thm. 3.6). In the class of functions that satisfy locally the Lipschitz condition proximal solutions are equivalent to bilateral solutions. Therefore, Theorem 4.6 would follow from the above result, if assumptions in this paper were more restrictive. Theorem 3.6 (Uniqueness) in [20] follows from theorems about invariance and viability proven in [20]. The condition (H4) is not a problem since proofs of mentioned theorems about viability and invariance can be done with little changes using Lemma 3.1. The problem is that Plaskacz and Quincampoix [20] assumed the Lipschitz-type condition with respect to the time variable and we assume only continuity. The Lipschitz-type condition appears only in the theorem about invariance. Proving this theorem with assumptions from this paper we solve the problem.

By z we denote (x,u). Assume that a bilateral solution U of the equation (1.1) satisfies locally the Lipschitz condition. Then for every $(s,y)\in \operatorname{epi} U$, $s\in(0,T)$ there exists a neighbourhood O of a point (s,y) such that

$$\forall\,(t,z)\in \operatorname{epi} U\cap O\quad \forall\,(n_t,n_z)\in \widehat{N}_{\operatorname{epi} U}(t,z)\quad \forall\, f\in Q(t,z)\;:\; -n_t-\langle f,n_z\rangle\leqslant 0 \tag{4.2}$$

Proposition 4.7. *Suppose that L satisfies the epi-continuity property and* (L1)−(L6) *and g is finite. Let U satisfy the local Lipschitz condition, $U(T,x)=g(x)$ and for every $(s,y)\in \operatorname{epi} U$, $s\in(0,T)$ there exists a neighbourhood O of a point (s,y) such that the condition* (4.2) *holds. Then for every L-solution $\overline{z}(\cdot)$ satisfying the Lipschitz condition on $[t_0,T]$ and $\overline{z}(T)\in \operatorname{epi} g(\cdot)$ the inclusion $\overline{z}(t)\in \operatorname{epi} U(t,\cdot)$ is fulfilled for every $t\in[t_0,T]$.*

The above proposition is some version of the theorem about invariance from [20]. We prove it in Appendix. If $(t_0,x_0)\notin \operatorname{dom} V$, then $V(t_0,x_0)\geqslant U(t_0,x_0)$. Let $(t_0,x_0)\in \operatorname{dom} V$, then from results of this paper it follows that at the point (t_0,x_0) there exists a minimizer $\overline{z}(\cdot):=(\overline{x},\overline{u})(\cdot)$ satisfying the Lipschitz condition on $[t_0,T]$. Using Proposition 4.7 we have that $\overline{z}(t_0)\in \operatorname{epi} U(t_0,\cdot)$, so $V(t_0,x_0)\geqslant U(t_0,x_0)$. From the theorem about viability (see [20], Thm. 3.19) the opposite inequality holds $V(t_0,x_0)\leqslant U(t_0,x_0)$. Therefore, $V(t,x)=U(t,x)$ for every $(t,x)\in[0,T]\times\mathbb{R}^n$, which proves Theorem 4.6.

5. Necessary conditions of optimality

Consider the problem of optimization given in the following form

$$\begin{array}{ll}\text{minimize} & \Lambda(x(a),u(a),x(b),u(b))\\ \text{subject to} & (\dot{x}(t),\dot{u}(t))\in Q(t,x(t),u(t))\ \ a.e.\ t\in[a,b],\end{array} \tag{$\mathcal{P}$}$$

where we are to minimize ($\mathcal{P}$) over $\mathcal{A}([a,b],\mathbb{R}^n\times\mathbb{R})$, and Q is given by formula (1.3) and $[a,b]\subset[0,T]$.

Theorem 5.1 ([18], Thm. 3.1)**.** *Suppose that L satisfies the epi-continuity property and* (L1)−(L6)*. Let the function Λ be proper and lsc. If $(\overline{x},\overline{u})(\cdot)$ is a solution of the problem ($\mathcal{P}$), then there exists a function $(p,q)(\cdot)\in \mathcal{A}([a,b],\mathbb{R}^n\times\mathbb{R})$ such that*

(a) $(\dot{p}(t),\dot{q}(t))\in \operatorname{co}\big\{(w,r):(w,r,p(t),q(t))\in N_{gph\,Q(t,\cdot,\cdot)}(\overline{x}(t),\overline{u}(t),\dot{\overline{x}}(t),\dot{\overline{u}}(t))\big\}$ *a.e.* $t\in[a,b]$,
(b) *one of the following two conditions holds:*
 (i) $(p(a),q(a),-p(b),-q(b))\in\partial\Lambda(\overline{x}(a),\overline{u}(a),\overline{x}(b),\overline{u}(b))$,
 (ii) $(p,q)(\cdot)\not\equiv 0$ *and* $(p(a),q(a),-p(b),-q(b))\in\partial^\infty\Lambda(\overline{x}(a),\overline{u}(a),\overline{x}(b),\overline{u}(b))$.

Consider the function Λ given by the formula

$$\Lambda(x_a,u_a,x_b,u_b):=u_a+f(x_a)+\Psi_{\operatorname{epi} g}(x_b,u_b). \tag{5.1}$$

Theorem 5.2. *Suppose that L satisfies the epi-continuity property and* (L1)−(L6)*. Let functions f and g be proper and lsc. If $(\overline{x},\overline{u})(\cdot)$ is a solution of the problem ($\mathcal{P}$) with Λ given by* (5.1)*, then there exists $(p,q)(\cdot)\in \mathcal{A}([a,b],\mathbb{R}^n\times\mathbb{R})$ such that*

(a) $(\dot{p}(t),\dot{q}(t))\in \operatorname{co}\big\{(w,r):(w,r,p(t),-q(t))\in N_{\operatorname{epi} L(t,\cdot,\cdot,\cdot)}(\overline{x}(t),\overline{u}(t),\dot{\overline{x}}(t),-\dot{\overline{u}}(t))\big\}$ *a.e.* $t\in[a,b]$,
(b) $(-p(b),-q(b))\in N_{\operatorname{epi} g}(\overline{x}(b),\overline{u}(b))$, *moreover, one of the following two conditions holds:*
 (i) $(p(a),q(a))\in\partial f(\overline{x}(a))\times\{1\}$,
 (ii) $(p,q)(\cdot)\not\equiv 0$ *and* $(p(a),q(a))\in\partial^\infty f(\overline{x}(a))\times\{0\}$.

Proof. Notice that the point (a) follows from (a) in Theorem 5.1, because for a.e. $t\in[a,b]$

$$\begin{aligned}(\dot{p}(t),\dot{q}(t))&\in \operatorname{co}\big\{(w,r):(w,r,p(t),q(t))\in N_{gph\,Q(t,\cdot,\cdot)}(\overline{x}(t),\overline{u}(t),\dot{\overline{x}}(t),\dot{\overline{u}}(t))\big\}\\ &=\operatorname{co}\big\{(w,r):(w,r,p(t),-q(t))\in N_{\operatorname{epi} L(t,\cdot,\cdot,\cdot)}(\overline{x}(t),\overline{u}(t),\dot{\overline{x}}(t),-\dot{\overline{u}}(t))\big\}\end{aligned}$$

Besides, notice that because of (i) in Theorem 5.1 and ([23], Prop. 10.5) we obtain

$$\begin{aligned}(p(a),q(a),-p(b),-q(b)) &\in \partial \Lambda(\overline{x}(a),\overline{u}(a),\overline{x}(b),\overline{u}(b))\\ &= \{(\alpha,1,\beta,\gamma) : \alpha\in\partial f(\overline{x}(a)), (\beta,\gamma)\in\partial\Psi_{\mathrm{epi}\,g}(\overline{x}(b),\overline{u}(b))\}\\ &= \{(\alpha,1,\beta,\gamma) : \alpha\in\partial f(\overline{x}(a)), (\beta,\gamma)\in N_{\mathrm{epi}\,g}(\overline{x}(b),\overline{u}(b))\}.\end{aligned}$$

Finally, from (ii) in Theorem 5.1 and ([23], Exercise 8.14, Prop. 10.5) we have $(p,q)(\cdot)\not\equiv 0$ and

$$\begin{aligned}(p(a),q(a),-p(b),-q(b)) &\in \partial^\infty \Lambda(\overline{x}(a),\overline{u}(a),\overline{x}(b),\overline{u}(b))\\ &= \{(\alpha,0,\beta,\gamma) : \alpha\in\partial^\infty f(\overline{x}(a)), (\beta,\gamma)\in\partial^\infty\Psi_{\mathrm{epi}\,g}(\overline{x}(b),\overline{u}(b))\}\\ &= \{(\alpha,0,\beta,\gamma) : \alpha\in\partial^\infty f(\overline{x}(a)), (\beta,\gamma)\in N_{\mathrm{epi}\,g}(\overline{x}(b),\overline{u}(b))\}\end{aligned}$$

□

Definition 5.3. Let $(\overline{x},\overline{u})(\cdot)$ be a solution of the problem $(\mathcal{P})$. Define $R := 1+\max\{\,|(\overline{x},\overline{u})(t)| : t\in[a,b]\,\}$. If the condition (GS) holds, then there exists $C>0$ such that the inequality in (GS) is satisfied for every $(t,x,u)\in[0,T]\times R\mathbb{B}\times[-R,R]$ and $v\in\mathbb{R}^n$. Let $C_1,C_2\geqslant 0$ be constants like in (L4). We define $k(t) = C\left[\,1+2C_1+2C_2R+|\dot{\overline{x}}(t)|-\dot{\overline{u}}(t)\,\right]$ for a.e. $t\in[a,b]$.

Lemma 5.4. *Assume that L satisfies* (L1), (L4) *and* (GS). *Let $(\overline{x},\overline{u})(\cdot)$, R and $k(t)$ be as in Definition 5.3. If $(w,r,p(t),-q(t))\in N_{\mathrm{epi}\,L(t,\cdot,\cdot,\cdot)}(\overline{x}(t),\overline{u}(t),\dot{\overline{x}}(t),-\dot{\overline{u}}(t))$, then $|(w,r)|\leqslant 2k(t)|(p(t),q(t))|$.*

Proof. If $(w,r,p(t),-q(t))\in N_{\mathrm{epi}\,L(t,\cdot,\cdot,\cdot)}(\overline{x}(t),\overline{u}(t),\dot{\overline{x}}(t),-\dot{\overline{u}}(t))$, then there exists sequences $(\overline{x}_n,\overline{u}_n,\overline{v}_n,-\overline{\eta}_n)\to(\overline{x}(t),\overline{u}(t),\dot{\overline{x}}(t),-\dot{\overline{u}}(t))$ and $(\overline{w}_n,\overline{r}_n,\overline{p}_n,-\overline{q}_n)\to(w,r,p(t),-q(t))$ with $-\overline{\eta}_n\geqslant L(t,\overline{x}_n,\overline{u}_n,\overline{v}_n)$ and $(\overline{w}_n,\overline{r}_n,\overline{p}_n,-\overline{q}_n)\in\widehat{N}_{\mathrm{epi}\,L(t,\cdot,\cdot,\cdot)}(\overline{x}_n,\overline{u}_n,\overline{v}_n,-\eta_n)$. Let $\overline{L}_n := L(t,\overline{x}_n,\overline{u}_n,\overline{v}_n)$, then

$$\forall\, n\in\mathbb{N},\qquad (\overline{w}_n,\overline{r}_n,\overline{p}_n,-\overline{q}_n)\in\widehat{N}_{\mathrm{epi}\,L(t,\cdot,\cdot,\cdot)}(\overline{x}_n,\overline{u}_n,\overline{v}_n,\overline{L}_n). \tag{5.2}$$

From the inclusion (5.2) there exist sequences $\{(x_n,u_n,v_n)\}$ with $\max\{|x_n-\overline{x}_n|,|u_n-\overline{u}_n|,|v_n-\overline{v}_n|\}\leqslant 1/n$ and $\{(w_n,r_n,p_n,q_n)\}$ with $\max\{|w_n-\overline{w}_n|,|r_n-\overline{r}_n|,|p_n-\overline{p}_n|,|q_n-\overline{q}_n|\}\leqslant 1/n$, satisfying ($L_n := L(t,x_n,u_n,v_n)$)

$$\forall\, n\in\mathbb{N},\qquad (w_n,r_n,p_n,-q_n)\in\widehat{N}_{\mathrm{epi}\,L(t,\cdot,\cdot,\cdot)}(x_n,u_n,v_n,L_n),\qquad |L_n-\overline{L}_n|\leqslant 1/n,\qquad q_n>0, \tag{5.3}$$

Indeed, if we fix $n\in\mathbb{N}$ then from the inclusion (5.2) we have $\overline{q}_n>0$ or $\overline{q}_n=0$. In the first case $\overline{q}_n>0$ we define $(x_n,u_n,v_n):=(\overline{x}_n,\overline{u}_n,\overline{v}_n)$ and $(w_n,r_n,p_n,q_n):=(\overline{w}_n,\overline{r}_n,\overline{p}_n,\overline{q}_n)$. Consider the second case $\overline{q}_n=0$. Then from the inclusion (5.2) we have $(\overline{w}_n,\overline{r}_n,\overline{p}_n)\in\partial^\infty L(t,\overline{x}_n,\overline{u}_n,\overline{v}_n)$. Therefore there exist sequences $(x'_k,u'_k,v'_k)\to(\overline{x}_n,\overline{u}_n,\overline{v}_n)$ with $L(t,x'_k,u'_k,v'_k)\to_k L(t,\overline{x}_n,\overline{u}_n,\overline{v}_n)$ and $(w'_k,r'_k,p'_k)\in\widehat{\partial}L(t,x'_k,u'_k,v'_k)$ with $\tau_k(w'_k,r'_k,p'_k)\to_k(\overline{w}_n,\overline{r}_n,\overline{p}_n)$ for some sequence $\tau_k\to 0+$. So $(\tau_k w'_k,\tau_k r'_k,\tau_k p'_k,-\tau_k)\in\widehat{N}_{\mathrm{epi}\,L(t,\cdot,\cdot,\cdot)}(x'_k,u'_k,v'_k,L'_k)$ for all $k\in\mathbb{N}$, where $L'_k := L(t,x'_k,u'_k,v'_k)$, see ([23], Thm. 8.9). Now taking $k_n\in\mathbb{N}$ large enough we define $(x_n,u_n,v_n):=(x'_{k_n},u'_{k_n},v'_{k_n})$ and $(w_n,r_n,p_n,q_n):=(\tau_{k_n}w'_{k_n},\tau_{k_n}r'_{k_n},\tau_{k_n}p'_{k_n},\tau_{k_n})$.

Note that from the inclusion (5.3) we have $(w_n/q_n,r_n/q_n,p_n/q_n,-1)\in\widehat{N}_{\mathrm{epi}\,L(t,\cdot,\cdot,\cdot)}(x_n,u_n,v_n,L_n)$. Therefore $(w_n/q_n,r_n/q_n,p_n/q_n)\in\widehat{\partial}L(t,x_n,u_n,v_n)\subset\partial L(t,x_n,u_n,v_n)$. From the condition (GS) for $n\in\mathbb{N}$ large enough we obtain

$$\begin{aligned} & |(w_n/q_n,r_n/q_n)| \leqslant C(1+|v_n|+|L(t,x_n,u_n,v_n)|)(1+|p_n/q_n|),\\ \Rightarrow\quad & |(w_n,r_n)| \leqslant C(1+|v_n|+|L(t,x_n,u_n,v_n)|)(q_n+|p_n|)\\ &\leqslant C(1+|v_n|+2C_1+2C_2R+L_n)(q_n+|p_n|)\\ &\leqslant C(1+|v_n|+2C_1+2C_2R+1/n+\overline{L}_n)(q_n+|p_n|)\\ &\leqslant C(1+2C_1+2C_2R+|v_n|+1/n-\overline{\eta}_n)(q_n+|p_n|)\\ &\leqslant 2C(1+2C_1+2C_2R+|v_n|+1/n-\overline{\eta}_n)|(p_n,q_n)|.\end{aligned}$$

In the above inequality, when $n\to+\infty$, we get $|(w,r)|\leqslant 2k(t)|(p(t),q(t))|$. □

Proposition 5.5. *Suppose that L satisfies* (L1)−(L5) *and* (GS). *Let $k(t)$ be as in Definition 5.3. If the function $(p,q)(\cdot)\in\mathcal{A}([a,b],\mathbb{R}^n\times\mathbb{R})$ fulfills the inclusion* (a) *in Theorem 5.2, then the function $q(\cdot)$ is nonincreasing and the inequality holds:*

$$|(\dot p(t),\dot q(t))|\leqslant 2k(t)|(p(t),q(t))| \ \text{ for a.e. } t\in[a,b]. \tag{5.4}$$

Proof. Assume that $(p,q)(\cdot)$ satisfies the inclusion (a) in Theorem 5.2. Let t be an arbitrary fixed element of an adequate subset of the full measure of $[a,b]$. Then for $i\in\{1,2,\ldots,n+2\}$ there exist (w_i,r_i) and $\alpha_i\geqslant 0$ such that $\sum_{i=1}^{n+2}\alpha_i=1$ and $(\dot p(t),\dot q(t))=\sum_{i=1}^{n+2}\alpha_i(w_i,r_i)$, moreover

$$(w_i,r_i,p(t),-q(t))\in N_{\operatorname{epi}L(t,\cdot,\cdot,\cdot)}(\overline{x}(t),\overline{u}(t),\dot{\overline{x}}(t),-\dot{\overline{u}}(t)). \tag{5.5}$$

First, we notice that from Lemma 5.4 we have

$$|(w_i,r_i)|\leqslant 2k(t)|(p(t),q(t))| \ \text{ for all } i\in\{1,2,\ldots,n+2\}. \tag{5.6}$$

Second, from the inequality (5.6) we obtain

$$|(\dot p(t),\dot q(t))|\leqslant\sum_{i=1}^{n+2}\alpha_i|(w_i,r_i)|\leqslant 2k(t)|(p(t),q(t))|. \tag{5.7}$$

It is not difficult to show that from the condition (L3) and the inclusion (5.5) we have $r_i\leqslant 0$ for all $i\in\{1,2,\ldots,n+2\}$. So we obtain the inequality:

$$\dot q(t)=\sum_{i=1}^{n+2}\alpha_i r_i\leqslant 0. \tag{5.8}$$

The variable t is an arbitrary fixed element of the subset of the full measure of $[a,b]$. Therefore, inequalities (5.7) and (5.8) hold for a.e. $t\in[a,b]$. So, the inequality (5.4) is satisfied. Moreover, the function $q(\cdot)$ is nonincreasing, because $q(\cdot)\in\mathcal{A}([a,b],\mathbb{R})$ and $\dot q(t)\leqslant 0$ *a.e.* $t\in[a,b]$. □

5.1. Lipschitzian minimizers

The consequence of results in this subsection is Theorem 4.1. Suppose that $(\overline{x},\overline{u})(\cdot)$, R, C and $k(t)$ are as in Definition 5.3. Let M denote the Lipschitz constant of the function g on $R\mathbb{B}$. Define $\varDelta$ by the formula:

$$\varDelta=(1+M)\exp\left[\int_a^b 2k(s)\mathrm{d}s\right].$$

Theorem 5.6. *Suppose that L satisfies the epi-continuity property and* (L1)−(L6). *Let f be proper and lsc, and g be finite and satisfy locally the Lipschitz condition. If $(\overline{x},\overline{u})(\cdot)$ solves problem ($\mathcal{P}$) with $\varLambda$ given by* (5.1), *then there exists $(p,q)(\cdot)\in\mathcal{A}([a,b],\mathbb{R}^n\times\mathbb{R})$ such that*

(a) $q(t)\geqslant 1/\varDelta$ *and* $|p(t)/q(t)|\leqslant\varDelta^2$ *for all* $t\in[a,b]$;
(b) $\dot{\overline{x}}(t)\in\partial_p H(t,\overline{x}(t),\overline{u}(t),p(t)/q(t))$ *a.e.* $t\in[a,b]$.

Proof. By the inclusion (b) in Theorem 5.2 and continuity of g we have

$$(-p(b),-q(b))\in N_{\operatorname{epi}g}(\overline{x}(b),\overline{u}(b))\subset N_{\operatorname{epi}g}(\overline{x}(b),g(\overline{x}(b))) \tag{5.9}$$

From inclusion (5.9) we obtain $q(b)\geqslant 0$. Moreover, from Proposition 5.5 the function $q(\cdot)$ is nonincreasing. Therefore, the following inequalities hold:

$$q(a)\geqslant q(t)\geqslant q(b)\geqslant 0 \quad \text{for all} \quad t\in[a,b]. \tag{5.10}$$

By Proposition 5.5 the inequality (5.4) also holds. Using Gronwall's Lemma we obtain the inequality:

$$|(p(t), q(t))| \leqslant |(p(b), q(b))| \exp\left[\int_a^b 2k(s)\mathrm{d}s\right] \quad \text{for all } \; t \in [a,b]. \tag{5.11}$$

Now we prove that with our assumptions the condition (ii) in Theorem 5.2 does not hold. On the contrary, if it holds, then $(p,q)(\cdot) \not\equiv 0$ and $q(a) = 0$. Therefore, by the inequality (5.10) we have $q(b) = 0$. So the inclusion (5.9) implies $-p(b) \in \partial^\infty g(\overline{x}(b))$. The function g is finite and satisfies locally the Lipschitz condition, so $\partial^\infty g(\overline{x}(b)) = \{0\}$. Therefore, we have $p(b) = 0$. Since $|(p(b), q(b)| = 0$, then we have from the inequality (5.11) that $(p,q)(\cdot) \equiv 0$, which is impossible, because $(p,q)(\cdot) \not\equiv 0$.

If the condition (ii) in Theorem 5.2 is not satisfied, then the condition (i) in Theorem 5.2 has to be fulfilled and then $q(a) = 1$. We show that $q(b) \neq 0$. On the contrary, if $q(b) = 0$, then by similar reasoning as above we have $p(b) = 0$. Since $|(p(b), q(b))| = 0$, then from the inequality (5.11) we have $(p,q)(\cdot) \equiv 0$, which is impossible, because $q(a) = 1$.

Therefore, we know that $q(a) = 1$ and $q(b) > 0$, so now we can prove the property (a) in Theorem 5.6. By the inclusion (5.9) we have $-p(b)/q(b) \in \partial\, g(\overline{x}(b))$. Since the function g is finite and satisfies the Lipschitz condition on $R\mathbb{B}$ with the constant M, then $-p(b)/q(b) \in M\mathbb{B}$. Therefore, we have $|p(b)| \leqslant Mq(b)$. Moreover, using the inequality (5.11) for every $t \in [a,b]$ we obtain

$$\max\{\, |p(t)|, |q(t)|\,\} \leqslant (M+1)q(b) \exp\left[\int_a^b 2k(s)\mathrm{d}s\right] \quad \text{for all } \; t \in [a,b]. \tag{5.12}$$

If we put $t := a$ in the inequality (5.12), then we have $1 = q(a) \leqslant \Delta\, q(b)$. Moreover, by the inequality (5.10) we obtain $q(t) \geqslant q(b) \geqslant 1/\Delta$ for every $t \in [a,b]$. Besides, by the inequality (5.12) we also have $|p(t)| \leqslant \Delta\, q(b)$. Therefore, by the equality $q(a) = 1$ and the inequality (5.10) we have $|p(t)| \leqslant \Delta$. Consequently, we have the inequality $|p(t)/q(t)| \leqslant \Delta^2$ for all $t \in [a,b]$. It finishes the proof of the property (a).

Now we prove the property (b). By the inclusion (a) of Theorem 5.2 for a.e. $t \in [a,b]$ there exist points (w_t, r_t) such that

$$(w_t, r_t, p(t), -q(t)) \in N_{\operatorname{epi} L(t,\cdot,\cdot,\cdot)}(\overline{x}(t), \overline{u}(t), \dot{\overline{x}}(t), -\dot{\overline{u}}(t)) \quad a.e. \;\; t \in [a,b].$$

Let t be arbitrary fixed element of an adequate subset of the full measure of $[a,b]$. Then from the above inclusion there exists sequences $(x_n, u_n, v_n, -\eta_n) \to (\overline{x}(t), \overline{u}(t), \dot{\overline{x}}(t), -\dot{\overline{u}}(t))$ and $(w_n, r_n, p_n, -q_n) \to (w_t, r_t, p(t), -q(t))$ with $-\eta_n \geqslant L(t, x_n, u_n, v_n)$ and $(w_n, r_n, p_n, -q_n) \in \widehat{N}_{\operatorname{epi} L(t,\cdot,\cdot,\cdot)}(x_n, u_n, v_n, -\eta_n)$. Let $L_n := L(t, x_n, u_n, v_n)$, then

$$\forall\, n \in \mathbb{N} \qquad (w_n, r_n, p_n, -q_n) \in \widehat{N}_{\operatorname{epi} L(t,\cdot,\cdot,\cdot)}(x_n, u_n, v_n, L_n).$$

Note that $q_n > 0$ for large $n \in \mathbb{N}$, because $q(t) > 0$. Therefore, from the above inclusion $(w_n/q_n, r_n/q_n, p_n/q_n) \in \widehat{\partial}\, L(t, x_n, u_n, v_n)$ for large $n \in \mathbb{N}$. Hence $p_n/q_n \in \widehat{\partial}_p\, L(t, x_n, u_n, v_n)$ for large $n \in \mathbb{N}$. Using results of the convex analysis we obtain $v_n \in \widehat{\partial}_p\, H(t, x_n, u_n, p_n/q_n)$. From continuity of the Hamiltonian H and the convex analysis, when $n \to +\infty$, we get

$$\dot{\overline{x}}(t) \in \widehat{\partial}_p\, H(t, \overline{x}(t), \overline{u}(t), p(t)/q(t)) = \partial_p\, H(t, \overline{x}(t), \overline{u}(t), p(t)/q(t)).$$

The variable t is an arbitrary fixed element of the subset of the full measure of $[a,b]$. Therefore, we have $\dot{\overline{x}}(t) \in \partial_p\, H(t, \overline{x}(t), \overline{u}(t), p(t)/q(t))$ for almost all $t \in [a,b]$, which completes the proof. □

Proposition 5.7. *If assumptions of Theorem 5.6 are satisfied and $(\overline{x}, \overline{u})(\cdot)$ is the solution of the problem ($\mathcal{P}$), then there exists $K > 0$ such that*

$$\max\{\, |\dot{\overline{x}}(t)|,\; |L(t, \overline{x}(t), \overline{u}(t), \dot{\overline{x}}(t))|\,\} < K \quad \text{for a.e. } \; t \in [a,b]. \tag{5.13}$$

Proof. By our assumptions we have $(t,\overline{x}(t),\overline{u}(t))\in[0,T]\times R\mathbb{B}\times R\mathbb{B}$ for every $t\in[a,b]$. By (a) of Theorem 5.6 we have $p(t)/q(t)\in\Delta^2\mathbb{B}$ for all $t\in[a,b]$. Therefore, by continuity of H, convexity of $H(t,x,u,\cdot)$ and Theorem 5.6 (b) there exists $D>0$ such that $|\dot{\overline{x}}(t)|<D$ a.e. $t\in[a,b]$. Using the convex analysis and the inclusion (b) of Theorem 5.6 for a.e. $t\in[a,b]$ we have

$$\begin{aligned}&L(t,\overline{x}(t),\overline{u}(t),\dot{\overline{x}}(t))=\langle\,\dot{\overline{x}}(t),p(t)/q(t)\,\rangle-H(t,\overline{x}(t),\overline{u}(t),p(t)/q(t)),\\ \Rightarrow\quad &|L(t,\overline{x}(t),\overline{u}(t),\dot{\overline{x}}(t))|\leqslant|\dot{\overline{x}}(t)||p(t)/q(t)|+|H(t,\overline{x}(t),\overline{u}(t),p(t)/q(t))|\\ &\qquad\leqslant D\Delta^2+|H(t,\overline{x}(t),\overline{u}(t),p(t)/q(t))|.\end{aligned}$$

From the last inequality and boundedness of the Hamiltonian H on the compact set we have boundedness of the Lagrangian $L(t,\overline{x}(t),\overline{u}(t),\dot{\overline{x}}(t))$ a.e. on $[a,b]$, that finishes the proof. □

Remark 5.8. The function $(\overline{x},\overline{u})(\cdot)$ being a minimizer from Definition 3.3 is a solution of the problem $(\mathcal{P})$ with Λ given by the formula (5.1), where $f=\Psi_{\{x_0\}}$ and $[t_0,T]=[a,b]$.

Proposition 5.9. *Suppose that L satisfies* (L1)−(L5) *and g is lsc and bounded from below. Let V be the value function associated with g and L. If at the point $(t_0,x_0)\in\operatorname{dom}V$, $t_0\in[0,T)$, the function $(\overline{x},\overline{u})(\cdot)$ is a minimizer such that for some constant $K>0$ the inequality* (5.13) *holds, then there exists a function $\overline{\eta}(\cdot)$ such that at the point (t_0,x_0) a function $(\overline{x},\overline{\eta})(\cdot)$ is a minimizer satisfying the Lipschitz condition.*

Proof. Since at the point (t_0,x_0) a function $(\overline{x},\overline{u})(\cdot)$ is a minimizer, we can define the function $\overline{\eta}(\cdot)\in\mathcal{A}([t_0,T],\mathbb{R})$ by the formula:

$$\overline{\eta}(t)=V(t_0,x_0)-\int_{t_0}^{t}L(s,\overline{x}(s),\overline{u}(s),\dot{\overline{x}}(s))\,\mathrm{d}s. \tag{5.14}$$

Notice that $\dot{\overline{\eta}}(t)=-L(t,\overline{x}(t),\overline{u}(t),\dot{\overline{x}}(t))$ a.e. $t\in[t_0,T]$. Therefore, from (5.13) the derivative $(\dot{\overline{x}},\dot{\overline{\eta}})(t)$ is bounded a.e. on $[t_0,T]$. Therefore, a function $(\overline{x},\overline{\eta})(\cdot)$ satisfies the Lipschitz condition on $[t_0,T]$. To finish the proof we are to show that the function $(\overline{x},\overline{\eta})(\cdot)$ is L-solution satisfying properties (3.5). By the formula (5.14) we have $\overline{\eta}(t_0)=V(t_0,x_0)$. Since the function $(\overline{x},\overline{u})(\cdot)$ is L-solution, so

$$-L(t,\overline{x}(t),\overline{u}(t),\dot{\overline{x}}(t))\;\geqslant\;\dot{\overline{u}}(t)\quad a.e.\;\; t\in[t_0,T]. \tag{5.15}$$

Using the inequality (5.15) and the equality (5.14) we have that

$$\overline{\eta}(t)\geqslant V(t_0,x_0)+\overline{u}(t)-\overline{u}(t_0)=\overline{u}(t)\;\text{ for all }\; t\in[t_0,T]. \tag{5.16}$$

By the property (3.5) of the function $(\overline{x},\overline{u})(\cdot)$ we have $\overline{u}(T)\geqslant g(\overline{x}(T))$. Therefore, by (5.16) we obtain $\overline{\eta}(T)\geqslant g(\overline{x}(T))$. Finally, we notice that by the condition (L3) and the inequality (5.16) we have

$$-\dot{\overline{\eta}}(t)=L(t,\overline{x}(t),\overline{u}(t),\dot{\overline{x}}(t))\geqslant L(t,\overline{x}(t),\overline{\eta}(t),\dot{\overline{x}}(t))\quad a.e.\;\; t\in[t_0,T].$$ □

5.2. Equi-Lipschitzian minimizers

In the previous subsection we have proved that solutions $(\overline{x},\overline{u})(\cdot)$ of the problem $(\mathcal{P})$ include functions $\overline{x}(\cdot)$ satisfying the Lipschitz condition. Now we show that for some family of solutions $\{(\overline{x},\overline{u})(\cdot)\}$ of the problem $(\mathcal{P})$ functions $\{\overline{x}(\cdot)\}$ satisfy the Lipschitz condition with the same constant. This fact is used in the proof of local Lipschitz continuity of the value function.

Definition 5.10. By $\mathfrak{F}(\check{R})$ we denote the set of L-solutions $(x,u)(\cdot)$ on intervals $[a,b]\subset[0,T]$ and satisfying properties:

$$|(x,u)(t)|<\check{R}/2\;\text{ for all }\; t\in[a,b],\qquad \int_a^b|\dot{x}(t)|\,\mathrm{d}t<\check{R}.$$

Definition 5.11. By $\mathfrak{R}(\hat{R})$ we denote the set of L-solutions $(x,u)(\cdot)$ on intervals $[a,b]\subset[0,T]$ and satisfying properties:

$$|(x,u)(t)|<\hat{R}/2 \text{ for all } t\in[a,b], \qquad |\dot{x}(t)|<\hat{R} \text{ a.e. } t\in[a,b].$$

Theorem 5.12. *Suppose that L satisfies the epi-continuity property and* (L1)−(L6). *Let g be finite and satisfy the local Lipschitz condition. Then for every $\check{R}>0$ there exists $\hat{R}>0$ such that for all solutions $(\overline{x},\overline{u})(\cdot)$ of the problem* ($\mathcal{P}$) *the following implication holds*

$$(\overline{x},\overline{u})(\cdot)\in\mathfrak{F}(\check{R}) \implies (\overline{x},\overline{u})(\cdot)\in\mathfrak{R}(\hat{R}). \tag{5.17}$$

Proof. We fix $\check{R}>0$ and put $R:=\check{R}/2+1$. We notice that for every function $(x,u)(\cdot)\in\mathfrak{F}(\check{R})$ the following inequality $R_{(x,u)}=1+\max\{|(x,u)(t)|\,;t\in[a,b]\}\leqslant R$ holds. From Corollary 2.6 the condition (GS) is satisfied. Therefore, there exists a constant $C_R>0$ such that the inequality in (GS) holds for every $(t,x,u)\in[0,T]\times R\mathbb{B}\times[-R,R]$ and every $v\in\mathbb{R}^n$. Let $C_1,C_2\geqslant 0$ be constants from (L4). For all functions $(x,u)(\cdot)\in\mathfrak{F}(\check{R})$ we put $k_{(x,u)}(t):=C_R[\,1+2C_1+2C_2R+|\dot{x}(t)|-\dot{u}(t)\,]$ *a.e.* $t\in[a,b]$. Then for every function $(x,u)(\cdot)\in\mathfrak{F}(\check{R})$ the following inequality hold:

$$\begin{aligned}\int_a^b k_{(x,u)}(t)\,\mathrm{d}t &= \int_a^b C_R[\,1+2C_1+2C_2R+|\dot{x}(t)|-\dot{u}(t)\,]\,\mathrm{d}t\\ &\leqslant C_R[\,T(1+2C_1+2C_2R)+4R\,].\end{aligned} \tag{5.18}$$

Let M_R denote the Lipschitz constant of the function g on $R\mathbb{B}$. Then by the inequality (5.18) we obtain for every function $(x,u)(\cdot)\in\mathfrak{F}(\check{R})$ that

$$\begin{aligned}\Delta_{(x,u)} &= (1+M_R)\exp\left[\int_a^b 2k_{(x,u)}(s)\mathrm{d}s\right]\\ &\leqslant (1+M_R)\exp[\,C_R[\,T(1+2C_1+2C_2R)+4R\,]\,] \ =:\ \Delta_R.\end{aligned}$$

For every $(\overline{x},\overline{u})(\cdot)\in\mathfrak{F}(\check{R})$ solution of the problem ($\mathcal{P}$), by Theorem 5.6, there exists a function $(p,q)_{(\overline{x},\overline{u})}(\cdot)\in\mathcal{A}([a,b],\mathbb{R}^n\times\mathbb{R})$ such that

($\star$) $q_{(\overline{x},\overline{u})}(t)\geqslant 1/\Delta_R$ and $|p_{(\overline{x},\overline{u})}(t)/q_{(\overline{x},\overline{u})}(t)|\leqslant\Delta_R^2$ for all $t\in[a,b]$;
($\star\star$) $\dot{\overline{x}}(t)\in\partial_p H(t,\overline{x}(t),\overline{u}(t),p_{(\overline{x},\overline{u})}(t)/q_{(\overline{x},\overline{u})}(t))$ a.e. $t\in[a,b]$.

If we assume that $(\overline{x},\overline{u})(\cdot)\in\mathfrak{F}(\check{R})$, then $(t,\overline{x}(t),\overline{u}(t))\in[0,T]\times R\mathbb{B}\times R\mathbb{B}$ for every $t\in[a,b]$. By ($\star$) we get $p_{(\overline{x},\overline{u})}(t)/q_{(\overline{x},\overline{u})}(t)\in\Delta_R^2\mathbb{B}$ for all $t\in[a,b]$. Therefore, from continuity of H, convexity of $H(t,x,u,\cdot)$ and ($\star\star$) there exists $D_{R,H}>0$ such that $|\dot{\overline{x}}(t)|<D_{R,H}$ a.e. $t\in[a,b]$. So for every solution $(\overline{x},\overline{u})(\cdot)\in\mathfrak{F}(\check{R})$ of the problem ($\mathcal{P}$) we have $(\overline{x},\overline{u})(\cdot)\in\mathfrak{R}(\hat{R})$, where $\hat{R}:=2R+D_{R,H}$. □

5.3. Lipschitzian minimizers in the neighbourhood of t_0

The consequence of results in this subsection is Theorem 4.2.

Theorem 5.13. *Suppose that L satisfies the epi-continuity property and* (L1)−(L6). *Let g be proper and lsc, and f be finite and satisfy the local Lipschitz condition. If $(\overline{x},\overline{u})(\cdot)$ solves problem* ($\mathcal{P}$) *with Λ given by* (5.1), *then there exists $(p,q)(\cdot)\in\mathcal{A}([a,b],\mathbb{R}^n\times\mathbb{R})$ and $\tau\in(a,b]$ such that*

(a) $q(t)\geqslant 1/2$ *for all* $t\in[a,\tau]$;
(b) $\dot{\overline{x}}(t)\in\partial_p H(t,\overline{x}(t),\overline{u}(t),p(t)/q(t))$ *a.e.* $t\in[a,\tau]$.

Proof. Let R, C and $k(t)$ be as in Definition 5.3. From Proposition 5.5 we have the inequality (5.4). Applying to it Gronwall's Lemma we get

$$|(p(t), q(t))| \leqslant |(p(a), q(a))| \exp\left[\int_a^b 2k(s)\mathrm{d}s\right] \text{ for all } t \in [a, b]. \tag{5.19}$$

Now we prove that with our assumptions the condition (ii) in Theorem 5.2 does not hold. On the contrary, if it holds, then $(p, q)(\cdot) \not\equiv 0$ and $q(a) = 0$. The function f is finite and satisfies locally the Lipschitz condition, so $\partial^\infty f(\overline{x}(a)) = \{0\}$. Therefore, by the inclusion (ii) in Theorem 5.2 we have $p(a) = 0$. Since $|(p(a), q(a)| = 0$, so from the inequality (5.19) we get $(p, q)(\cdot) \equiv 0$, that is impossible, because $(p, q)(\cdot) \not\equiv 0$.

If the condition (ii) of Theorem 5.2 does not hold, then the condition (i) of Theorem 5.2 has to be satisfied, so $q(a) = 1$. Therefore, from continuity of the function $q(\cdot)$ there exists $\tau \in (a, b]$ such that $q(t) \geqslant 1/2$ for all $t \in [a, \tau]$.

The property (b) can be proved similarly to the property (b) of Theorem 5.6 if we take into consideration that the inequality $q(t) > 0$ holds only for $t \in [a, \tau]$. □

Remark 5.14. Using Theorem 5.13 we can prove properties of Proposition 5.7 and 5.9 only on small intervals $[a, \tau]$ and $[t_0, \tau_0]$, correspondingly.

Lemma 5.15. *Suppose that the function $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ is bounded from below. If $d\,\varphi(x_0)(0) = 0$, then there exists a function $\psi : \mathbb{R}^n \to \mathbb{R}$, that satisfies the Lipschitz condition, $\varphi(x_0) = \psi(x_0)$ and $\varphi(x) \geqslant \psi(x)$ for every $x \in \mathbb{R}^n$.*

Proof. Let $C > 0$ be such that $\varphi(\cdot) > -C$. By ([23], Prop. 8.32), there exists a constant $k_1 > 0$ and a neighbourhood $\mathcal{W}$ of a point x_0 such that the inequality $-k_1|x - x_0| \leqslant \varphi(x) - \varphi(x_0)$ holds for every $x \in \mathcal{W}$. Let $k_2 > 0$ be large enough so $x_0 + [(\varphi(x_0) + C)/k_2]\mathbb{B} \subset \mathcal{W}$. Put $k_0 = \max\{k_1, k_2\}$ and

$$\psi(x) = \inf \{ \varphi(z) + k_0|z - x| : z \in \mathbb{R}^n \}.$$

We show that the function $\psi(\cdot)$ is as in the conclusion of Lemma 5.15. From the definition of this function it is not difficult to notice that $\psi(\cdot)$ is finite and Lipschitz, and $\varphi(x) \geqslant \psi(x)$ for all $x \in \mathbb{R}^l$. We are to prove that the equality $\varphi(x_0) = \psi(x_0)$ holds. To this end it is enough to show that $\psi(x_0) \geqslant \varphi(x_0)$. First, we notice that if $z \notin x_0 + [(\varphi(x_0) + C)/k_0]\mathbb{B}$, then we have the inequality $\varphi(x_0) < -C + k_0|z - x_0|$, that implies

$$\psi(x_0) = \inf\{ \varphi(z) + k_0|z - x_0| : z \in x_0 + [(\varphi(x_0) + C)/k_0]\mathbb{B} \}.$$

Now we can prove that the inequality $\psi(x_0) \geqslant \varphi(x_0)$ holds.

$$\begin{aligned}\psi(x_0) &\geqslant \inf \{ \varphi(z) + k_0|z - x_0| : z \in x_0 + [(\varphi(x_0) + C)/k_2]\mathbb{B} \} \\ &\geqslant \inf \{ \varphi(z) + k_0|z - x_0| : z \in \mathcal{W} \} \\ &\geqslant \inf \{ \varphi(x_0) + (k_0 - k_1)|z - x_0| : z \in \mathcal{W} \} \\ &\geqslant \varphi(x_0),\end{aligned}$$

that finishes the proof. □

Proposition 5.16. *Suppose that L satisfies the epi-continuity property and* (L1)–(L6)*. Let the function g be proper, lsc and bounded from below. Fix $(t_0, x_0) \in \operatorname{dom} V$, $t_0 \in [0, T)$ and assume that there exists a function $\psi(\cdot, \cdot)$ that is finite on $[0, T] \times \mathbb{R}^n$ and $V(t_0, x_0) = \psi(t_0, x_0)$, besides $V(t, x) \geqslant \psi(t, x)$ for every $(t, x) \in [0, T] \times \mathbb{R}^n$. Then at the point (t_0, x_0) the function $(\overline{x}, \overline{u})(\cdot)$ being a minimizer from Definition 3.3 is a solution of the problem $(\mathcal{P})$ with the function Λ given by the formula* (5.1)*, where $f(\cdot) = -\psi(t_0, \cdot)$ and $[t_0, T] = [a, b]$.*

Proof. First, we show that for every L-solution $(x,u)(\cdot)$ on $[t_0,T]$ the following inequality is satisfied.

$$\begin{aligned} 0 &\leqslant \Lambda(x(t_0),u(t_0),x(T),u(T)) \\ &= u(t_0) - \psi(t_0,x(t_0)) + \Psi_{\mathrm{epi}\,g}(x(T),u(T)). \end{aligned} \tag{5.20}$$

We consider two cases $u(T) < g(x(T))$ and $u(T) \geqslant g(x(T))$. If $u(T) < g(x(T))$, then the inequality (5.20) is obvious, because $\Lambda(x(t_0),u(t_0),x(T),u(T)) = +\infty$. Let $u(T) \geqslant g(x(T))$, then $u(t_0) \geqslant V(t_0,x(t_0))$, moreover

$$\begin{aligned} \Lambda(x(t_0),u(t_0),x(T),u(T)) &= u(t_0) - \psi(t_0,x(t_0)) \\ &\geqslant V(t_0,x(t_0)) - \psi(t_0,x(t_0)) \;\; \geqslant \;\; 0. \end{aligned}$$

Now we prove that if at the point (t_0,x_0) the function $(\overline{x},\overline{u})(\cdot)$ is a minimizer from Definition 3.3, then

$$0 \;=\; \Lambda(\overline{x}(t_0),\overline{u}(t_0),\overline{x}(T),\overline{u}(T)). \tag{5.21}$$

Indeed, since the function $(\overline{x},\overline{u})(\cdot)$ is a minimizer, then $\overline{u}(T) \geqslant g(\overline{x}(T))$ and $(\overline{x},\overline{u})(t_0) = (x_0,V(t_0,x_0))$. Therefore we obtain

$$\begin{aligned} \Lambda(\overline{x}(t_0),\overline{u}(t_0),\overline{x}(T),\overline{u}(T)) &= \overline{u}(t_0) - \psi(t_0,\overline{x}(t_0)) \\ &= V(t_0,x_0) - \psi(t_0,x_0) \;\; = \;\; 0. \end{aligned}$$

From the inequality (5.20) and the equality (5.21) we have that at the point (t_0,x_0) the function $(\overline{x},\overline{u})(\cdot)$ being a minimizer from Definition 3.3 is a solution of the problem $(\mathcal{P})$ with the function Λ given by the formula (5.1), where $f(\cdot) = -\psi(t_0,\cdot)$ and $[t_0,T] = [a,b]$. □

6. Existence of bilateral solutions

In this section we prove Proposition 6.2 and 6.6, where the value function is a subsolution and supersolution of (1.1), correspondingly. The supersolution (1.1) is defined the same way as the bilateral solution of (1.1), but with the equality in (1.6) replaced by the weak inequality $\geqslant$. Similarly we define a subsolution replacing in (1.6) the equality by the inequality $\leqslant$. If a solution is simultaneously sub- and supersolution of (1.1), then it is bilateral solution of (1.1). Therefore, the consequence of Proposition 6.2 and 6.6 is Theorem 4.3.

6.1. Supersolution

The properties (L1)$-$(L5) imply that the multivalued function Q given by formula (1.3) has the following property ([5], Sect. 8.5.A):

$$Q(t,x,u) = \bigcap_{\varepsilon>0} \mathrm{cl}\,\mathrm{co}\, Q(t,x,u;\varepsilon), \tag{Q}$$

where
$$Q(t,x,u;\varepsilon) := \bigcup_{|t-t'|<\varepsilon,\ |x-x'|<\varepsilon,\ |u-u'|<\varepsilon} Q(t',x',u').$$

Proposition 6.1. *Suppose that the Lagrangian L satisfies* (L1)$-$(L5)*. Let g be proper, lsc and bounded from below, and V be the value function associated with g and L. If at the point $(t_0,x_0) \in \mathrm{dom}\,V$, $t_0 \in [0,T)$ there exists a minimizer $(\overline{x},\overline{u})(\cdot)$ such that the function $\overline{x}(\cdot)$ satisfies the Lipschitz condition on $[t_0,t_0+\tau]$ for some $\tau \in (0,T-t_0]$, then there exists $v_0 \in \mathbb{R}^n$ such that*

$$d\,V(t_0,x_0)(1,v_0) \leqslant -L(t_0,x_0,V(t_0,x_0),v_0). \tag{6.1}$$

Proof. Let $(\overline{x},\overline{u})(\cdot)$ be a minimizer at (t_0,x_0) satisfying the Lipschitz condition on $[t_0,t_0+\tau]$. Therefore there exists a sequence $h_n\in(0,\tau]$ such that $h_n\to 0+$ and

$$\frac{1}{h_n}\int_{t_0}^{t_0+h_n}\dot{\overline{x}}(t)\,\mathrm{d}t=\frac{\overline{x}(t_0+h_n)-\overline{x}(t_0)}{h_n}\to v_0. \tag{6.2}$$

From the definition of the value function we have the inequality $\overline{u}(t)\geqslant V(t,\overline{x}(t))$ for every $t\in[t_0,T]$. Choose v_n such that $\overline{x}(t_0+h_n)=x_0+v_nh_n$. Then we obtain $(1,v_n)\to(1,v_0)$. Therefore, the following inequalities are satisfied:

$$\begin{aligned} d\,V(t_0,x_0)(1,v_0)&\leqslant\liminf_n\frac{V(t_0+h_n,\overline{x}(t_0+h_n))-V(t_0,x_0)}{h_n}\\ &\leqslant\liminf_n\frac{\overline{u}(t_0+h_n)-\overline{u}(t_0)}{h_n}\\ &\leqslant\liminf_n\frac{1}{h_n}\int_{t_0}^{t_0+h_n}-L(t,\overline{x}(t),\overline{u}(t),\dot{\overline{x}}(t))\,\mathrm{d}t. \end{aligned}\tag{6.3}$$

We consider two cases:

Case 1. If $d\,V(t_0,x_0)(1,v_0)=-\infty$, then the inequality (6.1) holds obviously.

Case 2. If $d\,V(t_0,x_0)(1,v_0)>-\infty$, then denoting by η_0 the right hand side of the inequality (6.3), then using the assumption (L4), we obtain that $\eta_0\in\mathbb{R}$. Fix $\varepsilon>0$ and choose $n_0\in\mathbb{N}$ such that for every $n>n_0$ the following property holds

$$\forall\, t\in[t_0,t_0+h_n]\quad |t_0-t|<\varepsilon,\ |x_0-\overline{x}(t)|<\varepsilon,\ |u_0-\overline{u}(t)|<\varepsilon, \tag{6.4}$$

where $(\overline{x},\overline{u})(t_0)=(x_0,u_0)$. We notice that for a.e. $t\in[t_0,T]$ we have

$$(\dot{\overline{x}}(t),-L(t,\overline{x}(t),\overline{u}(t),\dot{\overline{x}}(t)))\in Q(t,\overline{x}(t),\overline{u}(t)),$$

so using (6.4) for a.e. $t\in[t_0,t_0+h_n]$ we obtain

$$(\dot{\overline{x}}(t),-L(t,\overline{x}(t),\overline{u}(t),\dot{\overline{x}}(t)))\in Q(t_0,x_0,u_0;\varepsilon).$$

Therefore, for a.e. $t\in[t_0,t_0+h_n]$ we get

$$(\dot{\overline{x}}(t),-L(t,\overline{x}(t),\overline{u}(t),\dot{\overline{x}}(t)))\in \operatorname{cl}\operatorname{co}Q(t_0,x_0,u_0;\varepsilon),$$

so for every $n>n_0$ we have

$$\frac{1}{h_n}\int_{t_0}^{t_0+h_n}\big(\dot{\overline{x}}(t),-L(t,\overline{x}(t),\overline{u}(t),\dot{\overline{x}}(t))\big)\,\mathrm{d}t\ \in\ \operatorname{cl}\operatorname{co}Q(t_0,x_0,u_0;\varepsilon). \tag{6.5}$$

From (6.2), the definition of η_0 and (6.5) for a subsequence, if $n\to+\infty$, we get

$$(v_0,\eta_0)\in\operatorname{cl}\operatorname{co}Q(t_0,x_0,u_0;\varepsilon).$$

The number ε is arbitrary, so we obtain

$$(v_0,\eta_0)\in\bigcap_{\varepsilon>0}\operatorname{cl}\operatorname{co}Q(t_0,x_0,u_0;\varepsilon).$$

Therefore, from the property (Q) we see that $(v_0,\eta_0)\in Q(t_0,x_0,u_0)$, that together with (6.3) gives us

$$d\,V(t_0,x_0)(1,v_0)\leqslant\eta_0\leqslant -L(t_0,x_0,u_0,v_0).$$

□

Proposition 6.2. *Suppose that H is continuous and satisfies* (H1)−(H6). *Let g be proper, lsc and bounded from below, and V be the value function associated with g and L, where L is given by* (1.2). *Then the value function V is the supersolution.*

Proof. Let $(t_0, x_0) \in \operatorname{dom} V$, $t_0 \in (0, T)$. We are to show that for $(p_t, p_x) \in \widehat{\partial} V(t_0, x_0)$ the following inequality holds $-p_t + H(t_0, x_0, V(t_0, x_0), -p_x) \geqslant 0$. If $(p_t, p_x) \in \widehat{\partial} V(t_0, x_0)$, then using the property ([23], s. 301), for every $v \in \mathbb{R}^n$ we see that the following inequality is satisfied

$$\langle\, (p_t, p_x), (1, v)\,\rangle \leqslant d\,V(t_0, x_0)(1, v). \tag{6.6}$$

Moreover, $d\,V(t_0, x_0)(0, 0) = 0$, because the set $\widehat{\partial} V(t_0, x_0)$ is nonempty. Therefore assumptions of Theorem 4.2 are satisfied, so from Theorem 3.4 at the point (t_0, x_0) there exists a minimizer $(\overline{x}, \overline{u})(\cdot)$ such that the function $\overline{x}(\cdot)$ satisfies the Lipschitz condition in the neighbourhood of t_0. This fact allows us to use Proposition 6.1, so there exists $v_0 \in \mathbb{R}^n$ such that the inequality (6.1) holds. Putting inequalities (6.6) and (6.1) together we have that $v_0 \in \operatorname{dom} L(t_0, x_0, V(t_0, x_0), \cdot)$ and

$$\begin{aligned} 0 &\leqslant -p_t + \langle -p_x, v_0\rangle - L(t_0, x_0, V(t_0, x_0), v_0) \\ &\leqslant -p_t + H(t_0, x_0, V(t_0, x_0), -p_x). \end{aligned}$$

□

6.2. Subsolution

Recall that if the Lagrangian L satisfies the epi-continuity property and $L(t, x, u, \cdot)$ is proper, then a multivalued function Q is lsc in the sense of Kuratowski (see Rem. 2.3).

Proposition 6.3. *Let a multivalued function $F : \mathbb{R}^d \rightsquigarrow \mathbb{R}^d$ be lsc in the sense of Kuratowski and have closed, nonempty and convex values. Then for every $z_0 \in \mathbb{R}^d$, $y_0 \in F(z_0)$ there exists $\tau > 0$ and a function z: $(t_0 - \tau, t_0 + \tau) \to \mathbb{R}^d$ from the class C^1 such that $\dot{z}(t) \in F(z(t))$, $z(t_0) = z_0$, $\dot{z}(t_0) = y_0$.*

Proof. Fix $z_0 \in \mathbb{R}^d$ and $y_0 \in F(z_0)$. Using Michael's theorem (about a continuous selector [23], Cor. 5.59), we get existence of a continuous function $f : \mathbb{R}^d \to \mathbb{R}^d$ such that $f(z_0) = y_0$ and $f(z) \in F(z)$ for every $z \in \mathbb{R}^d$. From Peano's theorem there exists $\tau > 0$ and a function $z : (t_0 - \tau, t_0 + \tau) \to \mathbb{R}^d$ from the class C^1 such that $\dot{z}(t) = f(z(t))$ for every $t \in (t_0 - \tau, t_0 + \tau)$ and $z(t_0) = z_0$. We notice that $\dot{z}(t) = f(z(t)) \in F(z(t))$ and $\dot{z}(t_0) = f(z(t_0)) = f(z_0) = y_0$, what finishes the proof. □

Corollary 6.4. *Suppose that L satisfies the epi-continuity property and* (L1)−(L5) *hold. If $t_0 \in [0, T]$ and $v_0 \in \operatorname{dom} L(t_0, x_0, u_0, \cdot)$, then there exists $\tau > 0$ and a function $(x, u) : \Delta(t_0, \tau) \to \mathbb{R}^n \times \mathbb{R}$ of the class C^1 such that $(x, u)(t_0) = (x_0, u_0)$ and $-\dot{u}(t) \geqslant L(t, x(t), u(t), \dot{x}(t))$ for all $t \in \Delta(t_0, \tau)$, moreover $(\dot{x}, \dot{u})(t_0) = (v_0, -L(t_0, x_0, u_0, v_0))$.*

Proof. Fix $t_0 \in [0, T]$ and $v_0 \in \operatorname{dom} L(t_0, x_0, u_0, \cdot)$. Let us extend the multivalued function Q on the set $[0, T] \times \mathbb{R}^n \times \mathbb{R}$ to the multivalued function $\overline{Q}$ on the set R^{1+n+1} in the following way $\overline{Q}(t, x, u) = Q(T, x, u)$ for $t > T$ and $\overline{Q}(t, x, u) = Q(0, x, u)$ for $t < 0$. Let $F(t, x, u) = \{1\} \times \overline{Q}(t, x, u)$. Then by Proposition 6.3, there exists $\tau > 0$ and a function $z : (t_0 - 2\tau, t_0 + 2\tau) \to \mathbb{R}^d$ of the class C^1 such that $\dot{z}(t) \in F(z(t))$, $z(t_0) = (t_0, x_0, u_0)$, $\dot{z}(t_0) = (1, v_0, -L(t_0, x_0, u_0, v_0))$. If $z(t) = (r(t), x(t), u(t))$, then by the definition of F we have $r(t) = t$, moreover $-\dot{u}(t) \geqslant L(t, x(t), u(t), \dot{x}(t))$ for $t \in \Delta(t_0, \tau)$. □

Proposition 6.5. *Suppose that L satisfies the epi-continuity property and* (L1)−(L5). *Let g be an lsc function bounded from below, and V be the value function associated with g and L. Then for every $(t, x) \in \operatorname{dom} V$, $t \in (0, T)$, and every $v \in \operatorname{dom} L(t, x, V(t, x), \cdot)$ the inequality*

$$D_{\uparrow} V(t, x)(-1, -v) \leqslant L(t, x, V(t, x), v), \tag{6.7}$$

holds. Moreover, the equality (4.1) *is satisfied.*

Proof. The proof is similar to ([20], Prop. 3.15), with such a difference that instead of using ([20], Prop. 3.14), we use Proposition 6.3. The proof of ([20], Prop. 3.14), demands the pseudo-Lipschitz property of F with respect to the variable t. Proposition 6.3 is proven only, with the use of lower semicontinuity of F. Indeed, to prove Proposition 6.5 it is enough to apply Corollary 6.4. □

Proposition 6.6. *Suppose that H is continuous and satisfies* (H1)−(H5). *Let g be lsc and bounded from below, and V be the value function associated with g and L, where L is given by* (1.2). *Then the value function V is a subsolution, and moreover the equality* (4.1) *holds.*

Proof. Let $(t,x) \in \operatorname{dom}(V)$, $t \in (0,T)$. We are to prove that for $(p_t,p_x) \in \widehat{\partial} V(t,x)$ the following inequality holds $-p_t + H(t,x,V(t,x),-p_x) \leqslant 0$. If $(p_t,p_x) \in \widehat{\partial} V(t,x)$, then by the property ([23], p. 301), the inequality holds

$$\langle (p_t,p_x),(-1,-v) \rangle \leqslant \mathrm{d}\, V(t,x)(-1,-v) \;\text{ for every }\; v \in \mathbb{R}^n. \tag{6.8}$$

By Proposition 6.5 the inequality (6.7) is satisfied. Putting inequalities (6.8) and (6.7) together we obtain for every $v \in \operatorname{dom} L(t,x,V(t,x),\cdot)$,

$$\langle (p_t,p_x),(-1,-v) \rangle \leqslant L(t,x,V(t,x),v).$$

Therefore

$$\begin{aligned} 0 &\geqslant -p_t + \sup_{v\in\mathbb{R}^n} \langle -p_x, v\rangle - L(t,x,V(t,x),v) \\ &= -p_t + H(t,x,V(t,x),-p_x). \end{aligned}$$

On the other hand, the equality (4.1) is a consequence of Proposition 6.5. □

7. Regularity of value functions

First, we state the lemma ([20], Lem. 3.17), that concerns a relation between solutions of differential inclusions and initial conditions.

Suppose that the Lagrangian L satisfies (L1)−(L6). Let a function $(x,u)(\cdot)$ be L-solution on $[a,b]$. Choose $R>0$ such that $\max\{|x(t)|,|u(t)|\} < R/2$ for every $t \in [a,b]$. Next choose $C>0$ such that inequalities in (L6) are satisfied for R and $C/2$. Let $C_1, C_2 \geqslant 0$ be constants from (L4). Put

$$l(t) = C\,[\,1 + 2C_1 + 2C_2R + |\dot{x}(t)| - \dot{u}(t)\,]\,.$$

We notice that $l(t)>0$ for a.e. $t\in[a,b]$. Fix $M>1$ and choose $\varepsilon>0$ such that

$$\exp\left(C\varepsilon \exp\left(\int_a^b Ml(s)\mathrm{d}s \right)\right) \;<\; \min\left\{ M,\, \exp\left(\frac{RC}{2}\right)\right\}. \tag{7.1}$$

In ([20], Lem. 3.17), the solutions of inclusions $(\dot{x},\dot{u}) \in Q(t,x,u)$ are considered, "backward in time". It can be proven in a similar way, ([20], Lem. 3.17), that solutions of inclusions $(\dot{x},\dot{u}) \in Q(t,x,u)$ are, "forward in time":

Lemma 7.1 ([20], Lem. 3.17)**.** *Suppose that the Lagrangian L, constants C, M, ε and functions $l(\cdot)$, $(x,u)(\cdot)$ are as before. Let $(x,u)(a) = (x_a,u_a)$ and $|\pi_a - x_a| < \varepsilon/2$, $|\eta_a - u_a| < \varepsilon/2$. Then there exists L-solution $(\pi,\eta)(\cdot)$ on $[a,b]$ such that $(\pi,\eta)(a) = (\pi_a,\eta_a)$ and*

(i) $|\pi(t)-x(t)|+|\eta(t)-u(t)| \leqslant (|\pi_a-x_a|+|\eta_a-u_a|)\exp\left(\int_a^t Ml(s)\mathrm{d}s\right)$ for all $t\in[a,b]$

(ii) $|\dot{\pi}(t)-\dot{x}(t)|+|\dot{\eta}(t)-\dot{u}(t)| \leqslant M\,l(t)\,\dfrac{R}{2}$ for *a.e.* $t\in[a,b]$.

Consider $\check{R} > 0$ and a family of functions $\mathfrak{F}(\check{R})$. Choose $C_{\check{R}} > 0$ such that inequalities in (L6) are satisfied for $\check{R}$ and $C_{\check{R}}/2$. If $(x,u)(\cdot) \in \mathfrak{F}(\check{R})$, then for

$$l_{(x,u)}(t) = C_{\check{R}}\left[\,1 + 2C_1 + 2C_2\check{R} + |\dot{x}(t)| - \dot{u}(t)\,\right]$$

we can put a bound on the integral:

$$\begin{aligned}\int_a^b l_{(x,u)}(s)\,\mathrm{d}s &\leqslant C_{\check{R}}\left[(1+2C_1+2C_2\check{R})T + \int_a^b |\dot{x}(s)|\,\mathrm{d}s - \int_a^b \dot{u}(s)\,\mathrm{d}s\right]\\ &\leqslant C_{\check{R}}\left[(1+2C_1+2C_2\check{R})T + 2\check{R}\right] \;=: \check{K}.\end{aligned}$$

Fix $\check{M} > 1$ and choose $\check{\varepsilon} > 0$ such that

$$\exp\left[\check{\varepsilon}C_{\check{R}}\exp(\check{M}\check{K})\right] < \min\left\{\check{M}, \exp(\check{R}C_{\check{R}}/2)\right\}.$$

We notice that for $\check{\varepsilon}$ choose in this way the inequality (7.1) is satisfied for every function $l_{(x,u)}(\cdot)$ such that $(x,u)(\cdot) \in \mathfrak{F}(\check{R})$. Therefore, by Lemma 7.1 we get the corollary:

Corollary 7.2. *Suppose that the Lagrangian L satisfies* (L1)$-$(L6)*. Then for every $R > 0$ there exists $\varepsilon > 0$ and a constant $N > 0$ such that for every function $(x,u)(\cdot) \in \mathfrak{F}(R)$ on $[a,b]$ satisfying $(x,u)(a) = (x_a,u_a)$ and every point (π_a,η_a) satisfying $|\pi_a - x_a| < \varepsilon/2$, $|\eta_a - u_a| < \varepsilon/2$ there exists on L-solution $(\pi,\eta)(\cdot)$ on $[a,b]$ such that $(\pi,\eta)(a) = (\pi_a,\eta_a)$ and*

$$|\pi(t)-x(t)|+|\eta(t)-u(t)| \leqslant (|\pi_a-x_a|+|\eta_a-u_a|)N \;\text{ for all }\; t \in [a,b].$$

Lemma 7.3. *Suppose that L satisfies the epi-continuity property and* (L1)$-$(L5) *hold. Let g be lsc and bounded from below. If V is the value function associated with g and L, then for every $(s_0,y_0) \in \operatorname{dom} V$ there exist $\tau > 0$ and L-solution $(x,u)(\cdot)$ on $[(s_0-\tau)_+, T]$ such that*

$$(x,u)(s_0) = (y_0, V(s_0,y_0)), \qquad u(T) \geqslant g(x(T)). \tag{7.2}$$

Proof. If $(s_0,z_0) \in \operatorname{dom}(V)$, $s_0 \in (0,T]$, then by Corollary 6.4 there exist $\tau \in (0,s_0)$ and a function $(y,\eta)(\cdot) \in \mathcal{C}^1([s_0-\tau,s_0],\mathbb{R}^n \times \mathbb{R})$ such that $(y,\eta)(s_0) = (y_0,V(s_0,y_0))$ and $(\dot{y}(t),\dot{\eta}(t)) \in Q(t,y(t),\eta(t))$ for every $t \in [s_0-\tau,s_0]$. If $(s_0,z_0) \in \operatorname{dom}(V)$, $s_0 \in [0,T)$, then by Theorem 3.4, at the point (s_0,y_0) there exists L-solution $(\overline{x},\overline{u})(\cdot)$ such that $(\overline{x},\overline{u})(s_0) = (y_0,V(s_0,y_0))$ and $\overline{u}(T) \geqslant g(\overline{x}(T))$. The function $(x,u)(\cdot)$ on the interval $[s_0-\tau,T]$ is given by the formula $(x,u)(t) := (y,\eta)(t)$ for $t \in [s_0-\tau,s_0]$ and $(x,u)(t) := (\overline{x},\overline{u})(t)$ for $t \in [s_0,T]$. Therefore, $(x,u)(\cdot)$ is the L-solution on $t \in [s_0-\tau,T]$ such that $u(T) \geqslant g(x(T))$ and $(x,u)(s_0) = (y_0,V(s_0,y_0))$. ☐

To simplify the notation in the next subsections we introduce notations $\varDelta^*(s,r) := \varDelta(s,r)\backslash\{T\}$ and $O_r^*(s,y) := \varDelta^*(s,r)\times\mathbb{B}(y,r)$, where $\varDelta(t_0,\tau) := [t_0-\tau,t_0+\tau]\cap[0,T]$ and $O_r(s,y) := \varDelta(s,r)\times\mathbb{B}(y,r)$.

7.1. Continuity of value functions

Continuity of value functions (Thm. 4.4) is a consequence of Theorem 3.4 and the following one:

Theorem 7.4. *Suppose that L satisfies the epi-continuity property and* (L1)$-$(L6) *hold. Let the function g be finite, continuous and bounded from below. If V is the value function associated with g and L, then for every $(s_0,y_0) \in \operatorname{dom}(V)$ and $\mu > 0$, there exists $\delta > 0$ such that*

$$V(s,y) \leqslant V(s_0,y_0) + \mu \;\text{ for all }\; (s,y) \in O_\delta(s_0,y_0).$$

Proof. Fix $(s_0, y_0) \in \operatorname{dom} V$. By Lemma 7.3 there exist $\tau > 0$ and L-solution $(x,u)(\cdot)$ on $[(s_0-\tau)_+, T]$ satisfying (7.2). Let $R > 0$ be large enough to satisfy inequalities:

$$|(x,u)(t)| < R/2 \;\text{ for }\; t \in [(s_0-\tau)_+, T], \qquad \int_{(s_0-\tau)_+}^{T} |\dot{x}(t)|\,\mathrm{d}t < R.$$

Next to R we choose $\varepsilon > 0$ and a constant $N > 0$ such that claim of Corollary 7.2 holds. Let $\omega(\cdot)$ be a modulus of uniform continuity of a function g on $2R\mathbb{B}$. Because of the continuity of a function $x(\cdot)$ there exists r such that $0 < r \leqslant \min\{\tau, \varepsilon/5, R/2N\}$ and $|x(t) - x(s_0)| \leqslant \min\{\varepsilon/5, R/2N\}$ for every $t \in [s_0 - r, s_0 + r] \cap [0,T]$.

To prove the theorem, it is enough to show that for $\overline{\omega}(\theta) := \omega(\theta) + \theta$ and every $(s,y) \in O_r(s_0,y_0)$ the following inequality holds

$$V(s,y) - V(s_0,y_0) \leqslant u(s) - u(s_0) + \overline{\omega}(|y-y_0|N + |x(s) - x(s_0)|N).$$

Indeed, fix $(t_0, x_0) \in O_r(s_0, y_0)$. We notice that the following inequalities are satisfied:

$$\begin{aligned} |x_0 - x(t_0)| &\leqslant |x_0 - y_0| + |x(s_0) - x(t_0)| \\ &\leqslant r + \min\{\varepsilon/5, R/2N\} \\ &\leqslant \min\{2\varepsilon/5, R/N\}. \end{aligned} \tag{7.3}$$

Because of $(x,u)(\cdot) \in \mathfrak{F}(R)$, the function $(x,u)(\cdot)$ reduced to the set $[t_0,T]$ also belongs to $\mathfrak{F}(R)$. From the inequality (7.3) we have $|x_0 - x(t_0)| < \varepsilon/2$. Moreover $|u(t_0) - u(t_0)| = 0 < \varepsilon/2$. Therefore, hypotheses of Corollary 7.2 are satisfied, so there exists on L-solution $(\pi,\eta)(\cdot)$ on $[t_0,T]$ such that $(\pi,\eta)(t_0) = (x_0, u(t_0))$ and

$$|\pi(t) - x(t)| + |\eta(t) - u(t)| \;\leqslant\; |x_0 - x(t_0)|N \quad \text{for all } \; [t_0, T]. \tag{7.4}$$

From the inequality (7.3) and (7.4) we have $|\pi(T) - x(T)| \leqslant R$. Since $|x(T)| \leqslant R$, we get $\pi(T), x(T) \in 2R\mathbb{B}$. So $|g(\pi(T)) - g(x(T))| \leqslant \omega(|\pi(T) - x(T)|)$.

We define an absolutely continuous function $\widetilde{\eta}(\cdot)$ by the formula $\widetilde{\eta}(t) := \eta(t) + \alpha_0$ on $[t_0,T]$, where a nonnegative number α_0 is given by

$$\alpha_0 := \max\{0, g(\pi(T)) - \eta(T)\}.$$

We notice that $\widetilde{\eta}(t) \geqslant \eta(t)$ for every $t \in [t_0,T]$. Therefore, from the condition (L3) for a.e. $t \in [t_0,T]$

$$-\dot{\widetilde{\eta}}(t) = -\dot{\eta}(t) \geqslant L(t, \pi(t), \eta(t), \dot{\pi}(t)) \geqslant L(t, \pi(t), \widetilde{\eta}(t), \dot{\pi}(t)).$$

So $(\pi, \widetilde{\eta})(\cdot)$ is the L-solution on $[t_0, T]$ such that

$$(\pi, \widetilde{\eta})(t_0) = (x_0, u(t_0) + \alpha), \qquad \widetilde{\eta}(T) \geqslant g(\pi(T)).$$

By the definition of the value function $V(t_0, x_0) \leqslant u(t_0) + \alpha_0$. Therefore,

$$\begin{aligned} V(t_0,x_0) - V(s_0,y_0) &\leqslant u(t_0) + \alpha_0 - u(s_0) \\ &\leqslant u(t_0) - u(s_0) + |u(T) - g(x(T)) + g(\pi(T)) - \eta(T)| \\ &\leqslant u(t_0) - u(s_0) + |u(T) - \eta(T)| + |g(\pi(T)) - g(x(T))| \\ &\leqslant u(t_0) - u(s_0) + |\eta(T) - u(T)| + \omega(|\pi(T) - x(T)|) \\ &\leqslant u(t_0) - u(s_0) + \overline{\omega}(|x_0 - x(t_0)|N) \\ &\leqslant u(t_0) - u(s_0) + \overline{\omega}(|x_0 - y_0|N + |x(s_0) - x(t_0)|N) \end{aligned}$$

and that finishes the proof. ∎

7.2. Lipschitz continuity of value functions

The lipschitz continuity of value functions V (Thm. 4.5) is obtained as follows: first we prove Lipschitz continuity of the function $V(t,\cdot)$ uniformly in t, and then Lipschitz continuity of the function $V(\cdot,x)$ uniformly in x.

Definition 7.5. By $\mathfrak{M}(s_0,y_0,r)$ we denote a family of functions $(\overline{x},\overline{u})(\cdot)$ whose at points $(t_0,x_0)\in O_r^*(s_0,y_0)$, $(t_0,x_0)\in \operatorname{dom} V$ are minimizers from Definition 3.3.

Proposition 7.6. *Suppose that L satisfies* (L1)$-$(L5). *Let a function g be finite, lsc and bounded from below. If V is the value function associated with g and L, bounded on the set $O_r(s_0,y_0)$, then there exists $\check{R}>0$ such that $\mathfrak{M}(s_0,y_0,r)\subset\mathfrak{F}(\check{R})$.*

The proof is contained in Appendix.

The consequence of Theorem 5.12 and Proposition 7.6 is the corollary:

Corollary 7.7. *Suppose that L satisfies the epi-continuity property and* (L1)$-$(L6). *Let g be finite, bounded from below and satisfy the local Lipschitz condition. If V is the value function associated with g and L, bounded on the set $O_r(s_0,y_0)$, then there exists $\hat{R}>0$ such that $\mathfrak{M}(s_0,y_0,r)\subset\mathfrak{R}(\hat{R})$.*

Theorem 7.8. *Suppose that L satisfies the epi-continuity property and* (L1)$-$(L6). *Let g be finite, bounded from below and satisfy locally the Lipschitz condition. If V is the value function associated with g and L, then for every $(s_0,y_0)\in\operatorname{dom}V$ there exists $r_0>0$ such that V is finite on $O_{r_0}(s_0,y_0)$. Moreover, there exists $k_0>0$ such that*

$$|V(t,y)-V(t,x)|\leqslant k_0|y-x|\quad\text{for all }\ t\in\varDelta(s_0,r_0),\ \ y,x\in\mathbb{B}(y_0,r_0). \tag{7.5}$$

Proof. By Theorem 4.4 there exists $r>0$ such that a function V is finite and continuous on the set $O_r(s_0,y_0)$. Therefore, by Proposition 7.6 there exists $R>0$ such that $\mathfrak{M}(s_0,y_0,r)\subset\mathfrak{F}(R)$. Next we choose $\varepsilon>0$ and $N\geqslant 0$ to R in such a way that the claim of Corollary 7.2 is satisfied. Put $r_0:=\min\{r,\varepsilon/5\}$ and $\lambda:=2r_0N+R+r_0+|y_0|$. Let k be the Lipschitz constant of the function g on the ball $\lambda\mathbb{B}$. Define $k_0:=N(1+k)+k$.

Now we prove that for r_0 and k_0 given above the inequality (7.5) holds. Indeed, if we fix $t_0\in\varDelta^*(s_0,r_0)$, $x_0,\overline{x}_0\in\mathbb{B}(y_0,r_0)$ then by Theorem 3.4, at the point $(t_0,\overline{x}_0)$ there exists a minimizer $(\overline{x},\overline{u})(\cdot)$ on $[t_0,T]$. Since $(\overline{x},\overline{u})(\cdot)\in\mathfrak{M}(s_0,y_0,r)$, then $(\overline{x},\overline{u})(\cdot)\in\mathfrak{F}(R)$. Moreover, $|x_0-\overline{x}_0|\leqslant 2r_0<\varepsilon/2$ and $|\overline{u}(t_0)-\overline{u}(t_0)|=0<\varepsilon/2$. By Corollary 7.2 there exists an L-solution $(x,u)(\cdot)$ on $[t_0,T]$ such that $(x,u)(t_0)=(x_0,\overline{u}(t_0))$. Besides,

$$|x(t)-\overline{x}(t)|+|u(t)-\overline{u}(t)|\leqslant|x_0-\overline{x}_0|N\ \text{ for all }\ [t_0,T]. \tag{7.6}$$

As $(\overline{x},\overline{u})(\cdot)\in\mathfrak{F}(R)$, we have $|\overline{x}(T)|<R$. Therefore from the inequality (7.6) we get $|x(T)|\leqslant\lambda$, so $\overline{x}(T),x(T)\in\lambda\mathbb{B}$ that implies $|g(x(T))-g(\overline{x}(T))|\leqslant k|x(T)-\overline{x}(T)|$. Define a function $\eta(\cdot)\in\mathcal{A}([t_0,T],\mathbb{R})$ by the formula $\eta(t):=u(t)+\alpha_0$, where $\alpha_0:=\max\{0,g(x(T))-u(T)\}$. Similarly like in the proof of Theorem 7.4 we show that the function $(x,\eta)(\cdot)$ is an L-solution on $[t_0,T]$ and satisfies properties:

$$(x,\eta)(t_0)=(x_0,u(t_0)+\alpha_0),\qquad \eta(T)\geqslant g(x(T)).$$

From the definition of the value function $V(t_0,x_0)\leqslant u(t_0)+\alpha_0$. Thus,

$$\begin{aligned} V(t_0,x_0)-V(t_0,\overline{x}_0)&\leqslant u(t_0)+\alpha_0-\overline{u}(t_0)\ =\ \alpha_0\\ &\leqslant|\overline{u}(T)-g(\overline{x}(T))+g(x(T))-u(T)|\\ &\leqslant|\overline{u}(T)-u(T)|+|g(x(T))-g(\overline{x}(T))|\\ &\leqslant|x_0-\overline{x}_0|N+k|x(T)-\overline{x}(T)|\\ &\leqslant k_0|x_0-\overline{x}_0|. \end{aligned}\tag{7.7}$$

If $t_0 = T$ and $x_0, \overline{x}_0 \in \mathbb{B}(y_0, r_0)$, then $x_0, \overline{x}_0 \in \lambda\mathbb{B}$. Moreover,

$$V(t_0, x_0) - V(t_0, \overline{x}_0) \;=\; g(x_0) - g(\overline{x}_0) \;\leqslant\; k_0|x_0 - \overline{x}_0|. \tag{7.8}$$

Since the points t_0, $x_0, \overline{x}_0$ are arbitrary elements of the sets $\Delta(s_0, r_0)$, $\mathbb{B}(y_0, r_0)$, correspondingly, from the inequalities (7.7) and (7.8) we have the inequality (7.5). □

Lemma 7.9. *Suppose that L satisfies the epi-continuity property and* (L1)−(L6)*. Let g be finite, bounded from below and satisfy the local Lipschitz condition. If V is the value function associated with g and L, then for every $(s_0, y_0) \in \operatorname{dom}(V)$ there exist $r, k > 0$ such that for every $t_0, \overline{t}_0 \in \Delta(s_0, r)$, where $t_0 < \overline{t}_0$ and $x_0 \in \mathbb{B}(y_0, r)$, there exists L-solution $(x, u)(\cdot)$ on $[t_0, \overline{t}_0]$ that satisfies properties:*

$$(x, u)(\overline{t}_0) = (x_0, V(\overline{t}_0, x_0)), \qquad |(\dot{x}(t), \dot{u}(t))| \leqslant k \;\; a.e.\; t \in [t_0, \overline{t}_0]. \tag{7.9}$$

For the proof see Appendix.

Theorem 7.10. *Suppose that L satisfies the epi-continuity property and* (L1)−(L6)*. Let g be finite, bounded from below and satisfy the local Lipschitz condition. If V is the value function associated with g and L, then for every $(s_0, y_0) \in \operatorname{dom}(V)$ there exists $r_0 > 0$ such that V is finite on $O_{r_0}(s_0, y_0)$. Moreover, there exists $k_0 > 0$ such that*

$$|V(s, x) - V(t, x)| \leqslant k_0|s - t| \quad \text{for all } \; s, t \in \Delta(s_0, r_0), \;\; x \in \mathbb{B}(y_0, r_0). \tag{7.10}$$

Proof. From Theorem 4.4 there exists $r > 0$ such that the function V is finite and continuous on the set $O_r(s_0, y_0)$, so there exists $D > 0$ such that $|V(t, x)| \leqslant D$ for every $(t, x) \in O_r(s_0, y_0)$. Moreover, by Lemma 7.9 for r (decreased if necessary) we obtain $k > 0$ such that for every $t_0, \overline{t}_0 \in \Delta(s_0, r)$, where $t_0 < \overline{t}_0$, and $x_0 \in \mathbb{B}(y_0, r)$ there exists L-solution $(x, u)(\cdot)$ on $[t_0, \overline{t}_0]$ satisfying properties (7.9). By Corollary 7.7 there exists $\hat{R} > 0$ such that $\mathfrak{M}(s_0, y_0, r) \subset \mathfrak{R}(\hat{R})$. Let $C_1, C_2 > 0$ be constants from (L4) and let $C_3 := C_1 + C_2\hat{R}$. Next we choose $\varepsilon > 0$ and a constant $N > 0$ to $R := (1 + T)(\hat{R} + 6(2kr + D + r + |y_0|) + k)$ in such a way that the claim of Corollary 7.2 is satisfied. Take r_0 such that $0 < r_0 < \min\{r, \varepsilon/4\hat{R}, 1/2N, \varepsilon/4k, \hat{R}/2kN\}$. Let $l > 0$ be a Lipschitz constant of g on the ball $2\hat{R}\mathbb{B}$. Define k_0 to be $C_3 + (1 + l)N\hat{R} + k + (1 + l)Nk$.

Now we prove that for r_0 and k_0 given above the inequality (7.10) holds. We fix $t_0, \overline{t}_0 \in \Delta(s_0, r_0)$, $x_0 \in \mathbb{B}(y_0, r_0)$ and consider two cases, namely $\overline{t}_0 < t_0$ and $t_0 < \overline{t}_0$.

Case 1. Let $\overline{t}_0 < t_0$. By Theorem 3.4, at the point$(\overline{t}_0, x_0)$ there exists a minimizer $(\overline{x}, \overline{u})(\cdot)$ on $[\overline{t}_0, T]$. Since $(\overline{x}, \overline{u})(\cdot) \in \mathfrak{M}(s_0, y_0, r)$, we have $(\overline{x}, \overline{u})(\cdot) \in \mathfrak{R}(\hat{R})$. Moreover, by (L4) we get $\dot{\overline{u}}(t) \leqslant C_1 + C_2\hat{R} = C_3$ for a.e. $t \in [\overline{t}_0, T]$. We notice that

$$|x_0 - \overline{x}(t_0)| \leqslant \int_{\overline{t}_0}^{t_0} |\dot{x}(t)| \,\mathrm{d}t \leqslant |t_0 - \overline{t}_0|\hat{R} \leqslant 2r_0\hat{R} < \min\{\varepsilon/2, \hat{R}/N\}. \tag{7.11}$$

Since $(\overline{x}, \overline{u})(\cdot) \in \mathfrak{R}(\hat{R})$, we have $(\overline{x}, \overline{u})(\cdot) \in \mathfrak{F}(R)$. Therefore, the function $(\overline{x}, \overline{u})(\cdot)$ on the set $[t_0, T]$ also belongs to $\mathfrak{F}(R)$. By the inequality (7.11) we have $|x_0 - \overline{x}(t_0)| < \varepsilon/2$ and $|\overline{u}(t_0) - \overline{u}(t_0)| = 0 < \varepsilon/2$, so by Corollary 7.2 there exists the L-solution $(x, u)(\cdot)$ on $[t_0, T]$ such that $(x, u)(t_0) = (x_0, \overline{u}(t_0))$ and

$$|x(t) - \overline{x}(t)| + |u(t) - \overline{u}(t)| \leqslant |x_0 - \overline{x}(t_0)|N \;\text{ for all }\; t \in [t_0, T]. \tag{7.12}$$

From inequalities (7.11) and (7.12) we obtain $|x(T) - \overline{x}(T)| \leqslant \hat{R}$. Since $|\overline{x}(T)| < \hat{R}$, we have $\overline{x}(T), x(T) \in 2\hat{R}B$, that implies $|g(x(T)) - g(\overline{x}(T))| \leqslant l|x(T) - \overline{x}(T)|$. We define a function $\eta(\cdot) \in \mathcal{A}([t_0, T], \mathbb{R})$ by the formula $\eta(t) := u(t) + \alpha_0$, where $\alpha_0 := \max\{0, g(x(T)) - u(T)\}$. Similarly to the proof of Theorem 7.4 we show that the function $(x, \eta)(\cdot)$ is L-solution on $[t_0, T]$ and satisfies

$$(x, \eta)(t_0) = (x_0, \overline{u}(t_0) + \alpha_0), \qquad \eta(T) \geqslant g(x(T)).$$

By the definition of the value function $V(t_0,x_0) \leqslant \overline{u}(t_0)+\alpha_0$. Thus

$$\begin{aligned} V(t_0,x_0)-V(\overline{t}_0,x_0) &\leqslant \overline{u}(t_0)+\alpha_0-\overline{u}(\overline{t}_0)\\ &\leqslant \overline{u}(t_0)-\overline{u}(\overline{t}_0)+|\overline{u}(T)-g(\overline{x}(T))+g(x(T))-u(T)|\\ &\leqslant \overline{u}(t_0)-\overline{u}(\overline{t}_0)+|\overline{u}(T)-u(T)|+l|x(T)-\overline{x}(T)|\\ &\leqslant \overline{u}(t_0)-\overline{u}(\overline{t}_0)+(1+l)N|x_0-\overline{x}(t_0)|\\ &\leqslant \int_{\overline{t}_0}^{t_0}\dot{\overline{u}}(t)\,\mathrm{d}t+(1+l)N\hat{R}|t_0-\overline{t}_0|\\ &\leqslant |t_0-\overline{t}_0|C_3+(1+l)N\hat{R}|t_0-\overline{t}_0|\\ &\leqslant k_0|t_0-\overline{t}_0|. \end{aligned}$$

Case 2. Let $t_0<\overline{t}_0$. By Theorem 3.4 at the point $(\overline{t}_0,x_0)$ such that $\overline{t}_0<T$, there exists $(\overline{x},\overline{u})(\cdot)$ on $[\overline{t}_0,T]$. Since $(\overline{x},\overline{u})(\cdot)\in\mathfrak{M}(s_0,y_0,r)$, we have $(\overline{x},\overline{u})(\cdot)\in\mathfrak{R}(\hat{R})$. By Lemma 7.9 for $t_0<\overline{t}_0\leqslant T$ there exists the L-solution $(x,u)(\cdot)$ on $[t_0,\overline{t}_0]$ that satisfies properties (7.9). We put $(\pi,\eta)(t):=(x,u)(t)$ for $t\in[t_0,\overline{t}_0]$ and $(\pi,\eta)(t):=(\overline{x},\overline{u})(t)$ for $t\in[\overline{t}_0,T]$ and notice that the function $(\pi,\eta)(\cdot)$ is the L-solution on $[t_0,T]$ satisfying

$$(\pi,\eta)(\overline{t}_0)=(x_0,V(\overline{t}_0,x_0)),\qquad \eta(T)\geqslant g(\pi(T)).$$

Moreover, $|(\dot\pi(t),\dot\eta(t))|\leqslant k$ for a.e. $t\in[t_0,\overline{t}_0]$. In particular the function $(\pi,\eta)(\cdot)$ satisfies the Lipschitz condition on $[t_0,\overline{t}_0]$ with the constant k. We notice that

$$\begin{aligned} |x_0-\pi(t_0)| \leqslant |\pi(\overline{t}_0)-\pi(t_0)| &\leqslant k|t_0-\overline{t}_0|\\ \leqslant \qquad 2r_0k &< \min\{\varepsilon/2,\hat{R}/N\}. \end{aligned} \tag{7.13}$$

It is not difficult to prove that $(\pi,\eta)(\cdot)\in\mathfrak{R}(R/(1+T))$, so $(\pi,\eta)(\cdot)\in\mathfrak{F}(R)$. From the inequality (7.13) we have $|x_0-\pi(t_0)|<\varepsilon/2$ and $|\eta(t_0)-\eta(t_0)|=0<\varepsilon/2$. By Corollary 7.2 there exists the L-solution $(p,q)(\cdot)$ on $[t_0,T]$ such that $(p,q)(t_0)=(x_0,\eta(t_0))$, moreover

$$|p(t)-\pi(t)|+|q(t)-\eta(t)|\leqslant |x_0-\pi(t_0)|N \ \text{ for all }\ t\in[t_0,T]. \tag{7.14}$$

Using inequalities (7.13) and (7.14) we obtain $|p(T)-\pi(T)|\leqslant\hat{R}$. Since $|\pi(T)|=|\overline{x}(T)|<\hat{R}$, we have $p(T),\pi(T)\in 2\hat{R}B$ that implies $|g(p(T))-g(\pi(T))|\leqslant l|p(T)-\pi(T)|$. We define the function $\widetilde{q}(\cdot)\in\mathcal{A}([t_0,T],\mathbb{R})$ by the formula $\widetilde{q}(t):=q(t)+\alpha_0$, where $\alpha_0:=\max\{0,g(p(T))-q(T)\}$. Similarly to the proof of Theorem 7.4 we show that the function $(p,\widetilde{q})(\cdot)$ is the L-solution on $[t_0,T]$ satisfying

$$(p,\widetilde{q})(t_0)=(x_0,\eta(t_0)+\alpha_0),\qquad \widetilde{q}(T)\geqslant g(p(T)).$$

By the definition of the value function $V(t_0,x_0)\leqslant\eta(t_0)+\alpha_0$. Therefore

$$\begin{aligned} V(t_0,x_0)-V(\overline{t}_0,x_0) &\leqslant \eta(t_0)+\alpha_0-\eta(\overline{t}_0)\\ &\leqslant \eta(t_0)-\eta(\overline{t}_0)+|\eta(T)-g(\pi(T))+g(p(T))-q(T)|\\ &\leqslant \eta(t_0)-\eta(\overline{t}_0)+|\eta(T)-q(T)|+l|p(T)-\pi(T)|\\ &\leqslant \eta(t_0)-\eta(\overline{t}_0)+(1+l)N|x_0-\pi(t_0)|\\ &\leqslant k|t_0-\overline{t}_0|+(1+l)Nk|t_0-\overline{t}_0|\\ &\leqslant k_0|t_0-\overline{t}_0|, \end{aligned}$$

that finishes the proof. □

Appendix A.

Proof of Proposition 2.5. Notice that replacing the multivalued function Q by $E(t,z) := \operatorname{epi} L(t,z,\cdot)$ in (SL) we obtain equivalent conditions. Analysing the proof ([18], Thm. 5.1), we find that the condition $(\overline{v},\overline{L}) \in E(t,x)$ in that theorem is not necessary. Therefore, we may set $(\overline{v},\overline{L}) = (0,0)$. By ([18], Thm 5.1), it is enough to show that the condition (L6) is equivalent to the following

$$\begin{aligned}&\text{For every } R>0 \text{ there exists } C>0 \text{ and } \delta_0>0 \text{ such that for all } t\in[0,T],\ z',z\in R\mathbb{B}\\ &\text{and } \delta\geqslant\delta_0 \text{ the following condition is satisfied:}\\ &\quad(\star)\ \text{For any } v\in\delta\mathbb{B} \text{ obeying } L(t,z,v)\leqslant\delta \text{ there exists } v'\in\mathbb{R}^l \text{ satisfying}\\ &\qquad\text{(i) } |v'-v|\leqslant C\delta|z'-z|;\\ &\qquad\text{(ii) } L(t,z',v')\leqslant\max\{-\delta,L(t,z,v)\}+C\delta|z'-z|.\end{aligned} \tag{A.1}$$

Let us assume that (L6) is fulfilled. We fix $R>0$ and choose $C>0$ such that for every $t\in[0,T]$, $z',z\in R\mathbb{B}$ conditions (a) and (b) of (L6) are satisfied. Let $\delta_0 = 1+2C_1+2C_2R$, where $C_1,C_2>0$ are from (L4). Then for every $\delta\geqslant\delta_0$ we obtain that if $v\in\delta\mathbb{B}$ and $L(t,z,v)\leqslant\delta$, then there exists $v'\in\mathbb{R}^l$ such that (a) and (b) of (L6) are satisfied. Therefore,

$$\begin{aligned}|v'-v| &\leqslant C(1+|v|+|L(t,z,v)|)|z'-z|\\ &\leqslant C(1+|v|+2C_1+2C_2R+L(t,z,v))|z'-z| \leqslant 3C\delta|z'-z|\end{aligned}$$

and

$$\begin{aligned}L(t,z',v') &\leqslant L(t,z,v)+C(1+|v|+|L(t,z,v)|)|z'-z|\\ &\leqslant \max\{-\delta,L(t,z,v)\}+3C\delta|z'-z|.\end{aligned}$$

Since the condition $(\star)$ is fulfilled, we have (A.1).

Conversely, we assume that the condition (A.1) is satisfied. We fix $R>0$ and choose $C>0$ and $\delta_0>0$ such that for every $t\in[0,T]$, $z',z\in R\mathbb{B}$ and $\delta\geqslant\delta_0$ the condition $(\star)$ is fulfilled. If $v\in\operatorname{dom} L(t,z,\cdot)$ and $\delta = 1+\delta_0+|v|+|L(t,z,v)|$, then we see that $\delta\geqslant\delta_0$ and $v\in\delta\mathbb{B}$. Moreover, $L(t,z,v)\leqslant\delta$. Since assumptions $(\star)$ are satisfied, conditions (i), (ii) are fulfilled, so

$$|v'-v|\leqslant C\delta|z'-z|\leqslant C(1+\delta_0)(1+|v|+|L(t,z,v)|)|z'-z|,$$

and

$$\begin{aligned}L(t,z',v') &\leqslant \max\{-\delta,L(t,z,v)\}+3C\delta|z'-z| = L(t,z,v)+3C\delta|z'-z|\\ &\leqslant L(t,z,v)+C(1+\delta_0)(1+|v|+|L(t,z,v)|)|z'-z|).\end{aligned}$$

Therefore, conditions (a) and (b) of (L6) are satisfied. The proof is finished. □

Proof of Proposition 4.7. Let $\overline{z}(\cdot) := (\overline{x},\overline{u})(\cdot)$ be the L-solution on $[t_0,T]$ that satisfies the Lipschitz condition on $[t_0,T]$ and $\overline{z}(T)\in\operatorname{epi} g(\cdot)$. We know that the function g is finite and the function U satisfies locally the Lipschitz condition. Moreover $U(T,\overline{x}(T)) = g(\overline{x}(T))$. Therefore, there exists $\tau_n\to T-$ and $x_n\to\overline{x}(T)$ such that $U(\tau_n,x_n)\to g(\overline{x}(T))$. Since $\overline{u}(T)\geqslant g(\overline{x}(T))$, we can choose a sequence $u_n\geqslant U(\tau_n,x_n)$ such that $u_n\to\overline{u}(T)$. For large $n\in\mathbb{N}$ we have $|x_n-\overline{x}(\tau_n)|<\varepsilon$ and $|u_n-\overline{u}(\tau_n)|<\varepsilon$. From Lemma 7.1, "backward in time" there exist L-solutions $\overline{z}_n(\cdot) := (\overline{x}_n,\overline{u}_n)(\cdot)$ on $[t_0,\tau_n]$ such that $\overline{z}_n(\tau_n) = (x_n,u_n)$. In addition to this they satisfy properties (i) and (ii). By property (i) we obtain $\overline{z}_n(t)\to\overline{z}(t)$ for every $t\in[t_0,T)$, and by the property (ii) the function $\overline{z}_n(\cdot)$ satisfies the Lipschitz condition on $[t_0,\tau_n]$. Using the fact that $\overline{z}_n(\tau_n)\in\operatorname{epi} U(\tau_n,\cdot)$ we can show the following inclusion

$$\overline{z}_n(t)\in\operatorname{epi} U(t,\cdot) \quad \text{for all} \quad t\in[t_0,\tau_n]. \tag{A.2}$$

Letting $n\to+\infty$ in (A.2), by the closedness of $\operatorname{epi} U$, we obtain $\overline{z}(t)\in\operatorname{epi} U(t,\cdot)$ for all $t\in[t_0,T]$. Therefore, the proof is over if we prove the inclusion (A.2). In order to do it we need the following Lemma.

Lemma A.1. *Let L and U satisfies the assumptions of Proposition 4.7. Suppose that $z(\cdot)$ is the L-solution on $[t_0,\tau]$, where $t_0<\tau<T$, that satisfy the Lipschitz condition and $z(\tau)\in\operatorname{epi} U(\tau,\cdot)$. Then there exists $\upsilon\in(t_0,\tau)$ such that $z(t)\in\operatorname{epi} U(t,\cdot)$ for all $t\in[\upsilon,\tau]$.*

The proof of Proposition 4.7 is finished if we show Lemma A.1. □

Proof of Lemma A.1. Let $R = 2 + \max\{\, |z(t)| : t \in [t_0, \tau]\,\}$. By the epi-continuity property of L there exists $\rho_1 > 0$ such that $Q(t,z) \cap \rho_1 \mathbb{B} \neq \emptyset$ for every $(t,z) \in [0,T] \times R\mathbb{B}$. Let ρ_2 be such that $\max\{|\dot{z}(t)|, \rho_1\} < \rho_2$ for a.e. $t \in [t_0, \tau]$. We define a multivalued function $Q_\rho(t,z) := Q(t,z) \cap \rho\mathbb{B}$ on $[0,T] \times R\mathbb{B}$. If $\rho > \rho_2$, then the inclusion

$$\dot{z}(t) \in Q_\rho(t, z(t)) \quad \text{a.e.} \quad t \in [t_0, \tau], \tag{A.3}$$

holds. Since L is lsc and satisfies the epi-continuity property, the multivalued function Q is lower semicontinuous in the sense of Kuratowski and has a closed graph. So the multivalued function Q_ρ for $\rho > \rho_2$ is continuous in the sense of Hausdorff (see [16], Prop. 15.6). Moreover, for $\rho > 2\rho_2$ the multivalued function $Q_\rho(t,\cdot)$ is Lipschitz in sense of Hausdorff on $R\mathbb{B}$ uniformly with respect to t, because Q satisfies (SL) (see [23], Prop. 4.39). Besides, notice that Q_ρ satisfies the condition (4.2).

Therefore, we reduce the proof to the following problem considered by Frankowska [12]. Let $\Pi(\cdot)$ be a projection of $\mathbb{R}^{n+1}$ on $R\mathbb{B}$. We define a multivalued function $\widehat{Q}_\rho(t,z) := Q_\rho(t, \Pi(z))$ on $[0,T] \times \mathbb{R}^{n+1}$. Proceeding similarly to the proof of ([12], Thm. 3.3), we choose a continuous function $f : [0,T] \times \mathbb{R}^{n+1} \times \mathbb{B} \mapsto \mathbb{R}^{n+1}$ such that $f(t,\cdot,\cdot)$ satisfies the Lipschitz condition and $\widehat{Q}_\rho(t,z) = f(t,z,\mathbb{B})$. There exists a measurable function $b(\cdot)$ such that $\dot{z}(t) = f(t, z(t), b(t))$ for a.e. $t \in [t_0, \tau]$. We consider a sequence of continuous maps $b_n(\cdot)$ converging to $b(\cdot)$ in $L^1([t_0,\tau], \mathbb{B})$ and let $z_n(\cdot)$ denote the solution of

$$\dot{z}_n(t) = f(t, z_n(t), b_n(t)) \;\; \text{a.e.}\; t \in [t_0, \tau], \quad z_n(\tau) = z(\tau). \tag{A.4}$$

By properties of the function f we have $z_n(t) \to z(t)$ for all $t \in [t_0, \tau]$. Since derivatives $\dot{z}_n(\cdot)$ are equi-bounded on $[t_0, \tau]$, there exists $\upsilon \,:\, t_0 < \upsilon < \tau$ such that $z_n(t) \in (R-1)\mathbb{B}$ for all $t \in [\upsilon, \tau]$ and $n \in \mathbb{N}$. Consider a multivalued function $(t,z) \rightsquigarrow \{f(t,z,b_n(t))\}$, that is continuous and satisfies the condition (4.2) on the neighbourhoods of the points $(s,y) \in \operatorname{epi} U \cap [(0,T) \times (R-1)\mathbb{B}]$. Applying to this multivalued function the theorem about viability we have $z_n(t) \in \operatorname{epi} U(t,\cdot)$ for all $t \in [\upsilon, \tau]$ and $n \in \mathbb{N}$. In the limit $z(t) \in \operatorname{epi} U(t,\cdot)$ for all $t \in [\upsilon, \tau]$. □

Proof of Proposition 7.6. Let $D > 0$ be such that $|V(t_0, x_0)| \leqslant D$ for every $(t_0, x_0) \in O_r(s_0, y_0)$. Let $\{(x_\lambda, u_\lambda)(\cdot) : \lambda \in \Lambda\} = \mathfrak{M}(s_0, y_0, r)$. Then L-solutions $(x_\lambda, u_\lambda)(\cdot)$ on $[t_{\lambda 0}, T]$ satisfy properties:

$$(x_\lambda, u_\lambda)(t_{\lambda 0}) = (x_{\lambda 0}, V(t_{\lambda 0}, x_{\lambda 0})), \quad u_\lambda(T) \geqslant g(x_\lambda(T)) \quad \text{for every} \;\; \lambda \in \Lambda.$$

Moreover, $x_\lambda(t_{\lambda 0}) = x_{\lambda 0} \in \mathbb{B}(y_0, r)$. Let $K > 0$ be such that $g(x) \geqslant -K$ for every $x \in \mathbb{R}^n$. Then, for $M := \max\{D, K\}$ and every $\lambda \in \Lambda$ we have $u_\lambda(T) \geqslant -M$ and $u_\lambda(t_{\lambda 0}) \leqslant M$. Similarly to above we prove the inequality (3.3) for $\{x_\lambda(\cdot)\}_{\lambda \in \Lambda}$. Therefore, the assumptions of Tonelli–Nagumo Theorem ([20], Prop. 2.6), are satisfied. As a consequence, the family $\{\dot{x}_\lambda(\cdot)\}_{\lambda \in \Lambda}$ is equi-absolutely integrable. Therefore, the family $\{x_\lambda(\cdot)\}_{\lambda \in \Lambda}$ has to be equi-bounded. So there exists a constant $C > 0$ such that the following inequalities hold for every $\lambda \in \Lambda$

$$|x_\lambda(t)| \leqslant C \;\; \text{for all} \;\; t \in [t_{\lambda 0}, T] \qquad \text{and} \qquad \int_{t_{\lambda 0}}^{T} |\dot{x}_\lambda(t)| \, \mathrm{d}t \leqslant C.$$

By we have (L4) $\dot{u}_\lambda(t) \leqslant C_1 + C_2 u_\lambda(t)_+$ for a.e. $t \in [t_{\lambda 0}, T]$. Therefore, by points (i) and (ii) in Lemma 3.1 for every $\lambda \in \Lambda$ we obtain

$$-M - 2\mathrm{e}^{2T[C_1 + C_2]}[M+1] \leqslant u_\lambda(t) \leqslant [M + TC_1]\mathrm{e}^{C_2 T} \;\; \text{for all} \;\; t \in [t_{\lambda 0}, T].$$

Taking $\check{R}$ sufficiently large, we get the claim. □

Proof of Lemma 7.9. By Theorem 4.4 there exists $\nu > 0$ such that a function V is finite and continuous on the set $O_\nu(s_0, y_0)$. Moreover, by Corollary 6.4 there exist $\tau < \nu$ and $(y, \eta)(\cdot) \in \mathcal{C}^1(\Delta(s_0, \tau), \mathbb{R}^n \times \mathbb{R})$ such that $(y, \eta)(s_0) = (y_0, V(s_0, y_0))$ and $(\dot{y}(t), \dot{\eta}(t)) \in Q(t, y(t), \eta(t))$ for all $t \in \Delta(s_0, \tau)$. Let $R > 0$ be such that

$|(y,\eta)(t)| < R/2$ and $|(\dot{y},\dot{\eta})(t)| < R$ for all $t \in \Delta(s_0,\tau)$. We choose $C > 0$ in such a way that inequalities in (L6) are satisfied for R and $C/2$. Let $C_1, C_2 > 0$ be constants like in (L4). Notice that for all $t \in \Delta(s_0,\tau)$

$$\begin{aligned} l(t) &:= C[\,1 + 2C_1 + 2C_2R + |\dot{y}(t)| - \dot{\eta}(t)\,] \\ &\leqslant C[\,1 + 2C_1 + 2C_2R + 2R\,] \quad =: \quad K. \end{aligned}$$

We fix $M > 1$ and choose $\varepsilon > 0$ such that

$$\exp\left[\varepsilon C \exp(MK)\right] < \min\{M, \exp(RC/2)\}.$$

By the above we have that to the function $(y,\eta)(\cdot)$ in the version "backward in time" Lemma 7.1 can be applied. By continuity of the functions $V(\cdot,\cdot)$ and $(y,\eta)(\cdot)$ there exists r such that $0 < r < \{\tau, \varepsilon/4\}$ and for every $(s,y) \in O_r(s_0,y_0)$ the following inequality holds

$$\max\{\,|y(s) - y(s_0)|,\ |\eta(s) - \eta(s_0)|,\ |V(s,y) - V(s_0,y_0)|\,\} < \varepsilon/4.$$

Now, we show that for r given above and $k := 2(MK+1)R$ the claim of Lemma 7.9 holds. Indeed, fix $t_0, \bar{t}_0 \in \Delta(s_0,r)$, where $t_0 < \bar{t}_0$ and $x_0 \in \mathbb{B}(y_0,r)$. Since the following inequalities hold:

$$\begin{aligned} &|y(\bar{t}_0) - x_0| \leqslant |y(\bar{t}_0) - y(s_0)| + |y_0 - x_0| < \varepsilon/2, \\ &|\eta(\bar{t}_0) - V(\bar{t}_0,x_0)| \leqslant |\eta(\bar{t}_0) - \eta(s_0)| + |V(s_0,y_0) - V(\bar{t}_0,x_0)| < \varepsilon/2, \end{aligned}$$

by Lemma 7.1 in version, "backward in time" there exists L-solution $(x,u)(\cdot)$ on $[t_0,\bar{t}_0]$ such that $(x,u)(\bar{t}_0) = (x_0, V(\bar{t}_0,x_0))$. Moreover,

$$|\dot{x}(t) - \dot{y}(t)| + |\dot{u}(t) - \dot{\eta}(t)| \leqslant M\,l(t)\,\frac{R}{2} \leqslant MKR \quad \text{a.e. } t \in [t_0,\bar{t}_0].$$

Therefore, $|(\dot{x}(t),\dot{u}(t))| \leqslant k$ for a.e. $t \in [t_0,\bar{t}_0]$. So $(x,u)(\cdot)$ satisfies property (7.9), that ends the proof. □

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