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POINTWISE CONSTRAINED RADIALLY INCREASING MINIMIZERS IN THE QUASI-SCALAR CALCULUS OF VARIATIONS*

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Abstract. We prove *uniform continuity* of *radially symmetric* vector minimizers $u_A(x) = U_A(|x|)$ to multiple integrals $\int_{B_R} L^{**}(u(x), |Du(x)|)\ \mathrm{d}x$ on a *ball* $B_R \subset \mathbb{R}^d$, among the Sobolev functions $u(\cdot)$ in $A + W_0^{1,1}(B_R, \mathbb{R}^m)$, using a *jointly convex lsc* $L^{**} : \mathbb{R}^m \times \mathbb{R} \to [0, \infty]$ with $L^{**}(S, \cdot)$ *even* and *superlinear*. Besides such basic hypotheses, $L^{**}(\cdot,\cdot)$ is assumed to satisfy also a *geometrical* constraint, which we call *quasi − scalar*; the simplest example being the *biradial* case $L^{**}(|u(x)|, |Du(x)|)$. Complete liberty is given for $L^{**}(S, \lambda)$ to take the ∞ value, so that our minimization problem implicitly also represents *e.g. distributed-parameter optimal control* problems, on *constrained domains*, under PDEs or inclusions in explicit or implicit form. While generic radial functions $u(x) = U(|x|)$ in this Sobolev space oscillate wildly as $|x| \to 0$, our minimizing *profile-curve* $U_A(\cdot)$ is, in contrast, *absolutely continuous* and *tame*, in the sense that its "*static level*" $L^{**}(U_A(r), 0)$ always increases with r, a original feature of our result.

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1. Introduction

Aim of this research is to extend into the vectorial $m > 1$ case, under the weakest possible hypotheses, the regularity results appearing, for scalar situations, in our previous paper [1], namely: to reach a radial minimizer whose profile satisfies (1.7) below. While a reader familiar with the contents of [1] may now jump directly to (1.11) and proceed from there, for completeness let us recall such contents along the next paragraphs.

In [1] we have proved existence of a *radial* (or radially symmetric) minimizer $u_A^0(x) = U_A^0(|x|)$ to the convex vectorial multiple integral

$$\int_{B_R} L^{**}\left(u(x), |Du(x)|\ \ \rho_1(|x|)\right)\ \ .\ \ \rho_2(|x|)\ \mathrm{d}x \quad \text{on} \quad W_A^{1,1}(B_R, \mathbb{R}^m), \tag{1.1}$$

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where the lagrangian

$$L^{**} : \mathbb{R}^m \times \mathbb{R} \to [0, \infty] \quad \text{is} \quad \textit{jointly convex lsc} \quad \text{with} \quad L^{**}(S, \cdot) \quad \textit{even} \tag{1.2}$$

$$\text{and} \quad \rho_1 \,\&\, \rho_2 : [\,0, R\,] \to [\,c_0, c_\infty\,] \subset (0, \infty) \qquad \text{are} \quad \textit{Borel measurable}, \tag{1.3}$$

e.g. $\rho_1(\cdot) \equiv 1 \equiv \rho_2(\cdot)$; while the class of functions in competition is the usual Sobolev space

$$W_A^{1,1} := A + W_0^{1,1}(B_R, \mathbb{R}^m) \tag{1.4}$$

of those $u(\cdot)$ taking the constant value $A \in \mathbb{R}^m$ along the boundary ∂B_R of the ball $B_R := \{ x \in \mathbb{R}^d \,:\, |x| < R \}$; and $|\,D\,u(x)\,|$ is the euclidian norm of the $m \times d - \textit{gradient matrix}$.

Many research books and papers (see *e.g.* [5, 8, 10, 11]) contain applications-oriented mathematical models which motivate such search for general hypotheses on the lagrangian $L^{**}(\cdot, \cdot)$ ensuring existence of minimizers to integrals like (1.1). On the other hand, several previous theoretical results already proved existence of radial relaxed minimizers (see *e.g.* [3, 5–7, 9, 10]) for integrals having separation of state from gradient variables,

$$\int_{B_R} g\,(|x|\,, u(x)) + h^{**}\,(|x|\,, |\,D\,u(x)\,|) \;\mathrm{d}x\,; \quad \text{and at least } g\,(t, S) \;\; \textit{finite} \quad \forall\, t, S. \tag{1.5}$$

Naturally it is also quite helpful, for applications, to guarantee nice regularity properties which these minimizers must necessarily satisfy, besides belonging to $W_A^{1,1}$ and being radial; namely to secure some specific *geometric behaviour* of the optimal radial $\textit{profile} - \textit{curve}$ $\;U_A : [\,0, R\,] \to \mathbb{R}^m$. Indeed, one should bear in mind that: even reinforcing superlinearity into p-growth with $p > 1$, while radial functions $u(x) = U\,(|x|)$ in $W_A^{1,p}(B_R, \mathbb{R}^m)$ are, as one easily checks, necessarily *Hölder continuous* away from zero (*e.g.* $u\,(\cdot) \in C_{loc}^{0,1/7}(B_R \setminus \{0\}, \mathbb{R}^m)$ whenever $p \geq 7/6$ & $d > 1$), they generically turn wildly discontinuous as $|x| \to 0$, to the point of mapping arbitrarily small balls $B\,(0, \varepsilon)$ onto the whole of $\mathbb{R}^m$! A simple and striking example (using $p = 7/6$) is

$$u(x) := |x|^{-1/4} \left| \sin\left(|x|^{-1/4}\right) \right| \qquad \left(\cos\left(\,|x|^{-1/4}\right), \sin\left(|x|^{-1/4}\right)\right), \tag{1.6}$$

in which $U\,((0, 1/i)) = \mathbb{R}^2 \quad \forall\, i \in \mathbb{N}$. (Clearly $|\,U(r)\,| \sim r^{-1/4}$ & $|\,U\,'(r)\,|^p \quad r^{d-1} \sim r^{-3/4} \; r^{d-2}$ both belong to $L^1(0, R)$, while $|\,U\,'(r)\,| \sim r^{-3/2}$ does not.)

In contrast with (1.6) and with previous results by other authors, our minimizer $u_A(\cdot)$ is *uniformly continuous*, more precisely its profile

$$U_A(\cdot) \quad \text{is } AC \text{ with } \textit{increasing static} - \textit{level} \quad L^{**}\,(U_A(\cdot), 0) \tag{1.7}$$

in the scalar $m = 1$ case of our previous paper [1]. (Notice that $U_A(\cdot)$ being AC (*absolutely continuous*) is the same as having $U_A(\cdot) \in W^{1,1}([\,0, R\,], \mathbb{R}^m)$.) We need no growth hypotheses on $L^{**}\,(S, \cdot)$, sufficing the knowledge of existence of minimum to the integral (1.1); which is automatic *e.g.* whenever $L^{**}\,(\cdot, \cdot)$ satisfies the usual superlinear growth

$$\frac{\inf\, L^{**}\,(\mathbb{R}^m, \lambda)}{\lambda} \to \infty \qquad \text{as} \qquad \lambda \to \infty. \tag{1.8}$$

On the other hand, again in contrast with previous results by other authors, recall (1.5), our lagrangians $L^{**}\,(\,S, |\xi| \quad \rho_1\,(|x|)) \quad . \quad \rho_2\,(\,|x|)$ satisfy two novelties: they jointly depend on *state* & *gradient* (though joint-convexity must be enforced here; and possibly in a nonautonomous way, provided their nonautonomous part appears under such factorized form); with value $L^{**}\,(S, \lambda) = \infty$ freely allowed. In particular, implicitly included is the possibility of imposing *state* & *gradient pointwise constraints* at will, *e.g.* under the form of partial differential equations or inclusions (in explicit or implicit form), so that *e.g. optimal control* problems on *constrained domains* are also (theoretically) included in our optimization problem for the integral (1.1).

Provided, of course, their *variational* reformulation has the form (1.1), with (1.2) and (1.3), and our extra hypotheses are satisfied, as explained below.

Now just two simple observations before formally stating our previous result: for jointly depending $L^{**}(\cdot,\cdot)$, clearly the general hypothesis (1.2) is anyway needed, in order to apply *Jensen inequality* so as to reach radial minimizers; and the simple hypothesis

$$\exists \ \min L^{**}(\mathbb{R}^m, 0) \tag{1.9}$$

holds true not only whenever $L^{**}(\cdot, 0)$ has bounded sublevel sets

$$\Sigma_A := \{ S \in \mathbb{R}^m : \quad L^{**}(S,0) \le L^{**}(A,0) \}, \tag{1.10}$$

but also in case its set of minimizers is unbounded, *e.g.* a half-space.

Here is, finally, a first version of our previous result:

Proposition 1.1 (See [1], Thms. 1 and 2)**.** *Under (1.2) and (1.3), the vectorial multiple integral*

$$(1.1) \qquad \text{has radial minimizers} \qquad u_A^0(x) = U_A^0(|x|) \qquad \text{provided it has minimizers;} \tag{1.11}$$

and (1.1) must have minimizers e.g. whenever $L^{**}(\cdot,\cdot)$ is superlinear, as in (1.8).

Moreover, its optimal profile $U_A^0 : [0,R] \to \mathbb{R}^m$ in (1.11) satisfies the extra regularity:

$$m = 1 \ \& \ \exists \min L^{**}(\mathbb{R}, 0) \qquad \Rightarrow \qquad U_A^0(\cdot) \ \text{is AC monotone} \ \& \ L^{**}\big(U_A^0(\cdot), 0\big) \ \text{increases.} \tag{1.12}$$

Aim of the present paper is to extend into the vectorial $m > 1$ case this extra regularity, namely to reach a radial minimizer $u_A(\cdot)$ having profile satisfying (1.7). In order to fulfill such aim, we would first need to overcome the main difficulty blocking our road in [1]: a complete lack of info on the spatial path followed by the given minimizing vector profile-curve $U_A^0(\cdot)$; in particular, full ignorance on whether it did, or did not, always remain inside the sublevel region Σ_A defined in (1.10).

To overcome such obstacle, we have in this research decided to experiment with an original strategy: to gradient-flow, in $\mathbb{R}^m$, starting from the given final boundary-data point A, towards decreasing $L^{**}(\cdot, 0)$; thus ensuring, by construction, permanence within Σ_A, and wishfully hoping to again obtain, in this novel way, a (new) minimizing profile-curve $U_A(\cdot)$.

Although our geometric intuition was, finally, correct, it turned out much harder to prove rigorously than anticipated. Indeed, to our great surprise, it was quite hard to discover the adequate hidden precise previous steps appropriate to open the gate towards a successful proof of the crucial geometric inequalities (3.64) and (3.65), hence to what we regard as our main technical achievement in this paper: the statement and proof of the heavy Claim 1 − consisting in a chain of fifteen affirmations, (3.56) to (3.70) − which will become a useful technical tool in future researches.

We would never have expected uniform continuity of radial minimizers to hold necessarily true again in the much more complex vectorial case under the same amazingly simple basic hypotheses (1.2), (1.3) and (1.9) previously used by us in the scalar case. But we do succeed here in proving such extension into the vectorial $m > 1$ case by adding just three very precise *extra hypotheses*. First, a trivial one: we assume

$$\text{the} \quad \textit{subdifferential} \quad \text{of} \quad L^{**}(\cdot, 0) \quad \text{at} \quad A \quad \text{to be} \quad \textit{nonempty}. \tag{1.13}$$

Second, and defining

$$\partial^0 L^{**}(S,0) := \text{the} \quad \textit{minimal norm element} \quad \text{of the subdifferential of} \quad L^{**}(\cdot,0) \quad \text{at} \quad S, \tag{1.14}$$

unless $L^{**}(\cdot,0) \equiv L^{**}(A,0)$ we ask that $\exists\, \mu_L > 0$ for which, at any $S \in \mathbb{R}^m$ having $\partial L^{**}(S,0) \neq \emptyset$,

$$\partial^0 L^{**}(S,0) \neq 0 \qquad \Rightarrow \qquad \big|\, \partial^0 L^{**}(S,0) \,\big| \ge \mu_L > 0. \tag{1.15}$$

(While (1.13) generally holds true in real-life applications, clearly the effect of (1.15) is to reinforce (1.9), due to (1.2), by imposing a mild geometrical restriction on approaching *min points*: the *slope* of $L^{**}(\cdot,0)$ cannot approach smoothly zero, on the contrary, if this slope ever moves away from zero, it must do it by jumping *nonsmoothly*).

Finally, our third extra hypothesis turns out to be our main restrictive hypothesis, to the point of inspiring the title of this paper. Its most obvious instance, for which we would expect useful geometric applications, is the special case in which the *static-level* only depends − see (1.25) − on the distance to a set and $L^{**}(\cdot,\lambda)$ is a scalar function of this distance. We are still not sure whether there may exist other possibilities, essentially different from this one; or whether it is, instead, a simple geometric necessity which is somehow intrinsically encoded in our analytic definition.

Such hypothesis, concerning any pair S, S' of points along the same level "line" of the static-level, was hidden in the scalar or biradial case by being trivially true: for any $S \,\&\, S'$ in $\mathbb{R}^m$,

$$\begin{aligned} &\inf\, L^{**}(\mathbb{R}^m,0) < L^{**}(S,0) = L^{**}(S',0) \le L^{**}(A,0) \qquad \Rightarrow \\ &\Rightarrow \left|\partial^0 L^{**}(S,0)\right| = \left|\partial^0 L^{**}(S',0)\right| \quad \& \quad L^{**}(S,\lambda) = L^{**}(S',\lambda)\ \forall\,\lambda. \end{aligned} \tag{1.16}$$

Definition 1.2. Under (1.2) and (1.14), we call

$$L^{**}(\cdot,\cdot) \qquad \textit{quasi-scalar} \qquad \text{whenever} \quad (1.16) \quad \text{is satisfied.} \tag{1.17}$$

In order to analyze the meaning of (1.16), let us consider those $L^{**}(\cdot,\cdot)$ of the special form

$$L^{**}(S,\lambda) := \begin{cases} \ell^{**}(L^{**}(S,0),\lambda) & \text{where} \quad L^{**}(S,0) < \infty \\ \infty & \text{elsewhere,} \end{cases} \tag{1.18}$$

$$\text{for some} \qquad \ell^{**}: I_L\times\mathbb{R} \to [0,\infty] \qquad \text{yielding a} \quad L^{**}(\cdot,\cdot) \quad \text{as in (1.2) \& (1.9),} \tag{1.19}$$

$$\text{using the interval} \qquad I_L := [\,\min\, L^{**}(\mathbb{R}^m,0)\,,\, \sup\, L^{**}(\mathbb{R}^m,0)\,] \cap \mathbb{R}, \tag{1.20}$$

so that in particular

$$\ell^{**}(p,\cdot) \quad \text{is} \quad \textit{even} \quad \text{with} \qquad \ell^{**}(p,0) = p\ \ \forall\, p \in I_L. \tag{1.21}$$

As is natural, along the next paragraphs we always assume the given

$$L^{**}(\cdot,0): \mathbb{R}^m \to [0,\infty] \qquad \text{to be} \ \ \textit{convex lsc} \qquad \text{and satisfy} \qquad (1.13)\ \&\ (1.9). \tag{1.22}$$

(Then, by picking − as seems natural − any convex *lsc* $\ell^{**}: I_L\times\mathbb{R} \to [0,\infty]$ satisfying (1.21), the easiest recipe to ensure that such $\ell^{**}(\cdot,\cdot)$ yields − through (1.18) − a $L^{**}(\cdot,\cdot)$ satisfying (1.2) is by imposing $\ell^{**}(\cdot,\lambda)$ to increase along I_L, $\forall\,\lambda$, as is easily checked.)

Clearly the last equality in (1.16) trivially holds true for any $L^{**}(\cdot,\cdot)$ as in (1.18); while, reciprocally, the last equality in (1.16), satisfied by a $L^{**}(\cdot,\cdot)$ as in (1.2), implies existence of an adequate

$$\ell^{**}: I_{L,A}\times\mathbb{R} \to [0,\infty], \qquad I_{L,A} := [\,\min\, L^{**}(\mathbb{R}^m,0)\,,\, L^{**}(A,0)\,] \tag{1.23}$$

to satisfy (1.21) $\forall\, p \in I_{L,A}$ together with

$$L^{**}(S,\lambda) = \ell^{**}(L^{**}(S,0),\lambda) \qquad \text{whenever} \qquad L^{**}(S,0) \le L^{**}(A,0)\,. \tag{1.24}$$

Indeed, just define $\ell^{**}(p,\lambda) := L^{**}(S,\lambda)$, at any $p \in I_{L,A}$, by picking any point S of the given level set $L^{**}(\cdot,0)^{-1}(p)$. (Notice, by the way, that the inequality $\le L^{**}(A,0)$ (resp. the interval $I_{L,A}$) can be replaced by the inequality $< \infty$ in (1.16), hence in (1.24) (resp. by the interval I_L in (1.23)), since this just means to assume a stronger hypothesis).

Thus the last equality in (1.16), for such $L^{**}(\cdot,\cdot)$, simply tells us that $L^{**}(\cdot,\cdot)$ has the form (1.24). On the contrary, the first equality in (1.16) poses an intrinsic geometrical constraint on the *graph* of the *static-level*: $L^{**}(\cdot,0)$ must have constant slope (= length of the gradient) along each one of its level sets, as happens notably in case

$$L^{**}(S,0) := \ (signed)\ distance \quad \text{from} \quad S \quad \text{to a set} \quad C; \tag{1.25}$$

in particular whenever $L^{**}(S,0) = |\,S\,|$ or $L^{**}(\cdot,0)$ is affine, or whenever $m = 1$ (using $L^{**}(s,0) = s$ and the set $(-\infty,0)$). Another example: an adequate function $f(\cdot)$ − *e.g.* $f(d) := \max\left\{-1/2, \mathrm{e}^d - 1\right\}$ − of the (signed) distance to an open convex set, the simplest nontrivial examples being the interior of an ellipse or triangle. (The *signed distance* to an open set becomes negative inside such set, equal to minus the distance to its boundary.)

While our third extra hypothesis (1.16) may now seem a bit restrictive, a remarkable feature of our approach is as follows. We did not impose, from the start, to treat just the simple biradial case (as in the Abstract) or just the more general case (1.18), (1.21) and (1.25), in which $L^{**}(\cdot,\lambda)$ is a scalar function of the (signed) distance to a set. On the contrary, we have begun by developing first our original solution method; and only afterwards did we push its hypotheses to their weakest possibility. Thus our third extra hypothesis (1.16) is not a convenience one to facilitate proofs; rather it arose intrinsically as a minimal necessity imposed on us by the solution method which was our starting point. So, while it is nice to have joint dependence $L^{**}(S,\lambda)$ on state & gradient yielding uniform continuity of vector minimizers with increasing static-level, maybe (1.16) is a unavoidable price to be paid for such convenience. To settle such question it would certainly be good to find a counterexample featuring discontinuous or nonincreasing minimizers.

Besides the nonautonomous vectorial multiple integral (1.1), we also deal here with the problems of minimizing three other auxiliary nonautonomous scalar and vectorial single convex integrals (as stated in the next section), for which we prove both existence of AC minimizers (unknown before, we feel, even in the superlinear case (1.8)) and regularity as well.

Here is a rough description of the proof: after defining the raw, possibly wild, static-level $w_A^0(\cdot)$ of a given radial minimizer $u_A^0(\cdot)$ to the integral (1.1), we begin by regularizing it, thus reaching an *increasing* AC static-level $w_A(\cdot)$, from which we build a new, more regular, minimizer $u_A(\cdot)$ to (1.1), as follows. Starting from the final endpoint A, we build $u_A(\cdot)$ by gradient-flowing in $\mathbb{R}^m$, downwards, using $w_A(\cdot)$ as guide; and show, moreover, that also $w_A(\cdot)$ itself minimizes a (new) integral, an intuitive idea which seems to us quite original. Indeed, we have thus reduced our minimization problem (1.1), of a vectorial multiple integral, to the (*e.g.* numerical) minimization of a scalar single integral. Moreover, since the spatial path followed by our minimizer now becomes known as soon as one fixates the problem-data, it clearly suffices to − roughly speaking − assume hypotheses along that path only; or, in other words, one is given complete freedom to increase $L^{**}(\cdot,\lambda)$ − *e.g.* so as to become ∞ − away from such path without affecting our minimizer. This is a feature which might prove useful *e.g.* in optimal control.

After having decided on our new strategy, much of the proof turned out relatively straightforward, even if unavoidably long and technical; the main exception being the above-mentioned crucial inequality (3.65), whose conquest took us months of faith, ingenuity and perseverance, leading to the whole Claim 1.

Since this paper is already long, its nonconvex version [2], in which $L^{**}(S,\cdot)$ in (1.1) is replaced by a *nonconvex* $L(S,\cdot)$, appears elsewhere.

2. Statement of the integrals to be minimized and their spaces of functions in competition

Starting from a given

$$u_A^0(\cdot) \qquad (e.g.\ radial)\ \ minimizer \quad \text{to the} \quad integral \quad (1.1), \tag{2.1}$$

we construct below, under the basic hypotheses (1.2), (1.3), (1.9), (1.13), (1.15) and (1.16), and recalling the notation $W_A^{1,1}$ in (1.4), a radial minimizer $u_A(\cdot)$ to the vectorial convex multiple integral

$$\int_{B_R} L^{**}\left(u(x), |\, D\, u(x)\,| \quad \rho_1\left(|x|\right)\right). \quad \rho_2\left(|x|\right)\; \mathrm{d}x \qquad\qquad \text{on} \quad W^{1,1}_{A\nearrow}, \tag{2.2}$$

$$W^{1,1}_{A\nearrow} := \left\{ u(\cdot) \in W_A^{1,1} \cap C^0\left(\overline{B}_R, \mathbb{R}^m\right) : \quad \exists\, U(\cdot) \in \mathcal{U}^{0,R}_{A\nearrow} \qquad \text{with} \qquad u(x) = U\left(|x|\right)\; \forall\, x \right\}, \tag{2.3}$$

$$\mathcal{U}^{0,R}_{A\nearrow} := \left\{ U(\cdot) \in W^{1,1}\left([\,0,R\,]\,, \mathbb{R}^m\right) : \quad L^{**}\left(U(\cdot), 0\right) \quad \textit{increases} \qquad \& \qquad U(R) = A \right\}. \tag{2.4}$$

However, besides this main pair of vectorial convex multiple integrals (1.1) and (2.2), we also consider below the problems of minimizing two new pairs of auxiliary single integrals (the second in each pair being, again, the same integral but defined over a more regular class of functions in competition). First such pair:

$$\alpha_d \int_0^R L^{**}\left(U(r), |\, U'(r)\,| \quad \rho_1(r)\right). \quad \rho_2(r)\; r^{d-1}\; \mathrm{d}r \quad \text{on} \quad \mathcal{U}^{0,R}_{A} \tag{2.5}$$

$$\alpha_d \int_0^R L^{**}\left(U(r), |\, U'(r)\,| \quad \rho_1(r)\right). \quad \rho_2(r)\, r^{d-1}\; \mathrm{d}r \quad \text{on} \quad \mathcal{U}^{0,R}_{A\nearrow}, \tag{2.6}$$

where α_d is the Hausdorff measure in dimension $d-1$ of the unit sphere $S^d := \left\{ x \in \mathbb{R}^d : \quad |x| = 1 \right\}$, while $\mathcal{U}^{0,R}_{A\nearrow}$ has been defined in (2.4) and

$$\mathcal{U}^{0,R}_{A} := \left\{ U(\cdot) \in W^{1,1}_{loc}\left((0,R\,]\,, \mathbb{R}^m\right) : \quad r \to |\, U'(r)\,|\;\; r^{d-1} \in L^1(0,R) \qquad \& \qquad U(R) = A \right\}. \tag{2.7}$$

Second such pair of auxiliary single integrals:

$$\alpha_d \int_0^a L^{**}(z(t), |\, z'(t)\,| \quad \rho(t))\; \mathrm{d}t \qquad\qquad \text{on} \quad Z^{0,a}_{A} \tag{2.8}$$

$$\alpha_d \int_0^a L^{**}(z(t), |\, z'(t)\,| \quad \rho(t))\; \mathrm{d}t \qquad\qquad \text{on} \quad Z^{0,a}_{A\nearrow}, \tag{2.9}$$

using the following definitions:

$$a := \int_0^R \rho_2(r) \quad r^{d-1}\; \mathrm{d}r \quad \& \quad \rho(t) := \rho_1\left(\gamma(t)\right)\; \rho_2\left(\gamma(t)\right)\; \gamma(t)^{d-1}, \tag{2.10}$$

$$\gamma : [0,a] \to [\,0,R\,]\,,\; r = \gamma(t), \qquad \text{being the} \quad \textit{inverse function} \quad \text{of} \tag{2.11}$$

$$\gamma^{-1}(\cdot) :\; r \mapsto t = \gamma^{-1}(r) := \int_0^r \rho_2(\alpha) \quad \alpha^{d-1}\; d\,\alpha, \tag{2.12}$$

$$Z^{0,a}_{A} := \left\{ z(\cdot) \in W^{1,1}_{loc}\left((0,a], \mathbb{R}^m\right) : \quad |\, z'(\cdot)\,|\;\; \gamma(\cdot)^{d-1} \in L^1(0,a) \qquad \& \qquad z(a) = A \right\}, \tag{2.13}$$

$$Z^{0,a}_{A\nearrow} := \left\{ z(\cdot) \in W^{1,1}\left([0,a], \mathbb{R}^m\right) : \quad L^{**}\left(z(\cdot), 0\right) \quad \textit{increases} \qquad \& \qquad z(a) = A \right\}. \tag{2.14}$$

Another related pair of single integrals will be introduced near the end of this paper (in (4.1) and (4.2)).

3. Radially increasing minimizers to vectorial quasi-scalar convex multiple integrals

Here is our first existence and regularity result (to be complemented by (4.6) below):

Theorem 3.1. *Assume (1.2), (1.3), (1.9), (1.13), (1.15) and (1.16) together with*

$$\text{either} \quad (1.8) \quad \text{or else} \qquad \exists \text{ minimum to } (1.1) \text{ or } (2.5) \text{ or } (2.8). \tag{3.1}$$

Then

$$\exists \text{ radial } \; u_A(x) = U_A(|x|) \; \text{ minimizing both } (1.1) \;\&\; (2.2), \tag{3.2}$$

$$\exists \; U_A(r) \; \text{ minimizing both } (2.5) \;\&\; (2.6), \tag{3.3}$$

$$\exists \; z_A(t) \; \text{ minimizing both } (2.8) \;\&\; (2.9). \tag{3.4}$$

Moreover: the minimum value to all these integrals is the same and the following equivalences hold true

$$(3.1) \;\Leftrightarrow\; (3.2) \;\Leftrightarrow\; (3.3) \;\Leftrightarrow\; (3.4). \tag{3.5}$$

Proof. More precisely than in (1.11), the first part of our previous result is:

Proposition 3.2 (See [1], Thm.1).
Assume (1.2) and (1.3). Then the following four statements are all equivalent:

$$\text{the integral } (1.1) \text{ has minimum} \qquad (\text{which is implied e.g. by } (1.8)) \tag{3.6}$$

$$\exists \text{ a radial minimizer } \; u_A^0(x) = U_A^0(|x|) \; \text{ to } (1.1) \tag{3.7}$$

$$\exists \text{ a minimizer } \; U_A^0(r) \; \text{ to } (2.5) \tag{3.8}$$

$$\exists \text{ a minimizer } \; z_A^0(t) \; \text{ to } (2.8). \tag{3.9}$$

Moreover the minimum value to all these integrals is the same and

$$U_A^0(\cdot) = z_A^0\left(\gamma^{-1}(\cdot)\right), \quad \text{with} \quad \gamma^{-1}(\cdot) \quad \text{Lipschitz increasing} \tag{3.10}$$

(see (2.12) and (1.3)), in the sense that: by picking a minimizer $U_A^0(\cdot)$ to (2.5) (resp. $z_A^0(\cdot)$ to (2.8)) and applying the formula (3.10) one gets a minimizer $z_A^0(\cdot)$ to (2.8) (resp. $U_A^0(\cdot)$ to (2.5)).

Thus, by the above equivalences, all the implications

$$(3.7) \;\Leftrightarrow\; (3.1) \;\Leftrightarrow\; (3.8) \;\Leftrightarrow\; (3.9) \;\Rightarrow\; (3.4) \tag{3.11}$$

are proved, except for the last one, (3.9) $\Rightarrow$ (3.4), which will be established only in (3.74) & (3.78), after lengthy, though unavoidable, technical preliminaries. (This lengthy first step almost exhausts the whole proof of Theorem 3.1, remaining only minor steps.)

To begin with take (3.9), namely a

$$z_A^0(\cdot) \in Z_A^{0,a} \;\; \textit{minimizer} \;\; \text{to} \;\; (2.8), \tag{3.12}$$

define its "lsc optimal level"

$$w_A^0(0) := \inf\, L^{**}\left(z_A^0\left((0,a]\right), 0\right) \qquad \& \qquad w_A^0(t) := L^{**}\left(z_A^0(t), 0\right) \quad \text{for} \quad t \in (0,a]; \tag{3.13}$$

and its "increasing continuous optimal level"

$$w_A(t) := \min\, w_A^0([t,a]) \qquad \forall\, t \in [0,a]. \tag{3.14}$$

Then clearly

$$0 \le w_A(\cdot) \le w_A^0(\cdot) \in W_{loc}^{1,1}((0,a]), \tag{3.15}$$

by (3.13), (3.14), (2.13) and (1.2), since $L^{**}(\cdot,0)$ is convex hence locally Lipschitz there; while since $w_A(\cdot) \in C^0([0,a])$, $w_A(\cdot)$ increases and $w_A(\cdot) \in W_{loc}^{1,1}((0,a])$ (due to remaining constant where it differs from $w_A^0(\cdot)$), by [12], Theorem 13.8, we have:

$$w_A(\cdot) \in W^{1,1}([0,a]) \qquad \& \qquad w_A(\cdot) \quad \text{increases.} \tag{3.16}$$

Hence, setting

$$p_A^{\min} := w_A(0) \qquad \& \qquad p_A^{\max} := w_A(a), \tag{3.17}$$

one gets

$$0 \le w_A(0) = p_A^{\min} = \min\, w_A([0,a]) = \min\, w_A^0([0,a]) = \inf\, L^{**}\left(z_A^0((0,a]), 0\right) \le w_A(\cdot) \tag{3.18}$$

$$\& \qquad w_A(\cdot) \le w_A(a) = p_A^{\max} = L^{**}(A,0) = \max\, w_A([0,a]). \tag{3.19}$$

Defining now

$$b := \max\left\{ t \in [0,a]: \quad w_A(t) = w_A(0) \right\} \qquad (b \in [0,a]) \tag{3.20}$$

$$a' := \min\left\{ t \in [0,a]: \quad w_A(t) = w_A(a) \right\} \qquad (b < a' \in [0,a]), \tag{3.21}$$

notice first that we indeed have $b < a'$ because, excluding the trivial case $p_A^{\min} = p_A^{\max}$ in which we just pick $u_A(\cdot) \equiv A$ as our minimizer to (2.2) and (1.1), we assume, in this proof,

$$p_A^{\min} \;<\; p_A^{\max}. \tag{3.22}$$

Note then that

$$w_A(\cdot) \equiv p_A^{\min} \quad \text{on} \quad [0,b], \quad w_A((b,a')) = \left(p_A^{\min}, p_A^{\max}\right) \;\&\; w_A(\cdot) \equiv p_A^{\max} \;\text{on}\; [a',a], \tag{3.23}$$

set

$$\Sigma_A^{<} := \left\{ S \in \mathbb{R}^m : \quad p_A^{\min} \le L^{**}(S,0) < p_A^{\max} \right\} \tag{3.24}$$

and, using the notation (1.14), define the vector orthogonal to the level sets and pointing downwards:

$$V_A : \Sigma_A^{<} \to \mathbb{R}^m, \;\; V_A(S) := -\,\partial^0 L^{**}(S,0). \tag{3.25}$$

Since we are assuming (1.13),

$$\partial^0 L^{**}(A,0) \qquad \textit{exists} \quad \text{and is} \quad \ne 0 \qquad (\textit{see} \quad (1.14),\ (3.22)\ \&\ (3.19)). \tag{3.26}$$

Therefore, by (1.2), (3.26) and *e.g.* [4], there exists a unique solution, in $W^{1,2}((0,\infty), \mathbb{R}^m)$, to the ordinary differential equation

$$\sigma_A'(\tau) = V_A(\sigma_A(\tau)) \qquad \text{for a.e.} \quad \tau \in \left[0, \tau_A^0\right] \qquad \& \quad \sigma_A(0) = A, \tag{3.27}$$

with

$$\tau_A^0 := \min\left\{\,\tau \in (0,\infty) : \qquad L^{**}\left(\sigma_A(\tau),0\right) = p_A^{\min}\,\right\}; \tag{3.28}$$

so that, setting

$$p_A : \left[\,0, \tau_A^0\,\right] \to \left[\,p_A^{\min},\, p_A^{\max}\,\right], \qquad\qquad p_A(\tau) := L^{**}\left(\sigma_A(\tau),0\right), \tag{3.29}$$

we get:

$$-\,p_A'(\tau) = \left|\,\partial^{\,0} L^{**}\left(\sigma_A(\tau),0\right)\right|^2 = \left|\,V_A\left(\sigma_A(\tau)\right)\right|^2 > 0 \qquad \text{a.e. on } \left[\,0,\tau_A^0\,\right], \tag{3.30}$$

$$p_A(\cdot) \quad \text{is} \quad \textit{decreasing convex} \qquad\qquad (\text{since} \quad p_A'(\cdot) \quad \textit{increases}), \tag{3.31}$$

$$p_A(0) = p_A^{\max} \qquad \& \qquad p_A\left(\tau_A^0\right) = p_A^{\min}, \tag{3.32}$$

$$L^{**}\left(\sigma_A\left(\tau_A^0\right),0\right) = p_A^{\min} = w_A(0) = w_A^0(0) \qquad (\text{see also } (3.18)). \tag{3.33}$$

In particular, by (3.27) and (3.30), $|\,\sigma_A'(\cdot)\,| = |\,V_A\left(\,\sigma_A(\cdot)\right)| = |\,p_A'(\cdot)\,|^{1/2}$ decreases, hence

$$|\,\sigma_A'(\cdot)\,| \le \,|\,p_A'(0)\,|^{1/2} = \left|\,\partial^{\,0} L^{**}\left(A,0\right)\right| < \infty \qquad\quad \text{and} \qquad\quad \sigma_A(\cdot) \quad \text{is } \textit{Lipschitz}. \tag{3.34}$$

(To check that

$$\tau_A^0(\cdot) \in (0,\infty) \qquad\quad \text{is well-defined by} \quad (3.28), \tag{3.35}$$

recall (3.22) and notice that since, by (3.30) & (1.15), $p_A'(\tau) \le -\mu_L^2$ on $\left(0,\,\tau_A^0\right)$, we would have $p_A(\tau) < 0 \le p_A^{\min}$ whenever $\tau > p_A^{\max}\,\mu_L^{-2} < \infty$).

Obviously $p_A(\cdot)$ has, due to (3.35), (3.29) and (3.30), continuous inverse

$$\tau_A : \left[\,p_A^{\min}, p_A^{\max}\,\right] \subset [0,\infty) \to \left[\,0,\tau_A^0\,\right] \subset [0,\infty), \qquad \tau = \tau_A(p) \qquad \Leftrightarrow \qquad p = p_A(\tau); \tag{3.36}$$

and clearly

$$-\frac{1}{\mu_L^2} \le \tau_A'(p) = \frac{1}{p_A'\left(\tau_A(p)\right)} = \frac{-1}{\left|\,\partial^{\,0} L^{**}\left(\sigma_A\left(\tau_A(p)\right),0\right)\right|^2} < 0, \tag{3.37}$$

$$\tau_A\left(p_A^{\min}\right) = \tau_A^0 \qquad\qquad \& \qquad\qquad \tau_A\left(p_A^{\max}\right) = 0, \tag{3.38}$$

$$\tau_A'(\cdot) \quad \textit{increases} \qquad\quad \text{and} \qquad\quad \tau_A(\cdot) \quad \text{is} \quad \textit{Lipschitz convex decreasing}, \tag{3.39}$$

a simple specific instance of (3.36), (3.31) & (3.39) being *e.g.*

$$p_A : [\,0,1\,] \to [\,2,\,5\,], \qquad p_A(\tau) := 1 + (\tau-2)^2 = p \qquad \Leftrightarrow \qquad \tau = \tau_A(p) = 2 - \sqrt{p-1}. \tag{3.40}$$

Our next step consists in using these tools in order to define new and useful functions, as follows. First take

$$g_A : \left[\,p_A^{\min},\, p_A^{\max}\,\right] \to (0,\infty), \quad g_A(p) := \frac{1}{\left|\,\partial^{\,0} L^{**}\left(\sigma_A\left(\tau_A(p)\right),0\right)\right|} \tag{3.41}$$

for $p > p_A^{\min}$, with $g_A\left(p_A^{\min}\right) := 1/\mu_L$, so that

$$\begin{gathered} g_A(p)^2 = -\,\tau_A'(p) \qquad \& \qquad g_A(\cdot) \qquad \textit{decreases}, \\ 0 < \tfrac{1}{|\,\partial^{\,0} L^{**}(A,0)\,|} \le g_A(p) = |\,\tau_A'(p)\,|^{1/2} \le \tfrac{1}{\mu_L} < \infty \qquad\qquad (\text{due to } (1.15)), \end{gathered} \tag{3.42}$$

$$g_A(\cdot) \qquad \text{is } \textit{bounded} \text{ away from } 0 \ \& \ \infty, \tag{3.43}$$

$$\tau_A(\cdot) \ \text{ and } \ p_A(\cdot) \qquad \text{are both } \textit{Lipschitz}. \tag{3.44}$$

Then define

$$Q_A : \left[p_A^{\min}, p_A^{\max} \right] \to \mathbb{R}^m, \qquad Q_A(p) := \sigma_A\left(\tau_A(p)\right) \quad (\text{see } (3.27) \ \& \ (3.36)) \tag{3.45}$$

$$\ell_A^{**} : \left[p_A^{\min}, p_A^{\max} \right] \times \mathbb{R} \to [0,\infty], \qquad \ell_A^{**}(p,\lambda) := L^{**}\left(Q_A(p), \lambda\right). \tag{3.46}$$

Finally, the next function will become our new desired minimizer to (2.8):

$$z_A(t) := Q_A\left(w_A(t)\right) \qquad \text{(using (3.14) \& (3.45))}. \tag{3.47}$$

(The reader may at this point wish to pause, recall our starting point (3.12) and preview our destination (3.75).

Clearly

$$Q_A(\cdot) \ \text{ is } \ \textit{Lipschitz} \qquad \& \qquad z_A(\cdot) \in W^{1,1}\left([0,a], \mathbb{R}^m\right), \tag{3.48}$$

by (3.45), (3.34), (3.44), (3.47) and (3.16); while by (3.47), (3.45), (3.29) & (3.36),

$$L^{**}\left(z_A(t), 0\right) = L^{**}\left(\sigma_A\left(\tau_A\left(w_A(t)\right)\right), 0\right) = p_A\left(\tau_A\left(w_A(t)\right)\right) = w_A(t) \tag{3.49}$$

so that, by (3.49), (3.15), (3.13) & (1.2),

$$L^{**}\left(z_A(\cdot), 0\right) = w_A(\cdot) \le w_A^0(\cdot) = L^{**}\left(z_A^0(\cdot), 0\right) \le L^{**}\left(z_A^0(\cdot), \left| z_A^{0\,\prime}(\cdot) \right| \quad \rho(\cdot)\right) \qquad \text{on } \ (0,a]; \tag{3.50}$$

and, by (3.47), (3.45), (3.27), (3.37), (3.41) and (3.25),

$$z_A'(t) = -\frac{V_A\left(z_A(t)\right)}{\left| V_A\left(z_A(t)\right) \right|} \ \ g_A\left(w_A(t)\right) \ w_A'(t) \qquad \text{a.e. on } \ (b, a') \tag{3.51}$$

hence

$$\left| z_A'(t) \right| = g_A\left(w_A(t)\right) \ w_A'(t) \quad \& \quad w_A'(t) = \left| z_A'(t) \right| \ . \ \left| \partial^0 L^{**}\left(z_A(t), 0\right) \right| \qquad \text{a.e. on } \ (b, a'). \tag{3.52}$$

Moreover, by (3.46), (3.45), (3.29) and (3.36), for any $p \in \left[p_A^{\min}, p_A^{\max} \right]$,

$$\ell_A^{**}(p, 0) := L^{**}\left(Q_A(p), 0\right) = p_A\left(\tau_A(p)\right) = p. \tag{3.53}$$

By (3.15), (3.16), (3.12), (2.13), (3.48), (3.20) and (3.21) one may define the set

$$\mathcal{T}_+ := \left\{ t \in (b, a') \ : \ \exists\, w_A'(t) > 0, \ \exists\, w_A^{0\,\prime}(t), \ \exists\, z_A^{0\,\prime}(t) \ \& \ \exists\, z_A'(t) \right\}, \tag{3.54}$$

whose crucial properties are condensed in the next claim, the main technical result of this paper:
Claim 1. *Extending the definitions in* (3.13) and (3.14) *by*

$$w_A(t) = w_A^0(t) := p_A^{\min}, \ \ t < 0 \qquad \& \qquad w_A(t) = w_A^0(t) := p_A^{\max}, \ \ t > a, \tag{3.55}$$

then each one of the following fifteen numbered statements, (3.56) to (3.70), *holds true*:

$$L^{**}\left(z_A^0(t), 0\right) = w_A^0(t) = w_A(t) = L^{**}\left(z_A(t), 0\right) \qquad \forall\, t \in \mathcal{T}_+ \tag{3.56}$$

$$w_A^0(t) < w_A^0(t+h) \quad \forall\, t \in \mathcal{T}_+ \quad \forall\, h > 0 \tag{3.57}$$

$$w_A(t) < w_A(t+h) \quad \forall\, t \in \mathcal{T}_+ \quad \forall\, h > 0 \tag{3.58}$$

$$\forall\, t \in \mathcal{T}_+ \ \exists\, (h_k) \searrow 0 \ : \ w_A(t+h_k) = w_A^0(t+h_k) \tag{3.59}$$

$$0 < w_A'(t) = w_A^{0\,\prime}(t) = |\, z_A'(t)\,| \ . \ \left|\, \partial^0 L^{**}\,(z_A(t), 0)\,\right| \quad \forall\, t \in \mathcal{T}_+ \tag{3.60}$$

$$\forall\, t \in \mathcal{T}_+ \quad \exists\, \delta > 0 \ : \ w_A^0(t-h) < w_A^0(t) \quad \forall\, h \in (0, \delta) \tag{3.61}$$

$$w_A(t-h) < w_A(t) \quad \forall\, t \in \mathcal{T}_+ \quad \forall\, h > 0 \tag{3.62}$$

$$\begin{aligned} t \in \mathcal{T}_+ \ \Rightarrow \ 0 < w_A'(t) = w_A^{0\,\prime}(t) = |\, z_A'(t)\,| \ . \ \left|\, \partial^0 L^{**}\,(\, z_A(t), 0)\,\right| \ \leq \\ \leq \ \left|\, z_A^{0\,\prime}(t)\,\right| \ . \ \left|\, \partial^0 L^{**}\left(z_A^0(t), 0\right)\right| \ \leq \ \left|\, z_A^{0\,\prime}(t)\,\right| \ . \ \left|\, \partial^0 L^{**}\,(A, 0)\,\right| \end{aligned} \tag{3.63}$$

$$0 < |\, z_A'(t)\,| \leq \left|\, z_A^{0\,\prime}(t)\,\right| \quad \forall\, t \in \mathcal{T}_+ \tag{3.64}$$

$$L^{**}\,(z_A(t), |\, z_A'(t)\,| \quad \rho(t)) \ \leq \ L^{**}\left(z_A^0(t), \left|\, z_A^{0\,\prime}(t)\,\right| \quad \rho(t)\right) \quad \forall\, t \in \mathcal{T}_+ \tag{3.65}$$

$$0 = w_A'(t) = |\, z_A'(t)\,| \leq \left|\, z_A^{0\,\prime}(t)\,\right| \quad a.e. \ on \ \ [\,0, a\,] \setminus \mathcal{T}_+ \tag{3.66}$$

$$L^{**}\left(z_A^0(t), 0\right) = w_A^0(t) = w_A(t) = L^{**}\,(z_A(t), 0) \qquad \forall\, t \in [0, a] \tag{3.67}$$

$$L^{**}\left(z_A^0(t), \left|\, z_A^{0\,\prime}(t)\,\right| \quad \rho(t)\right) \ = \ L^{**}\,(z_A(t), |\, z_A'(t)\,| \quad \rho(t)) \quad a.e. \ on \ \ [\,0, a\,] \tag{3.68}$$

$$L^{**}\left(z_A^0(t), \left|\, z_A^{0\,\prime}(t)\,\right| \quad \rho(t)\right) \ = \ L^{**}\left(z_A^0(t), 0\right) \quad a.e. \ on \ \ [\,0, a\,] \setminus \mathcal{T}_+ \tag{3.69}$$

$$L^{**}\,(S, \lambda) > L^{**}\,(\,S, 0) \quad \forall\, S \in \Sigma_A^{<} \ \ \forall\, \lambda > 0 \quad \Rightarrow \quad \left|\, z_A^{0\,\prime}(\cdot)\,\right| = 0 \quad a.e. \ on \ \ [\,0, a\,] \setminus \mathcal{T}_+. \tag{3.70}$$

The proof of this long chain of statements − the main one being the crucial inequality (3.65) – follows the next corresponding chain of reasonings. To begin with, by (3.50), the denial of (3.56), *i.e.* $w_A^0(t) > w_A(t)$, would imply, by (3.14) and (3.15),

$$\exists\, \delta > 0 \ : \quad w_A(t) = w_A(t+h) = w_A(t+\delta) = w_A^0(t+\delta) \quad \forall\, h \in (0, \delta) \tag{3.71}$$

hence $t \notin \mathcal{T}_+$, by (3.54), thus proving (3.56), by (3.50). On the other hand, denying (3.57) we would get, by (3.56), (3.15) and (3.50),

$$\exists\, \delta > 0 : \quad w_A(t) = w_A^0(t) \geq w_A^0(t+\delta) \geq w_A(t+\delta) \geq w_A(t)$$

so that these coincide and again (3.71) holds and $t \notin \mathcal{T}_+$, which proves (3.57); while (3.58) is still easier to prove.
Since denying (3.59) would yield, by (3.15),

$$\exists\, \delta_1 > 0 : \qquad w_A(t+h) < w_A^0(t+h) \qquad \forall\, h \in (0, \delta_1),$$

then by (3.14) $\exists\, \delta \geq \delta_1 > 0$ for which again (3.71) holds and $t \notin \mathcal{T}_+$. Thus also (3.59) is proved. Moreover, since, for $t \in \mathcal{T}_+$, by (3.54), (3.56) and (3.59),

$$0 < w_A'(t) = \lim \frac{w_A\,(t+h_k) - w_A(t)}{h_k} = \lim \frac{w_A^0\,(t+h_k) - w_A^0(t)}{h_k} = w_A^{0\,\prime}(t);$$

which, by (3.52), proves (3.60). On the other hand, denial of (3.61) would imply the existence of some sequence

$$(h_k) \searrow 0 : \quad w_A^0(t) - w_A^0\,(t - h_k) \leq 0$$

so that, by (3.60), we would reach the contradicting inequalities

$$0 < w_A'(t) = w_A^{0\,\prime}(t) = \lim \frac{w_A^0(t) - w_A^0\,(t-h_k)}{h_k} \leq 0.$$

Such absurd proves (3.61). As to (3.62), it is still easier to prove.
In order to prove (3.63) consider now the inequality associated to the fact of $\partial^0 L^{**}\left(z_A^0(t), 0\right)$ being in the subdifferential of $L^{**}(\cdot, 0)$ at $z_A^0(t)$ (recall (1.14)), namely

$$L^{**}\left(z_A^0\,(t-h)\,,0\right) \geq L^{**}\left(z_A^0(t),0\right) + \left\langle\, \partial^0 L^{**}\left(z_A^0(t),0\right), z_A^0\,(\,t-h) - z_A^0(t)\,\right\rangle.$$

Together with (3.61) and (3.13), such inequality yields some $\delta > 0$ for which

$$\begin{aligned} 0 < w_A^0(t) - w_A^0\,(t-h) &\leq \left\langle\, \partial^0 L^{**}\left(z_A^0(t),0\right), z_A^0(t) - z_A^0\,(t-h)\,\right\rangle \\ &\leq \left|\,\partial^0 L^{**}\left(z_A^0(t),0\right)\right| \;.\; \left|\,z_A^0(t) - z_A^0\,(t-h)\,\right| \qquad \forall\, h \in (0,\delta), \end{aligned}$$

so that (3.54) and (3.60) implies (3.63). Moreover, by (3.56) and (1.16),

$$\left|\,\partial^0 L^{**}\,(z_A(\cdot),0)\,\right| = \left|\,\partial^0 L^{**}\left(z_A^0(\cdot),0\right)\right| \qquad \text{on} \quad \mathcal{T}_+ \tag{3.72}$$

$$L^{**}\,(z_A(\cdot), |\,z_A'(\cdot)\,| \quad \rho(\cdot)) = L^{**}\left(z_A^0(\cdot), |\,z_A'(\cdot)\,| \quad \rho(\cdot)\right) \qquad \text{on} \quad \mathcal{T}_+ \tag{3.73}$$

i.e., by (3.72) and (3.63), we have proved (3.64). On the other hand, by (3.64), (3.73) and (1.2), we get (3.65). Finally, a.e. on $[0,a] \setminus (b,a')$ we have, by (3.23) and (3.47), $w_A'(\cdot) = 0 = |\,z_A'(\cdot)\,|$; while on $(b,a') \setminus \mathcal{T}_+$, by (3.54) and (3.52), $\exists\, w_A'(t) = 0 = |\,z_A'(t)\,|$, proving (3.66). Thus claim 1 is proved, except for (3.67) to (3.70) which will be proved below (in (3.77)).

Claim 2.

$$(3.9) \quad \Rightarrow \quad (3.4). \tag{3.74}$$

Before proceeding to prove this claim, we briefly remind the reader − after such lengthy preliminaries and proof of Claim 1 − of what has been achieved up to this point and of what is the meaning of (3.74). Namely, having assumed (3.9), *i.e.* having taken, in (3.12), a $z_A^0(\cdot)$ minimizing (2.8); and having constructed from it a new $z_A(\cdot)$ by the formula (3.47) (using $z_A^0(\cdot)$ through (3.13) and (3.14)), we now claim (3.4), namely that this

$$z_A(\cdot) \qquad \textit{minimizes both} \quad (2.8) \quad \& \quad (2.9). \tag{3.75}$$

Thus Claim 2 will be proved as soon as one checks (3.75). To begin with clearly, by (3.48), (3.49), (3.16), (3.19), (3.45), (3.38), (3.27), (2.14) and (2.13),

$$z_A(\cdot) \in Z^{0,a}_{A\nearrow} \quad \subset \quad Z^{0,a}_{A}. \tag{3.76}$$

On the other hand, by (3.66) and (3.50),

$$\int_{[0,a]\setminus\mathcal{T}_+} L^{**}\left(z_A(t), |\, z_A'(t)\,| \quad \rho(t)\right)\, \mathrm{d}t = \int_{[0,a]\setminus\mathcal{T}_+} L^{**}\left(z_A(t), 0\right)\, \mathrm{d}t$$

$$= \int_{[0,a]\setminus\mathcal{T}_+} w_A(t)\, \mathrm{d}t \le \int_{[0,a]\setminus\mathcal{T}_+} w_A^0(t)\, \mathrm{d}t = \int_{[0,a]\setminus\mathcal{T}_+} L^{**}\left(z_A^0(t), 0\right)\, \mathrm{d}t$$

$$\le \int_{[0,a]\setminus\mathcal{T}_+} L^{**}\left(z_A^0(t), \left|\, z_A^{0\prime}(t)\,\right| \quad \rho(t)\right)\, \mathrm{d}t;$$

and adding this inequality to the inequality (3.65), by (3.12) and (3.76) the proof of (3.75), hence of (3.74), claim 2 and (3.11), is complete.

Now we just need to fix a pending matter, before proceeding to the final steps of the proof. Indeed, stepping back to the final comment before (3.74), let us return for a moment to the statements (3.67) to (3.70). To begin with, by (3.50) and (3.56) the proof of (3.67) reduces to showing that

$$w_A^0(t) \le w_A(t) \qquad\qquad \forall\, t \in [0,a] \setminus \mathcal{T}_+; \tag{3.77}$$

but denial of (3.77) would yield $w_A(\cdot) < w_A^0(\cdot)$ along a nonempty open interval $\subset (0,a] \setminus \mathcal{T}_+$, hence the first inequality after (3.76) would be strict, in contradiction with (3.12) and (3.76). Exactly the same would happen if we did not have (3.68), by (3.65) together with, by (3.66) and (3.67),

$$L^{**}\left(z_A(t), |z_A'(t)|\;\; \rho(t)\right) = L^{**}\left(z_A^0(t), 0\right) \le L^{**}\left(z_A^0(t), \left|z_A^{0\prime}(t)\right|\;\; \rho(t)\right) \qquad \text{a.e.} \quad \text{on} \quad [0,a] \setminus \mathcal{T}_+. \tag{3.78}$$

But the same reasoning also proves (3.69), hence (3.70), thus finally completing the proof of claim 1.

Having thus proved Claim 1 and Claim 2 (essentially (3.68) and (3.74)) we now complete the proof of the equivalences (3.5) $-$ *i.e.* of Theorem 3.1 $-$ by complementing the chain (3.11) of implications with a few more:

$$(3.4) \quad \Rightarrow \quad (3.3) \quad \Rightarrow \quad (3.2) \quad \Rightarrow \quad (3.1). \tag{3.79}$$

To begin with, obviously (3.4) $\Rightarrow$ (3.3): taking a minimizer $z_A(\cdot)$ to both (2.8) and (2.9), then $z_A(\cdot) \in Z^{0,a}_{A\nearrow}$; and setting $U_A(r) := z_A\left(\gamma^{-1}(r)\right)$ one gets (since $\gamma^{-1}(\cdot)$ is Lipschitz increasing, see (3.10)) $U_A(\cdot) \in \mathcal{U}^{0,a}_{A\nearrow} \subset \mathcal{U}^{0,a}_{A}$ (recall (2.14), (2.4), (2.7) & [1, (95)]); while $U_A(\cdot)$ minimizes (2.5), by the comment after (3.10), so that it also minimizes (2.6).

Similarly (3.3) $\Rightarrow$ (3.2): taking a minimizer $U_A(\cdot)$ to both (2.5) and (2.6) then $U_A(\cdot) \in \mathcal{U}^{0,a}_{A\nearrow}$; while setting $u_A(x) := U_A(|x|)$ one gets $u_A(\cdot) \in W^{1,1}_{A\nearrow}$ (see (2.3) and (2.4)); so that (recalling what is stated just after (3.10)) such $u_A(\cdot)$ has to minimize (1.1) hence (2.2) also.

Trivially (3.2) $\Rightarrow$ (3.1): existence of a minimizer to (1.1) implies at least existence of minimum to (1.1).

This proves (3.79), hence, by (3.11), also (3.5) *i.e.* Theorem 3.1. □

4. Reducing a $m \times d$-dim problem to a 1×1-dim problem

Clearly Theorem 3.1 yields a nice simplification: we have thus reduced the question of minimizing the multiple integral (1.1), among vectorial functions of vectorial variable in competition, to an equivalent minimization problem involving any one of four new single integrals – those in (2.5) to (2.9) – among vectorial functions of scalar variable in competition.

However, we aim higher hence present now a further huge simplification, this one never achieved by previous authors, to our knowledge: we reduce such vectorial $m \times d$- or $m \times 1$-dimensional minimizations to a new scalar 1×1-dim problem, of minimizing an adequate single integral among scalar functions of scalar variable in competition.

Indeed, considering the functions $g_A(\cdot)$ & $\ell_A^{**}(\cdot)$ defined above – in (3.41) and (3.46) – and recalling the space $Z_A^{0,a}$ defined in (2.13), we now define a third pair of single integrals:

$$\alpha_d \int_0^a \ell_A^{**}\left(w(t),\, g_A\left(w(t)\right)\ \left|w'(t)\right| \quad \rho(t)\right)\ \mathrm{d}t \qquad\qquad \text{on} \quad \mathcal{W}_A^{0,a} \tag{4.1}$$

$$\alpha_d \int_0^a \ell_A^{**}\left(w(t),\, g_A\left(w(t)\right)\ \left|w'(t)\right| \quad \rho(t)\right)\ \mathrm{d}t \qquad\qquad \text{on} \quad \mathcal{W}_{A\nearrow}^{0,a}, \tag{4.2}$$

$$\mathcal{W}_A^{0,a} := \left\{ w(\cdot) \in W_{loc}^{1,1}\left((0,a]\right) : \quad \exists\, z(\cdot) \in Z_A^{0,a} \,:\, w(\cdot) = L^{**}\left(z(\cdot), 0\right) \quad \& \quad w'(\cdot)\ \gamma(\cdot)^{d-1} \in L^1(0,a) \right\}, \tag{4.3}$$

$$\mathcal{W}_{A\nearrow}^{0,a} := \left\{ w(\cdot) \in W^{1,1}\left([0,a]\right) : \quad w(\cdot) \ \ \textit{increases} \qquad \& \qquad w(a) = L^{**}(A,0) \right\}. \tag{4.4}$$

Theorem 4.1. *Assume (1.2), (1.3), (1.9), (1.13), (1.15) and (1.16) together with: either (3.1) or else*

$$\exists \ \textit{minimum} \ \ \textit{to} \ \ (4.1). \tag{4.5}$$

Then, besides (3.2) to (3.4),

$$\exists\ w_A(t) \ \ \textit{minimizing} \ \ \textit{both} \ \ (4.1) \ \ \& \ \ (4.2). \tag{4.6}$$

Moreover: the minimum value to all these integrals is the same; the equivalences (3.5) are complemented by

$$(3.1) \quad \Leftrightarrow \quad (4.5) \quad \Leftrightarrow \quad (4.6) \quad \Leftrightarrow \quad (3.2); \tag{4.7}$$

and – recalling $\gamma^{-1}(\cdot)$ & $Q_A(\cdot)$ from (2.12) and (3.45) – all these minimizers are related by the equalities:

$$u_A(x) = U_A(|x|) = z_A\left(\gamma^{-1}(|x|)\right) \qquad \& \qquad w_A(t) = L^{**}\left(z_A(t),\, 0\right) \tag{4.8}$$

$$z_A(t) = Q_A\left(w_A(t)\right) \qquad \& \qquad U_A(r) = Q_A\left(w_A\left(\gamma^{-1}(r)\right)\right). \tag{4.9}$$

Proof. Here, besides what has already been proved in (3.5), we will – at (4.12), (4.23) and (4.29) – establish the further implications

$$(3.4) \qquad \Leftrightarrow \qquad (4.6) \tag{4.10}$$

$$(4.5) \qquad \Rightarrow \qquad (4.6); \tag{4.11}$$

hence the equivalences in (4.7), thus proving Theorem 4.1.

Let us begin by proving the implication (3.4) ⇒ (4.6) in (4.10), through showing that our $w_A(\cdot)$, as in (3.49), satisfies

$$w_A(\cdot) \in \mathcal{W}^{0,a}_{A\nearrow} \qquad \subset \qquad \mathcal{W}^{0,a}_{A} \tag{4.12}$$

$$\text{and} \qquad\qquad w_A(\cdot) \ \textit{minimizes} \ \text{ both } (4.1) \ \& \ (4.2). \tag{4.13}$$

Indeed, to begin with, by (4.3), (4.4), (3.16), (3.76), (3.19) and (3.49), obviously $w_A(\cdot)$ belongs to both spaces in (4.12); while, on the other hand, picking any generic $w_1(\cdot)$ in $\mathcal{W}^{0,a}_{A\nearrow}$ then exactly the same arguments as above (namely in (3.47) to (3.76), with $w_A(\cdot)$ replaced by this $w_1(\cdot)$) yield a corresponding $z_1(\cdot) := Q_A\,(w_1(\cdot))$ in $Z^{0,a}_{A\nearrow} \subset Z^{0,a}_{A}$ (see (3.47) and (3.76)), thus showing that such generic $w_1(\cdot)$ also belongs to $\mathcal{W}^{0,a}_{A}$, see (4.3), hence proving the general inclusion in (4.12).

To complete the proof of (4.13) assume, by contradiction, the existence of some

$$w^0_0(\cdot) \in \mathcal{W}^{0,a}_{A} \qquad \text{for which} \qquad (\text{recalling} \ \ (3.41) \ \ \& \ \ (3.46)) \tag{4.14}$$

$$\int_0^a \ell^{**}_A\left(w^0_0(t),\, g_A\left(w^0_0(t)\right)\ \ {w^0_0}'(t) \qquad \rho(t)\right)\ \mathrm{d}t < \int_0^a \ell^{**}_A\left(\, w_A(t),\, g_A\left(w_A(t)\right)\ \ w'_A(t) \qquad \rho(t)\right)\ \mathrm{d}t. \tag{4.15}$$

Then, redefining (recall (4.3) and (3.13))

$$w^0_0(0) \quad \text{to become} \ := \inf w^0_0\left((0,a]\right) \tag{4.16}$$

and setting (recall (3.14) and (3.47))

$$w_0(t) := \min w^0_0\left([t,a]\right) \quad \text{for} \quad t \in [0,a] \tag{4.17}$$

$$z_0(t) := Q_A\left(w_0(t)\right) \quad \text{for} \quad t \in [0,a], \tag{4.18}$$

one, by using the same arguments as above, would reach (similarly to (3.76), (3.52) and (3.15))

$$z_0(\cdot) \in Z^{0,a}_{A\nearrow} \qquad \subset \qquad Z^{0,a}_{A} \tag{4.19}$$

$$|\,z'_0(t)| = g_A\left(w_0(t)\right)\ \ w'_0(t) \qquad\qquad \& \qquad\qquad w^0_0(t) \le w_0(t). \tag{4.20}$$

Moreover, reasoning as after (3.76) but now with $z_A(\cdot)$, $w_A(\cdot)$, $w^0_A(\cdot)$ & $z^0_A(\cdot)$ replaced by $z_0(\cdot)$, $w_0(\cdot)$, $w^0_0(\cdot)$ & $z^0_0(\cdot)$, one reaches, by (3.56), (3.60), (3.66), (3.53) and (4.20), the inequality:

$$\int_0^a \ell^{**}_A\left(w_0(t),\, g_A\left(\, w_0(t)\right)\ \ w'_0(t) \qquad \rho(t)\right)\ \mathrm{d}t \ \le \ \int_0^a \ell^{**}_A\left(\, w^0_0(t),\, g_A\left(w^0_0(t)\right)\ \ {w^0_0}'(t) \qquad \rho(t)\right)\ \mathrm{d}t. \tag{4.21}$$

Therefore, by (4.18), (4.20), (3.46) and (4.15),

$$\begin{aligned}
\int_0^a L^{**}\left(z_0(t), |\,z'_0(t)\,| \qquad \rho(t)\right)\ \mathrm{d}t &= \int_0^a L^{**}\left(Q_A\left(w_0(t)\right),\, g_A\left(\, w_0(t)\right)\ \ w'_0(t) \qquad \rho(t)\right)\ \mathrm{d}t \\
= \int_0^a \ell^{**}_A\left(w_0(t),\, g_A\left(\, w_0(t)\right)\ \ w'_0(t) \qquad \rho(t)\right)\ \mathrm{d}t &\le \int_0^a \ell^{**}_A\left(\, w^0_0(t),\, g_A\left(w^0_0(t)\right)\ \ {w^0_0}'(t) \qquad \rho(t)\right)\ \mathrm{d}t \\
< \int_0^a \ell^{**}_A\left(w_A(t),\, g_A\left(\, w_A(t)\right)\ \ w'_A(t) \qquad \rho(t)\right)\ \mathrm{d}t &= \int_0^a L^{**}\left(z_A(t), |\,z'_A(t)\,| \qquad \rho(t)\right)\ \mathrm{d}t,
\end{aligned} \tag{4.22}$$

by a new application of (3.46), (3.47) and (3.52), thus contradicting (3.75), by (4.19). Such absurd denies the possibility of existence of a $w_0^0(\cdot)$ as in (4.14) and (4.15) and proves (4.13), due to (4.12), hence the right-wing implication in (4.10) aimed at just before (4.12).

To complete the proof of (4.10) we now prove the remaining implication

$$(3.4) \quad \Leftarrow \quad (4.6). \tag{4.23}$$

Namely, taking a

$$w_A(t) \in \mathcal{W}_{A\nearrow}^{0,a} \quad \textit{minimizer} \quad \text{to both} \quad (4.1)\ \&\ (4.2) \tag{4.24}$$

and setting $z_A(t) := Q_A\left(w_A(t)\right)$, as in (3.47), we now claim that $z_A(\cdot) \in Z_{A\nearrow}^{0,a} \subset Z_A^{0,a}$, *i.e.* (see (2.13) and (2.14))

$$z_A(\cdot) \in W^{1,1}\left([0,a], \mathbb{R}^m\right), \quad z_A(a) = A \ \ \& \ \ L^{**}\left(z_A(\cdot), 0\right) \ \textit{increases}. \tag{4.25}$$

Indeed, since $w_A(\cdot) \in W^{1,1}\left([0,a]\right)$, $w_A(\cdot)$ increases and $w_A(a) = L^{**}\left(A, 0\right)$, by (4.4); and since, by (3.48), $Q_A(\cdot)$ in (3.45) is Lipschitz, we have $z_A(\cdot) \in W^{1,1}\left(\,[0,a], \mathbb{R}^m\right)$. On the other hand, by (3.17), (3.45), (3.38) and (3.27), $z_A(a) = Q_A\left(w_A(a)\right) = \sigma_A\left(\tau_A\left(p_A^{\max}\right)\right) = \sigma_A(0) = A$; and, by (3.49), $L^{**}\left(z_A(\cdot), 0\right) = L^{**}\left(Q_A\left(w_A(\cdot)\right), 0\right) = w_A(\cdot)$ increases.

Thus (4.25) is proved; and we now claim that such

$$z_A(\cdot) \quad \textit{minimizes both} \quad (2.8) \quad \& \quad (2.9). \tag{4.26}$$

Indeed assume, by contradiction, that

$$\exists\ z_0^0(\cdot) \in Z_A^{0,a} \quad \text{for which} \tag{4.27}$$

$$\int_0^a L^{**}\left(z_0^0(t), \left|\, {z_0^0}'(t)\,\right| \quad \rho(t)\right)\ \mathrm{d}t \ < \ \int_0^a L^{**}\left(z_A(t), \left|\, z_A'(t)\,\right| \quad \rho(t)\right)\ \mathrm{d}t. \tag{4.28}$$

Then defining, as in (3.13),

$$w_0^0(t) := L^{**}\left(z_0^0(t), 0\right) \quad \text{for} \quad t \in (0,a] \qquad \& \qquad w_0^0(0) := \inf\ L^{**}\left(z_0^0\left((0,a]\right), 0\right),$$

obtain from $w_0^0(t)$ the new functions $w_0(\cdot)$ & $z_0(\cdot)$ as in (4.17) and (4.18), hence satisfying (4.19) & (4.20); so that, by (3.46), (4.18), (4.20) & (4.28),

$$\int_0^a \ell_A^{**}\left(w_0(t), g_A\left(\,w_0(t)\right)\ w_0'(t) \quad \rho(t)\right)\ \mathrm{d}t \ = \ \int_0^a L^{**}\left(Q_A\left(\,w_0(t)\right), g_A\left(w_0(t)\right)\ w_0'(t) \quad \rho(t)\right)\ \mathrm{d}t$$

$$= \int_0^a L^{**}\left(z_0(t), \left|\, z_0'(t)\,\right| \quad \rho(t)\right)\ \mathrm{d}t \ < \ \int_0^a L^{**}\left(z_A(t), \left|\, z_A'(t)\,\right| \quad \rho(t)\right)\ \mathrm{d}t$$

$$= \int_0^a L^{**}\left(Q_A\left(w_A(t)\right), g_A\left(w_A(t)\right)\ w_A'(t) \quad \rho(t)\right)\ \mathrm{d}t = \int_0^a \ell_A^{**}\left(\,w_A(t), g_A\left(w_A(t)\right)\ w_A'(t) \quad \rho(t)\right)\ \mathrm{d}t,$$

a contradiction to (4.24) showing that no such $z_0^0(\cdot)$ as in (4.27) and (4.28) can be; and proving (4.26), hence (4.23) and (4.10), due to (3.76).

To prove (4.11) assume (4.5) namely

$$\exists\ w_A^0(\cdot) \in \mathcal{W}_A^{0,a} \quad \textit{minimizing} \quad (4.1). \tag{4.29}$$

Then since, by (4.3), $\exists\, z^0_A(\cdot) \in Z^{0,a}_A$ with $L^{**}\left(z^0_A(\cdot), 0\right) = w^0_A(\cdot)$, one may redefine $w^0_A(0)$ as in (3.13), and define $w_A(\cdot)$ & $z_A(\cdot)$ as in (3.14) and (3.47). We claim that:

$$w_A(\cdot) \in \mathcal{W}^{0,a}_{A\nearrow} \qquad \subset \qquad \mathcal{W}^{0,a}_A \tag{4.30}$$

$$0 \le w_A(t) = w^0_A(t) \quad \& \quad 0 < w_A'(t) = w^0_A{}'(t), \qquad \forall\, t \in \mathcal{T}_+ \tag{4.31}$$

$$\ell^{**}_A\left(w_A(t), g_A\left(w_A(t)\right)\ w_A'(t)\quad \rho(t)\right) \le \ell^{**}_A\left(w^0_A(t), g_A\left(w^0_A(t)\right)\ w^0_A{}'(t)\quad \rho(t)\right) \quad \text{for a.e. } t \in [0,a] \setminus \mathcal{T}_+ \tag{4.32}$$

and

$$w_A(t) \quad minimizes \quad \text{both} \quad (4.1) \quad \& \quad (4.2). \tag{4.33}$$

Indeed, one may prove (4.30) as in (4.12); while (4.31) follows as in (3.56) and (3.63); and, finally, to prove (4.32) (and noticing that the proof of (3.67) requires the use of (3.12), see (3.77)) we have, a.e. on $[0,a] \setminus \mathcal{T}_+ :$ $0 = w_A'(t) \le \left| w^0_A{}'(t)\right|$, by (3.66) hence, by (3.53) and (3.15),

$$\begin{aligned}
\ell^{**}_A\left(w_A(t), g_A\left(w_A(t)\right)\ w_A'(t)\quad \rho(t)\right) \quad &= \quad \ell^{**}_A\left(w_A(t), 0\right) = w_A(t) \le w^0_A(t) = \ell^{**}_A\left(w^0_A(t), 0\right)\right) \le \\
&\le \quad \ell^{**}_A\left(w^0_A(t), g_A\left(w^0_A(t)\right)\ w^0_A{}'(t)\quad \rho(t)\right),
\end{aligned}$$

so that indeed (4.32) does hold true. Since the inequality in (4.32) also holds true (trivially) on $\mathcal{T}_+$, by (4.31), we get (4.33), by (4.29) and (4.30), thus proving (4.11). □

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