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DIMENSION REDUCTION FOR $-\Delta_1$

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Abstract. A 3D-2D dimension reduction for $-\Delta_1$ is obtained. A power law approximation from $-\Delta_p$ as $p \to 1$ in terms of Γ-convergence, duality and asymptotics for least gradient functions has also been provided.

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1. Introduction

Recently a great deal of attention has been devoted to thin structures because of the many applications they find in the applied sciences. A wide literature, concerning mathematical problems defined in thin structures and modelled through partial differential equations and integral functionals, is available both in the Sobolev and BV settings. To our knowledge little is known when one wants to investigate the relations between problems dealing with thin structures whose deformation fields are functions of bounded variation and the analogous problems modelled through Sobolev fields. This issue has been in fact pointed out also by [8], in the context of applications dealing with approximations of yield sets in Plasticity and for models dealing with dielectric breakdown.

The aim of this paper consists, in fact, in determining the asymptotic behaviour, both for $\varepsilon \to 0$ and $p \to 1$ of $p-$*harmonic* functions in thin domains of the type $\Omega_\varepsilon : \omega \times \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$, with prescribed boundary data v_0 on the lateral boundary of $\Omega_\varepsilon := \partial\omega \times \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$, *i.e.*

$$\begin{cases} -\Delta_p v := -\mathrm{div}(|\nabla v|^{p-2}\nabla v) = 0 & \text{in } \Omega_\varepsilon, \\ v \equiv v_0 & \text{on } \partial\omega \times \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right), \\ |\nabla v|^{p-2}\nabla v \cdot \nu = 0 & \text{on } \omega \times \left\{-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right\}, \end{cases} \tag{1.1}$$

where ν denotes the unit normal to the top and the bottom of the cylinder.

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We emphasize the fact that the thin domain is a cylinder, with cross section ω, satisfying suitable regularity requirements, that will be clearly stated in the sequel (see in particular Sect. 5). We assume in our subsequent analysis that the boundary is indeed piecewise C^1 (see beginning of Sect. 3).

Equivalently one may think of studying as $\varepsilon \to 0$ and $p \to 1$, the associated Dirichlet integral, namely

$$\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} |\nabla v|^p \mathrm{d}x \tag{1.2}$$

among all the fields $v \in W^{1,p}(\Omega_\varepsilon)$, with $v \equiv v_0$ on $\partial\omega \times \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$.

Several issues appear at this point, (see for instance [24] for a recent survey on the asymptotics as $p \to 1$): varying domains Ω_ε, meaning of the equation (1.1) for $p = 1$, the possibility and the order with respect to which one may take the limits as $\varepsilon \to 0$ and $p \to 1$.

We start by rescaling our problem, thus eliminating the varying domains, transferring the dependence on ε to the expression of the equation and its associated variational functional.
To this end, we fix our notations: let $\omega \subset \mathbb{R}^2$ be a bounded smooth domain which is piecewise C^1 (or whose boundary $\partial\omega$ has positive mean curvature (*cf.* [29] and Thm. 5.2 below)) and let u_0 be in a suitable trace space to be defined later according to the different formulations of the problems.

For every $\varepsilon > 0$, let Ω_ε be a cylindrical domain of cross section $\omega \subset \mathbb{R}^2$ and thickness ε, namely $\Omega_\varepsilon := \omega \times \left(-\frac{\varepsilon}{2}; \frac{\varepsilon}{2}\right)$. We reformulate (1.2), considering a $\frac{1}{\varepsilon}$–dilation in the transverse direction x_3.

$$\begin{aligned} &\Omega := \Omega_1 = \omega \times \left(-\tfrac{1}{2}, \tfrac{1}{2}\right), \\ &u(x_1, x_2, x_3) := v(x_1, x_2, \varepsilon x_3), \\ &u_0(x_1, x_2) = v_0(x_1, x_2). \end{aligned} \tag{1.3}$$

In the sequel we will denote the planar variables (x_1, x_2) by x_α and for every $\xi_1, \xi_2, \xi_3 \in \mathbb{R}$, the vector (ξ_1, ξ_2, ξ_3) will be denoted by $(\xi_\alpha|\xi_3)$.
Thus for every $p > 1$, (1.2) is replaced by $I_{p,\varepsilon} : W^{1,p}(\Omega) \to \mathbb{R}^+$, defined as

$$I_{p,\varepsilon}(u) := \int_\Omega \left|\left(\nabla_\alpha u \left| \frac{1}{\varepsilon}\nabla_3 u\right.\right)\right|^p \mathrm{d}x_\alpha \mathrm{d}x_3. \tag{1.4}$$

We can consider the following variational problem

$$\mathcal{P}_{p,\varepsilon} := \min\left\{ I_{p,\varepsilon}(u) : u \in W^{1,p}(\Omega), u \equiv u_0 \text{ on } \partial\omega \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right\}. \tag{1.5}$$

The Euler–Lagrange equation associated to (1.5) is

$$\begin{cases} -\Delta_{p,\varepsilon} u = 0 & \text{in } \Omega, \\ u \equiv u_0 & \text{on } \partial\omega \times \left(-\frac{1}{2}, \frac{1}{2}\right), \\ |Id_\varepsilon \nabla u \cdot \nabla u|^{\frac{p-2}{2}} (Id_\varepsilon \nabla u) \cdot \nu = 0 & \text{on } \omega \times \left\{-\frac{1}{2}, \frac{1}{2}\right\}, \end{cases} \tag{1.6}$$

where $Id_\varepsilon \in \mathbb{R}^{3\times 3}$ is the matrix defined as

$$(Id_\varepsilon)_{i,j} = \begin{cases} \frac{1}{\varepsilon^2} & \text{if } i = j = 3, \\ \delta_{i,j} & \text{otherwise}, \end{cases} \tag{1.7}$$

and $\Delta_{p,\varepsilon}$ is the simple anisotropic p,ε–Laplace operator defined as

$$\Delta_{p,\varepsilon} u = \operatorname{div}\left(|Id_\varepsilon \nabla u \cdot \nabla u|^{\frac{p-2}{2}} \; Id_\varepsilon \nabla u\right)\cdot$$

We are interested in the asymptotic behaviour of $\mathcal{P}_{p,\varepsilon}$ and $\operatorname{argmin}\mathcal{P}_{p,\varepsilon}$, (namely the behaviour of the weak solutions of (1.6)) both in the order ($p \to 1$, $\varepsilon \to 0$) and in the reverse one, *i.e.* ($\varepsilon \to 0$, $p \to 1$).
In order to exploit pre-existing results in the Γ–convergence setting, we will discuss first the case $\varepsilon \to 0$ before $p \to 1$.
For $\varepsilon = 0$ we may introduce the $3D$ problem in terms of PDE's

$$\begin{cases} -\Delta_{\alpha,p,0} u := -\operatorname{div}_\alpha(|\nabla_\alpha u|^{p-2}\nabla_\alpha u) = 0 & \text{in } \Omega, \\ \nabla_3 u = 0 & \text{in } \Omega, \\ u = u_0 & \text{in } \partial\omega \times \left(-\frac{1}{2}, \frac{1}{2}\right), \end{cases} \tag{1.8}$$

where the index α means that the derivatives are taken only with respect to x_α.

Let $I_{p,0} : W^{1,p}(\omega) \to \mathbb{R}^+$, be the functional defined as

$$I_{p,0}(u) := \int_\omega |\nabla_\alpha u|^p \,\mathrm{d}x, \tag{1.9}$$

and define the minimum problem

$$\mathcal{P}_{p,0} := \min\left\{I_{p,0}(u) : u \in W^{1,p}(\Omega), u \equiv u_0 \text{ on } \partial\omega\right\}. \tag{1.10}$$

It is well known since the pioneering papers [1,26] that, for every $p > 1$, $\mathcal{P}_{p,\varepsilon}$ converges as $\varepsilon \to 0$ to $\mathcal{P}_{p,0}$, namely the functionals $I_{p,\varepsilon}$ Γ–converge with respect to L^p strong topology, as $\varepsilon \to 0$ to $I_{p,0}$, (*cf.* Sect. 3.1). In particular, it has to be observed that the convexity of the space functions in (1.5) and (1.10), the strict convexity and the coerciveness of $I_{p,\varepsilon}$ and $I_{p,0}$, due to the choice $p > 1$, ensure that $\mathcal{P}_{p,\varepsilon}$ and $\mathcal{P}_{p,0}$ admit a unique solution, which, in turn is a weak solution of (1.6) and (1.8), respectively, for instance when $u_0 \in W^{\frac{p-1}{p},p}(\partial\omega)$ (*cf.* Sect. 2 for the definition of trace spaces).

At this point it is worth, identifying the fields in $W^{1,p}(\Omega)$ with $\nabla_3 u = 0$ with the fields in $W^{1,p}(\omega)$, to observe that (1.8) admits the equivalent $2D$ formulation

$$\begin{cases} -\Delta_{p,0} u := -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 & \text{in} \quad \omega, \\ u = u_0 & \text{on} \quad \partial\omega. \end{cases} \tag{1.11}$$

For every fixed $\varepsilon > 0$ and $p = 1$, one can also define the following variational problems

$$\mathcal{P}_{1,\varepsilon} := \inf\left\{I_{1,\varepsilon}(u) : u \in W^{1,1}(\Omega), u \equiv u_0 \text{ on } \partial\omega \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right\}, \tag{1.12}$$

where $I_{1,\varepsilon} : W^{1,1}(\Omega) \to \mathbb{R}^+$, is defined as

$$I_{1,\varepsilon}(u) := \int_\Omega \left|\left(\nabla_\alpha u \middle| \frac{1}{\varepsilon}\nabla_3 u\right)\right| \mathrm{d}x. \tag{1.13}$$

In principle $I_{1,\varepsilon}$ may not admit a solution in the Sobolev setting, because of many reasons, first of all the lack of coerciveness, but, as we shall see in Section 5, also the choice of the trace space and the regularity of the set Ω_ε play a crucial role.

Consequently in order to guarantee a correct formulation for problem $\mathcal{P}_{1,\varepsilon}$ one needs to extend $I_{1,\varepsilon}$ (with abuse of notations) on the space of functions with bounded variation $BV(\Omega)$, taking care of the fact that $u = u_0$ outside the lateral boundary of Ω, thus considering

$$I_{1,\varepsilon}(u) := \left|\left(D_\alpha u \left| \frac{1}{\varepsilon} D_3 u\right.\right)\right|(\Omega) \tag{1.14}$$

where the derivatives are intended in the sense of distributions and the integral is replaced by the total variation. Hence the minimum problem, after a relaxation procedure (*cf.* [28], Thm. 3.4), becomes

$$\mathcal{P}_{1,\varepsilon} = \min\left\{ \left|\left(D_\alpha u \left| \frac{1}{\varepsilon} D_3 u\right.\right)\right|(\Omega) + \int_{\partial\omega\times\left(-\frac{1}{2},\frac{1}{2}\right)} |u-u_0| \mathrm{d}\mathcal{H}^2, u \in BV(\Omega)\right\}. \tag{1.15}$$

Analogously one may consider the problem $\mathcal{P}_{p,\varepsilon}$ for $p = 1$ and $\varepsilon = 0$, thus formally obtaining

$$\mathcal{P}_{1,0} = \min\left\{ |D_\alpha u|(\Omega) + \int_{\partial\omega\times\left(-\frac{1}{2},\frac{1}{2}\right)} |u-u_0| \mathrm{d}\mathcal{H}^2, u \in BV(\Omega), D_3 u = 0\right\}, \tag{1.16}$$

which arises from the relaxation in $BV(\Omega)$ (see [2, 18]) of the functional $I_{1,0} : \mathcal{U} \to \mathbb{R}$, where $\mathcal{U} := \{u \in W^{1,1}(\Omega) : \nabla_3 u = 0, u \equiv u_0 \text{ on } \partial\omega \times \left(-\frac{1}{2}, \frac{1}{2}\right)\}$, defined as

$$I_{1,0}(u) := \int_\Omega |\nabla_\alpha u| \, \mathrm{d}x, \tag{1.17}$$

whose related minimum problem in $\mathcal{U}$ is

$$\mathcal{P}_{1,0} := \inf\left\{I_{1,0}(u) : u \in \mathcal{U}\right\}. \tag{1.18}$$

Also the asymptotic behaviour of $I_{1,\varepsilon}$ as $\varepsilon \to 0$ is a consequence of the results in [6], *cf.* Section 3.1, where we state the Γ–convergence of $I_{1,\varepsilon}$ to the functional in problem (1.16).

The asymptotics in terms of Γ–convergence for $p \to 1$ are indeed one of the targets of this paper. Namely in Theorems 3.8 and 3.11. we prove the convergence of $\mathcal{P}_{p,0}$ to $\mathcal{P}_{1,0}$ and of $\mathcal{P}_{p,\varepsilon}$ to $\mathcal{P}_{1,\varepsilon}$ respectively. In the above mentioned analysis it is assumed that the prescribed boundary datum u_0 is in the space $W^{1-\frac{1}{\overline{p}},\overline{p}}(\partial\omega)$, for a certain $\overline{p} > 1$.

We emphasize that a different view to the limit $p \to 1$ of problems $\mathcal{P}_{p,\varepsilon}$ and $\mathcal{P}_{p,0}$ can be provided in terms of equations, namely, besides the asymptotic analysis in terms of Γ–convergence, mentioned above, *via* Duality theory we define in a precise way the anisotropic $-\Delta_1$ and $-\Delta_1$, thus giving a clear meaning to (1.6) and (1.11) when $p = 1$.

Our analysis focuses also on the study of least gradient problems in dimensional reduction, in connection with $\mathcal{P}_{1,0}$ and $\mathcal{P}_{1,\varepsilon}$. In this framework the minimum problems can be stated essentially in the same way but test fields are assumed in BV, thus in order to ensure existence of solutions a crucial role is played by the regularity of the domain ω and the boundary datum u_0.

The paper is organized as follows. Section 2 is devoted to preliminary results about Γ–convergence, measures, functions of bounded variation, trace spaces and duality theory. In Section 3, we first discuss in Section 3.1 the asymptotics as $\varepsilon \to 0$, for every $p \geq 1$ by means of recalls to the existing literature, we then provide sufficient conditions in order to pass to the limit as $p \to 1$ for every $\varepsilon \geq 0$ (*cf.* Sects. 3.2 and 3.3). Finally in Section 3.4 we conclude that the limits $p \to 1$ and $\varepsilon \to 0$ commute (*cf.* the diagram therein). In Section 4 through Proposition 4.1 a meaning to 1–Laplacian and anisotropic 1–Laplacian operators is given and we state a rigorous connection, for a suitable choice of the boundary datum u_0, between the differential problems (1.6) and the integral ones *via* the duality when $p = 1$, see Remark 4.2 and Proposition 4.4.

Connections with the least gradient problem will be addressed in Section 5, see Theorems 5.7 and 5.8. This latter approach reveals its importance in determining the existence of solutions to the limit problems (as $p \to 1$) of (1.1). In fact, in spite of possible lack of coerciveness of Problems 1.15 and 1.16 below, the solution exists provided suitable geometrical regularity assumptions on the cross section ω of the cylinder Ω_ε.

2. Preliminary results

In the following subsections we give a brief survey of Γ–convergence, functions of bounded variation and trace spaces. For a detailed treatment of these subjects, we refer to [3, 4, 9, 10] respectively.

2.1. Γ–convergence

Let (X, d) be a metric space.

Definition 2.1 (Γ–convergence for a sequence of functionals)**.** Let $\{J_n\}$ be a sequence of functionals defined on X with values in $\overline{\mathbb{R}}$. The functional $J : X \to \overline{\mathbb{R}}$ is said to be the $\Gamma - \liminf$ (resp. $\Gamma - \limsup$) of $\{J_n\}$ with respect to the metric d if for every $u \in X$

$$J(u) = \inf\left\{\liminf_{n\to\infty} J_n(u_n) : u_n \in X, u_n \to u \text{ in } X\right\} \ \left(\text{resp. } \limsup_{n\to\infty}\right).$$

Thus we write

$$J = \Gamma - \liminf_{n\to\infty} J_n \ \left(\text{resp. } J = \Gamma - \limsup_{n\to\infty} J_n\right).$$

Moreover, the functional J is said to be the $\Gamma-$limit of $\{J_n\}$ if

$$J = \Gamma - \liminf_{n\to\infty} J_n = \Gamma - \limsup_{n\to\infty} J_n,$$

and we may write

$$J = \Gamma - \lim_{n\to\infty} J_n.$$

For every $\varepsilon > 0$, let J_ε be a functional over X with values in $\overline{\mathbb{R}}$, $J_\varepsilon : X \to \overline{\mathbb{R}}$.

Definition 2.2 (Γ–convergence for a family of functionals)**.** A functional $J : X \to \overline{\mathbb{R}}$ is said to be the Γ–liminf (resp. Γ–limsup or Γ–limit) of $\{J_\varepsilon\}$ with respect to the metric d, as $\varepsilon \to 0^+$, if for every sequence $\varepsilon_n \to 0^+$

$$J = \Gamma - \liminf_{n\to\infty} J_{\varepsilon_n} \left(\text{resp. } J = \Gamma - \limsup_{n\to\infty} J_{\varepsilon_n} \text{ or } J = \Gamma - \lim_{n\to\infty} J_{\varepsilon_n}\right),$$

and we write

$$J = \Gamma - \liminf_{\varepsilon\to 0^+} J_\varepsilon \left(\text{resp. } J = \Gamma - \limsup_{\varepsilon\to 0^+} J_\varepsilon \text{ or } J = \Gamma - \lim_{\varepsilon\to 0^+} J_\varepsilon\right).$$

Next we state the Urysohn property for Γ–convergence in a metric space.

Proposition 2.3. *Given $J : X \to \overline{\mathbb{R}}$ and $\varepsilon_n \to 0^+$, $J = \Gamma - \lim_{n\to\infty} J_{\varepsilon_n}$ if and only if for every subsequence $\left\{\varepsilon_{n_j}\right\} \equiv \{\varepsilon_j\}$ there exists a further subsequence $\left\{\varepsilon_{n_{j_k}}\right\} \equiv \{\varepsilon_k\}$ such that $\{J_{\varepsilon_k}\}$ $\Gamma-$converges to J.*

In addition, if the metric space is also separable the following compactness property holds.

Proposition 2.4. *Each sequence $\varepsilon_n \to 0^+$ has a subsequence $\left\{\varepsilon_{n_j}\right\} \equiv \{\varepsilon_j\}$ such that $\Gamma - \lim_{j\to\infty} J_{\varepsilon_j}$ exists.*

Proposition 2.5. *If $J = \Gamma - \liminf_{\varepsilon\to 0^+} J_\varepsilon$ (or $\Gamma - \limsup_{\varepsilon\to 0^+} J_\varepsilon$) then J is lower semicontinuous (with respect to the metric d).*

We conclude with a result dealing with the convergence of minimizers and minimum points, [10], Corollary 7.17.

Theorem 2.6. *For every $\varepsilon \in \mathbb{N}$, let $\{x_\varepsilon\}$ be a minimizer of J_ε in X. If $\{x_\varepsilon\}$ converge to x in X, then x is a minimizer of $\Gamma - \liminf_\varepsilon J_\varepsilon$ and $\Gamma - \limsup_\varepsilon J_\varepsilon$ in X and*

$$\left(\Gamma - \liminf_{\varepsilon} J_\varepsilon\right)(x) = \liminf_{\varepsilon} J_\varepsilon\left(x_\varepsilon\right), \qquad \left(\Gamma - \limsup_{\varepsilon} J_\varepsilon\right)(x) = \limsup_{\varepsilon} J_\varepsilon\left(x_\varepsilon\right).$$

2.2. Measures

We start this subsection by recalling a result that may be found in [12].

Proposition 2.7. *Let O be a bounded open set in $\mathbb{R}^N$, and for every sequence $p > 1$, let $\{\mu_p\}_p$ and μ be non-negative Borel measures on O such that*

$$\begin{cases} \limsup_{p\to 1} \mu_p(O) \leq \mu(O) < +\infty, \\ \limsup_{p\to 1} \mu_p(A) \geq \mu(A) \text{ for every open subset } A \text{ of } O. \end{cases}$$

Then for every $\varphi \in C(\overline{O})$ we have

$$\lim_{p\to 1} \int_O \varphi \mathrm{d}\mu_p = \int_O \varphi \mathrm{d}\mu.$$

Let O be an open subset of $\mathbb{R}^N$, we denote by $\mathcal{M}(O)$ the space of all signed Radon measures in O with bounded total variation. By the Riesz Representation Theorem, $\mathcal{M}(O)$ can be identified with the dual of the separable space $\mathcal{C}_0(O)$ of continuous functions on the closure of O vanishing on the boundary ∂O. The N–dimensional Lebesgue measure in $\mathbb{R}^N$ is designated as $\mathcal{L}^N$ while $\mathcal{H}^{N-1}$ denotes the $(N-1)$–dimensional Hausdorff measure. If $\mu \in \mathcal{M}(O)$ and $\lambda \in \mathcal{M}(O)$ is a nonnegative Radon measure, we denote by $\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}$ the Radon–Nikodým derivative of μ with respect to λ. By a generalization of the Besicovitch Differentiation Theorem (see [2], Prop. 2.2), it can be proved that there exists a Borel set $E \subset O$ such that $\lambda(E) = 0$ and

$$\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}(x) = \lim_{\rho\to 0^+} \frac{\mu(x+\rho\, C)}{\lambda(x+\rho\, C)} \text{ for all } x \in \text{Supp } \lambda \setminus E$$

and any open convex set C containing the origin. (Recall that the set E is independent of C.)

2.3. Functions of bounded variation

We say that $u \in L^1(O;\mathbb{R}^d)$ is a function of bounded variation, and we write $u \in BV(O;\mathbb{R}^d)$, if all its first distributional derivatives $D_j u_i$ belong to $\mathcal{M}(O)$ for $1 \leq i \leq d$ and $1 \leq j \leq N$. We refer to [3] for a detailed analysis of BV functions. The matrix–valued measure whose entries are $D_j u_i$ is denoted by Du and $|Du|$ stands for its total variation. By the Lebesgue Decomposition Theorem we can split Du into the sum of two mutually singular measures $D^a u$ and $D^s u$ where $D^a u$ is the absolutely continuous part of Du with respect to the Lebesgue measure $\mathcal{L}^N$, while $D^s u$ is the singular part of Du with respect to $\mathcal{L}^N$. By ∇u we denote the Radon–Nikodým derivative of $D^a u$ with respect to the Lebesgue measure so that we can write

$$Du = \nabla u \mathcal{L}^N + D^s u.$$

The set S_u of points where u does not have an approximate limit is called the approximated discontinuity set, while $J_u \subseteq S_u$ is the so-called jump set of u defined as the set of points $x \in O$ such that there exist $u^{\pm}(x) \in \mathbb{R}^d$ (with $u^+(x) \neq u^-(x)$) and $\nu_u(x) \in \mathbb{S}^{N-1}$ satisfying

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^N} \int_{\{y\in B_\varepsilon(x):(y-x)\cdot\nu_u(x)>0\}} |u(y) - u^+(x)| \,\mathrm{d}y = 0,$$

and

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^N} \int_{\{y\in B_\varepsilon(x):(y-x)\cdot\nu_u(x)<0\}} |u(y) - u^-(x)| \,\mathrm{d}y = 0.$$

2.4. Trace spaces

If O is an open set with Lipschitz boundary ∂O and $u \in BV(O)$, we denote by u_o the null extension of u to $\mathbb{R}^N$ defined by

$$\begin{cases} u(x) & \text{if } x \in O, \\ 0 & \text{if } x \in \mathbb{R}^N \setminus O, \end{cases}$$

for $\mathcal{L}^N$ a.e. $x \in \mathbb{R}^N$. It turns out that $u_o \in BV(\mathbb{R}^N)$, and we define the trace $\gamma_O(u)$ of u on ∂O as

$$\gamma_O(u) = (u_o)^+ - (u_o)^-.$$

It results that for $\mathcal{H}^{N-1}$-a.e. $x \in \partial O$, the vector $\nu_{u_o}(x)$ agrees with the exterior (interior) normal $\nu(x)$ to ∂O at x, moreover $u_o^+(x) = 0$ or $u_o^-(x) = 0$ and $\gamma_O(u)(x) = u_o^+$ or $\gamma_O(u)(x) = u_o^-$. We observe that

$$\gamma_O(u)(x) = u(x)$$

for every $u \in W^{1,p}(O) \cap C(\overline{O})$ and for $\mathcal{H}^{N-1}$-a.e. $x \in \partial O$. We also recall that (see [32])

$$\lim_{r^N \to 0} \frac{1}{r^N} \int_{O \cap B_r(x_0)} |u(x) - \gamma_O(u)(x_0)|^{\frac{N}{N-1}} \mathrm{d}x = 0 \text{ for } \mathcal{H}^{N-1} - \text{a.e. } x_0 \in \partial O.$$

Let $O \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary, $p \geq 1$, there is a well defined continuous trace operator from $W^{1,p}(O)$ (resp. $BV(O)$) into $L^p(\partial O)$ (resp. $L^1(\partial O)$) satisfying the following integration by parts formula

$$\int_O u \operatorname{div} \phi \mathrm{d}x = -\int_O \nabla u \cdot \phi \mathrm{d}x + \int_{\partial O} \phi \gamma_O(u) \cdot \nu \mathrm{d}\mathcal{H}^{N-1},$$

for every $u \in W^{1,p}(O)$ (resp. $u \in BV(O)$), $\phi \in C_c^1(\mathbb{R}^N)^N$.

Then the trace space of $W^{1,p}(O)$, meaning that there is a continuous surjection whose kernel is $W_0^{1,p}(O)$, is denoted by $W^{1-\frac{1}{p},p}(\partial O)$, and it turns out that for $p = 1$ $W^{0,1}(\partial O) = L^1(\partial O)$.

Namely the following inequalities hold

$$\|\gamma_O(u)\|_{W^{1-\frac{1}{p},p}(\partial O)} \leq C_0 \|u\|_{W^{1,p}(O)} \text{ for every } u \in W^{1,p}(O), \tag{2.1}$$

and, conversely, for every $\varphi \in W^{1-\frac{1}{p},p}(\partial O)$ there exists $u \in W^{1,p}(O)$ such that $\gamma_O(u) = \varphi$ and

$$\|u\|_{W^{1,p}(O)} \leq C_1 \|\varphi\|_{W^{1-\frac{1}{p},p}(\partial O)}, \tag{2.2}$$

for suitable constants $C_0, C_1 \geq 0$.

The following result (*cf.* [31], Prop. 1.1) allows us to extend the previous considerations and inequality (2.1) to $\mathbb{R}^N \setminus \overline{O}$, provided O is bounded.

Proposition 2.8. *Let $p > 1$, let O be a bounded open set with Lipschitz boundary, then there exists $C_2' > 0$ such that for every $\varphi \in W^{1-\frac{1}{p},p}(\partial O)$ there exists $u \in W^{1,p}(\mathbb{R}^N \setminus \overline{O})$ such that $\gamma_{\mathbb{R}^N \setminus \overline{O}}(u) = \varphi$ and*

$$\|u\|_{W^{1,p}(\mathbb{R}^N \setminus \overline{O})} \leq C_2' \|\varphi\|_{W^{1-\frac{1}{p},p}(\partial O)}.$$

For every $p \in [1, +\infty[$, let I be a bounded open set in $\mathbb{R}^N$ with Lipschitz boundary such that $\Gamma := \partial O \cap I \neq \emptyset$ and suppose that $\mathcal{H}^{N-1}(\overline{\Gamma} \setminus \Gamma) = 0$. We denote by $W_{0,\Gamma}^{1,p}(O)$ the space $\{u \in W^{1,p}(O) : u = 0 \, \mathcal{H}^{N-1} - \text{a.e. on } \Gamma\}$, $W_{0,\partial O}^{1,p}(O) = W_0^{1,p}(O)$. In the sequel, for every $u_1 \in W_{\mathrm{loc}}^{1,p}(\mathbb{R}^N)$ we denote $u_1 + W_{0,\Gamma}^{1,p}(O)$ by $W_{u_1,\Gamma}^{1,p}(O)$, and $u_1 + W_0^{1,p}(O)$ by $W_{u_1}^{1,p}(O)$.

Moreover with an abuse of notation, we will identify (the restriction of) a function u with its trace on ∂O (or part of ∂O), $\gamma_O(u)$.

2.5. Duality

We end this section by recalling a result due to Ekeland and Temam (*cf.* [17], Thm. 4.1 Chap. III) that will be exploited in the sequel, we refer to the version mentioned in [16], Theorem 2.

Theorem 2.9. *Suppose that X and Y are Banach spaces, that Λ is a linear and continuous operator which sends X into Y, that F and G are convex functions on X and Y, respectively. We denote F^* and G^* their Fenchel conjugates, defined, respectively, on X^* and Y^*, by Λ^* the adjoint operator of Λ. Then*

$$\inf_{u\in X}\left\{F(u)+G(\Lambda u)\right\}\geq \sup_{p^*\in Y^*}\left\{-F^*(\Lambda^* p^*)-G^*(-p^*)\right\}.$$

Suppose that there exists $u_0\in X$, such that $F(u_0)<\infty$, and G is continuous on Λu_0. Then,

$$\inf_{u\in X}\left\{F(u)+G(\Lambda u)\right\}= \sup_{p^*\in Y^*}\left\{-F^*(\Lambda^* p^*)-G^*(-p^*)\right\},$$

and the dual problem on the right-hand side of the above possesses at least one solution.

3. Asymptotics in terms of Γ–convergence

In order to study the asymptotics for $\varepsilon\to 0$ and $p\to 1$ of problems $\mathcal{P}_{p,\varepsilon}$ and $\mathcal{P}_{p,0}$ in (1.5) and (1.10) respectively, we will invoke previous results and prove more general ones for generic open sets $O\subset\mathbb{R}^N$. Finally we will apply these lemmata to the specific open sets $\Omega\subset\mathbb{R}^3$ and $\omega\in\mathbb{R}^2$ involved in problems $\mathcal{P}_{p,0}$ and $\mathcal{P}_{p,\varepsilon}$. We will assume, in all the following statements, that ω is a bounded open set in $\mathbb{R}^2$, which is piecewise C^1, on the other hand we will weaken this assumption in some particular cases as below specified. We conjecture that it is possible, in the general framework, to assume ω with Lipschitz boundary, but, since our aim consists of providing Γ–convergence results in dimension reduction for $-\Delta_1$, connecting our results, in the last section, with 'Least Gradient' theory, we do not focus on the regularity assumptions for the boundary $\partial\omega$.

3.1. Asymptotics as $\varepsilon\to 0$

The first part of this section is devoted to recall the results available in literature for the asymptotics as $\varepsilon\to 0$ of problems $\mathcal{P}_{p,\varepsilon}$ in (1.5) for $p>1$ and $\mathcal{P}_{1,\varepsilon}$ in (1.12). Within this subsection $\omega\subset\mathbb{R}^2$ will be a bounded open set with Lipschitz boundary and $\Omega:=\omega\times\left(-\frac{1}{2},\frac{1}{2}\right)\cdot$

First we refer to the Sobolev case, *i.e.* $p>1$, to this end we state the following result due to Le Dret and Raoult (*cf.* [26], Thm. 2 where also loadings are considered). Their result deals with the hyperelastic case, besides some technical restrictions have been imposed. For the scalar case one may refer to [1], where the $3D-1D$ dimension reduction has been performed under mechanically consistent hypotheses.

Theorem 3.1. *Let $u_0\in W^{1,p}(\omega;\mathbb{R}^3)$, let $f:\mathbb{R}^{3\times 3}\to[0,+\infty[$ be a continuous function satisfying the following growth and coercivity condition*

$$C_1|\xi|^p-C_2\leq f(\xi)\leq C_3(1+|\xi|^p)$$

for every $\xi\in\mathbb{R}^{3\times 3}$ and for some $C_1,C_3>0$, and $C_2\geq 0$. Then the family of functionals $E_\varepsilon: L^p(\Omega;\mathbb{R}^3)\to[0,+\infty]$ defined by

$$E_\varepsilon(u)=\begin{cases}\displaystyle\int_\Omega f\left(\nabla_\alpha u\left|\frac{1}{\varepsilon}\nabla_3 u\right.\right)\mathrm{d}x & \text{if } u\in W^{1,p}_{u_0,\partial\omega\times\left(-\frac{1}{2},\frac{1}{2}\right)}(\Omega;\mathbb{R}^3),\\[2mm] +\infty & \text{otherwise,}\end{cases}$$

Γ–converges, with respect to the $L^p(\Omega;\mathbb{R}^3)$ convergence, as $\varepsilon \to 0$ to the functional $E_0 : L^p(\Omega;\mathbb{R}^3) \to [0,+\infty]$ defined by

$$E_0(u) = \begin{cases} \displaystyle\int_\omega \mathcal{Q}f_0(\nabla_\alpha u)\mathrm{d}x_\alpha & \text{if } u \in W^{1,p}_{u_0}(\omega;\mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases}$$

where $W^{1,p}_{u_0}(\omega;\mathbb{R}^3)$ has been identified with $\{u \in W^{1,p}_{u_0,\partial\omega\times(-\frac{1}{2},\frac{1}{2})}(\Omega;\mathbb{R}^3) : \nabla_3 u = 0\}$ and $f_0 : \mathbb{R}^{3\times 2} \to [0,+\infty[$ is defined as

$$f_0(z) := \inf_{c\in\mathbb{R}^3} f(z,c)$$

and $\mathcal{Q}f_0 : \mathbb{R}^{3\times 2} \to [0,+\infty)$ is the quasiconvexification of f_0, viz

$$\mathcal{Q}f_0(z) = \inf\left\{\frac{1}{|D|}\int_D f_0(z+\nabla\varphi)\mathrm{d}x : \varphi \in W^{1,\infty}_0(D;\mathbb{R}^2)\right\} \tag{3.1}$$

with $D \subset \mathbb{R}^3$.

Remark 3.2. The above result applies to the family $I_{p,\varepsilon}$ in (1.4), just replacing the density $f(\cdot)$ by $|\cdot|^p$ as in (1.4), providing the Γ–convergence, as $\varepsilon \to 0$, to $I_{p,0}$ in (1.9) (observe that $\mathcal{Q}(|\cdot|^p)_0 = |\cdot|^p$).

Analogously, in the linear case, *i.e.* $p = 1$, from [6], Theorem 3.2, where the SBV setting has been considered, the following result can be deduced.

Theorem 3.3. *Let $f : \mathbb{R}^{3\times 3} \to [0,+\infty[$ be a continuous function satisfying the following growth and coercivity condition*

$$|\xi| \leq f(\xi) \leq C(1+|\xi|)$$

for every $\xi \in \mathbb{R}^{3\times 3}$ and for some $C > 0$. Assume also that there exist constants $C, L > 0$, $0 < r < 1$, such that

$$\left| f^\infty(\xi) - \frac{f(t\xi)}{t}\right| \leq C\frac{1}{t^r}$$

for every $\xi \in \mathbb{R}^{3\times 3}$ with $|\xi| = 1$ and for all $t > 0$ and $t > L$. Then the family of functionals $J_\varepsilon : L^1(\Omega;\mathbb{R}^3) \to [0,+\infty]$ defined by

$$J_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega f\left(\nabla_\alpha u, \left|\frac{1}{\varepsilon}\nabla_3 u\right.\right)\mathrm{d}x & \text{if } u \in W^{1,1}(\Omega;\mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases}$$

Γ–converges, with respect to the $L^1(\Omega;\mathbb{R}^3)$ convergence, as $\varepsilon \to 0$ to the functional $J_0 : L^1(\Omega;\mathbb{R}^3) \to [0,+\infty]$ defined by

$$J_0(u) = \begin{cases} \displaystyle\int_\omega \mathcal{Q}f_0(\nabla_\alpha u)\mathrm{d}x_\alpha + \int_\omega (\mathcal{Q}f_0)^\infty\left(\frac{\mathrm{d}D^s_\alpha u}{\mathrm{d}|D^s_\alpha u|}\right)\mathrm{d}|D^s_\alpha u| & \text{if } u \in BV(\Omega;\mathbb{R}^3), D_3 u = 0, \\ +\infty & \text{otherwise,} \end{cases}$$

where $(\mathcal{Q}f_0)^\infty$ represents the recession function of the quasiconvexification of f_0 in (3.1), namely

$$(\mathcal{Q}f_0)^\infty(v) := \limsup_{t\to+\infty}\frac{\mathcal{Q}f_0(tv)}{t}\cdot$$

Let $W_\varepsilon : \mathbb{R}^3 \to \mathbb{R}$ be the function defined as $W_\varepsilon(\xi) = W_\varepsilon(\xi_1, \xi_2, \xi_3) = \left|\left(\xi_\alpha\middle|\frac{1}{\varepsilon}\xi_3\right)\right|$. We recall the functionals $I_{1,\varepsilon} : BV(\Omega) \to \mathbb{R}$, introduced in (1.14), as

$$I_{1,\varepsilon}(u) := \left|\left(D_\alpha u\middle|\frac{1}{\varepsilon}D_3 u\right)\right|(\Omega) + \int_{\partial\omega\times\left(-\frac{1}{2},\frac{1}{2}\right)} W_\varepsilon((u-u_0)\nu)\mathrm{d}\mathcal{H}^2, \tag{3.2}$$

where ν is the unit exterior normal to $\partial\omega \times \left(-\frac{1}{2}, \frac{1}{2}\right)\cdot$

We observe that the restriction of $I_{1,\varepsilon}$ to $W^{1,1}_{u_0,\partial\omega\times\left(-\frac{1}{2},\frac{1}{2}\right)}(\Omega)$ is given by (1.13).
Moreover, for every $\varepsilon > 0$, let $G_{1,\varepsilon} : BV(\Omega) \to [0, +\infty)$ be the functionals defined as

$$G_{1,\varepsilon}(u) := \begin{cases} \displaystyle\int_\Omega \left|\left(\nabla_\alpha u\middle|\frac{1}{\varepsilon}\nabla_3 u\right)\right| \mathrm{d}x & \text{if } u \in W^{1,1}_{u_0,\partial\omega\times\left(-\frac{1}{2},\frac{1}{2}\right)}(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.3}$$

Then, their relaxed functionals (with respect to L^1- strong topology) coincide with the functionals $I_{1,\varepsilon}$ in (3.2) (*cf.* [28], Thm. 3.4).

We point out that entirely similar arguments to those adopted in the proof of Theorem 3.3 (*cf.* also [7] where bending moments are taken into account) allow to consider the case with fields u clamped on the lateral boundary, thus leading to the following result.

Proposition 3.4. *The family of functionals $\{I_{1,\varepsilon}\}$ in (1.13), defined in $\left\{u \in W^{1,1}(\Omega) : u \equiv u_0 \text{ on } \partial\omega\times\left(-\frac{1}{2}, \frac{1}{2}\right)\right\}$, Γ–converges as $\varepsilon \to 0$, with respect to L^1 strong convergence, to $\overline{I_{1,0}}(u) = |Du|(\omega) + \int_{\partial\omega} |u - u_0|\mathrm{d}\mathcal{H}^1$, where this latter functional describes in $\{u \in BV(\Omega) : D_3 u = 0\}$, the relaxed functional, with respect to the L^1–strong convergence, of $I_{1,0}$ in (1.17).*

Remark 3.5. We recall that the Γ–convergence result as $\varepsilon \to 0$, stated in Proposition 3.4 is the same either if we consider the family of functionals $\{G_{1,\varepsilon}\}_\varepsilon$ in (3.3) or their relaxed ones $\{I_{1,\varepsilon}\}_\varepsilon$ in (3.2) (*cf.* [10], Prop. 6.11).

3.2. Asymptotics as $p \to 1$ in the reduced $2D$ model

Let $\omega \subset \mathbb{R}^2$ be a bounded open set, piecewise C^1, let $\overline{p} \geq p > 1$, let $u_0 \in W^{1-\frac{1}{\overline{p}},\overline{p}}(\partial\omega)$ and let $H_{p,0} : BV(\omega) \to \mathbb{R}$ be the family of functionals defined as

$$H_{p,0}(u) := \begin{cases} \left(\displaystyle\int_\omega W^p(\nabla u)\mathrm{d}x\right)^{\frac{1}{p}} & \text{if } u \in W^{1,p}_{u_0}(\omega), \\ +\infty & \text{otherwise,} \end{cases} \tag{3.4}$$

where $W : \mathbb{R}^2 \to [0, +\infty[$ is convex, positively 1–homogeneous and verifies (3.8).

The target of this subsection is to study the asymptotic behaviour as $p \to 1$ of (3.4) in terms of Γ–convergence. We start by observing that the regularity of ω, and the fact that $u_0 \in W^{1-\frac{1}{\overline{p}},\overline{p}}(\partial\omega)$ allow us to apply Proposition 2.8 and thus deduce that u_0 can be naturally extended as a $W^{1,p}(\mathbb{R}^2)$ function. Consequently in this subsection we will implicitly assume that $u_0 \in W^{1,p}(\mathbb{R}^2)$ and prove the following result.

Theorem 3.6. *The family of functionals $\{H_{p,0}\}_p$ defined in (3.4), Γ–converges, as p tends to 1 and with respect to L^1 strong convergence, to the functional $H_{1,0} : BV(\omega) \to \mathbb{R}$ defined as*

$$H_{1,0}(u) := \int_\omega W\left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}\right)\mathrm{d}|Du| + \int_{\partial\omega} W((u_0 - u)\nu)\mathrm{d}\mathcal{H}^{N-1}, \tag{3.5}$$

where ν denotes the unit exterior normal to $\partial\omega$.

This result will be achieved by several steps: first we will consider the case $W(\cdot) := |\cdot|$, stating first the upper bound inequality in Proposition 3.7 for any dimension N and achieving full Γ–convergence in Theorem 3.8. Then we will treat the case of W convex and positively 1–homogeneous, proving the upper bound inequality in Proposition 3.9 and arguing, in the proof of Theorem 3.6 exactly as in Theorem 3.8.

We start by recalling the following result that can be found in [14, 21].

Proposition 3.7. *Let $O \subset \mathbb{R}^N$ be some bounded open set, which is piecewise C^1. Let $u_1 \in L^1(\partial O)$. Suppose that $u_p \in W^{\frac{p-1}{p},p}(\partial O)$ converges in $L^1(\partial O)$ to u_1. Then for every $u \in BV(O)$, there exists $U_p \in W^{1,p}(O)$, $U_p = u_p$ on ∂O, such that*

$$\lim_{p\to 1}\int_O |\nabla U_p|^p \mathrm{d}x = |Du|(O) + \int_O |u - u_1| \mathrm{d}\mathcal{H}^{N-1},$$

$$\lim_{p\to 1}\int_O |U_p - u|^{1^*} \mathrm{d}x = 0,$$

where $1^* = \frac{N}{N-1}$.

We restate the above result in terms of Γ–convergence with respect to L^1–strong convergence.

Let $F_{p,0} : BV(\omega) \to \mathbb{R}$ be the functional defined as

$$F_{p,0}(u) := \begin{cases} \left(\displaystyle\int_\omega |\nabla u|^p \mathrm{d}x\right)^{\frac{1}{p}} & \text{if } u \in W^{1,p}_{u_0}(\omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.6}$$

Let $F_{1,0} : BV(\omega) \to \mathbb{R}$ be defined as

$$F_{1,0}(u) := |Du|(\overline{\omega}) = |Du|(\omega) + \int_{\partial\omega} |u - u_0| \mathrm{d}\mathcal{H}^1. \tag{3.7}$$

We can prove the following theorem

Theorem 3.8. *Let $\{F_{p,0}\}_p$ be the family of functionals introduced in (3.6), then $\{F_{p,0}\}_p$ Γ–converges, with respect to the $L^1(\omega)$ strong topology, to $F_{1,0}$.*

Proof. The lower bound is trivially obtained if $\{u_p\}_p$ is such that $\lim_{p\to 1} F_{p,0}(u_p) = +\infty$. Let $\{u_p\}_p$ strongly converge in $L^1(\omega)$ to $u \in BV(\omega)$ and assume also that it is a sequence with equibounded energy, namely there exists $C > 0$ such that

$$F_{p,0}(u_p) = \left(\int_\omega |\nabla u_p|^p \mathrm{d}x\right)^{\frac{1}{p}} \leq C.$$

By Hölder inequality, and the fact that $u_p \in W^{1,p}_{u_0}(\omega)$ it results that

$$|Du_p|(\overline{\omega}) \leq \left(\int_\omega |\nabla u_p|^p \mathrm{d}x\right)^{\frac{1}{p}} |\omega|^{1-\frac{1}{p}} \leq C' \text{ for every } 1 \leq p \leq \bar{p}.$$

Observe that, by virtue of Poincaré inequality, any sequence with equibounded energy $\{u_p\}_p$ admits a further subsequence, converging weakly $*$ in $BV(\omega)$ to $u \in BV(\omega)$.

Now by the observations made at the beginning of Section 3.2, u_0 can be assumed as a $W^{1,\overline{p}}(\mathbb{R}^2 \setminus \overline{\omega})$-function, whence the regularity assumptions on $\partial\omega$ ensure that we can extend $u \in BV(\omega)$ by u_0 in $\mathbb{R}^2 \setminus \overline{\omega}$, thus obtaining a $BV(\mathbb{R}^2)$ function, still denoted by u. In the same way we may extend, with an abuse of notations, any u_p, by $u_0 \in \mathbb{R}^2 \setminus \overline{\omega}$, getting $u_p \in W^{1,p}(\mathbb{R}^2)$.

Clearly $\{u_p\}_p$ weakly $*$ converges to u in $BV(\omega')$ for any bounded open set $\omega' \supset\supset \omega$. Consequently the lower semicontinuity of the total variation with respect to the weak $*$ topology in BV, and Hölder inequality provide the following chain of inequalities

$$
\begin{aligned}
|Du|(\omega') &\le \liminf_{p\to 1} |Du_p|(\omega') \le \liminf_{p\to 1} \left(\int_{\omega'} |\nabla u_p|^p \mathrm{d}x\right)^{\frac{1}{p}} |\omega'|^{1-\frac{1}{p}} \\
&= \liminf_{p\to 1} \left(\int_{\omega} |\nabla u_p|^p \mathrm{d}x + \int_{\omega'\setminus\overline{\omega}} |\nabla u_0|^p \mathrm{d}x\right)^{\frac{1}{p}}, \text{ for every } p \le \overline{p}.
\end{aligned}
$$

As ω' shrinks to ω, by (3.7), we obtain the so-called Γ–liminf inequality

$$
|Du|(\overline{\omega}) \le \liminf_{p\to 1} \left(\int_{\omega} |\nabla u_p|^p \mathrm{d}x\right)^{\frac{1}{p}}, \text{ for every } p \le \overline{p}.
$$

For what concerns the upper bound, we invoke Proposition 3.7 with $N = 2$, thus for every $u \in BV(\omega)$ we get the existence of a sequence $\{u_p\}_p \in W^{1,p}_{u_0}(\omega)$ such that

$$
\lim_{p\to 1} \int_{\omega} |u_p - u|^{1*} \mathrm{d}x = 0,
$$

$$
\lim_{p\to 1} \left(\int_{\omega} |\nabla u_p|^p \mathrm{d}x\right)^{\frac{1}{p}} = |Du|(\omega) + \int_{\partial\omega} |u - u_0| \mathrm{d}\mathcal{H}^1,
$$

and this concludes the proof. □

The following result carries Proposition 3.7 over to more general integrands. To this end we will consider bounded open subsets O of $\mathbb{R}^N$, with piecewise C^1 boundary and boundary datum $u_1 \in W^{1-\frac{1}{\overline{p}},\overline{p}}(\partial O)$ for some $\overline{p} > 1$. The same argument invoked at the beginning of Section 3.2, namely the regularity of O and Proposition 2.8, lead us, without loss of generality, to assume that $u_1 \in W^{1,\overline{p}}(\mathbb{R}^N)$.

Proposition 3.9. *Let $O \subset \mathbb{R}^N$ be a bounded open set, with piecewise C^1 boundary. Let $W : \mathbb{R}^N \to [0, +\infty[$ be a continuous, positively 1–homogeneous function such that*

$$
\frac{1}{C}|\xi| \le W(\xi) \le C|\xi| \text{ for every } \xi \in \mathbb{R}^N, \tag{3.8}
$$

for a suitable positive constant C. Let $u_1 \in W^{1-\frac{1}{\bar{p}},\bar{p}}(\partial O)$, for some $\bar{p} > 1$. Then, for every $u \in BV(O)$, and for every $1 < p \le \overline{p}$, there exists $U_p \in W^{1,p}(O)$, $U_p = u_1$ on ∂O, such that

$$
\begin{aligned}
&\lim_{p\to 1} \int_O (W(\nabla U_p))^p \mathrm{d}x = \int_O W\left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}\right) \mathrm{d}|Du| + \int_{\partial O} W((u_1 - u)\nu) \mathrm{d}\mathcal{H}^{N-1}, \\
&\lim_{p\to 1} \int_O |U_p - u|^{1^*} \mathrm{d}x = 0,
\end{aligned}
$$

where ν is the unit exterior normal to ∂O, and $1^* = \frac{N}{N-1}$.

Proof. Let $u \in BV(O)$, first we claim that for every sequence $\{p\}$ converging to 1, with $p \ge 1$, it is possible to find a subsequence, still denoted by $\{p\}$ and a sequence $\{v_p\} \subset W^{1,p}(O) \cap C^\infty(O)$, with $v_p = u_1$ on ∂O such that

$$
\lim_{p\to 1} \int_O |v_p - u|^{1^*} \mathrm{d}x = 0 \tag{3.9}
$$

and

$$\lim_{p\to 1}\int_O W(\nabla v_p)\mathrm{d}x=\int_O W\left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}\right)\mathrm{d}|Du|+\int_{\partial O}W((u_1-u)\nu)\mathrm{d}\mathcal{H}^{N-1}. \tag{3.10}$$

To prove the claim we observe that [15], Proposition 2 ensures that there exists a sequence $\{v_p\}_p$ such that $v_p\in W^{1,p}(O)\cap C^\infty(O)$, and $v_p=u_1$ on ∂O, (3.9) holds, $\lim\limits_{p\to 1}\int_O|v_p-u|^{1^*}\mathrm{d}x=0$ and $\lim\limits_{p\to 1}\int_O|\nabla v_p|^p\mathrm{d}x=|Du|(O)+\int_{\partial O}|u-u_1|\mathrm{d}\mathcal{H}^{N-1}$. This in turn, by virtue of Hölder inequality, implies that $\lim\limits_{p\to 1}\int_O|\nabla v_p|\mathrm{d}x\le\lim\limits_{p\to 1}\left(\int_O|\nabla v_p|^p\mathrm{d}x\right)^{\frac{1}{p}}|O|^{1-\frac{1}{p}}=|Du|(O)+\int_{\partial O}|u-u_1|\mathrm{d}\mathcal{H}^{N-1}$.

The opposite inequality follows by well known relaxation results, see [20], where the functional $|Dv|(O)+\int_{\partial O}|v-u_1|\mathrm{d}\mathcal{H}^{N-1}$ turns out to be the relaxed functional (with respect to $L^1(O)$ strong convergence) of $\begin{cases}\int_O|\nabla v|\mathrm{d}x & \text{if } v\in W^{1,1}_{u_1}(O),\\ +\infty & \text{if } v\in BV(O)\setminus W^{1,1}_{u_1}(O).\end{cases}$

Now, observing that, without loss of generality u_1 can be considered a $W^{1,p}(\mathbb{R}^N\setminus O)$ function, we can extend v_p and u by u_1 outside O, thus obtaining a $W^{1,p}(\mathbb{R}^N)$ function and a $BV(\mathbb{R}^N)$ one (*cf.* Prop. 2.8 and [3], Cor. 3.89), respectively. Consequently for every open set $O'\supset\supset O$, applying Reshetnyak's continuity theorem ([3], Thm. 2.39), it results

$$\begin{aligned}\lim_{p\to 1}\int_{O'}|\nabla v_p|\mathrm{d}x&=\lim_{p\to 1}\left(\int_{O'\setminus O}|\nabla u_1|\mathrm{d}x+\int_O|\nabla v_p|\mathrm{d}x\right)\\&=|Du|(O)+\int_{\partial O}|u-u_1|\mathrm{d}\mathcal{H}^{N-1}+\int_{O'\setminus O}|\nabla u_1|\mathrm{d}x.\end{aligned}$$

Thus, as O' shrinks to O, we obtain, invoking again Reshetnyak's continuity theorem, (3.10) and this proves the claim.

Again, *via* an extension argument to any open set $O'\supset\supset O$, we can assume that all the functions are extended as u_1 to all O'.

Next, the density of smooth functions in $W^{1,p}(O)$, with respect to strong $W^{1,p}$ convergence, the Sobolev embedding theorems and the continuity of W imply that there exists a further sequence $\{w_q\}_q\in W^{1,p}(O)\cap C^\infty(O)$, with $w_q\equiv u_1$ on ∂O, converging strongly in $W^{1,p}(O)$ to v_p as $q\to 1$, such that $\lim\limits_{q\to 1}\int_O|v_p-w_q|^{1^*}\mathrm{d}x=0$, ∇w_q and $W^q(\nabla v_q)$ pointwise converge a.e. to ∇v_p and $W(\nabla v_p)$, respectively, as $q\to 1$.

The growth from above in (3.8), and Hölder inequality entail that $W^q(\nabla w_q)$ is equi-integrable, thus we can conclude that $\int_O W^q(\nabla w_q)\mathrm{d}x$ converges to $\int_O W(\nabla v_p)\mathrm{d}x$ as $q\to 1$.

Finally a diagonal argument guarantees that there exists another sequence in $W^{1,p}(O)\cap C^\infty(O)$, denoted by $\{U_p\}$ such that $U_p\equiv u_1$ on ∂O, (3.9) holds and

$$\lim_{p\to 1}\int_O W^p(\nabla U_p)\mathrm{d}x=\int_O W\left(\frac{\mathrm{d}DU}{\mathrm{d}|DU|}\right)\mathrm{d}|DU|+\int_{\partial O}W((u_1-U)\nu)\mathrm{d}\mathcal{H}^{N-1}. \tag{3.11}$$

By the arbitrariness of the sequence $\{p\}$ the thesis follows. □

Proof of Theorem 3.6. The proof develops along the same lines as Theorem 3.8. Namely the lower bound can be proved arguing exactly as in the latter theorem, just exploiting the lower semicontinuity with respect to BV–weak $*$ convergence, of the functional $H_{1,0}$ as proven in [19]. On the other hand the upper bound is an immediate consequence of Proposition 3.9. □

Remark 3.10. Let $O\subset\mathbb{R}^N$ be any bounded open set with piecewise C^1 boundary, $1<p<\bar{p}$, and let $u_1\in W^{1-\frac{1}{\bar{p}},\bar{p}}(\partial O)$. The results expressed by Proposition 3.7 and the arguments in the first part of that proof, allow us to prove Γ–convergence, as $p \to 1$, with respect to L^1–strong convergence of the functionals $\{G_p\}_p$: $u \in W^{1,p}_{u_1}(O) \to \int_O W^p(\nabla u)\mathrm{d}x$ to $G_1 : u \in BV(O) \to \int_O W\left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}\right)\mathrm{d}|Du| + \int_{\partial O} W(|u-u_1|\nu)\mathrm{d}\mathcal{H}^{N-1}$ (ν being the unit exterior normal to ∂O) for any $W : \mathbb{R}^N \to [0,+\infty)$ convex, positively 1–homogeneous, satisfying a linear growth condition as (3.8)

3.3. Asymptotics as $p \to 1$ in the original $3D$ model

As in the previous subsections we recall that we are assuming $\overline{p} > 1$, ω a bounded open subset of $\mathbb{R}^2$ with piecewise C^1 boundary, and $\Omega := \omega \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ and let $u_0 \in W^{1-\frac{1}{\overline{p}},\overline{p}}(\partial\omega)$. Clearly the same arguments used at the beginning of Section 3.2 about the regularity of ω, and the possibility of applying Proposition 2.8 ensure that u_0 can be naturally extended to a function in $W^{1,p}(\mathbb{R}^2)$, in turn with an abuse of notations, this latter function can be regarded as a function depending also on x_3, $u \in W^{1,p}(\mathbb{R}^3)$.

Having in mind the functionals $\{I_{p,\varepsilon}\}_{p,\varepsilon}$ quoted in (1.4), we define, for every $p > 1$, with $p \leq \bar{p}$ and $\varepsilon > 0$, $F_{p,\varepsilon} : BV(\Omega) \to \mathbb{R}$ as the functionals

$$F_{p,\varepsilon}(u) := \begin{cases} \left(\displaystyle\int_\Omega W^p\left(\nabla_\alpha u \Big| \frac{1}{\varepsilon}\nabla_3 u\right)\mathrm{d}x\right)^{\frac{1}{p}} & \text{if } u \in W^{1,p}_{u_0,\partial\omega\times\left(-\frac{1}{2},\frac{1}{2}\right)}(\Omega), \\ +\infty & \text{otherwise,} \end{cases} \tag{3.12}$$

where $W : \mathbb{R}^3 \to [0,+\infty[$ is a continuous and positively 1–homogeneous function satisfying (3.8) and the space $W^{1,p}_{u_0,\partial\omega\times\left(-\frac{1}{2},\frac{1}{2}\right)}(\Omega)$ has been introduced in Section 2 (*cf.* Sect. 2.4 and observe that $\mathcal{H}^2\left(\overline{\partial\omega\times\left(-\frac{1}{2},\frac{1}{2}\right)} \setminus \partial\omega\times\left(-\frac{1}{2},\frac{1}{2}\right)\right) = 0$).

The main result of this subsection is stated in the following theorem.

Theorem 3.11. *Let $\{F_{p,\varepsilon}\}_p$ be the functionals introduced in* (3.12)*. Then $\{F_{p,\varepsilon}\}_p$ Γ–converges as $p \to 1$, with respect to the $L^1(\Omega)$ strong topology, to*

$$\int_\Omega W_\varepsilon\left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}\right)\mathrm{d}|Du| + \int_{\partial\omega\times\left(-\frac{1}{2},\frac{1}{2}\right)} W_\varepsilon((u-u_0)\nu)\mathrm{d}\mathcal{H}^2,$$

where $W_\varepsilon(\xi_1,\xi_2,\xi_3) := W\left(\xi_1,\xi_2,\frac{1}{\varepsilon}\xi_3\right)$.

Remark 3.12. This theorem provides Γ–convergence of the functionals $\{I_{p,\varepsilon}\}_{p,\varepsilon}$ in (1.4) towards the functional $I_{1,\varepsilon}$ in (3.2) as $p \to 1$, just replacing the function $W(\cdot)$ in (3.12) by $|\cdot|$.

To prove the Γ–convergence of $\{F_{p,\varepsilon}\}_p$ to $I_{1,\varepsilon}$ in (3.2) as $p \to 1$, we need some preliminary results in the same spirit of those proposed in [28], which need the assumption $\mathcal{H}^2\left(\overline{\partial\omega\times\left(-\frac{1}{2},\frac{1}{2}\right)} \setminus \partial\omega\times\left(-\frac{1}{2},\frac{1}{2}\right)\right) = 0$. We also observe that, having in mind the subsequent applications to $-\Delta_1$-type equations, and for the sake of simplicity in the exposition of the proof, we consider an energy density W positively 1–homogeneous, but analogous results hold replacing W with its recession function W^∞ where necessary.

The Γ–convergence result of Theorem 3.11 will be obtained through several steps. First in Lemma 3.13 we prove the lower bound inequality, then *via* intermediate results we will achieve the upper bound inequality in Lemma 3.17. The main difficulty consists in fact of the construction of recovery sequence, realized through Lemmas 3.14, 3.15 and 3.16. Indeed the imposed mixed boundary conditions require, roughly speaking, to glue three 'recovery sequences', one for the Dirichlet part, one for the Neumann and one for the open subset far from the boundary. In this "gluing" procedure it is important the requirement of essential closedness of the Dirichlet part of the boundary.

Lemma 3.13. *Let $\overline{p}$, ω, Ω and u_0 be as above and let $W : \mathbb{R}^3 \to [0,+\infty[$ be a convex and positively 1–homogeneous function, satisfying (3.8). Then for every $u \in BV(\Omega)$, for every $1 < p \leq \bar{p}$ it results*

$$\int_\Omega W\left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}\right)\mathrm{d}|Du| + \int_{\partial\omega\times\left(-\frac{1}{2},\frac{1}{2}\right)} W((u_0-u)\nu)\mathrm{d}\mathcal{H}^2 \leq \liminf_{p\to 1}\left(\int_\Omega W^p(\nabla u_p)\mathrm{d}x\right)^{\frac{1}{p}} \tag{3.13}$$

for every sequence $\{u_p\}_p$ with $u_p \in W^{1,p}_{u_0,\partial\omega\times\left(-\frac{1}{2},\frac{1}{2}\right)}(\Omega)$, such that $u_p \to u$ in $L^1(\Omega)$.

Proof. The result easily follows from the lower semicontinuity with respect to $L^1(\Omega)$ strong topology of the left hand side of (3.13) as proven in [28], Proposition 3.1, and the Hölder inequality. □

Now we introduce the following notations, already adopted in [11, 28]. We say that an open set $O \subset \mathbb{R}^N$ is cone-shaped if and only if there exists $x_0 \in \mathbb{R}^N$, $S \subset \mathbb{R}^N$, such that

$$O = \{(1-t)x_0 + tx : x \in S, t \in]0,1[\}.$$

We call x_0 the vertex of O, S the basis of O and observe that, if $t \in]0,1[$, then

$$x_0 + t(O - x_0) \subset O, \quad x_0 + t(S - x_0) \subset O.$$

Let $x_0 \in \mathbb{R}^N$ and $S \subset \mathbb{R}^N$ we denote by $C_{x_0,S}$ the cone

$$C_{x_0,S} = \{(1-t)x_0 + tx : x \in S, t > 0\}.$$

In what follows we will consider cone-shaped sets of vertex x_0 and basis S such that for any fixed $x \in S$, one has,

$$\{(1-t)x_0 + tx : t \in [0,1]\} \cap S = \{x\}. \tag{3.14}$$

The following lemma develops along the lines of [11], Lemma 2.1.

Lemma 3.14. *Let $1 < \overline{p}$ and let $W : \mathbb{R}^N \to [0,+\infty[$ be convex, positively 1–homogeneous and verify (3.8). Let $u_1 \in W^{1,\overline{p}}_{\mathrm{loc}}(\mathbb{R}^N)$, O be an open set with piecewise C^1 boundary. Let A, B be open sets such that $A \subseteq O, A \subset\subset B, O \setminus \overline{B} \neq \emptyset$, and let us assume that $O \cap B$ has piecewise C^1 boundary. Let $u \in BV(O)$, with $u = u_1$ a.e. in $O \setminus A$, then there exists $\{u_p\}_p$ such that $u_p \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ and $u_p \equiv u_1$ a.e. in $O \setminus B$ for every $1 < p \leq \bar{p}$ and*

$$\lim_{p\to 1}\int_O |u_p - u|^{1^*}\mathrm{d}x = 0,$$

and

$$\lim_{p\to 1}\int_O W^p(\nabla u_p)\mathrm{d}x \leq \int_{O\cap B} W(\nabla u)\mathrm{d}x$$

$$+ \int_A W\left(\frac{\mathrm{d}D^s u}{\mathrm{d}|D^s u|}\right)\mathrm{d}|D^s u| + \int_{O\cap\partial A} W((u_1-u)\nu)\mathrm{d}\mathcal{H}^{N-1} + \int_{O\setminus A} W(\nabla u_1)\mathrm{d}x.$$

Proof. Since $O \cap B$ has Lipschitz boundary, by virtue of Proposition 3.9, applied to $O \cap B$, and since $u \equiv u_1$ a.e. in $O \setminus A$ we know that for every $\overline{p} \geq p > 1$ there exists $\{v_p\}_p$ with $v_p \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$, such that

$$\lim_{p\to 1}\int_{O\cap B} |v_p - u|^{1^*}\mathrm{d}x = 0,$$

and

$$\lim_{p\to 1}\int_{O\cap B}(W(\nabla v_p))^p \mathrm{d}x = \int_{O\cap B} W\left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}\right)\mathrm{d}|Du|. \tag{3.15}$$

For every sequence $p>1$, $k\in\mathbb{N}$, let $\chi_k:\mathbb{R}\to\mathbb{R}$ be a smooth function such that $0\le \chi_k'\le 1$ with

$$\chi_k(t)=\begin{cases} -(k+1) & \text{if } t\le -(k+2),\\ t & \text{if } -k\le t\le k,\\ k+1 & \text{if } t\ge k+2,\end{cases}$$

and set

$$\begin{aligned}\hat{v}_{k,p} &= u_1+\chi_k(v_p-u_1),\\ \hat{v}_k &= u_1+\chi_k(u-u_1).\end{aligned}$$

Let $\varphi\in C_0^\infty(B)$ with $\varphi=1$ in A and define, for $t\in]0,1[$,

$$\begin{aligned} w_{t,k,p} &= t^2(2-t)[\varphi\hat{v}_{k,p}+(1-\varphi)u_1]+(1-t)(1+t-t^2)u_1,\\ w_{t,k} &= t^2(2-t)[\varphi\hat{v}_k+(1-\varphi)u_1]+(1-t)(1+t-t^2)u_1,\\ w_t &= t^2(2-t)[\varphi u+(1-\varphi)u_1]+(1-t)(1+t-t^2)u_1\end{aligned}$$

Clearly, for every $p>1$, $k\in\mathbb{N}$, $t\in]0,1[, w_{t,k,p}\in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ and $w_{t,k,p}=u_1$ a.e. in $O\setminus B$. By the convexity of W, the convexity and increasing monotonicity of $s\in\mathbb{R}^+\to s^p\in\mathbb{R}^+$, we get

$$\begin{aligned} &\int_O W^p(\nabla w_{t,k,p})\mathrm{d}x\\ &\le t\int_O W^p(t(2-t)(\varphi\nabla\hat{v}_{k,p}+(1-\varphi)\nabla u_1+(\hat{v}_{k,p}-u_1)\nabla\varphi))\mathrm{d}x\\ &\quad + j(1-t)\int_O W^p((1+t-t^2)\nabla u_1)\mathrm{d}x\\ &\le t^2(2-t)\int_O W^p(\varphi\nabla\hat{v}_{h,k}+(1-\varphi)\nabla u_1)\mathrm{d}x\\ &\quad + t(1-t(2-t))\int_O W^p\left(\frac{t(2-t)}{1-t(2-t)}(\hat{v}_{k,p}-u_1)\nabla\varphi\right)\\ &\quad + (1-t)\int_O W^p((1+t-t^2)\nabla u_1)\mathrm{d}x,\end{aligned} \tag{3.16}$$

for every $p>1$, $k\in\mathbb{N}, t\in]0,1[$.

The estimate of the first term in the right hand side of (3.16), gives, since $W^p(\cdot)$ is convex,

$$
\begin{aligned}
&\int_O W^p(\varphi\nabla\hat{v}_{k,p}+(1-\varphi)\nabla u_1)\mathrm{d}x\\
&=\int_A W^p(\nabla\hat{v}_{k,p})\mathrm{d}x+\int_{O\cap(B\setminus A)}W^p(\varphi\nabla\hat{v}_{k,p}+(1-\varphi)\nabla u_1)\mathrm{d}x+\int_{O\setminus B}W^p(\nabla u_1)\mathrm{d}x\\
&\le\int_A W^p(\nabla\hat{v}_{k,p})\mathrm{d}x+\int_{O\cap(B\setminus A)}\varphi W^p(\nabla\hat{v}_{k,p})\mathrm{d}x\\
&\quad+\int_{O\cap(B\setminus A)}(1-\varphi)W^p(\nabla u_1)\mathrm{d}x+\int_{O\setminus B}W^p(\nabla u_1)\mathrm{d}x\\
&\le\int_{O\cap B}W^p(\nabla\hat{v}_{k,p})\mathrm{d}x+\int_{O\setminus A}W^p(\nabla u_1)\mathrm{d}x\\
&=\int_{O\cap B\cap\{|v_p-u_1|\ge k+2\}}W^p(\nabla u_1)\mathrm{d}x+\int_{O\cap B\cap\{|v_p-u_1|\le k\}}W^p(\nabla v_p)\mathrm{d}x\\
&\quad+\int_{O\cap B\cap\{k<|v_p-u_1|<k+2\}}W^p(\chi_k'(v_p-u_1)\nabla v_p+(1-\chi_k'(v_p-u_1))\nabla u_1)\mathrm{d}x\\
&\quad+\int_{O\setminus A}W^p(\nabla u_1)\mathrm{d}x\\
&\le\int_{O\cap B\cap\{|v_p-u_1|\ge k+2\}}W^p(\nabla u_1)\mathrm{d}x+\int_{O\cap B\cap\{|v_p-u_1|\le k\}}W^p(\nabla v_p)\mathrm{d}x\\
&\quad+\int_{O\cap B\cap\{k<|v_p-u_1|<k+2\}}\left[\chi_k'(v_p-u_1)W^p(\nabla v_p)+(1-\chi_k'(v_p-u_1))W^p(\nabla u_1)\right]\mathrm{d}x\\
&\quad+\int_{O\setminus A}W^p(\nabla u_1)\mathrm{d}x\\
&\le\int_{O\cap B\cap\{|v_p-u_1|<k+2\}}W^p(\nabla v_p)\mathrm{d}x+\int_{O\cap B\cap\{|v_p-u_1|>k\}}W^p(\nabla u_1)\mathrm{d}x+\int_{O\setminus A}W^p(\nabla u_1)\mathrm{d}x,
\end{aligned}
\tag{3.17}
$$

for every $p>1$, $k\in\mathbb{N}$, $t\in]0,1[$. The growth condition on W, expressed in (3.8), the fact that $u_1\in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ entail that

$$
\limsup_{p\to1}\int_{O\cap B\cap\{|v_p-u_1|>k\}}W^p(\nabla u_1)\mathrm{d}x\le\int_{O\cap B\cap\{|u-u_1|\ge k\}}W(\nabla u_1)\mathrm{d}x. \tag{3.18}
$$

On the other hand, since we want an upper bound we can estimate the asymptotics as $p\to1$ of the term $\int_{O\cap B\cap\{|v_p-u_1|<k+2\}}W^p(\nabla v_p)\mathrm{d}x$ as follows

$$
\begin{aligned}
&\limsup_{p\to1}\int_{O\cap B\cap\{|v_p-u_1|<k+2\}}W^p(\nabla v_p)\mathrm{d}x\le\limsup_{p\to1}\int_{O\cap B}W^p(\nabla v_p)\mathrm{d}x=\int_{O\cap B}W\left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}\right)\mathrm{d}|Du|,\\
&\int_{O\cap B}W(\nabla u)\mathrm{d}x+\int_A W\left(\frac{\mathrm{d}D^su}{\mathrm{d}|D^su|}\right)\mathrm{d}|D^su|
\end{aligned}
\tag{3.19}
$$

where we have used (3.15) and the fact that u coincides with u_1 outside A.

Again Lebesgue's dominated convergence theorem implies that

$$\lim_{p\to 1}\int_{O\setminus A} W^p(\nabla u_1)\mathrm{d}x = \int_{O\setminus A} W(\nabla u_1)\mathrm{d}x. \tag{3.20}$$

Consequently by (3.17), (3.19), (3.18), (3.20), the fact that u coincides with the Sobolev function u_1 a.e. in $O\setminus A$, we obtain

$$\begin{aligned}
&\limsup_{p\to 1}\int_O W^p(\varphi\nabla\hat{v}_{k,p}+(1-\varphi)\nabla u_1)\mathrm{d}x\\
&\leq \int_{O\cap B} W\left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}\right)\mathrm{d}|Du|\\
&\quad+\int_{O\cap B\cap\{|u-u_1|\geq k\}} W(\nabla u_1)\mathrm{d}x+\int_{O\setminus A} W(\nabla u_1)\mathrm{d}x\\
&=\int_{O\cap B} W(\nabla u)\mathrm{d}x+\int_A W\left(\frac{\mathrm{d}D^s u}{\mathrm{d}|D^s u|}\right)\mathrm{d}|D^s u|+\int_{O\cap\partial A} W((u_1-u)\nu)\mathrm{d}\mathcal{H}^{N-1}\\
&\quad+\int_{O\cap B\cap\{|u-u_1|\geq k\}} W(\nabla u_1)\mathrm{d}x+\int_{O\setminus A} W(\nabla u_1)\mathrm{d}x,
\end{aligned} \tag{3.21}$$

for every $k\in\mathbb{N}, t\in]0,1[$.

Let us fix $k\in\mathbb{N}$, $t\in]0,1[$ and observe that $\|\hat{v}_{k,p}-u_1\|_{L^\infty(O\cap B)}\leq k+2$ for every $p>1$. Therefore, the growth condition on W (3.8), its convexity and the fact that $\frac{t(2-t)}{1-t(2-t)}(\hat{v}_{k,p}-u_1)\nabla\varphi\in L^\infty(O\cap B)$ converges pointwise a.e. in $O\cap B$ to $\frac{t(2-t)}{1-t(2-t)}(\hat{v}_k-u_1)\nabla\varphi$, lead us, *via* Lebesgue's dominated convergence theorem, to get

$$\lim_{p\to 1}\int_{O\cap B} W^p\left(\frac{t(2-t)}{1-t(2-t)}(\hat{v}_{k,p}-u_1)\nabla\varphi\right)\mathrm{d}x=\int_{O\cap B} W\left(\frac{t(2-t)}{1-t(2-t)}(\hat{v}_k-u_1)\nabla\varphi\right)\mathrm{d}x \tag{3.22}$$

for every $k\in\mathbb{N}$, $t\in]0,1[$.

Consequently by (3.16), (3.21), (3.22), we obtain

$$\begin{aligned}
&\limsup_{p\to 1}\int_O W^p(\nabla w_{t,k,p})\mathrm{d}x\\
&\leq t^2(2-t)\left[\int_{O\cap B} W(\nabla u)\mathrm{d}x+\int_A W\left(\frac{\mathrm{d}D^s u}{\mathrm{d}|D^s u|}\right)\mathrm{d}|D^s u|\right.\\
&\quad+\int_{O\cap\partial A} W((u_1-u)\nu)\mathrm{d}\mathcal{H}^{N-1}\\
&\quad\left.+\int_{O\cap B\cap\{|u-u_1|\geq k\}} W(\nabla u_1)\mathrm{d}x+\int_{O\setminus A} W(\nabla u_1)\mathrm{d}x\right]\\
&\quad+t(1-t(2-t))\int_O W\left(\frac{t(2-t)}{1-t(2-t)}(\hat{v}_k-u_1)\nabla\varphi\right)\mathrm{d}x+(1-t)\int_O W\left((1+t-t^2)\nabla u_1\right)\mathrm{d}x
\end{aligned}$$

for every $k\in\mathbb{N}, t\in]0,1[$.

The proof from now on is identical to that of Lemma 2.1 in [11] and we omit the details. We just observe that the positive 1–homogeneity of W allows us to replace the recession function W^∞ in [11] by W.

Thus we have that

$$\begin{aligned}&\limsup_{t\to 1}\limsup_{k\to+\infty}\limsup_{p\to 1}\int_O W^p(\nabla w_{t,k,p})\mathrm{d}x\\ &\le \int_{O\cap B} W(\nabla u)\mathrm{d}x+\int_A W\left(\frac{\mathrm{d}D^s u}{\mathrm{d}|D^s u|}\right)\mathrm{d}|D^s u|\\ &\quad+\int_{O\cap\partial A} W((u_1-u)\nu)\mathrm{d}\mathcal{H}^{N-1}+\int_{O\setminus A} W(\nabla u_1)\mathrm{d}x.\end{aligned}\tag{3.23}$$

By (3.23) the thesis follows by a standard diagonal argument once we observe that $w_{t,k,p}\to w_{t,k}$ in $L^{1^*}(O)$ for every $k\in\mathbb{N}$, $t\in]0,1[$ as $p\to 1$, and $w_{t,k}\to w_t$ in $L^{1^*}(O)$ for every $t\in]0,1[$ as $k\to+\infty$ and $w_t\to u$ in $L^{1^*}(O)$ as $t\to 1$. □

The result stated below is analogous to [11], Lemma 2.2.

Lemma 3.15. *Let $\bar p>1$, let O be a cone-shaped open set with piecewise C^1 boundary, with vertex x_0 and basis S, $W:\mathbb{R}^N\to[0,+\infty[$ be convex, positively 1–homogeneous and verifying (3.8), $u_1\in W^{1,\bar p}(\mathbb{R}^N)$, and $u\in BV(O)$. Then there exists $\{u_p\}_p$, $\bar p\ge p>1$, such that $u_p\in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ and $u_p\equiv u_1$ a.e. in $C_{x_0,S}\setminus O$ for every $p>1$, $u_p\to u$ in $L^{1^*}(O)$ and*

$$\limsup_{p\to 1}\int_O W^p(\nabla u_p)\mathrm{d}x\le\int_O W(\nabla u)\mathrm{d}x+\int_O W\left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}\right)\mathrm{d}|Du|+\int_S W((u_1-u)\nu)\mathrm{d}\mathcal{H}^{N-1}.$$

Proof. The proof is very similar to that of Lemma 2.2 in [11]. We do not propose it in its entirety but we just outline the main steps and differences.

First we extend $u\in C_{x_0,S}$ by defining $u=u_1$ a.e. in $C_{x_0,S}\setminus O$. Let $t\in]1,+\infty[$ and $\tau\in]0,1[$, set $O_t=x_0+\frac{(O-x_0)}{t}$ and define $u_{t,\tau}=u_1+\frac{\tau}{t}(u-u_1)(x_0+t(\cdot-x_0))$. We have that if $x\notin O_t$, then $x_0+t(x-x_0)\notin O$, hence being $u=u_1$ a.e. in $C_{x_0,S}\setminus O$, it turns out that $u_{t,\tau}\in BV(A)$ for every bounded subset A of $C_{x_0,S}$, $u_{t,\tau}=u_1$ a.e. in $C_{x_0,S}\setminus O_t$ and

$$\nabla u_{t,\tau}(x)=\nabla u_1(x)+\tau\nabla(u-u_1)(x_0+t(x-x_0))\text{ for a.e. }x\in C_{x_0,S}.$$

We claim that $u_{t,\tau}\to u$ in $L^{1^*}(O)$ first as $t\to 1$ and then $\tau\to 1$.

Indeed, by Lemma 3.14, applied to $A=O_t$ and B a bounded open set satisfying $O_t\subset\subset B$, $O\setminus\overline{B}\neq\emptyset$ and such that $O\cap B$ is piecewise C^1, there exists $\{u_p^{t,\tau}\}_{p,t,\tau}$ such that $u_p^{t,\tau}\in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ and verifies $u_p^{t,\tau}\to u_{t,\tau}$ in $L^{1^*}(O)$, $u_p^{t,\tau}=u_1$ a.e. in $O\setminus B$ for every $p>1$, and

$$\begin{aligned}&\limsup_{p\to 1}\int_O W^p(\nabla u_p^{t,\tau})\mathrm{d}x\\ &\le\int_{O\cap B} W(\nabla u_{t,\tau})\mathrm{d}x+\int_{O_t} W\left(\frac{\mathrm{d}D^s u_{t,\tau}}{\mathrm{d}|D^s u_{t,\tau}|}\right)\mathrm{d}|D^s u_{t,\tau}|\\ &\quad+\int_{O\cap\partial O_t} W((u_1-u_{t,\tau})\nu)\mathrm{d}\mathcal{H}^{N-1}+\int_{O\setminus O_t} W(\nabla u_1)\mathrm{d}x.\end{aligned}\tag{3.24}$$

Observe also that it is not restrictive to assume that $u_p^{t,\tau}=u_1$ a.e. in $C_{x_0,S}\setminus O$ for every $p>1$.

Then, exploiting the convexity and the positive 1–homogeneity of W and the change of variable $y=x_0+t(x-x_0)$ the proof develops along the same lines of [11], Lemma 2.1, thus we omit it.

In conclusion, taking first the limit as $t \to 1$ and then letting τ go to 1, we have,

$$\limsup_{\tau\to 1}\limsup_{t\to 1}\left\{\int_{O\cap B} W(\nabla u_{t,\tau})\mathrm{d}x + \int_{O_t} W\left(\frac{\mathrm{d}D^s u_{t,\tau}}{\mathrm{d}|D^s u_{t,\tau}|}\right)\mathrm{d}|D^s u_{t,\tau}|\right.$$
$$\left. + \int_{O\cap\partial O_t} W((u_1-u)\nu)\mathrm{d}\mathcal{H}^{N-1} + \int_{O\setminus O_t} W(\nabla u_1)\mathrm{d}x\right\} \leq \tag{3.25}$$
$$\leq \int_O W(\nabla u)\mathrm{d}x + \int_O W\left(\frac{\mathrm{d}D^s u}{\mathrm{d}|D^s u|}\right)\mathrm{d}|D^s u| + \int_S W((u_1-u)\nu)\mathrm{d}\mathcal{H}^{N-1}.$$

By (3.24), (3.25) and a diagonal argument, the thesis follows. $\square$

The following result, developed in the same spirit of [28], Lemma 3.2 will be exploited in the sequel.

Lemma 3.16. *Let O be a cone-shaped open set in $\mathbb{R}^N$ with piecewise C^1 boundary with vertex x_0 and basis S, satisfying (3.14). Let I be another bounded open set in $\mathbb{R}^N$ piecewise C^1 such that $\Gamma := S\cap I \neq \emptyset$, and assume that $\mathcal{H}^{N-1}(\bar\Gamma\setminus\Gamma)=0$. Let $W:\mathbb{R}^N\to[0.+\infty[$ be a convex, positively 1–homogeneous function satisfying (3.8). Let $u\in BV(O)$ and $u_1\in W^{1,\bar p}_{\rm loc}(\mathbb{R}^N)$ for some $\bar p>1$. Then there exists a sequence $\{u_p\}_p$ such that for every $1<p<\bar p$, $u_p\in W^{1,p}_{\rm loc}(\mathbb{R}^N)$ and $u_p=u_1$ on $C_{x_0,\Gamma}\setminus O$, $u_p\to u_1$ in $L^{1^*}(O)$ and*

$$\limsup_{p\to 1}\int_O W^p(\nabla u_p)\mathrm{d}x \leq \int_O W\left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}\right)\mathrm{d}|Du| + \int_\Gamma W((u_1-u)\nu)\mathrm{d}\mathcal{H}^{N-1}.$$

Proof. Without loss of generality we may assume that the right hand side of the above formula is finite.

Let $\{B^\varepsilon\}_{\varepsilon>0}$ be a decreasing family of open subsets of O with Lipschitz boundary such that, setting $\Gamma^\varepsilon = B^\varepsilon\cap S$, one has,

i) $\Gamma^\varepsilon \supset \bar\Gamma$;
ii) $\cap_{\varepsilon>0}\Gamma^\varepsilon = \bar\Gamma$.

Let A be a cone-shaped set of basis Γ and vertex x_A with $x_A\in \mathrm{int}(O)$, and for every ε denote by A^ε a cone-shaped set of basis Γ^ε and vertex x_ε, with $x_\varepsilon\in\mathrm{int}(O\setminus A)$ suitably chosen (a convenient choice is to take x_ε along the line which provide the distance between x_A and Γ with a bigger distance from Γ). Assume that $\{A^\varepsilon\}_{\varepsilon>0}$ is a decreasing family of sets such that $\cap_{\varepsilon>0}A^\varepsilon = \bar A$. (3.8) allows us to apply Lemma 3.15. Hence there exists a sequence $\{u^\varepsilon_p\}_p$ with $u^\varepsilon_p\in W^{1,p}_{\rm loc}(\mathbb{R}^N)$, $u^\varepsilon_p = u_1$ in $C_{x_\varepsilon,\Gamma^\varepsilon}\setminus A^\varepsilon$, such that $u^\varepsilon_p\to u$ in $L^{1^*}(A^\varepsilon)$ and

$$\limsup_{p\to 1}\int_{A^\varepsilon} W^p(\nabla u^\varepsilon_p)\mathrm{d}x \leq \int_{A^\varepsilon} W\left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}\right)\mathrm{d}|Du| + \int_{\Gamma^\varepsilon} W((u_1-u)\nu)\mathrm{d}\mathcal{H}^{N-1}. \tag{3.26}$$

Moreover an argument analogous to that exploited in the proof of Proposition 3.9 guarantees that there exists a sequence $\{v_p\}_p$ such that $v_p\in W^{1,p}_{\rm loc}(\mathbb{R}^N)$, $\lim_{p\to 1}\int_{O\setminus\bar A}|v_p-u|^{1^*}\mathrm{d}x = 0$ and

$$\lim_{p\to 1}\int_{O\setminus\bar A} W^p(\nabla v_p)\mathrm{d}x = \int_{O\setminus\bar A} W\left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}\right)\mathrm{d}|Du|. \tag{3.27}$$

For every $\varepsilon > 0$, let $0 \le \varphi^\varepsilon \le 1$ be a smooth function such that

$$\varphi^\varepsilon : x \in O \to \begin{cases} 0 & \text{if } x \in C_{x_A,\Gamma}, \\ 1 & \text{if } x \in C_{x_0,S} \setminus C_{x_\varepsilon,\Gamma^\varepsilon}, \end{cases}$$

and set $w_p^\varepsilon = (1-\varphi^\varepsilon)u_p^\varepsilon + \varphi^\varepsilon v_p$.

We observe that, by definition, $w_p^\varepsilon \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ and that $w_p^\varepsilon = u_1$ in $C_{x_0,\Gamma} \setminus O$. Let us fix $\varepsilon > 0$ and observe that

$$\begin{aligned} \limsup_{p\to 1} \int_O W^p(\nabla w_p^\varepsilon)\mathrm{d}x &= \limsup_{p\to 1} \left\{ \int_A W^p(\nabla u_p^\varepsilon)\mathrm{d}x \right. \\ &\left. + \int_{A^\varepsilon\setminus A} W^p([(1-\varphi^\varepsilon)\nabla u_p^\varepsilon + \varphi^\varepsilon \nabla v_p + \nabla\varphi^\varepsilon(v_p - u_p^\varepsilon)])\mathrm{d}x + \int_{O\setminus A^\varepsilon} W^p(\nabla v_p)\mathrm{d}x \right\}. \end{aligned} \tag{3.28}$$

Next, by exploiting the convexity of W and (3.8), we obtain the following inequality due to the local Lipschitz continuity of W^p (the constant C below may vary from line to line, being uniformly bounded in p as p tends to 1).

$$\begin{aligned} &\int_{A^\varepsilon\setminus \bar{A}} W^p([(1-\varphi^\varepsilon)\nabla u_p^\varepsilon + \varphi^\varepsilon \nabla v_p + \nabla\varphi^\varepsilon(v_p - u_p^\varepsilon)])\mathrm{d}x \\ &\le \int_{A^\varepsilon\setminus A} W^p((1-\varphi^\varepsilon)\nabla u_p^\varepsilon + \varphi^\varepsilon \nabla v_p)\mathrm{d}x \\ &\quad + C\int_{A^\varepsilon\setminus A} \left(\left|[(1-\varphi^\varepsilon)\nabla u_p^\varepsilon + \varphi^\varepsilon \nabla v_p + \nabla\varphi^\varepsilon(v_p - u_p^\varepsilon)]\right|^{p-1} + \left|(1-\varphi^\varepsilon)\nabla u_p^\varepsilon + \varphi^\varepsilon \nabla v_p\right|^{p-1} \right) \\ &\quad \times \left|\nabla\varphi^\varepsilon(v_p - u_p^\varepsilon)\right| \mathrm{d}x. \end{aligned}$$

By exploiting again the convexity of W and Hölder inequality we obtain

$$\begin{aligned} &\int_{A^\varepsilon\setminus \bar{A}} W^p([(1-\varphi^\varepsilon)\nabla u_p^\varepsilon + \varphi^\varepsilon \nabla v_p + \nabla\varphi^\varepsilon(v_p - u_p^\varepsilon)])\mathrm{d}x \\ &\le \int_{A^\varepsilon\setminus A} W^p(\nabla u_p^\varepsilon)\mathrm{d}x + \int_{A^\varepsilon\setminus A} W^p(\nabla v_p)\mathrm{d}x \\ &\quad + C\left(\int_{A^\varepsilon\setminus A} \left(\left|[(1-\varphi^\varepsilon)\nabla u_p^\varepsilon + \varphi^\varepsilon \nabla v_p + \nabla\varphi^\varepsilon(v_p - u_p^\varepsilon)]\right|^{p-1} + \left|(1-\varphi^\varepsilon)\nabla u_p^\varepsilon + \varphi^\varepsilon \nabla v_p\right|^{p-1} \right)^{\frac{p}{p-1}} \mathrm{d}x \right)^{\frac{p-1}{p}} \cdot \\ &\left(\int_{A^\varepsilon\setminus A} \left|\nabla\varphi^\varepsilon(v_p - u_p^\varepsilon)\right|^p \mathrm{d}x \right)^{\frac{1}{p}}. \end{aligned}$$

Thus, the last inequality and (3.28) provide

$$
\begin{aligned}
&\limsup_{p\to 1}\int_O W^p(\nabla w_p^\varepsilon)\mathrm{d}x\\
&\leq \limsup_{p\to 1}\Bigg[\int_{A^\varepsilon} W^p(\nabla u_p^\varepsilon)\mathrm{d}x+\int_{O\setminus A} W^p(\nabla v_p)\mathrm{d}x\\
&+C\left(\int_{A^\varepsilon\setminus A}\left(\left|[(1-\varphi^\varepsilon)\nabla u_p^\varepsilon+\varphi^\varepsilon\nabla v_p+\nabla\varphi^\varepsilon(v_p-u_p^\varepsilon)]\right|^{p-1}+\left|(1-\varphi^\varepsilon)\nabla u_p^\varepsilon+\varphi^\varepsilon\nabla v_p\right|^{p-1}\right)^{\frac{p}{p-1}}\mathrm{d}x\right)^{\frac{p-1}{p}}\cdot\\
&\left(\int_{A^\varepsilon\setminus A}\left|\nabla\varphi^\varepsilon(v_p-u_p^\varepsilon)\right|^p\mathrm{d}x\right)^{\frac{1}{p}}\Bigg].
\end{aligned}
$$

Exploiting (3.26),(3.27), the bounds on $\int_{A^\varepsilon\setminus A}|\nabla u_p^\varepsilon|^p\mathrm{d}x$ and $\int_{A^\varepsilon\setminus A}|\nabla v_p|^p\mathrm{d}x$ following from (3.26) and (3.27) and the growth from below of W in (3.8), and since both u_p^ε and $v_p\in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ converge to u in $L^{1^*}(A^\varepsilon)$ and $L^{1^*}(O\setminus A)$ respectively, we can conclude, passing to the limit as $p\to 1$, that

$$\lim_{p\to 1}\int_O\left|w_p^\varepsilon-u\right|^{1^*}\mathrm{d}x=0,$$

obtaining also

$$
\begin{aligned}
\limsup_{p\to 1}\int_O W^p(\nabla w_p^\varepsilon)\mathrm{d}x\leq \int_O W\left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}\right)\mathrm{d}|DU|+\int_{\Gamma^\varepsilon}W((u_1-u)\nu)\mathrm{d}\mathcal{H}^{N-1}\\
+\int_{A^\varepsilon\setminus A}W(\nabla u)\mathrm{d}x+\int_{A^\varepsilon\setminus\bar{A}}W\left(\frac{\mathrm{d}D^s u}{\mathrm{d}|D^s u|}\right)\mathrm{d}|D^s u|.
\end{aligned}
$$

We also observe that

$$\int_{\Gamma^\varepsilon}W((u_1-u)\nu)\mathrm{d}\mathcal{H}^{N-1}=\int_{\Gamma}W((u_1-u)\nu)\mathrm{d}\mathcal{H}^{N-1}+\int_{\Gamma^\varepsilon\setminus\Gamma}W((u_1-u)\nu)\mathrm{d}\mathcal{H}^{N-1},$$

where this latter term is finite as a consequence of (3.8). Then the thesis follows exploiting again the growth condition and the fact that $\mathcal{H}^{N-1}(\bar{\Gamma}\setminus\Gamma)=0$ as A^ε shrinks to A. □

The following result, analogous to [28], Lemma 3.3, allows us to obtain the upper bound inequality for the desired Γ–convergence. We emphasize that the arguments below essentially rely on the application of a partition of unity to glue the recovery sequences (in $L^{1^*}(\Omega)$ and not just in $L^1(\Omega)$ as in [28]) for the Neumann and Dirichlet parts of Ω, *i.e.* "lateral boundary" and 'bases' of the domain and exploit the local p–Lipschitz continuity of W^p and the fact that in the above lemmas, the "almost" recovery sequences converge in L^{1^*} and not only in L^1.

Lemma 3.17. *Let $\overline{p},\omega,\Omega$ and u_0 be as above. Let $W:\mathbb{R}^3\to[0,+\infty[$ be a convex, positively 1–homogeneous function verifying (3.8). Let $u\in BV(\omega)$. Then there exists a sequence $\{u_p\}_p$ such that $u_p\in W^{1,p}_{u_0,\partial\omega\times(-\frac{1}{2},\frac{1}{2})}(\Omega)$, with $1<p<\bar{p}$ such that*

$$\lim_{p\to 1}\int_\Omega|u_p-u|^{1^*}\mathrm{d}x=0,$$

and

$$\lim_{p\to 1}\int_{\Omega} W^p(\nabla u_p)\mathrm{d}x \le \int_{\Omega} W\left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}\right)\mathrm{d}|Du| + \int_{\partial\omega\times\left(-\frac12,\frac12\right)} W((u_0-u)\nu)\mathrm{d}\mathcal{H}^2. \tag{3.29}$$

Proof. We start by recalling, as always in this section, that u_0 can be considered with an abuse of notations as a function in $W^{1,\bar p}_{\mathrm{loc}}(\mathbb{R}^3)$.

Without loss of generality we may assume the right hand side of (3.29) is finite, otherwise there is nothing to prove.

Let $\gamma\left(\partial\omega\times\left(-\frac12,\frac12\right)\right)$ be the boundary of Γ in the topology induced on $\partial\Omega$. Take a finite open covering $S_j, j=1,\dots,i$ of $\gamma\left(\partial\omega\times\left(-\frac12,\frac12\right)\right)$, made up of balls in $\mathbb{R}^3$ centered in $\gamma\left(\partial\omega\times\left(-\frac12,\frac12\right)\right)$, and let $S_j, j=i{+}1,\dots l$ be a finite covering of the remaining part of $\partial\omega\times\left(-\frac12,\frac12\right)$, made up of balls centered on Γ, such that $S_j\cap\partial\Omega$ is the graph of a piecewise C^1 function for $j=1,\dots,l$.

Let $x_j\in\partial S_j\cap\left(\partial\omega\times\left(-\frac12,\frac12\right)\right)$, and, for every $j=1,\dots,l$, define A_j as the cone-shaped set of basis $S_j\cap\left(\partial\omega\times\left(-\frac12,\frac12\right)\right)$ with vertex x_j. Clearly A_j has piecewise C^1 boundary. Let $r>1$ and consider $B_j = x_j + r(A_j-x_j)$ for every $j=1,\dots,l$, then we obtain an open covering of $\bar\Gamma$ with cone-shaped open sets. Moreover for $j=l+1,\dots,m$ let B_j be a finite family of balls such that $\{B_j\}_{j=1,\dots,m}$ is an open covering of $\overline{\omega\times\left(-\frac12,\frac12\right)}$. Let $\{\varphi_j\}_{j=1,\dots,m}$ be a partition of the unity, relative to this covering, namely $\varphi_j\in C_0^\infty(B_j)$, $0\le\varphi_j\le 1$, for $j=1,\dots,m$ and $\sum_{j=1}^m\varphi_j(x)=1$ in $\overline{\Omega}$.

We also observe that, by Lemma 3.16, applied to the cone-shaped set A_j, for $j=1,\dots,i$, there exists a sequence $\{v_p^j\}$ such that $v_p^j\in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ and $v_p^j=u_0$ on $C_{x_j,S_j\cap\left(\partial\omega\times\left(-\frac12,\frac12\right)\right)}\setminus A_j$, $\int_{A_j}|v_p^j-u|^{1^*}\mathrm{d}x\to 0$ as $p\to 1$, and

$$\limsup_{p\to 1}\int_{A_j} W^p(\nabla v_p^j)\mathrm{d}x \le \int_{A_j} W\left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}\right)\mathrm{d}|Du| + \int_{S_j\cap\left(\partial\omega\times\left(-\frac12,\frac12\right)\right)} W((u_0-u)\nu)\mathrm{d}\mathcal{H}^2, \tag{3.30}$$

for every $j=1,\dots,i$. Moreover the growth condition (3.8) allows us to apply Lemma 3.15 to the cone-shaped set A_j for $j=i+1,\dots,l$, hence we can conclude that there exists a sequence $\{v_p^j\}_p$, with $v_p^j\in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ such that $v_p^j=u_0$ on $C_{x_j,S_j\cap\left(\partial\omega\times\left(-\frac12,\frac12\right)\right)}\setminus A_j$, $\int_{A_j}|v_p^j-u|^{1^*}\mathrm{d}x\to 0$ as $p\to 1$, and

$$\limsup_{p\to 1}\int_{A_j} W^p(\nabla v_p^j)\mathrm{d}x \le \int_{A_j} W\left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}\right)\mathrm{d}|Du| + \int_{S_j\cap\left(\partial\omega\times\left(-\frac12,\frac12\right)\right)} W((u_0-u)\nu)\mathrm{d}\mathcal{H}^2, \tag{3.31}$$

for every $j=i+1,\dots,l$.

Concerning the remaining part of the domain Ω, we recall that an argument entirely similar to that of Proposition 3.9, guarantees that for all $j=l+1,\dots,m$ there exists $\{v_p^j\}_p$ with $v_p^j\in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ such that $v_p^j\to u$ in $\mathcal{L}^{1^*}(B_j\cap\Omega)$ and

$$\lim_{p\to 1}\int_{B_j\cap\Omega} W^p(\nabla v_p^j)\mathrm{d}x = \int_{B_j\cap\Omega} W\left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}\right)\mathrm{d}|Du|. \tag{3.32}$$

Now, for every $p>1$ we may define the function $w_p\in W^{1,p}_{u_0,\partial\omega\times\left(-\frac12,\frac12\right)}(\Omega)$ as

$$w_p = \sum_{j=1}^m \varphi_j v_p^j.$$

The convexity of W^p and, the growth condition (3.8) entail the local p–Lipschitz property for W^p. Thus exploiting the convexity of W^p, the Hölder inequality we get

$$
\begin{aligned}
&\int_\Omega W^p(\nabla w_p)\mathrm{d}x = \int_\Omega W^p\left(\sum_{j=1}^m(\varphi_j\nabla v_p^j + v_p^j\nabla\varphi_j)\right)\mathrm{d}x\\
&\leq \int_\Omega W^p\left(\sum_{j=1}^m \varphi_j\nabla v_p^j\right)\mathrm{d}x + C\int_\Omega\left(\left|\sum_{j=1}^m(\varphi_j\nabla v_p^j+\varphi_p^j\nabla v_p^j)\right|^{p-1}+\left|\sum_{j=1}^m\varphi_j\nabla v_p^j\right|^{p-1}\right)\left|\sum_{j=1}^m v_p^j\nabla\varphi_j\right|\mathrm{d}x\\
&\leq \int_\Omega W^p\left(\sum_{j=1}^m \varphi_j\nabla v_p^j\right)\mathrm{d}x + C\left(\int_\Omega\left(\left|\sum_{j=1}^m(\varphi_j\nabla v_p^j+\varphi_p^j\nabla v_p^j)\right|^{p-1}+\left|\sum_{j=1}^m\varphi_j\nabla v_p^j\right|^{p-1}\right)^{\frac{p}{p-1}}\mathrm{d}x\right)^{\frac{p-1}{p}}\cdot\\
&\left(\int_\Omega\left|\sum_{j=1}^m v_p^j\nabla\varphi_j\right|^p\mathrm{d}x\right)^{\frac{1}{p}}\\
&\leq \sum_{j=1}^m\int_\Omega \varphi_j W^p(\nabla v_p^j)\mathrm{d}x + C\left(\int_\Omega\left(\left|\sum_{j=1}^m(\varphi_j\nabla v_p^j+\varphi_p^j\nabla v_p^j)\right|^{p-1}+\left|\sum_{j=1}^m\varphi_j\nabla v_p^j\right|^{p-1}\right)^{\frac{p}{p-1}}\mathrm{d}x\right)^{\frac{p-1}{p}}\cdot\\
&\left(\int_\Omega\left|\sum_{j=1}^m v_p^j\nabla\varphi_j\right|^p\mathrm{d}x\right)^{\frac{1}{p}}.
\end{aligned}
\tag{3.33}
$$

We notice that $v_p^j\in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$, $\sum_{j=1}^m\nabla\varphi_j=0$ and $v_p^j\to u$ in $L^{1^*}(A_j)$ as $p\to1$, for $j=1,\dots,m$. Hence

$$
\lim_{p\to1}\int_\Omega\left|\sum_{j=1}^m v_p^j\nabla\varphi_j\right|^p\mathrm{d}x=0. \tag{3.34}
$$

Next we want to show that

$$
\lim_{p\to1}\int_\Omega\varphi_jW^p(\nabla v_p^j)\mathrm{d}x\leq\int_\Omega\varphi_jW\left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}\right)\mathrm{d}|Du|+\int_{S_j\cap\left(\partial\omega\times\left(-\frac12,\frac12\right)\right)}\varphi_jW((u_0-u)\nu)\mathrm{d}\mathcal{H}^2, \tag{3.35}
$$

for $j=1,\dots,l$.

To this aim we start by fixing $j\in\{1,\dots,i\}$ and consider the set $C_j=B_j\setminus\Big(C_{x_j,(S_j\cap\partial\Omega)\setminus\partial\omega\times\left(-\frac12,\frac12\right)}\setminus A_j\Big)$. We may assume that C_j is an open set with piecewise C^1 boundary (the general case always reduce to this one by considering a set C_j^* with piecewise C^1 boundary such that $A_j\cup\left(\partial\omega\times\left(-\frac12,\frac12\right)\right)\subset C_j^*\subset C_j$, and apply the following argument to C_j^*). Thus, we extend u to C_j setting $u=u_0$ in $C_j\setminus A_j$. Clearly $u\in BV(C_j)$ and by (3.30) we can write

$$
\begin{aligned}
&\limsup_{p\to 1}\int_{C_j} W^p(\nabla v_p^j)\mathrm{d}x = \limsup_{p\to 1}\left[\int_{A_j} W^p(\nabla v_p^j)\mathrm{d}x + \int_{C_j\setminus A_j} W^p(\nabla u_0)\mathrm{d}x\right]\\
&\le \int_{A_j} W\left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}\right)\mathrm{d}|Du| + \int_{S_j\cap\left(\partial\omega\times\left(-\frac12,\frac12\right)\right)} W((u_0-u)\nu)\mathrm{d}\mathcal{H}^2 \qquad (3.36)\\
&\quad + \int_{C_j\setminus A_j} W(\nabla u_0)\mathrm{d}x = \int_{C_j} W\left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}\right)\mathrm{d}|Du|.
\end{aligned}
$$

The growth condition on W (3.8), the regularity of u_0, namely the fact that $u_0\in W^{1,\bar p}_{\mathrm{loc}}(\mathbb{R}^3)$, entail that the right hand side of (3.36) is finite.

On the other hand, well known lower semicontinuity results, and Hölder inequality provide

$$
\int_A W\left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}\right)\mathrm{d}|Du| \le \liminf_{p\to 1}\int_A W^p(\nabla v_p^j)\mathrm{d}x \qquad (3.37)
$$

for every open subset A of C_j. By (3.36) and (3.37) we can apply Proposition 2.7 to C_j for $\mu_p=\int_\cdot W^p(\nabla v_p^j)\mathrm{d}x$ and $\mu=\int_\cdot W\left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}\right)\mathrm{d}|Du|$ to obtain (3.35) for $j=1,\ldots,i$ as follows

$$
\begin{aligned}
&\lim_{p\to 1}\int_\Omega \varphi_j W^p(\nabla v_p^j)\mathrm{d}x = \lim_{p\to 1}\left[\int_{C_j}\varphi_j W^p(\nabla v_p^j)\mathrm{d}x - \int_{C_j\setminus A_j}\varphi_j W^p(\nabla u_0)\mathrm{d}x\right]\\
&= \int_{C_j} W\left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}\right)\mathrm{d}|Du| - \int_{C_j\setminus A_j} W(\nabla u_0)\mathrm{d}x\\
&= \int_{A_j} W\left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}\right)\mathrm{d}|Du| + \int_{S_j\cap\left(\partial\omega\times\left(-\frac12,\frac12\right)\right)} W((u_0-u)\nu)\mathrm{d}\mathcal{H}^2.
\end{aligned}
$$

Next, we fix $j\in\{i+1,\ldots,l\}$ and extend u to B_j setting $u=u_0$ in $B_j\setminus A_j$. Clearly $u\in BV(B_j)$ and we can reason as in the previous case taking B_j in place of C_j to obtain (3.35) for $j=i+1,\ldots,l$. Now, notice that for fixed $j\in\{l+1,\ldots,m\}$, (3.37) still holds for every subset A of $B_j\cap\partial\Omega$. Moreover by (3.32) we get that

$$
\lim_{p\to 1}\int_{B_j\cap\Omega} W^p(\nabla v_p^j)\mathrm{d}x = \int_{B_j\cap\Omega} W\left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}\right)\mathrm{d}|Du| < +\infty.
$$

Thus applying again Proposition 2.7 to $B_j\cap\Omega$ with $\mu_p=\int_\cdot W^p(\nabla v_p^j)\mathrm{d}x$ and $\mu=\int_\cdot W\left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}\right)\mathrm{d}|Du|$ we obtain

$$
\lim_{p\to 1}\int_\Omega \varphi_j W^p(\nabla v_p^j)\mathrm{d}x \le \int_\Omega \varphi_j W\left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}\right)\mathrm{d}|Du|, \qquad (3.38)
$$

for $j=l+1,\ldots,m$. Thus, putting together (3.33)$\div$ (3.35) and (3.38), since W is convex we conclude that

$$
\begin{aligned}
&\limsup_{p\to 1}\int_\Omega W^p(\nabla w_p)\mathrm{d}x \le \sum_{j=1}^m\left(\int_\Omega \varphi_j W\left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}\right)\mathrm{d}|Du| + \int_{\partial\omega\times\left(-\frac12,\frac12\right)}\varphi_j W((u_0-u)\nu)\mathrm{d}\mathcal{H}^2\right)\\
&= \int_\omega W\left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}\right)\mathrm{d}|Du| + \int_{\partial\omega\times\left(-\frac12,\frac12\right)} W((u-u_0)\nu)\mathrm{d}\mathcal{H}^2,
\end{aligned} \qquad (3.39)
$$

and this concludes the proof. □

Proof of Theorem 3.11. We start observing that Lemma 3.17 guarantees the existence of a sequence $\{u_p\}_p$ such that $u_p \in W^{1,p}_{u_0,\partial\omega\times\left(-\frac{1}{2},\frac{1}{2}\right)}(\Omega)$,

$$\lim_{p\to 1}\int_\Omega |u_p - u|^{1^*}\mathrm{d}x = 0,$$

and

$$\lim_{p\to 1}\int_\Omega W_\varepsilon^p(\nabla u_p)\mathrm{d}x = \int_\Omega W_\varepsilon\left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}\right)\mathrm{d}|Du| + \int_{\partial\omega\times\left(-\frac{1}{2},\frac{1}{2}\right)} W_\varepsilon((u-u_0)\nu)\mathrm{d}\mathcal{H}^2.$$

These prove the upper bound. For what concerns the lower bound it is enough to invoke Lemma 3.13. This concludes the proof. □

Remark 3.18. We observe that in Theorem 3.11 and in the preliminary lemmata, we have chosen a function W positively 1–homogeneous, having in mind the applications to the $-\Delta_1$ type equations, but the Γ–convergence results hold similarly without this assumption, introducing the recession function W^∞ in the integrals dealing with the singular part of Du.

3.4. Summary of the results

Let $\omega \subset \mathbb{R}^2$ be a bounded open set, with piecewise C^1 boundary, and let $u_0 \in W^{1-\frac{1}{\bar{p}},\bar{p}}(\partial\omega)$, for some $\bar{p} > 1$; recall the families of problems $\{\mathcal{P}_{p,\varepsilon}\}_{p,\varepsilon}$, $\{\mathcal{P}_{1,\varepsilon}\}_\varepsilon$, $\{\mathcal{P}_{p,0}\}_p$ and $\mathcal{P}_{1,0}$ in (1.5), (1.12), (1.10) and (1.16), respectively.

As a consequence of the above results we obtain that the dimensional reduction, *i.e.* the asymptotics as $\varepsilon \to 0$, and the so-called power law approximation, namely the convergence as $p \to 1$, commute in the sense of Γ-convergence with respect to $L^1(\Omega)$–strong convergence, as summarized by the following diagram:

$$\begin{tikzcd}
\mathcal{P}_{p,\varepsilon} \arrow[r, "p\,\longrightarrow\,1"'] \arrow[d, "\varepsilon\,\longrightarrow\,0"] & \mathcal{P}_{1,\varepsilon} \arrow[d, "\varepsilon\,\longrightarrow\,0"] \\
\mathcal{P}_{p,0} \arrow[r, "p\,\longrightarrow\,1"'] & \mathcal{P}_{1,0}
\end{tikzcd}$$

Indeed, the left vertical arrow is a consequence of Remark 3.2, the right vertical arrow has been proven in Proposition 3.4, the upper horizontal arrow has been proved in Theorem 3.11 while the lower horizontal arrow follows from Theorem 3.8.

Other types of analysis of solutions to problems $\mathcal{P}_{p,\varepsilon}$ as $p \to 1$ and $\varepsilon \to 0$ will be discussed in the following sections.

In the following remark we point out a result which turns out to be a byproduct of our previous Γ–convergence analysis.

Remark 3.19. Let $\omega \subset \mathbb{R}^2$ be a bounded open set, piecewise C^1, with $\mathcal{L}^2(\omega) = 1$ for convenience, let $\Omega := \omega \times \left(-\frac{1}{2},\frac{1}{2}\right)$, let $W : \mathbb{R}^3 \to [0,+\infty[$ be a continuous function, positively 1–homogeneous and verifying (3.8). Fix $\overline{p} > 1$, and let $u_0 \in W^{1,\overline{p}}_{\mathrm{loc}}(\mathbb{R}^2)$. For every $1 < p \le \overline{p}$ we can define the functionals

$$\mathcal{F}_p(u) = \begin{cases} \|W(\nabla u)\|_{L^p(\Omega)} & \text{if } u \in W^{1,p}_{u_0,\partial\omega\times\left(-\frac{1}{2},\frac{1}{2}\right)}(\Omega), \\ +\infty & \text{otherwise in } BV(\Omega). \end{cases}$$

It is easily verified that $\mathrm{dom}(\mathcal{F}_p) \supset \mathrm{dom}(\mathcal{F}_q)$ whenever $1 < p < q$ and if $u \in \mathrm{dom}(\mathcal{F}_q)$ then $\mathcal{F}_p(u) \le \mathcal{F}_q(u)$.

Let $\mathcal{F} : BV(\Omega) \to [0, +\infty]$ be the functional defined as

$$\mathcal{F}(u) = \begin{cases} \|W(\nabla u)\|_{L^1(\Omega)} & \text{if } u \in \bigcup_{p>1} W^{1,p}_{u_0, \partial\omega \times \left(-\frac{1}{2}, \frac{1}{2}\right)}(\Omega), \\ \\ +\infty & \text{if } u \in BV(\Omega) \setminus \bigcup_{p>1} W^{1,p}_{u_0, \partial\omega \times \left(-\frac{1}{2}, \frac{1}{2}\right)}(\Omega). \end{cases}$$

The monotonicity of $\{\mathcal{F}_p\}_p$ provides pointwise convergence as $p \to 1$ of $\mathcal{F}_p(u)$ to $\mathcal{F}(u)$, for every $u \in BV(\Omega)$. On the other hand it is easy to verify that $\mathcal{F}$ is not lower semicontinuous with respect to $L^1(\Omega)$ strong convergence. Thus [10], Proposition 5.7 ensures Γ–convergence, with respect to $L^1(\Omega)$ strong convergence, of $\mathcal{F}_p$, as $p \to 1$, to the lower semicontinuous envelope of $\mathcal{F}$, denoted by $\overline{\mathcal{F}}$. On the other hand Lemma 3.13 and Lemma 3.17 guarantee that $\{\mathcal{F}_p\}_p$ Γ – converges, with respect to $L^1(\Omega)$ strong convergence, as $p \to 1$, to the functional

$$\mathcal{F}_1(u) = \int_\Omega W(\nabla u)\mathrm{d}x + \int_\Omega W\left(\frac{\mathrm{d}D^s u}{\mathrm{d}|D^s u|}\right)\mathrm{d}|D^s u| + \int_{\partial\omega \times \left(-\frac{1}{2}, \frac{1}{2}\right)} W((u - u_0)\nu)\mathrm{d}\mathcal{H}^2$$

for every $u \in BV(\Omega)$. Consequently we have proven that $\overline{\mathcal{F}}(u) = \mathcal{F}_1(u)$ for every $u \in BV(\Omega)$.

4. The case $p = 1$ in terms of differential problems

The aim of this section is to provide another view to the limit problems of (1.6) and (1.11) as $p \to 1$, by means of duality. As it is well known that (1.6) and (1.11) represent the Euler–Lagrange equations associated to (1.5) and (1.10) respectively, in the sequel we state some results which allow us to regard the limiting equations (4.1) and (4.2) as the counterparts in duality of the limit functionals (3.2) and (3.7) respectively, achieved *via* Γ–convergence in Section 3.

Formally, putting $p = 1$ in (1.6) and (1.11), one obtains

$$\begin{cases} -\Delta_{1,\varepsilon} u = -\mathrm{div}\left(|Id_\varepsilon \nabla u \cdot \nabla u|^{\frac{-1}{2}} Id_\varepsilon \nabla u\right) = 0 & \text{in } \omega, \\ \\ u = u_0 & \text{on } \partial\omega \times \left(-\frac{1}{2}, \frac{1}{2}\right), \\ \\ |Id_\varepsilon \nabla u \cdot \nabla u|^{\frac{-1}{2}} (Id_\varepsilon \nabla u) \cdot \nu = 0 & \text{on } \omega \times \left\{-\frac{1}{2}, \frac{1}{2}\right\} \end{cases} \tag{4.1}$$

where Id_ε has been defined in (1.7), and

$$\begin{cases} -\Delta_{1,0} u = -\mathrm{div}\left(\dfrac{\nabla u}{|\nabla u|}\right) = 0 & \text{in } \omega, \\ \\ u = u_0 & \text{on } \partial\omega. \end{cases} \tag{4.2}$$

Clearly the above equations are meaningless in $W^{1,1}(\omega)$. In order to deal with problems (1.15) and (1.16) in terms of PDE's it is useful to approach them using the theory developed by Ekeland and Temam in the context of variational problems (see [17]).

The following proposition is stated in [25], Proposition 1.1 and, with the purpose of applications to 1–Laplace equations quoted also in [13–15]. A proof can be found in [25], Theorem 3.2 in the context of Hencky's Plasticity theory, *cf.* also [5], Section 4 for a proof in the scalar case.

Proposition 4.1. *Let $O \subset \mathbb{R}^N$ be an open set. Suppose that $u \in BV(O)$ and $\sigma \in L^\infty(O;\mathbb{R}^N)$ is such that* $\mathrm{div}\sigma \in L^N(O)$*. One defines the distribution $\sigma \cdot Du$ by the following*

1. *For every $\varphi \in C_0^\infty(O)$*
$$< \sigma \cdot Du, \varphi >= -\int_O \mathrm{div}(\sigma) u\varphi \mathrm{d}x - \int_O \sigma \cdot (D\varphi) u \mathrm{d}x.$$
Then, the distribution $\sigma \cdot Du$ hence defined is a bounded measure in O, absolutely continuous with respect to $|Du|$, with
$$|\sigma \cdot Du| \leq |Du||\sigma|_\infty. \tag{4.3}$$
2. *Suppose that O is piecewise C^1. The following generalized Green's formula holds for $\varphi \in C_0^\infty(\mathbb{R}^N)$*
$$< \sigma \cdot Du, \varphi >= -\int_O \mathrm{div}(\sigma) u\varphi \mathrm{d}x - \int_O \sigma \cdot D\varphi u \mathrm{d}x + \int_{\partial O} \sigma \cdot \nu u \varphi \mathrm{d}\mathcal{H}^{N-1}, \tag{4.4}$$
where ν denotes the unit outer normal to ∂O and $\mathcal{H}^{N-1}$ the $N-1$ dimensional Hausdorff measure.

By virtue of Proposition 4.1 applied to $O = \omega$, with $u_0 \in W^{1-\frac{1}{\overline{p}},\overline{p}}(\partial\omega)$ for a suitable $\overline{p} > 1$ one may consider the following equation which provides a rigorous meaning to (4.2).

$$\begin{cases} -\mathrm{div}\sigma = 0, & \text{in } \omega, \\ \sigma \cdot Du = |Du| & \text{in } \omega, \\ \sigma \cdot \nu(u - u_0) = |u - u_0| & \text{on } \partial\omega. \end{cases} \tag{4.5}$$

Applying again Proposition 4.1 to $O = \omega \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ we can give a meaning to the anisotropic $-\Delta_{1,\varepsilon}$ operator appearing in dimension reduction, and we can also consider it as the "Euler–Lagrange equation" associated to (1.15).

$$\begin{cases} -\mathrm{div}\sigma_\varepsilon = 0 & \text{in } \omega \times \left(-\frac{1}{2}, \frac{1}{2}\right), \\ \sigma_\varepsilon \cdot \nabla u = |Id_\varepsilon \nabla u \cdot \nabla u|^{\frac{1}{2}} & \text{in } \omega \times \left(-\frac{1}{2}, \frac{1}{2}\right), \\ \sigma_\varepsilon \cdot \nu(u - u_0) = |u - u_0| & \text{on } \partial\omega \times \left(-\frac{1}{2}, \frac{1}{2}\right), \\ \sigma_\varepsilon \cdot \nu = 0 & \text{on } \omega \times \left\{-\frac{1}{2}, \frac{1}{2}\right\}, \end{cases} \tag{4.6}$$

where ν represents the unit outer normal vector to $\partial\omega \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ and Id_ε is as in (1.7).

Via the duality theory the solutions to (4.5) and (4.6) are in correspondence with the minimizers of $\mathcal{P}_{1,\varepsilon}$ in (1.15) and $\mathcal{P}_{1,0}$ in (1.16), according to the regularity assumptions on u_0.

In fact we can invoke Theorem 2.9 and apply it to (4.5) and (4.6). Namely, having in mind the notations of Theorem 2.9 in the first case we can set $X = W^{1,1}(\omega)$ and $Y = (L^1(\omega))^2$, the linear operator Λ maps $u \in X$ to $\nabla u \in Y$, G and F are defined as

$$G(\mathbf{p}) = \int_\omega \left(\sum_{i=1}^2 p_i^2\right)^{\frac{1}{2}} \mathrm{d}x_1 \mathrm{d}x_2,$$

with $\mathbf{p} = (p_1, p_2)$

$$F(u) = \begin{cases} 0 & \text{if } u \equiv u_0 \text{ in } \partial\omega, \\ +\infty & \text{otherwise,} \end{cases}$$

where the equality is intended, as usual, in the sense of traces, recalling that $u_0 \in W^{1-\frac{1}{\bar p},\bar p}(\partial\omega)$, for some $\bar p > 1$.

Thus it easily checked that the dual Problem of $\mathcal{P}_{1,0}$ is

$$\mathcal{D}_0 = \sup_{\substack{\sigma \in L^\infty(\omega;\mathbb{R}^3),\\ \mathrm{div}\sigma = 0, |\sigma| \le 1}} \left\{ -\int_{\partial\omega} \sigma \cdot \nu u_0 \mathrm{d}\mathcal{H}^1 \right\}, \tag{4.7}$$

where in fact σ is exactly as in (4.5).

Analogously in the ε–dependent case, by assuming $X = W^{1,1}\left(\omega \times \left(-\frac{1}{2},\frac{1}{2}\right)\right)$ and $Y = (L^1\left(\omega \times \left(-\frac{1}{2},\frac{1}{2}\right)\right))^3$ and $\Lambda : u \in X \to \left(\nabla_\alpha u \middle| \frac{1}{\varepsilon}\nabla_3 u\right) \in Y$, let G be given by

$$G(\mathbf{p}) = \int_\Omega \left(\sum_{i=1}^3 p_i^2\right)^{\frac{1}{2}} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3,$$

(where we kept track of the factor $\frac{1}{\varepsilon}$ in the space of admissible functions rather than in the integrand), and

$$F(u) = \begin{cases} 0 & \text{if } u \equiv u_0 \text{ in } \partial\omega \times \left(-\frac{1}{2},\frac{1}{2}\right), \\ +\infty & \text{otherwise.} \end{cases}$$

The dual problem becomes

$$\mathcal{D}_{1,\varepsilon} = \sup_{\substack{\sigma_\varepsilon \in L^\infty(\Omega;\mathbb{R}^3),\\ \mathrm{div}\sigma_\varepsilon = 0, \left|Id_{\frac{1}{\varepsilon}}\sigma_\varepsilon\right| \le 1,\\ \sigma_\varepsilon\cdot\nu = 0 \text{ on } \omega\times\left\{-\frac{1}{2},\frac{1}{2}\right\}}} \left\{ -\int_{\partial\omega\times\left(-\frac{1}{2},\frac{1}{2}\right)} \sigma_\varepsilon \cdot \nu u_0 \mathrm{d}\mathcal{H}^2 \right\}.$$

Remark 4.2. We observe that the application of Theorem 2.9 entails the existence of the solution only to the dual problems, related to anisotropic almost 1–Laplacian and almost 1–Laplacian, namely to (4.5) and (4.6). On the other hand the regularity of u_0, namely the fact that it is in some suitable trace space, guarantees the application of our Γ–convergence results, Theorem 3.8 and 3.11. On the other hand by virtue of Theorem 2.6, the same arguments exploited to exhibit the recovery sequence for the upper bound in Theorems 3.8 and 3.11 guarantee the convergence of the minimizers at p–level of $\mathcal{P}_{p,0}$ and $\mathcal{P}_{p,\varepsilon}$ (that exist for convexity reasons) to the minimum points in the original problems $\mathcal{P}_{1,0}$ and $\mathcal{P}_{1,\varepsilon}$ respectively as $p \to 1$, in spite of the lack of coerciveness of $I_{1,0}$ and $I_{1,\varepsilon}$.

A direct proof of existence of minimizers to $\mathcal{P}_{1,0}$ and $\mathcal{P}_{1,\varepsilon}$ will be provided in the last section.

The relations between the extremal points in the dual problems $\mathcal{P}_{1,0}$ and $\mathcal{D}_{1,0}$ are stated in Proposition 4.3, while the relations between $\mathcal{P}_{1,\varepsilon}$ and $\mathcal{D}_{1,\varepsilon}$ are stated in 4.4. We omit the proofs of these results for the sake of brevity.

Proposition 4.3. *Suppose that $u \in BV(\omega)$, and $\sigma \in L^\infty(\omega;\mathbb{R}^3)$, with $\mathrm{div}\sigma = 0$ and $|\sigma| \le 1$ a.e. in ω. Then u and σ are extremal for $\mathcal{P}_{1,0}$ and $\mathcal{D}_0$, respectively if and only if*

$$-\sigma \cdot Du = |Du| \textit{ as measures on } \omega, \tag{4.8}$$

and

$$\sigma \cdot \nu = \frac{u-u_0}{|u-u_0|} \textit{ on } \partial\omega \cap \{u \ne u_0\}. \tag{4.9}$$

Proposition 4.4. *Suppose that $u \in BV\left(\omega \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right)$, and $\sigma \in L^\infty\left(\omega \times \left(-\frac{1}{2}, \frac{1}{2}\right); \mathbb{R}^3\right)$, with $\mathrm{div}\sigma = 0$ and $\left|Id_{\frac{1}{\varepsilon}}\sigma\right| \leq 1$ a.e. in $\omega \times \left(-\frac{1}{2}, \frac{1}{2}\right)$. Then u and σ are extremal for $\mathcal{P}_{1,\varepsilon}$ and $\mathcal{D}_{1,\varepsilon}$, respectively if and only if*

$$-\sigma \cdot Du = |Id_\varepsilon Du \cdot Du|^{\frac{1}{2}} \text{ as measures on } \omega \times \left(-\frac{1}{2}, \frac{1}{2}\right),$$

and

$$\sigma \cdot \nu = \frac{u - u_0}{|u - u_0|} \text{ on } \partial\omega \times \left(-\frac{1}{2}, \frac{1}{2}\right) \cap \{u \neq u_0\},$$

and

$$\sigma \cdot \nu = 0 \text{ on } \omega \times \left\{-\frac{1}{2}, \frac{1}{2}\right\}.$$

5. Asymptotics in terms of least gradient problem

The target of this section consists of discussing asymptotics as $\varepsilon \to 0$ and $p \to 1$ for problems (1.6) when the imposed boundary datum has a regularity, in principle different from that required in the previous Γ–convergence analysis, but a more stringent requirement is imposed on the domain $\omega \times \left(-\frac{1}{2}, \frac{1}{2}\right)$. Under this new set of assumptions we will prove that the problems $\mathcal{P}_{1,\varepsilon}$ in (1.15) and $\mathcal{P}_{1,0}$ in (1.16) indeed admit a solution. Consequently in the light of Propositions 4.3 and 4.4, there exist solutions to the anisotropic almost 1–Laplacian and almost 1–Laplacian in (4.1) and (4.2), respectively, when both the assumptions introduced in Section 3 and Section 5 are imposed. We recall that the symbols for the domains Ω and ω denote the same sets as in (1.3), namely $\omega \subset \mathbb{R}^2$ is a bounded open set and $\Omega = \omega \times \left(-\frac{1}{2}, \frac{1}{2}\right)\cdot$

As already observed in the Introduction there is equivalence, in the sense of "Euler–Lagrange", between problems (1.6) and (1.11) and their variational formulation (1.5) and (1.10) respectively when $p > 1$ and the boundary datum u_0 is in a suitable trace space. Analogously, with the same regularity assumptions on u_0, and *via* the duality argument invoked in Section 4, there is 'equivalence' between (4.6) ((4.5) respectively) and the minimum problems (1.15) ((1.16) respectively). This fact may be no longer true if one requires u_0 to be a continuous function on $\partial\omega$, *cf.* [23].

On the other hand, as already emphasized, the problems $\mathcal{P}_{p,\varepsilon}$ and $\mathcal{P}_{p,0}$ may exhibit other behaviors when $p = 1$, and the integral formulation can be understood in different ways. Besides the duality approach quoted in Section 4 in the present section we aim to make a link in terms of least gradient functions, which will allow us to determine other sufficient conditions for the existence of solutions to $\mathcal{P}_{1,0}$ and $\mathcal{P}_{1,\varepsilon}$.

We start by focusing on the case $p > 1$ and $\varepsilon = 0$, and we recall the definition of *p–harmonic* functions following [23], Definition 2.2, namely local weak solutions of (1.11), when $u_0 \in C(\partial\omega)$. First we give this definition on any generic open set $O \subset \mathbb{R}^n$. Then we formulate the least gradient problems, *cf.* (5.1) and (5.2). We recall the results available in literature in which there have been provided sufficient conditions for the existence and uniqueness of solutions to the least gradient (*cf.* [29]). Moreover we recall the approximation result due to Juutinen, where p–harmonic functions approach locally uniformly functions of least gradient. Essentially these latter results represent another asymptotic analysis as $p \to 1$ for problems $\{\mathcal{P}_{p,0}\}$. Then, for what concerns the asymptotics as $p \to 1$ of problems $\{\mathcal{P}_{p,\varepsilon}\}$ we prove explicitly in Lemma 5.6 a uniqueness result for p–harmonic functions $u_{p,\varepsilon}$ in cylindrical domains of the type $\omega \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ with mixed boundary conditions: Neumann conditions on the basis and Dirichlet ones on the lateral boundary. Finally in Theorem 5.7 we prove that these latter functions $u_{p,\varepsilon}$ are not dependent on x_3, thus p–harmonic in ω. We deduce also the trivial limiting behavior of $u_{p,\varepsilon}$ as $\varepsilon \to 0$ to a function of least gradient in ω, *i.e.* still independent on x_3.

Definition 5.1. Let $1 < p < \infty$, a continuous function $u \in W^{1,p}_{\mathrm{loc}}(O)$ is *p–harmonic* in O if

$$\int_O |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \mathrm{d}x = 0$$

for every $\varphi \in C_0^\infty(O)$.

The continuity in Definition 5.1 is redundant as shown in [23].

It is useful also to recall (see [23]) that a continuous function $u \in W^{1,p}_{\mathrm{loc}}(O)$ is $p-$*harmonic* in O if and only if

$$\int_{O_0} |\nabla u|^p \mathrm{d}x \leq \int_{O_0} |\nabla v|^p \mathrm{d}x \text{ whenever } O_0 \text{ open set } \subset\subset O \text{ and } u - v \in W^{1,p}_0(O_0).$$

Now we recall some results deeply connected with $-\Delta_1$, problem (1.16) and its approximating ones (1.5) (as $p \to 1$ and $\varepsilon \to 0$). The analysis we present will be mainly concerned with differential problems defined in the cross section ω, when the boundary datum u_0 is regular. To this end we will recall the notion of functions of least gradient in a generic open set $O \subset \mathbb{R}^n$.

Let $O \subset \mathbb{R}^n$ be an open set, following [29], we say that a function $u \in BV(O)$, with prescribed boundary value $u_0 \in C(\partial O)$ is of least gradient if it is a solution of

$$\inf_{u \in BV(O)} \{|Du|(O), u \equiv u_0 \text{ on } \partial O\}. \tag{5.1}$$

It has been established in [30] that the existence of such a function is deeply related with the regularity of O, the regularity of the trace u_0 and the sense in which this trace must be understood, indeed this latter fact plays a crucial role.

In fact one may also consider

$$\inf_{u \in BV(O) \cap C(\overline{O})} \{|Du|(O), u \equiv u_0 \text{ on } \partial O\}. \tag{5.2}$$

Clearly in this latter problem the trace is intended in the classical sense (restriction), and the equality $u = u_0$ is understood pointwise in ∂O. On the contrary in (5.1) the equality $u = u_0$ on ∂O has to be taken in the sense of traces for $BV-$functions (see Sect. 2.4).

The following result has been proven in [29].

Theorem 5.2. *Let $O \subset \mathbb{R}^n$ be a bounded Lipschitz open domain such that ∂O has non-negative mean curvature (in a weak sense) and is not locally area-minimizing. If $u_0 \in C(\partial O)$, then there exists a unique function of least gradient $u \in BV(O) \cap C(\overline{O})$ such that $u \equiv u_0$ on ∂O, namely u is the unique solution of* (5.2).

The assumptions in Theorem 5.2 read as

- For every $x \in \partial O$ there exists $\varepsilon_0 > 0$ such that for every set of finite perimeter $A \subset\subset B(x, \varepsilon_0)$

$$P(O; \mathbb{R}^n) \leq P(O \cup A; \mathbb{R}^n) \tag{5.3}$$

- For every $x \in \partial O$, and every $\eta > 0$ there exists a set of finite perimeter $A \subset\subset B(x, \eta)$ such that

$$P(O, B(x, \eta)) > P(O \setminus A, B(x, \eta)), \tag{5.4}$$

where $P(\cdot; \mathbb{R}^n)$ denotes the perimeter in $\mathbb{R}^n$. Examples showing that neither (5.3) nor (5.4) can be dropped are given in [29].

On the other hand in [30], (to which we refer for the precise assumptions) it has been established the following result.

Theorem 5.3. *Let $O \subset \mathbb{R}^n$ be a bounded Lipschitz open domain satisfying, the same assumptions of Theorem 5.2, namely* (5.3) *and* (5.4). *Assume also that a uniform interior ball condition of radius R holds. Then there is at most one solution to the least gradient problem (5.1).*

Combining both the assumptions in Theorems 5.2 and 5.3, and observing that any solution of (5.2) (which, in this setting, exists and is unique) solves also (5.1) (which, in turn, with these hypotheses admits one solution), we conclude that the solutions of problems (5.1) and (5.2) are unique and coincide.

In order to deal with the asymptotics as $p \to 1$ of the $-\Delta_p$- equations, Juutinen in [23], Theorem 3.1 has proven the following theorem (*cf.* also Rem. 3.4 therein).

Theorem 5.4. *Let $O \subset \mathbb{R}^n$ be a bounded smooth open domain whose boundary has positive mean curvature and $u_0 \in C(\partial O)$, and let $u \in BV(O) \cap C(\overline{O})$ be the unique function of least gradient such that $u = u_0$ on ∂O. Then if $u_p \in W^{1,p}_{\mathrm{loc}}(O) \cap C(\overline{O})$ is the unique p–harmonic function satisfying $u_p = u_0$ on ∂O, it results*

$$u_p \to u \ \text{ locally uniformly in } O, \ \text{ as } p \to 1.$$

Remark 5.5. We recall that the existence and uniqueness of the solution u_p mentioned in Theorem 5.4 do not rely on 'classical' Calculus of Variations arguments, since the boundary datum u_0 may not be the trace of a Sobolev function. Namely for a generic open set O when $u_0 \in C(\partial O)$, one cannot conclude that $u_0 \in W^{1-\frac{1}{\overline{p}},\overline{p}}(\partial O)$ for some $\overline{p} \geq p > 1$ and thus the existence of a p–harmonic function cannot be deduced by minimizing an integral functional of the type (1.4). The exploited techniques in [23] to have a unique p–harmonic function u_p with boundary datum u_0 are those suitably employed in the context of Nonlinear PDEs, *cf.* [22]. On the other hand we underline the fact that if $p \to 1$, namely it is $1 < p < 2$, and O is as in Theorem 5.4 the unique p–harmonic function $u_p \in W^{1,p}(O)$ with boundary datum u_0 (for instance in $C^{1,1}(\partial O)$), solves $\{P\}_{p,0}$ and converges to the solution of the least gradient problem.

In the next lemma, by means of duality products in terms of Lions–Magenes spaces and trace spaces, exploiting the monotonicity of the p–laplacian operator (see Eq. (3.8) in [22]) and arguing as in [22], Theorem 3.17 and Lemma 3.18 we will deduce the following uniqueness result.

First we recall that for $p > 1$ if $u \in W^{1,p}(O)$, then $\Delta_p u \in [W^{1,p}(O)]^*$ (the dual space of $W^{1,p}(O)$) and $|\nabla u|^{p-2}\nabla u$ has a normal trace, denoted by $|\nabla u|^{p-2}\nabla u \cdot \nu$, (where ν denotes the normal to ∂O), such that

$$|\nabla u|^{p-2}\nabla u \cdot \nu \in W^{-\frac{1}{p'},p'}(\partial O) := [W^{\frac{1}{p'},p'}(\partial O)]^*,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover if $\Gamma_1 \subset \partial O$ is open in the relative topology, then the restriction of $|\nabla u|^{p-2}\nabla u \cdot \nu$ to Γ_1, denoted by $|\nabla u|^{p-2}\nabla u \cdot \nabla u|_{\Gamma_1}$ satisfies

$$|\nabla u|^{p-2}\nabla u \cdot \nu|_{\Gamma_1} \in W^{-\frac{1}{p'},p'}(\Gamma_1) := [W^{\frac{1}{p'},p}_{00}(\Gamma_1)]^*,$$

where $W^{\frac{1}{p'},p}_{00}(\Gamma_1)$ is the Lions–Magenes space of all functions $u \in L^p(\Gamma_1)$ whose extension by 0 on $\partial O \setminus \Gamma_1$ belongs to $W^{\frac{1}{p'},p}(\partial O)$.

Lemma 5.6. *Let O be a smooth open domain with $\partial O = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$. Let $u, v \in W^{1,p}(O)$ satisfy*

$$-\Delta_p u \geq -\Delta_p v$$

in O, in the sense of $[W^{1,p}(O)]^*$,

$$|\nabla u|^{p-2}\nabla u \cdot \nu \geq |\nabla v|^{p-2}\nabla v \cdot \nu$$

on Γ_1, in the sense of $W^{-\frac{1}{p'},p'}(\Gamma_1)$ and

$$u \geq v$$

on $\mathcal{H}^{N-1}$ a.e. on Γ_2, in the sense of traces. Then $u \geq v$ in $\overline{O}$.

Proof. In order to prove that $u \geq v$ in $\overline{O}$ it is enough to show that

$$(u - v)^- \equiv 0,$$

where for any couple of functions f and g, $(f - g)^- := \min(f - g, 0)$.

To this purpose, given $\varepsilon > 0$, we observe that $(u+\varepsilon-v)^-$ belongs to $W^{1,p}(O)$ and its trace on Γ_2 is 0 $\mathcal{H}^{N-1}$ almost everywhere. Thus $(u+\varepsilon+v)^-|_{\Gamma_1} \in W_{00}^{-\frac{1}{p'},p}(\Gamma_1)$ and thus we can take it as a non positive test function in the inequality

$$-(\Delta_p u - \Delta_p v) \geq 0, \text{ in } [W^{1,p}(O)]^*$$

namely we deduce that

$$< -(\Delta_p u - \Delta_p v), (u+\varepsilon-v)^- >_{|[W^{1,p}(O)]^*, W^{1,p}(O)} \leq 0.$$

Then by the definition of the normal trace we obtain

$$\begin{aligned} &\int_O \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right) \nabla (u+\varepsilon-v)^- \,\mathrm{d}x - \\ &< \left(|\nabla u|^{p-2}\nabla u\cdot\nu - |\nabla v|^{p-2}\nabla v\cdot\nu\right), (u+\varepsilon-v)^- >_{W^{\frac{-1}{p'},p'}(\partial O), W^{\frac{1}{p'},p}(\partial O)} \leq 0. \end{aligned} \tag{5.5}$$

Recall that

$$\begin{aligned} &\int_O \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right) \nabla (u+\varepsilon-v)^- \,\mathrm{d}x \\ &= \int_{O\cap\{u+\varepsilon<v\}} \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right)\cdot(\nabla u - \nabla v)\,\mathrm{d}x \\ &= \int_{O\cap\{u+\varepsilon<v\}} |\nabla u|^{p-2}\nabla u\cdot(\nabla u-\nabla v)\,\mathrm{d}x - \int_{O\cap\{u+\varepsilon<v\}} |\nabla v|^{p-2}\nabla v\cdot(\nabla u-\nabla v)\,\mathrm{d}x. \end{aligned}$$

We use the following inequality (*cf.* [27], Lem. 4.2)

$$|x_2|^p - |x_1|^p \geq p|x_1|^{p-2}x_1\cdot(x_2-x_1) + c(p)\frac{|x_2-x_1|^p}{\left(|x_1|+|x_2|\right)^{2-p}}$$

and we get

$$|\nabla u|^{p-2}\nabla u\cdot(\nabla u-\nabla v) \geq \frac{1}{p}\left[|\nabla u|^p - |\nabla v|^p + c(p)\frac{|\nabla u-\nabla v|^p}{\left(|\nabla u|+|\nabla v|\right)^{2-p}}\right]$$

a.e. in O,

$$-|\nabla v|^{p-2}\nabla v\cdot(\nabla u-\nabla v) \geq \frac{1}{p}\left[|\nabla v|^p - |\nabla u|^p + c(p)\frac{|\nabla u-\nabla v|^p}{\left(|\nabla u|+|\nabla v|\right)^{2-p}}\right]$$

a.e. in O.
Hence

$$\left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right)\cdot(\nabla u-\nabla v) \geq \frac{2}{p}c(p)\frac{|\nabla u-\nabla v|^p}{\left(|\nabla u|+|\nabla v|\right)^{2-p}} > 0,$$

a.e. in O, when $\nabla u \neq \nabla v$, *i.e.* the monotonicity of the p–laplacian.

This proves the positivity of the first term in (5.5). For the second term, since $(u+\varepsilon+v)^-|_{\Gamma_1} \in W_{00}^{\frac{1}{p'},p}(\Gamma_1)$ we deduce that

$$< \left(|\nabla u|^{p-2}\nabla u \cdot \nu - |\nabla v|^{p-2}\nabla v \cdot \nu\right), (u+\varepsilon-v)^- >_{W^{-\frac{1}{p'},p'}(\partial O), W^{\frac{1}{p'},p}(\partial O)}$$

$$= < \left(|\nabla u|^{p-2}\nabla u \cdot \nu|_{\Gamma_1} - |\nabla v|^{p-2}\nabla v \cdot \nu|_{\Gamma_1}\right), (u+\varepsilon-v)^-_{\Gamma_1} >_{W^{-\frac{1}{p'},p'}(\Gamma_1), W_{00}^{-\frac{1}{p'},p}(\Gamma_1)} \leq 0.$$

We know that the function on Γ_1 is non positive since $(u+\varepsilon-v)^- \leq 0$, while the second term is zero. Then the left hand side of (5.5) is strictly positive. Again the first term in (5.5) turns out both ≥ 0 and ≤ 0. This together with the monotonicity of the p–laplacian, ensures that $\nabla(u+\varepsilon-v) \equiv 0$, so $v = u + C$ in the set $\{u+\varepsilon < v\}$. Hence $v \leq u+\varepsilon$ for any ε, and then $u \geq v$. □

We can now prove the following result.

Theorem 5.7. *Let $\Omega := \omega \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ and assume that $\omega \subset \mathbb{R}^2$ is a bounded smooth open domain whose boundary has positive mean curvature, and let $u_0 \in C(\partial\omega)$. Then the unique weak solutions of (1.6) u_p, in the sense that they are in $C(\overline{\Omega}) \cap W^{1,p}_{\mathrm{loc}}(\Omega)$ and*

$$\int_\Omega \left(|Id_\varepsilon \nabla u_p \cdot \nabla u_p|^{\frac{p-2}{2}}\ Id_\varepsilon \nabla u_p\right) \cdot \nabla\varphi \mathrm{d}x = 0$$

for every $\varphi \in C_0^\infty(\Omega)$, for every $\varepsilon > 0$, are also $p-$harmonic functions referred to (1.8) and (1.11), and thus independent on x_3. Moreover they converge locally uniformly as $p \to 1$ to the unique function of least gradient in ω with datum u_0.

Proof. For $p > 1$, the existence and uniqueness of $p-$*harmonic* solutions (independent on x_3) to (1.8) and (1.11) can be deduced as already observed in Theorem 5.4. For $p = 1$, we observe that Theorem 5.2 applied to ω ensures that there exists a unique function u of least gradient with datum u_0. Moreover again Theorem 5.4 and [23], Remark 3.4 provide the locally uniform convergence of the above u_p to this solution u. To conclude the proof it remains to show that u_p are also unique among the functions in $W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$. This latter fact follows from the lemma 5.6 applied to the unrescaled domain Ω_ε and the well-posedness of the problem (*cf.* [22,23]). □

Now we can introduce the least gradient problem in the thin domain, taking into account the rescaling in (1.3)

$$\inf_{u \in BV(\Omega)} \left\{ \left| \left(D_\alpha u \Big| \frac{1}{\varepsilon} D_3 u \right) \right| (\Omega), u \equiv u_0 \text{ on } \partial\omega \times \left(-\frac{1}{2}, \frac{1}{2}\right) \right\}, \tag{5.6}$$

and its version on the class $BV(\Omega) \cap C(\overline{\Omega})$,

$$\inf_{u \in BV(\Omega) \cap C(\overline{\Omega})} \left\{ \left| \left(D_\alpha u \Big| \frac{1}{\varepsilon} D_3 u \right) \right| (\Omega), u \equiv u_0 \text{ on } \partial\omega \times \left(-\frac{1}{2}, \frac{1}{2}\right) \right\}. \tag{5.7}$$

In order to provide sufficient conditions ensuring that both problems (5.6) and (5.7) admit a unique solution, we prove the following theorem.

Theorem 5.8. *Let $\Omega : \omega \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ with $\omega \subset \mathbb{R}^2$ a bounded open domain, piecewise C^1, and verifying (5.3), (5.4) and a uniform interior ball condition as in Theorem 5.3. Let $u_0 \in C^{1,1}(\partial\omega)$. Then problems (5.6) and (5.7) admit a unique coincident solution, independent on x_3, obtained as limit for $p \to 1$ in $L^1(\Omega)$–strong topology and locally uniformly in Ω of $\{u_{p,\varepsilon}\}$, where the latter is the unique solution of (1.6).*

We observe that in the statement of this theorem we denoted the unique solution of (1.6) by $u_{p,\varepsilon}$ while in Theorem 5.7 we denoted it simply by u_p. The present choice is due to the fact that we want to stress the fact that $u_{p,\varepsilon} = u_p$ and it solves (1.6) for every $\varepsilon > 0$. We emphasize that Theorem 5.8 holds whenever $u_0 \in C(\partial\omega) \cap W^{1-\frac{1}{\overline{p}},\overline{p}}(\partial\omega)$ for $\overline{p} \geq p > 1$, while if $u_0 \in C(\partial\omega)$ it is possible to deduce just the locally uniform convergence.

Proof. We start by observing that the assumptions on ω ensure that, as can be deduced from [23], Theorem 3.1 and already emphasized in Remark 5.5, $u_0 \in W^{1-\frac{1}{p},p}(\partial\omega)$ for $1 < p < 2$. Consequently for every $1 < p < 2$ there exists a unique function $u_p \in W^{1,p}(\omega)$ solution of (1.11). The fact that u_p is independent of x_3, implies that u_p solves also (1.8) and (1.6) for every $\varepsilon > 0$. On the other hand theorem 5.7 says also that u_p is the unique solution of (1.6). Thus we can denote this solution u_p also as $u_{p,\varepsilon}$. Next we can observe, by virtue of the strict convexity of $I_{p,\varepsilon}$ in (1.4) and $I_{p,0}$ in (1.9), that for every $1 < p < 2$ and for every $\varepsilon > 0$, $u_p \equiv u_{p,\varepsilon}$ is also the unique minimum point of $\mathcal{P}_{p,0}$ and $\mathcal{P}_{p,\varepsilon}$. On the other hand Theorem 5.7 guarantees that $u_{p,\varepsilon} = u_p$ converges locally uniformly in Ω to the unique solution u of (5.1) and (5.2). It is easily seen that the function u is admissible also for problems (5.6) and (5.7). Moreover the fact that u_0 is an admissible boundary datum for the Γ–convergence Theorems 3.8 and 3.11, leads us to conclude that the common minimum values u_p of $\mathcal{P}_{p,\varepsilon}$ and $\mathcal{P}_{p,0}$ converge to the minimum of $\mathcal{P}_{1,0}$ and $\mathcal{P}_{1,\varepsilon}$. Consequently exploiting Theorem 2.6 we can say that u (the strong $L^1(\Omega)$ limit of $u_{p,\varepsilon} = u_p$ as $p \to 1$) is a minimum both for (5.6) and (5.7). This concludes the proof. $\square$

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