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UNIFORMLY EXPONENTIALLY OR POLYNOMIALLY STABLE APPROXIMATIONS FOR SECOND ORDER EVOLUTION EQUATIONS AND SOME APPLICATIONS

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Abstract. In this paper, we consider the approximation of second order evolution equations. It is well known that the approximated system by finite element or finite difference is not uniformly exponentially or polynomially stable with respect to the discretization parameter, even if the continuous system has this property. Our goal is to damp the spurious high frequency modes by introducing numerical viscosity terms in the approximation scheme. With these viscosity terms, we show the exponential or polynomial decay of the discrete scheme when the continuous problem has such a decay and when the spectrum of the spatial operator associated with the undamped problem satisfies the generalized gap condition. By using the Trotter–Kato Theorem, we further show the convergence of the discrete solution to the continuous one. Some illustrative examples are also presented.

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1. Introduction and main results

Let H be a complex Hilbert space with norm and inner product denoted respectively by $\|.\|$ and $(.,.)$. Let $A : \mathcal{D}(A) \to H$ be a densely defined self-adjoint and positive operator with a compact inverse in H. Let $V = \mathcal{D}(A^{\frac{1}{2}})$ be the domain of $A^{\frac{1}{2}}$. Denote by $\mathcal{D}(A^{\frac{1}{2}})'$ the dual space of $\mathcal{D}(A^{\frac{1}{2}})$ obtained by means of the inner product in H.

Furthermore, let U be a complex Hilbert space (which will be identified to its dual space) with norm and inner product denoted respectively by $\|.\|_U$ and $(.,.)_U$ and let $B \in \mathcal{L}(U, H)$. We consider the closed loop system

$$\begin{gathered} \ddot{\omega}(t) + A\omega(t) + BB^*\dot{\omega}(t) = 0, \\ \omega(0) = \omega_0,\, \dot{\omega}(0) = \omega_1, \end{gathered} \tag{1.1}$$

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where $t \in [0, \infty)$ represents the time, $\omega : [0, \infty) \to H$ is the state of the system. Most of the linear equations modeling the vibrations of elastic structures with feedback control (corresponding to collocated actuators and sensors) can be written in the form (1.1), where ω represents the displacement field.

We define the energy of system (1.1) at time t by

$$E(t) = \frac{1}{2}\left(\|\dot{\omega}(t)\|^2 + \left\|A^{\frac{1}{2}}\omega(t)\right\|^2\right).$$

Simple formal calculations give

$$E(0) - E(t) = \int_0^t (BB^*\dot{\omega}(s),\, \dot{\omega}(s))\, \mathrm{d}s, \quad \forall t \geq 0.$$

This obviously means that the energy is non-increasing.

In many applications, the system (1.1) is approximated by finite dimensional systems but usually if the continuous system is exponentially or polynomially stable, the discrete ones do no more inherit of this property due to spurious high frequency modes. Several remedies have been proposed and analyzed to overcome this difficulties. Let us quote the Tychonoff regularization [18, 19, 31, 34], a bi-grid algorithm [16, 28], a mixed finite element method [6, 10, 11, 17, 27], or filtering the high frequencies [22, 25, 35] (both methods providing good numerical results).

As in [31, 34] our goal is to damp the spurious high frequency modes by introducing a numerical viscosity in the approximation schemes. Though our paper is inspired from [31], it differs from that paper on the following points:

(i) Contrary to [31] where the standard gap condition is required, we only assume that the spectrum of the operator $A^{1/2}$ satisfies the generalized gap condition, allowing to treat more general concrete systems;
(ii) we analyze the polynomial decay of the discrete schemes when the continuous problem has such a decay;
(iii) we prove a result about uniform polynomial stability for a family of semigroups of operators;
(iv) by using a general version of the Trotter–Kato Theorem proved in [23], we show that the discrete solution tends to the solution of (1.1) as the discretization parameter goes to zero and if the discrete initial data are well chosen.

Before stating our main results, let us introduce some notations and assumptions.

We denote by $\|.\|_V$ the norm

$$\|\varphi\|_V = \sqrt{(A^{\frac{1}{2}}\varphi,\, A^{\frac{1}{2}}\varphi)},\ \forall \varphi \in V.$$

Remark that

$$\|\varphi\|_V = \sqrt{(A\varphi,\, \varphi)},\ \forall \varphi \in \mathcal{D}(A).$$

We now assume that $(V_h)_{h>0}$ is a sequence of finite dimensional subspaces of $\mathcal{D}(A^{\frac{1}{2}})$. The inner product in V_h is the restriction of the inner product of H and it is still denoted by $(.,.)$ (since V_h can be seen as a subspace of H). We define the operator $A_h : V_h \to V_h$ by

$$(A_h\varphi_h,\, \psi_h) = (A^{\frac{1}{2}}\varphi_h,\, A^{\frac{1}{2}}\psi_h),\ \forall \varphi_h,\, \psi_h \in V_h. \tag{1.2}$$

Let $a(.,.)$ be the sesquilinear form on $V_h \times V_h$ defined by

$$a(\varphi_h,\, \psi_h) = (A^{\frac{1}{2}}\varphi_h,\, A^{\frac{1}{2}}\psi_h),\ \forall (\varphi_h,\, \psi_h) \in V_h \times V_h. \tag{1.3}$$

We also define the operators $B_h : U \to V_h$ by

$$B_h u = j_h B u, \quad \forall u \in U, \tag{1.4}$$

where j_h is the orthogonal projection of H into V_h with respect to the inner product in H.

The adjoint B_h^* of B_h is then given by the relation

$$B_h^*\varphi_h = B^*\varphi_h, \quad \forall \varphi_h \in V_h.$$

We also suppose that the family of spaces $(V_h)_h$ approximates the space $V = \mathcal{D}(A^{\frac{1}{2}})$. More precisely, if π_h denotes the orthogonal projection of $V = \mathcal{D}(A^{\frac{1}{2}})$ onto V_h, we suppose that there exist $\theta > 0$, $h^* > 0$ and $C_0 > 0$ such that, for all $h \in (0, h^*)$, we have:

$$\|\pi_h\varphi - \varphi\|_V \le C_0 h^\theta \|A\varphi\|, \forall \varphi \in \mathcal{D}(A), \tag{1.5}$$

$$\|\pi_h\varphi - \varphi\| \le C_0 h^{2\theta} \|A\varphi\|, \forall \varphi \in \mathcal{D}(A). \tag{1.6}$$

Assumptions (1.5) and (1.6) are, in particular, satisfied in the case of standard finite element approximations of Sobolev spaces.

Denote by $\{\lambda_k\}_{k\ge1}$ the set of eigenvalues of $A^{\frac{1}{2}}$ counted with their multiplicities (*i.e.* we repeat the eigenvalues according to their multiplicities). We further rewrite the sequence of eigenvalues $\{\lambda_k\}_{k\ge1}$ as follows:

$$\lambda_{k_1} < \lambda_{k_2} < \ldots < \lambda_{k_i} < \ldots$$

where $k_1 = 1$, k_2 is the lowest index of the second distinct eigenvalue, k_3 is the lowest index of the third distinct eigenvalue, etc. For all $i \in \mathbb{N}^*$, let l_i be the multiplicity of the eigenvalue λ_{k_i}, *i.e.*

$$\lambda_{k_{i-1}} < \lambda_{k_i} = \lambda_{k_i+1} = \ldots = \lambda_{k_i+l_i-1} < \lambda_{k_i+l_i} = \lambda_{k_{i+1}}.$$

We have $k_1 = 1$, $k_2 = 1+l_1$, $k_3 = 1+l_1+l_2$, etc. Let $\{\varphi_{k_i+j}\}_{0\le j\le l_i-1}$ be the orthonormal eigenvectors associated with the eigenvalue λ_{k_i}.

Now, we assume that the following generalized gap condition holds:

$$\exists M \in \mathbb{N}^*, \exists \gamma_0 > 0, \forall k \ge 1, \lambda_{k+M} - \lambda_k \ge M\gamma_0. \tag{1.7}$$

Fix a positive real number $\gamma_0' \le \gamma_0$ and denote by A_k, $k = 1, \ldots, M$ the set of natural numbers k_m satisfying (see for instance [5])

$$\begin{cases} \lambda_{k_m} - \lambda_{k_{m-1}} \ge \gamma_0' \\ \lambda_{k_n} - \lambda_{k_{n-1}} < \gamma_0' & \text{for } m+1 \le n \le m+k-1, \\ \lambda_{k_{m+k}} - \lambda_{k_{m+k-1}} \ge \gamma_0'. \end{cases}$$

Then one easily checks that

$$\{k_{m+j} + l \,|k_m \in A_k,\, k \in \{1, \ldots, M\}, j \in \{0, \ldots, k-1\}, l \in \{0, \ldots, l_{m+j} - 1\}\} = \mathbb{N}^*.$$

Notice that some sets A_k may be empty because, for the generalized gap condition, the choice of M takes into account multiple eigenvalues. For $k_n \in A_k$, we define $B_{k_n} = (B_{k_n,ij})_{1\le i,j\le k}$ the matrix of size $k \times k$ by

$$B_{k_n,ij} = \begin{cases} \displaystyle\prod_{\substack{q=n\\ q\ne n+i-1}}^{n+j-1} (\lambda_{k_{n+i-1}} - \lambda_{k_q})^{-1} & \text{if} \quad i \le j,\, (i,j) \ne (1,1), \\ 1 & \text{if} \quad (i,j) = (1,1), \\ 0 & \text{else.} \end{cases}$$

More explicitly, we have

$$B_{k_n} = \begin{pmatrix} 1 & \frac{1}{\lambda_{k_n}-\lambda_{k_{n+1}}} & \frac{1}{(\lambda_{k_n}-\lambda_{k_{n+1}})(\lambda_{k_n}-\lambda_{k_{n+2}})} & \cdots & \frac{1}{(\lambda_{k_n}-\lambda_{k_{n+1}})\cdots(\lambda_{k_n}-\lambda_{k_{n+k-1}})} \\ 0 & \frac{1}{\lambda_{k_{n+1}}-\lambda_{k_n}} & \frac{1}{(\lambda_{k_{n+1}}-\lambda_{k_n})(\lambda_{k_{n+1}}-\lambda_{k_{n+2}})} & \cdots & \frac{1}{(\lambda_{k_{n+1}}-\lambda_{k_n})\cdots(\lambda_{k_{n+1}}-\lambda_{k_{n+k-1}})} \\ 0 & 0 & \frac{1}{(\lambda_{k_{n+2}}-\lambda_{k_n})(\lambda_{k_{n+2}}-\lambda_{k_{n+1}})} & \cdots & \frac{1}{(\lambda_{k_{n+2}}-\lambda_{k_n})\cdots(\lambda_{k_{n+2}}-\lambda_{k_{n+k-1}})} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{(\lambda_{k_{n+k-1}}-\lambda_{k_n})\cdots(\lambda_{k_{n+k-1}}-\lambda_{k_{n+k-2}})} \end{pmatrix}.$$

Lemma 1.1. *The inverse matrix of B_{k_n} is given by*

$$B^{-1}_{k_n,\, ij} = \begin{cases} \prod_{q=n}^{n+i-2} (\lambda_{k_{n+j-1}} - \lambda_{k_q}) & \text{if} \quad i \le j,\, i \ne 1, \\ 1 & \text{if} \quad i = 1, \\ 0 & \text{else,} \end{cases}$$

that is to say

$$B^{-1}_{k_n} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & (\lambda_{k_{n+1}} - \lambda_{k_n}) & \cdots & (\lambda_{k_{n+k-1}} - \lambda_{k_n}) \\ 0 & 0 & \cdots & (\lambda_{k_{n+k-1}} - \lambda_{k_n})(\lambda_{k_{n+k-1}} - \lambda_{k_{n+1}}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (\lambda_{k_{n+k-1}} - \lambda_{k_n}) \cdots (\lambda_{k_{n+k-1}} - \lambda_{k_{n+k-2}}) \end{pmatrix},$$

and therefore

$$B^{-1}_{k_n} \to \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 0 \end{pmatrix}, \text{ when} \quad n \to +\infty.$$

Proof. The form of $B^{-1}_{k_n}$ is obtained by induction on the size k of B_{k_n}. The generalized gap condition (1.7) implies that $\lambda_{k_{n+j}} - \lambda_{k_n} \to 0$ as $n \to +\infty$, $\forall 0 \le j \le k-1$. This leads to the convergence of $B^{-1}_{k_n}$. □

Now, for $k_n \in A_k$, we define the matrix Φ_{k_n} with coefficients in U and size $k \times L_n$, where $L_n = \sum_{i=1}^{k} l_{n+i-1}$, as follows: for all $i = 1, \ldots, k$, we set

$$(\Phi_{k_n})_{ij} = \begin{cases} B^* \varphi_{k_{n+i-1}+j-L_{n,\, i-1}-1} & \text{if } L_{n,\, i-1} < j \le L_{n,\, i}, \\ 0 & \text{else,} \end{cases}$$

where

$$L_{n,\, 0} = 0, \quad L_{n,\, i} = \sum_{i'=1}^{i} l_{n+i'-1} \text{ for } i \ge 1. \tag{1.8}$$

For a vector $c = (c_l)_{l=1}^m$ in U^m, we set $\|c\|_{U,\, 2}$ its norm in U^m defined by

$$\|c\|^2_{U,\, 2} = \sum_{l=1}^{m} \|c_l\|^2_U .$$

In this paper, we prove two results. The first result gives a necessary and sufficient condition to have the exponential stability of the family of systems

$$\begin{aligned} \ddot{\omega}_h(t) + A_h \omega_h(t) + B_h B_h^* \dot{\omega}_h(t) + h^\theta A_h \dot{\omega}_h(t) = 0 \\ \omega_h(0) = \omega_{0h} \in V_h,\, \dot{\omega}_h(0) = \omega_{1h} \in V_h, \end{aligned} \tag{1.9}$$

in the absence of the standard gap condition assumed in [31]. Here and below ω_{0h} (resp. ω_{1h}) is an approximation of ω_0 (resp. ω_1) in V_h. For that purpose, we need to make the following assumption

$$\exists \alpha_0 > 0,\, \forall k \in \{1, \ldots, M\},\, \forall k_n \in A_k,\, \forall C \in \mathbb{R}^{L_n},\, \left\| B^{-1}_{k_n} \Phi_{k_n} C \right\|_{U,\, 2} \ge \alpha_0 \left\| C \right\|_2 , \tag{1.10}$$

where $\|.\|_2$ is the euclidian norm. The first main result is the following

Theorem 1.2. *Suppose that the generalized gap condition* (1.7) *and the assumption* (1.10) *are verified. Assume that the family of subspaces* (V_h) *satisfies* (1.5) *and* (1.6)*. Then the family of systems* (1.9) *is uniformly exponentially stable, in the sense that there exist constants* M, α, $h^* > 0$ *(independent of* h, ω_{0h}, ω_{1h}*) such that for all* $h \in (0, h^*)$*:*

$$\|\dot{\omega}_h(t)\|^2 + a(\omega_h(t), \omega_h(t)) \leq M\mathrm{e}^{-\alpha t}(\|\omega_{1h}\|^2 + a(\omega_{0h}, \omega_{0h})), \forall t \geq 0.$$

Remark 1.3. If the standard gap condition

$$\exists \gamma_0 > 0, \forall n \geq 1, \lambda_{k_{n+1}} - \lambda_{k_n} \geq \gamma_0 \tag{1.11}$$

holds, then $A_1 = \mathbb{N}^*$ and $B_1 = 1$. In this case, the assumption (1.10) becomes

$$\exists \alpha_0 > 0, \forall k_n \geq 1, \forall C \in \mathbb{R}^{L_n}, \|\Phi_{k_n} C\|_U \geq \alpha_0 \|C\|_2 .$$

Moreover, if the standard gap condition (1.11) holds and if the eigenvalues are simple, the assumption (1.10) becomes

$$\exists \alpha_0 > 0, \forall k \geq 1, \|B^* \varphi_k\|_U \geq \alpha_0. \tag{1.12}$$

These assumptions are assumed in [31].

Remark 1.4. Note that Theorem 1.2 is the discrete counterpart of the exponential decay of the solution of the continuous problem (1.1) under the assumptions (1.7) and (1.10), which follows Theorem 2.2 of [3] (see also [29]). Note that the assumption (H) from [3] here holds since A is a positive selfadjoint operator with a compact resolvent and B is bounded.

Remark 1.5. The uniform exponential stability of the family of systems (1.9) has been already proved in Theorem 7.1 of [14] without any assumption on the spectrum of A and the dimension of the space. The proof of this theorem is based on decoupling of low and high frequencies. More precisely, the author combines a uniform observability estimate for filtered initial data corresponding to low frequencies (see [14], Thm. 1.3) together with a result of [15]. Indeed, in [15], after adding the numerical viscosity term, another uniform observability estimate is obtained for the high frequency components. The two established observability inequalities yield the uniform exponential decay of (1.9).

If the condition (1.10) is not satisfied, we may look at a weaker version. Namely if we assume that

$$\exists l \in \mathbb{N}^*, \exists \alpha_0 > 0, \forall k \in \{1, \ldots, M\}, \forall k_n \in A_k, \forall C \in \mathbb{R}^{L_n}, \left\|B_{k_n}^{-1} \Phi_{k_n} C\right\|_{U,2} \geq \frac{\alpha_0}{\lambda_{k_n}^l} \|C\|_2 , \tag{1.13}$$

then we will obtain a polynomial stability for the family of systems

$$\begin{gathered} \ddot{\omega}_h(t) + \left(1 + h^\theta\right)^{-2} \left(I + h^\theta A_h^{\frac{l}{2}}\right)^2 A_h \omega_h(t) + \left(I + h^\theta A_h^{\frac{l}{2}}\right) \left(B_h B_h^* + h^\theta A_h^{1+\frac{l}{2}}\right) \left(I + h^\theta A_h^{\frac{l}{2}}\right)^{-1} \dot{\omega}_h(t) = 0, \\ \omega_h(0) = \omega_{0h} \in V_h, \, \dot{\omega}_h(0) = \left(1 + h^\theta\right)^{-1} \left(I + h^\theta A_h^{\frac{l}{2}}\right) \omega_{1h} \in V_h. \end{gathered} \tag{1.14}$$

The structure of the above discrete system has been inspired from the one introduced in [31] for the exponential stability case where the authors have used system (1.9) corresponding to $l = 0$. In both cases, this choice is motivated by the corresponding observability estimates. The numerical viscosity term $(I + h^\theta A_h^{\frac{l}{2}})(B_h B_h^* + h^\theta A_h^{1+\frac{l}{2}})(I+h^\theta A_h^{\frac{l}{2}})^{-1}\dot{\omega}_h(t)$ is added to damp the high frequency modes and as the set of high frequency modes is larger in the polynomial case, the viscosity term is naturally stronger. In the case $l \geq 2$ the powers of $(I+h^\theta A_h^{\frac{l}{2}})$ have been added to guarantee the boundedness of the resolvent of $\tilde{A}_{l,h}$ (defined below) near zero. The question of the optimality of these viscosity terms remains open.

The second main result of our paper is the following one.

Theorem 1.6. *Suppose that the generalized gap condition (1.7) and the Assumption* (1.13) *are verified with* $l \in \mathbb{N}^*$ *even. Assume that the family of subspaces* (V_h) *satisfies* (1.5) *and* (1.6)*. Then the family of systems* (1.14) *is uniformly polynomially stable, in the sense that there exist constants* C, $h^* > 0$ *(independent of* h, ω_{0h}, ω_{1h}*) such that for all* $h \in (0, h^*)$*:*

$$\left\| \left(I + h^{\theta} A_h^{\frac{l}{2}} \right)^{-1} \dot{\omega}_h(t) \right\|^2 + a\left(\omega_h(t),\, \omega_h(t)\right) \le \frac{C}{t^2} \left\| (\omega_{0h}, \omega_{1h}) \right\|^2_{D(\tilde{A}^{2l}_{l,h})},$$

$$\left\| \left(I + h^{\theta} A_h^{\frac{l}{2}} \right)^{-1} \dot{\omega}_h(t) \right\|^2 + a\left(\omega_h(t),\, \omega_h(t)\right) \le \frac{C}{t^{\frac{1}{l}}} \left\| (\omega_{0h}, \omega_{1h}) \right\|^2_{D(\tilde{A}_{l,h})},\ \forall t > 0,\ \forall\, (\omega_{0h}, \omega_{1h}) \in V_h \times V_h,$$

where for $q \in \mathbb{N}^*$, $\|.\|_{D(\tilde{A}^q_{l,h})}$ *is the graph norm of the matrix operator* $\tilde{A}^q_{l,h}$ *given in* (4.1) *of Section* 4 *below.*

For a technical reason, we assume l to be even (see Lem. 4.4). If (1.13) holds for l odd, then it is also true for $l+1$ and we can apply the previous result with $l+1$.

Remark 1.7. If the standard gap condition (1.11) holds, the Assumption (1.13) becomes

$$\exists l \in \mathbb{N}^*,\ \exists \alpha_0 > 0,\ \forall k_n \ge 1,\ \forall C \in \mathbb{R}^{L_n},\ \left\| \Phi_{k_n} C \right\|_U \ge \frac{\alpha_0}{\lambda^l_{k_n}} \left\| C \right\|_2.$$

Moreover, if the standard gap condition (1.11) holds and if the eigenvalues are simple, the Assumption (1.13) becomes

$$\exists l \in \mathbb{N}^*,\ \exists \alpha_0 > 0,\ \forall k \ge 1,\ \left\| B^* \varphi_k \right\|_U \ge \frac{\alpha_0}{\lambda^l_k}\cdot \tag{1.15}$$

Remark 1.8. As before, Theorem 1.6 is the discrete counterpart of the polynomial decay of the solution of the continuous problem (1.1) under the Assumptions (1.7) and (1.13), that follows from Theorem 2.4 of [3] (see also [29]).

The paper is organized as follows: In Section 2, we show that the generalized gap condition and the observability conditions (1.10) and (1.13) remain valid for filtered eigenvalues. Section 3 first recalls a result about uniform exponential stability for a family of semigroup of operators, and then extends such a result to the case of uniform polynomial stability. Some technical lemmas are proved in Section 4. Sections 5 and 6 are devoted to the proof of Theorem 1.2 and 1.6 respectively. In Section 7, we show that the solution ω_h (resp. $\left(I + h^{\theta} A_h^{\frac{l}{2}})^{-1}\dot{\omega}_h\right)$ tends to ω, the solution of (1.1), (resp. $\dot{\omega}$) in V (resp. in H) as h goes to zero and if the discrete initial data are well chosen. Finally, we illustrate our results by presenting different examples in Section 8.

2. Spectral analysis of the discretized problem

The eigenvalue problem of the discretized problem is the following one: find $\lambda_{k,\,h} \in]0, +\infty[$, $\varphi_{k,\,h} \in V_h$, such that

$$a(\varphi_{k,\,h},\, \psi_h) = \lambda^2_{k,\,h}(\varphi_{k,\,h},\, \psi_h),\ \forall \psi_h \in V_h. \tag{2.1}$$

Let $N(h)$ be the dimension of V_h. We denote by $\{\lambda^2_{k,\,h}\}_{1 \le k \le N(h)}$ the set of eigenvalues of (2.1) counted with their multiplicities. Let $\{\varphi_{k,\,h}\}_{1 \le k \le N(h)}$ be the orthonormal eigenvectors associated with the eigenvalue $\lambda^2_{k,\,h}$.

In this Section, we show that the generalized gap condition (1.7) and the observability conditions (1.10) and (1.13) still hold for the approximate problem (uniformly in h), provided that we consider only "low frequencies". More precisely, we first have the following result:

Proposition 2.1. *Suppose that the generalized gap condition* (1.7) *and the Assumption* (1.10) *are verified. Then, there exist two constants* $\epsilon > 0$ *and* $h^* > 0$*, such that, for all* $0 < h < h^*$ *and for all* $k \in \{1, \ldots, N(h)\}$ *satisfying*

$$h^{\theta} \lambda^2_k \le \epsilon, \tag{2.2}$$

we have

$$\exists M \in \mathbb{N}^*,\, \exists \gamma > 0,\, \lambda_{k+M,\,h} - \lambda_{k,\,h} \geq M\gamma \tag{2.3}$$

and

$$\exists \alpha > 0,\, \forall p \in \{1, \ldots, M\},\, \forall k_n \in A_{p,h},\, \forall C \in \mathbb{R}^{L_n},\, \left\| B_{k_n}^{-1} \Phi_{k_n,\,h} C \right\|_{U,\,2} \geq \alpha \left\| C \right\|_2, \tag{2.4}$$

where α is independent of h, and where the matrix $\Phi_{k_n,\,h} \in \mathcal{M}_{p,\,L_n}(U)$, with coefficients in U, is defined as follows: for all $i = 1, \ldots, p$, we set

$$(\Phi_{k_n,\,h})_{ij} = \begin{cases} B_h^* \varphi_{k_{n+i-1}+j-L_{n,\,i-1}-1,\,h} & \textit{if} \quad L_{n,\,i-1} < j \leq L_{n,\,i}, \\ 0 & \textit{else,} \end{cases}$$

where $L_{n,\,i-1}$ is defined by (1.8) and

$$A_{p,h} = \{k_n \in A_p \textit{ satisfying } (2.2) \textit{ and s.t.} k_{n+p-1} + l_{n+p-1} - 1 \leq N(h)\}.$$

For the proof of this proposition, we need a result proved by Babuska and Osborn in [4]. For that purpose, we introduce $\epsilon_h(n,\, j)$ such that

$$\epsilon_h(n,\, j) = \inf_{\varphi \in M_j(\lambda_{k_n})} \inf_{v_h \in V_h} \left\| \varphi - v_h \right\|_V,$$

where $M_j(\lambda_{k_n}) = \{\varphi \in M(\lambda_{k_n}) \,:\, a(\varphi,\, \varphi_{k_n,\,h}) = \ldots = a(\varphi,\, \varphi_{k_n+j-2,\,h}) = 0\}$ and $M(\lambda_{k_n}) = \{\varphi \,:\, \varphi \text{ is an eigenvector of } A^{\frac{1}{2}} \text{ corresponding to } \lambda_{k_n},\, \|\varphi\| = 1\}$. The restrictions $a(\varphi,\, \varphi_{k_n,\,h}) = \ldots = a(\varphi,\, \varphi_{k_n+j-2,\,h}) = 0$ are not imposed if $j = 1$. Then, we have the following estimate about the eigenvalue and eigenvector errors for the Galerkin method in terms of the approximability quantities $\epsilon_h(n,\, j)$.

Theorem 2.2. *There are positive constants C and h_0 such that*

$$\lambda_{k_n+j,\,h} - \lambda_{k_n+j} \leq C\epsilon_h^2(n,\, j), \quad \forall 0 < h \leq h_0,\, j = 0, \ldots, l_n - 1,\, k_n + j \leq N(h),\, n \in \mathbb{N}^* \tag{2.5}$$

and such that the eigenvectors $\{\varphi_{k_n+j}\}_{0 \leq j \leq l_n - 1}$ of $A^{\frac{1}{2}}$ can be chosen so that

$$\left\| \varphi_{k_n+j,\,h} - \varphi_{k_n+j} \right\|_V \leq C\epsilon_h(n,\, j), \quad \forall 0 < h \leq h_0,\, j = 0, \ldots, l_n - 1,\, k_n + j \leq N(h),\, n \in \mathbb{N}^*. \tag{2.6}$$

This result is proved by Babuska and Osborn in [4], p. 702. because

$$\lambda^2_{k_n+j,\,h} - \lambda^2_{k_n+j} = (\lambda_{k_n+j,\,h} - \lambda_{k_n+j})(\lambda_{k_n+j,\,h} + \lambda_{k_n+j}) \geq 2\lambda_1(\lambda_{k_n+j,\,h} - \lambda_{k_n+j}).$$

Remark 2.3. Notice that for every $\varphi \in M_j(\lambda_{k_n})$ we have

$$\begin{aligned} \epsilon_h(n,\, j) &\leq \inf_{v_h \in V_h} \left\| \varphi - v_h \right\|_V \\ &\leq C_0 h^\theta \left\| A\varphi \right\| \text{ by } (1.5) \\ &\leq C_0 h^\theta \lambda^2_{k_n} \left\| \varphi \right\| = C_0 h^\theta \lambda^2_{k_n+j}. \end{aligned} \tag{2.7}$$

Proof of Proposition 2.1. We begin with the proof of the generalized gap condition for the approximate eigenvalues $\lambda_{k,\,h}$. First, we use the Min-Max principle (see [32]) to obtain

$$\lambda_k \leq \lambda_{k,\,h},\, \forall k \in \{1, \ldots, N(h)\}. \tag{2.8}$$

Second, we use the estimates (2.5) and (2.7) and we have

$$\lambda_{k,\,h} \leq \lambda_k + C(C_0 h^\theta \lambda_k^2)^2 \leq \lambda_k + C(C_0\epsilon)^2 \leq \lambda_k + CC_0^2\epsilon, \tag{2.9}$$

for all $k \in \{1, \ldots, N(h)\}$ verifying (2.2) and $\epsilon \leq 1$. Therefore, we may write

$$\lambda_{k+M,\,h} - \lambda_{k,\,h} \geq \lambda_{k+M} - \lambda_k - CC_0^2\epsilon \geq M\gamma_0 - CC_0^2\epsilon \geq M\frac{\gamma_0}{2} =: M\gamma$$

for all $k \in \{1, \ldots, N(h)\}$ satisfying (2.2) and for $\epsilon \leq \frac{M\gamma_0}{2CC_0^2}$.

Now, we prove the estimate (2.4) which is the approximated version of (1.10). Notice that

$$\begin{aligned}\|\Phi_{k_n,h}-\Phi_{k_n}\|_U &\le C \max_{i=0,\dots,p-1} \sum_{j=0}^{l_{n+i}-1} \left\|B^*\varphi_{k_{n+i}+j,h}-B^*\varphi_{k_{n+i}+j}\right\|_U \\ &\le C \max_{i=0,\dots,p-1} \sum_{j=0}^{l_{n+i}-1} \|B^*\|_{\mathcal{L}(H,U)} \left\|\varphi_{k_{n+i}+j,h}-\varphi_{k_{n+i}+j}\right\| \\ &\le C \max_{i=0,\dots,p-1} \sum_{j=0}^{l_{n+i}-1} \|B^*\|_{\mathcal{L}(H,U)} \left\|\varphi_{k_{n+i}+j,h}-\varphi_{k_{n+i}+j}\right\|_V \\ &\le C \max_{i=0,\dots,p-1} \sum_{j=0}^{l_{n+i}-1} \epsilon_h(n+i,\,j) \text{ by (2.6)} \\ &\le C \max_{i=0,\dots,p-1} \sum_{j=0}^{l_{n+i}-1} h^\theta \lambda^2_{k_{n+i}+j} \text{ by (2.7)}.\end{aligned}$$

Thus, by (2.2), we get

$$\|\Phi_{k_n,h}-\Phi_{k_n}\|_U \le C\epsilon. \tag{2.10}$$

Therefore the triangular inequality leads to

$$\begin{aligned}\left\|B^{-1}_{k_n}\Phi_{k_n,h}C\right\|_{U,2} &= \left\|B^{-1}_{k_n}\Phi_{k_n}C + B^{-1}_{k_n}(\Phi_{k_n,h}-\Phi_{k_n})C\right\|_{U,2} \\ &\ge \left\|B^{-1}_{k_n}\Phi_{k_n}C\right\|_{U,2} - \left\|B^{-1}_{k_n}(\Phi_{k_n,h}-\Phi_{k_n})C\right\|_{U,2} \\ &\ge \alpha_0\|C\|_2 - \left\|B^{-1}_{k_n}(\Phi_{k_n,h}-\Phi_{k_n})C\right\|_{U,2}\end{aligned}$$

by (1.10). But, as $B^{-1}_{k_n} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 0\end{pmatrix} + R_{k_n}$, with $R_{k_n}\to 0$, when $k_n\to+\infty$ (see Lem. 1.1), we obtain

$$\begin{aligned}\left\|B^{-1}_{k_n}(\Phi_{k_n,h}-\Phi_{k_n})C\right\|_{U,2} &\le \left\|\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 0\end{pmatrix}(\Phi_{k_n,h}-\Phi_{k_n})C\right\|_{U,2} \\ &\quad + \|R_{k_n}(\Phi_{k_n,h}-\Phi_{k_n})C\|_{U,2} \\ &\le C\|\Phi_{k_n,h}-\Phi_{k_n}\|_U\|C\|_2 + \eta_n\|\Phi_{k_n,h}-\Phi_{k_n}\|_U\|C\|_2 \\ &\le C\epsilon(1+\eta_n)\|C\|_2,\end{aligned} \tag{2.11}$$

where $\eta_n = \|R_{k_n}\| \to 0$. Thus

$$\left\|B^{-1}_{k_n}\Phi_{k_n,h}C\right\|_{U,2} \ge (\alpha_0 - C\epsilon(1+\eta_n))\|C\|_2 \ge \frac{\alpha_0}{2}\|C\|_2$$

for $\epsilon \le \frac{\alpha_0}{2C(1+\max\limits_n(1+\eta_n))}$.

For the polynomial stability, we have the same kind of result, but more filtering is necessary in order to have the discrete counterpart of the observability condition (1.13) (uniformly in h). □

Proposition 2.4. *Suppose that the generalized gap condition (1.7) and the Assumption (1.13) are verified. Then, there exist two constants $\epsilon > 0$ and $h^* > 0$, such that, for all $0 < h < h^*$ and for all $k \in \{1,\dots,N(h)\}$, satisfying*

$$h^\theta\lambda_k^2 \le \frac{\epsilon}{\lambda_k^l}, \tag{2.12}$$

we have (2.3) *and*

$$\exists\alpha > 0,\, \forall p \in \{1,\dots,M\},\, \forall k_n \in A^{(l)}_{p,h},\, \forall C \in \mathbb{R}^{L_n},\, \left\|B^{-1}_{k_n}\Phi_{k_n,\,h}C\right\|_{U,\,2} \geq \frac{\alpha}{\lambda^l_{k_n}}\left\|C\right\|_2, \tag{2.13}$$

where $A^{(l)}_{p,h} = \{k_n \in A_p$ *satisfying* (2.12) *and s.t.* $k_{n+p-1} + l_{n+p-1} - 1 \leq N(h)\}$.

Proof. The generalized gap condition for the approximate eigenvalues $\lambda_{k,\,h}$ is a consequence of Proposition 2.1, because $\lambda_k \geq \lambda_1 > 0$.

To prove the estimate (2.13) we notice that

$$\|\Phi_{k_n,\,h} - \Phi_{k_n}\|_U \leq C \max_{i=0,\dots,p-1} \sum_{j=0}^{l_{n+i}-1} h^\theta \lambda^2_{k_{n+i}+j} \leq C\, h^\theta \lambda^2_{k_{n+p-1}}.$$

Moreover, by the triangular inequality and (1.13), we have

$$\left\|B^{-1}_{k_n}\Phi_{k_n,\,h}C\right\|_{U,\,2} \geq \tfrac{\alpha_0}{\lambda^l_{k_n}}\left\|C\right\|_2 - \left\|B^{-1}_{k_n}(\Phi_{k_n,\,h} - \Phi_{k_n})C\right\|_{U,\,2}.$$

By (2.11) and (2.12), we obtain

$$\begin{aligned}\left\|B^{-1}_{k_n}\Phi_{k_n,\,h}C\right\|_{U,\,2} &\geq \left(\tfrac{\alpha_0}{\lambda^l_{k_n}} - \tfrac{C(1+\eta_n)\epsilon}{\lambda^l_{k_{n+p-1}}}\right)\left\|C\right\|_2\\ &\geq \left(\tfrac{\alpha_0}{\lambda^l_{k_n}} - \tfrac{C\epsilon}{\lambda^l_{k_n}+\rho_n}(1+\eta_n)\right)\left\|C\right\|_2, \text{ with } \rho_n = \lambda^l_{k_{n+p-1}} - \lambda^l_{k_n} \to 0\\ &\geq \tfrac{\alpha}{\lambda^l_{k_n}}\left\|C\right\|_2\end{aligned}$$

for an appropriate choice of $\epsilon > 0$. □

3. Uniform stability results

3.1. Exponential stability result

The Proof of Theorem 1.2 is based on the following result (see Thm. 7.1.3 in [26]):

Theorem 3.1. *Let* $(T_h)_{h>0}$ *be a family of semigroups of contractions on the Hilbert spaces* $(X_h)_{h>0}$ *and let* $(\tilde{A}_h)_{h>0}$ *be the corresponding infinitesimal generators. The family* $(T_h)_{h>0}$ *is uniformly exponentially stable, that is to say there exist constants* $M > 0$, $\alpha > 0$ *(independent of* $h \in (0,\, h^*)$*) such that*

$$\|T_h(t)\|_{\mathcal{L}(X_h)} \leq M\mathrm{e}^{-\alpha t},\, \forall t \geq 0,$$

if and only if the two following conditions are satisfied:

(*i*) *For all* $h \in (0,\, h^*)$, $i\mathbb{R}$ *is contained in the resolvent set* $\rho(\tilde{A}_h)$ *of* $\tilde{A}_h$,

(*ii*) $\sup_{h\in(0,h^*),\omega\in\mathbb{R}} \left\|(i\omega - \tilde{A}_h)^{-1}\right\|_{\mathcal{L}(X_h)} < +\infty.$

3.2. Polynomial stability result

The proof of Theorem 1.6 is based on the results presented in this section by adapting the results from [9] and from [24] to obtain the (uniform) polynomial stability of the discretized problem (1.14). Throughout this section, let $(T_h(t))_{\substack{t\geq 0\\ h\in(0,h^*)}}$ be a family of uniformly bounded C_0 semigroups on the associated Hilbert spaces $(X_h)_{h\in(0,h^*)}$ (*i.e.*, $\exists M > 0$, $\forall h \in (0, h^*)$, $\|T_h(t)\|_{\mathcal{L}(X_h)} \leq M$) and let $(\tilde{A}_h)_{h\in(0,h^*)}$ be the corresponding infinitesimal generators.

In the following, for shortness, we denote by $R(\lambda, \tilde{A}_h)$ the resolvent $(\lambda - \tilde{A}_h)^{-1}$; moreover, for any operator mapping X_h into X_h, we skip the index $\mathcal{L}(X_h)$ in its norm, since in the whole section we work in X_h.

Definition 3.2. Assuming that

$$i\mathbb{R} \subseteq \rho(\tilde{A}_h), \quad \forall h \in (0, h^*), \tag{3.1}$$

and that for all $m \geq 1$, there exists $c = c(m) > 0$ such that

$$\sup_{\substack{h\in(0,h^*)\\|s|\leq m}} \|R(is, \tilde{A}_h)\|_{\mathcal{L}(X_h)} \leq c, \tag{3.2}$$

we define the fractional power $\tilde{A}_h^{-\alpha}$ for $\alpha > 0$ and $h \in (0, h^*)$, according to [2] and [13], as

$$\tilde{A}_h^{-\alpha} = \frac{1}{2\pi i} \int_\Gamma \lambda^{-\alpha}(\lambda - \tilde{A}_h)^{-1} \mathrm{d}\lambda,$$

where $\lambda^{-\alpha} = \mathrm{e}^{-\alpha \log \lambda}$ and $\mathbb{R}^+$ is taken as the cut branch of the complex log function and where the curve $\Gamma = \Gamma_1 \cup \Gamma_2$ is given by

$$\Gamma = \{-\epsilon + t\mathrm{e}^{i\theta},\ t \in [0, +\infty)\} \cup \{-\epsilon - t\mathrm{e}^{-i\theta},\ t \in (-\infty, 0]\} \tag{3.3}$$

for some $\epsilon > 0$ small enough independent of h and θ is a fixed angle in $\left(0, \dfrac{\pi}{4}\right)$.

Remark 3.3. Throughout this section, whenever $\tilde{A}_h^{-\alpha}$ is mentioned, the Assumptions (3.1) and (3.2) are directly taken into consideration since otherwise $\tilde{A}_h^{-\alpha}$ is not well defined.

In fact, under the Assumptions (3.1) and (3.2), for all $m > 0$ there exists $\epsilon = \epsilon(m) > 0$ such that

$$-\mu + i\beta \in \rho(\tilde{A}_h), \quad \forall h \in (0, h^*),\ \forall 0 \leq \mu \leq \epsilon, \forall |\beta| \leq m.$$

Indeed, for all $m > 0$ such that $|\beta| \leq m$, we have

$$(-\mu + i\beta - \tilde{A}_h)^{-1} = (i\beta - \tilde{A}_h)^{-1}[I_h - \mu(i\beta - \tilde{A}_h)^{-1}]^{-1}$$

and

$$\|\mu(i\beta - \tilde{A}_h)^{-1}\| \leq \mu c.$$

Hence, if $|\beta| \leq m$ and $\mu \leq \epsilon \leq \frac{1}{2c}$, then $(-\mu + i\beta - \tilde{A}_h)$ is invertible and we have

$$\|(-\mu + i\beta - \tilde{A}_h)^{-1}\| \leq 2\|(i\beta - \tilde{A}_h)^{-1}\| \leq 2c,\ \forall h \in (0, h^*). \tag{3.4}$$

We choose $m = \Im(-\epsilon + t\mathrm{e}^{i\theta}) = \epsilon \tan\theta$ when $\Re(-\epsilon + t\mathrm{e}^{i\theta}) = 0$, *i.e.* when $t = \frac{\epsilon}{\cos\theta}$. Therefore, by (3.4), Assumptions (3.1) and (3.2) imply that there exists $\epsilon > 0$ independent of h such that the curve Γ is included in $\rho(\tilde{A}_h)$ for any $h \in (0, h^*)$, and hence $\tilde{A}_h^{-\alpha}$ is well defined. In fact, if $\xi \in \Gamma$ such that $\Re\xi > 0$, then, by the Hille Yosida theorem, $\xi \in \rho(\tilde{A}_h)$, while if $-\epsilon \leq \xi \leq 0$, then, by (3.4), $\xi \in \rho(\tilde{A}_h)$.

Proposition 3.4. *If, in addition to Assumptions* (3.1) *and* (3.2)*, we have*

$$\sup_{h\in(0,h^*)} \|R(is, \tilde{A}_h)\|_{\mathcal{L}(X_h)} = O(|s|^\alpha),\ |s| \to \infty, \tag{3.5}$$

then $\tilde{A}_h^{-\alpha}$ is uniformly bounded independently of $h \in (0, h^*)$.

Proof. We have

$$\tilde{A}_h^{-\alpha} = \frac{1}{2\pi i}\int_0^{+\infty}(-\epsilon + t\mathrm{e}^{i\theta})^{-\alpha}(-\epsilon + t\mathrm{e}^{i\theta} - \tilde{A}_h)^{-1}\mathrm{e}^{i\theta}\mathrm{d}t$$
$$+ \frac{1}{2\pi i}\int_{-\infty}^{0}(-\epsilon - t\mathrm{e}^{-i\theta})^{-\alpha}(-\epsilon - t\mathrm{e}^{-i\theta} - \tilde{A}_h)^{-1}(-\mathrm{e}^{-i\theta})\mathrm{d}t. \quad (3.6)$$

Since $(T_h(t))_{\substack{t\geq 0\\ h\in(0,h^*)}}$ is bounded, then by Hille–Yosida Theorem (see Thm. I.3.1 of [30]) we get

$$\|R(\lambda,\tilde{A}_h)\| \leq \frac{M}{\mathrm{Re}\lambda},\ \forall Re\lambda > 0.$$

For $-\epsilon \leq Re\lambda \leq 0$, we have $|\Im\lambda| \leq m$ and therefore, by (3.4), we get

$$\|R(\lambda,\tilde{A}_h)\| \leq 2c.$$

Let $t_0 > 0$ be such that $-\epsilon \leq \mathrm{Re}(-\epsilon + t\mathrm{e}^{i\theta}) \leq 0,\ \forall 0 \leq t \leq t_0 = \dfrac{\epsilon}{\cos\theta}$ and $\mathrm{Re}(-\epsilon + t\mathrm{e}^{i\theta}) \geq 0,\ \forall t \geq t_0$ and let $t_1 = -\dfrac{\epsilon}{\cos\theta} \leq 0$ be such that $\mathrm{Re}(-\epsilon - t\mathrm{e}^{-i\theta}) \leq 0,\ \forall t_1 \leq t \leq 0$ and $\mathrm{Re}(-\epsilon - t\mathrm{e}^{-i\theta}) \geq 0,\ \forall t \leq t_1$. Therefore, split the integrals in (3.6) then use (3.4) in case of $0 \leq t \leq t_0$ or $t_1 \leq t \leq 0$; in addition to (3.5) in case of $t \geq t_0$ or $t \leq t_1$ to get the uniform boundedness of $\tilde{A}_h^{-\alpha}$. □

The proof of the polynomial stability of $(T_h(t))_{t\geq 0}$ (see Thm. 3.8 below) is based on the following three lemmas. The first lemma is the discretized version of Lemma 3.2 in [24] and the other ones are the discrete versions of similar results of Lemmas 2.1 and 2.3 in [9].

Lemma 3.5. *Let $S = \{\lambda \in \mathbb{C} :\ a \leq Re\lambda \leq b\}$ be a subset of $\rho(\tilde{A}_h)$ for all $h \in (0,h^*)$ where $0 \leq a < b$. Then if (3.1)–(3.5) are satisfied and if for some positive constants α and M we have*

$$\sup_{\substack{h\in(0,h^*)\\ \lambda\in S}} \frac{\|R(\lambda,\tilde{A}_h)\|}{1+|\lambda|^{\alpha}} \leq M,$$

then there exists a constant $c > 0$ independent of h such that

$$\sup_{\substack{h\in(0,h^*)\\ \lambda\in S}} \|R(\lambda,\tilde{A}_h)\tilde{A}_h^{-\alpha}\| \leq c.$$

Proof. There exists $c > 0$ and $\varphi_0,\ 0 < \varphi_0 < \dfrac{\pi}{2}$, such that

$$|\mu - \mathrm{e}^{i\varphi}| \geq c|\mu|,\quad \forall \mu \in \Gamma,\ \forall \varphi_0 < |\varphi| < \pi - \varphi_0 \quad (3.7)$$

where the curve Γ is given by (3.3).

Since b is finite, choose N large enough such that whenever $\lambda \in S$ and $|\lambda| > N$ we get both $\varphi_0 < |arg\lambda| < \pi - \varphi_0$ and λ does not belong to the sector bounded by the curve $|\lambda|\Gamma = \{-\epsilon|\lambda| + t|\lambda|\mathrm{e}^{i\theta},\ t \in [0,+\infty)\} \cup \{-\epsilon|\lambda| - t|\lambda|\mathrm{e}^{-i\theta},\ t \in (-\infty, 0]\}$.

For all such choice of $\lambda \in S$, we have according to (3.7)

$$|\mu - \mathrm{e}^{\mathrm{i}\arg\lambda}| \geq c|\mu|\ \ \forall \mu \in \Gamma. \quad (3.8)$$

Consider the following integral for all $\lambda \in S$ with $|\lambda| > N$

$$I_\lambda = \int_\Gamma \frac{\mu^{-\alpha}}{\mu - \lambda}\mathrm{d}\mu.$$

By the above choice of λ, we have $\lambda \notin \Gamma$ and $\lambda \notin |\lambda|\Gamma$. Consequently, the integral has no singular points between Γ and $|\lambda|\Gamma$. Therefore, by the Cauchy theorem, we have

$$I_\lambda = \int_{|\lambda|\Gamma} \frac{\mu^{-\alpha}}{\mu - \lambda} \mathrm{d}\mu = \frac{1}{|\lambda|^\alpha} \int_\Gamma \frac{\mu^{-\alpha}}{\mu - \mathrm{e}^{\mathrm{i}\arg\lambda}} \mathrm{d}\mu.$$

Therefore, by (3.8), we get

$$|I_\lambda| \leq \frac{c}{|\lambda|^\alpha}.$$

Now, for $|\lambda| > N$ with $\lambda \in S$, we have by the resolvent identity

$$\begin{aligned} R(\lambda, \tilde{A}_h)\tilde{A}_h^{-\alpha} &= \frac{1}{2\pi i} \int_\Gamma \mu^{-\alpha} R(\lambda, \tilde{A}_h) R(\mu, \tilde{A}_h) \mathrm{d}\mu \\ &= \frac{1}{2\pi i} \int_\Gamma \frac{\mu^{-\alpha}}{\mu - \lambda} R(\lambda, \tilde{A}_h) \mathrm{d}\mu - \frac{1}{2\pi i} \int_\Gamma \frac{\mu^{-\alpha}}{\mu - \lambda} R(\mu, \tilde{A}_h) \mathrm{d}\mu \\ &= \frac{1}{2\pi i} I_\lambda R(\lambda, \tilde{A}_h) - \frac{1}{2\pi i} \int_\Gamma \frac{\mu^{-\alpha}}{\mu - \lambda} R(\mu, \tilde{A}_h) \mathrm{d}\mu. \end{aligned}$$

On the other hand, similar to the proof of Proposition 3.4,

$$\left| \int_\Gamma \frac{\mu^{-\alpha}}{\mu - \lambda} R(\mu, \tilde{A}_h) \mathrm{d}\mu \right| \leq c \int_\Gamma \frac{1}{|\mu|^{\alpha+1}} \|R(\mu, \tilde{A}_h)\| \mathrm{d}\mu \leq c',$$

where c is independent of h. Therefore for all $\lambda \in S$, with $|\lambda| > N$, we have

$$\|R(\lambda, \tilde{A}_h)\tilde{A}_h^{-\alpha}\| \leq \frac{c}{|\lambda|^\alpha} \|R(\lambda, \tilde{A}_h)\| + c \leq c \frac{1 + |\lambda|^\alpha}{|\lambda|^\alpha} + c \leq c''.$$

Now, for $\lambda \in S$ such that $|\lambda| \leq N$, we have

$$\|R(\lambda, \tilde{A}_h)\tilde{A}_h^{-\alpha}\| \leq \|R(\lambda, \tilde{A}_h)\| \|\tilde{A}_h^{-\alpha}\| \leq c(1 + |\lambda|^\alpha) \leq c(1 + N^\alpha),$$

which completes the proof with Proposition 3.4. □

Lemma 3.6. *If (3.1)–(3.5) are satisfied, then there exists $c > 0$ independent of h such that*

$$\sup_{\substack{h \in (0,h^*) \\ Re\lambda > 0}} \|R(\lambda, \tilde{A}_h)\tilde{A}_h^{-\alpha}\| \leq c. \tag{3.9}$$

Proof. For all $h \in (0, h^*)$, $m > 0$, and $B > \max\{2m, 1\}$, consider $F_h(\lambda) = R(\lambda, \tilde{A}_h)\lambda^{-\alpha}(1 - \frac{\lambda^2}{B^2})$ on the domain $D = \left\{ \lambda \in \mathbb{C} : \ Re\lambda > 0, \ m < |\lambda| \leq \frac{B}{2} \right\}$. F_h, by the maximum principle, attains its maximum for $|\lambda| = \frac{B}{2}$. Therefore,

$$|F_h(\lambda)| \leq \frac{c}{Re\lambda}.$$

If there exists $\epsilon > 0$ such that $Re\lambda > \epsilon$, then $|F_h(\lambda)| \leq c$.

Otherwise, for $0 < Re\lambda < \epsilon$, using the resolvent identity

$$R(\lambda, \tilde{A}_h) = R(i\mathrm{Im}\lambda, \tilde{A}_h) - Re\lambda R(i\mathrm{Im}\lambda, \tilde{A}_h) R(\lambda, \tilde{A}_h) \tag{3.10}$$

then, as $|\mathrm{Im}\lambda| \geq m - \epsilon$ for all $m > 0$, we have

$$\|R(\lambda, \tilde{A}_h)\| \leq c|\mathrm{Im}\lambda|^\alpha.$$

Therefore,

$$|F_h(\lambda)| \leq c|\mathrm{Im}\lambda|^{\alpha}|\lambda|^{-\alpha}\left|1-\frac{\lambda^2}{B^2}\right| \leq c.$$

Hence, in all cases, there exists $c > 0$ independent of B such that

$$|F_h(\lambda)| \leq c.$$

As a result, for all $\lambda \in D$,

$$\|R(\lambda, \tilde{A}_h)\| \leq \frac{c|\lambda|^{\alpha}}{\left|1-\frac{\lambda^2}{B^2}\right|} \leq c|\lambda|^{\alpha} \leq c(1+|\lambda|^{\alpha}).$$

If $0 < Re\lambda \leq |\lambda| \leq m$, then by (3.10) and Assumption (3.2), we get

$$\|R(\lambda, \tilde{A}_h)\| \leq c\|R(i\mathrm{Im}\lambda, \tilde{A}_h)\| \leq c \leq c(1+|\lambda|^{\alpha}).$$

Letting $B \to +\infty$ yields

$$\|R(\lambda, \tilde{A}_h)\| \leq c(1+|\lambda|^{\alpha}),\ \forall \mathrm{Re}\lambda > 0.$$

Applying Lemma 3.5, we get for $0 \leq \mathrm{Re}\lambda \leq m$,

$$\|R(\lambda, \tilde{A}_h)\tilde{A}_h^{-\alpha}\| \leq c.$$

In addition, if $\mathrm{Re}\lambda \geq m$, by the Hille Yosida theorem and Proposition 3.4, there exists some positive constants c_1 and c_2 such that

$$\|R(\lambda, \tilde{A}_h)\tilde{A}_h^{-\alpha}\| \leq c_1\frac{\|\tilde{A}_h^{-\alpha}\|}{Re\lambda} \leq c_2.$$

In all cases, we get (3.9). □

The last lemma in this section gives the necessary and sufficient conditions for the boundedness of any family of C_0 semigroups $(S_h(t))_{\substack{t\geq 0\\ h\in(0,h^*)}}$.

Lemma 3.7. *Let $(S_h(t))_{\substack{t\geq 0\\ h\in(0,h^*)}}$ be a family of C_0 semigroups on the associated Hilbert spaces $(Y_h)_{h\in(0,h^*)}$ and let $(\tilde{E}_h)_{h\in(0,h^*)}$ be the corresponding infinitesimal generators. Then $(S_h(t))_{\substack{t\geq 0\\ h\in(0,h^*)}}$ is uniformly bounded if and only if*

(*i*) $\{\lambda \in \mathbb{C} :\ Re\lambda > 0\} \subseteq \rho(\tilde{E}_h),\ \forall h \in (0, h^*)$
(*ii*) *There exists $c > 0$ independent of h such that*

$$\sup_{\substack{\xi>0\\ h\in(0,h^*)}} \xi\int_{\mathbb{R}}(\|R(\xi+i\eta, \tilde{E}_h)\|^2 + \|R(\xi+i\eta, \tilde{E}_h^*)\|^2)\mathrm{d}\eta \leq c.$$

Proof. First, we assume that $(S_h(t))$ is uniformly bounded. Then (i) holds by the Hille–Yosida theorem. As for (ii), we only need to prove that

$$\sup_{\substack{\xi>0\\ h\in(0,h^*)}} \xi\int_{\mathbb{R}}\|R(\xi+i\eta, \tilde{E}_h)x_h\|^2\mathrm{d}\eta \leq c\|x_h\|^2,\ \forall x_h \in Y_h \tag{3.11}$$

because according to the theory of adjoint semigroups, (see [30]), $S^*(t)$ is a C_0 semigroup with the same properties as $S(t)$.

Similar to the proof of Theorem 1.1 in [21], we have for all $h \in (0, h^*)$, $x_h \in Y_h$

$$\|R(\xi + i\eta, \tilde{E}_h)x_h\|^2 = \int_{\mathbb{R}} \mathrm{e}^{-i\eta s} f_h(s)\mathrm{d}s,$$

where

$$f_h(s) = \int_{\max\{0,-s\}}^{+\infty} \mathrm{e}^{-\xi(s+2u)} < S_h(u+s)x_h, S_h(u)x_h >_{Y_h,Y_h} \mathrm{d}u.$$

For $s \geq 0$, since $(S_h(t)_{h\in(0,h^*)}$ is uniformly bounded, *i.e.* $\sup_{h\in(0,h^*)} \|S_h(t)\| \leq M$, we have

$$|f_h(s)| \leq \int_0^{+\infty} M^2\|x_h\|^2 \mathrm{e}^{-\xi(s+2u)}\mathrm{d}u = \frac{M^2\|x_h\|^2}{2\xi}\mathrm{e}^{-\xi s} \leq \frac{M^2\|x_h\|^2}{2\xi}\cdot$$

For $s < 0$, we have

$$|f_h(s)| \leq \int_{-s}^{+\infty} M^2\|x_h\|^2 \mathrm{e}^{-\xi(s+2u)}\mathrm{d}u = \frac{M^2\|x_h\|^2\mathrm{e}^{\xi s}}{2\xi} \leq \frac{M^2\|x_h\|^2}{2\xi}\cdot$$

Hence, $f_h \in \mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^\infty(\mathbb{R})$ and

$$\mathfrak{F}(f_h(s)) = \frac{1}{\sqrt{2\pi}}\|R(\xi + i\eta, \tilde{E}_h)x_h\|^2.$$

Using Lemma 21.50 in [20], it follows that

$$\frac{1}{2\pi}\int_{\mathbb{R}} \|R(\xi + i\eta, \tilde{E}_h)x_h\|^2\mathrm{d}\eta = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathfrak{F}(f_h)(\tau)d\tau \leq c\|f_h\|_{L^\infty} \leq \frac{cM^2\|x_h\|^2}{2\xi}\cdot$$

Hence, (3.11) is verified.

As for the sufficient condition, since $\{\lambda \in \mathbb{C} : \ \mathrm{Re}\lambda > 0\} \subseteq \rho(\tilde{E}_h)$, with $\sigma = \frac{1}{t}$, we get for all $x_h \in Y_h$

$$\begin{aligned} S_h(t)x_h &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathrm{e}^{\lambda t}(\lambda - \tilde{E}_h)^{-1}x_h\mathrm{d}\lambda, \\ &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\mathrm{e}^{\lambda t}}{t}(\lambda - \tilde{E}_h)^{-2}x_h\mathrm{d}\lambda + \frac{\mathrm{e}^{\lambda t}}{t}(\lambda - \tilde{E}_h)^{-1}x_h \mid_{\sigma-i\infty}^{\sigma+i\infty}. \end{aligned}$$

But $\frac{\mathrm{e}^{\lambda t}}{t}(\lambda - \tilde{E}_h)^{-1}x_h|_{\sigma-i\infty}^{\sigma+i\infty} = 0$ since according to Lemma 2.1 of [33], under condition (ii), we have $\|R(\lambda, \tilde{E}_h)x_h\| \to 0$ as $|\lambda| \to +\infty$ whenever $Re\lambda > 0$. Therefore,

$$\begin{aligned} \langle S_h(t)x_h, y_h\rangle_{Y_h,Y_h} &= \left\langle \frac{1}{2\pi it}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathrm{e}^{\lambda t}(\lambda - \tilde{E}_h)^{-2}x_h\mathrm{d}\lambda, y_h\right\rangle_{Y_h,Y_h} \\ &= \frac{1}{2\pi it}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathrm{e}^{\lambda t}\left\langle(\lambda - \tilde{E}_h)^{-2}x_h, y_h\right\rangle_{Y_h,Y_h}\mathrm{d}\lambda. \end{aligned}$$

Let $\lambda = \frac{1}{t} + i\eta$ with $\eta \in \mathbb{R}$. Then

$$\langle S_h(t)x_h, y_h\rangle_{Y_h,Y_h} = \frac{e}{2\pi t}\int_{\mathbb{R}} \mathrm{e}^{i\eta t}\left\langle R^2\left(\frac{1}{t} + i\eta, \tilde{E}_h\right)x_h, y_h\right\rangle_{Y_h,Y_h}\mathrm{d}\eta.$$

Hölder's inequality yields

$$\begin{aligned}|\langle S_h(t)x_h, y_h\rangle_{Y_h,Y_h}| &= \left|\frac{e}{2\pi t}\int_{\mathbb{R}} \mathrm{e}^{i\eta t}\left\langle R\left(\frac{1}{t}+i\eta,\tilde{E}_h\right)x_h, R\left(\frac{1}{t}+i\eta,\tilde{E}_h^*\right)y_h\right\rangle_{Y_h,Y_h}\mathrm{d}\eta\right|\\ &\leq \frac{e}{2\pi t}\left(\int_{\mathbb{R}}\|R\left(\frac{1}{t}+i\eta,\tilde{E}_h\right)x_h\|^2\mathrm{d}\eta\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\|R\left(\frac{1}{t}+i\eta,\tilde{E}_h^*\right)y_h\|^2\mathrm{d}\eta\right)^{\frac{1}{2}}\\ &\leq c\|x_h\|\|y_h\|.\end{aligned}$$

Therefore

$$\|S_h(t)\| \leq c,\ \forall h \in (0, h^*).$$ □

Now, we display the main theorem which leads to the polynomial stability of the discretized problem (1.14).

Theorem 3.8. *Let* $(T_h(t))_{\substack{t\geq 0\\ h\in(0,h^*)}}$ *be a family of uniformly bounded* C_0 *semigroups on the associated Hilbert spaces* $(X_h)_{h\in(0,h^*)}$ *and let* $(\tilde{A}_h)_{h\in(0,h^*)}$ *be the corresponding infinitesimal generators such that* (3.1) *and* (3.2) *are satisfied. Then for a fixed* $\alpha > 0$, *the following statements are equivalent:*

(i)

$$\sup_{h\in(0,h^*)}\|R(is,\tilde{A}_h)\| = O(|s|^\alpha),\quad |s|\to\infty$$

(ii)

$$\sup_{h\in(0,h^*)}\|T_h(t)\tilde{A}_h^{-\alpha}\| = O(t^{-1}),\quad t\to+\infty$$

(iii)

$$\sup_{h\in(0,h^*)}\|T_h(t)\tilde{A}_h^{-1}\| = O(t^{\frac{-1}{\alpha}}),\quad t\to+\infty.$$

Proof. We begin to prove (ii) ⇔ (iii). We adapt the proof found in [7] Proposition 3.1 without the discretization parameter h. Given (ii), we have

$$\|T_h(t)\tilde{A}_h^{-\alpha n}\| = \left\|\left[T_h\left(\frac{t}{n}\right)\tilde{A}_h^{-\alpha}\right]^n\right\| \leq c\left(\frac{n}{t}\right)^n \leq c(n)t^{-n},\ \forall n\in\mathbb{N}^*,\ h\in(0,h^*),\ t\to+\infty.$$

According to the moment inequality in Theorem II.5.34 of [13], we remark that there exists a positive constant L independent of h such that, for all $\nu\in(0,1)$, we have

$$\begin{aligned}\|T_h(t)\tilde{A}_h^{-\alpha n\nu}\| &= \|\tilde{A}_h^{\alpha n(1-\nu)}T_h(t)\tilde{A}_h^{-\alpha n}\|\\ &\leq L\|\tilde{A}_h^{\alpha n}T_h(t)\tilde{A}_h^{-\alpha n}\|^{(1-\nu)}\|T_h(t)\tilde{A}_h^{-\alpha n}\|^{\nu}\\ &\leq LM^{1-\nu}c^{\nu}(n)t^{-n\nu}.\end{aligned}$$

Choose $\nu = \dfrac{1}{\alpha n}$ with $n > \dfrac{1}{\alpha}$ to get

$$\|T_h(t)\tilde{A}_h^{-1}\| \leq c(n)t^{-\frac{1}{\alpha}}.$$

Conversely, assume that (iii) holds. Then

$$\|T_h(t)\tilde{A}_h^{-n}\| = \|[T_h\left(\frac{t}{n}\right)\tilde{A}_h^{-1}]^n\| \leq \left(\frac{t}{n}\right)^{\frac{-n}{\alpha}} \leq n^{\frac{n}{\alpha}}t^{-\frac{n}{\alpha}},\ \forall n\in\mathbb{N}^*.$$

Therefore,

$$\begin{aligned}\|T_h(t)\tilde{A}_h^{-n\nu}\| &\leq c\|\tilde{A}_h^n T_h(t)\tilde{A}_h^{-n}\|^{1-\nu}\|T_h(t)\tilde{A}_h^{-n}\|^{\nu} \\ &\leq cM^{1-\nu}c(n)^{\nu}t^{-\frac{n\nu}{\alpha}}, \quad \forall \nu\in(0,1).\end{aligned}$$

Take $\nu = \dfrac{\alpha}{n}$ with $n > \alpha$ to get

$$\sup_{h\in(0,h^*)}\|T_h(t)\tilde{A}_h^{-\alpha}\| = O\left(t^{-1}\right).$$

Now, we prove the implication $(ii) \Rightarrow (i)$ (for the continuous case, see [8]). Given (ii), define

$$m_1(t) = \sup_{\substack{h\in(0,h^*)\\ s\geq t}}\|T_h(s)\tilde{A}_h^{-1}\|.$$

Notice that $m_1(t)$ is non increasing. Let $u_{0h}\in\mathcal{D}(\tilde{A}_h),\ f_{0h} = (-\tilde{A}_h + i\tau)u_{0h},\ \tau\in\mathbb{R}$, and let $v_h(t) = \mathrm{e}^{it\tau}u_{0h}$. We have

$$\begin{cases}\partial_t v_h - \tilde{A}_h v_h = i\tau\mathrm{e}^{it\tau}u_{0h} - \tilde{A}_h(\mathrm{e}^{it\tau}u_{0h}) = \mathrm{e}^{it\tau}f_{0h}\\ \qquad v_h(0) = u_{0h}.\end{cases}$$

By the Duhamel formula,

$$v_h = \mathrm{e}^{t\tilde{A}_h}u_{0h} + \int_0^t \mathrm{e}^{(t-s)\tilde{A}_h}\mathrm{e}^{i\tau s}f_{0h}\mathrm{d}s.$$

By the boundedness of the semigroup $(T_h(t))$ and the definition of m_1, we have

$$\begin{aligned}\|u_{0h}\| = \|v_h(t)\| &\leq \|T_h(t)\tilde{A}_h^{-1}\tilde{A}_h u_{0h}\| + c\ t\|f_{0h}\| \\ &\leq m_1(t)\|\tilde{A}_h u_{0h}\| + c\ t\|f_{0h}\| \\ &\leq m_1(t)(\|f_{0h}\| + |\tau|\|u_{0h}\|) + c\ t\|f_{0h}\|.\end{aligned}$$

Apply the above inequality with $t = G(|\tau|)$ where

$$G(\xi) = \begin{cases} m_{1r}^{-1}\left(\dfrac{1}{2(\xi+1)}\right) & \text{if} \quad \xi > 0 \text{ and } \dfrac{1}{2(\xi+1)} \leq m_1(0),\\ \qquad 0 & \text{if} \quad \xi > 0 \text{ and } \dfrac{1}{2(\xi+1)} > m_1(0),\end{cases}$$

where m_{1r}^{-1} is the right inverse of m_1. Therefore,

$$m_1(t)|\tau| = m_1(G(|\tau|))|\tau| \leq \frac{|\tau|}{2(|\tau|+1)} \leq \frac{1}{2}.$$

Hence,

$$\begin{aligned}\tfrac{1}{2}\|u_{0h}\| &\leq m_1(G(|\tau|))\|f_{0h}\| + c\ G(|\tau|)\|f_{0h}\| \\ &\leq \frac{\|f_{0h}\|}{2(|\tau|+1)} + c\ G(|\tau|)\|f_{0h}\| \\ &\leq \left(\tfrac{1}{2} + c\ G(|\tau|)\right)\|f_{0h}\|.\end{aligned}$$

Consequently,

$$\|(i\tau - \tilde{A}_h)^{-1}\| \leq 1 + 2c\ G(|\tau|),$$

i.e.,

$$\sup_{h\in(0,h^*)}\|(i\tau - \tilde{A}_h)^{-1}\| \leq 1 + 2c\ G(|\tau|).$$

Since, by (iii),

$$\sup_{h\in(0,h^*)} \|T_h(t)\tilde{A}_h^{-1}\| \leq Mt^{\frac{-1}{\alpha}}, \quad t \to +\infty,$$

then, as m_1 is non-increasing, we get

$$m_1(t) \leq Mt^{\frac{-1}{\alpha}}, \quad t \to +\infty.$$

Besides, as the inverse of $t^{\frac{-1}{\alpha}}$ is $t^{-\alpha}$, then

$$G(\xi) \leq m_{1r}^{-1}\left(\frac{1}{2(\xi+1)}\right) \leq C\left(\frac{1}{2(\xi+1)}\right)^{-\alpha} = C(2(\xi+1))^{\alpha} \leq c\xi^{\alpha}, \quad \xi \to +\infty.$$

Finally, we get

$$\sup_{h\in(0,h^*)} \|(i\tau - \tilde{A}_h)^{-1}\| \leq 1 + 2c|\tau|^{\alpha} \leq c|\tau|^{\alpha}, \quad |\tau| \to +\infty.$$

It remains to prove that $(i) \Rightarrow (ii)$. For this aim, for all $h \in (0,h^*)$, let $\mathbf{X_h} = X_h \times X_h$ and consider the operator $\tilde{\mathbf{A}}_\mathbf{h}$ given by the operator matrix

$$\tilde{\mathbf{A}}_\mathbf{h} = \begin{pmatrix} \tilde{A}_h & \tilde{A}_h^{-\alpha} \\ & \\ 0 & \tilde{A}_h \end{pmatrix},$$

where $\mathcal{D}(\tilde{\mathbf{A}}_\mathbf{h}) = \mathcal{D}(\tilde{A}_h) \times \mathcal{D}(\tilde{A}_h)$. For all $h \in (0,h^*)$ and all $\lambda_h \in \rho(\tilde{A}_h)$, we have

$$R(\lambda_h, \tilde{\mathbf{A}}_\mathbf{h}) = \begin{pmatrix} R(\lambda_h, \tilde{A}_h) & R^2(\lambda_h, \tilde{A}_h)\tilde{A}_h^{-\alpha} \\ 0 & R(\lambda_h, \tilde{A}_h) \end{pmatrix}.$$

Indeed,

$$R(\lambda_h, \tilde{\mathbf{A}}_\mathbf{h})(\lambda_h - \tilde{\mathbf{A}}_\mathbf{h}) = (\lambda_h - \tilde{\mathbf{A}}_\mathbf{h})R(\lambda_h, \tilde{\mathbf{A}}_\mathbf{h}) = \begin{pmatrix} I_h & 0 \\ 0 & I_h \end{pmatrix}.$$

Therefore, $\rho(\tilde{\mathbf{A}}_\mathbf{h}) = \rho(\tilde{A}_h)$ and for all $h \in (0,h^*)$, the operator $\tilde{\mathbf{A}}_\mathbf{h}$ is the generator of the C_0 semigroup $(\mathbf{T_h}(t))_{t\geq 0}$ on $\mathbf{X_h}$ defined by

$$\mathbf{T_h}(t) = \begin{pmatrix} T_h(t) & tT_h(t)\tilde{A}_h^{-\alpha} \\ 0 & T_h(t) \end{pmatrix}.$$

In fact,

$$\begin{aligned} \widehat{\mathbf{T_h}(t)} &= \begin{pmatrix} \widehat{T_h(t)} & \widehat{tT_h(t)}\tilde{A}_h^{-\alpha} \\ 0 & \widehat{T_h(t)} \end{pmatrix} \\ &= \begin{pmatrix} R(\lambda_h, \tilde{A}_h) & R^2(\lambda_h, \tilde{A}_h)\tilde{A}_h^{-\alpha} \\ 0 & R(\lambda_h, \tilde{A}_h) \end{pmatrix} \\ &= R(\lambda_h, \tilde{\mathbf{A}}_\mathbf{h}), \end{aligned}$$

where $\widehat{\mathbf{T_h}(t)}$ is the Laplace transform of $T_h(t)$. Since for all $h \in (0,h^*)$ we have

$$\|R(is, \tilde{A}_h)\| = O(|s|^{\alpha}), \text{ as } |s| \to +\infty,$$

then by Lemma 3.6 we get

$$\sup_{\substack{h\in(0,h^*)\\ Re\lambda>0}} \|R(\lambda, \tilde{A}_h)\tilde{A}_h^{-\alpha}\| \leq c.$$

Hence, for all $x_h = (x_{1h}, x_{2h}) \in \mathbf{X_h}$, and $Re\lambda_h > 0$, we have

$$\|R(\lambda_h, \tilde{\mathbf{A}}_{\mathbf{h}})x_h\|^2 = \left\| \begin{pmatrix} R(\lambda_h, \tilde{A}_h)x_{1h} + R^2(\lambda_h, \tilde{A}_h)\tilde{A}_h^{-\alpha}x_{2h} \\ R(\lambda_h, \tilde{A}_h)x_{2h} \end{pmatrix} \right\|^2 \leq c\left(\|R(\lambda_h, \tilde{A}_h)x_{1h}\|^2 + \|R(\lambda_h, \tilde{A}_h)x_{2h}\|^2\right).$$

Similarly, we have

$$\|R(\lambda_h, \tilde{\mathbf{A}}_{\mathbf{h}}^*)x_h\|^2 \leq c(\|R(\lambda_h, \tilde{A}_h^*)x_{1h}\|^2 + \|R(\lambda_h, \tilde{A}_h^*)x_{2h}\|^2).$$

Indeed, we have

$$\tilde{\mathbf{A}}_{\mathbf{h}}^* = \begin{pmatrix} \tilde{A}_h^* & (\tilde{A}_h^*)^{-\alpha} \\ 0 & \tilde{A}_h^* \end{pmatrix}.$$

In order to get

$$\sup_{\substack{h\in(0,h^*)\\ Re\lambda>0}} \|R(\lambda, \tilde{A}_h^*)(\tilde{A}_h^*)^{-\alpha}\| \leq c,$$

we must have at least

$$\|R(is, \tilde{A}_h^*)\| = O(|s|^\alpha), \text{ as } |s| \to +\infty.$$

Actually, we have

$$R(is, \tilde{A}_h^*) = [(is - \tilde{A}_h^*)]^{-1} = [(is - \tilde{A}_h)^*]^{-1} = R(is, \tilde{A}_h)^*.$$

Therefore, we get

$$\|R(is, \tilde{A}_h^*)\| \leq \|R(is, \tilde{A}_h)\| = O(|s|^\alpha), \text{ as } |s| \to +\infty.$$

Now, by Lemma 3.7, since for all $h \in (0, h^*)$, $T_h(t)$ is a uniformly bounded family of C_0 semigroups, we get

$$\sup_{\substack{\xi>0\\ h\in(0,h^*)}} \xi \int_{\mathbb{R}} (\|R(\xi + i\eta, \tilde{A}_h)x_h\|^2) + (\|R(\xi + i\eta, \tilde{A}_h^*)x_h\|^2)\mathrm{d}\eta < \infty, \quad \forall x_h \in X_h.$$

Hence,

$$\sup_{\substack{\xi>0\\ h\in(0,h^*)}} \xi \int_{\mathbb{R}} (\|R(\xi + i\eta, \tilde{\mathbf{A}}_{\mathbf{h}})x_h\|^2) + (\|R(\xi + i\eta, \tilde{\mathbf{A}}_{\mathbf{h}}^*)x_h\|^2)\mathrm{d}\eta < \infty, \quad \forall x_h \in \mathbf{X_h}.$$

Therefore, $(\mathbf{T_h}(t))_{\substack{t\geq 0\\ h\in(0,h^*)}}$ is uniformly bounded over $(\mathbf{X_h})_{h\in(0,h^*)}$ by Lemma 3.7. Since $(T_h(t))_{\substack{t\geq 0\\ h\in(0,h^*)}}$ is uniformly bounded over $(X_h)_{h\in(0,h^*)}$, the definition of $\mathbf{T_h}(t)$ implies that

$$\sup_{\substack{t\geq 0\\ h\in(0,h^*)}} \|tT_h(t)\tilde{A}_h^{-\alpha}\| < +\infty.$$

□

4. Preliminary lemmas

In this section, we fix $l \in \mathbb{N}$, l even. We introduce the Hilbert space $X_h = V_h \times V_h$ and the operator $\tilde{A}_{l,h} : X_h \to X_h$ defined by

$$\tilde{A}_{l,h} = \begin{pmatrix} 0 & (1+h^\theta)^{-1}\left(I + h^\theta A_h^{\frac{l}{2}}\right) \\ -(1+h^\theta)^{-1}(I + h^\theta A_h^{\frac{l}{2}})A_h & -h^\theta A_h^{1+\frac{l}{2}} - B_hB_h^* \end{pmatrix}. \tag{4.1}$$

The space X_h is here equipped with the inner product

$$\left(\begin{pmatrix} u_h \\ v_h \end{pmatrix}, \begin{pmatrix} \tilde{u}_h \\ \tilde{v}_h \end{pmatrix}\right)_{X_h} = a(u_h,\, \tilde{u}_h) + (v_h,\, \tilde{v}_h), \quad \forall (u_h,\, v_h),\, (\tilde{u}_h,\, \tilde{v}_h) \in X_h, \tag{4.2}$$

with associated norm $\|.\|_{X_h}$. Therefore, the system (1.14) is equivalent to the following first order system in X_h:

$$\dot{z}_h(t) = \tilde{A}_{l,h} z_h(t), \quad z_h(0) = z_{0h},$$

where $z_h(t) = \begin{pmatrix} \omega_h(t) \\ (1+h^\theta)(I + h^\theta A_h^{\frac{l}{2}})^{-1}\dot{\omega}_h(t) \end{pmatrix}$ and $z_{0h} = \begin{pmatrix} \omega_{0h} \\ \omega_{1h} \end{pmatrix}$. Note that we recover the system (1.9) in the particular case $l = 0$. We define the sesquilinear form $a^l(.,\, .)$ on V_h by

$$a^l(u_h,\, v_h) = \left(A_h^{1+\frac{l}{2}} u_h,\, v_h\right),\, \forall (u_h,\, v_h) \in V_h \times V_h,$$

i.e.

$$a^l(u_h,\, v_h) = \sum_{k=1}^{N(h)} c_k \overline{d_k} \lambda_{k,\,h}^{2+l},$$

for $u_h = \sum_{k=1}^{N(h)} c_k \varphi_{k,\,h}$ and $v_h = \sum_{k=1}^{N(h)} d_k \varphi_{k,\,h}$. Remark that $a^0(.,.) = a(.,.)$ defined in (1.3).

We easily prove that $\tilde{A}_{l,h}$ is maximal dissipative in X_h, hence $(T_{l,h}(t)) = (\mathrm{e}^{t\tilde{A}_{l,h}})$ forms a family of C_0 semigroups of contractions in X_h. In the sequel we prove that the family $(\tilde{A}_{l,h})_{h\in(0,h^*)}$ satisfies condition i) in Theorem 3.1 and the properties (3.1) and (3.2) of Subsection 3.2. Condition i) in Theorem 3.1 or (3.1) in Section 3.2 is satisfied due to the following lemma:

Lemma 4.1. *The spectrum of the operator $\tilde{A}_{l,h}$ contains no point on the imaginary axis.*

Proof. Suppose that $\begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix} \in X_h$ and $\omega \in \mathbb{R}$ are such that

$$\tilde{A}_{l,h} \begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix} = i\omega \begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix}.$$

Then, by using the definition (4.1) of $\tilde{A}_{l,h}$, we have

$$\begin{cases} \psi_h = i\omega(1+h^\theta)\left(I + h^\theta A_h^{\frac{l}{2}}\right)^{-1} \varphi_h \\ -(1+h^\theta)^{-1}\left(I + h^\theta A_h^{\frac{l}{2}}\right) A_h \varphi_h - i\omega(1+h^\theta)\left(h^\theta A_h^{1+\frac{l}{2}} + B_h B_h^*\right)\left(I + h^\theta A_h^{\frac{l}{2}}\right)^{-1} \varphi_h = \\ -\omega^2(1+h^\theta)(I + h^\theta A_h^{\frac{l}{2}})^{-1}\varphi_h. \end{cases} \tag{4.3}$$

Let $\chi_h = (1+h^\theta)(I + h^\theta A_h^{\frac{l}{2}})^{-1}\varphi_h$ then the second relation of (4.3) becomes

$$(1+h^\theta)^{-2}\left(I + h^\theta A_h^{\frac{l}{2}}\right)^2 A_h \chi_h + i\omega(h^\theta A_h^{1+\frac{l}{2}} + B_h B_h^*)\chi_h = \omega^2 \chi_h. \tag{4.4}$$

If $\omega = 0$, then taking the inner product of (4.4) with $\chi_h \in V_h$, we get $(I + h^\theta A_h^{\frac{l}{2}}) A_h^{\frac{1}{2}} \chi_h = 0$ and hence $\chi_h = 0$ which implies by the definition of χ_h that $\varphi_h = \psi_h = 0$.

It then remains to consider the case $\omega \neq 0$. In that case, we take the imaginary part of the inner product (in H) of (4.4) with $\chi_h \in V_h$ to obtain

$$\begin{aligned} 0 &= \omega h^\theta \left(A_h^{1+\frac{l}{2}} \chi_h,\, \chi_h \right) + \omega \left(B_h B_h^* \chi_h,\, \chi_h \right) \\ &= \omega h^\theta \left(A_h^{\frac{1}{2}+\frac{l}{4}} \chi_h,\, A_h^{\frac{1}{2}+\frac{l}{4}} \chi_h \right) + \omega \left(B_h^* \chi_h,\, B_h^* \chi_h \right)_U , \end{aligned}$$

that is to say

$$h^\theta \left\| A_h^{\frac{1}{2}+\frac{l}{4}} \chi_h \right\|^2 + \|B_h^* \chi_h\|_U^2 = 0.$$

This leads to $\chi_h = 0$, and hence $\varphi_h = \psi_h = 0$. □

Our main goal is to prove condition (ii) of Theorem 3.1 in the case $l = 0$ and condition (i) of Theorem 3.8 as well as (3.2) in the case $l \geq 2$ and $\alpha = 2l$. In that last case ($l \geq 2$), these two conditions are equivalent to

$$\sup_{h \in (0,h^*),\ s \in \mathbb{R}} (1 + |s|^{2l})^{-1} \|R(is, \tilde{A}_{l,h})\|_{\mathcal{L}(X_h)} < \infty. \tag{4.5}$$

To prove this above property, we use a contradiction argument. More precisely, we will assume that, for all $n \in \mathbb{N}$, there exist $h_n \in (0, h^*)$, $\omega_n \in \mathbb{R}$ and $z_n = \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} \in X_{h_n}$ such that

$$\|z_n\|_{X_{h_n}}^2 = a(\varphi_n, \varphi_n) + \|\psi_n\|^2 = 1, \forall n \in \mathbb{N}, \tag{4.6}$$

and

$$(1 + |\omega_n|^{2l}) \left\| i\omega_n z_n - \tilde{A}_{l,h_n} z_n \right\|_{X_{h_n}} \to 0, \text{ as } n \to \infty, \tag{4.7}$$

where $l = 0$ in the setting of Theorem 3.1.

Lemma 4.2. *Assume that the sequences (h_n), (ω_n), (z_n) satisfy (4.6) and (4.7). Then, we have*

$$(1 + |\omega_n|^{2l})(h_n^\theta a^l(\psi_n, \psi_n) + \left\|B_{h_n}^* \psi_n\right\|_U^2) \to 0, \text{ as } n \to \infty \tag{4.8}$$

and

$$\lim_{n\to\infty} a(\varphi_n, \varphi_n) = \lim_{n\to\infty} \|\psi_n\|^2 = \frac{1}{2}. \tag{4.9}$$

Proof. For (4.8), we take the inner product in X_{h_n} of $i\omega_n z_n - \tilde{A}_{l,h_n} z_n$ with z_n and take the real part. We obtain

$$\begin{aligned} &\Re \left(i\omega_n z_n - \tilde{A}_{l,h_n} z_n,\, z_n \right)_{X_{h_n}} \\ &= -\Re \left(\begin{pmatrix} (1+h_n^\theta)^{-1} \left(I + h_n^\theta A_{h_n}^{\frac{l}{2}} \right) \psi_n \\ -(1+h_n^\theta)^{-1} \left(I + h_n^\theta A_{h_n}^{\frac{l}{2}} \right) A_{h_n} \varphi_n - h_n^\theta A_{h_n}^{1+\frac{l}{2}} \psi_n - B_{h_n} B_{h_n}^* \psi_n \end{pmatrix}, \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} \right)_{X_{h_n}} \\ &= \Re \left(-(1+h_n^\theta)^{-1} \left(\left(I + h_n^\theta A_h^{\frac{l}{2}} \right) A_{h_n} \psi_n,\, \varphi_n \right) + (1+h_n^\theta)^{-1} \left(\left(I + h_n^\theta A_{h_n}^{\frac{l}{2}} \right) A_{h_n} \varphi_n,\, \psi_n \right) \right. \\ &\quad \left. + \left(h_n^\theta A_{h_n}^{1+\frac{l}{2}} \psi_n + B_{h_n} B_{h_n}^* \psi_n,\, \psi_n \right) \right) \\ &= \left(h_n^\theta A_{h_n}^{1+\frac{l}{2}} \psi_n + B_{h_n} B_{h_n}^* \psi_n,\, \psi_n \right). \end{aligned}$$

Then

$$\left(1 + |\omega_n|^{2l}\right) \Re \left(i\omega_n z_n - \tilde{A}_{l,h_n} z_n,\, z_n \right)_{X_{h_n}} = \left(1 + |\omega_n|^{2l}\right) \left(h_n^\theta a^l(\psi_n, \psi_n) + \left\|B_{h_n}^* \psi_n\right\|_U^2 \right) \to 0 \text{ by (4.7)}.$$

In order to prove (4.9), we introduce the operator

$$A_{1h_n} = (1 + h_n^\theta)^{-1}(I + h_n^\theta A_{h_n}^{\frac{l}{2}}) \begin{pmatrix} 0 & I \\ -A_{h_n} & 0 \end{pmatrix}. \tag{4.10}$$

We have

$$\tilde{A}_{l,h_n} \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} = A_{1h_n} \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} - \begin{pmatrix} 0 \\ h_n^\theta A_{h_n}^{1+\frac{l}{2}} \psi_n + B_{h_n} B_{h_n}^* \psi_n \end{pmatrix}, \forall \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} \in X_{h_n}.$$

We take the norm $\|.\|_{X_{h_n}}$ of $i\omega_n z_n - A_{1h_n} z_n + \begin{pmatrix} 0 \\ h_n^\theta A_{h_n}^{1+\frac{l}{2}} \psi_n \end{pmatrix}$ to obtain

$$\begin{aligned} &(1 + |\omega_n|^{2l}) \left\| i\omega_n z_n - A_{1h_n} z_n + \begin{pmatrix} 0 \\ h_n^\theta A_{h_n}^{1+\frac{l}{2}} \psi_n \end{pmatrix} \right\|_{X_{h_n}}^2 \\ &= \left(1 + |\omega_n|^{2l}\right) \left\| i\omega_n z_n - \tilde{A}_{l,h_n} z_n - \begin{pmatrix} 0 \\ B_{h_n} B_{h_n}^* \psi_n \end{pmatrix} \right\|_{X_{h_n}}^2 \\ &\leq 2 \left(1 + |\omega_n|^{2l}\right) \left(\left\| i\omega_n z_n - \tilde{A}_{l,h_n} z_n \right\|_{X_{h_n}}^2 + \|B_{h_n} B_{h_n}^* \psi_n\|^2 \right) \\ &\leq C \left(1 + |\omega_n|^{2l}\right) \left(\left\| i\omega_n z_n - \tilde{A}_{l,h_n} z_n \right\|_{X_{h_n}}^2 + \|B_{h_n}^* \psi_n\|_U^2 \right) \to 0, \end{aligned}$$

by (4.7) and (4.8). Therefore

$$\left(1 + |\omega_n|^{2l}\right) \left\| i\omega_n z_n - A_{1h_n} z_n + \begin{pmatrix} 0 \\ h_n^\theta A_{h_n}^{1+\frac{l}{2}} \psi_n \end{pmatrix} \right\|_{X_{h_n}}^2 \to 0. \tag{4.11}$$

We can now prove (4.9). If $l = 0$, then by Lemma 4.3 below there exists $n_0 \in \mathbb{N}$ such that the sequence $(|\omega_n|)_{n \geq n_0}$ is bounded away from zero. Hence, we may write

$$\begin{aligned} \Im \left(i\omega_n z_n - A_{1h_n} z_n + \begin{pmatrix} 0 \\ h_n^\theta A_{h_n}^{1+\frac{l}{2}} \psi_n \end{pmatrix}, \tfrac{1}{\omega_n} \begin{pmatrix} \varphi_n \\ -\psi_n \end{pmatrix} \right)_{X_{h_n}} &= \left(\begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix}, \begin{pmatrix} \varphi_n \\ -\psi_n \end{pmatrix} \right)_{X_{h_n}} \\ &= a(\varphi_n, \varphi_n) - \|\psi_n\|_{V_{h_n}}^2 \end{aligned}$$

and so, by (4.11) and (4.6), we have

$$\lim_{n\to\infty} (a(\varphi_n, \varphi_n) - \|\psi_n\|_{V_{h_n}}^2) = 0.$$

This relation and (4.6) lead to (4.9). □

Lemma 4.3. *Assume that* (4.6) *and* (4.7) *hold. Then there exists* $n_0 \in \mathbb{N}$ *such that the sequence* $(|\omega_n|)_{n \geq n_0}$ *is uniformly bounded away from zero.*

Proof. By a contradiction argument, we show that the sequence $(\omega_n)_n$ contains no subsequence converging to zero. Namely suppose that such a subsequence exists. For the sake of simplicity, we still denote it by $(\omega_n)_n$. Hence (4.11) implies that

$$-A_{1h_n} z_n + \begin{pmatrix} 0 \\ h_n^\theta A_{h_n}^{1+\frac{l}{2}} \psi_n \end{pmatrix} = \begin{pmatrix} -(1 + h_n^\theta)^{-1}(I + h_n^\theta A_{h_n}^{\frac{l}{2}})\psi_n \\ (1 + h_n^\theta)^{-1}(I + h_n^\theta A_{h_n}^{\frac{l}{2}}) A_{h_n} \varphi_n + h_n^\theta A_{h_n}^{1+\frac{l}{2}} \psi_n \end{pmatrix} \to 0 \text{ in } X_{h_n}. \tag{4.12}$$

Taking the inner product of first component in (4.12) with ψ_n, we get

$$\left(1+h_n^\theta\right)^{-1} a\left(\left(I+h_n^\theta A_{h_n}^{\frac{l}{2}}\right)\psi_n,\, \psi_n\right) = \left(1+h_n^\theta\right)^{-1}\left(a\left(\psi_n,\, \psi_n\right)+h_n^\theta a^l\left(\psi_n,\, \psi_n\right)\right) \to 0.$$

As $h_n \leq h^*$, then, by (4.8), we get

$$\left\|A_{h_n}^{\frac{1}{2}}\psi_n\right\|^2 = a(\psi_n,\, \psi_n) \to 0. \tag{4.13}$$

The convergence of the first component in (4.12) implies that

$$\left\|A_{h_n}^{\frac{1}{2}}\left(I+h_n^\theta A_{h_n}^{\frac{l}{2}}\right)\psi_n\right\|^2 \to 0.$$

Therefore, (4.13) yields

$$h_n^\theta A_{h_n}^{\frac{(1+l)}{2}}\psi_n \to 0 \ \text{ in } H. \tag{4.14}$$

The second component in (4.12) and the fact that $\alpha\|x\|^2 \leq \|A_h^{\frac{1}{2}}x\|^2 = a(x,x)$ for all $x \in V_h$ imply that

$$\left(1+h_n^\theta\right)^{-1}\left(I+h_n^\theta A_{h_n}^{\frac{l}{2}}\right)A_{h_n}^{\frac{1}{2}}\varphi_n + h_n^\theta A_{h_n}^{\frac{1+l}{2}}\psi_n \to 0 \ \text{ in } H,$$

which, by (4.14), yields

$$\left(1+h_n^\theta\right)^{-1}\left(I+h_n^\theta A_{h_n}^{\frac{l}{2}}\right)A_{h_n}^{\frac{1}{2}}\varphi_n \to 0 \ \text{ in } H.$$

Thus, as $h_n \leq h^*$, we get

$$a(\varphi_n,\, \varphi_n) \to 0.$$

This above relation and (4.13) contradict (4.6). □

According to the above lemma, we note that the coefficient $1+|\omega_n|^{2l}$ becomes equivalent to $|\omega_n|^{2l}$. Now, we introduce the operator D_{1h_n} defined by

$$D_{1h_n} = \begin{pmatrix} 0 & I \\ -A_{h_n} & 0 \end{pmatrix}.$$

Note that $A_{1h_n} = (1+h_n^\theta)^{-1}(I+h_n^\theta A_{h_n}^{\frac{l}{2}})D_{1h_n}$. We then use the following spectral basis of the operator D_{1h_n}. Namely, we extend the definitions of $\lambda_{k,\,h_n}$ and of $\varphi_{k,\,h_n}$ for $k \in \{-1,\ldots,-N(h_n)\}$ by setting $\lambda_{k,\,h_n} = -\lambda_{-k,\,h_n}$ and $\varphi_{k,\,h_n} = \varphi_{-k,\,h_n}$. Then an orthonormal basis of X_{h_n} formed by the eigenvectors of D_{1h_n} is given by

$$\Psi_{k,\,h_n} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\frac{i}{\lambda_{k,\,h_n}}\varphi_{k,\,h_n} \\ \varphi_{k,\,h_n} \end{pmatrix},\ 0<|k|\leq N(h_n), \tag{4.15}$$

of associated eigenvalue $i\lambda_{k,\,h_n}$, that is to say

$$D_{1h_n}\Psi_{k,\,h_n} = i\lambda_{k,\,h_n}\Psi_{k,\,h_n}.$$

Consequently, for all $n \in \mathbb{N}$, there exist complex coefficients $(c_k^n)_{0<|k|\leq N(h_n)}$ such that

$$z_n = \sum_{0<|k|\leq N(h_n)} c_k^n \Psi_{k,\,h_n}. \tag{4.16}$$

The normalization condition (4.6) implies that

$$\sum_{0<|k|\leq N(h_n)} |c_k^n|^2 = 1.$$

Let ϵ be the constant from Proposition 2.4 (if $l=0$, we recover the condition from Prop. 2.1). For any $n \in \mathbb{N}$, we define

$$M_l(h_n) = \max\left\{k \in \{1,\ldots,N(h_n)\} \,\middle|\, h_n^\theta(\lambda_k)^2 \le \frac{\epsilon}{\lambda_k^l}\right\}, \tag{4.17}$$

if $h_n^\theta(\lambda_1)^2 \le \frac{\epsilon}{\lambda_1^l}$ and $M_l(h_n) = 0$ otherwise.

Lemma 4.4. *Suppose that the sequences (h_n), (ω_n), (z_n) satisfy (4.6) and (4.7). Then, we have*

$$\psi_n = \frac{1}{\sqrt{2}} \sum_{k=1}^{N(h_n)} \left(c_k^n + c_{-k}^n\right) \varphi_{k,h_n}, \tag{4.18}$$

$$\sum_{M_l(h_n)<k\le N(h_n)} |\omega_n|^{2l} \left|c_k^n + c_{-k}^n\right|^2 \to 0, \tag{4.19}$$

and

$$\sum_{0<|k|\le M_l(h_n)} |\omega_n|^{2l} \left|\omega_n - \left(1+h_n^\theta\right)^{-1}\left(\lambda_{k,h_n} + h_n^\theta \lambda_{k,h_n}^{1+l}\right)\right|^2 |c_k^n|^2 \to 0. \tag{4.20}$$

Proof. Relation (4.18) follows directly by taking the second component in (4.16) and by using (4.15) and the fact that $\varphi_{k,h} = \varphi_{-k,h}$.

On the other hand, we use (4.16) and the fact that Ψ_{k,h_n} is an eigenvector of D_{1h_n} associated with eigenvalue $i\lambda_{k,h_n}$ to obtain for all $\tilde\psi_{h_n} \in X_{h_n}$

$$\left(i\omega_n z_n - A_{1h_n} z_n, \tilde\psi_{h_n}\right)_{X_{h_n}} = \sum_{0<|k|\le N(h_n)} i\left(\omega_n - \left(1+h_n^\theta\right)^{-1}\left(\lambda_{k,h_n} + h_n^\theta\lambda_{k,h_n}^{1+l}\right)\right) c_k^n \left(\Psi_{k,h_n}, \tilde\psi_{h_n}\right)_{X_{h_n}}. \tag{4.21}$$

From (4.8) and (4.18), it follows that

$$|\omega_n|^{2l} h_n^\theta a^l(\psi_n,\psi_n) = \frac{1}{2}\sum_{k=1}^{N(h_n)} h_n^\theta \lambda_{k,h_n}^{2+l} |\omega_n|^{2l} \left|c_k^n + c_{-k}^n\right|^2 \to 0. \tag{4.22}$$

As we have $\lambda_k \le \lambda_{k,h_n}$ for all $k \in \{1,\ldots,N(h_n)\}$ and by the definition (4.17), we obtain (4.19).

By (2.9), we have

$$h_n^\theta \lambda_{k,h_n}^2 \le h_n^\theta(\lambda_k + (Ch_n^\theta\lambda_k^2)^2)^2 \le 2h_n^\theta\lambda_k^2 + 2C^4 h_n^\theta(h_n^\theta\lambda_k^2)^4 \le C\frac{\epsilon}{\lambda_k^l} + C\frac{\epsilon^4}{\lambda_k^{4l}} \le C'\frac{\epsilon}{\lambda_k^l} \tag{4.23}$$

for $h_n^\theta(\lambda_k)^2 \le \frac{\epsilon}{\lambda_k^l}$. So, by using (4.22) and again (2.9), there exists a constant C independent of h_n such that

$$\begin{aligned} h_n^{2\theta}\sum_{k=1}^{M_l(h_n)} \lambda_{k,h_n}^{4+2l} |\omega_n|^{2l}\left|c_k^n + c_{-k}^n\right|^2 &\le C\sum_{k=1}^{M_l(h_n)} \epsilon h_n^\theta \lambda_{k,h_n}^{2+l}|\omega_n|^{2l}\left|c_k^n + c_{-k}^n\right|^2 \\ &\le C\epsilon \sum_{k=1}^{M_l(h_n)} h_n^\theta\lambda_{k,h_n}^{2+l}|\omega_n|^{2l}\left|c_k^n + c_{-k}^n\right|^2 \to 0. \end{aligned} \tag{4.24}$$

We also have for all $\tilde\psi_{h_n} \in X_{h_n}$

$$\left(\begin{pmatrix} 0 \\ h_n^\theta A_{h_n}^{1+\frac{l}{2}}\psi_n \end{pmatrix}, \tilde\psi_{h_n}\right)_{X_{h_n}} = \sum_{0<|k|\le N(h_n)} \frac{h_n^\theta}{2}\lambda_{k,h_n}^{2+l}(c_k^n + c_{-k}^n)\left(\Psi_{k,h_n}, \tilde\psi_{h_n}\right)_{X_{h_n}} \tag{4.25}$$

because l is even. Relations (4.24) and (4.25) imply that for all $\tilde{\psi}_{h_n} \in X_{h_n}$

$$|\omega_n|^l \left(\begin{pmatrix} 0 \\ h_n^\theta A_{h_n}^{1+\frac{l}{2}} \psi_n \end{pmatrix} - \sum_{M_l(h_n)<|k|\le N(h_n)} \frac{h_n^\theta}{2} \lambda_{k,\,h_n}^{2+l} (c_k^n + c_{-k}^n) \Psi_{k,\,h_n},\, \tilde{\psi}_{h_n} \right)_{X_{h_n}} \to 0.$$

So, we obtain with (4.11), (4.21) and the above relation, for all $\tilde{\psi}_{h_n} \in X_{h_n}$, that the inner product in X_{h_n} of $\tilde{\psi}_{h_n}$ with

$$\sum_{0<|k|\le N(h_n)} i\,|\omega_n|^l \left(\omega_n - \left(1+h_n^\theta\right)^{-1} \left(\lambda_{k,\,h_n} + h_n^\theta \lambda_{k,\,h_n}^{1+l}\right)\right) c_k^n \Psi_{k,\,h_n}$$
$$+ \sum_{M_l(h_n)<|k|\le N(h_n)} \tfrac{h_n^\theta}{2} |\omega_n|^l \lambda_{k,\,h_n}^{2+l} \left(c_k^n + c_{-k}^n\right) \Psi_{k,\,h_n}$$

tends to zero. Taking $\tilde{\psi}_{h_n} \in X_{h_n}$ to be equal to the same above relation and as the family $(\Psi_{k,\,h_n})$ is orthogonal, the above relation implies (4.20). □

5. Proof of Theorem 1.2

We use the results of the previous section with $l = 0$ and set, for shortness, $\tilde{A}_h := \tilde{A}_{0,h}$ and $M(h_n) := M_0(h_n)$.

Proof of Theorem 1.2. This proof is based on Theorem 3.1. First, for all $h \in (0, h^*)$, the family $(\mathrm{e}^{t\tilde{A}_h})$ forms a contraction semigroup. The family $(\tilde{A}_h)$ satisfies the condition i) in Theorem 3.1 owing to Lemma 4.1. To show that the family $(\tilde{A}_h)$ also satisfies the condition ii) in Theorem 3.1, we use a contradiction argument. Let $(h_n)_n$, $(\omega_n)_n$ and $z_n = \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} \in \mathcal{D}(\tilde{A}_{h_n})$ be three sequences satisfying (4.6) and (4.7). Notice that for $k_m \in A_k$, we have

$$\begin{aligned} \lambda_{k_m,h_n} - \lambda_{k_{m-1}+l_{m-1}-1,h_n} &\ge \lambda_{k_m} - \lambda_{k_{m-1}+l_{m-1}-1} - c\epsilon \\ &= \lambda_{k_m} - \lambda_{k_{m-1}} - c\epsilon \ge \gamma_0' - c\epsilon \\ &\ge \tfrac{\gamma_0'}{2} =: \gamma' \end{aligned}$$

for $\epsilon \le \frac{\gamma_0'}{2c}$ by (2.8) and (2.9). We now introduce the set

$$\mathcal{F} = \Big\{ n \in \mathbb{N} \,\big|\, \exists k(n) \in \{1,\dots,M\},\, \exists k_{m(n)} \in A_{k(n)},\, \left|k_{m(n)}\right| \le M(h_n) \text{ and}$$
$$\left|k_{m(n)+k(n)-1} + l_{m(n)+k(n)-1}\right| \le N(h_n) \text{ such that } \left|\omega_n - \lambda_{k_{m(n)},\,h_n}\right| < \frac{\gamma'}{2} \Big\}. \quad (5.1)$$

We distinguish two cases.

First case. The set $\mathcal{F}$ is infinite. Then, without loss of generality, we can suppose that $\mathcal{F} = \mathbb{N}$ (otherwise we take a subsequence of (ω_n)). Then, by reducing the value of γ' if needed, we can assume that for all $n \in \mathbb{N}$, we have that for all $k_m \in A_{k'}$, $k' = 1,\dots,M$ with $m \ne m(n)$,

$$\left|\omega_n - \lambda_{k_{m+j}+l,\,h_n}\right| \ge \frac{\gamma'}{2},\ \forall j = 0,\dots,k'-1,\ \forall l = 0,\dots,l_{m+j}-1.$$

By using (4.20), we obtain that

$$\sum_{k=1}^{M} \sum_{\substack{k_m \in A_k \\ m \ne m(n)}} \sum_{j=0}^{k-1} \sum_{\substack{l=0 \\ 0<|k_{m+j}+l_{m+j}-1|\le M(h_n)}}^{l_{m+j}-1} \left|c_{k_{m+j}+l}^n\right|^2 \to 0. \quad (5.2)$$

Define now

$$\tilde{\psi}_n = \frac{1}{\sqrt{2}} \sum_{j=0}^{k(n)-1} \sum_{l=0}^{l_{m(n)+j}-1} c^n_{k_{m(n)+j}+l} \varphi_{k_{m(n)+j}+l,\, h_n}. \tag{5.3}$$

We have, by (4.18),

$$\psi_n = \frac{1}{\sqrt{2}} \sum_{k=1}^{M} \sum_{k_m \in A_k} \sum_{\substack{j=0 \\ 1 \le k_{m+j}+l \le N(h_n)}}^{k-1} \sum_{l=0}^{l_{m+j}-1} (c^n_{k_{m+j}+l} + c^n_{-(k_{m+j}+l)}) \varphi_{k_{m+j}+l,\, h_n},$$

and so, by (5.2) and (4.19), we obtain

$$\left\| \tilde{\psi}_n - \psi_n \right\| \to 0. \tag{5.4}$$

Thus, since $(\|B_h^*\|_{\mathcal{L}(V_h,\, U)})_{h \in (0,\, h^*)}$ is bounded, we deduce that

$$\left\| B^*_{h_n} (\tilde{\psi}_n - \psi_n) \right\|_U \to 0.$$

The above relation and (4.8) imply that

$$\left\| B^*_{h_n} \tilde{\psi}_n \right\|_U \to 0. \tag{5.5}$$

But

$$\begin{aligned} \left\| B^*_{h_n} \tilde{\psi}_n \right\|_U &= \tfrac{1}{\sqrt{2}} \left\| \sum_{j=0}^{k(n)-1} \sum_{l=0}^{l_{m(n)+j}-1} c^n_{k_{m(n)+j}+l} B^*_{h_n} \varphi_{k_{m(n)+j}+l,\, h_n} \right\|_U \\ &= \tfrac{1}{\sqrt{2}} \left\| (B^*_{h_n} \varphi_{k_{m(n)},\, h_n} \cdots B^*_{h_n} \varphi_{k_{m(n)+k(n)-1} + l_{m(n)+k(n)-1}-1,\, h_n}) C \right\|_U \\ &= \tfrac{1}{\sqrt{2}} \left\| (1 \cdots 1) \Phi_{k_{m(n)},\, h_n} C \right\|_U, \end{aligned}$$

where $C = (c_{k_{m(n)}} \cdots c_{k_{m(n)}+l_{m(n)}-1}\ c_{k_{m(n)+1}} \cdots c_{k_{m(n)+k(n)-1}+l_{m(n)+k(n)-1}-1})^T$. So, we have

$$\left\| B^*_{h_n} \tilde{\psi}_n \right\|_U = \tfrac{1}{\sqrt{2}} \left\| \begin{pmatrix} 1\ 1 \cdots 1 \\ 0\ 0 \cdots 0 \\ \vdots \quad \ddots \ \vdots \\ 0 \quad \cdots\ 0 \end{pmatrix} \Phi_{k_{m(n)},\, h_n} C \right\|_{U,\, 2}.$$

We now use Lemma 1.1 to have

$$\begin{aligned} \left\| B^*_{h_n} \tilde{\psi}_n \right\|_U &\ge c \left\| B^{-1}_{k_{m(n)}} \Phi_{k_{m(n)},\, h_n} C \right\|_{U,\, 2} \text{ for } n \text{ large enough} \\ &\ge c\alpha \|C\|_2 \text{ by Proposition 2.1.} \end{aligned} \tag{5.6}$$

Gathering (5.3)–(5.6), we obtain that $\tilde{\psi}_n \to 0$ in H. Therefore, by (5.4), $\psi_n \to 0$, which contradicts (4.9).

Second case. the set $\mathcal{F}$ is finite. Then, we can assume, without loss of generality, that $\mathcal{F}$ is empty (otherwise we take off the finite number of (ω_n)), *i.e.*, that for all $n \in \mathbb{N}$, we have that

$$|\omega_n - \lambda_{k,\, h_n}| \ge \frac{\gamma'}{2} \quad \text{if } 0 < |k| \le M(h_n).$$

Thus, by (4.20) and the above relation, we obtain that

$$\sum_{0 < |k| \le M(h_n)} |c_k^n|^2 \to 0.$$

Therefore, by (4.18), (4.19) and the above relation, we have $\psi_n \to 0$ in H, which contradicts (4.9).

In conclusion, the family $(\tilde{A}_h)$ satisfies the condition (ii) in Theorem 3.1 and so the family of systems (1.9) is uniformly exponentially stable. □

6. Proof of Theorem 1.6

Here we use the results of Section 4 with $l > 0$ and l even. Without loss of generality, we may assume that $0 < h < h^* = 1$.

Proof of Theorem 1.6 and of (3.2) This proof is based on Theorem 3.8. First, for all $h \in (0, h^*)$, $(e^{t\tilde{A}_{l,h}})$ forms a family of contraction semigroups and the family $(\tilde{A}_{l,h})_h$ satisfies (3.1). To apply the results of Theorem 3.8, the family $(\tilde{A}_{l,h})$ must also satisfy condition i) of Theorem 3.8 with $\alpha = 2l$ and condition (3.2) or equivalently condition (4.5). We again use a contradiction argument to prove this last condition. Let $(h_n)_n$, $(\omega_n)_n$ and $z_n = \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} \in X_h$ be three sequences satisfying (4.6) and (4.7). Notice that for $k_m \in A_k$, we have

$$\begin{aligned} \lambda_{k_m,h} - \lambda_{k_{m-1}+l_{m-1}-1,h} &\geq \lambda_{k_m} - \lambda_{k_{m-1}+l_{m-1}-1} - \tfrac{c\epsilon}{\lambda_{k_{m-1}}^{2l}} \\ &\geq \lambda_{k_m} - \lambda_{k_{m-1}} - \tfrac{c\epsilon}{\lambda_{k_1}^{2l}} \geq \gamma_0' - \tfrac{c\epsilon}{\lambda_{k_1}^{2l}} \\ &\geq \tfrac{\gamma_0'}{2} =: \gamma' \end{aligned}$$

for $\epsilon \leq \frac{\gamma_0' \lambda_{k_1}^{2l}}{2c}$ by (2.8), (2.9) and because $\lambda_{k_m} \geq \lambda_{k_1} > 0$. We introduce the set $\mathcal{F}_2$ like

$$\mathcal{F}_2 = \Big\{ n \in \mathbb{N} \,\big|\, \exists k(n) \in \{1, \ldots, M\},\ \exists k_{m(n)} \in A_{k(n)},\ \left|k_{m(n)}\right| \leq M_l(h_n) \text{ and}$$

$$\left|k_{m(n)+k(n)-1} + l_{m(n)+k(n)-1}\right| \leq N(h_n) \text{ such that } \left|\omega_n - (1+h_n^\theta)^{-1}(\lambda_{k_{m(n)},h_n} + h_n^\theta \lambda_{k_{m(n)},h_n}^{1+l})\right| < \frac{\gamma'}{4} \Big\}. \tag{6.1}$$

We distinguish two cases.

First case. the set $\mathcal{F}_2$ is infinite. Then, without loss of generality, we can suppose that $\mathcal{F}_2 = \mathbb{N}$ (otherwise we take a subsequence of $(\omega_n)_n$). Then, by reducing the value of γ' if needed, we can assume that for all $n \in \mathbb{N}$, we have that for all $k_m \in A_{k'}$, $k' = 1, \ldots, M$ with $m \neq m(n)$, and for all $|k_{m+j} + l| \leq M_l(h_n)$

$$\left|\omega_n - \left(1+h_n^\theta\right)^{-1}\left(\lambda_{k_{m+j}+l,h_n} + h_n^\theta \lambda_{k_{m+j}+l,h_n}^{1+l}\right)\right| \geq \frac{\gamma'}{8},\ \forall j = 0, \ldots, k'-1,\ \forall l = 0, \ldots, l_{m+j}-1. \tag{6.2}$$

Indeed, similar to (4.23), we have

$$\begin{aligned} \left|\omega_n - \left(1+h_n^\theta\right)^{-1}\left(\lambda_{k_{m+j}+l,h_n} + h_n^\theta \lambda_{k_{m+j}+l,h_n}^{1+l}\right)\right| &\geq \left(1+h_n^\theta\right)^{-1}\left|\lambda_{k_{m+j}+l,h_n} - \lambda_{k_{m(n)},h_n}\right| \\ &\quad - \left|\omega_n - \left(1+h_n^\theta\right)^{-1}\left(\lambda_{k_{m(n)},h_n} + h_n^\theta \lambda_{k_{m(n)},h_n}^{1+l}\right)\right| \\ &\quad - \left(1+h_n^\theta\right)^{-1}\left(h_n^\theta \lambda_{k_{m(n)},h_n}^{1+l} + h_n^\theta \lambda_{k_{m+j}+l,h_n}^{1+l}\right) \\ &\geq \frac{\gamma'}{2} - \frac{\gamma'}{4} - \frac{2C\epsilon}{\lambda_{k_1}}. \end{aligned}$$

So choose again $\epsilon \leq \dfrac{\gamma' \lambda_{k_1}}{16C}$ to get (6.2). By using (4.20), we obtain that

$$\sum_{k=1}^{M} \sum_{\substack{k_m \in A_k \\ m \neq m(n)}} \sum_{j=0}^{k-1} \sum_{\substack{l=0 \\ 0 < |k_{m+j}+l_{m+j}-1| \leq M_l(h_n)}}^{l_{m+j}-1} |\omega_n|^{2l} \left|c_{k_{m+j}+l}^n\right|^2 \to 0. \tag{6.3}$$

Define now

$$\tilde{\psi}_n = \frac{1}{\sqrt{2}} \sum_{j=0}^{k(n)-1} \sum_{l=0}^{l_{m(n)+j}-1} c_{k_{m(n)+j}+l}^n \varphi_{k_{m(n)+j}+l,h_n}. \tag{6.4}$$

We have, by (4.18),

$$\psi_n = \frac{1}{\sqrt{2}} \sum_{k=1}^{M} \sum_{k_m \in A_k} \sum_{\substack{j=0 \\ 1 \le k_{m+j}+l \le N(h_n)}}^{k-1} \sum_{l=0}^{l_{m+j}-1} \left(c^n_{k_{m+j}+l} + c^n_{-(k_{m+j}+l)} \right) \varphi_{k_{m+j}+l,\, h_n},$$

and so, by (6.3) and (4.19), we obtain

$$|\omega_n|^l \left\| \tilde{\psi}_n - \psi_n \right\| \to 0. \tag{6.5}$$

Thus, since $(\|B_h^*\|_{\mathcal{L}(V_h,\, U)})_{h \in (0,\, h^*)}$ is bounded, we deduce that

$$|\omega_n|^l \left\| B^*_{h_n} (\tilde{\psi}_n - \psi_n) \right\|_U \to 0.$$

The above relation and (4.8) imply that

$$|\omega_n|^l \left\| B^*_{h_n} \tilde{\psi}_n \right\|_U \to 0. \tag{6.6}$$

But

$$\begin{aligned} |\omega_n|^l \left\| B^*_{h_n} \tilde{\psi}_n \right\|_U &= \tfrac{|\omega_n|^l}{\sqrt{2}} \left\| \sum_{j=0}^{k(n)-1} \sum_{l=0}^{l_{m(n)+j}-1} c^n_{k_{m(n)+j}+l} B^*_{h_n} \varphi_{k_{m(n)+j}+l,\, h_n} \right\|_U \\ &= \tfrac{|\omega_n|^l}{\sqrt{2}} \left\| \left(B^*_{h_n} \varphi_{k_{m(n)},\, h_n} \cdots B^*_{h_n} \varphi_{k_{m(n)+k(n)-1} + l_{m(n)+k(n)-1} - 1,\, h_n} \right) C \right\|_U \\ &= \tfrac{|\omega_n|^l}{\sqrt{2}} \left\| (1 \cdots 1) \Phi_{k_{m(n)},\, h_n} C \right\|_U, \end{aligned}$$

where $C = \left(c_{k_{m(n)}} \cdots c_{k_{m(n)} + l_{m(n)} - 1}\; c_{k_{m(n)+1}} \cdots c_{k_{m(n)+k(n)-1} + l_{m(n)+k(n)-1} - 1} \right)^T$. So, we have

$$|\omega_n|^l \left\| B^*_{h_n} \tilde{\psi}_n \right\|_U = \tfrac{|\omega_n|^l}{\sqrt{2}} \left\| \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 0 \end{pmatrix} \Phi_{k_{m(n)},\, h_n} C \right\|_{U,\, 2}.$$

We now use Lemma 1.1 to have

$$\begin{aligned} |\omega_n|^l \left\| B^*_{h_n} \tilde{\psi}_n \right\|_U &\ge c\, |\omega_n|^l \left\| B^{-1}_{k_{m(n)}} \Phi_{k_{m(n)},\, h_n} C \right\|_{U,\, 2} \text{ for } n \text{ large enough} \\ &\ge c \tfrac{\alpha |\omega_n|^l}{\lambda^l_{k_{m(n)}}} \|C\|_2 \text{ by Proposition 2.4.} \end{aligned}$$

But, ω_n verifies $\left| \omega_n - (1 + h_n^\theta)^{-1} (\lambda_{k_{m(n)},\, h_n} + h_n^\theta \lambda^{1+l}_{k_{m(n)},\, h_n}) \right| < \frac{\gamma'}{4}$ by definition (6.1) of $\mathcal{F}_2$, thus $|\omega_n| \ge (1 + h_n^\theta)^{-1} (\lambda_{k_{m(n)},\, h_n} + h_n^\theta \lambda^{1+l}_{k_{m(n)},\, h_n}) - \frac{\gamma'}{4} \ge \frac{1}{2} \lambda_{k_{m(n)},\, h_n} - \frac{\gamma'}{4}$. Therefore, we have

$$\begin{aligned} |\omega_n|^l \left\| B^*_{h_n} \tilde{\psi}_n \right\|_U &\ge \frac{c\alpha}{2^l} \frac{\left(\lambda_{k_{m(n)},\, h_n} - \frac{\gamma'}{2} \right)^l}{\lambda^l_{k_{m(n)}}} \|C\|_2 \\ &\ge \frac{c\alpha}{2^{2l}} \frac{\lambda^l_{k_{m(n)},\, h_n}}{\lambda^l_{k_{m(n)}}} \|C\|_2 \text{ for } n \text{ large enough} \\ &\ge \frac{c\alpha}{2^{2l}} \|C\|_2 \text{ by (2.8).} \end{aligned} \tag{6.7}$$

Gathering (6.4)–(6.7), we obtain that $\tilde{\psi}_n \to 0$ in H. Therefore, by (6.5), $\psi_n \to 0$, which contradicts (4.9).

Second case. the set $\mathcal{F}_2$ is finite. We proceed similar to the proof of the second case of Theorem 1.2.

In conclusion, the family $(\tilde{A}_{l,h})$ satisfies (4.5); *i.e.*, the condition (i) in Theorem 3.8 with $\alpha = 2l$ when l is even and property (3.2) of Section 3.2.

7. A convergence result

Here we want to prove that the solution ω_h of the discrete problem (1.14) tends to the solution ω of the continuous problem (1.1) in $X := V \times H$ as h goes to zero and if the discrete initial data are well chosen. This is obtained with the help of a general version of the Trotter–Kato Theorem proved in [23] that is appropriated when the approximated semi-groups are defined in proper subspaces of the limit one. The basic idea is that the convergence of the semi-groups is equivalent to the convergence of the resolvent, hence we prove such a convergence result for the resolvents.

Before going on we recall that (1.1) is equivalent to

$$\dot{z}(t) = \tilde{A}z(t) \text{ in } X, \quad z(0) = (\omega_0, \omega_1)^\top,$$

where $z(t) = (\omega(t), \dot{\omega}(t))^\top$ and

$$\tilde{A}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -Au - BB^*v \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A & -BB^* \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}.$$

It is easy to check that $\tilde{A}$ with domain $\mathcal{D}(\tilde{A}) = \mathcal{D}(A) \times V$ is a maximal dissipative operator in X, equipped with the inner product

$$\left((u,v)^\top, (u^*,v^*)^\top\right)_X = a\,(u,u^*) + (v,v^*) \quad \forall (u,v)^\top, (u^*,v^*)^\top \in X.$$

Moreover, $\tilde{A}$ has no eigenvalues on the imaginary axis. We will denote by $T(t), t \geq 0$ the strongly continuous semi-group of contractions generated by $\tilde{A}$.

Let us start with some preliminary results.

Lemma 7.1. *Let $l \in \mathbb{N}, l \geq 2$. If $f \in V = \mathcal{D}(A^{\frac{1}{2}})$, then*

$$\|(1+h^\theta)(I + h^\theta A_h^{\frac{l}{2}})^{-1}\pi_h f - \pi_h f\|_H \leq C h^{\frac{\theta}{l}} \|f\|_V, \tag{7.1}$$

for some $C > 0$.

Proof. We write

$$\pi_h f = \sum_{k=1}^{N(h)} f_k \varphi_{k,h},$$

with $f_k \in \mathbb{C}$. Hence

$$v_h = (1+h^\theta)(I + h^\theta A_h^{\frac{l}{2}})^{-1}\pi_h f,$$

can be written

$$v_h = \sum_{k=1}^{N(h)} v_k \varphi_{k,h},$$

with $v_k = (1+h^\theta)(1+h^\theta \lambda_{k,h}^l)^{-1} f_k$. Consequently we have

$$\begin{aligned}\|v_h - \pi_h f\|_H^2 &= \sum_{k=1}^{N(h)} |f_k|^2 \Big((1+h^\theta)(1+h^\theta\lambda_{k,h}^l)^{-1} - 1\Big)^2 \\ &= h^{2\theta} \sum_{k=1}^{N(h)} |f_k|^2 \left(\frac{1-\lambda_{k,h}^l}{1+h^\theta \lambda_{k,h}^l}\right)^2 \\ &\le ch^{2\theta} \sum_{k=1}^{N(h)} |f_k|^2 \frac{\lambda_{k,h}^{2l}}{(1+h^\theta\lambda_{k,h}^l)^2} \\ &\le ch^{2\theta} \sum_{k=1}^{N(h)} \lambda_{k,h}^2 |f_k|^2 (g(\lambda_{k,h}))^2\end{aligned}$$

for some $c > 0$ independent of h, where the function $g : [0,\infty) \mapsto \mathbb{R}$ is given by $g(\lambda) = \dfrac{\lambda^{l-1}}{(1+h^\theta\lambda^l)}$. As the maximum of g is attained at $\lambda_0 > 0$ given by

$$h^\theta \lambda_0^l = l - 1,$$

we get that

$$\|v_h - \pi_h f\|_H^2 \le cc_2^2 h^{\frac{2\theta}{l}} \sum_{k=1}^{N(h)} |f_k|^2 \lambda_{k,h}^2$$

since $\lambda_0 = c_1 h^{-\frac{\theta}{l}}$ and $g(\lambda_0) = c_2 h^{-\frac{\theta(l-1)}{l}}$ with c_1, c_2 two positive constants independent of h. This proves the first estimate since

$$\sum_{k=1}^{N(h)} |f_k|^2 \lambda_{k,h}^2 = \|A_h^{\frac{1}{2}} \pi_h f\|_H^2 = a(\pi_h f, \pi_h f) \le a(f,f) = \|A^{\frac{1}{2}} f\|_H^2. \qquad \square$$

Corollary 7.2. *Let $l \in \mathbb{N}, l \ge 2$, then for any $f_h \in V_h$ we have*

$$\|(1+h^\theta)(I + h^\theta A_h^{\frac{l}{2}})^{-1} f_h - f_h\|_{\mathcal{D}(A_h^{-\frac{1}{2}})} \le Ch^{\frac{\theta}{l}} \|f_h\|_H, \tag{7.2}$$

for some $C > 0$.

Proof. As in the previous lemma, we have

$$\begin{aligned}\|(1+h^\theta)(I + h^\theta A_h^{\frac{l}{2}})^{-1} f_h - f_h\|_{\mathcal{D}(A_h^{-\frac{1}{2}})}^2 &= \|A_h^{-\frac{1}{2}} \Big((1+h^\theta)(I+h^\theta A_h^{\frac{l}{2}})^{-1} f_h - f_h\Big)\|_H^2 \\ &= h^{2\theta} \sum_{k=1}^{N(h)} \lambda_{k,h}^{-2} |f_k|^2 \left(\frac{1-\lambda_{k,h}^l}{1+h^\theta\lambda_{k,h}^l}\right)^2 \\ &\le ch^{2\theta} \sum_{k=1}^{N(h)} |f_k|^2 (g(\lambda_{k,h}))^2,\end{aligned}$$

when

$$f_h = \sum_{k=1}^{N(h)} f_k \varphi_{k,h}.$$

We then conclude as before. $\square$

Lemma 7.3. *Let $l \in \mathbb{N}, l \geq 2$ and let $f \in \mathcal{D}(A)$, then*

$$h^\theta \|A_h^{1+\frac{l}{2}}(I + h^\theta A_h^{\frac{l}{2}})^{-2}\pi_h f\|_{\mathcal{D}(A_h^{-\frac{1}{2}})} \leq C h^{\frac{\theta}{l}} \|f\|_{\mathcal{D}(A)}, \tag{7.3}$$

for some $C > 0$.

Proof. We easily see that

$$\begin{aligned} h^{2\theta} \|A_h^{1+\frac{l}{2}}(I + h^\theta A_h^{\frac{l}{2}})^{-2}\pi_h f\|^2_{\mathcal{D}(A_h^{-\frac{1}{2}})} &= h^{2\theta} \|A_h^{-\frac{1}{2}} A_h^{1+\frac{l}{2}}(I + h^\theta A_h^{\frac{l}{2}})^{-2}\pi_h f\|^2_H \\ &= h^{2\theta} \sum_{k=1}^{N(h)} |f_k|^2 \frac{\lambda_{k,h}^{2l+2}}{(1 + h^\theta \lambda_{k,h}^l)^4} \\ &\leq h^{2\theta} \sum_{k=1}^{N(h)} |f_k|^2 \lambda_{k,h}^4 (g(\lambda_{k,h}))^2, \end{aligned}$$

and we conclude as before. □

Lemma 7.4. *Let $l \in \mathbb{N}, l \geq 2$ and let $f \in V$, then*

$$\|(1 + h^\theta)(I + h^\theta A_h^{\frac{l}{2}})^{-1} B_h B_h^*(1 + h^\theta)(I + h^\theta A_h^{\frac{l}{2}})^{-1}\pi_h f - B_h B_h^* \pi_h f\|_{\mathcal{D}(A_h^{-\frac{1}{2}})} \leq C h^{\frac{\theta}{l}} \|f\|_V, \tag{7.4}$$

for some $C > 0$.

Proof. As in Lemma 7.1, we set

$$v_h = (1 + h^\theta)(I + h^\theta A_h^{\frac{l}{2}})^{-1}\pi_h f.$$

First, we notice that

$$\|B_h B_h^*(v_h - \pi_h f)\|_H \leq C\|v_h - \pi_h f\|_H,$$

and by Lemma 7.1 we get

$$\|B_h B_h^*(v_h - \pi_h f)\|_H \leq C h^{\frac{\theta}{l}} \|f\|_V.$$

Second, by Corollary 7.2, we have

$$\begin{aligned} \|(1 + h^\theta)(I + h^\theta A_h^{\frac{l}{2}})^{-1} B_h B_h^* v_h - B_h B_h^* v_h\|_{\mathcal{D}(A_h^{-\frac{1}{2}})} &\leq C h^{\frac{\theta}{l}} \|B_h B_h^* v_h\|_H \\ &\leq C h^{\frac{\theta}{l}} (\|B_h B_h^*(v_h - \pi_h f)\|_H + \|B_h B_h^* \pi_h f\|_H) \\ &\leq C h^{\frac{\theta}{l}} \|f\|_V, \end{aligned}$$

where we use the fact that $\|\pi_h f\|_H \leq c\|\pi_h f\|_V \leq c\|f\|_V$. The conclusion follows from the two above estimates. □

Theorem 7.5. *If $z = (f, g)^\top \in \mathcal{D}(A) \times \mathcal{D}(A)$, then*

$$\|(\tilde{A}_{l,h})^{-1}(\pi_h f, \pi_h g)^\top - \tilde{A}^{-1}(f, g)^\top\|_X \to 0 \text{ as } h \to 0.$$

Proof. By the definition of $\tilde{A}_{l,h}$ and $\tilde{A}$, we have

$$(u_h, v_h)^\top = (\tilde{A}_{l,h})^{-1}(\pi_h f, \pi_h g)^\top,$$

and

$$(u,v)^\top = \tilde{A}^{-1}(f,g)^\top,$$

if and only if

$$\begin{cases} v_h & = (1+h^\theta)\left(I+h^\theta A_h^{\frac{l}{2}}\right)^{-1}\pi_h f \\ -A_h u_h & = (1+h^\theta)\left(I+h^\theta A_h^{\frac{l}{2}}\right)^{-1}\left(h^\theta A_h^{1+\frac{l}{2}}+B_hB_h^*\right)v_h + (1+h^\theta)\left(I+h^\theta A_h^{\frac{l}{2}}\right)^{-1}\pi_h g, \end{cases}$$

and

$$\begin{cases} v = f \\ -Au = BB^*v + g. \end{cases}$$

Therefore, we can write

$$-A_h u_h = \pi_h g + B_hB_h^*\pi_h f + r_h,$$

where $r_h \in V_h$ is given by

$$\begin{aligned} r_h = &(1+h^\theta)\left(I+h^\theta A_h^{\frac{l}{2}}\right)^{-1}\pi_h g - \pi_h g \\ &+ (1+h^\theta)h^\theta\left(I+h^\theta A_h^{\frac{l}{2}}\right)^{-1}A_h^{1+\frac{l}{2}}v_h \\ &+ (1+h^\theta)\left(I+h^\theta A_h^{\frac{l}{2}}\right)^{-1}B_hB_h^*v_h - B_hB_h^*\pi_h f. \end{aligned}$$

By the previous Lemmas, r_h satisfies

$$\|r_h\|_{\mathcal{D}(A_h^{-\frac{1}{2}})} \le Ch^{\frac{\theta}{l}}\|(f,g)^\top\|_{\mathcal{D}(A)\times V}. \tag{7.5}$$

Therefore, $u_h \in V_h$ can be seen as the unique solution of

$$a(u_h,w_h) = -(\pi_h g, w_h) - (B_hB_h^*\pi_h f, w_h) - \langle r_h; w_h\rangle \quad \forall w_h \in V_h, \tag{7.6}$$

where $\langle ; \rangle$ denotes the dual product in $D(A_h^{-\frac{1}{2}})$. Since $u \in V$ is solution of

$$a(u,w) = -(g,w) - (BB^*f,w) \quad \forall w \in V,$$

we get (recalling that $V_h \subset V$)

$$a(u,w_h) = -(g,w_h) - (BB^*f,w_h) \quad \forall w_h \in V_h.$$

Hence, taking the difference of this identity with (7.6), we obtain

$$a(u-u_h,w_h) = (\pi_h g - g, w_h) + (B^*(\pi_h f - f), B^*w_h)_U + \langle r_h; w_h\rangle \quad \forall w_h \in V_h.$$

Consequently, taking $w_h = \pi_h u - u_h$, we get

$$\begin{aligned} a(u-u_h,u-u_h) &= a(u-u_h,u-\pi_h u) + a(u-u_h,\pi_h u - u_h) \\ &= a(u-u_h,u-\pi_h u) + (\pi_h g - g, \pi_h u - u_h) \\ &\quad + (B^*(\pi_h f - f), B^*(\pi_h u - u_h))_U + \langle r_h; \pi_h u - u_h\rangle. \end{aligned}$$

Hence, by Cauchy-Schwarz's inequality and the boundedness of B^*, we obtain

$$\begin{aligned} \|u-u_h\|_V^2 &= a(u-u_h,u-u_h) \\ &\le \|u-u_h\|_V\|u-\pi_h u\|_V + C(\|\pi_h g - g\|_H + \|\pi_h f - f\|_H + \|r_h\|_{\mathcal{D}(A_h^{-\frac{1}{2}})})\|\pi_h u - u_h\|_V. \end{aligned}$$

Now, using the triangle inequality, we get

$$\|u-u_h\|_V^2 \le C\Big((\|u-\pi_h u\|_V + \|\pi_h g - g\|_H + \|\pi_h f - f\|_H + \|r_h\|_{\mathcal{D}(A_h^{-\frac{1}{2}})})\|u-u_h\|_V + (\|\pi_h g - g\|_H + \|\pi_h f - f\|_H + \|r_h\|_{\mathcal{D}(A_h^{-\frac{1}{2}})})\|u-\pi_h u\|_V\Big).$$

Hence, by Young's inequality, we arrive at

$$\|u-u_h\|_V^2 \le C\Big(\|u-\pi_h u\|_V^2 + \|\pi_h g - g\|_H^2 + \|\pi_h f - f\|_H^2 + \|r_h\|^2_{\mathcal{D}(A_h^{-\frac{1}{2}})} + (\|\pi_h g - g\|_H + \|\pi_h f - f\|_H + \|r_h\|_{\mathcal{D}(A_h^{-\frac{1}{2}})})\|u-\pi_h u\|_V\Big).$$

The estimates (1.5), (1.6), and (7.5) then yield

$$\|u-u_h\|_V^2 \le C\Big(h^{2\theta}\|u\|^2_{\mathcal{D}(A)} + h^{4\theta}\|f\|^2_{\mathcal{D}(A)} + h^{4\theta}\|g\|^2_{\mathcal{D}(A)} + h^{\frac{2\theta}{l}}\|(f,g)^\top\|^2_{\mathcal{D}(A)\times V} + (h^{2\theta}\|f\|_{\mathcal{D}(A)} + h^{2\theta}\|g\|_{\mathcal{D}(A)} + h^{\frac{\theta}{l}}\|(f,g)^\top\|_{\mathcal{D}(A)\times V})h^\theta\|u\|_{\mathcal{D}(A)}\Big). \tag{7.7}$$

For $v - v_h$, we notice that

$$v - v_h = f - (1+h^\theta)(I + h^\theta A_h^{\frac{l}{2}})^{-1}\pi_h f = f - \pi_h f + \pi_h f - (1+h^\theta)(I + h^\theta A_h^{\frac{l}{2}})^{-1}\pi_h f,$$

and we conclude that it tends to zero in H due to the estimate (1.5) and Lemma 7.1. □

Corollary 7.6. *If $z = (f,g)^\top \in V \times H$, recalling that j_h is the projection from H into V_h, we have*

$$\|(\tilde{A}_{l,h})^{-1}(\pi_h f, j_h g)^\top - \tilde{A}^{-1}(f,g)^\top\|_X \to 0 \text{ as } h \to 0.$$

Proof. First for $z = (f,g)^\top \in \mathcal{D}(A) \times \mathcal{D}(A)$, then

$$\|(\tilde{A}_{l,h})^{-1}(\pi_h f, j_h g)^\top - \tilde{A}^{-1}(f,g)^\top\|_X \le \|(\tilde{A}_{l,h})^{-1}(\pi_h f, \pi_h g)^\top - \tilde{A}^{-1}(f,g)^\top\|_X + \|(\tilde{A}_{l,h})^{-1}(0, j_h g - \pi_h g)^\top\|_X.$$

The first term of this right-hand side tends to zero as h goes to zero by the previous Theorem. On the other hand for the second term, as $\tilde{A}_{l,h}$ satisfies (3.2) (see Sect. 6), there exists $C > 0$ (independent of h) such that for all $h < h^*$

$$\|(\tilde{A}_{l,h})^{-1}(0, j_h g - \pi_h g)^\top\|_X \le C\|j_h g - \pi_h g\|_H.$$

Hence, by the triangle inequality and the property $\|g - j_h g\|_H \le \|g - \pi_h g\|_H$ (as j_h in the projection on V_h in H), we get

$$\|(\tilde{A}_{l,h})^{-1}(0, j_h g - \pi_h g)^\top\|_X \le 2C\|g - \pi_h g\|_H.$$

By the estimate (1.6), we then conclude that this second term tends also to zero as h goes to zero.

If $z = (f,g)^\top$ is only in $V \times H$, then for an arbitrary $\varepsilon > 0$, we use the density of $\mathcal{D}(A) \times \mathcal{D}(A)$ into $V \times H$ to get $(F,G)^\top \in \mathcal{D}(A) \times \mathcal{D}(A)$ such that

$$\|(f,g)^\top - (F,G)^\top\|_X \le \varepsilon.$$

Now, by the triangle inequality, we have

$$\|(\tilde{A}_{l,h})^{-1}(\pi_h f, j_h g)^\top - \tilde{A}^{-1}(f,g)^\top\|_X \le \|(\tilde{A}_{l,h})^{-1}(\pi_h(f-F), j_h(g-G))^\top\|_X + \|\tilde{A}^{-1}(f-F, g-G)^\top\|_X + \|(\tilde{A}_{l,h})^{-1}(\pi_h F, j_h G)^\top - \tilde{A}^{-1}(F,G)^\top\|_X.$$

By the first step, there exists h_ε small enough such that

$$\|(\tilde{A}_{l,h})^{-1}(\pi_h F, j_h G)^\top - \tilde{A}^{-1}(F,G)^\top\|_X \leq \varepsilon, \forall 0 < h < h_\varepsilon.$$

For the second term, by the boundedness of $\tilde{A}^{-1}$, we may write

$$\|\tilde{A}^{-1}(f-F, g-G)^\top\|_X \leq C\|(f-F, g-G)^\top\|_X \leq C\varepsilon.$$

Finally for the first term, using the property (3.2) and the fact that π_h (resp. j_h) is a projection from V (resp. from H) into V_h, we get for all $h < h^*$

$$\|(\tilde{A}_{l,h})^{-1}(\pi_h(f-F), j_h(g-G))^\top\|_X \leq C\|(\pi_h(f-F), j_h(g-G))^\top\|_X \leq C\|(f-F, g-G)^\top\|_X \leq C\varepsilon.$$

All together we have obtained that

$$\|(\tilde{A}_{l,h})^{-1}(\pi_h f, j_h g)^\top - \tilde{A}^{-1}(f,g)^\top\|_X \leq (1+2C)\varepsilon, \quad \forall 0 < h < \min\{h_\varepsilon, h^*\}.$$

This proves the result. □

We are now ready to state the convergence result.

Theorem 7.7. *If* $(\omega_0, \omega_1)^\top \in V \times H$, *then*

$$\|T_{l,h}(t)(\pi_h\omega_0, j_h\omega_1)^\top - T(t)(\omega_0,\omega_1)^\top\|_X \to 0 \text{ as } h \to 0. \tag{7.8}$$

Proof. We use Theorem 2.1 of [23] with $X = Z = V \times H$, $X_n = V_h \times V_h$, and $P_n : X \to X_n$ defined by

$$P_n(f,g)^\top = (\pi_h f, j_h g)^\top, \forall (f,g)^\top \in X,$$

and $E_n = P_n^*$ that is here the canonical injection of $V_h \times V_h$ into $V \times H$. The Assumptions (A1) and (A3) of [23] are trivially satisfied, while the assumption (A2) is a consequence of (1.5), (1.6) and the density of $\mathcal{D}(A) \times \mathcal{D}(A)$ into $V \times H$.

Since Corollary 7.6 shows that point (a) of Theorem 2.1 of [23] holds, we conclude that point (b) of this Theorem, namely (7.8), holds. □

8. Examples

8.1. Two coupled wave equations

We consider the following system

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + \alpha y(x,t) + \beta(x)u_t(x,t) = 0 & \text{in } (0,1)\times\mathbb{R}_+, \\ y_{tt}(x,t) - y_{xx}(x,t) + \alpha u(x,t) + \gamma(x)y_t(x,t) = 0 & \text{in } (0,1)\times\mathbb{R}_+, \\ u(0,t) = u(1,t) = y(0,t) = y(1,t) = 0 & \forall t > 0, \\ u(\cdot,0) = u_0, u_t(\cdot,0) = u_1, y(\cdot,0) = y_0, y_t(\cdot,0) = y_1 & \text{in } (0,1), \end{cases} \tag{8.1}$$

when $\alpha \in \mathbb{R}$ such that $\alpha > 0$ is small enough (see below), $\beta(.)$ and $\gamma(.)$ are two non-negative bounded functions such that $\beta(x) \geq \beta > 0$ for $x \in I_\beta \subseteq (0,1)$ and $\gamma(x) \geq \gamma > 0$ for $x \in I_\gamma \subseteq (0,1)$ where I_β and I_γ are two open sets such that their measures do not vanish simultaneously. Hence, (8.1) is written in the form (1.1) with the following choices: Take $H = L^2(0,1)^2$, the operator B as follows:

$$B\omega = \sqrt{\beta(.)}\begin{pmatrix} u \\ 0 \end{pmatrix} + \sqrt{\gamma(.)}\begin{pmatrix} 0 \\ y \end{pmatrix}, \tag{8.2}$$

when $\omega = \begin{pmatrix} u \\ y \end{pmatrix}$, which is a bounded operator from H into itself (*i.e.* $U = H$) and the operator A defined by

$$\mathcal{D}(A) = V \cap H^2(0,1)^2,$$

when $V = H_0^1(0,1)^2$ and

$$A\omega = \begin{pmatrix} -u_{xx} + \alpha y \\ -y_{xx} + \alpha u \end{pmatrix}.$$

If α is small enough, namely if $\alpha < \pi^2$, this operator A is a positive selfadjoint operator in H, since it is the Friedrichs extension of the triple (H, V, a), where the sesquilinear form a is defined by

$$a(\omega, \omega^*) = \int_0^1 (u_x(\overline{u^*})_x + y_x(\overline{y^*})_x + \alpha y\overline{u^*} + \alpha u\overline{y^*})\,\mathrm{d}x, \forall \omega = \begin{pmatrix} u \\ y \end{pmatrix}, \omega^* = \begin{pmatrix} u^* \\ y^* \end{pmatrix} \in V.$$

Indeed a is clearly a continuous symmetric sesquilinear form on V and is coercive if $\alpha < \pi^2$ due to Poincaré's inequality

$$\int_0^1 |u_x|^2\,\mathrm{d}x \geq \pi^2 \int_0^1 |u|^2\,\mathrm{d}x, \quad \forall u \in H_0^1(0,1).$$

Furthermore, A has a compact resolvent since $\mathcal{D}(A)$ is compactly embedded into H.

Let us now check that the generalized gap condition (1.7) and the Assumptions (1.10) or (1.13) are satisfied for our system (8.1). We start by the determination of the spectrum of the operator A. Hence we are looking for $\omega = (u, y)^\top \in V \cap H^2(0,1)^2$ different from 0 and $\lambda^2 > 0$ solution of

$$-u_{xx} + \alpha y = \lambda^2 u \text{ in } (0,1),$$
$$-y_{xx} + \alpha u = \lambda^2 y \text{ in } (0,1).$$

If such a pair exists, we can set

$$s = \frac{u+y}{2}, \quad d = \frac{u-y}{2},$$

and notice that s and d belong to $H_0^1(0,1) \cap H^2(0,1)$ and are solution of

$$-s_{xx} + \alpha s = \lambda^2 s \text{ in } (0,1),$$
$$-d_{xx} - \alpha d = \lambda^2 d \text{ in } (0,1).$$

Hence s (resp. d) is an eigenvector of the Laplace operator $-\frac{\mathrm{d}}{\mathrm{d}x^2}$ with Dirichlet boundary condition of eigenvalue $\lambda^2 - \alpha$ (resp. $\lambda^2 + \alpha$). A first choice is then to have for all $k \in \mathbb{N}^*$: $\lambda^2 = k^2\pi^2 + \alpha$, $s = \sin(k\pi\cdot)$ and $d = 0$. Coming back to (u, y), we find (since $u = s + d$ and $y = s - d$) a sequence of eigenvalues $\lambda_{+,k}^2 = k^2\pi^2 + \alpha$ of associated eigenvector

$$\omega_{+,k} = (\sin(k\pi\cdot), \sin(k\pi\cdot)).$$

Note that each eigenvalue is simple and that $\omega_{+,k}$ is of norm 1 in H.

A second choice is to take for all $k \in \mathbb{N}^*$: $\lambda^2 = k^2\pi^2 - \alpha$ (which is meaningful since $\alpha < \pi^2$), $s = 0$ and $d = \sin(k\pi\cdot)$. Again coming back to (u, y), we find a sequence of eigenvalues $\lambda_{-,k}^2 = k^2\pi^2 - \alpha$ of associated eigenvector

$$\omega_{-,k} = (\sin(k\pi\cdot), -\sin(k\pi\cdot)).$$

As before each eigenvalue is simple and $\omega_{-,k}$ is of norm 1 in H.

Now we remark that the sequence $\{\omega_{+,k}\}_{k\in\mathbb{N}^*} \cup \{\omega_{-,k}\}_{k\in\mathbb{N}^*}$ is an orthonormal basis of H (because $\omega_{+,k} + \omega_{-,k} = 2(\sin(k\pi\cdot), 0)$ and $\omega_{+,k} - \omega_{-,k} = 2(0, \sin(k\pi\cdot)))$ and therefore we have found all possible eigenvectors of A. We have then shown that the spectrum of A is given by

$$\mathrm{Sp}(A) = \{\lambda_{+,k}^2\}_{k\in\mathbb{N}^*} \cup \{\lambda_{-,k}^2\}_{k\in\mathbb{N}^*},$$

and that each eigenvalue is simple (because the assumption $\alpha < \pi^2$ implies that $k^2\pi^2 + \alpha < (k+1)^2\pi^2 - \alpha$).

We now need to estimate the distance between the consecutive eigenvalues of $A^{1/2}$. We have two different cases to consider:

1. For all $k \in \mathbb{N}^*$, we need to look at the distance between $\lambda_{+,k}$ and $\lambda_{-,k}$. Since

$$\lambda_{+,k} - \lambda_{-,k} = \sqrt{k^2\pi^2 + \alpha} - \sqrt{k^2\pi^2 - \alpha} = \frac{2\alpha}{\sqrt{k^2\pi^2 + \alpha} + \sqrt{k^2\pi^2 - \alpha}},$$

we see that this distance goes to zero as k goes to infinity.
2. For all $k \in \mathbb{N}^*$, we look at the distance between $\lambda_{+,k}$ and $\lambda_{-,k+1}$. Here we have

$$\lambda_{-,k+1} - \lambda_{+,k} = \sqrt{(k+1)^2\pi^2 - \alpha} - \sqrt{k^2\pi^2 + \alpha} = \frac{2k\pi^2 + \pi^2 - 2\alpha}{\sqrt{(k+1)^2\pi^2 - \alpha} + \sqrt{k^2\pi^2 + \alpha}},$$

which tends to π as k goes to infinity.

This shows that the generalized gap condition (1.7) is satisfied with $M = 2$. With the terminology of Section 1, we see that $A_1 = \emptyset$ and $A_2 = \mathbb{N}^*$.

In order to check (1.10) or (1.13), for all $k \in \mathbb{N}^*$, we set

$$\alpha_k = \lambda_{+,k} - \lambda_{-,k},$$

that behaves like k^{-1} or equivalently like $\lambda_{-,k}^{-1}$. We further need to use the matrix (see Lem. 1.1)

$$B_k^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & \alpha_k \end{pmatrix},$$

as well as the matrix Φ_k which here takes the form

$$\Phi_k = \begin{pmatrix} B^*\omega_{-,k} & 0 \\ 0 & B^*\omega_{+,k} \end{pmatrix}.$$

Hence for all $C = (c_1, c_2)^\top \in \mathbb{R}^2$, we have

$$B_k^{-1}\Phi_k C = \begin{pmatrix} c_1 B^*\omega_{-,k} + c_2 B^*\omega_{+,k} \\ \alpha_k c_2 B^*\omega_{+,k} \end{pmatrix},$$

and consequently

$$\begin{aligned}\|B_k^{-1}\Phi_k C\|_{U,2}^2 &= \|c_1 B^*\omega_{-,k} + c_2 B^*\omega_{+,k}\|_2^2 + |\alpha_k|^2|c_2|^2\|B^*\omega_{+,k}\|_2^2 \\ &= |c_1 + c_2|^2 \int_0^1 \beta(x)\sin^2(k\pi x)\mathrm{d}x + |c_2 - c_1|^2 \int_0^1 \gamma(x)\sin^2(k\pi x)\mathrm{d}x \\ &\quad + |\alpha_k|^2|c_2|^2 \int_0^1 (\beta(x) + \gamma(x))\sin^2(k\pi x)\mathrm{d}x.\end{aligned}$$

We have two different cases to consider:

First case. $I_\beta \neq \emptyset$ and $I_\gamma \neq \emptyset$.

In this case, we have

$$\begin{aligned}\|B_k^{-1}\Phi_k C\|_{U,2}^2 &\geq \min\{\beta, \gamma\} \min\left\{\int_{I_\beta} \sin^2(k\pi x)\mathrm{d}x, \int_{I_\gamma} \sin^2(k\pi x)\mathrm{d}x\right\}\left((c_1 + c_2)^2 + (c_2 - c_1)^2\right) \\ &= 2\min\{\beta, \gamma\} \min\left\{\int_{I_\beta} \sin^2(k\pi x)\mathrm{d}x, \int_{I_\gamma} \sin^2(k\pi x)\mathrm{d}x\right\}(c_1^2 + c_2^2)\end{aligned}$$

and hence (1.10) holds since $\min\left\{\int_{I_\beta}\sin^2(k\pi x)\mathrm{d}x, \int_{I_\gamma}\sin^2(k\pi x)\mathrm{d}x\right\}$ is uniformly bounded from below. Indeed, as $I_\gamma \neq \emptyset$, there exists $a \in (0,1)$ and $\epsilon > 0$ such that $(a, a+\epsilon) \subset I_\gamma$, and therefore

$$\int_{I_\gamma}\sin^2(k\pi x)\mathrm{d}x \geq \frac{\epsilon}{2} + \frac{1}{4k\pi}\left(\sin(2k\pi a) - \sin(2k\pi(a+\epsilon))\right) \geq \frac{\epsilon}{2} - \frac{1}{2k\pi} \geq \frac{\epsilon}{4},$$

for $k \geq \dfrac{2}{\epsilon\pi}\cdot$ On the other hand, we clearly have

$$\min_{1\leq k< \dfrac{2}{\epsilon\pi}} \int_{I_\gamma}\sin^2(k\pi x)\mathrm{d}x > 0,$$

which shows that $\int_{I_\gamma}\sin^2(k\pi x)\mathrm{d}x$ is uniformly bounded from below.

Second case. $I_\beta = \emptyset$ or $I_\gamma = \emptyset$ (but not empty together). For instance, suppose that $I_\beta = \emptyset$ and $I_\gamma \neq \emptyset$. As $|\alpha_k| \sim \lambda_{-,k}^{-1}$, we deduce that

$$\|B_k^{-1}\Phi_k C\|_{U,2} \geq \alpha_0\lambda_{-,k}^{-1}\|C\|_2,$$

for a positive constant α_0, and shows that (1.13) holds with $l = 1$.

As stated before, in the first case the system (8.1) is exponentially stable, while in the second case (8.1) is polynomially stable. We refer to Theorem 2.4 of [3] or to [1, 29] for the proof of these results.

As approximated space V_h, we use the standard one based on $P1$ finite elements. More precisely, for $N \in \mathbb{N}$ and $h = \frac{1}{N+1}$, we define the points $x_j = jh$, $j = 0, 1, \ldots, N+1$. The space V_h is the linear span of the family of hat functions $(e_i, e_j)_{i,j\in\{1,\ldots,N\}}$ such that

$$e_j(x) = \left[1 - \frac{|x - x_j|}{h}\right]^+, \text{ for } j = 1, \ldots, N.$$

Then, we define the operators A_h and B_h by (1.2) and (1.4). It is well-known (see [12]) that the operator A and the space V_h satisfy conditions (1.5) and (1.6) with $\theta = 1$.

Consequently, in the first case ($I_\beta \neq \emptyset$ and $I_\gamma \neq \emptyset$), we can apply Theorem 1.2 and thus the family of systems (1.9) is uniformly exponentially stable, in the sense that there exist constants M, α, $h^* > 0$ (independent of h, u_{0h}, u_{1h}, y_{0h}, y_{1h}) such that for all $h \in (0, h^*)$:

$$\|\dot\omega_h(t)\|^2 + a(\omega_h(t), \omega_h(t)) \leq M\mathrm{e}^{-\alpha t}(\|\omega_{1h}\|^2 + a(\omega_{0h}, \omega_{0h})), \forall t \geq 0,$$

where $\omega_h = (u_h, y_h)$, and $\omega_{0h} = (u_{0h}, y_{0h}) \in V_h$ (resp. $\omega_{1h} = (u_{1h}, y_{1h}) \in V_h$) is an approximation of $\omega_0 = (u_0, y_0)$ (resp. $\omega_1 = (u_1, y_1)$).

In the second case ($I_\beta = \emptyset$ and $I_\gamma \neq \emptyset$), we can apply Theorem 1.6 with $l = 2$ and thus the family of systems (1.14) is uniformly polynomially stable, in the sense that, there exist constants C, $h^* > 0$ (independent of h, u_{0h}, u_{1h}, y_{0h}, y_{1h}) such that for all $h \in (0, h^*)$:

$$\left\|(I + hA_h)^{-1}\dot\omega_h(t)\right\|^2 + a(\omega_h(t), \omega_h(t)) \leq \frac{C}{\sqrt{t}}\|(\omega_{0h}, \omega_{1h})\|^2_{D(\tilde A_{2,h})} \;\forall t > 0, \tag{8.3}$$

where $\tilde A_{2,h}$ is given as in (4.1) with $l = 2$, $\theta = 1$, and the the graph norm $\|\cdot\|_{D(\tilde A_{2,h})}$ is defined by

$$\|(\omega_{0h}, \omega_{1h})\|^2_{D(\tilde A_{2,h})} = \|(\omega_{0h}, \omega_{1h})\|^2_{X_h} + \|\tilde A_{2,h}(\omega_{0h}, \omega_{1h})\|^2_{X_h}.$$

8.2. Two boundary coupled wave equations

We consider the following system

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } (0,1)\times\mathbb{R}_+,\\ y_{tt} - y_{xx} + \beta y_t = 0 & \text{in } (0,1)\times\mathbb{R}_+,\\ u(0,t) = y(0,t) = 0 & \forall t>0,\\ y_x(1,t) = \alpha u(1,t) & \forall t>0,\\ u_x(1,t) = \alpha y(1,t) & \forall t>0,\\ u(\cdot,0) = 0, u_t(\cdot,0) = u_1, y(\cdot,0) = 0, y_t(\cdot,0) = y_1 & \text{in } (0,1), \end{cases} \tag{8.4}$$

when $\alpha, \beta \in \mathbb{R}$ with $\beta > 0$ and $\alpha > 0$ small enough (see below). Hence it is written in the form (1.1) with the following choices: take $H = L^2(0,1)^2$, the operator B as follows:

$$B\omega = \sqrt{\beta}\begin{pmatrix} 0\\ y\end{pmatrix},$$

when $\omega = \begin{pmatrix} u\\ y\end{pmatrix}$, which is a bounded operator from H into itself (*i.e.* $U = H$) and the operator A defined by

$$\mathcal{D}(A) = \left\{(u,y) \in V \cap H^2(0,1)^2 : \, y_x(1) = \alpha u(1); \, u_x(1) = \alpha y(1)\right\}$$

when $V = \left\{\omega \in H^1(0,1)^2 : \, \omega(0) = 0\right\}$ and

$$A\omega = \begin{pmatrix} -u_{xx}\\ -y_{xx}\end{pmatrix}.$$

If α is small enough, namely if $\alpha < 1$, this operator A is a positive selfadjoint operator in H, since it is the Friedrichs extension of the triple (H, V, a), where the sesquilinear form a is defined by

$$a(\omega, \omega^*) = \int_0^1 (u_x(\overline{u^*})_x + y_x(\overline{y^*})_x)\,\mathrm{d}x - \alpha u(1)\overline{y^*}(1) - \alpha\overline{u^*}(1)y(1), \quad \forall \omega = \begin{pmatrix} u\\ y\end{pmatrix}, \omega^* = \begin{pmatrix} u^*\\ y^*\end{pmatrix} \in V.$$

Indeed a is clearly a continuous symmetric sesquilinear form on V and is coercive if $\alpha < 1$ due to the trace theorem

$$u(1)^2 \le \int_0^1 |u_x|^2\,\mathrm{d}x, \quad \forall u \in V.$$

In addition to that, the operator A admits a compact resolvent as $\mathcal{D}(A)$ is compactly embedded in H.

Let us now check that the generalized gap condition (1.7) and the Assumption (1.13) are satisfied for our system (8.4). We start by the determination of the spectrum of the operator A. Hence we are looking for $\omega = (u,y)^\top \in \mathcal{D}(A)$ different from 0 and $\lambda^2 > 0$ solution of

$$-u_{xx} = \lambda^2 u \text{ in } (0,1),$$
$$-y_{xx} = \lambda^2 y \text{ in } (0,1).$$

Then

$$u(x) = a\sin(\lambda x) \text{ in } (0,1),$$
$$y(x) = b\sin(\lambda x) \text{ in } (0,1).$$

The coupling condition in (8.4) gives

$$\begin{cases} a\lambda\cos\lambda = \alpha b\sin\lambda\\ b\lambda\cos\lambda = \alpha a\sin\lambda. \end{cases}$$

Since it is not possible to have $\sin\lambda = 0$ (otherwise $a = b = 0$), we obtain

$$a = \frac{b\lambda\cos\lambda}{\alpha\sin\lambda}, \tag{8.5}$$

and then

$$\tan\lambda = \pm\frac{\lambda}{\alpha}, \tag{8.6}$$

because $b \neq 0$ (otherwise $u = y = 0$).

We then have two sequences of eigenvalues defined by

$$\lambda_{-,k} = \frac{\pi}{2} + k\pi - \epsilon_{-,k}$$

with $\lim_{k\to+\infty}\epsilon_{-,k} = 0$ and $\epsilon_{-,k} > 0$ for all $k \in \mathbb{N}$, and

$$\lambda_{+,k} = \frac{\pi}{2} + k\pi + \epsilon_{+,k}$$

with $\lim_{k\to+\infty}\epsilon_{+,k} = 0$ and $\epsilon_{+,k} > 0$ for all $k \in \mathbb{N}$. Moreover as $\lambda_{-,k}$ and $\lambda_{+,k}$ satisfies (8.6), we can verify that

$$\epsilon_{-,k} = \arctan\left(\frac{\alpha}{\lambda_{-,k}}\right) \text{ and } \epsilon_{+,k} = \arctan\left(\frac{\alpha}{\lambda_{+,k}}\right)\cdot$$

By (8.5) and (8.6), the eigenvector associated with the eigenvalue $\lambda_{+,k}$ is given by

$$\omega_{+,k} = b_{+,k}\sin(\lambda_{+,k}\cdot)(-1,\,1)^T,$$

and the eigenvector associated with the eigenvalue $\lambda_{-,k}$ is given by

$$\omega_{-,k} = b_{-,k}\sin(\lambda_{-,k}\cdot)(1,\,1)^T,$$

where $b_{+,k}$, $b_{-,k}$ are chosen to normalize the eigenvectors.

Since we have found all possible eigenvectors of A, we have shown that the spectrum of A is given by

$$\mathrm{Sp}(A) = \{\lambda_{+,k}^2\}_{k\in\mathbb{N}^*} \cup \{\lambda_{-,k}^2\}_{k\in\mathbb{N}^*},$$

and that each eigenvalue is simple.

We again need to estimate the distance between the consecutive eigenvalues of $A^{1/2}$ and as before we consider two different cases:

1. For all $k \in \mathbb{N}^*$, we need to look at the distance between $\lambda_{+,k}$ and $\lambda_{-,k}$. Since

$$\lambda_{+,k} - \lambda_{-,k} = \epsilon_{+,k} + \epsilon_{-,k} = \arctan\left(\frac{\alpha}{\lambda_{+,k}}\right) + \arctan\left(\frac{\alpha}{\lambda_{-,k}}\right),$$

we see that this distance goes to zero as k goes to infinity.

2. For all $k \in \mathbb{N}^*$, we look at the distance between $\lambda_{+,k}$ and $\lambda_{-,k+1}$. Here we have

$$\lambda_{-,k+1} - \lambda_{+,k} = \pi - (\epsilon_{+,k} + \epsilon_{-,k+1}),$$

which tends to π as k goes to infinity.

This shows that the generalized gap condition (1.7) is satisfied with $M = 2$.

In order to check (1.13), for all $k \in \mathbb{N}^*$, we set

$$\alpha_k = \lambda_{+,k} - \lambda_{-,k},$$

that behaves like k^{-1} or equivalently like $\lambda_{-,k}^{-1}$. As in the previous subsection for all $C=(c_1,c_2)^\top\in\mathbb{R}^2$, we have

$$B_k^{-1}\Phi_k C=\begin{pmatrix} c_1B^*\omega_{-,k}+c_2B^*\omega_{+,k}\\ \alpha_k c_2 B^*\omega_{+,k}\end{pmatrix},$$

and consequently

$$\begin{aligned}\|B_k^{-1}\Phi_k C\|_{U,2}^2&=\|c_1B^*\omega_{-,k}+c_2B^*\omega_{+,k}\|_H^2+|\alpha_k|^2|c_2|^2\|B^*\omega_{+,k}\|_H^2\\ &=\beta\int_0^1\left(b_{-,k}c_1\sin(\lambda_{-,k}x)+b_{+,k}c_2\sin(\lambda_{+,k}x)\right)^2\mathrm{d}x\\ &\quad+\beta|\alpha_k|^2|c_2|^2b_{+,k}^2\int_0^1\sin^2(\lambda_{+,k}x)\mathrm{d}x.\end{aligned}$$

By using Young's inequality with $\epsilon>0$ and the fact that the eigenvectors are normalized (by the choice of $b_{\pm,k}$), we obtain

$$\begin{aligned}\|B_k^{-1}\Phi_k C\|_{U,2}^2&\ge\beta\left(1-\frac{1}{\epsilon}\right)c_1^2b_{-,k}^2\int_0^1\sin^2(\lambda_{-,k}x)\mathrm{d}x+\beta\left(1-\epsilon\right)c_2^2b_{+,k}^2\int_0^1\sin^2(\lambda_{+,k}x)\mathrm{d}x\\ &\quad+\beta|\alpha_k|^2|c_2|^2b_{+,k}^2\int_0^1\sin^2(\lambda_{+,k}x)\mathrm{d}x\\ &=\frac{\beta}{2}\left(\left(1-\frac{1}{\epsilon}\right)c_1^2+\left(1+\alpha_k^2-\epsilon\right)c_2^2\right).\end{aligned}$$

We then take $\epsilon=1+\alpha_k^2/2$, which implies

$$1+\alpha_k^2-\epsilon=\frac{\alpha_k^2}{2}\quad\text{ and }\quad 1-\frac{1}{\epsilon}>\frac{\alpha_k^2}{4},$$

(since $\alpha_k^2<2$). Consequently

$$\|B_k^{-1}\Phi_k C\|_{U,2}^2\ge\frac{\beta}{8}\alpha_k^2(c_1^2+c_2^2).$$

As $|\alpha_k|\sim\lambda_{-,k}^{-1}$, we deduce that

$$\|B_k^{-1}\Phi_k C\|_{U,2}\ge\alpha_0\lambda_{-,k}^{-1}\|C\|_2,$$

for a positive constant α_0, and shows that (1.13) holds with $l=1$.

We construct the space V_h like in the previous subsection, *i.e.* it is the span of $(e_i,e_j)_{i,j\in\{1,\dots,N+1\}}$, that still satisfies (1.5) and (1.6) with $\theta=1$.

Consequently, we can apply Theorem 1.6 with $l=2$ and thus the family of systems (1.14) is uniformly polynomially stable, in the sense that the estimate (8.3) holds.

8.3. A more general wave type system

We consider the following more general system: let $\omega=(\omega_1,\cdots,\omega_N)^T$ be a solution of

$$\begin{cases}\omega_{tt}-\omega_{xx}+M\omega+BB^*\omega_t=0 & \text{in } (0,1)^N\times\mathbb{R}_+,\\ \omega(0,t)=\omega(1,t)=0 & \forall t>0,\\ \omega(\cdot,0)=\omega^{(0)},\ \omega_t(\cdot,0)=\omega^{(1)} & \text{in } (0,1)^N,\end{cases}\tag{8.7}$$

where $M\in\mathcal{M}_N(\mathbb{R})$ is symmetric and such that A_0+M is positive definite in $H=L^2(0,1)^N$, when A_0 is the operator of domain $\mathcal{D}(A_0)=H_0^1(0,1)^N\cap H^2(0,1)^N$ and such that $A_0u=-u_{xx}$, for all $u\in\mathcal{D}(A_0)$; $B\in\mathcal{L}(U,H)$, with U a complex Hilbert space.

Hence it is written in the form (1.1) with the self-adjoint positive operator A defined by $A = A_0 + M$ and $\mathcal{D}(A) = \mathcal{D}(A_0) = V \cap H^2(0,1)^N$, when $V = H_0^1(0,1)^N$. We remark that A admits a compact resolvent since $\mathcal{D}(A)$ is compactly embedded into H.

As M is symmetric, M can be diagonalized by an orthogonal matrix, *i.e.* there exist a real orthogonal matrix O and a diagonal matrix D such that $O^T M O = D$. We denote by d_i $(i = 1, \cdots, N)$ the coefficients of the diagonal matrix D.

We start by the determination of the spectrum of the operator A. Hence we are looking for $\omega \in V \cap H^2(0,1)^N$ different from 0 and $\lambda^2 > 0$ solution of

$$-\omega_{xx} + M\omega = \lambda^2 \omega.$$

If we denote by $U = O^T \omega$, then $U = (u_1, \cdots, u_N)^T$ satisfies

$$-U_{xx} + DU = \lambda^2 U,$$

which is equivalent to

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} u_i = (\lambda^2 - d_i) u_i, \quad \text{in } (0,1), \quad \forall i = 1, \cdots, N.$$

Hence there exists $c_i \in \mathbb{C}$ such that

$$u_i = \sqrt{2} c_i \sin(k\pi .), \quad \lambda_{i,k}^2 = k^2 \pi^2 + d_i, \quad i = 1, \cdots, N.$$

Therefore we have found N families of eigenvectors and eigenvalues:

$$U_{i,k} = \sqrt{2} f_i \sin(k\pi .), \quad \lambda_{i,k}^2 = k^2 \pi^2 + d_i, \quad i = 1, \cdots, N,$$

where $(f_i)_{i \in \{1, \cdots, N\}}$ is the canonical basis of $\mathbb{C}^N$. Coming back to the initial eigenvalue problem, we have N families of eigenvectors given by

$$\omega_{i,k} = O U_{i,k}, \quad i = 1, \cdots, N, \tag{8.8}$$

and the spectrum of A is given by

$$\mathrm{Sp}(A) = \{\lambda_{1,k}^2\}_{k \in \mathbb{N}^*} \cup \cdots \cup \{\lambda_{N,k}^2\}_{k \in \mathbb{N}^*}.$$

For simplicity we now assume that all d_i are different and, for instance that

$$d_1 < d_2 < \cdots < d_N.$$

We still estimate the distance between the consecutive eigenvalues of $A^{1/2}$:

1. For all $k \in \mathbb{N}^*$, we need to look at the distance between $\lambda_{i,k}$ and $\lambda_{j,k}$ $(i \neq j)$. Since

$$\lambda_{i,k} - \lambda_{j,k} = \sqrt{k^2\pi^2 + d_i} - \sqrt{k^2\pi^2 + d_j} = \frac{d_i - d_j}{\sqrt{k^2\pi^2 + d_i} + \sqrt{k^2\pi^2 + d_j}},$$

we see that this distance goes to zero as k goes to infinity.

2. For all $k \in \mathbb{N}^*$, we look at the distance between $\lambda_{N,k}$ and $\lambda_{1,k+1}$. Here we have

$$\lambda_{1,k+1} - \lambda_{N,k} = \sqrt{(k+1)^2\pi^2 + d_1} - \sqrt{k^2\pi^2 + d_N} = \frac{2k\pi^2 + \pi^2 + d_1 - d_N}{\sqrt{(k+1)^2\pi^2 + d_1} + \sqrt{k^2\pi^2 + d_N}},$$

which tends to π as k goes to infinity.

This shows that the generalized gap condition (1.7) is satisfied with $M = N$. With the terminology of Section 1, we see that $A_1 = \cdots = A_{N-1} = \emptyset$ and $A_N = \mathbb{N}^*$. Hence, for $N > 1$, our previous results will allow to obtain stability results for system (8.7).

If the eigenvalues are simple (a necessary condition is that all d_i are different), then in order to verify (1.10) or (1.13), we have to bound from below $\left\|B_k^{-1}\Phi_k C\right\|_{U,2}^2$ with $C=(c_1,\cdots,c_N)\in\mathbb{R}^N$, B_k^{-1} defined in Lemma 1.1 and Φ_k given by

$$\Phi_k=\begin{pmatrix} B^*\omega_{1,k} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & B^*\omega_{N,k}\end{pmatrix}.$$

Such a lower bound can only be made on some particular examples.

Note that, if $N=2$, B is defined by (8.2) and

$$M=\alpha\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

with $\alpha>0$, then we are back to the setting of Section 8.1. Indeed M is symmetric with A_0+M positive definite for α small enough, and diagonalized by the orthogonal matrix

$$O=\frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}\left(\text{with } D=\alpha\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\right).$$

We then finish this subsection by considering another example. Take $N=3$ and

$$B\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix}=\sqrt{\beta}\begin{pmatrix}\omega_1\\ 0\\ 0\end{pmatrix}+\sqrt{\gamma}\begin{pmatrix}0\\ \omega_2\\ 0\end{pmatrix}+\sqrt{\delta}\begin{pmatrix}0\\ 0\\ \omega_3\end{pmatrix},$$

with non negative real numbers β, γ, δ, which is a bounded operator from H into itself (*i.e.* $U=H$). We chose the matrix M defined by

$$M=\alpha\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix},\quad \alpha>0$$

which is obviously symmetric. As previously we can verify that A_0+M is positive definite if $\alpha<\pi^2/2$. Moreover M can be diagonalized by the orthogonal matrix

$$O=\frac{1}{2}\begin{pmatrix}1 & \sqrt{2} & 1\\ -\sqrt{2} & 0 & \sqrt{2}\\ 1 & -\sqrt{2} & 1\end{pmatrix},$$

into

$$D=\begin{pmatrix}-\sqrt{2}\alpha & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \sqrt{2}\alpha\end{pmatrix}.$$

Then the spectrum of $A=A_0+M$ is given by

$$\mathrm{Sp}(A)=\{k^2\pi^2-\sqrt{2}\alpha\}_{k\in\mathbb{N}^*}\cup\{k^2\pi^2\}_{k\in\mathbb{N}^*}\cup\{k^2\pi^2+\sqrt{2}\alpha\}_{k\in\mathbb{N}^*},$$

and the eigenvalues are simple (because the assumption $\alpha<\pi^2/2$ implies that $k^2\pi^2+\sqrt{2}\alpha<(k+1)^2\pi^2-\sqrt{2}\alpha$). Moreover, as we have shown previously, the generalized gap condition (1.7) is satisfied with $M=3$. Thanks to (8.8) the normalized eigenvectors are given by

$$\omega_{1,k}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -\sqrt{2}\\ 1\end{pmatrix}\sin(k\pi\cdot),\quad \omega_{2,k}=\frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{2}\\ 0\\ -\sqrt{2}\end{pmatrix}\sin(k\pi\cdot),\quad \omega_{3,k}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \sqrt{2}\\ 1\end{pmatrix}\sin(k\pi\cdot).$$

We set

$$\alpha_k^{(1,2)} = \lambda_{2,k} - \lambda_{1,k}, \quad \alpha_k^{(1,3)} = \lambda_{3,k} - \lambda_{1,k}, \quad \alpha_k^{(2,3)} = \lambda_{3,k} - \lambda_{2,k}.$$

Therefore, for all $C = (c_1, c_2, c_3)^T \in \mathbb{R}^3$, we have

$$\begin{aligned} \left\|B_k^{-1}\Phi_k C\right\|_{U,2}^2 &= \left\| \begin{pmatrix} 1 & 1 & 1 \\ 0 & \alpha_k^{(1,2)} & \alpha_k^{(1,3)} \\ 0 & 0 & \alpha_k^{(1,3)}\alpha_k^{(2,3)} \end{pmatrix} \begin{pmatrix} B^*\omega_{1,k} & 0 & 0 \\ 0 & B^*\omega_{2,k} & 0 \\ 0 & 0 & B^*\omega_{3,k} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \right\|_{U,2}^2 \\ &= \|c_1 B^*\omega_{1,k} + c_2 B^*\omega_{2,k} + c_3 B^*\omega_{3,k}\|_H^2 + \left\| c_2\alpha_k^{(1,2)} B^*\omega_{2,k} + c_3\alpha_k^{(1,3)} B^*\omega_{3,k} \right\|_H^2 \\ &\quad + |c_3|^2 \left|\alpha_k^{(1,3)}\alpha_k^{(2,3)}\right|^2 \|B^*\omega_{3,k}\|_H^2 . \end{aligned}$$

After some calculations, we obtain

$$\begin{aligned} \left\|B_k^{-1}\Phi_k C\right\|_{U,2}^2 &= \frac{\beta}{4}\left(c_1 + \sqrt{2}c_2 + c_3\right)^2 + \frac{\gamma}{2}(c_3 - c_1)^2 + \frac{\delta}{4}\left(c_1 - \sqrt{2}c_2 + c_3\right)^2 + \frac{\beta}{4}\left(\sqrt{2}\alpha_k^{(1,2)}c_2 + \alpha_k^{(1,3)}c_3\right)^2 \\ &\quad + \frac{\gamma}{2}\left|c_3\alpha_k^{(1,3)}\right|^2 + \frac{\delta}{2}\left(-\sqrt{2}\alpha_k^{(1,2)}c_2 + \alpha_k^{(1,3)}c_3\right)^2 + \frac{|c_3|^2}{2}\left|\alpha_k^{(1,3)}\alpha_k^{(2,3)}\right|^2\left(\frac{\beta+\delta}{2} + \gamma\right). \end{aligned}$$

Hence different decay results can be obtained for system (8.7) according to the values of β, γ and δ.

First if $\beta, \gamma, \delta > 0$, then we have

$$\left\|B_k^{-1}\Phi_k C\right\|_{U,2}^2 \geq C(c_1^2 + c_2^2 + c_3^2)$$

for $C > 0$, which shows that (1.10) holds and therefore system (8.7) is exponentially stable.

Second if $\gamma = 0$ and $\beta, \delta > 0$, we have

$$\begin{aligned} \left\|B_k^{-1}\Phi_k C\right\|_{U,2}^2 &\geq \tfrac{\min\{\beta,\delta\}}{4}\left(2c_1^2 + 4c_2^2 + 2c_3^2 + 4c_1c_3 + \min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2 (4c_2^2 + 2c_3^2)\right. \\ &\qquad \left. + \min\left\{\alpha_k^{(1,3)}, \alpha_k^{(2,3)}\right\}^4 c_3^2\right) \\ &\geq \tfrac{\min\{\beta,\delta\}}{4}\left(\left(2 - \tfrac{2}{\epsilon}\right)c_1^2 + 4\left(1 + \min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2\right)c_2^2\right. \\ &\qquad \left. + \left(2 - 2\epsilon + 2\min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2\right)c_3^2\right), \end{aligned}$$

by Young's inequality with $\epsilon > 0$. We then take $\epsilon = 1 + \min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2/2$, which implies

$$2 - \frac{2}{\epsilon} > \frac{\min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2}{2}, \quad 2 - 2\epsilon + 2\min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2 = \min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2,$$

if k is large enough. Consequently if k is large enough, we have obtained that

$$\begin{aligned} \left\|B_k^{-1}\Phi_k C\right\|_{U,2}^2 &\geq \tfrac{\min\{\beta,\delta\}}{4}\left(\tfrac{\min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2}{2}c_1^2 + 4\min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2 c_2^2 + \min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2 c_3^2\right) \\ &\geq \tfrac{\min\{\beta,\delta\}}{8}\min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2 (c_1^2 + c_2^2 + c_3^2), \end{aligned}$$

which shows that (1.13) holds with $l = 1$, since $\min\left\{\alpha_k^{(1,2)}, \alpha_k^{(1,3)}\right\}^2 \sim \lambda_{1,k}^{-2}$.

We construct the space V_h like in the previous subsection, *i.e.* it is the span of $(e_i, e_j, e_k)_{i,j,k\in\{1,...,N\}}$, that still satisfies (1.5) and (1.6) with $\theta = 1$.

Consequently, in the first case ($\beta, \gamma, \delta > 0$), we can apply Theorem 1.2 and thus the family of systems (1.9) is uniformly exponentially stable. In the second case ($\beta, \delta > 0$ and $\gamma = 0$), we can apply Theorem 1.6 with $l = 2$ and thus the family of systems (1.14) is uniformly polynomially stable, in the sense that (8.3) holds.

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