ESAIM: COCV 18 (2012) 930–940
DOI: 10.1051/cocv/2011189

**ESAIM: Control, Optimisation and Calculus of Variations**
www.esaim-cocv.org

# MULTIPLICITY OF SOLUTIONS FOR THE NONCOOPERATIVE $p$-LAPLACIAN OPERATOR ELLIPTIC SYSTEM WITH NONLINEAR BOUNDARY CONDITIONS

Sihua Liang[1] and Jihui Zhang[2]

**Abstract.** In this paper, we study the multiplicity of solutions for a class of noncooperative $p$-Laplacian operator elliptic system. Under suitable assumptions, we obtain a sequence of solutions by using the limit index theory.

**Mathematics Subject Classification.** 35J70, 35B20.

Received May 5, 2011.
Published online 16 January 2012.

## 1. Introduction

In this paper we deal with the existence and multiplicity of solutions to the following $p$-Laplacian operator elliptic system with nonlinear boundary conditions.

$$\begin{cases} \Delta_p u - |u|^{p-2}u = F_u(x,u,v), & \text{in } \Omega, \\ -\Delta_p v + |v|^{p-2}v = F_v(x,u,v), & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = |u|^{p^*-2}u, \quad |\nabla v|^{p-2}\frac{\partial v}{\partial \nu} = |v|^{p^*-2}v, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $1 < p < N$, $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is a $p$-Laplacian operator and $\frac{\partial}{\partial \nu}$ is the outer normal derivative, $F = F(x,u,v)$, $F_u = \frac{\partial F}{\partial u}$, $F_v = \frac{\partial F}{\partial v}$, $p^* = Np/(N-p)$ is the critical exponent according to the Sobolev embedding.

In recent years, the existence and multiplicity of solutions for a noncooperative elliptic system have been obtained by many papers. In [1], Benci assumed $X$ is a Hilbert space, $f$ satisfies $(PS)$-condition and is the form

$$f(u) = \frac{1}{2}\langle Lu, u\rangle + \Phi(u),$$

where $L$ is bounded self-adjoint operator and $\Phi'$ is compact.

*Keywords and phrases.* $p$-Laplacian operator, limit index, critical growth, concentration-compactness principle.

[1] College of Mathematics, Changchun Normal University, Changchun 130032, Jilin, P.R. China. liangsihua@163.com

[2] Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing, Jiangsu 210046, P.R. China. jihuiz@jlonline.com

When $p = 2$ (a constant) with Dirichlet boundary condition, Lin and Li [9] considered the following system

$$\begin{cases} \Delta u = |u|^{2^*-2}u + F_u(x,u,v), & \text{in } \Omega, \\ -\Delta v = |v|^{2^*-2}v + F_v(x,u,v), & \text{in } \Omega, \\ u = 0, \quad v = 0, & \text{on } \partial\Omega, \end{cases}$$

by applying the Limit Index Theory, they obtained the existence of multiple solutions under some assumptions on nonlinear part.

When $p \neq 2$, Huang and Li [6] considered the following the system of elliptic equations involving the $p$-Laplacian in the unbounded domain of $\mathbb{R}^N$ by applying the Limit Index Theory,

$$\begin{cases} \Delta_p u - |u|^{p-2}u = F_u(x,u,v), & \text{in } \mathbb{R}^N, \\ -\Delta_p v + |v|^{p-2}v = F_v(x,u,v), & \text{in } \mathbb{R}^N, \\ u, v \in W^{1,p}(\mathbb{R}^N), \end{cases}$$

where $1 < p < N$ and they extended some results of [8].

We note that these papers deal with Dirichlet boundary condition [2,7]. However, nonlinear boundary conditions have only been considered in recent years. For the Laplace operator with nonlinear boundary conditions see for example [3,14]. For elliptic systems with nonlinear boundary conditions see [5]. For previous work for the $p$-Laplacian with nonlinear boundary conditions of different type see [4,13].

Motivated by papers above, a natural question arises whether the existence and multiplicity of solutions to the $p$-Laplacian operator elliptic system with nonlinear boundary conditions (1.1) can be obtained. In this paper we deal with the problem (1.1). Throughout this paper, we assume that $F(x,u,v)$ satisfies the following conditions:

($H_1$) $F \in C(\overline{\Omega} \times R^2, R^+)$ and $F(x,s,t) = F(x,-s,-t)$ for all $(x,s,t) \in \Omega \times \in \mathbb{R}^2$;
($H_2$) $\lim_{|t|\to\infty} \frac{F_t(x,s,t)}{|t|^{p-1}} = 0$ uniformly for $x \in \Omega$;
($H_3$) $sF_s(x,s,t) \geq 0$ for all $(x,s,t) \in \overline{\Omega} \times R^2$.

Under assumptions ($H_1$) and ($H_2$), we have

$$F_v(x,u,v)v = o\left(|v|^p\right),$$

which means that, for all $\varepsilon > 0$, there exist $a(\varepsilon)$, $b(\varepsilon) > 0$ such that

$$|F(x,0,v)v| \leq a(\varepsilon) + \varepsilon|v|^p \tag{1.2}$$

and

$$|F_v(x,u,v)v| \leq b(\varepsilon) + \varepsilon|v|^p. \tag{1.3}$$

Hence, together with condition (1.2), (1.3) and the mean value theorem for any constants $\beta$ and fixed $u$ we have

$$|F(x,u,v) - \beta F_v(x,u,v)v| \leq c(\varepsilon) + \varepsilon|v|^p, \tag{1.4}$$

for some $c(\varepsilon) > 0$.

Furthermore, we assume that $F(x,u,v)$ satisfies condition:

($H_4$) There exist $L > 0$ (where $L$ will be determined later) and

$$\xi < |\Omega|^{-1} \min\left\{0, \frac{1}{N}S^{p^*/(p^*-p)} - c\left(\frac{1}{2N}\right)|\Omega|\right\}$$

such that $F(x,s,t)t \geq L|t|^p - \xi$, for every $(x,s,t) \in \overline{\Omega} \times \mathbb{R}^2$.

**Notation.** Weak (resp. strong) convergence is denoted by $\rightharpoonup$ (resp., $\to$). $|\cdot|_p$ is the usual norm in $L^p(\Omega)$. $L^p_2(\Omega) = L^p(\Omega)\times L^p(\Omega)$ with the norm $|(u,v)|_p := (|u|_p^p + |v|_p^p)^{\frac{1}{p}}$. $E := W^{1,p}(\Omega)$ with the norm $\|u\|_p := \int_\Omega(|\nabla u|^p + |u|^p)\mathrm{d}x$. $Y = E\times E$, $X_n = E\times E_n$. $c_i$ denote a positive constant and can be determined in concrete conditions.

According to [15], there exists a Schauder basis $\{e_n\}_{n=1}^\infty$ for $E$. Furthermore, since $E$ is reflexive, $\{e_n^*\}_{n=1}^\infty$ the biorthogonal functionals associated to the basis $\{e_n\}_{n=1}^\infty$ which are characterized by the relations

$$e_n^*(e_m) = \delta_{n,m} = \begin{cases} 1, & \text{if } n = m,\\ 0, & \text{if } n \neq m,\end{cases}$$

form a basis for $E^*$ with the following properties (*cf.* [10] Prop. 1.b.1 and Thm. 1.b.5). Denote

$$E_n = \operatorname{span}\{e_1,\dots,e_n\},\quad E_n^\perp = \overline{\operatorname{span}\{e_{n+1},\dots\}}$$

and

$$E_n^* = \operatorname{span}\{e_1^*,\dots,e_n^*\}.$$

Let $P_n : E\to E_n$ be the projector corresponding to decomposition $E = E_n\oplus E_n^\perp$ and $P_n^* : E^*\to E_n^*$ be the projector corresponding to the decomposition $E^* = E_n^*\oplus(E_n^*)^\perp$. Then $P_n u\to u$, $P_n^* v^*\to v^*$ for any $u\in E$, $v^*\in E^*$ as $n\to\infty$ and $\langle P_n^* v^*, u\rangle = \langle v^*, P_n u\rangle$. Let $\tau : E\to E^*$ be the mapping given by

$$\langle \tau u, \widetilde{u}\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\widetilde{u}\mathrm{d}x,\quad \text{for all } u,\widetilde{u}\in E.$$

It is easy to check that the operator $\tau$ is bounded, continuous. And if $u_n\to\widetilde{u}$ in $E$ and $\langle \tau u_n - \tau\widetilde{u}, u_n - \widetilde{u}\rangle\to 0$, then $u_n\to\widetilde{u}$ in $E$ (see [6,8])

The energy functional corresponding to problem (1.1) is defined as follows,

$$\begin{aligned} J(u,v) = &-\frac{1}{p}\int_\Omega(|\nabla u|^p + |u|^p)\mathrm{d}x + \frac{1}{p}\int_\Omega(|\nabla v|^p + |v|^p)\mathrm{d}x\\ &-\frac{1}{p^*}\int_{\partial\Omega}|u|^{p^*}\mathrm{d}\sigma - \frac{1}{p^*}\int_{\partial\Omega}|v|^{p^*}\mathrm{d}\sigma - \int_\Omega F(x,u,v)\mathrm{d}x. \end{aligned}\tag{1.5}$$

The main result of this paper is as follows.

**Theorem 1.1.** *Suppose that $F(x,u,v)$ satisfies conditions $(H_1)$–$(H_4)$. Then there exists $k_0 > 1$ such that (1.1) possesses at least $k_0 - 1$ pairs weak nontrivial solutions.*

**Remark 1.2.** There are two difficulties in considering the elliptic problem (1.1). One is the functional $J(u,v)$ is strongly indefinite. Therefore one cannot apply the symmetric Mountain pass theorem of the functional $J(u,v)$. The other one in solving the problem is a lack of compactness which can be illustrated by the fact that the embedding of $W^{1,p}(\Omega)$ into $L^{p^*}(\partial\Omega)$ is no longer compact.

**Remark 1.3.** Theorem 1.1 is new as far as we know. We mainly follow the way in [8] to prove our main result.

## 2. Preliminaries and lemmas

First of all, we recall the limit index theory due to Li [8]. In order to do that, we introduce the following definitions.

**Definition 2.1.** [8,16] The action of a topological group $G$ on a normed space $Z$ is a continuous map

$$G\times Z\to Z : [g,z]\mapsto gz$$

such that

$$1 \cdot z = z, \quad (gh)z = g(hz) \quad z \mapsto gz \ \text{ is linear}, \quad \forall\, g, h \in G.$$

The action is isometric if

$$\|gz\| = \|z\|, \quad \forall\, g \in G, \quad z \in Z.$$

And in this case $Z$ is called $G$-space.

The set of invariant points is defined by

$$\mathrm{Fix}G := \{z \in Z : gz = z, \ \forall\, g \in G\}.$$

A set $A \subset Z$ is invariant if $gA = A$ for every $g \in G$. A function $\varphi : Z \to R$ is invariant $\varphi \circ g = \varphi$ for every $g \in G$, $z \in Z$. A map $f : Z \to Z$ is equivariant if $g \circ f = f \circ g$ for every $g \in G$.

Suppose $Z$ is a $G$-Banach space, that is, there is a $G$ isometric action on $Z$. Let

$$\Sigma := \{A \subset Z : A \text{ is closed and } gA = A, \forall\, g \in G\}$$

be a family of all $G$-invariant closed subset of $Z$, and let

$$\Gamma := \left\{h \in C^0(Z, Z) : h(gu) = g(hu), \quad \forall\, g \in G\right\}$$

be the class of all $G$-equivariant mapping of $Z$. Finally, we call the set

$$O(u) := \{gu : g \in G\}$$

$G$-orbit of $u$.

**Definition 2.2.** [8] An index for $(G, \Sigma, \Gamma)$ is a mapping $i : \Sigma \to \mathcal{Z}_+ \cup \{+\infty\}$ (where $\mathcal{Z}_+$ is the set of all nonnegative integers) such that for all $A, B \in \Sigma$, $h \in \Gamma$, the following conditions are satisfied:

(1) $i(A) = 0 \Leftrightarrow A = \emptyset$;
(2) (monotonicity) $A \subset B \Rightarrow i(A) \le i(B)$;

(3) (subadditivity) $i(A \cup B) \le i(A) + i(B)$;

(4) (supervariance) $i(A) \le i(\overline{h(A)}), \forall\, h \in \Gamma$;

(5) (continuity) If $A$ is compact and $A \cap \mathrm{Fix}G = \emptyset$, then $i(A) < +\infty$ and there is a $G$-invariant neighbourhood $N$ of $A$ such that $i(\overline{N}) = i(A)$;

(6) (normalization) If $x \notin \mathrm{Fix}G$, then $i(O(x)) = 1$.

**Definition 2.3.** [1] An index theory is said to satisfy the $d$-dimension property if there is a positive integer $d$ such that

$$i(V^{dk} \cap S_1) = k$$

for all $dk$-dimensional subspaces $V^{dk} \in \Sigma$ such that $V^{dk} \cap \mathrm{Fix}G = \{0\}$, where $S_1$ is the unit sphere in $Z$.

Suppose $U$ and $V$ are $G$-invariant closed subspaces of $Z$ such that

$$Z = U \oplus V,$$

where $V$ is infinite dimensional and

$$V = \overline{\bigcup_{j=1}^{\infty} V_j},$$

where $V_j$ is a $dn_j$-dimensional $G$-invariant subspace of $V$, $j = 1, 2, \ldots$, and $V_1 \subset V_2 \subset \ldots \subset V_n \subset \ldots$ Let

$$Z_j = U \bigoplus V_j,$$

and $\forall\ A \in \Sigma$, let

$$A_j = A \bigoplus Z_j.$$

**Definition 2.4.** [8] Let $i$ be an index theory satisfying the $d$-dimension property. A limit index with respect to $(Z_j)$ induced by $i$ is a mapping

$$i^\infty : \Sigma \to \mathcal{Z} \cup \{-\infty, +\infty\}$$

given by

$$i^\infty(A) = \lim\sup_{j\to\infty} (i(A_j) - n_j).$$

**Proposition 2.5.** [8] *Let $A, B \in \Sigma$. Then $i^\infty$ satisfies:*
(1) $A = \emptyset \Rightarrow i^\infty = -\infty$;
(2) *(monotonicity)* $A \subset B \Rightarrow i^\infty(A) \le i^\infty(B)$;
(3) *(subadditivity)* $i^\infty(A \cup B) \le i^\infty(A) + i^\infty(B)$;
(4) *If* $V \cap \mathrm{Fix}G = \{0\}$, *then* $i^\infty(S_\rho \cap V) = 0$, *where* $S_\rho = \{z \in Z : \|z\| = \rho\}$;
(5) *If* $Y_0$ *and* $\widetilde{Y_0}$ *are* $G$*-invariant closed subspaces of* $V$ *such that* $V = Y_0 \oplus \widetilde{Y_0}$, $\widetilde{Y_0} \subset V_{j_0}$ *for some* $j_0$ *and* $\dim\widetilde{Y_0} = dm$, *then* $i^\infty(S_\rho \cap Y_0) \ge -m$.

**Definition 2.6.** [16] A functional $J \in C^1(Z, R)$ is said to satisfy the condition $(PS)_c^*$ if any sequence $\{u_{n_k}\}$, $u_{n_k} \in Z_{n_k}$ such that

$$J(u_{n_k}) \to c, \quad \mathrm{d}J_{n_k}(u_{nk}) \to 0, \quad \text{as } k \to \infty$$

possesses a convergent subsequence, where $Z_{n_k}$ is the $n_k$-dimension subspace of $Z$, $J_{n_k} = J|_{Z_{n_k}}$.

**Theorem 2.7.** [8] *Assume that*

(B$_1$) $J \in C^1(Z, R)$ *is* $G$*-invariant;*
(B$_2$) *there are* $G$*-invariant closed subspaces* $U$ *and* $V$ *such that* $V$ *is infinite dimensional and* $Z = U \oplus V$*;*
(B$_3$) *there is a sequence of* $G$*-invariant finite dimensional subspaces*

$$V_1 \subset V_2 \subset \cdots \subset V_j \subset \cdots, \quad \dim V_j = dn_j,$$

*such that* $V = \overline{\cup_{j=1}^\infty V_j}$*;*
(B$_4$) *there is an index theory* $i$ *on* $Z$ *satisfying the* $d$*-dimension property;*
(B$_5$) *there are* $G$*-invariant subspaces* $Y_0$, $\widetilde{Y_0}$, $Y_1$ *of* $V$ *such that* $V = Y_0 \oplus \widetilde{Y_0}$, $Y_1, \widetilde{Y_0} \subset V_{j_0}$ *for some* $j_0$ *and* $\dim\widetilde{Y_0} = \mathrm{d}m < \mathrm{d}k = \dim Y_1$*;*
(B$_6$) *there are* $\alpha$ *and* $\beta$, $\alpha < \beta$ *such that* $f$ *satisfies* $(PS)_c^*$, $\forall\ c \in [\alpha, \beta]$*;*
(B$_7$)

$$\begin{cases} (a)\ \textit{either}\ \mathrm{Fix}G \subset U \oplus Y_1, \quad \textit{or} \quad \mathrm{Fix}G \cap V = \{0\}, \\ (b)\ \textit{there is}\ \rho > 0\ \textit{such that}\ \forall\ u \in Y_0 \cap S_\rho, f(z) \ge \alpha, \\ (c)\ \forall\ z \in U \oplus Y_1,\ f(z) \le \beta, \end{cases}$$

*if* $i^\infty$ *is the limit index corresponding to* $i$, *then the numbers*

$$c_j = \inf_{i^\infty(A)\ge j} \sup_{z\in A} f(u), \quad -k+1 \le j \le -m,$$

*are critical values of* $f$, *and* $\alpha \le c_{-k+1} \le \cdots \le c_{-m} \le \beta$. *Moreover, if* $c = c_l = \cdots = c_{l+r}$, $r \ge 0$, *then* $i(\mathbb{K}_c) \ge r + 1$, *where* $\mathbb{K}_c = \{z \in Z : \mathrm{d}f(z) = 0, f(z) = c\}$.

## 3. Local Palais-Smale condition

To prove Theorem 1.1, noting the lack of compactness, in the inclusion $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\partial\Omega)$, we can no longer expect the Palais-Smale condition to hold. Anyway we can prove a local Palais-Smale condition that will hold for $J(u,v)$ below a certain value of energy. Let $u_n$ be a bounded sequence in $W^{1,p}(\Omega)$ then there exists a subsequence that we still denote $u_n$ such that

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{weakly in } W^{1,p}(\Omega),\\ &u_n \to u \quad \text{strongly in } L^r(\Omega),\ 1 \le r < p^*,\\ &|\nabla u_n|^p \rightharpoonup \mathrm{d}\mu, \quad |u_n|_{\partial\Omega}|^{p^*} \rightharpoonup \mathrm{d}\eta, \end{aligned}$$

weakly-$^*$ in the sense of measures. Observe that $\mathrm{d}\eta$ is a measure supported on $\partial\Omega$.

If we consider $\phi \in C^\infty(\overline{\Omega})$, from the Sobolev trace inequality we obtain, passing to the limit

$$\left(\int_{\partial\Omega} |\phi|^{p^*}\mathrm{d}\eta\right)^{\frac{1}{p^*}} S^{\frac{1}{p}} \le \left(\int_\Omega |\phi|^p \mathrm{d}\mu + \int_\Omega |u|^p|\nabla\phi|^p \mathrm{d}x + \int_\Omega |\phi u|^p \mathrm{d}x\right)^{\frac{1}{p}}, \tag{3.1}$$

where $S$ is the best constant in the Sobolev trace embedding theorem. From (3.1) we observe that if $u = 0$ we get a reverse Hölder-type inequality (but it involves one integral over $\Omega$) between the two measures $\mu$ and $\eta$.

Similar to the proof of [11,12], we have the following lemma.

**Lemma 3.1.** [4] *Let $u_j$ be a weakly convergent sequence in $W^{1,p}(\Omega)$ with weak limit $u$ such that*

$$|\nabla u_j|^p \rightharpoonup \mathrm{d}\mu, \quad \text{and} \quad |u_j|_{\partial\Omega}|^{p^*} \rightharpoonup \mathrm{d}\eta,$$

*weakly-$^*$ in the sense of measures. Then there exists $x_1, \dots, x_l \in \partial\Omega$ such that*

*(i)* $\mathrm{d}\eta = |u|^{p^*} + \sum_{j=1}^l \eta_j \delta_{x_j}$, $\eta_j > 0$;

*(ii)* $\mathrm{d}\mu \ge |\nabla u|^p + \sum_{j=1}^l \mu_j \delta_{x_j}$, $\mu_j > 0$;

*(iii)* $(\eta_j)^{\frac{p}{p^*}} \le \frac{\mu_j}{S}$.

Similar to [6,16], it is easy to obtain the following lemma:

**Lemma 3.2.** *Assume $1 \le \theta_1, \theta_2, \theta < \infty$, $I \in C(\overline{\Omega} \times R^2, R)$ and*

$$I(x,u,v) \le C\left(|u|^{\frac{\theta_1}{\theta}} + |v|^{\frac{\theta_2}{\theta}}\right).$$

*Then for every $(u,v) \in L^{\theta_1}(\Omega) \times L^{\theta_2}(\Omega)$, $I(\cdot,u,v) \in L^\theta(\Omega)$ and the operator*

$$T : (u,v) \mapsto I(x,u,v)$$

*is a continuous map from $L^{\theta_1}(\Omega) \times L^{\theta_2}(\Omega)$ to $L^\theta(\Omega)$.*

**Lemma 3.3.** *Suppose that $F(x,u,v)$ satisfies conditions $(H_1)$–$(H_3)$. Then*

*(i)* $J \in C^1(X,R)$;

*(ii)*

$$\begin{aligned} \langle \mathrm{d}J(u,v),(\widehat{u},\widehat{v})\rangle = &-\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\widehat{u} + |u|^{p-2}u\widehat{u}\mathrm{d}x - \int_{\partial\Omega} |u|^{p^*-2}u\widehat{u}\mathrm{d}\sigma\\ &+ \int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla\widehat{v} + \ll |v|^{p-2}v\widehat{v}\mathrm{d}x - \int_{\partial\Omega} |v|^{p^*-2}v\widehat{v}\mathrm{d}\sigma\\ &- \int_\Omega F_u(x,u,v)\widehat{u}\mathrm{d}x - \int_\Omega F_v(x,u,v)\widehat{v}\mathrm{d}x; \end{aligned}$$

*(iii) A critical point of $J$ is a weak solution of system (1.1).*

Now set

$$X = U \oplus V, \quad U = E \times \{0\}, \quad V = \{0\} \times E,$$
$$Y_0 = \{0\} \times E_1^{\perp}, \quad V = Y_0 \oplus \widetilde{Y_0},$$
$$Y_1 = \{0\} \times E_{k_0}, \quad E_{k_0} = \operatorname{span}\{e_1, \ldots, e_{k_0}\},$$

then $\dim \widetilde{Y_0} = 1$, $\dim Y_1 = k_0$.

Define a group action $G_2 = \{1, \tau\} \cong \mathcal{Z}_2$ by setting $\tau(u, v) = (-u, -v)$, then $\mathrm{Fix}G = \{0\} \times \{0\}$ (also denote $\{0\}$). It is clear that $U$ and $V$ are $G$-invariant closed subspaces of $X$, and $Y_0$, $\widetilde{Y_0}$ and $Y_1$ are $G$-invariant subspace of $V$. Set

$$\Sigma := \{A \subset X \setminus \{0\} : \; A \text{ is closed in } X \text{ and } (u, v) \in A \Rightarrow (-u, -v) \in A\}.$$

Define an index $\gamma$ on $\Sigma$ by:

$$\gamma(A) = \begin{cases} \min\{N \in \mathcal{Z} : \exists\, h \in C(A, \mathbb{R}^N \backslash \{0\} \text{ such that } h(-u, -v) = h(u, v))\}, \\ 0, \quad \text{if } A = \emptyset, \\ +\infty, \quad \text{if such } h \text{ does not exist.} \end{cases}$$

Then we have the following proposition from [6]: $\gamma$ is an index satisfying the properties given in Definition 2.2. Moreover, $\gamma$ satisfies the one-dimension property. According to Definition 2.4 we can obtain a limit index $\gamma^\infty$ with respect to $(X_n)$ from $\gamma$.

Now we turn to prove local Palais-Smale condition.

**Lemma 3.4.** *Assume condition $(H_1)$–$(H_3)$ hold, Then the functional $J$ satisfies the local $(PS)_c$ condition in*

$$c \in \left(-\infty, \frac{1}{N} S^{p^*/(p^*-p)} - c\left(\frac{1}{2N}\right) |\Omega|\right),$$

*in the following sense: if*

$$J(u_{n_k}, v_{n_k}) \to c \in \left(-\infty, \frac{1}{N} S^{p^*/(p^*-p)} - c\left(\frac{1}{2N}\right) |\Omega|\right), \quad \mathrm{d}J_{n_k}(u_{n_k}, v_{n_k}) \to 0, \quad \textit{as } k \to \infty,$$

*where $J_{n_k} = J|_{X_{n_k}}$. Then $\{(u_{n_k}, v_{n_k})\}$ contains a subsequence converging strongly in $X$.*

*Proof.* First, we show that $\{(u_{n_k}, v_{n_k})\}$ is bounded in $X$.

We note that by condition $(H_3)$,

$$\begin{aligned} o(1)\|u_{n_k}\|_p &\geq \langle -\mathrm{d}J_{n_k}(u_{n_k}, v_{n_k}), (u_{n_k}, 0)\rangle \\ &= \int_\Omega |\nabla u_{n_k}|^p + |u_{n_k}|^p \mathrm{d}x + \int_{\partial\Omega} |u_{n_k}|^{p^*} \mathrm{d}\sigma + \int_\Omega F_u(x, u_{n_k}, v_{n_k}) u_{n_k} \mathrm{d}x \\ &\geq \int_\Omega |\nabla u_{n_k}|^p + |u_{n_k}|^p \mathrm{d}x + \int_{\partial\Omega} |u_{n_k}|^{p^*} \mathrm{d}x \\ &\geq \|u_{n_k}\|_p^p, \end{aligned} \tag{3.2}$$

since $p > 1$, from (3.2), we know that $\|u_{n_k}\|_p$ is bounded. On the one hand, we have

$$\begin{aligned} &J_{n_k}(0, v_{n_k}) - \frac{1}{p^*}\langle \mathrm{d}J_{n_k}(u_{n_k}, v_{n_k}), (0, v_{n_k})\rangle \\ &= \left(\frac{1}{p} - \frac{1}{p^*}\right) \int_\Omega (|\nabla v_{n_k}|^p + |v_{n_k}|^p)\, \mathrm{d}x - \int_\Omega \left[F(x, u_{n_k}, v_{n_k}) - \frac{1}{p^*} F_v(x, u_{n_k}, v_{n_k}) v_{n_k}\right] \mathrm{d}x \\ &= c + o(1)\|v_n\|_p, \end{aligned}$$

*i.e.*

$$\frac{1}{N}\int_\Omega (|\nabla v_{n_k}|^p + |v_{n_k}|^p)\,\mathrm{d}x = \int_\Omega \left[F(x,u_{n_k},v_{n_k}) - \frac{1}{p^*}F_v(x,u_{n_k},v_{n_k})v_{n_k}\right]\mathrm{d}x + c + o(1)\|v_{n_k}\|_p.$$

Then by (1.4), we have

$$\left(\frac{1}{N}-\varepsilon\right)\|v_{n_k}\|_p^p \le c(\varepsilon)|\Omega| + c + o(1)\|v_{n_k}\|_p,$$

where $|\cdot|$ denote by Lebesgue measure. Setting $\varepsilon = 1/2N$, we get

$$\|v_{n_k}\|_p^p \le M + o(1)\|v_{n_k}\|_p, \tag{3.3}$$

where $o(1)\to 0$ and $M$ is a some positive number. Thus (3.3) implies that $\{v_{n_k}\}$ is bounded in $W^{1,p}(\Omega)$. This implies $\|u_{n_k}\|_p + \|v_{n_k}\|_p$ is bounded in $X$.

Next, we prove that $\{(u_{n_k},v_{n_k})\}$ contains a subsequence converging strongly in $X$.

We note that $\{u_{n_k}\}$ is bounded in $E$. Hence, up to a subsequence, $u_{n_k}\rightharpoonup u$ weakly in $E$ and $u_{n_k}(x)\to u(x)$, a.e. in $\mathbb{R}^N$. We claim that $u_{n_k}\to u$ strongly in $E$. In fact, note that

$$\int_\Omega |\nabla u_{n_k}-\nabla u|^p + |u_{n_k}-u|^p\mathrm{d}x + \int_{\partial\Omega}|u_{n_k}-u|^{p^*}\mathrm{d}\sigma + \int_\Omega F_u(x,u_{n_k}-u,v_{n_k})(u_{n_k}-u)\mathrm{d}x = \langle -\mathrm{d}J_{n_k}(u_{n_k}-u,v_{n_k}),(u_{n_k}-u,0)\rangle \to 0 \quad \text{as } n\to\infty,$$

and condition $(H_3)$ imply that

$$u_{n_k}\to u \quad \text{strongly in } E. \tag{3.4}$$

In the following we will prove that there exists $v\in E$ such that

$$v_{n_k}\to v \quad \text{strongly in } E. \tag{3.5}$$

By Lemma 3.1 and (3.3) there exists a subsequence, there exists a subsequence, that we still denote $v_{n_k}$ such that

$$\begin{aligned}
&v_{n_k}\rightharpoonup v \quad \text{weakly in } W^{1,p}(\Omega),\\
&v_{n_k}\to v \quad \text{strongly in } L^r(\Omega),\ 1\le r<p^*,\ \text{and a.e. in } \Omega\\
&|\nabla v_{n_k}|^p \rightharpoonup \mathrm{d}\mu \ge |\nabla v|^p + \sum_{k=1}^{l}\mu_k\delta_{x_k},\\
&|v_{n_k}|_{\partial\Omega}|^{p^*} \rightharpoonup \mathrm{d}\eta = |v|_{\partial\Omega}|^{p^*} + \sum_{k=1}^{l}\eta_k\delta_{x_k}.
\end{aligned}$$

Let $\phi(x)\in C^\infty(\Omega)$ such that $\phi(x)\equiv 1$ in $B(x_k,\varepsilon)$, $\phi(x)\equiv 0$ in $\Omega\setminus(x_k,2\varepsilon)$ and $|\nabla\phi|\le 2/\varepsilon$, where $x_k$ belongs to the support of $\mathrm{d}\eta$. Consider Then $\{\phi v_{n_k}\}$ is bounded in $E$, Obviously, $\langle \mathrm{d}J_{n_k}(u_{n_k},v_{n_k}),(0,v_{n_k}\phi)\rangle\to 0$, *i.e.*

$$\lim_{n\to\infty}\left[\int_\Omega (|\nabla v_{n_k}|^p + |v_{n_k}|^p)\,\phi\mathrm{d}x - \int_{\partial\Omega}|v_{n_k}|^{p^*}\phi\mathrm{d}\sigma - \int_\Omega F_v(x,u_{n_k},v_{n_k})v_{n_k}\phi\mathrm{d}x\right] = -\lim_{n\to\infty}\int_\Omega \left(v_{n_k}|\nabla v_{n_k}|^{p-2}\nabla v_{n_k}\nabla\phi\right)\mathrm{d}x. \tag{3.6}$$

On the other hand, by Hölder inequality and weak convergence, we obtain

$$\begin{aligned}
0 &\le \lim_{\varepsilon\to 0}\lim_{n\to\infty}\left|\int_\Omega v_{n_k}|\nabla v_{n_k}|^{p-2}\nabla v_{n_k}\nabla\phi \mathrm{d}x\right| \\
&\le \lim_{\varepsilon\to 0}\lim_{n\to\infty}\left(\int_\Omega |v_{n_k}|^p|\nabla\phi|^p\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_\Omega |\nabla v_{n_k}|^q\mathrm{d}x\right)^{\frac{p-1}{p}} \\
&\le C\lim_{\varepsilon\to 0}\left(\int_\Omega |v|^p|\nabla\phi|^p\mathrm{d}x\right)^{\frac{1}{p}} \\
&\le C\lim_{\varepsilon\to 0}\left(\int_{B(x_j,\varepsilon)} |\nabla\phi|^N\mathrm{d}x\right)^{\frac{1}{N}}\left(\int_{B(x_j,\varepsilon)} |v|^{p^*}\mathrm{d}x\right)^{\frac{1}{p^*}} \\
&\le C\lim_{\varepsilon\to 0}\left(\int_{B(x_j,\varepsilon)} |v|^{p^*}\mathrm{d}x\right)^{\frac{1}{p^*}} = 0. \qquad (3.7)
\end{aligned}$$

From (3.6) and (3.7), we have

$$0 = \lim_{\varepsilon\to 0}\left[\int_{\partial\Omega}\phi\mathrm{d}\eta - \int_\Omega \phi\mathrm{d}\mu - \int_\Omega |v|^p\phi\mathrm{d}x - \int_\Omega F_v(x,u,v)v\phi\mathrm{d}x\right] = \eta_k - \mu_k. \qquad (3.8)$$

Combing this with Lemma 3.1, we obtain $(\mu_k)^{p/p^*}S \le \mu_k$. This result implies that

$$\mu_k = 0 \qquad \text{or} \qquad \mu_k \ge S^{p^*/(p^*-p)}.$$

If the second case $\mu_k \ge S^{p^*/(p^*-p)}$ holds, for some $k \in J$, then by using Lemma 3.1 and the Hölder inequality, we have that

$$\begin{aligned}
c &= \lim_{n\to\infty}\left(J_{n_k}(0,v_{n_k}) - \frac{1}{p^*}\langle \mathrm{d}J_{n_k}(u_{n_k},v_{n_k}),(0,v_{n_k})\rangle\right) \\
&= \left(\frac{1}{p}-\frac{1}{p^*}\right)\int_\Omega (|\nabla v_{n_k}|^p + |v_{n_k}|^p)\,\mathrm{d}x - \int_\Omega\left[F(x,u_{n_k},v_{n_k}) - \frac{1}{p^*}F_v(x,u_{n_k},v_{n_k})v_{n_k}\right]\mathrm{d}x \\
&\ge \frac{1}{N}\int_\Omega \mathrm{d}\mu - c\left(\frac{1}{2N}\right)|\Omega| \\
&\ge \frac{1}{N}\int_\Omega |\nabla v_{n_k}|^p\mathrm{d}x + \frac{1}{N}S^{p^*/(p^*-p)} - c\left(\frac{1}{2N}\right)|\Omega| \\
&\ge \frac{1}{N}S^{p^*/(p^*-p)} - c\left(\frac{1}{2N}\right)|\Omega|,
\end{aligned}$$

where $\varepsilon = 1/2N$. This is impossible. Consequently, $\mu_k = 0$ for all $k \in J$. From (3.8) we know that $\eta_k = 0$ for all $k \in J$ and hence

$$\int_{\partial\Omega}|v_{n_k}|^{p^*}\mathrm{d}\sigma \to \int_{\partial\Omega}|v|^{p^*}\mathrm{d}\sigma.$$

Now $v_{n_k} \rightharpoonup v$ in $E$ and Brezis-Lieb lemma [2] implies that

$$\lim_{n\to\infty}\int_{\partial\Omega}|v_{n_k} - v|^{q^*}\mathrm{d}\sigma = 0.$$

Thus, we have

$$\begin{aligned} o(1)\|v_{n_k}\|_p &= \|v_{n_k}\|_p^p - \int_\Omega |v_{n_k}|^{p^*}\mathrm{d}\sigma - \int_\Omega F_v(x,u_{n_k},v_{n_k})v_{n_k}\mathrm{d}x \\ &= \|v_{n_k}-v\|_p^p + \|v\|_p^p - \int_\Omega |v|^{p^*}\mathrm{d}\sigma - \int_\Omega F_v(x,u,v)v\mathrm{d}x \\ &= \|v_{n_k}-v\|_p^p + o(1)\|v\|_p, \end{aligned}$$

since $\mathrm{d}J_{n_k}(0,v)=0$. Thus we prove that $\{v_{n_k}\}$ strongly converges to $v$ in $E$. Thus (3.5) holds. (3.4) and (3.5) imply the conclusion of Lemma 3.4 follows. □

## 4. Proof of Theorem 1.1

*Proof of Theorem 1.1.* Now we shall verify the conditions of Theorem 2.7. Obviously, $(B_1)$, $(B_2)$, $(B_4)$ in Theorem 2.7 are satisfied. Set $V_j = E_j = \mathrm{span}\{e_1, e_2, \dots, e_j\}$, then $(B_3)$ is also satisfied. Since $1 = \dim \widetilde{Y_0} < k_0 = \dim Y_1$, $(B_5)$ is satisfied. In the following we verify the conditions in $(B_7)$. Since $\mathrm{Fix}G \cap V = 0$, that is $(a)$ of $(B_7)$ holds. It remains to verify $(b)$, $(c)$ of $(B_7)$. Choose a number $\alpha$ such that

$$\alpha < \min\left\{0, \frac{1}{N}S^{p^*/(p^*-p)} - c\left(\frac{1}{2N}\right)|\Omega|, \frac{1}{N}2^{\frac{p^*}{p-p^*}}S^{\frac{pp^*}{p-p^*}} - b\left(\frac{1}{2p}\right)|\Omega|\right\}. \tag{4.1}$$

(i) If $(0,v) \in Y_0 \cap S_\rho$ (where $\rho$ is to be determined) then by $(H_2)$,

$$\begin{aligned} J(0,v) &= \frac{1}{p}\int_\Omega |\nabla v|^p + |v|^p\mathrm{d}x - \frac{1}{p^*}\int_{\partial\Omega}|v|^{p^*}\mathrm{d}\sigma - \int_\Omega F(x,0,v)\mathrm{d}x \\ &\geq \left(\frac{1}{p}-\varepsilon\right)\cdot\int_\Omega |\nabla v|^p + |v|^p\mathrm{d}x - \frac{1}{p^*}\int_{\partial\Omega}|v|^{p^*}\mathrm{d}\sigma - b(\varepsilon)|\Omega| \\ &\geq \frac{1}{2p}\|v\|_p^p - \frac{1}{p^*}S^{p^*}\|v\|_p^{p^*} - b\left(\frac{1}{2p}\right)|\Omega|, \end{aligned} \tag{4.2}$$

where $\varepsilon = \frac{1}{2p}$. Since

$$\max_{t\in\mathbb{R}}\left(\frac{1}{2p}t^p - \frac{1}{p^*}S^{p^*}t^{p^*} - b\left(\frac{1}{2p}\right)|\Omega|\right) = \frac{1}{N}2^{\frac{p^*}{p-p^*}}S^{\frac{pp^*}{p-p^*}} - b\left(\frac{1}{2p}\right)|\Omega|,$$

Therefore, there exists $\rho > 0$ such that $J(0,v) \geq \alpha$ for every $\|v\|_p = \rho$, that is $(b)$ of $(B_7)$ holds.

(ii) For each $(u,v) \in U \oplus Y_1$, by condition $(H_4)$, we have

$$\begin{aligned} J(u,v) &= -\frac{1}{p}\int_\Omega (|\nabla u|^p + |u|^p)\mathrm{d}x + \frac{1}{p}\int_\Omega (|\nabla v|^p + |v|^p)\mathrm{d}x \\ &\quad -\frac{1}{p^*}\int_{\partial\Omega}|u|^{p^*}\mathrm{d}\sigma - \frac{1}{p^*}\int_{\partial\Omega}|v|^{p^*}\mathrm{d}\sigma - \int_\Omega F(x,u,v)\mathrm{d}x \\ &\leq \frac{1}{p}\|v\|_p^p - L|v|_p^p + \xi|\Omega| \\ &\leq \max_{v\in E_{k_0}}\left(\frac{1}{p}\|v\|_p^p - L|v|_p^p\right) + \xi|\Omega| \\ &= \max_{\{t\geq 0, v\in\partial B_1(0)\cap E_{k_0}\}}\left[t^p\left(\frac{1}{p} - L|v|_p^p\right)\right] + \xi|\Omega|. \end{aligned} \tag{4.3}$$

Let $r = \min\{\int_\Omega |v|^p \mathrm{d}x : v \in \partial B_1(0) \cap E_{k_0}\}$. By taking $L \geq \frac{1}{pr}$, we have

$$\frac{1}{p} - L|v|_p^p \leq \frac{1}{p} - Lr \leq 0. \tag{4.4}$$

It follows from (4.3), (4.4) and $(H_4)$ that

$$J(u,v) \leq \xi|\Omega| \leq \min\left\{0, \frac{1}{N}S^{p^*/(p^*-p)} - c\left(\frac{1}{2N}\right)|\Omega|\right\}.$$

Let $\beta = \xi|\Omega|$, so we get $(c)$ in $(B_7)$. By Lemma 3.4, for any $c \in [\alpha, \beta]$, $J(u,v)$ satisfies the condition of $(PS)_c^*$, then $(B_6)$ in Theorem 2.7 holds. So according to Theorem 2.7,

$$c_j = \inf_{i^\infty(A) \geq j} \sup_{z \in A} f(u), \quad -k_0 + 1 \leq j \leq -1,$$

are critical values of $J$, $\alpha \leq c_{-k_0+1} \leq \cdots \leq c_{-1} \leq \beta < 0$ and $J$ has at least $k_0 - 1$ pairs critical points. □

*Acknowledgements.* The Project is supported by NSFC (10871096), Research Fundation during the 12st Five-Year Plan Period of Department of Education of Jilin Province, China (Grant [2011] No. 196), Natural Science Foundation of Changchun Normal University.

## References

[1] V. Benci, On critical point theory for indefinite functionals in presence of symmetries. *Trans. Amer. Math. Soc.* **274** (1982) 533–572.
[2] H. Brézis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical exponents. *Comm. Pure Appl. Math.* **34** (1983) 437–477.
[3] M. Chipot, I. Shafrir and M. Fila, On the solutions to some elliptic equations with nonlinear boundary conditions. *Advances Differential Equations* **1** (1996) 91–110.
[4] J. Fernández Bonder and J.D. Rossi, Existence results for the $p$-Laplacian with nonlinear boundary conditions. *J. Math. Anal. Appl.* **263** (2001) 195–223.
[5] J. Fernández Bonder, J.P. Pinasco and J.D. Rossi, Existence results for a Hamiltonian elliptic system with nonlinear boundary conditions. *Electron. J. Differential Equations* **1999** (1999) 1–15.
[6] D.W. Huang and Y.Q. Li, Multiplicity of solutions for a noncooperative $p$-Laplacian elliptic system in $\mathbb{R}^N$. *J. Differential Equations* **215** (2005) 206–223.
[7] W. Krawcewicz and W. Marzantowicz, Some remarks on the Lusternik-Schnirelman method for non-differentiable functionals invariant with respect to a finite group action. *Rocky Mt. J. Math.* **20** (1990) 1041–1049.
[8] Y.Q. Li, A limit index theory and its application. *Nonlinear Anal.* **25** (1995) 1371–1389.
[9] F. Lin and Y.Q. Li, Multiplicity of solutions for a noncooperative elliptic system with critical Sobolev exponent. *Z. Angew. Math. Phys.* **60** (2009) 402–415.
[10] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I.* Springer, Berlin (1977).
[11] P.L. Lions, The concentration-compactness principle in the caculus of variation: the limit case, I. *Rev. Mat. Ibero.* **1** (1985) 45–120.
[12] P.L. Lions, The concentration-compactness principle in the caculus of variation: the limit case, II. *Rev. Mat. Ibero.* **1** (1985) 145–201.
[13] K. Pflüger, Existence and multiplicity of solutions to a $p$-Laplacian equation with nonlinear boundary condition, *Electron. J. Differential Equations* **10** (1998) 1–13.
[14] S. Terraccini, Symmetry properties of positive solutions to some elliptic equations with nonlinear boundary conditions. *Differential Integral Equations* **8** (1995) 1911–1922.
[15] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators.* North- Holland, Amsterdam (1978).
[16] M. Willem, *Minimax Theorems.* Birkhäuser, Boston (1996).