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HOMOGENIZATION OF QUASILINEAR OPTIMAL CONTROL PROBLEMS INVOLVING A THICK MULTILEVEL JUNCTION OF TYPE 3 : 2 : 1*

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Abstract. We consider quasilinear optimal control problems involving a thick two-level junction Ω_ε which consists of the junction body Ω_0 and a large number of thin cylinders with the cross-section of order $\mathcal{O}(\varepsilon^2)$. The thin cylinders are divided into two levels depending on the geometrical characteristics, the quasilinear boundary conditions and controls given on their lateral surfaces and bases respectively. In addition, the quasilinear boundary conditions depend on parameters $\varepsilon, \alpha, \beta$ and the thin cylinders from each level are ε-periodically alternated. Using the Buttazzo–Dal Maso abstract scheme for variational convergence of constrained minimization problems, the asymptotic analysis (as $\varepsilon \to 0$) of these problems are made for different values of α and β and different kinds of controls. We have showed that there are three qualitatively different cases. Application for an optimal control problem involving a thick one-level junction with cascade controls is presented as well.

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Introduction and statement of the problem

A thick junction of type $k : p : d$ is the union of some domain in $\mathbb{R}^n$, which is called the junction's body, and a large number of ε-periodically situated thin domains along some manifold on the boundary of the junction's body. This manifold is called the joint zone. Here ε is a small parameter, which characterizes the distance between neighboring thin domains and their thickness. The type $k : p : d$ of a thick junction refers to the limiting dimensions of the body, the joint zone, and each of the attached thin domains, respectively.

This classification of thick junctions was given in [17,18,21–23], where rigorous mathematical methods were developed (homogenization, approximation, asymptotic expansions) for analyzing the main boundary-value problems in thick junctions of different types. It was pointed out that qualitative properties of solutions

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essentially depend on the junction type and on the conditions given on the boundaries of the attached thin domains.

Various constructions of thick junction type are successfully used in nanotechnologies (*e.g.* [13,14]), microtechnique (*e.g.* [16]), modern engineering constructions (microstrip radiator, ferrite-filled rod radiator), as well as many physical and biological systems such as, for example, the structure of the intestine lining with different levels of absorption of nutrients on different part of the tissues.

Therefore boundary-value problems in thick junctions of different types are very extensively investigated at present (see [1–3,6,8,10], [19,20,24,25] and references therein). The aim of these researches is to develop rigorous methods to study the asymptotic behavior of solutions when the number of the attached thin domains of a thick junction infinitely increases and their thickness vanishes.

It should be noted here that such problems lose coercitivity in the limit passage. Secondly, thick junctions have special character of the connectedness and, as a result, there are no extension operators that would be bounded uniformly in the corresponding Sobolev spaces. At the same time the availability of an uniformly bounded family of extension operators is typical supposition in overwhelming majority of the existing homogenization schemes for problems in perforated domains with the Neumann boundary conditions (see *e.g.* [12]). Thirdly, thick junctions are non-convex domains with non-smooth boundaries. Therefore, solutions of boundary-value problems in such domains have only minimal H^1-smoothness and we have to take admissible boundary controls with more smoothness (for comparison see [15], where the Dirichlet boundary controls belong to L^2 but the boundary is smooth). All these factors create special difficulties in the asymptotic investigation.

A thick multilevel junction is a thick junction in which the thin domains are divided into a finite number of levels depending on their geometrical and other characteristics (boundary conditions and controls for our problem); in addition the thin domains from each level are ε-periodically alternated along the joint zone. In [8,10,24] it was shown that processes in thick multi-level junctions behave as a "many-phase system" in the region which is filled up by the thin domains from each level in the limit passage.

In this paper we continue our investigation of optimal control problems in thick multilevel junctions, which we have begun in [10]. Here we improve and generalize our results in the case of the perturbed nonlinear boundary multi-phase interactions and more complicated structure of a thick multilevel junction.

There are two different approaches to homogenize optimal control problems. One consists in the passage to the limit in the corresponding adjoint problem and then recover an optimal control problem which is called the homogenized control problem to the initial one (see *e.g.* [7,11,12]). The other one (so-called *direct method*) is based on the theory of Γ-convergence (see [4,5]) and is more expedient since it keeps convergence of the optimal solutions of the initial problem to the similar characteristics of the corresponding homogenized optimal control problem. The main difficulty of the second approach consists in the mathematical description of the homogenized optimal control problem and in the identification of the effective set of its cost functional. This approach was improved by Denkowski and Mortola in [9] using the Kuratovski convergence of solution sets and applying the Buttazzo–Dal Maso abstract scheme [5].

The main assumption in [9] is G-convergence (or PG-convergence for parabolic problem) of the sequence operators, which describe the sequence of perturbed boundary-value problems in concrete case. Therefore, *the crucial point* and the first step in the homogenization of an optimal control problem involving perturbed domains is the proof of the convergence theorem for the state. On the second step, with the help of the convergence theorem we get Kuratowskii convergence of solution sets which is equivalent to the Γ-convergence of the corresponding indicator functions. Then we deduce the Γ-convergence of cost functionals. Finally, applying the Buttazzo–Dal Maso abstract scheme, we obtain results for the asymptotic behavior of the optimal solutions, thereby we correctly define the homogenized optimal control problem, and results for the convergence of minimal values, thereby we prove the stability result of this direct method.

Statement of the problem. Let B be a finite union of smooth plane domains which are not crossed and touched. In addition, the set B is strongly situated in the square $\square := \{\xi' = (\xi_1, \xi_2) : 0 < \xi_1 < 1, 0 < \xi_2 < 1\}$. Let us arbitrarily divide B into two classes: $B^{(1)} = \bigcup_{k=1}^{K_1} B_k^{(1)}$ and $B^{(2)} = \bigcup_{k=1}^{K_2} B_k^{(2)}$ (see Fig. 1).

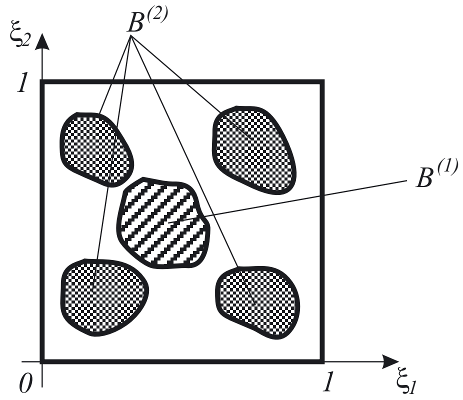

FIGURE 1. Subsets of B.

A model thick two-level junction Ω_ε of type $3:2:1$ consists of the junction's body

$$\Omega_0 = \{x \in \mathbb{R}^3 : \ x' = (x_1, x_2) \in Q := (0,a) \times (0,a), \quad 0 < x_3 < \gamma(x')\},$$

and a large number of the thin cylinders

$$G_\varepsilon^{(m)} = \bigcup\nolimits_{k=1}^{K_m} G_\varepsilon^{(m)}(k), \quad m = 1, 2,$$

where $\gamma \in C^1(\overline{Q})$ and $\min_{x' \in \overline{Q}} \gamma(x') = \gamma_0 > 0$,

$$G_\varepsilon^{(m)}(k) = \bigcup\nolimits_{i,j=0}^{N-1} \Big\{ x : \ \Big(\frac{x_1}{\varepsilon} - i, \frac{x_2}{\varepsilon} - j \Big) \in B_k^{(m)}, \ x_3 \in (-d_m, 0] \Big\}.$$

Here N is a large natural number, $\varepsilon = a/N$ is a small discrete parameter that characterizes the distance between nearby thin cylinders and their thickness; $0 < d_2 \le d_1$. Thus,

$$\Omega_\varepsilon = \Omega_0 \bigcup G_\varepsilon, \quad G_\varepsilon = G_\varepsilon^{(1)} \bigcup G_\varepsilon^{(2)}.$$

The thin cylinders G_ε are divided into two levels $G_\varepsilon^{(1)}$ and $G_\varepsilon^{(2)}$. Cylinders $G_\varepsilon^{(m)}(k)$ are ε-periodically alternated along the Ox_1-direction and Ox_2-direction and they are joined with Ω_0 over the ε-homothetic images $\varepsilon\big((i,j) + B_k^{(m)}\big)$, $i, j = 0, 1, \dots, N-1$, of $B_k^{(m)} \subset B^{(m)}$, $m = 1, 2$, $k = 1, \dots, K_m$. Some example of the cell of the alternation is shown in Figure 2.

Denote by $S_\varepsilon^{(m)}(k)$ the union of the lateral surfaces of the thin cylinders $G_\varepsilon^{(m)}(k)$, by $\Gamma_\varepsilon^{(m)}(k)$ the union of the bases of $G_\varepsilon^{(m)}(k)$; in addition, $S_\varepsilon^{(m)} := \bigcup_{k=0}^{K_m} S_\varepsilon^{(m)}(k)$, $\Gamma_\varepsilon^{(m)} := \bigcup_{k=0}^{K_m} \Gamma_\varepsilon^{(m)}(k)$, $m = 1, 2$.

Consider two classes of admissible controls

$$\mathcal{K}_\varepsilon^{(1)} = \Big\{ \theta \,|_{\Gamma_\varepsilon^{(1)}} : \ \theta \in H^{\delta_1}(\Gamma_{d_1}), \quad \|\theta\|_{H^{\delta_1}(\Gamma_{d_1})} \le \mathbf{C_{d_1}} \Big\}, \tag{0.1}$$

$$\mathcal{K}_\varepsilon^{(2)} = \Big\{ \vartheta \,|_{\Gamma_\varepsilon^{(2)}} : \ \vartheta \in H^{\delta_2}(\Gamma_{d_2}), \quad \|\vartheta\|_{H^{\delta_2}(\Gamma_{d_2})} \le \mathbf{C_{d_2}} \Big\}, \tag{0.2}$$

where $\Gamma_{d_m} = \{x : \ x' \in Q, x_3 = -d_m\}$ $(m = 1, 2)$, $\delta_1 > 1$ and $\delta_2 > 1$, $\mathbf{C_{d_1}}$ and $\mathbf{C_{d_2}}$ are some fixed positive constants that independent of ε.

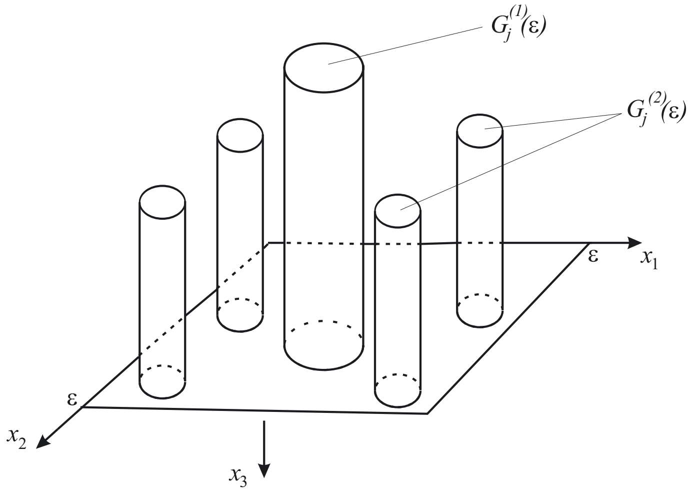

FIGURE 2. The cell of alternation.

Let f_0, ϱ_1 and ϱ_2 be given functions such that $f_0 \in L^2(\Omega_0)$, $\varrho_m : \mathbb{R} \to \mathbb{R}, m = 1, 2$, are Lipschitz continuous (it is equivalent that $\varrho_m \in W^{1,\infty}_{\text{loc}}(\mathbb{R})$) and such that

$$\exists\, c_1 > 0 \ \ \exists\, c_2 > 0 : \quad c_1 \le \varrho_m' \le c_2 \quad \text{a.e. in } \mathbb{R} \quad (m = 1, 2). \tag{0.3}$$

Using the approach of paper [19], we can state that for fixed values of the parameters ε, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, and for every function $\theta_\varepsilon \in \mathcal{K}_\varepsilon^{(1)}$ and $\vartheta_\varepsilon \in \mathcal{K}_\varepsilon^{(2)}$ there exists a unique weak solution u_ε to the following problem

$$\begin{array}{rcll} -\Delta_x u_\varepsilon(x) & = & f_0(x), & x \in \Omega_0; \\ -\Delta_x u_\varepsilon(x) & = & 0, & x \in G_\varepsilon; \\ -\partial_\nu u_\varepsilon(x) & = & \varepsilon^\alpha \varrho_1\big(u_\varepsilon(x)\big), & x \in S_\varepsilon^{(1)}; \\ -\partial_\nu u_\varepsilon(x) & = & \varepsilon^\beta \varrho_2\big(u_\varepsilon(x)\big), & x \in S_\varepsilon^{(2)}; \\ u_\varepsilon(x', -d_1) & = & \theta_\varepsilon(x'), & (x', -d_1) \in \Gamma_\varepsilon^{(1)}; \\ u_\varepsilon(x', -d_2) & = & \vartheta_\varepsilon(x'), & (x', -d_2) \in \Gamma_\varepsilon^{(2)}; \\ u_\varepsilon(x) & = & 0, & x \in \Upsilon_1; \\ \partial_\nu u_\varepsilon(x) & = & 0, & x \in \Upsilon_\varepsilon; \\ [u_\varepsilon]_{|x_3=0} = [\partial_{x_3} u_\varepsilon]_{|x_3=0} & = & 0, & x' \in Q_\varepsilon, \end{array} \tag{0.4}$$

where $\partial_\nu = \partial/\partial\nu$ is the outward normal derivative, $\Upsilon_1 = \{x : \ x_3 = \gamma(x'), \ x' \in Q\}$, $\Upsilon_\varepsilon = \partial\Omega_\varepsilon \setminus \big(\partial G_\varepsilon \cup \Upsilon_1\big)$, $Q_\varepsilon = \partial\Omega_0 \cap \partial G_\varepsilon$; the brackets denote the jump of the enclosed quantities.

Recall that a function $u_\varepsilon \in H^1(\Omega_\varepsilon; \Upsilon_1) = \{u \in H^1(\Omega_\varepsilon) : \ u|_{\Upsilon_1} = 0\}$ is a weak solution to problem (0.4) if for any function $\psi \in H^1(\Omega_\varepsilon; \Upsilon_1)$ such that $\psi|_{\Gamma_\varepsilon^{(1)} \cup \Gamma_\varepsilon^{(2)}} = 0$ the following integral identity

$$\int\limits_{\Omega_\varepsilon} \nabla_x u_\varepsilon \cdot \nabla_x \psi \, \mathrm{d}x \, + \, \varepsilon^\alpha \int\limits_{S_\varepsilon^{(1)}} \varrho_1\big(u_\varepsilon\big)\, \psi \, \mathrm{d}\sigma_x \, + \, \varepsilon^\beta \int\limits_{S_\varepsilon^{(2)}} \varrho_2\big(u_\varepsilon\big)\, \psi \, \mathrm{d}\sigma_x = \int\limits_{\Omega_0} f_0\, \psi \, \mathrm{d}x \tag{0.5}$$

holds and the traces of u_ε on $\Gamma_\varepsilon^{(1)}$ and $\Gamma_\varepsilon^{(2)}$ are equal to θ_ε and ϑ_ε respectively.

As usual, functions $\theta_\varepsilon, \vartheta_\varepsilon$ are called *"controls"*, the corresponding solution $u_\varepsilon = u_\varepsilon(x, \theta_\varepsilon, \vartheta_\varepsilon)$ is said *"state"* of the system to be controlled. We look for optimal controls $\theta^*_\varepsilon \in \mathcal{K}^{(1)}_\varepsilon$ and $\vartheta^*_\varepsilon \in \mathcal{K}^{(2)}_\varepsilon$, which with the corresponding state u^*_ε, minimize the following cost functional

$$J_\varepsilon\left(\theta_\varepsilon, \vartheta_\varepsilon\right) = \frac{1}{2}\int\limits_{\Omega_0} (u_\varepsilon - q_0)^2 \mathrm{d}x + \frac{N_1}{2}\int\limits_{\Gamma^{(1)}_\varepsilon} \big(\theta_\varepsilon(x') - \eta_1(x')\big)^2 \mathrm{d}x' + \frac{N_2}{2}\int\limits_{\Gamma^{(2)}_\varepsilon} \big(\vartheta_\varepsilon(x') - \eta_2(x')\big)^2 \mathrm{d}x' + \mathbb{E}_\varepsilon(u_\varepsilon; \alpha, \beta), \quad (0.6)$$

that is,

$$J_\varepsilon\left(\theta^*_\varepsilon, \vartheta^*_\varepsilon\right) = \inf_{\theta_\varepsilon \in \mathcal{K}^{(1)}_\varepsilon,\ \vartheta_\varepsilon \in \mathcal{K}^{(2)}_\varepsilon} J_\varepsilon\left(\theta_\varepsilon, \vartheta_\varepsilon\right). \quad (0.7)$$

In other words, we want to optimally control the state in Ω_0 and the energy

$$\mathbb{E}_\varepsilon(u_\varepsilon; \alpha, \beta) = \frac{N_3}{2}\left(\int\limits_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \,\mathrm{d}x + \varepsilon^\alpha \int\limits_{S^{(1)}_\varepsilon} \big(\varrho_1(u_\varepsilon) - \varrho_1(0)\big)\, u_\varepsilon \,\mathrm{d}\sigma_x + \varepsilon^\beta \int\limits_{S^{(2)}_\varepsilon} \big(\varrho_2(u_\varepsilon) - \varrho_2(0)\big)\, u_\varepsilon \,\mathrm{d}\sigma_x\right)$$

of the all complex system using controls from the different classes $\mathcal{K}^{(1)}_\varepsilon$ and $\mathcal{K}^{(2)}_\varepsilon$ given respectively on the different bases $\Gamma^{(1)}_\varepsilon$ and $\Gamma^{(2)}_\varepsilon$ of the thin cylinders that are ε-periodically alternated along the surface Q. Here N_1, N_2, N_3 are positive constants, the given functions $q_0 \in L^2(\Omega_0)$ and $\eta_i \in L^2(Q),\ i = 1, 2$. This optimal control problem will be denoted by CP_ε.

The aim of this paper is: (i) to find the corresponding homogenized optimal control problem CP_0 for problem CP_ε as $\varepsilon \to 0$ ($N \to +\infty$), *i.e.*, when the number of attached thin cylinders from each level infinitely increases and their thickness vanishes; (ii) to prove that the optimal solutions to CP_ε converge to an optimal solution to the homogenized problem CP_0 and that the minimal values of J_ε converge to the minimal value of J_0 as $\varepsilon \to 0$.

Remark 0.1. Recall that $\varepsilon = a/N$ is the small discrete parameter and we always mean the elements of this sequence when we write $\varepsilon \to 0$ or $\varepsilon > 0$.

In a typical interpretation the solution to the boundary-value problem (0.4) represents the density of some quantity (chemical concentration, temperature, electronic potential) at equilibrium within the junction Ω_ε. We considered the nonlinear Fourier boundary conditions on the boundaries of the thin cylinders. These conditions mean that there is a flux of this quantity through the surfaces of the cylinders. In fact very small activity holds always on the surface of some material (therefore the Fourier boundary conditions are more natural for applied mathematical problems).

For instance, in [13] the following experimental data were obtained: the electron microphotographs of the surface of a thick absorber (its structure has the form of a thick junction) have shown that these structures exhibit the chemical activity without interference of an external fields in reactions of degradation of organic solutes in water; in addition, the analysis of the absorption spectra has shown that mainly oxidative degradation of organic molecules takes place. Mathematical justification of this fact is presented in [19] with due regard for the very small chemical activity between the surface of a thick absorber and water.

Therefore to study the influence of the boundary interactions on the asymptotic behavior of problem CP_ε we introduce special intensity factors ε^α and ε^β in the Fourier boundary conditions on the lateral surfaces of the thin cylinders from the first and second level respectively. We will show that there are three qualitatively different cases: **(1)** $\alpha \geq 1$ and $\beta \geq 1$; **(2)** $\alpha \geq 1$ and $\beta < 1$ (the same when $\alpha < 1$ and $\beta \geq 1$); and **(3)** $\alpha < 1$ and $\beta < 1$.

To homogenize problem CP_ε in the first case we will use the direct method and approach of our previous paper [10], where we studied a linear optimal control problem involving a plane thick two-level junction of type 2 : 1 : 1. Nevertheless, here we essentially simplify this approach, namely, we prove the convergence results without any special constructions of multilevel extension operators and without any additional assumptions of smoothness for the right-hand sides and controls. These studies we do in Section 3.

The second case is characterized by the fact that we are only partially able to control our system for very small values of ε. This case is considered in Section 4.

In the third case, even the formulation of the optimal control problem becomes meaningless. In fact, we can not completely control our system for ε small enough. Therefore in Section 5, we prove only the convergence of the corresponding solutions of problem (0.4) and the convergence of the energy integrals.

All these results are discussed in the last Section 6, where application for an optimal control problem involving a thick one-level junction with cascade controls is presented as well.

1. Auxiliary integral identities and estimates

In what follows we will often use the following identities (see [18])

$$\varepsilon \int_{S_\varepsilon^{(m)}(k)} v \, \mathrm{d}\sigma_x = \frac{l_k^{(m)}}{|B_k^{(m)}|} \int_{G_\varepsilon^{(m)}(k)} v \, \mathrm{d}x + \varepsilon \int_{G_\varepsilon^{(m)}(k)} \nabla_{\xi'} Y_k^{(m)}(\xi')|_{\xi'=\frac{x'}{\varepsilon}} \cdot \nabla_{x'} v \, \mathrm{d}x \tag{1.1}$$

for arbitrary function $v \in H^1\big(G_\varepsilon^{(m)}(k)\big),\ k = 1, \ldots, K_m,\ m = 1, 2$. Here $|B_k^{(m)}|$ is the area of the plane domain $B_k^{(m)}$, $l_k^{(m)}$ is the perimeter of $B_k^{(m)}$, $Y_k^{(m)}$ is the unique solution to the following problem

$$\begin{cases} \Delta_\xi Y_k^{(m)}(\xi') = l_k^{(m)} \left|B_k^{(m)}\right|^{-1}, & \xi' = (\xi_1, \xi_2) \in B_k^{(m)}, \\ \partial_{\nu(\xi')} Y_k^{(m)}(\xi') = 1, & \xi' \in \partial B_k^{(m)}, \\ \int_{B_k^{(m)}} Y_k^{(m)}(\xi') d\xi' = 0, & \end{cases}$$

and then it is 1-periodically continued in ξ_1 and ξ_2. Due to the regularity properties of solutions to elliptic problems we have

$$\sup_{\xi' \in B_k^{(m)}} |\nabla_{\xi'} Y_k^{(m)}(\xi')| \le c_{k,m}. \tag{1.2}$$

Using Cauchy's inequality with δ ($ab \le \delta a^2 + \frac{b^2}{4\delta}, a, b, \delta > 0$) and (1.2), we deduce from (1.1) the following estimates

$$\varepsilon \int_{S_\varepsilon^{(m)}(k)} v^2 \, \mathrm{d}\sigma_x \le C_1 \left(\varepsilon^2 \int_{G_\varepsilon^{(m)}(k)} |\nabla_{x'} v|^2 \, \mathrm{d}x + \int_{G_\varepsilon^{(m)}(k)} v^2 \, \mathrm{d}x \right), \tag{1.3}$$

$$\int_{G_\varepsilon^{(m)}(k)} v^2 \, \mathrm{d}x \le C_2 \left(\varepsilon^2 \int_{G_\varepsilon^{(m)}(k)} |\nabla_{x'} v|^2 \, \mathrm{d}x + \varepsilon \int_{S_\varepsilon^{(m)}(k)} v^2 \, \mathrm{d}\sigma_x \right) \tag{1.4}$$

for arbitrary function $v \in H^1\big(G_\varepsilon^{(m)}(k)\big),\ k = 1, \ldots, K_m,\ m = 1, 2.$

Remark 1.1. In (1.3), (1.4) and in what follows all constants $\{C_i\}$ and $\{c_i\}$ in inequalities are independent of the parameter ε.

Similarly as in [19] we get from (0.3) the following inequalities

$$c_1 t^2 + \varrho_m(0)\, t \le \varrho_m(t)\, t \le c_2 t^2 + \varrho_m(0)\, t \quad \forall t \in \mathbb{R}, \quad m = 1, 2. \tag{1.5}$$

With standard approach and with the help of (1.3), (1.4) and (1.5) we deduce the following *a priori* estimate for the solution to problem (0.4):

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C_3\Big\{\|f_0\|_{L^2(\Omega_0)} + \varepsilon^{\alpha-1}|\varrho_1(0)| + \varepsilon^{\beta-1}|\varrho_2(0)| + \|\theta_\varepsilon\|_{H^1(\Gamma_\varepsilon^{(1)})} + \|\vartheta_\varepsilon\|_{H^1(\Gamma_\varepsilon^{(2)})}$$
$$+ \varepsilon^{\alpha-1}\big(1+|\varrho_1(0)|\big)\left(\varepsilon\|\nabla_{x'}\theta_\varepsilon\|_{L^2(\Gamma_\varepsilon^{(1)})} + \|\theta_\varepsilon\|_{L^2(\Gamma_\varepsilon^{(1)})}\right)$$
$$+ \varepsilon^{\beta-1}\big(1+|\varrho_2(0)|\big)\left(\varepsilon\|\nabla_{x'}\vartheta_\varepsilon\|_{L^2(\Gamma_\varepsilon^{(2)})} + \|\vartheta_\varepsilon\|_{L^2(\Gamma_\varepsilon^{(2)})}\right)\Big\}, \tag{1.6}$$

where the constant C_3 is independent both of ε and of u_ε, f_0, θ_ε, ϑ_ε, u_ε.

From (1.6) it follows that we have to consider different values of the parameter α and β, namely $\alpha < 1$, $\alpha = 1$, $\alpha > 1$ (similarly for β), to study the asymptotic behavior of problem CP_ε.

2. Properties of problem CP_ε for a fixed value ε

Obviously, that the sets $\mathcal{K}_\varepsilon^{(1)}$ and $\mathcal{K}_\varepsilon^{(2)}$ are non-empty and convex. Let us show that they are closed with respect to the weak topology of $H^{\delta_1}(\Gamma_\varepsilon^{(1)})$ and $H^{\delta_2}(\Gamma_\varepsilon^{(2)})$ respectively. We do this for $\mathcal{K}_\varepsilon^{(1)}$. Let $\{\theta_\varepsilon^{(n)}\}_{n\in\mathbb{N}} \subset \mathcal{K}_\varepsilon^{(1)}$ be a sequence of control functions such that $\theta_\varepsilon^{(n)} \to \theta_\varepsilon^*$ weakly in $H^{\delta_1}(\Gamma_\varepsilon^{(1)})$ as $n \to +\infty$. From definitions of the sets $\mathcal{K}_\varepsilon^{(1)}$ and $\mathcal{K}_\varepsilon^{(2)}$ (see (0.1), (0.2)) it follows that these controls are restrictions of functions from the following sets

$$\mathcal{K}_0^{(1)} = \left\{\theta \in H^{\delta_1}(\Gamma_{d_1}): \quad \|\theta\|_{H^{\delta_1}(\Gamma_{d_1})} \le \mathbf{C_{d_1}}\right\}, \tag{2.1}$$
$$\mathcal{K}_0^{(2)} = \left\{\vartheta \in H^{\delta_2}(\Gamma_{d_2}): \quad \|\vartheta\|_{H^{\delta_2}(\Gamma_{d_2})} \le \mathbf{C_{d_2}}\right\}, \tag{2.2}$$

on $\Gamma_\varepsilon^{(1)}$ and $\Gamma_\varepsilon^{(2)}$ respectively. Therefore, there exists a sequence $\{\widehat{\theta}_\varepsilon^{(n)}\}_{n\in\mathbb{N}} \subset \mathcal{K}_0^{(1)}$ such that $\big(\widehat{\theta}_\varepsilon^{(n)}\big)|_{\Gamma_\varepsilon^{(1)}} = \theta_\varepsilon^{(n)}$. Note that such a sequence is not unique. Then for any such sequence we can choose a subsequence such that $\widehat{\theta}_\varepsilon^{(n_k)} \to \widehat{\theta}_\varepsilon^* \in \mathcal{K}_0^{(1)}$ weakly in $H^{\delta_1}(\Gamma_{d_1})$ as $k \to +\infty$. From this we immediately conclude that the sequence of corresponding restrictions $\{\theta_\varepsilon^{(n_k)}\}_{k\in\mathbb{N}}$ is weakly convergent to $\big(\widehat{\theta}_\varepsilon^*\big)|_{\Gamma_\varepsilon^{(1)}}$ in $H^{\delta_1}(\Gamma_\varepsilon^{(1)})$. Therefore, $\theta_\varepsilon^* = \big(\widehat{\theta}_\varepsilon^*\big)|_{\Gamma_\varepsilon^{(1)}}$, *i.e.*, θ_ε^* belongs to the set $\mathcal{K}_\varepsilon^{(1)}$.

Proposition 2.1. *If for some sequences of controls* $\{\theta_\varepsilon^{(n)}\}_{n\in\mathbb{N}} \subset \mathcal{K}_\varepsilon^{(1)}$ *and* $\{\vartheta_\varepsilon^{(n)}\}_{n\in\mathbb{N}} \subset \mathcal{K}_\varepsilon^{(2)}$

$$\theta_\varepsilon^{(n)} \to \theta_\varepsilon^* \quad \textit{weakly in } H^{\delta_1}(\Gamma_\varepsilon^{(1)}) \quad \textit{as} \quad n \to +\infty,$$
$$\vartheta_\varepsilon^{(n)} \to \vartheta_\varepsilon^* \quad \textit{weakly in } H^{\delta_2}(\Gamma_\varepsilon^{(2)}) \quad \textit{as} \quad n \to +\infty,$$

then the corresponding sequence of states $\{u_\varepsilon^{(n)}\}_{n\in\mathbb{N}}$ *weakly in* $H^1(\Omega_\varepsilon;\Upsilon_1)$ *converges to a function* u_ε^* *that is the unique state corresponding to the controls* θ_ε^* *and* ϑ_ε^*.

Proof. Let $u_\varepsilon^{(n)}$ be a state that corresponds to the controls $\theta_\varepsilon^{(n)} \in \mathcal{K}_\varepsilon^{(1)}$ and $\vartheta_\varepsilon^{(n)} \in \mathcal{K}_\varepsilon^{(2)}$, $n \in \mathbb{N}$. Then from uniform estimate (1.6) and the compactness of trace operators it follows that there exists a subsequence $\{u_\varepsilon^{(n_k)}\}_{k\in\mathbb{N}}$ of $\{u_\varepsilon^{(n)}\}_{n\in\mathbb{N}}$ such that $u_\varepsilon^{(n_k)} \to u_\varepsilon^*$ weakly in $H^1(\Omega_\varepsilon;\Upsilon_1)$ and $u_\varepsilon^{(n_k)}|_{S_\varepsilon^{(m)}} \to u_\varepsilon^*|_{S_\varepsilon^{(m)}}$ a.e. in $S_\varepsilon^{(m)}$ $(m = 1, 2)$ as $k \to +\infty$. Since functions ϱ_1 and ϱ_2 are continuous,

$$\varrho_m\big(u_\varepsilon^{(n_k)}|_{S_\varepsilon^{(m)}}\big) \to \varrho_m\big(u_\varepsilon^*|_{S_\varepsilon^{(m)}}\big) \quad \text{as} \quad k \to +\infty \quad (m = 1, 2).$$

Now, passing to the limit in the integral identity (0.5) for $u_\varepsilon^{(n_k)}$, we conclude that u_ε^* is the weak solution to problem (0.4) with the Dirichlet condition $u_\varepsilon^* = \theta_\varepsilon^*$ on $\Gamma_\varepsilon^{(1)}$ and $u_\varepsilon^* = \vartheta_\varepsilon^*$ on $\Gamma_\varepsilon^{(2)}$. Due to the uniqueness of the weak solution to problem (0.4), the above arguments hold for any subsequence $\{u_\varepsilon^{(n_k)}\}_{k\in\mathbb{N}}$ of $\{u_\varepsilon^{(n)}\}_{n\in\mathbb{N}}$ chosen at the beginning of the proof. Therefore, the proposition is proved. □

From definition of the sets of admissible controls and Proposition 2.1 it follows the compactness property of problem CP_ε.

Proposition 2.2. *For any sequences of controls* $\{(\theta_\varepsilon^{(n)}, \vartheta_\varepsilon^{(n)})\}_{n\in\mathbb{N}} \subset \mathcal{K}_\varepsilon^{(1)} \times \mathcal{K}_\varepsilon^{(2)}$ *there exist subsequences such that*

$$\theta_\varepsilon^{(n_k)} \to \theta_\varepsilon^* \quad \textit{weakly in} \quad H^{\delta_1}(\Gamma_\varepsilon^{(1)}) \quad \textit{as} \quad k \to +\infty, \tag{2.3}$$

$$\vartheta_\varepsilon^{(n_k)} \to \vartheta_\varepsilon^* \quad \textit{weakly in} \quad H^{\delta_2}(\Gamma_\varepsilon^{(2)}) \quad \textit{as} \quad k \to +\infty, \tag{2.4}$$

and the corresponding sequence of the states $\{u_\varepsilon^{(n_k)}\}_{k\in\mathbb{N}}$ *weakly in* $H^1(\Omega_\varepsilon; \Upsilon_1)$ *converges to the state* u_ε^* *that corresponds to the controls* θ_ε^* *and* ϑ_ε^*.

Theorem 2.3. *For every value of* ε *the optimal control problem* CP_ε *has a solution,* i.e., *there exist a pair of controls* $\left(\theta_\varepsilon^*, \vartheta_\varepsilon^*\right) \subset \mathcal{K}_\varepsilon^{(1)} \times \mathcal{K}_\varepsilon^{(2)}$ *and the corresponding unique state such that the equality* (0.7) *is satisfied.*

Proof. Let $\{(\theta_\varepsilon^{(n)}, \vartheta_\varepsilon^{(n)})\}_{n\in\mathbb{N}}$ be any minimizing sequence for the cost functional (0.6). Due to the compactness property of problem CP_ε (Prop. 2.2) there exist subsequences such that (2.3) and (2.4) hold and the corresponding sequence of the states $\{u_\varepsilon^{(n_k)}\}_{k\in\mathbb{N}}$ weakly in $H^1(\Omega_\varepsilon; \Upsilon_1)$ converges to the state u_ε^* that corresponds to the controls θ_ε^* and ϑ_ε^*. In virtue of the weakly lower-semicontinuity of the cost functional J_ε, we have

$$\inf_{(\theta_\varepsilon,\vartheta_\varepsilon)\in\mathcal{K}_\varepsilon^{(1)}\times\mathcal{K}_\varepsilon^{(2)}} J_\varepsilon\left(\theta_\varepsilon, \vartheta_\varepsilon\right) = \lim_{k\to\infty} J_\varepsilon\left(\theta_\varepsilon^{(n_k)}, \vartheta_\varepsilon^{(n_k)}\right) \geq J_\varepsilon(\theta_\varepsilon^*, \vartheta_\varepsilon^*),$$

i.e. the equality (0.7) holds. □

3. The main results in the case $\alpha \geq 1$ and $\beta \geq 1$

It follows from (1.6) that if $\alpha \geq 1$ and $\beta \geq 1$ then

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C_1. \tag{3.1}$$

3.1. Convergence results for the admissible controls

Since the classes of admissible controls $\mathcal{K}_\varepsilon^{(1)}$ and $\mathcal{K}_\varepsilon^{(1)}$ are defined on variable spaces depending on ε, we should introduce the special convergence of controls.

Definition 3.1. We say that a sequence of control pairs $\left\{(\theta_\varepsilon, \vartheta_\varepsilon) \in \mathcal{K}_\varepsilon^{(1)} \times \mathcal{K}_\varepsilon^{(2)}\right\}$ is weakly convergent with respect to the space $H^{\delta_1}\left(\Gamma_{d_1}\right) \times H^{\delta_2}\left(\Gamma_{d_2}\right)$ as $\varepsilon \to 0$ (we will denote this convergence by $(\theta_\varepsilon, \vartheta_\varepsilon) \overset{w}{\rightsquigarrow} (\theta_0, \vartheta_0)$), if there are sequences $\{\widehat{\theta}_\varepsilon\}_{\varepsilon>0} \subset \mathcal{K}_0^{(1)}$ and $\{\widehat{\vartheta}_\varepsilon\}_{\varepsilon>0} \subset \mathcal{K}_0^{(2)}$ such that $\left(\widehat{\theta}_\varepsilon\right)|_{\Gamma_\varepsilon^{(1)}} = \theta_\varepsilon$, $\left(\widehat{\vartheta}_\varepsilon\right)|_{\Gamma_\varepsilon^{(2)}} = \vartheta_\varepsilon$ and

$$\widehat{\theta}_\varepsilon \xrightarrow{w} \theta_0 \quad \text{weakly in} \quad H^{\delta_1}\left(\Gamma_{d_1}\right) \quad \text{as} \quad \varepsilon \to 0, \tag{3.2}$$

$$\widehat{\vartheta}_\varepsilon \xrightarrow{w} \vartheta_0 \quad \text{weakly in} \quad H^{\delta_2}\left(\Gamma_{d_2}\right) \quad \text{as} \quad \varepsilon \to 0. \tag{3.3}$$

Obviously, that the pair $(\theta_0, \vartheta_0) \in \mathcal{K}_0^{(1)} \times \mathcal{K}_0^{(2)}$. To show the correctness of this definition we suppose that there are other sequences $\{\widetilde{\theta}_\varepsilon\}_{\varepsilon>0} \subset \mathcal{K}_0^{(1)}$ and $\{\widetilde{\vartheta}_\varepsilon\}_{\varepsilon>0} \subset \mathcal{K}_0^{(2)}$ such that $\left(\widetilde{\theta}_\varepsilon\right)|_{\Gamma_\varepsilon^{(1)}} = \theta_\varepsilon$, $\left(\widetilde{\vartheta}_\varepsilon\right)|_{\Gamma_\varepsilon^{(2)}} = \vartheta_\varepsilon$ and

$$\left(\widetilde{\theta}_\varepsilon, \widetilde{\vartheta}_\varepsilon\right) \overset{w}{\rightsquigarrow} (\widetilde{\theta}_0, \widetilde{\vartheta}_0) \quad \text{as} \quad \varepsilon \to 0.$$

Let us define 1-periodic functions χ_1 and χ_2 such that

$$\chi_m(\xi') = \begin{cases} 1, & \xi' \in B^{(m)}, \\ 0, & \xi' \in \square \setminus B^{(m)}, \end{cases} \qquad m = 1, 2.$$

It is well known that $\chi_m(\frac{x'}{\varepsilon}) \to |B^{(m)}|$ weakly in $L^2(Q)$ as $\varepsilon \to 0$, where $|B^{(m)}| = \sum_{k=1}^{K_1} |B_k^{(m)}|$, $|B_k^{(m)}|$ is the area of the plane domain $B_k^{(m)}, m = 1, 2$.

Then, passing to the limit $(\varepsilon \to 0)$ in the following integral identity

$$\int_Q \chi_1(x'/\varepsilon)\, \widetilde{\theta}_\varepsilon(x')\, \varphi(x')\, \mathrm{d}x' = \int_Q \chi_1(x'/\varepsilon)\, \widehat{\theta}_\varepsilon(x')\, \varphi(x')\, \mathrm{d}x' \quad \forall \varphi \in C_0^\infty(Q),$$

we get

$$|B^{(m)}| \int_Q \widetilde{\theta}_0\, \varphi\, \mathrm{d}x' = |B^{(m)}| \int_Q \theta_0\, \varphi\, \mathrm{d}x' \quad \forall \varphi \in C_0^\infty(Q),$$

i.e., $\widetilde{\theta}_0 = \theta_0$ a.e. in Γ_{d_1}. Similarly we can prove that $\widetilde{\vartheta}_0 = \vartheta_0$ a.e. in Γ_{d_2}.

From results obtained above we have as follows.

Proposition 3.2. *The following statements hold:*

(1) *Every sequence of control pairs $\{(\theta_\varepsilon, \vartheta_\varepsilon) \in \mathcal{K}_\varepsilon^{(1)} \times \mathcal{K}_\varepsilon^{(2)}\}_{\varepsilon>0}$ is compact with respect to the weak convergence introduced above and all its partial limits belong to $\mathcal{K}_0^{(1)} \times \mathcal{K}_0^{(2)}$.*

(2) *For any pair $(\theta_0, \vartheta_0) \in \mathcal{K}_0^{(1)} \times \mathcal{K}_0^{(2)}$ there exists a sequence of admissible control pairs $\{(\theta_\varepsilon, \vartheta_\varepsilon) \in \mathcal{K}_\varepsilon^{(1)} \times \mathcal{K}_\varepsilon^{(2)}\}_{\varepsilon>0}$ such that $(\theta_\varepsilon, \vartheta_\varepsilon) \overset{w}{\rightsquigarrow} (\theta_0, \vartheta_0)$ as $\varepsilon \to 0$.*

3.2. Convergence results for the states and the cost functionals

Let us introduce the following extensions by zero:

$$\widetilde{y_\varepsilon}^{(m,k)}(x) = \begin{cases} y_\varepsilon, & x \in G_\varepsilon^{(m)}(k), \\ 0, & x \in D_m \setminus G_\varepsilon^{(m)}(k), \end{cases} \qquad k = 1, \ldots, K_m, \quad m = 1, 2, \tag{3.4}$$

where $D_m = Q \times (-d_m, 0)$ is the parallelepiped that filled up with the thin cylinders $G_\varepsilon^{(m)}(k)$ in the limit passage as $\varepsilon \to 0$. It is obvious, that the extension $\widetilde{y_\varepsilon}^{(m,k)}$ belongs to the anisotropic Sobolev space $W^{0,0,1}(D_m) = \{v \in L^2(D_m) :\ \exists \text{ weak derivative } \partial_{x_3} v \in L^2(D_m)\}$.

Theorem 3.3 (the case $\alpha \geq 1$ and $\beta \geq 1$). *Let $\{(\theta_\varepsilon, \vartheta_\varepsilon) \in \mathcal{K}_\varepsilon^{(1)} \times \mathcal{K}_\varepsilon^{(2)}\}_{\varepsilon>0}$ be a sequence of admissible control pairs such that $(\theta_\varepsilon, \vartheta_\varepsilon) \overset{w}{\rightsquigarrow} (\theta_0, \vartheta_0)$ as $\varepsilon \to 0$ (obviously, $(\theta_0, \vartheta_0) \in \mathcal{K}_0^{(1)} \times \mathcal{K}_0^{(2)}$). Then the corresponding sequence of the solutions $\{u_\varepsilon\}_{\varepsilon>0}$ to problem (0.4) satisfies the following relations*

$$\left.\begin{array}{rcll} u_\varepsilon|_{\Omega_0} & \overset{w}{\longrightarrow} & v_0^+ & \text{weakly in } H^1(\Omega_0; \Upsilon_1), \\ \widetilde{u_\varepsilon}^{(m,k)} & \overset{w}{\longrightarrow} & |B_k^{(m)}|\, v_0^{(m,k)} & \text{weakly in } W^{0,0,1}(D_m), \\ \widetilde{\partial_{x_i} u_\varepsilon}^{(m,k)} & \overset{w}{\longrightarrow} & 0 & \text{weakly in } L^2(D_m),\ i = 1, 2, \end{array}\right\} \quad \text{as} \quad \varepsilon \to 0 \tag{3.5}$$

for $m = 1, 2$ and $k = 1, \ldots, K_m$. Here the multi-valued function

$$\mathbf{V}_0(x) = \begin{cases} v_0^+(x), & x \in \Omega_0, \\ v_0^{(m,k)}(x), & x \in D_m,\ k = 1, \ldots, K_m,\ m = 1, 2, \end{cases} \tag{3.6}$$

is a weak solution to the following problem

$$
\begin{aligned}
-\Delta v_0^+(x) &= f_0(x), && x\in\Omega_0,\\
v_0^+(x) &= 0, && x\in\Upsilon_1,\\
\partial_\nu v_0^+(x) &= 0, && x\in\partial\Omega_0\setminus(Q\cup\Upsilon_1);\\
-|B_k^{(1)}|\,\partial^2_{x_3}v_0^{(1,k)}(x)+\delta_{\alpha 1}l_k^{(1)}\varrho_1\big(v_0^{(1,k)}(x)\big) &= 0, && x\in D_1,\ k=1,\dots,K_1,\\
v_0^{(1,k)}(x',-d_1) &= \theta_0, && x'\in Q,\ k=1,\dots,K_1,\\
-|B_k^{(2)}|\,\partial^2_{x_3}v_0^{(2,k)}(x)+\delta_{\beta 1}l_k^{(2)}\varrho_2\big(v_0^{(2,k)}(x)\big) &= 0, && x\in D_2,\ k=1,\dots,K_2,\\
v_0^{(2,k)}(x',-d_2) &= \vartheta_0, && x'\in Q,\ k=1,\dots,K_2;\\
v_0^{(m,k)}(x',0) &= v_0^+(x',0), && x'\in Q,\ k=1,\dots,K_m,\\
\textstyle\sum_{m=1}^2\sum_{k=1}^{K_m}|B_k^{(m)}|\,\partial_{x_3}v_0^{(m,k)}(x',0) &= \partial_{x_3}v_0^+(x',0), && x'\in Q,
\end{aligned}
\tag{3.7}
$$

which is called homogenized problem *for problem* (0.4). *Here* $|B_k^{(m)}|$ *is the area of the plane domain* $B_k^{(m)}$, $l_k^{(m)}$ *is the perimeter of* $B_k^{(m)}$, $m=1,2$.

In addition, $\lim_{\varepsilon\to0} J_\varepsilon(\theta_\varepsilon,\vartheta_\varepsilon)=J_0(\theta_0,\vartheta_0)$, *where*

$$
\begin{aligned}
J_0(\theta_0,\vartheta_0) &= \frac12\int_{\Omega_0}(v_0^+-q_0)^2\mathrm{d}x+\frac{N_1}{2}|B^{(1)}|\int_{\Gamma_{d_1}}(\theta_0-\eta_1)^2\,\mathrm{d}x'+\frac{N_2}{2}|B^{(2)}|\int_{\Gamma_{d_2}}(\vartheta_0-\eta_2)^2\,\mathrm{d}x'\\
&+\frac{N_3}{2}\Bigg(\int_{\Omega_0}|\nabla v_0^+|^2\mathrm{d}x+\sum_{m=1}^{2}\sum_{k=1}^{K_m}|B_k^{(m)}|\int_{D_m}\big|\partial_{x_3}v_0^{(m,k)}\big|^2\mathrm{d}x\\
&+\delta_{\alpha,1}\sum_{k=1}^{K_1}l_k^{(1)}\int_{D_1}\big(\varrho_1\big(v_0^{(1,k)}(x)\big)-\varrho_1(0)\big)\,v_0^{(1,k)}(x)\,\mathrm{d}x\\
&+\delta_{\beta,1}\sum_{k=1}^{K_2}l_k^{(2)}\int_{D_2}\big(\varrho_2\big(v_0^{(2,k)}(x)\big)-\varrho_2(0)\big)\,v_0^{(2,k)}(x)\,\mathrm{d}x\Bigg).
\end{aligned}
\tag{3.8}
$$

Here $|B^{(m)}|=\sum_{k=1}^{K_m}|B_k^{(m)}|$, $m=1,2$.

Proof. It follows from (3.1) and (1.5) that the values $\|u_\varepsilon\|_{H^1(\Omega_0)}$, $\|\widetilde{u_\varepsilon}^{(m,k)}\|_{L^2(D_m)}$, $\|\widetilde{\partial_{x_i}u_\varepsilon}^{(m,k)}\|_{L^2(D_m)}$ ($i=1,2,3$), $\|\varrho_m\big(\widetilde{u}_\varepsilon^{(m,k)}\big)\|_{L^2(D_m)}$ ($k=1,\dots,K_m$, $m=1,2$) are uniformly bounded with respect to ε. Hence, there exists a subsequence $\{\varepsilon'\}\subset\{\varepsilon\}$, again denoted by $\{\varepsilon\}$, such that

$$
\left.
\begin{aligned}
u_\varepsilon|_{\Omega_0} &\stackrel{w}{\longrightarrow} v_0^+ && \text{in } H^1(\Omega_0;\Upsilon_1),\\
\widetilde{u_\varepsilon}^{(m,k)} &\stackrel{w}{\longrightarrow} |B_k^{(m)}|\big(|B_k^{(m)}|^{-1}v^{(m,k)}\big)=:|B_k^{(m)}|\,v_0^{(m,k)} && \text{in } L^2(D_m),\\
\widetilde{\partial_{x_i}u_\varepsilon}^{(m,k)} &\stackrel{w}{\longrightarrow} \gamma_i^{(m,k)} && \text{in } L^2(D_m),\\
\varrho_m\big(\widetilde{u}_\varepsilon^{(m,k)}\big) &\stackrel{w}{\longrightarrow} \zeta^{(m,k)} && \text{in } L^2(D_m),
\end{aligned}
\right\}\quad \text{as}\ \ \varepsilon\to0,
\tag{3.9}
$$

where v_0^+, $v_0^{(m,k)}$, $\gamma_i^{(m,k)}$, $\zeta^{(m,k)}$, $k=1,\dots,K_m, m=1,2$, $i=1,2,3$, are certain functions which will be determined in what follows.

1. At first we determine functions $\{\gamma_i^{(m,k)}\}$. Consider an arbitrary function $\psi \in C_0^\infty(D_m)$ $(m=1,2)$. Since $\partial_{x_3}\big(\widetilde{u_\varepsilon}^{(m,k)}\big) = \widetilde{\partial_{x_3} u_\varepsilon}^{(m,k)}, k=1,\dots,K_m, m=1,2$ (for curvilinear cylinder we should use special integral identities (see [19])),

$$\int_{D_m} \widetilde{\partial_{x_3} u_\varepsilon}^{(m,k)} \psi \,\mathrm{d}x = -\int_{D_m} \widetilde{u_\varepsilon}^{(m,k)} \partial_{x_3}\psi \,\mathrm{d}x \quad \forall \psi \in C_0^\infty(D_m). \tag{3.10}$$

Passing to the limit as $\varepsilon \to 0$ in this equality, we obtain the following integral identity

$$\int_{D_m} \gamma_3^{(m,k)} \psi \,\mathrm{d}x = -|B_k^{(m)}| \int_{D_m} v_0^{(m,k)} \partial_{x_3}\psi \,\mathrm{d}x \qquad \forall \psi \in C_0^\infty(D_m),$$

which implies that $v_0^{(m,k)}$ has the weak derivative in x_3 and $\gamma_3^{(m,k)} = |B_k^{(m)}|\, \partial_{x_3} v_0^{(m,k)}$ a.e. in $D_m,\ m=1,2,\ k=1,\dots,K_m$.

Let $(b_1^{(m)}(k), b_2^{(m)}(k))$ is the geometric center of gravity of the domain $B_k^{(m)}$. Consider the functions

$$Z_j^{(m,k)}(\xi_j) = -\xi_j + b_j^{(m)}(k) + [\xi_j], \quad k=1,\dots,K_m, \quad m=1,2, \quad j=1,2,$$

where $[t]$ is the integer part of t. With the help of this functions we determine the following test-functions

$$\Phi_j^{(m,k)}(x) = \begin{cases} 0, & x \in \Omega_\varepsilon \setminus G_\varepsilon^{(m)}(k), \\ \varepsilon Z_j^{(m,k)}\left(\frac{x_j}{\varepsilon}\right) \psi_m(x), & x \in G_\varepsilon^{(m)}(k), \end{cases}$$

$k=1,\dots,K_m,\ m=1,2,\ j=1,2,$ where ψ_1 and ψ_2 are arbitrary functions from $C_0^\infty(D_1)$ and $C_0^\infty(D_2)$ respectively. It is easy to see that functions $\{\Phi_j^{(m,k)}\}$ belong to $H^1(\Omega_\varepsilon;\Upsilon_1)$ and

$$\begin{aligned} \nabla\Phi_1^{(m,k)} &= (-\psi_m,0,0) + \varepsilon Z_1^{(m,k)} \nabla\psi_m \qquad \text{in } G_\varepsilon^{(m)}(k), \\ \nabla\Phi_2^{(m,k)} &= (0,-\psi_m,0) + \varepsilon Z_2^{(m,k)} \nabla\psi_m \qquad \text{in } G_\varepsilon^{(m)}(k). \end{aligned}$$

Substituting the functions $\{\Phi_j^{(1,k)}\}$ into the integral identity (0.5), we get

$$-\int\limits_{G_\varepsilon^{(1)}(k)} \partial_{x_j} u_\varepsilon\, \psi_1 \,\mathrm{d}x + \varepsilon \int\limits_{G_\varepsilon^{(1)}(k)} Z_j^{(1,k)}\, \nabla u_\varepsilon \cdot \nabla\psi_1 \,\mathrm{d}x = -\varepsilon^{1+\alpha} \int\limits_{S_\varepsilon^{(1)}(k)} \varrho_1\big(u_\varepsilon\big)\, Z_j^{(1,k)}\, \psi_1 \,\mathrm{d}\sigma_x \tag{3.11}$$

for $k=1,\dots,K_1,\ j=1,2$. Here $S_\varepsilon^{(m)}(k)$ is the union of the lateral surfaces of cylinders $G_\varepsilon^{(m)}(k) (m=1,2)$.

Since for any $k \in \{1,\dots,K_1\}$ and $j \in \{1,2\}$ $\max_{x' \in Q} |Z_j^{(1,k)}| \le 1$,

$$\varepsilon \left| \int_{G_\varepsilon^{(1)}(k)} Z_{j,k}^{(1)}\, \nabla u_\varepsilon \cdot \nabla\psi_1 \,\mathrm{d}x \right| \le c_1 \varepsilon \|\nabla u_\varepsilon\|_{L^2(G_\varepsilon^{(1)})} \|\psi_1\|_{L^2(D_1)} \le c_2\varepsilon. \tag{3.12}$$

Taking (1.2), (1.5) and (3.1) into account, with the help of (1.3) we estimate the right-hand side in (3.11):

$$\begin{aligned} \varepsilon^{1+\alpha} \left| \int_{S_\varepsilon^{(1)}(k)} \varrho_1(u_\varepsilon)\, Z_j^{(1,k)}\, \psi_1 \,\mathrm{d}\sigma_x \right| \;&\le\; \varepsilon^\alpha C_1 \int_{G_\varepsilon^{(1)}(k)} |\varrho_1(u_\varepsilon)\, \psi_1| \,\mathrm{d}x \\ &+\; \varepsilon^{1+\alpha} C_2 \int_{G_\varepsilon^{(1)}(k)} \big(|\varrho_1'(u_\varepsilon)|\, |\nabla_{x'} u_\varepsilon|\, |\psi_1| + |\varrho_1(u_\varepsilon)| |\nabla_{x'}\psi_1| \big) \,\mathrm{d}x \\ &\le\; \varepsilon^\alpha C_3 \|\psi_1\|_{L^2(D_1)} + \varepsilon^{1+\alpha} C_4 \|\psi_1\|_{H^1(D_1)} \le \varepsilon^\alpha C_5 \|\psi_1\|_{H^1(D_1)}. \end{aligned} \tag{3.13}$$

In virtue of (3.12) and (3.13) we deduce from (3.11) that

$$\left|\int_{D_1} \widetilde{\partial_{x_j} u_\varepsilon}^{(1,k)} \psi_1 \, \mathrm{d}x\right| \le \varepsilon\, C_5 \, \|\psi_1\|_{H^1(D_1)}, \quad j=1,2, \tag{3.14}$$

whence we get that $\gamma_1^{(1,k)} = \gamma_2^{(1,k)} = 0$ a.e. in $D_1,\ k=1,\dots,K_1$. Analogously, we obtain that $\gamma_1^{(2,k)} = \gamma_2^{(2,k)} = 0$ a.e. in $D_2,\ k=1,\dots,K_2$.

Thus the limits (3.5) hold up to subsequence. Now it remains to find $v_0^+, v_0^{(m,k)},\ k=1,\dots,K_m,\ m=1,2$.

2. At first we find the traces of $v_0^{(m,k)}$ at $x_3=-d_m,\ m=1,2,\ k=1,\dots,K_m$. Since $\big(\theta_\varepsilon,\vartheta_\varepsilon\big) \overset{w}{\rightsquigarrow} (\theta_0,\vartheta_0)$ as $\varepsilon\to 0$, we have

$$\int\limits_Q \chi_1^{(k)}\left(\frac{x'}{\varepsilon}\right)\theta_\varepsilon(x')\,\varphi(x')\,\mathrm{d}x' \to |B_k^{(1)}| \int\limits_Q \theta_0(x')\,\varphi(x')\,\mathrm{d}x' \quad \text{as} \quad \varepsilon\to 0 \tag{3.15}$$

for any function $\varphi \in C^\infty\big(\overline{Q}\big)$. Here $\{\chi_m^{(k)}(\xi'),\ \xi'\in\mathbb{R}^2 :\ k=1,\dots,K_m,\ m=1,2\}$ are 1-periodic functions such that

$$\chi_m^{(k)}(\xi') = \begin{cases} 1, & \xi' \in B_k^{(m)}, \\ 0, & \xi' \in \square \setminus B_k^{(m)}. \end{cases}$$

On the other hand, since $u_\varepsilon|_{\Gamma_\varepsilon^{(1)}} = \theta_\varepsilon$, we get

$$\int\limits_Q \chi_1^{(k)}\left(\frac{x'}{\varepsilon}\right)\theta_\varepsilon(x')\,\varphi(x')\,\mathrm{d}x' = -\frac{1}{d_1}\int\limits_{D_1}\widetilde{u_\varepsilon}^{(1,k)}\varphi(x')\,\mathrm{d}x - \frac{1}{d_1}\int\limits_{D_1} x_3\,\partial_{x_3}\widetilde{u_\varepsilon}^{(1,k)}\,\varphi(x')\,\mathrm{d}x. \tag{3.16}$$

Passing to the limit in (3.16) as $\varepsilon\to 0$ and taking (3.15) and the second relation in (3.5) into account, we obtain

$$|B_k^{(1)}|\int\limits_Q \theta_0(x')\,\varphi(x')\,\mathrm{d}x' = -\frac{|B_k^{(1)}|}{d_1}\int\limits_{D_1}\big(v_0^{(1,k)} + x_3\,\partial_{x_3}v_0^{(1,k)}\big)\varphi(x')\,\mathrm{d}x = |B_k^{(1)}|\int\limits_Q v_0^{(1,k)}(x',-d_1)\,\varphi(x')\,\mathrm{d}x'$$

for any function $\varphi\in C^\infty\big(\overline{Q}\big)$. This means that

$$v_0^{(1,k)}(x',-d_1) = \theta_0(x') \quad \text{for a.e. } x'\in Q,\quad k=1,\dots,K_1. \tag{3.17}$$

By the same arguments we can prove

$$v_0^{(2,k)}(x',-d_2) = \vartheta_0(x') \quad \text{for a.e. } x'\in Q,\quad k=1,\dots,K_2. \tag{3.18}$$

3. By virtue of the continuity of the trace operator, compact imbedding $H^{1/2}(Q)\subset L^2(Q)$ and the first relation in (3.9), we have

$$u_\varepsilon(x',0+0) \quad \overset{s}{\longrightarrow} \quad v_0^+(x',0) \quad \text{in } L^2(Q) \text{ as } \varepsilon\to 0. \tag{3.19}$$

Since $\widetilde{u_\varepsilon}^{(m,k)}(x',0-0) = \chi_m^{(k)}\left(\frac{x'}{\varepsilon}\right)u_\varepsilon(x',0+0)$ for a.e. $x'\in Q$,

$$\widetilde{u_\varepsilon}^{(m,k)}(x',0-0) \quad \overset{w}{\longrightarrow} \quad |B_k^{(m)}|\ v_0^+(x',0) \quad \text{weakly in } L^2(Q) \text{ as } \varepsilon\to 0,\ m=1,2.$$

On the other hand, for each $k\in\{1,\dots,K_m\}$, $m\in\{1,2\}$ and any $\psi\in C_0^\infty(Q)$

$$\int\limits_Q \widetilde{u_\varepsilon}^{(m,k)}(x',0)\psi(x')\,\mathrm{d}x' = \frac{1}{d_m}\int\limits_{D_m}\big(\widetilde{u_\varepsilon}^{(m,k)}\psi(x') + (x_3+d_m)\partial_{x_3}\widetilde{u_\varepsilon}^{(m,k)}\,\psi(x')\big)\,\mathrm{d}x.$$

Passing to the limit in these equalities and take the second relation in (3.5) into account, we find that

$$\widetilde{u_\varepsilon}^{(m,k)}(x',0) \xrightarrow{w} |B_k^{(m)}|\, v_0^{(m,k)}(x',0) \quad \text{weakly in} L^2(Q) \,\text{as}\, \varepsilon \to 0.$$

Thus,

$$v_0^+(x',0) = v_0^{(m,k)}(x',0) \quad \text{for a.e. } x' \in Q,\; k = 1,\dots,K_m,\; m = 1,2. \tag{3.20}$$

4. Consider the following space of multi-valued functions

$$\begin{aligned}\mathcal{C}^\infty(\Omega_0, D_1, D_2) := \Big\{ \mathbf{\Phi} = \Big(\varphi_0, \varphi_1^{(1)}, \dots, \varphi_{K_1}^{(1)}, \varphi_1^{(2)}, \dots, \varphi_{K_2}^{(2)}, \Big) :& \\ \varphi_0 \in C^\infty(\overline{\Omega}_0) \;\text{ and }\; \varphi_0|_{\Upsilon_1} = 0;& \\ \varphi_k^{(m)} \in C^\infty(\overline{D}_m) \;\text{ and }\; \varphi_k^{(m)}|_{\Gamma_{d_m}} = 0,\; k = 1,\dots,K_m,\; m = 1,2;& \\ \varphi_0^+(x',0) = \varphi_k^{(m)}(x',0) \quad \text{for }\; x' \in Q,\; k = 1,\dots,K_m,\; m = 1,2 \Big\}.&\end{aligned}$$

Obviously, the restriction

$$\Big(\varphi_0,\, \varphi_1^{(1)}|_{G_\varepsilon^{(1)}(1)}, \dots, \varphi_{K_1}^{(1)}|_{G_\varepsilon^{(1)}(K_1)},\, \varphi_1^{(2)}|_{G_\varepsilon^{(2)}(1)}, \dots, \varphi_{K_2}^{(2)}|_{G_\varepsilon^{(2)}(K_2)}\Big)$$

of any multi-valued function $\mathbf{\Phi}$ from $\mathcal{C}^\infty(\Omega_0, D_1, D_2)$ belongs to $H_\varepsilon^1(\Omega_\varepsilon; \Upsilon_1)$ and $\varphi_k^{(m)}|_{\Gamma_\varepsilon^{(m)}(k)} = 0$, $k = 1,\dots,K_m$, $m = 1,2$.

With the help of the identities (1.1) we rewrite the integral identity (0.5) in the following way

$$\begin{aligned}\int_{\Omega_0} f_0\, \varphi_0 \,\mathrm{d}x = \int_{\Omega_0} \nabla u_\varepsilon \cdot \nabla \varphi_0 \,\mathrm{d}x + \sum_{m=1}^{2} \sum_{k=1}^{K_m} \int_{D_m} \widetilde{\nabla u_\varepsilon}^{(m,k)} \cdot \nabla \varphi_k^{(m)} \,\mathrm{d}x& \\ + \sum_{k=1}^{K_1} \int_{G_\varepsilon^{(1)}(k)} \left(\varepsilon^{\alpha-1} \frac{l_k^{(1)}}{|B_k^{(1)}|}\, \varrho_1(u_\varepsilon)\, \varphi_k^{(1)} \;+\; \varepsilon^\alpha \nabla_\xi Y_k^{(1)}|_{\xi = \frac{x'}{\varepsilon}} \cdot \nabla_{x'} \big(\varrho_1(u_\varepsilon)\, \varphi_k^{(1)}\big) \right) \mathrm{d}x& \\ + \sum_{k=1}^{K_2} \int_{G_\varepsilon^{(2)}(k)} \left(\varepsilon^{\beta-1} \frac{l_k^{(2)}}{|B_k^{(2)}|}\, \varrho_2(u_\varepsilon)\, \varphi_k^{(2)} \;+\; \varepsilon^\beta \nabla_\xi Y_k^{(2)}|_{\xi = \frac{x'}{\varepsilon}} \cdot \nabla_{x'} \big(\varrho_2(u_\varepsilon)\, \varphi_k^{(2)}\big) \right) \mathrm{d}x&\end{aligned} \tag{3.21}$$

for each multi-valued function $\mathbf{\Phi} \in \mathcal{C}^\infty(\Omega_0, D_1, D_2)$.

Let us pass to the limit in (3.21) as $\varepsilon \to 0$. Taking into account (1.2) and that $\alpha \ge 1$ and $\beta \ge 1$, the integrals with factors ε^α and ε^β vanish in the limit passage. By virtue of results obtained in the previous items, we have

$$\begin{aligned}\int_{\Omega_0} f_0\, \varphi_0 \,\mathrm{d}x = \int_{\Omega_0} \nabla v_0^+ \cdot \nabla \varphi_0 \,\mathrm{d}x + \sum_{m=1}^{2} \sum_{k=1}^{K_m} |B_k^{(m)}| \int_{D_m} \partial_{x_3} v_0^{(m,k)}\, \partial_{x_3} \varphi_k^{(m)} \,\mathrm{d}x& \\ + \delta_{\alpha,1} \sum_{k=1}^{K_1} \frac{l_k^{(1)}}{|B_k^{(1)}|} \int_{D_1} \zeta^{(1,k)}\, \varphi_k^{(1)} \,\mathrm{d}x + \delta_{\beta,1} \sum_{k=1}^{K_2} \frac{l_k^{(2)}}{|B_k^{(2)}|} \int_{D_2} \zeta^{(2,k)}\, \varphi_k^{(2)} \,\mathrm{d}x.&\end{aligned} \tag{3.22}$$

Since the space $\mathcal{C}^\infty(\Omega_0, D_1, D_2)$ is dense in the anisotropic Sobolev space of multi-valued functions

$$\mathcal{H}(\Omega_0, D_1, D_2) := \Big\{ \mathbf{\Phi} = \Big(\varphi_0, \varphi_1^{(1)}, \ldots, \varphi_{K_1}^{(1)}, \varphi_1^{(2)}, \ldots, \varphi_{K_2}^{(2)}, \Big) : \quad \varphi_0 \in H^1(\Omega_0, \Upsilon_1);$$
$$\varphi_k^{(m)} \in W^{0,0,1}(D_m) \text{ and } \varphi_k^{(m)}|_{\Gamma_{d_m}} = 0, \ k = 1, \ldots, K_m, \ m = 1, 2;$$
$$\varphi_0^+(x', 0) = \varphi_k^{(m)}(x', 0) \quad \text{for a.e. } x' \in Q, \ k = 1, \ldots, K_m, \ m = 1, 2 \Big\},$$

the identity (3.22) is valid for each multi-valued function $\mathbf{\Phi} \in \mathcal{H}(\Omega_0, D_1, D_2)$.

5. Consider the following functions

$$q_\varepsilon^{(1)}(x) = \begin{cases} -x_3 \, d_1^{-1} \theta_\varepsilon(x'), & x \in D_1, \\ 0, & x \in \Omega_0, \end{cases} \quad \text{and} \quad q_\varepsilon^{(2)}(x) = \begin{cases} -x_3 \, d_2^{-1} \vartheta_\varepsilon(x'), & x \in D_2, \\ 0, & x \in \Omega_0. \end{cases}$$

Due to the definition of classes of admissible controls (see (0.1) and (0.2)) we can regard that θ_ε and ϑ_ε are defined respectively on Γ_{d_1} and Γ_{d_2}.

With the help of (0.5), (3.5), (3.14), (3.17), (3.18) and (3.22) we get that

$$\lim_{\varepsilon\to 0}\Big(\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \, \mathrm{d}x + \sum_{m=1}^{2} \varepsilon^{\alpha\delta_{1m}+\beta\delta_{2m}} \int_{S_\varepsilon^{(m)}} \varrho_m(u_\varepsilon) \, u_\varepsilon \mathrm{d}\sigma_x\Big) \tag{3.23}$$
$$= \lim_{\varepsilon\to 0}\Big(\int_{\Omega_0} |\nabla u_\varepsilon|^2 \, \mathrm{d}x + \sum_{m=1}^{2}\Big(\int_{G_\varepsilon^{(m)}} \nabla u_\varepsilon \cdot \nabla(u_\varepsilon - q_\varepsilon^{(m)}) \, \mathrm{d}x + \varepsilon^{\alpha\delta_{1m}+\beta\delta_{2m}} \int_{S_\varepsilon^{(m)}} \varrho_m(u_\varepsilon) \, (u_\varepsilon - q_\varepsilon^{(m)}) \, \mathrm{d}\sigma_x\Big)\Big)$$
$$+ \lim_{\varepsilon\to 0} \sum_{m=1}^{2}\Big(\int_{G_\varepsilon^{(m)}} \nabla u_\varepsilon \cdot \nabla q_\varepsilon^{(m)} \, \mathrm{d}x + \varepsilon^{\alpha\delta_{1m}+\beta\delta_{2m}} \int_{S_\varepsilon^{(m)}} \varrho_m(u_\varepsilon) \, q_\varepsilon^{(m)} \, \mathrm{d}\sigma_x\Big)$$
$$= \lim_{\varepsilon\to 0}\Big(\int_{\Omega_0} f_0 \, u_\varepsilon \, \mathrm{d}x\Big) + \sum_{m=1}^{2}\sum_{k=1}^{K_m} |B_k^{(m)}| \int_{D_m} \partial_{x_3} v_0^{(m,k)} \, \partial_{x_3} q_0^{(m)} \, \mathrm{d}x$$
$$+ \delta_{\alpha,1} \sum_{k=1}^{K_1} \frac{l_k^{(1)}}{|B_k^{(1)}|} \int_{D_1} \zeta^{(1,k)} \, q_0^{(1)} \, \mathrm{d}x + \delta_{\beta,1} \sum_{k=1}^{K_2} \frac{l_k^{(2)}}{|B_k^{(2)}|} \int_{D_2} \zeta^{(2,k)} \, q_0^{(2)} \, \mathrm{d}x$$
$$= \int_{\Omega_0} |\nabla v_0^+|^2 \, \mathrm{d}x + \sum_{m=1}^{2}\sum_{k=1}^{K_m} |B_k^{(m)}| \int_{D_m} |\partial_{x_3} v_0^{(m,k)}|^2 \mathrm{d}x$$
$$+ \delta_{\alpha,1} \sum_{k=1}^{K_1} \frac{l_k^{(1)}}{|B_k^{(1)}|} \int_{D_1} \zeta^{(1,k)} \, v_0^{(1,k)} \, \mathrm{d}x + \delta_{\beta,1} \sum_{k=1}^{K_2} \frac{l_k^{(2)}}{|B_k^{(2)}|} \int_{D_2} \zeta^{(2,k)} \, v_0^{(2,k)} \, \mathrm{d}x. \tag{3.24}$$

It should be commented here the passage to the limit in the integrals

$$\int_{G_\varepsilon^{(m)}(k)} \nabla u_\varepsilon \cdot \nabla q_\varepsilon^{(m)} \, \mathrm{d}x = \int_{D_m} \widetilde{\nabla u_\varepsilon}^{(m,k)} \cdot \nabla q_\varepsilon^{(m)} \, \mathrm{d}x, \quad k = 1, \ldots, K_m, \quad m = 1, 2.$$

For definiteness we take $m = 1$ and some $k \in \{1, \ldots, K_1\}$. Since $(\theta_\varepsilon, \vartheta_\varepsilon) \overset{w}{\rightsquigarrow} (\theta_0, \vartheta_0)$ as $\varepsilon \to 0$ and due to the compactness of the embedding $H^{\delta_1}(\Gamma_{d_1}) \subset H^1(\Gamma_{d_1})$ ($\delta_1 > 1$),

$$q_\varepsilon^{(m)} \to q_0^{(m)} \quad \text{strongly in } H^1(D_m) \text{ as } \varepsilon \to 0, \ m = 1, 2,$$

where $q_0^{(1)}(x) = -x_3\, d_1^{-1}\theta_0(x'),\ x \in D_1;\ \ q_0^{(2)}(x) = -x_3\, d_2^{-1}\vartheta_0(x'),\ x \in D_2$. Therefore, by virtue of the convergences obtained in the first item of the proof, we have

$$\lim_{\varepsilon\to 0}\int_{D_1} \widetilde{\nabla u_\varepsilon}^{(1,k)} \cdot \nabla q_\varepsilon^{(1)}\, \mathrm{d}x = |B_k^{(1)}| \int_{D_1} \partial_{x_3} v_0^{(1,k)}\, \partial_{x_3} q_0^{(1)}\, \mathrm{d}x. \tag{3.25}$$

6. Now it remains to determine the functions $\{\zeta^{(1,k)}\}$ if $\alpha = 1$ and $\{\zeta^{(2,k)}\}$ if $\beta = 1$. We will do this in the most complicate case $\alpha = \beta = 1$. For this we use the method of Browder and Minty, which somehow applies to the corresponding inequality of monotonicity to justify passing to a weak limit within a nonlinearity. Thanks to (0.3), the inequality of monotonicity in our case reads as follows

$$\begin{aligned}\int_{\Omega_0} |\nabla u_\varepsilon - \nabla\varphi_0|^2\, \mathrm{d}x + \int_{G_\varepsilon} |\nabla_{x'} u_\varepsilon|^2\, \mathrm{d}x + \sum_{m=1}^{2}\sum_{k=1}^{K_m} \int_{G_\varepsilon^{(m)}(k)} |\partial_{x_3} u_\varepsilon - \partial_{x_3}\varphi_k^{(m)}|^2\, \mathrm{d}x \\ + \varepsilon \sum_{m=1}^{2}\sum_{k=1}^{K_m} \int_{S_\varepsilon^{(m)}(k)} \big(\varrho_m(u_\varepsilon) - \varrho_m(\varphi_k^{(m)})\big)\big(u_\varepsilon - \varphi_k^{(m)}\big)\, \mathrm{d}\sigma_x \ge 0\end{aligned} \tag{3.26}$$

for any multi-valued function $\mathbf{\Phi} = \Big(\varphi_0, \varphi_1^{(1)}, \ldots, \varphi_{K_1}^{(1)}, \varphi_1^{(2)}, \ldots, \varphi_{K_2}^{(2)},\Big)$ such that $\varphi_0 \in H^1(\Omega_0, \Upsilon_1)$, $\varphi_k^{(m)} \in W^{0,0,1}(D_m)$, $\varphi_0(x', 0) = \varphi_k^{(m)}(x', 0)$ for a.e. $x' \in Q$ and $k = 1, \ldots, K_m,\ m = 1, 2$.

The inequality (3.26) is equivalent to

$$\begin{aligned}\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2\, \mathrm{d}x + \sum_{m=1}^{2} \varepsilon \int_{S_\varepsilon^{(m)}} \varrho_m(u_\varepsilon)\, u_\varepsilon \mathrm{d}\sigma_x - 2\int_{\Omega_0} \nabla u_\varepsilon \cdot \nabla\varphi_0\, \mathrm{d}x + \int_{\Omega_0} |\nabla\varphi_0|^2\, \mathrm{d}x \\ + \sum_{m=1}^{2}\sum_{k=1}^{K_m} \int_{D_m} \Big(\chi_m^{(k)}(\tfrac{x'}{\varepsilon})\, |\partial_{x_3}\varphi_k^{(m)}|^2 - 2\widetilde{\partial_{x_3} u_\varepsilon}^{(m,k)} \cdot \partial_{x_3}\varphi_k^{(m)}\Big)\mathrm{d}x \\ - \varepsilon \sum_{m=1}^{2}\sum_{k=1}^{K_m} \int_{S_\varepsilon^{(m)}(k)} \Big(\varrho_m\big(u_\varepsilon\big)\, \varphi_k^{(m)} + \varrho_m\big(\varphi_k^{(m)}\big)\, u_\varepsilon - \varrho_m\big(\varphi_k^{(m)}\big)\, \varphi_k^{(m)}\Big)\mathrm{d}\sigma_x \ge 0.\end{aligned} \tag{3.27}$$

The limit of the first three summands in the first line of (3.27) is given by (3.24). With regard to the results obtained above we know the limit of the other ones in the first and second line of (3.27). And with the help of (1.1) we can find the limits of the summands in the third line. As a result we get

$$\begin{aligned}\int_{\Omega_0} |\nabla v_0^+ - \nabla\varphi_0|^2\, \mathrm{d}x + \sum_{m=1}^{2}\sum_{k=1}^{K_m} |B_k^{(m)}| \int_{D_m} \big(\partial_{x_3} v_0^{(m,k)} - \partial_{x_3}\varphi_k^{(m)}\big)^2\, \mathrm{d}x \\ + \sum_{m=1}^{2}\sum_{k=1}^{K_m} \frac{l_k^{(m)}}{|B_k^{(m)}|} \int_{D_m} \big(\zeta^{(m,k)}(x) - |B_k^{(m)}|\, \varrho_m(\varphi_k^{(m)})\big)\, \big(v_0^{(m,k)} - \varphi_k^{(m)}\big)\, \mathrm{d}x \ge 0.\end{aligned} \tag{3.28}$$

Take arbitrary multi-valued function $\mathbf{\Psi} = \big(\psi_0, \psi_1^{(1)}, \ldots, \psi_{K_1}^{(1)}, \psi_1^{(2)}, \ldots, \psi_{K_2}^{(2)}\big)$ such that $\psi_0 \in C^\infty(\overline{\Omega_0})$, $\psi_0|_{\Upsilon_1} = 0$, $\psi_k^{(m)} \in C^\infty(\overline{D_m})$, $\psi_0(x', 0) = \psi_k^{(m)}(x', 0)$ for $x' \in Q$ and for $k = 1, \ldots, K_m,\ m = 1, 2$. Then substitute the following function $\mathbf{\Phi}_0 := \mathbf{V}_0 - \lambda\mathbf{\Psi}$ $(\lambda > 0)$ instead of $\mathbf{\Phi}$ in (3.28), where $\mathbf{V}_0$ is defined by formula (3.6). We then obtain

$$\lambda \int_{\Omega_0} |\nabla \psi_0|^2 \, \mathrm{d}x + \lambda \sum_{m=1}^{2} \sum_{k=1}^{K_m} |B_k^{(m)}| \int_{D_m} |\partial_{x_3} \psi_k^{(m)}|^2 \, \mathrm{d}x + \sum_{m=1}^{2} \sum_{k=1}^{K_m} \frac{l_k^{(m)}}{|B_k^{(m)}|} \int_{D_m} \Big(\zeta^{(m,k)}(x) - |B_k^{(m)}| \, \varrho_m \big(v_0^{(m,k)} - \lambda \psi_k^{(m)} \big) \Big) \, \psi_k^{(m)} \, \mathrm{d}x \ge 0.$$

In the limit (as $\lambda \to 0$) we get

$$\sum_{m=1}^{2} \sum_{k=1}^{K_m} \frac{l_k^{(m)}}{|B_k^{(m)}|} \int_{D_m} \Big(\zeta^{(m,k)}(x) - |B_k^{(m)}| \, \varrho_m \big(v_0^{(m,k)} \big) \Big) \, \psi_k^{(m)} \, \mathrm{d}x \ge 0.$$

Replacing the multi-valued function $\mathbf{\Psi}$ with $-\mathbf{\Psi}$ and taking into account that $\mathbf{\Psi}$ is arbitrary, we conclude that, in fact, the last inequality turns into the following equalities:

$$\zeta^{(m,k)}(x) = |B_k^{(m)}| \, \varrho_m \big(v_0^{(m,k)}(x) \big) \quad \text{for a.e. } x \in D_m, \quad k = 1, \ldots, K_m, \quad m = 1, 2. \tag{3.29}$$

7. Thus function $\mathbf{V}_0$ defined by (3.6) satisfies the following integral identity

$$\int_{\Omega_0} \nabla v_0^+ \cdot \nabla \varphi_0 \, \mathrm{d}x + \sum_{m=1}^{2} \sum_{k=1}^{K_m} |B_k^{(m)}| \int_{D_m} \partial_{x_3} v_0^{(m,k)} \, \partial_{x_3} \varphi_k^{(m)} \, \mathrm{d}x + \delta_{\alpha,1} \sum_{k=1}^{K_1} l_k^{(1)} \int_{D_1} \varrho_1 \big(v_0^{(1,k)}(x) \big) \, \varphi_k^{(1)} \, \mathrm{d}x + \delta_{\beta,1} \sum_{k=1}^{K_2} l_k^{(2)} \int_{D_2} \varrho_2 \big(v_0^{(2,k)}(x) \big) \, \varphi_k^{(2)} \, \mathrm{d}x = \int_{\Omega_0} f_0 \, \varphi_0 \, \mathrm{d}x \quad \forall \mathbf{\Phi} \in \mathcal{H}(\Omega_0, D_1, D_2). \tag{3.30}$$

Thanks to (3.17), (3.18) and (3.20) the integral identity (3.30) means that the multi-valued function $\mathbf{V}_0$ is a weak solution to the homogenized problem (3.7).

Assume that $\mathbf{V}_0$ and $\mathbf{U}_0$ are two weak solutions to problem (3.7). Then with the help of (0.3) we deduce that $\mathbf{V}_0 = \mathbf{U}_0$. Due to the uniqueness of the solution to problem (3.7), the above argumentations hold for any subsequence of $\{\varepsilon\}$ chosen at the beginning of the proof.

8. Now let us prove the convergence of the cost functionals. At first we note that from the first limit in (3.5) and the condition of this theorem, we get the limits of the first three summands in (0.6). The energy functional we can represent as follows

$$\mathbb{E}_\varepsilon(u_\varepsilon; \alpha, \beta) = \frac{N_3}{2} \left(\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \, \mathrm{d}x + \sum_{m=1}^{2} \varepsilon^{\alpha \delta_{1m} + \beta \delta_{2m}} \int_{S_\varepsilon^{(m)}} \varrho_m(u_\varepsilon) \, u_\varepsilon \, \mathrm{d}\sigma_x - \sum_{m=1}^{2} \varepsilon^{\alpha \delta_{1m} + \beta \delta_{2m}} \varrho_m(0) \int_{S_\varepsilon^{(m)}} u_\varepsilon \, \mathrm{d}\sigma_x \right).$$

The limit of the first three summands was found in the 5th and 6th items (see (3.24) and (3.29)). The limit of the last two integrals can be found with the help of (1.1) and the second limit in (3.5) (similarly as in (3.21) and (3.22)). As a result we get (3.8). □

From Theorem 3.3 and the second part of Proposition 3.2 the following statement ensues.

Proposition 3.4. *For any pair* $(\theta_0, \vartheta_0) \in \mathcal{K}_0^{(1)} \times \mathcal{K}_0^{(2)}$, *there exists a sequence of admissible control pairs* $\big\{ (\theta_\varepsilon, \vartheta_\varepsilon) \in \mathcal{K}_\varepsilon^{(1)} \times \mathcal{K}_\varepsilon^{(2)} \big\}_{\varepsilon > 0}$, *such that*

(1) $(\theta_\varepsilon, \vartheta_\varepsilon) \overset{w}{\rightsquigarrow} (\theta_0, \vartheta_0)$ *as* $\varepsilon \to 0$;

(2) *the sequence of the corresponding solutions* $\{u_\varepsilon(x,\theta_\varepsilon,\vartheta_\varepsilon),\ x\in\Omega_\varepsilon\}_{\varepsilon>0}$ *to problem* (0.4) *satisfies relations* (3.5), *where the multi-valued function* $\mathbf{V}_0$ *defined by* (3.6) *is the unique weak solution to the homogenized problem* (3.7) *such that* $v_0^{(1,k)}|_{\Gamma_{d_1}}=\theta_0,\ k=1,\dots,K_1$, *and* $v_0^{(2,k)}|_{\Gamma_{d_2}}=\vartheta_0,\ k=1,\dots,K_1$;

(3) $J_\varepsilon(\theta_\varepsilon,\vartheta_\varepsilon)\to J_0(\theta_0,\vartheta_0)$ *as* $\varepsilon\to0$.

3.3. Homogenized optimal control problem CP_0

The results obtained are crucial point in the asymptotic investigation of problem CP_ε. Using these results we define the following homogenized optimal control problem CP_0:

- *Find optimal controls* $\theta^*\in\mathcal{K}_0^{(1)}$, $\vartheta^*\in\mathcal{K}_0^{(2)}$ *and the corresponding multi-valued solution* $\mathbf{V}_*(\theta^*,\vartheta^*)$ *of the homogenized problem* (3.7) *to minimize the cost functional* J_0, i.e.,

$$J_0(\theta^*,\vartheta^*)=\inf_{(\theta,\vartheta)\in\mathcal{K}_0^{(1)}\times\mathcal{K}_0^{(2)}}J_0(\theta,\vartheta). \tag{3.31}$$

Here the control sets $\mathcal{K}_0^{(1)}$ and $\mathcal{K}_0^{(2)}$ are defined in (2.1) and (2.2) respectively, the cost functional J_0 is defined by (3.8).

Since $\mathcal{K}_0^{(1)}\times\mathcal{K}_0^{(2)}$ is convex and closed with respect to the weak topology in $H^1(\Gamma_{d_1})\times H^1(\Gamma_{d_2})$ and the cost functional J_0 is weakly lower-semicontinuous, we can prove by standard way (see for instance [15]) the existence of minimizer for problem CP_0. In the case if $\alpha>1$ and $\beta>1$ the cost functional J_0 is strictly convex and therefore the problem CP_0 has the unique solution.

To justify the definition of the homogenized problem for problem CP_ε, we prove the following theorem.

Theorem 3.5. **(1)** *If* $\alpha>1$ *and* $\beta>1$, *then for any sequence of the optimal control pairs* $\big\{(\theta_\varepsilon^*,\vartheta_\varepsilon^*)\in\mathcal{K}_\varepsilon^{(1)}\times\mathcal{K}_\varepsilon^{(2)}\big\}_{\varepsilon>0}$ *of problem* CP_ε the following convergences hold *(*as $\varepsilon\to0$*)*

$$(\theta_\varepsilon^*,\vartheta_\varepsilon^*)\overset{w}{\rightsquigarrow}(\theta_0^*,\vartheta_0^*),\qquad J_\varepsilon(\theta_\varepsilon^*,\vartheta_\varepsilon^*)\to J_0(\theta_0^*,\vartheta_0^*), \tag{3.32}$$

where $(\theta_0^*,\vartheta_0^*)$ *is the unique solution to problem* CP_0.

In addition, for the sequence of the corresponding solutions $\{u_\varepsilon^*:=u_\varepsilon(\theta_\varepsilon^*,\vartheta_\varepsilon^*)\}_{\varepsilon>0}$ *to problem* (0.4) *we have*

$$\left.\begin{aligned} u_\varepsilon^*|_{\Omega_0} &\overset{w}{\longrightarrow} v_*^+ && \text{weakly in } H^1(\Omega_0,\Gamma_1),\\ \widetilde{u_\varepsilon^*}^{(m,k)} &\overset{w}{\longrightarrow} |B_k^{(m)}|\,v_*^{(m,k)} && \text{weakly in } W^{0,0,1}(D_m),\\ \widetilde{\partial_{x_i}u_\varepsilon^*}^{(m,k)} &\overset{w}{\longrightarrow} 0 && \text{weakly in } L^2(D_m),\ i=1,2, \end{aligned}\right\}\quad \text{as}\ \ \varepsilon\to0 \tag{3.33}$$

for $k=1,\dots,K_m$, $m=1,2$, *where the multi-valued function*

$$\mathbf{V}_*(x)=\begin{cases} v_*^+(x), & x\in\Omega_0,\\ v_*^{(m,k)}(x), & x\in D_m,\quad m=1,2,\quad k=1,\dots,K_m,\end{cases} \tag{3.34}$$

is the unique weak solution to problem (3.7) *such that*

$$v_*^{(1,k)}|_{\Gamma_{d_1}}=\theta_0^*,\quad k=1,\dots,K_1;\qquad v_*^{(2,k)}|_{\Gamma_{d_2}}=\vartheta_0^*,\quad k=1,\dots,K_2. \tag{3.35}$$

(2) *In the other cases (when one or two of the parameters* α *and* β *can be equal one), for any sequence of the optimal control pairs* $\big\{(\theta_\varepsilon^*,\vartheta_\varepsilon^*)\in\mathcal{K}_\varepsilon^{(1)}\times\mathcal{K}_\varepsilon^{(2)}\big\}_{\varepsilon>0}$ *of problem* CP_ε *there exists a subsequence* $\big\{(\theta_{\varepsilon_n}^*,\vartheta_{\varepsilon_n}^*)\in\mathcal{K}_{\varepsilon_n}^{(1)}\times\mathcal{K}_{\varepsilon_n}^{(2)}\big\}$ *such that the limits* (3.32) *hold as* $\varepsilon_n\to0$ $(n\to+\infty)$ *and* $(\theta_0^*,\vartheta_0^*)$ *is a minimizer of problem* CP_0.

In addition, the sequence of the corresponding solutions $\{u^*_{\varepsilon_n} := u_{\varepsilon_n}(\theta^*_{\varepsilon_n}, \vartheta^*_{\varepsilon_n})\}$ *to problem* (0.4) *satisfies relations* (3.33) *as* $\varepsilon_n \to 0$ $(n \to +\infty)$ *and the corresponding multi-valued function* $\mathbf{V}_*$ *defined by* (3.34) *is the unique weak solution to problem* (3.7) *with the boundary conditions* (3.35).

Proof. **(1)** Let $\left\{(\theta^*_{\varepsilon_n}, \vartheta^*_{\varepsilon_n})\right\}_{\varepsilon_n>0}$ be any convergent subsequence of the sequence of minimizers $\left\{(\theta^*_\varepsilon, \vartheta^*_\varepsilon) \in \mathcal{K}^{(1)}_\varepsilon \times \mathcal{K}^{(2)}_\varepsilon\right\}_{\varepsilon>0}$ such that $(\theta^*_{\varepsilon_n}, \vartheta^*_{\varepsilon_n}) \overset{w}{\rightsquigarrow} (\theta_0, \vartheta_0)$ as $n \to \infty$. In view of Proposition 3.2 such choice is always possible and $(\theta_0, \vartheta_0) \in \mathcal{K}^{(1)}_0 \times \mathcal{K}^{(2)}_0$. In addition, due to Theorem 3.3 the corresponding solutions $\{u^*_{\varepsilon_n}\}$ to problem (0.4) satisfies relations (3.5), where the multi-valued function $\mathbf{V}_0$ defined by (3.6) is the unique weak solution to the homogenized problem (3.7) such that $v^{(1,k)}_0|_{\Gamma_{d_1}} = \theta_0,\ k = 1, \dots, K_1,\ v^{(2,k)}_0|_{\Gamma_{d_2}} = \vartheta_0,\ k = 1, \dots, K_1$, and

$$\lim_{n\to\infty} J_{\varepsilon_n}\left(\theta^*_{\varepsilon_n}, \vartheta^*_{\varepsilon_n}\right) = J_0\left(\theta_0, \vartheta_0\right) \geq \inf_{(\theta,\vartheta)\in\mathcal{K}^{(1)}_0\times\mathcal{K}^{(2)}_0} J_0\left(\theta, \vartheta\right) = J_0\left(\theta^*_0, \vartheta^*_0\right),$$

where $(\theta^*_0, \vartheta^*_0)$ is the unique solution of problem (3.31).

On the other hand, from Proposition 3.4 it follows that there exists a sequence of control pairs $\left\{(\theta^\flat_\varepsilon, \vartheta^\flat_\varepsilon) \in \mathcal{K}^{(1)}_\varepsilon \times \mathcal{K}^{(2)}_\varepsilon\right\}_{\varepsilon>0}$ such that $(\theta^\flat_\varepsilon, \vartheta^\flat_\varepsilon) \overset{w}{\rightsquigarrow} (\theta^*_0, \vartheta^*_0)$ and $J_\varepsilon\left(\theta^\flat_\varepsilon, \vartheta^\flat_\varepsilon\right) \to J_0\left(\theta^*_0, \vartheta^*_0\right)$ as $\varepsilon \to 0$, and the corresponding solutions $\{u_\varepsilon(x, \theta^\flat_\varepsilon, \vartheta^\flat_\varepsilon)\}_{\varepsilon>0}$ to problem (0.4) satisfies relations (3.33), where the multi-valued function $\mathbf{V}_*$ defined by (3.34) is the unique weak solution to problem (3.7) such that $v^{(1,k)}_0|_{\Gamma_{d_1}} = \theta^*_0,\ k = 1, \dots, K_1$, and $v^{(2,k)}_0|_{\Gamma_{d_2}} = \vartheta^*_0,\ k = 1, \dots, K_1$.

Taking these facts into account, we deduce

$$\begin{aligned}\min_{(\theta,\vartheta)\in\mathcal{K}^{(1)}_0\times\mathcal{K}^{(2)}_0} J_0\left(\theta, \vartheta\right) = J_0\left(\theta^*_0, \vartheta^*_0\right) = \lim_{\varepsilon\to0} J_\varepsilon\left(\theta^\flat_\varepsilon, \vartheta^\flat_\varepsilon\right) \geq \limsup_{\varepsilon\to0} \min_{(\theta_\varepsilon,\vartheta_\varepsilon)\in\mathcal{K}^{(1)}_\varepsilon\times\mathcal{K}^{(2)}_\varepsilon} J_\varepsilon\left(\theta_\varepsilon, \vartheta_\varepsilon\right)\\ \geq \limsup_{n\to\infty} \min_{(\theta_{\varepsilon_n},\vartheta_{\varepsilon_n})\in\mathcal{K}^{(1)}_{\varepsilon_n}\times\mathcal{K}^{(2)}_{\varepsilon_n}} J_\varepsilon\left(\theta_{\varepsilon_n}, \vartheta_{\varepsilon_n}\right) = \lim_{n\to\infty} J_{\varepsilon_n}\left(\theta^*_{\varepsilon_n}, \vartheta^*_{\varepsilon_n}\right) = J_0\left(\theta_0, \vartheta_0\right).\end{aligned}$$

Thus, $J_0\left(\theta_0, \vartheta_0\right) = J_0\left(\theta^*_0, \vartheta^*_0\right)$. Thanks to the uniqueness of the solutions to problem (3.31) and to problem (3.7), we have $\theta_0 = \theta^*_0$, $\vartheta_= \vartheta^*_0$, $\mathbf{V}_0 = \mathbf{V}_*$.

Since all these relations are valid for any converging subsequence chosen at the beginning of the proof, the convergences (3.32) and (3.33) hold.

(2) Due to the first statement of Proposition 3.2 we can extract from any sequence of the optimal control pairs $\left\{(\theta^*_\varepsilon, \vartheta^*_\varepsilon) \in \mathcal{K}^{(1)}_\varepsilon \times \mathcal{K}^{(2)}_\varepsilon\right\}_{\varepsilon>0}$ of problem CP_ε a subsequence $\left\{(\theta^*_{\varepsilon_n}, \vartheta^*_{\varepsilon_n}) \in \mathcal{K}^{(1)}_{\varepsilon_n} \times \mathcal{K}^{(2)}_{\varepsilon_n}\right\}$ such that $(\theta^*_{\varepsilon_n}, \vartheta^*_{\varepsilon_n}) \overset{w}{\rightsquigarrow} (\theta^*_0, \vartheta^*_0)$ as $\varepsilon_n \to 0$. By the same arguments as in the first item we can prove that $(\theta^*_0, \vartheta^*_0)$ is one of minimizers of problem CP_0. □

4. The main results in the case $\alpha \geq 1$ and $\beta < 1$

In this section we additionally assume that $\varrho_2(0) = 0$. Even under this assumption we cannot apply estimate (1.6) to obtain the uniform boundedness for the states as in the previous case (see (3.1)).

4.1. Reformulation of the problem

We have known that for every $\varepsilon > 0$ problem CP_ε has a solution $(\theta^*_\varepsilon, \vartheta^*_\varepsilon) \in \mathcal{K}^{(1)}_\varepsilon \times \mathcal{K}^{(2)}_\varepsilon$. Let $\left\{(\theta^*_{\varepsilon_n}, \vartheta^*_{\varepsilon_n})\right\}_{\varepsilon_n>0}$ be any convergent subsequence of the sequence of minimizers $\left\{(\theta^*_\varepsilon, \vartheta^*_\varepsilon)\right\}_{\varepsilon>0}$ such that

$$(\theta^*_{\varepsilon_n}, \vartheta^*_{\varepsilon_n}) \overset{w}{\rightsquigarrow} (\theta_0, \vartheta_0) \quad \text{as} \quad n \to \infty.$$

In view of Proposition 3.2 a such choice is always possible and $(\theta_0, \vartheta_0) \in \mathcal{K}^{(1)}_0 \times \mathcal{K}^{(2)}_0$.

For any ε_n let us take zero controls $\theta_0 = 0$ and $\vartheta_0 = 0$. Then the corresponding state $u^0_{\varepsilon_n}$ satisfies the following equality

$$\int_{\Omega_\varepsilon} |\nabla_x u^0_{\varepsilon_n}|^2 \,\mathrm{d}x + \varepsilon^\alpha \int_{S^{(1)}_\varepsilon} \big(\varrho_1\big(u^0_{\varepsilon_n}\big) - \varrho_1(0)\big)\, u^0_{\varepsilon_n} \,\mathrm{d}\sigma_x + \varepsilon^\beta \int_{S^{(2)}_\varepsilon} \varrho_2\big(u^0_{\varepsilon_n}\big)\, u^0_{\varepsilon_n} \,\mathrm{d}\sigma_x = \varepsilon^\alpha \int_{S^{(1)}_\varepsilon} \varrho_1\big(0\big)\, u^0_{\varepsilon_n} \,\mathrm{d}\sigma_x + \int_{\Omega_0} f_0\, u^0_{\varepsilon_n} \,\mathrm{d}x,$$

from where with the help of (1.3) and (1.5) we deduce at first that

$$\|\nabla_x u^0_{\varepsilon_n}\|_{L^2(\Omega_{\varepsilon_n})} \le C_1, \tag{4.1}$$

and then the following estimates

$$\varepsilon^\alpha \int_{S^{(1)}_\varepsilon} \big(\varrho_1\big(u^0_{\varepsilon_n}\big) - \varrho_1(0)\big)\, u^0_{\varepsilon_n} \,\mathrm{d}\sigma_x \le C_2, \qquad \varepsilon^\beta \int_{S^{(2)}_\varepsilon} \varrho_2\big(u^0_{\varepsilon_n}\big)\, u^0_{\varepsilon_n} \,\mathrm{d}\sigma_x \le C_2. \tag{4.2}$$

From these estimates it follows that $\mathbb{E}_\varepsilon(u^0_{\varepsilon_n}, \alpha, \beta) \le C_3$ and consequently

$$J_{\varepsilon_n}\left(\theta^*_{\varepsilon_n}, \vartheta^*_{\varepsilon_n}\right) \le J_{\varepsilon_n}(0,0) \le C_4,$$

from where we get

$$\|u^*_{\varepsilon_n}\|_{H^1(\Omega_{\varepsilon_n})} \le C_5, \qquad \varepsilon^\beta_n \int_{S^{(2)}_{\varepsilon_n}} \big|u^*_{\varepsilon_n}\big|^2 \,\mathrm{d}\sigma_x \le C_5, \tag{4.3}$$

where $u^*_{\varepsilon_n} := u_{\varepsilon_n}(x, \theta^*_{\varepsilon_n}, \vartheta^*_{\varepsilon_n}),\ x \in \Omega_{\varepsilon_n}$.

The first estimate in (4.3) means that there is a subsequence $\{\varepsilon'_n\} \subset \{\varepsilon_n\}$, again denoted by $\{\varepsilon_n\}$, such that

$$\left.\begin{array}{llll} u^*_{\varepsilon_n}|_{\Omega_0} & \stackrel{w}{\longrightarrow} & v^*_0 & \text{weakly in} \quad H^1(\Omega_0; \Upsilon_1), \\ \widetilde{u^*_{\varepsilon_n}}^{(m,k)} & \stackrel{w}{\longrightarrow} & |B^{(m)}_k|\, v^{(m,k)}_* & \text{weakly in} \quad L^2(D_m), \end{array}\right\} \quad \text{as} \ \ \varepsilon_n \to 0, \tag{4.4}$$

for $k = 1, \ldots, K_m,\ m = 1, 2,\ i = 1, 2, 3.$

With the help of (4.3) we deduce from (1.4) that

$$\int_{G^{(2)}_{\varepsilon_n}} \big|u^*_{\varepsilon_n}\big|^2 \,\mathrm{d}x \le C_5\, \varepsilon^{1-\beta}_n. \tag{4.5}$$

This means the strong convergence of $\big\{\widetilde{u^*_{\varepsilon_n}}^{(2,k)}\big\}_{n\in\mathbb{N}}$ to 0 in $L^2(D_2)$. Then, from (3.10) it follows the weak convergence of $\big\{\widetilde{\partial_{x_3} u^*_{\varepsilon_n}}^{(2,k)}\big\}_{n\in\mathbb{N}}$ to 0 in $L^2(D_2)$, $k = 1, \ldots, K_2$. Hence,

$$v^{(2,k)}_* = 0 \quad \text{a.e. in} \ \ D_2, \quad k = 1, \ldots, K_2. \tag{4.6}$$

Now from (3.16) we get

$$\int_{\Gamma^{(2)}_{\varepsilon_n}} \left(\vartheta^*_{\varepsilon_n}\right)^2 \mathrm{d}x' = -\frac{1}{d_2}\int_{G^{(2)}_{\varepsilon_n}} u^*_{\varepsilon_n}(x)\,\vartheta^*_{\varepsilon_n}(x')\,\mathrm{d}x - \frac{1}{d_2}\int_{G^{(2)}_{\varepsilon_n}} x_3\,\partial_{x_3} u_{\varepsilon_n}(x)\,\vartheta^*_{\varepsilon_n}(x')\,\mathrm{d}x. \tag{4.7}$$

Since $\vartheta^*_{\varepsilon_n} \overset{w}{\rightsquigarrow} \vartheta_0$ as $n\to\infty$ and $\widetilde{u^*_{\varepsilon_n}}^{(2,k)} \overset{w}{\longrightarrow} 0$ weakly in $W^{0,0,1}(D_2)$ $(k=1,\ldots,K_2)$, we have that

$$\lim_{n\to\infty}\int_{\Gamma^{(2)}_{\varepsilon_n}} \left(\vartheta^*_{\varepsilon_n}\right)^2 \mathrm{d}x' = 0.$$

This means that $\vartheta_0 = 0$.

From the first limit in (4.4) it follows that the sequence of the traces $\{u^*_{\varepsilon_n}(x', 0{+}0)\}$ strongly in $L^2(Q)$ converge to $v^*(x',0)$. Taking (4.6) into account, we deduce similarly as in the third item of the proof of Theorem 3.3 that

$$v^*_0(x',0) = v^{(1,k)}_*(x',0) = 0 \quad \text{for a.e. } x'\in Q,\ k=1,\ldots,K_1. \tag{4.8}$$

If we take a test-function ψ such that $\psi = 0$ on the thin cylinders G_{ε_n} in the integral identity (0.5) and then pass to the limit with regard to (4.4) and (4.8), we get that v^*_0 is the unique solution to the following boundary-value problem

$$\begin{cases} -\Delta_x\, v^*_0 = f_0 \,\text{in}\, \Omega_0, \\ v^*_0 = 0 \quad \text{on}\,\Gamma_1\cup Q, \qquad \partial_\nu v^*_0 = 0 \quad \text{on}\,\partial\Omega_0\setminus\big(\Gamma_1\cup Q\big). \end{cases} \tag{4.9}$$

Due to the uniqueness of the solution to problem (4.9), the above argumentations hold for any subsequence of $\{\varepsilon\}$ chosen at the beginning of this section. Hence

$$u^*_\varepsilon|_{\Omega_0} \overset{w}{\longrightarrow} v^*_0 \ \text{ weakly in } \ H^1(\Omega_0,\Gamma_1), \qquad \widetilde{u^*_\varepsilon}^{(2,k)} \overset{s}{\longrightarrow} 0 \ \text{ strongly in } \ L^2(D_2)\ (k=1,\ldots,K_2), \qquad \vartheta^*_\varepsilon \overset{w}{\rightsquigarrow} 0$$

as $\varepsilon\to 0$. These limits mean that we cannot control the state of the system in Ω_0 through the thin cylinders $G^{(2)}_\varepsilon$ if $\beta<1$ and ε is small enough. Therefore, we should reformulate problem CP_ε in the following way:

- *Find an optimal control $\theta^*_\varepsilon \in \mathcal{K}^{(1)}_\varepsilon$, which with the corresponding state u^*_ε, minimize the following cost functional*

$$J^{(1)}_\varepsilon(\theta_\varepsilon) = \frac{1}{2}\int_{\Omega_0}(u_\varepsilon - q_0)^2 \mathrm{d}x \,+\, \frac{N_1}{2}\int_{\Gamma^{(1)}_\varepsilon}(\theta_\varepsilon-\eta_1)^2\,\mathrm{d}x' \,+\, \mathbb{E}_\varepsilon(u_\varepsilon;\alpha,\beta), \tag{4.10}$$

where u_ε is the unique weak solution to problem (0.4) *with the following boundary conditions on the bases of the thin cylinders:*

$$u_\varepsilon(x',-d_1) = \theta_\varepsilon, \quad (x',-d_1)\in\Gamma^{(1)}_\varepsilon; \qquad u_\varepsilon(x',-d_2) = 0, \quad (x',-d_2)\in\Gamma^{(2)}_\varepsilon. \tag{4.11}$$

This new problem we denote by $\mathsf{CP}^{(1)}_\varepsilon$.

Obviously, we can repeat word-for-word the proofs of results from Section 2 and Section 3.1 for problem $\mathsf{CP}^{(1)}_\varepsilon$. In this case we should neglect all things connected to $\mathcal{K}^{(2)}_\varepsilon$. Next, due to the uniform Dirichlet conditions on $\Gamma^{(2)}_\varepsilon$ we obtain the following *a priori* estimate:

$$\begin{aligned}\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C_1\Big\{&\|f_0\|_{L^2(\Omega_0)} + \varepsilon^{\alpha-1}|\varrho_1(0)| + \|\theta_\varepsilon\|_{H^1(\Gamma^{(1)}_\varepsilon)} \\ &+ \varepsilon^{\alpha-1}\big(1+|\varrho_1(0)|\big)\Big(\varepsilon\|\nabla_{x'}\theta_\varepsilon\|_{L^2(\Gamma^{(1)}_\varepsilon)} + \|\theta_\varepsilon\|_{L^2(\Gamma^{(1)}_\varepsilon)}\Big)\Big\} \le C_2\end{aligned} \tag{4.12}$$

(see for comparison (1.6)).

4.2. Convergence theorem

Theorem 4.1 (the case $\alpha \geq 1$ and $\beta < 1$)**.** *Let* $\{\theta_\varepsilon\}_{\varepsilon>0}$ *be a sequence of admissible controls such that* $\theta_\varepsilon \overset{w}{\rightsquigarrow} \theta_0$ *as* $\varepsilon \to 0$ *(obviously* $\theta_0 \in \mathcal{K}_0^{(1)}$*). Then the corresponding sequence of the states* $\{u_\varepsilon\}_{\varepsilon>0}$ *satisfies the following relations*

$$\left.\begin{array}{llll} u_\varepsilon|_{\Omega_0} & \xrightarrow{w} & v_0^+ & \text{weakly in } H^1(\Omega_0,\Gamma_1), \\ \widetilde{u_\varepsilon}^{(1,k)} & \xrightarrow{w} & |B_k^{(1)}|\, v_0^{(1,k)} & \text{weakly in } W^{0,0,1}(D_1),\ k=1,\ldots,K_1, \\ \widetilde{\partial_{x_i} u_\varepsilon}^{(1,k)} & \xrightarrow{w} & 0 & \text{weakly in } L^2(D_2),\ i=1,2, \\ \widetilde{u_\varepsilon}^{(2,k)} & \xrightarrow{s} & 0 & \text{strongly in } L^2(D_2),\ k=1,\ldots,K_2, \end{array}\right\} \text{as } \varepsilon \to 0. \tag{4.13}$$

Here the function v_0^+ *is the unique weak solution to problem* (4.9) *and* $v_0^{(1,k)}$ *is the unique weak solution to the following problem*

$$\left\{ \begin{array}{rcl} -|B_k^{(1)}|\, \partial^2_{x_3} v_0^{(1,k)}(x) + \delta_{\alpha 1} l_k^{(1)}\, \varrho_1\big(v_0^{(1,k)}(x)\big) & = & 0, \quad x \in D_1, \\ v_0^{(1,k)}(x',0) = 0, \qquad v_0^{(1,k)}(x',-d_1) & = & \theta_0, \quad x' \in Q, \end{array}\right. \tag{4.14}$$

$k = 1,\ldots,K_1$, *which together with* (4.9) *are called the homogenized problem for problem* (0.4)*.*

Furthermore,

$$\lim_{\varepsilon\to 0} J_\varepsilon^{(1)}(\theta_\varepsilon) = \frac{1}{2}\int\limits_{\Omega_0} (v_0^+ - q_0)^2 \mathrm{d}x + \frac{N_3}{2}\int\limits_{\Omega_0} |\nabla v_0^+|^2 \mathrm{d}x + J_0^{(1)}(\theta_0), \tag{4.15}$$

where

$$J_0^{(1)}(\theta_0) = \frac{N_1}{2}|B^{(1)}| \int\limits_{\Gamma_{d_1}} (\theta_0 - \eta_1)^2\, \mathrm{d}x' + \frac{N_3}{2}\sum_{k=1}^{K_1}\Bigg(|B_k^{(1)}| \int\limits_{D_1} \big|\partial_{x_3} v_0^{(1,k)}\big|^2 \mathrm{d}x$$
$$+ \delta_{\alpha,1}\, l_k^{(1)} \int\limits_{D_1} \Big(\varrho_1\big(v_0^{(1,k)}(x)\big) - \varrho_1(0)\Big)\, v_0^{(1,k)}(x)\, \mathrm{d}x \Bigg). \tag{4.16}$$

Proof. Since the proof follows closely that of Theorem 3.3, we indicate only principal differences.

The state $u_\varepsilon \in H^1\big(\Omega_\varepsilon; \Upsilon_1 \cup \Gamma_\varepsilon^{(2)}\big) = \{u \in H^1(\Omega_\varepsilon) :\ u|_{\Upsilon_1 \cup \Gamma_\varepsilon^{(2)}} = 0\}$ satisfies the following integral identity

$$\int\limits_{\Omega_\varepsilon} \nabla_x u_\varepsilon \cdot \nabla_x \psi\, \mathrm{d}x + \varepsilon^\alpha \int\limits_{S_\varepsilon^{(1)}} \varrho_1\big(u_\varepsilon\big)\, \psi\, \mathrm{d}\sigma_x + \varepsilon^\beta \int\limits_{S_\varepsilon^{(2)}} \varrho_2\big(u_\varepsilon\big)\, \psi\, \mathrm{d}\sigma_x = \int\limits_{\Omega_0} f_0\, \psi\, \mathrm{d}x \tag{4.17}$$

for any function $\psi \in H^1\big(\Omega_\varepsilon; \Upsilon_1 \cup \Gamma_\varepsilon^{(1)} \cup \Gamma_\varepsilon^{(2)}\big)$. If we take in (4.17)

$$\psi(x) = \left\{ \begin{array}{ll} u_\varepsilon(x), & x \in \Omega_\varepsilon \cup G_\varepsilon^{(2)}, \\ \frac{1}{d_1}\,(x_3 + d_1)\, u_\varepsilon(x',0), & x \in G_\varepsilon^{(1)}, \end{array}\right.$$

and take (4.12) into account, we get similarly as before (see (4.5)) the following estimate

$$\int_{G_\varepsilon^{(2)}} |u_\varepsilon|^2\, \mathrm{d}x \leq C_1 \varepsilon^{1-\beta}. \tag{4.18}$$

From this it follows the last limit in (4.13).

Then, by the same way as in Section 4.1 we prove the first limit in (4.13), where the function v_0^+ is the unique weak solution to problem (4.9), and equalities (4.8) for functions v_0^+ and $v_0^{(1,k)}, k_1 = 1,\dots,K_1$, where the function $v_0^{(1,k)}$ is the limit of $\{\widetilde{u_\varepsilon}^{(1,k)}\}_{\varepsilon>0}$ under some subsequence of $\{\varepsilon\}$.

Now it remains to find the other relations which determine functions $v_0^{(1,k)}$, $k = 1,\dots,K_1$. Similarly as in the second item of the proof of Theorem 3.3, we prove that $v_0^{(1,k)}(x', -d_1) = \theta_0(x')$ for a.e. $x' \in Q$, $k = 1,\dots,K_1$.

In the fourth item of the proof of Theorem 3.3 we should take the subspace of $\mathcal{C}^\infty(\Omega_0, D_1, D_2)$, which consists of such multi-valued functions $\mathbf{\Phi} = \big(0, \varphi_1^{(1)},\dots,\varphi_{K_1}^{(1)}, 0,\dots,0,\big)$. Then repeating the arguments from this item, we obtain

$$\sum_{k=1}^{K_1} |B_k^{(1)}| \int_{D_1} \partial_{x_3} v_0^{(1,k)}\, \partial_{x_3}\varphi_k^{(1)}\,\mathrm{d}x + \delta_{\alpha,1} \sum_{k=1}^{K_1} \frac{l_k^{(1)}}{|B_k^{(1)}|} \int_{D_1} \zeta^{(1,k)}\,\varphi_k^{(1)}\,\mathrm{d}x = 0. \tag{4.19}$$

In the fifth item of the proof of Theorem 3.3 we have to take $q_\varepsilon^{(2)} \equiv 0$. Then we get

$$\lim_{\varepsilon\to 0}\Bigg(\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2\,\mathrm{d}x + \sum_{m=1}^{2} \varepsilon^{\alpha\delta_{1m}+\beta\delta_{2m}} \int_{S_\varepsilon^{(m)}} \varrho_m(u_\varepsilon)\,u_\varepsilon \mathrm{d}\sigma_x \Bigg)$$
$$= \int_{\Omega_0} |\nabla v_0^+|^2\,\mathrm{d}x + \sum_{k=1}^{K_1} |B_k^{(1)}| \int_{D_1} |\partial_{x_3} v_0^{(1,k)}|^2 \mathrm{d}x + \delta_{\alpha,1} \sum_{k=1}^{K_1} \frac{l_k^{(1)}}{|B_k^{(1)}|} \int_{D_1} \zeta^{(1,k)}\,v_0^{(1,k)}\,\mathrm{d}x. \tag{4.20}$$

To determine functions $\{\zeta^{(1,k)},\ k = 1,\dots,K_1\}$ in the case $\alpha = 1$, we consider the following inequality of monotonicity

$$\int_{\Omega_0} |\nabla u_\varepsilon - \nabla\varphi_0|^2\,\mathrm{d}x + \int_{G_\varepsilon^{(1)}} |\nabla_{x'} u_\varepsilon|^2\,\mathrm{d}x + \int_{G_\varepsilon^{(2)}} |\nabla u_\varepsilon|^2\,\mathrm{d}x + \sum_{k=1}^{K_1} \int_{G_\varepsilon^{(1)}(k)} |\partial_{x_3} u_\varepsilon - \partial_{x_3}\varphi_k^{(1)}|^2\,\mathrm{d}x$$
$$+\,\varepsilon \sum_{k=1}^{K_1} \int_{S_\varepsilon^{(1)}(k)} \big(\varrho_1(u_\varepsilon) - \varrho_1(\varphi_k^{(1)})\big)\,\big(u_\varepsilon - \varphi_k^{(1)}\big)\,\mathrm{d}\sigma_x + \varepsilon^\beta \int_{S_\varepsilon^{(2)}} \varrho_2(u_\varepsilon)\,u_\varepsilon\,\mathrm{d}\sigma_x \ge 0 \tag{4.21}$$

for any multi-valued function $\mathbf{\Phi} = \Big(\varphi_0, \varphi_1^{(1)},\dots,\varphi_{K_1}^{(1)}, 0,\dots,0\Big)$ such that $\varphi_0 \in H^1(\Omega_0, \Upsilon_1)$, $\varphi_k^{(1)} \in W^{0,0,1}(D_1)$, $\varphi_0(x',0) = \varphi_k^{(1)}(x',0) = 0$ for a.e. $x' \in Q$ and $k = 1,\dots,K_1$. Using (4.20), we pass to the limit in (4.21) similarly as the sixth item of the proof of Theorem 3.3. As a result we obtain

$$\int_{\Omega_0} |\nabla v_0^+ - \nabla\varphi_0|^2\,\mathrm{d}x + \sum_{k=1}^{K_1} |B_k^{(1)}| \int_{D_1} \big(\partial_{x_3} v_0^{(1,k)} - \partial_{x_3}\varphi_k^{(1)}\big)^2\,\mathrm{d}x$$
$$+ \sum_{k=1}^{K_1} \frac{l_k^{(1)}}{|B_k^{(1)}|} \int_{D_1} \big(\zeta^{(1,k)}(x) - |B_k^{(1)}|\,\varrho_1(\varphi_k^{(1)})\big)\,\big(v_0^{(1,k)} - \varphi_k^{(1)}\big)\,\mathrm{d}x \ge 0. \tag{4.22}$$

Now we take arbitrary multi-valued function $\mathbf{\Psi} = \big(\psi_0, \psi_1^{(1)},\dots,\psi_{K_1}^{(1)}, 0,\dots,0\big)$ such that $\psi_0 \in C^\infty(\overline{\Omega_0})$, $\psi_0|_{\Upsilon_1} = 0$, $\psi_k^{(1)} \in C^\infty(\overline{D_1})$, $\psi_0(x',0) = \psi_k^{(1)}(x',0) = 0$ for $x' \in Q$ and $k = 1,\dots,K_1$. Then substituting the following multi-valued function $\mathbf{\Phi}_0 := \mathbf{v}_0 - \lambda\mathbf{\Psi}$ instead of $\mathbf{\Phi}$ in (4.22), where $\mathbf{v}_0 = \big(v_0^+, v_0^{(1,1)},\dots,v_0^{(1,K_1)}, 0,\dots,0\big)$

and passing to the limit as $\lambda \to +0$, we get

$$\sum_{k=1}^{K_1} \frac{l_k^{(1)}}{|B_k^{(1)}|} \int_{D_1} \left(\zeta^{(1,k)}(x) - |B_k^{(1)}|\, \varrho_1\big(v_0^{(1,k)}\big) \right) \psi_k^{(1)} \, \mathrm{d}x \geq 0.$$

From this inequality it follows that

$$\zeta^{(1,k)}(x) = |B_k^{(1)}|\, \varrho_1\big(v_0^{(1,k)}(x)\big) \quad \text{for a.e. } x \in D_1, \quad k = 1, \dots, K_1. \tag{4.23}$$

The identity (4.19) and (4.23) mean that $v_0^{(1,k)}$ is the unique weak solution to problem (4.14), $k = 1, \dots, K_1$. Thus the limit in the second line of (4.13) also holds for the whole sequence $\{\varepsilon\}$.

By analogy to the proof of the limit (3.8), we deduce (4.15) with regard to the results obtained above. □

As before (see Prop. 3.4) the following statement ensues.

Proposition 4.2. *For any control $\theta_0 \in \mathcal{K}_0^{(1)}$ there exists a sequence of admissible controls $\left\{\theta_\varepsilon \in \mathcal{K}_\varepsilon^{(1)}\right\}_{\varepsilon>0}$, such that $\theta_\varepsilon \overset{w}{\rightsquigarrow} \theta_0$ as $\varepsilon \to 0$; the sequence of the corresponding solutions $\{u_\varepsilon(x, \theta_\varepsilon),\ x \in \Omega_\varepsilon\}_{\varepsilon>0}$ to problem (0.4) satisfies relations (4.13), where the functions v_0^+ and $v_0^{(1,k)}$, $k = 1, \dots, K_1$, are, respectively, the solutions to problems (4.9) and (4.14);*

$$\lim_{\varepsilon \to 0} J_\varepsilon^{(1)}(\theta_\varepsilon) = \frac{1}{2} \int_{\Omega_0} |v_0^+ - q_0|^2 \mathrm{d}x + \frac{N_3}{2} \int_{\Omega_0} |\nabla v_0^+|^2 \mathrm{d}x + J_0^{(1)}(\theta_0),$$

where $J_0^{(1)}(\theta_0)$ is defined in (4.16).

4.3. Homogenized optimal control problem $\mathsf{CP}_0^{(1)}$

Since we cannot control the state in the junction body Ω_0, we define the following homogenized optimal control problem $\mathsf{CP}_0^{(1)}$:

- *Find an optimal control $\theta^* \in \mathcal{K}_0^{(1)}$ and the corresponding solutions $v_*^{(1,k)}$, $k = 1, \dots, K_1$, of problems (4.14) to minimize the cost functional $J_0^{(1)}$,* i.e.,

$$J_0^{(1)}(\theta^*) = \inf_{\theta \in \mathcal{K}_0^{(1)}} J_0^{(1)}(\theta). \tag{4.24}$$

Here the control set $\mathcal{K}_0^{(1)}$ is defined in (2.1) and the cost functional $J_0^{(1)}$ is defined in (4.16).

Since $\mathcal{K}_0^{(1)}$ is closed with respect to the weak topology in $H^1(\Gamma_{d_1})$ and the cost functional $J_0^{(1)}$ is weakly lower-semicontinuous, we can prove by standard way (see for instance [15]) the existence of minimizer for problem $\mathsf{CP}_0^{(1)}$. In the case if $\alpha > 1$ the cost functional J_0 is strictly convex and therefore the problem $\mathsf{CP}_0^{(1)}$ has the unique solution.

Next reasoning as in the proof of Theorem 3.5, we can prove the following statement.

Theorem 4.3. **(1)** *If $\alpha > 1$, then for each sequence of the optimal controls $\left\{\theta_\varepsilon^* \in \mathcal{K}_\varepsilon^{(1)}\right\}_{\varepsilon>0}$ of problem $\mathsf{CP}_\varepsilon^{(1)}$ and for the sequence of the corresponding solutions $\{u_\varepsilon^* := u_\varepsilon(\theta_\varepsilon^*)\}_{\varepsilon>0}$ to problem (0.4) with the boundary* *conditions* (4.11) *the following limits hold (as* $\varepsilon \to 0$*)*

$$\theta^*_\varepsilon \overset{w}{\rightsquigarrow} \theta^*, \tag{4.25}$$

$$\left.\begin{array}{lcll} u^*_\varepsilon|_{\Omega_0} & \xrightarrow{w} & v_0^+ & \text{weakly in } H^1(\Omega_0,\Upsilon_1), \\ \widetilde{u^*_\varepsilon}^{(1,k)} & \xrightarrow{w} & |B_k^{(1)}|\, v_*^{(1,k)} & \text{weakly in } W^{0,0,1}(D_1),\ k=1,\ldots,K_1, \\ \widetilde{\partial_{x_i} u^*_\varepsilon}^{(1,k)} & \xrightarrow{w} & 0 & \text{weakly in } L^2(D_1),\ i=1,2,\quad k=1,\ldots,K_1, \\ \widetilde{u^*_\varepsilon}^{(2,k)} & \xrightarrow{s} & 0 & \text{strongly in } L^2(D_2),\ k=1,\ldots,K_2, \end{array}\right\} \tag{4.26}$$

$$J_\varepsilon^{(1)}\left(\theta^*_\varepsilon\right) \to \frac{1}{2}\int_{\Omega_0} (v_0^+ - q_0)^2 \mathrm{d}x + \frac{N_3}{2}\int_{\Omega_0} |\nabla v_0^+|^2 \mathrm{d}x + J_0^{(1)}\left(\theta^*\right), \tag{4.27}$$

where θ^* *is the unique solution to problem* $\mathsf{CP}_0^{(1)}$, v_0^+ *is the solution to problem* (4.9), *and* $v_*^{(1,k)}$ *is the solution to problem* (4.14) *such that* $v_*^{(1,k)}|_{\Gamma_{d_1}} = \theta^*, k=1,\ldots,K_1$.

(2) *If* $\alpha = 1$, *then for any sequence of the minimizers* $\left\{\theta^*_\varepsilon \in \mathcal{K}_\varepsilon^{(1)}\right\}_{\varepsilon>0}$ *of problem* $\mathsf{CP}_\varepsilon^{(1)}$ *there exists a subsequence* $\left\{\theta^*_{\varepsilon_n} \in \mathcal{K}_{\varepsilon_n}^{(1)}\right\}$ *such that the limits* (4.1) *and* (4.27) *hold as* $\varepsilon_n \to 0$ $(n \to +\infty)$ *and* θ^* *is a minimizer of problem* $\mathsf{CP}_0^{(1)}$; *for the sequence of the corresponding states* $\{u^*_\varepsilon := u_{\varepsilon_n}(\theta^*_\varepsilon)\}$ *we have*

$$\left.\begin{array}{lcll} u^*_\varepsilon|_{\Omega_0} & \xrightarrow{w} & v_0^+ & \text{weakly in } H^1(\Omega_0;\Upsilon_1), \\ \widetilde{u^*_\varepsilon}^{(2,k)} & \xrightarrow{s} & 0 & \text{strongly in } L^2(D_2),\ k=1,\ldots,K_2, \end{array}\right\} \quad \text{as} \ \ \varepsilon \to 0,$$

where v_0^+ *is the solution to problem* (4.9), *and*

$$\left.\begin{array}{lcll} \widetilde{u^*_{\varepsilon_n}}^{(1,k)} & \xrightarrow{w} & |B_k^{(1)}|\, v_*^{(1,k)} & \text{weakly in } W^{0,0,1}(D_1),\ k=1,\ldots,K_1, \\ \widetilde{\partial_{x_i} u^*_{\varepsilon_n}}^{(1,k)} & \xrightarrow{w} & 0 & \text{weakly in } L^2(D_1),\ i=1,2,\quad k=1,\ldots,K_1, \end{array}\right\} \quad \text{as} \ \ \varepsilon_n \to 0,$$

where $v_*^{(1,k)}$ *is the solution to problem* (4.14) *such that* $v_*^{(1,k)}|_{\Gamma_{d_1}} = \theta^*,\ k=1,\ldots,K_1$.

5. The case $\alpha < 1$ and $\beta < 1$

In this section we additionally assume that $\varrho_1(0) = \varrho_2(0) = 0$. In a way analogous to that made in Section 4.1 we can prove the following statement.

Theorem 5.1. *For the sequence of the optimal control pairs*

$$\left\{(\theta^*_\varepsilon, \vartheta^*_\varepsilon) \in \mathcal{K}_\varepsilon^{(1)} \times \mathcal{K}_\varepsilon^{(2)}\right\}_{\varepsilon>0}$$

of problem CP_ε *and for the sequence of the corresponding solutions* $\{u^*_\varepsilon := u_\varepsilon(\theta^*_\varepsilon, \vartheta^*_\varepsilon)\}_{\varepsilon>0}$ *to problem* (0.4) *the following limits hold (as* $\varepsilon \to 0$*)*

$$(\theta^*_\varepsilon, \vartheta^*_\varepsilon) \overset{w}{\rightsquigarrow} (0,0),$$

$$\left.\begin{array}{lcll} u^*_\varepsilon|_{\Omega_0} & \xrightarrow{w} & v_0^+ & \text{weakly in } H^1(\Omega_0;\Upsilon_1), \\ \widetilde{u^*_\varepsilon}^{(m,k)} & \xrightarrow{s} & 0 & \text{strongly in } L^2(D_m), \end{array}\right\} \tag{5.1}$$

for $k = 1, \dots, K_m$, $m = 1, 2$, *where* v_0^+ *is the unique solution to problem*

$$\begin{cases} -\Delta_x \, v_0^+ = f_0 \quad in\, \Omega_0, \\ v_0^+ = 0 \quad on\, \Upsilon_1 \cup Q, \qquad \partial_\nu v_0^+ = 0 \quad on\, \partial\Omega_0 \setminus \big(\Upsilon_1 \cup Q\big). \end{cases} \tag{5.2}$$

The statement of this theorem means that we cannot control the state of all system Ω_ε if $\alpha < 1$ and $\beta < 1$. In fact, even setting the optimal control problem makes no sense in this case for ε small enough. Nevertheless, similarly as was done in the proof of Theorems 3.3 and 4.1, we can prove the following theorem.

Theorem 5.2. *If* u_ε *is the weak solution to problem* (0.4) *with the uniform boundary conditions on the bases of the thin cylinders:* $u_\varepsilon(x', -d_1) = 0$, $(x', -d_1) \in \Gamma_\varepsilon^{(1)}$, *and* $u_\varepsilon(x', -d_2) = 0$, $(x', -d_2) \in \Gamma_\varepsilon^{(2)}$, *then the limits* (5.1) *hold and in addition*

$$\mathbb{E}_\varepsilon(u_\varepsilon; \alpha, \beta) \longrightarrow \frac{N_3}{2} \int_{\Omega_0} |\nabla v_0^+|^2 \, \mathrm{d}x \quad as \ \ \varepsilon \to 0,$$

where v_0^+ *is the unique solution to problem* (5.2).

6. Conclusions

1. In fact, the statements of Theorem 3.3 and Proposition 3.4 are sufficiently to apply the Buttazzo–Dal Maso abstract scheme based on the Γ-sequential convergence of functionals (see [5]) in the case $\alpha \geq 1$ and $\beta \geq 1$. The first statement of Theorem 3.3 and the first two statements of Proposition 3.4 mean the Kuratowski convergence of the solution sets (see Prop. 4.4 in [9]). This is the first condition in this scheme, which is equivalent to the Γ-convergence of the corresponding indicator functions.

The second statement of Theorem 3.3 and the first and third statements of Proposition 3.4 are equivalent to the Γ-convergence of the cost functionals (the second condition in the Buttazzo–Dal Maso scheme, see Prop. 4.2 and Rem. 4.2 in [9]). Applying this scheme to problems CP_ε and CP_0, we can directly obtain Theorem 3.5. Nevertheless, in the paper we proved this theorem independently.

2. It is evident from the results we have presented that the boundary conditions have a substantial influence on the asymptotic behavior of problem CP_ε (there are three qualitatively different cases).

At first glance it may seem that there is no difference between the boundary conditions

$$-\partial_\nu u_\varepsilon = \varepsilon^\alpha \varrho_1\big(u_\varepsilon\big) \quad \text{on}\, S_\varepsilon^{(1)}, \qquad -\partial_\nu u_\varepsilon = \varepsilon^\beta \varrho_2\big(u_\varepsilon\big) \quad \text{on}\, S_\varepsilon^{(2)}$$

and the homogeneous Neumann condition if $\alpha \geq 1$ and $\beta \geq 1$, since the terms $\varrho_m(u_\varepsilon)$, $m = 1, 2$, are multiplied by ε^α and ε^β respectively. It appears that this is true only for $\alpha > 1$ and $\beta > 1$. But if $\alpha = 1$ (or $\beta = 1$), then the term $\varepsilon\varrho_m(u_\varepsilon)$ from the boundary conditions transforms into the new blow-up terms $l_k^{(m)} \varrho_m\big(v_0^{(m,k)}\big)$, $k = 1, \dots, K_m$, in the homogenized equations in D_m, $m = 1, 2$. A similar phenomenon is observed in [19] for a boundary-value problem in a thick one-level junction.

Thus in this case ($\alpha \geq 1$ and $\beta \geq 1$) we have deduced the well-posed homogenized optimal control problem CP_0 for problem CP_ε. On the other hand, problem CP_0 is untypical since the corresponding state is the multi-valued solution to the nonstandard quasilinear boundary-value problem (3.7). It should be noted that problem CP_0 becomes linear if $\alpha > 1$ and $\beta > 1$.

Here we try to give some physical justification of a new qualitative property of problem (3.7). As a consequence of the difference of the local properties of conductivity on the lateral surfaces $S_\varepsilon^{(m)}(k)$ of the thin cylinders $G_\varepsilon^{(m)}(k)$, the flows of quantity (heat conductivity or any other physical entity) are different in both of the cylinder set $G_\varepsilon^{(m)}(k)$ ($k = 1, \dots, K_m, m = 1, 2$). But these set are connected through the junction's body Ω_0 and alternated along the joint zone Q. As a result, the global flow described by the multi-valued function $\mathbf{V_0}$ (see (3.6)) behaves as a *many-phase system* in the region which is filled up by the thin cylinder from each level in the limit passage as the parameter $\varepsilon \to 0$.

3. In the second case ($\alpha \geq 1$ and $\beta < 1$), interactions between the lateral surfaces $S_\varepsilon^{(2)}(k)$ of the thin cylinders $G_\varepsilon^{(2)}(k)$ and the medium plays a dominant role in the asymptotic behavior of all problem CP_ε. Note that this interaction is not necessarily too large locally for $\beta \in (0,1)$. However, such an effect takes place because of the total surface area of $S_\varepsilon^{(2)}(k),\ k = 1,\ldots,K_2$. As a result, we cannot control the state in the junction's body through the cylinder set $G_\varepsilon^{(2)} = \bigcup_{k=1}^{K_2} G_\varepsilon^{(2)}(k)$. Since the cylinders $G_\varepsilon^{(1)}$ and $G_\varepsilon^{(2)}$ are ε-periodically alternated, we cannot even control the state in the junction's body and the energy of whole system through the cylinders $G_\varepsilon^{(1)}$. Thus the optimal control problem CP_ε is degenerated as $\varepsilon \to 0$ into the homogenized optimal problem $\mathsf{CP}_0^{(1)}$.
4. In the third case ($\alpha < 1$ and $\beta < 1$) because of the reasons mentioned above, even the formulation of the optimal control problem becomes meaningless for ε small enough. Therefore we proved only the convergence results both for the solutions of the corresponding boundary-value problem and for the energy integrals.

6.1. **Application**

Here we present the application of our results for an optimal control problem involving a thick one-level junction with cascade controls.

Let $d_1 = d_2,\ \varrho_1 \equiv \varrho_2,\ \alpha = \beta \geq 1$ and $K_1 = K_2 = 1$. In addition, we assume that $B^{(1)}$ is congruent to $B^{(2)}$; this means that $|B^{(1)}| = |B^{(2)}|$ and $l^{(1)} := l_1^{(1)} = l_1^{(2)}$. Then the corresponding optimal control problem CP_ε involving the thick one-level junction Ω_ε in which $\Gamma_\varepsilon^{(1)} \cup \Gamma_\varepsilon^{(2)} \subset \Gamma_{d_1} = \Gamma_{d_2} = \{x :\ x' \in Q,\ x_3 = -d_1\}$ and admissible controls are taken from the different classes $\mathcal{K}_\varepsilon^{(1)}$ and $\mathcal{K}_\varepsilon^{(2)}$ that are ε-periodically alternated (so-called *cascade controls*).

Nevertheless, the homogenized optimal control problem for problem CP_ε is the corresponding problem CP_0 involving the multi-valued state

$$\mathbf{V}_0(x) = \begin{cases} v_0^+(x), & x \in \Omega_0, \\ v_0^{(m)}(x), & x \in D_1 = D_2, \quad m = 1,2, \end{cases} \tag{6.1}$$

that is the weak solution to the following problem

$$\begin{array}{rcll} -\Delta v_0^+(x) &=& f_0(x), & x \in \Omega_0, \\ v_0^+(x) &=& 0, & x \in \Upsilon_1, \\ \partial_\nu v_0^+(x) &=& 0, & x \in \partial\Omega_0 \setminus (Q \cup \Upsilon_1); \\ -|B^{(1)}|\,\partial_{x_3}^2 v_0^{(m)}(x) + \delta_{\alpha,1}\, l^{(1)} \varrho_1\big(v_0^{(m)}(x)\big) &=& 0, & x \in D_1,\ m = 1,2, \\ v_0^{(1)}(x', -d_1) &=& \theta_0, & x' \in Q, \\ v_0^{(2)}(x', -d_1) &=& \vartheta_0, & x' \in Q, \\ v_0^{(m)}(x', 0) &=& v_0^+(x', 0), & x' \in Q,\ m = 1,2, \\ |B^{(1)}| \sum_{m=1}^{2} \partial_{x_3} v_0^{(m)}(x', 0) &=& \partial_{x_3} v_0^+(x', 0), & x' \in Q. \end{array} \tag{6.2}$$

5 From this result it follows that we cannot take in problem CP_ε the Dirichlet controls bounded in L^2. To argue this statement we assume that $\alpha > 1$ and $\eta_1 \equiv \eta_2 \equiv 0$ for simplicity and consider problem CP_ε in the thick one-level junction with the Dirichlet controls from a set

$$\mathcal{L}_\varepsilon := \{\theta \in H^{\delta_0}(\Gamma_\varepsilon^{(1)} \cup \Gamma_\varepsilon^{(2)}) : \quad \|\theta\|_{L^2(\Gamma_\varepsilon^{(1)} \cup \Gamma_\varepsilon^{(2)})} \leq \mathbf{C_0}\} \qquad (\delta_0 = \delta_1 = \delta_2 > 1).$$

Let us show that this optimal control problem cannot have any reasonable limit as $\varepsilon \to 0$.

Indeed, the homogenized optimal control problem for problem CP_ε with the Dirichlet controls from the set

$$\mathcal{A}_\varepsilon^{(0)} := \{\theta|_{\Gamma_\varepsilon^{(1)} \cup \Gamma_\varepsilon^{(2)}} :\ \theta \in H^{\delta_0}(\Gamma_{d_1}), \quad \|\theta\|_{H^{\delta_0}(\Gamma_{d_1})} \leq \mathbf{C_0}\}$$

(in this case $\mathcal{A}_\varepsilon^{(0)} \subset \mathcal{L}_\varepsilon$) is as follows:

- *Find an optimal control* $\theta \in \mathcal{A}_0^{(0)} = \{\theta \in H^{\delta_0}(\Gamma_{d_1}) : \ \|\theta\|_{H^{\delta_0}(\Gamma_{d_1})} \le \mathbf{C_0}\}$ *and the corresponding solution to the problem*

$$\begin{array}{rcll} -\Delta v_0^+(x) &=& f_0(x), & x \in \Omega_0, \\ v_0^+(x) &=& 0, & x \in \Upsilon_1, \\ \partial_\nu v_0^+(x) &=& 0, & x \in \partial\Omega_0 \setminus (Q \cup \Upsilon_1); \\ -|B^{(1)}|\,\partial_{x_3}^2 v_0^-(x) &=& 0, & x \in D_1, \\ v_0^-(x', -d_1) &=& \theta_0, & x' \in Q, \\ v_0^-(x', 0) &=& v_0^+(x', 0), & x' \in Q, \\ 2|B^{(1)}|\partial_{x_3} v_0^-(x', 0) &=& \partial_{x_3} v_0^+(x', 0), & x' \in Q, \end{array}$$

to minimize the cost functional

$$J_0(\theta_0) = \frac{1}{2}\int\limits_{\Omega_0} (v_0^+ - q_0)^2 \mathrm{d}x + \frac{N_1 + N_2}{2}|B^{(1)}| \int\limits_{\Gamma_{d_1}} \theta_0^2 \,\mathrm{d}x' + \frac{N_3}{2}\left(\int\limits_{\Omega_0} |\nabla v_0^+|^2 \mathrm{d}x + 2|B^{(1)}| \int\limits_{D_1} \left|\partial_{x_3} v_0^-\right|^2 \mathrm{d}x \right).$$

On the other hand, the homogenized optimal control problem for problem CP_ε but with the following Dirichlet controls on $\Gamma_\varepsilon^{(1)}$ and $\Gamma_\varepsilon^{(2)}$:

$$\mathcal{A}_\varepsilon^{(1)} := \left\{ \theta|_{\Gamma_\varepsilon^{(1)}} : \ \theta \in H^{\delta_0}(\Gamma_{d_1}), \quad \|\theta\|_{H^{\delta_0}(\Gamma_{d_1})} \le C_1, \quad \|\theta\|_{L^2(\Gamma_{d_1})} \le \mathbf{C_0}/\sqrt{2} \right\}$$
$$\mathcal{A}_\varepsilon^{(2)} := \left\{ \vartheta|_{\Gamma_\varepsilon^{(2)}} : \ \vartheta \in H^{\delta_0}(\Gamma_{d_1}), \quad C_1 < C_2 \le \|\vartheta\|_{H^{\delta_0}(\Gamma_{d_1})} \le C_3, \quad \|\vartheta\|_{L^2(\Gamma_{d_1})} \le \mathbf{C_0}/\sqrt{2} \right\},$$

where C_1, C_2, C_3 are some fixed constants (in this case also $\mathcal{A}_\varepsilon^{(1)} \cup \mathcal{A}_\varepsilon^{(2)} \subset \mathcal{L}_\varepsilon$) is as follows:

- *Find*

$$\inf_{(\theta,\vartheta) \in \mathcal{A}_0^{(1)} \times \mathcal{A}_0^{(2)}} J_0(\theta, \vartheta),$$

where

$$\mathcal{A}_0^{(1)} = \left\{ \theta \in H^{\delta_0}(\Gamma_{d_1}) : \ \|\theta\|_{H^{\delta_0}(\Gamma_{d_1})} \le C_1, \quad \|\theta\|_{L^2(\Gamma_{d_1})} \le \mathbf{C_0}/\sqrt{2} \right\}$$
$$\mathcal{A}_0^{(2)} = \left\{ \vartheta \in H^{\delta_0}(\Gamma_{d_1}) : \ C_1 < C_2 \le \|\vartheta\|_{H^{\delta_0}(\Gamma_{d_1})} \le C_3, \quad \|\vartheta\|_{L^2(\Gamma_{d_1})} \le \mathbf{C_0}/\sqrt{2} \right\},$$

the corresponding cost functional $J_0(\theta, \vartheta)$ *is defined in* (3.8) *and the multi-valued function* $\mathbf{V}_0$ *defined in* (6.1) *is the solution to problem* (6.2).

Thus, for the different admissible control sets, which are subset of $\mathcal{L}_\varepsilon$, we have obtained two different homogenized optimal control problems.

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