ESAIM: COCV 17 (2011) 801–835
DOI: 10.1051/cocv/2010026

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

LOGARITHMIC DECAY OF THE ENERGY FOR AN HYPERBOLIC-PARABOLIC COUPLED SYSTEM

Ines Kamoun Fathallah[1]

Abstract. This paper is devoted to the study of a coupled system which consists of a wave equation and a heat equation coupled through a transmission condition along a steady interface. This system is a linearized model for fluid-structure interaction introduced by Rauch, Zhang and Zuazua for a simple transmission condition and by Zhang and Zuazua for a natural transmission condition. Using an abstract theorem of Burq and a new Carleman estimate proved near the interface, we complete the results obtained by Zhang and Zuazua and by Duyckaerts. We prove, without a Geometric Control Condition, a logarithmic decay of the energy.

Mathematics Subject Classification. 37L15, 35B37, 74F10, 93D20.

Received December 24, 2008. Revised November 14, 2009 and March 31, 2010.
Published online August 6, 2010.

1. Introduction and results

In this work, we are interested in a linearized model for fluid-structure interaction introduced by Zhang and Zuazua in [15] and Duyckaerts in [6]. This model consists of a wave equation and a heat equation coupled through an interface by suitable transmission conditions. Our purpose is to analyze the stability of this system and therefore to determine the decay rate of the energy of solutions as $t \to \infty$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary $\Gamma = \partial\Omega$. Let Ω_1 and Ω_2 be two bounded open sets with smooth boundary such that $\Omega_1 \subset \Omega$ and $\Omega_2 = \Omega \backslash \overline{\Omega}_1$. We denote by $\gamma = \partial\Omega_1 \cap \partial\Omega_2$ the interface, $\gamma \subset\subset \Omega$, $\Gamma_j = \partial\Omega_j \backslash \gamma$, $j = 1, 2$, and we suppose that $\Gamma_2 \neq \emptyset$. Let ∂_n and $\partial_{n'}$ the unit outward normal vectors of Ω_1 and Ω_2 respectively. We recall that $\partial_{n'} = -\partial_n$ on γ.

$$\begin{cases} \partial_t u - \triangle u = 0 & \text{in } (0,\infty) \times \Omega_1, \\ \partial_t^2 v - \triangle v = 0 & \text{in } (0,\infty) \times \Omega_2, \\ u = 0 & \text{on } (0,\infty) \times \Gamma_1, \\ v = 0 & \text{on } (0,\infty) \times \Gamma_2, \\ u = \partial_t v, \quad \partial_n u = -\partial_{n'} v & \text{on } (0,\infty) \times \gamma, \\ u|_{t=0} = u_0 \in L^2(\Omega_1) & \text{in } \Omega_1, \\ v|_{t=0} = v_0 \in H^1(\Omega_2), \quad \partial_t v|_{t=0} = v_1 \in L^2(\Omega_2) & \text{in } \Omega_2. \end{cases} \tag{1.1}$$

Keywords and phrases. Fluid-structure interaction, wave-heat model, stability, logarithmic decay.

[1] Laboratoire LMV, Université de Versailles Saint-Quentin-en-Yvelines, 45 Avenue des États-Unis, Bâtiment Fermat, 78035 Versailles, France. ines.fathallah@math.uvsq.fr

Article published by EDP Sciences
© EDP Sciences, SMAI 2010

In this system, u may be viewed as the velocity of fluid; while v and $\partial_t v$ represent respectively the displacement and velocity of the structure. That's why the transmission condition $u = \partial_t v$ is considered as the natural condition. For the discussion of this model, we refer to [12,15].

System (1.1) is introduced by Zhang and Zuazua [15]. The same system is considered by Rauch *et al.* in [12] but for the simplified transmission condition $u = v$ on the interface instead of $u = \partial_t v$. They prove, under a suitable Geometric Control Condition (GCC) (see [1]), a polynomial decay result. Zhang and Zuazua in [15] prove, without the GCC, a logarithmic decay result. Duyckaerts in [6] improves these results.

For system (1.1), Zhang and Zuazua in [15], prove the lack of uniform decay and, under the GCC, a polynomial decay result. Without geometric conditions, they analyze the difficulty to prove the logarithmic decay result. This difficulty is mainly due to the lack of regularity gain of the wave component v near the interface γ (see [15], Rem. 19) which means that the embedding of the domain $D(\mathcal{A})$ of the dissipative operator in the energy space is not compact (see [15], Thm. 1). In [6], Duyckaerts improves the polynomial decay result under the GCC and confirms the same obstacle to proving the logarithmic decay for solution of (1.1) without the GCC. In this paper we are interested in this problem.

There is an extensive literature on the stabilization of PDEs and on the Logarithmic decay of the energy ([2–4,7,9,11,13] and the references cited therein) and this paper uses part of the idea developed in [3].

Here we recall the mathematical framework for this problem (see [15]).

Define the energy space H and the operator $\mathcal{A}$ on H with domain $D(\mathcal{A})$ by

$$H = \left\{ F = (f_1, f_2, f_3) \in L^2(\Omega_1) \times H^1_{\Gamma_2}(\Omega_2) \times L^2(\Omega_2) \right\}$$

where $H^1_{\Gamma_2}(\Omega_2)$ is defined as the space

$$\begin{aligned} H^1_{\Gamma_2}(\Omega_2) &= \left\{ f \in H^1(\Omega_2),\ f|_{\Gamma_2} = 0 \right\}, \\ \mathcal{A}F &= (\triangle f_1, f_3, \triangle f_2) \\ D(\mathcal{A}) &= \{ F \in H,\ f_1 \in H^1(\Omega_1),\ \triangle f_1 \in L^2(\Omega_1), \\ f_3 \in H^1_{\Gamma_2}(\Omega_2),\ &\triangle f_2 \in L^2(\Omega_2),\ f_1|_\gamma = f_3|_\gamma,\ \partial_n f_1|_\gamma = -\partial_n f_2|_\gamma \}. \end{aligned}$$

Thus system (1.1) may be rewritten as an abstract Cauchy problem in H as

$$\begin{cases} \partial_t U(t) = \mathcal{A}U(t), \quad t > 0, \\ U(0) = U_0, \end{cases} \tag{1.2}$$

where $U(t) = (u(t), v(t), \partial_t v(t))$ and $U_0 = (u_0, v_0, v_1)$.

The operator $\mathcal{A}$ is the generator of a strongly continuous semi-group (see [15], Thm. 1).

In our case, *i.e.* when $\Gamma_2 \neq \emptyset$, the energy of any solution $U = (u, v, \partial_t v)$ of (1.2) is defined as one half of the square of a norm on H and we have

$$E(U(t)) = \frac{1}{2} \left(\int_{\Omega_1} |u(t)|^2 \,\mathrm{d}x + \int_{\Omega_2} |\partial_t v(t)|^2 \,\mathrm{d}x + \int_{\Omega_2} |\nabla v(t)|^2 \,\mathrm{d}x \right).$$

When $\Gamma_2 = \emptyset$, we refer to [6,15].

By means of the classical energy method, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} E(U(t)) = -\int_{\Omega_1} |\nabla u|^2 \,\mathrm{d}x.$$

Therefore the energy of (1.2) is decreasing with respect to t, the dissipation coming from the heat component u. Our main goal is to prove a logarithmic decay without the GCC assumption.

As Duyckaerts [6] did for the simplified model, the idea is, first, to use a known result of Burq (see [5]) which links, for dissipative operators, logarithmic decay to resolvent estimates with exponential loss; secondly to prove, following the work of Bellassoued in [3], a new Carleman inequality near the interface γ.

The main results are the following Theorem 1.1 concerning the resolvent and Theorem 1.2 concerning the decay.

Theorem 1.1. *There exists $C > 0$, such that for every $\mu \in \mathbb{R}$, we have*

$$\left\|(\mathcal{A} - \mathrm{i}\mu)^{-1}\right\|_{\mathcal{L}(H)} \le C\mathrm{e}^{C|\mu|}. \tag{1.3}$$

Theorem 1.2. *There exists $C > 0$, such that for all $U_0 \in D(\mathcal{A})$, we have*

$$\sqrt{E(U(t))} \le \frac{C}{\log(t+2)} \left\|U_0\right\|_{D(\mathcal{A})}. \tag{1.4}$$

Remark 1.1. *To simplify, we assumed that $\Gamma_2 \neq \emptyset$. When Γ_2 is empty, the constant functions $(0, c, 0)$, where c is arbitrary, are solutions of system (1.2). Therefore it is necessary to consider the decay of solutions orthogonal to $(0, c, 0)$ in H (for more details we refer to Thm. 1 in [15]).*

Burq in [5], Theorem 3, and Duyckaerts in [6], Section 7, show that to prove Theorem 1.2 it suffices to prove Theorem 1.1.

The strategy of the proof of Theorem 1.1, when $\mu \neq 0$, is the following. A new Carleman estimate proved near the interface γ implies an interpolation inequality given by Theorem 2.2. Theorem 2.2 implies Theorem 2.1 which gives an estimate of the wave component by the heat one and which is the key point of the proof of Theorem 1.1.

The rest of this paper is organized as follows. In Section 2, we prove Theorem 1.1, for $\mu \neq 0$, from Theorem 2.1 and we explain how Theorem 2.2 implies Theorem 2.1. For $\mu = 0$, the proof of Theorem 1.1 is given in Appendix C. In Section 3, we begin by stating the new Carleman estimate and we explain how this estimate implies Theorem 2.2. Then we give the proof of this Carleman estimate. Section 4 is devoted to the proof of important estimates, stated in Theorem 3.2, in the proof of this Carleman estimate. Appendices A and B are devoted to prove some technical results used along the paper.

2. Proof of Theorem 1.1

For $\mu = 0$, the proof of Theorem 1.1 is given in Appendix C. For $\mu \neq 0$, we start by stating Theorem 2.1. Then we will explain how this theorem implies Theorem 1.1. Finally, we give the proof of Theorem 2.1.

Let $\mu_0 > 0$, small enough, for any μ such that $|\mu| \ge \mu_0$, we assume

$$F = (\mathcal{A} - \mathrm{i}\mu)U, \quad U = (u_0, v_0, v_1) \in D(\mathcal{A}), \qquad F = (f_0, g_0, g_1) \in H. \tag{2.1}$$

Equation (2.1) yields

$$\left\{\begin{array}{rcll} (\triangle - \mathrm{i}\mu)u_0 &=& f_0 & \text{in } \Omega_1, \\ (\triangle + \mu^2)v_0 &=& g_1 + \mathrm{i}\mu g_0 & \text{in } \Omega_2, \\ v_1 &=& g_0 + \mathrm{i}\mu v_0 & \text{in } \Omega_2, \end{array}\right. \tag{2.2}$$

with the following boundary conditions

$$\left\{\begin{array}{rcl} u_0|_{\Gamma_1} &=& 0, \qquad v_0|_{\Gamma_2} = 0 \\ u_0 - \mathrm{i}\mu v_0 &=& g_0|_\gamma, \\ \partial_n u_0 - \partial_n v_0 &=& 0|_\gamma. \end{array}\right. \tag{2.3}$$

Theorem 2.1. *Let $U = (u_0, v_0, v_1) \in D(\mathcal{A})$ satisfy equations (2.2) and (2.3). Then there exist constants $C > 0$, $c_1 > 0$ and $\mu_0 > 0$ such that for any $|\mu| \geq \mu_0$ we have the following estimate*

$$\|v_0\|^2_{H^1(\Omega_2)} \leq C e^{c_1|\mu|} \Big(\|f_0\|^2_{L^2(\Omega_1)} + \|g_1 + i\mu g_0\|^2_{L^2(\Omega_2)} + \|g_0\|^2_{H^1(\Omega_2)} + \|u_0\|^2_{H^1(\Omega_1)} \Big). \tag{2.4}$$

Moreover, from the first equation of system (2.2), we have

$$\int_{\Omega_1} (-\triangle + i\mu) u_0 \overline{u_0} \mathrm{d}x = \|\nabla u_0\|^2_{L^2(\Omega_1)} + i\mu \|u_0\|^2_{L^2(\Omega_1)} - \int_\gamma \partial_n u_0 \overline{u_0} \mathrm{d}\sigma.$$

Since $u_0|_\gamma = g_0 + i\mu v_0$ and $\partial_n u_0 = -\partial_{n'} v_0$, then

$$\int_{\Omega_1} (-\triangle + i\mu) u_0 \overline{u}_0 \mathrm{d}x = \|\nabla u_0\|^2_{L^2(\Omega_1)} + i\mu \|u_0\|^2_{L^2(\Omega_1)} - i\mu \int_\gamma \partial_{n'} v_0 \overline{v}_0 \mathrm{d}\sigma + \int_\gamma \partial_{n'} v_0 \overline{g}_0 \mathrm{d}\sigma. \tag{2.5}$$

From the second equation of system (2.2) and multiplying by $(-i\mu)$, we obtain

$$i\mu \int_{\Omega_2} (\triangle + \mu^2) v_0 \overline{v}_0 \mathrm{d}x = -i\mu \|\nabla v_0\|^2_{L^2(\Omega_2)} + i\mu^3 \|v_0\|^2_{L^2(\Omega_2)} + i\mu \int_\gamma \partial_{n'} v_0 \overline{v}_0 \mathrm{d}\sigma. \tag{2.6}$$

Adding (2.5) and (2.6), we obtain

$$\begin{aligned}\int_{\Omega_1} (-\triangle + i\mu) u_0 \overline{u}_0 \mathrm{d}x &+ i\mu \int_{\Omega_2} (\triangle + \mu^2) v_0 \overline{v}_0 \mathrm{d}x \\ &= i\mu \|u_0\|^2_{L^2(\Omega_1)} + \|\nabla u_0\|^2_{L^2(\Omega_1)} - i\mu \|\nabla v_0\|^2_{L^2(\Omega_2)} + i\mu^3 \|v_0\|^2_{L^2(\Omega_2)} + \int_\gamma \partial_{n'} v_0 \overline{g}_0 \mathrm{d}\sigma.\end{aligned}$$

Taking the real part of this expression, we get

$$\|\nabla u_0\|^2_{L^2(\Omega_1)} \leq \|(\triangle - i\mu) u_0\|_{L^2(\Omega_1)} \|u_0\|_{L^2(\Omega_1)} + \left\|(\triangle + \mu^2) v_0\right\|_{L^2(\Omega_2)} \|v_0\|_{L^2(\Omega_2)} + \left| \int_\gamma \partial_{n'} v_0 \overline{g}_0 \mathrm{d}\sigma \right|. \tag{2.7}$$

Recalling that $\triangle v_0 = g_1 + i\mu g_0 - \mu^2 v_0$ and using the trace lemma (Lem. 3.4 in [6]), we obtain

$$\|\partial_n v_0\|_{H^{-\frac{1}{2}}(\gamma)} \leq C \Big(\mu^2 \|v_0\|_{H^1(\Omega_2)} + \|g_1 + i\mu g_0\|_{L^2(\Omega_2)} \Big).$$

Combining with (2.7), we obtain

$$\begin{aligned}\|\nabla u_0\|^2_{L^2(\Omega_1)} \leq \|f_0\|_{L^2(\Omega_1)} \|u_0\|_{L^2(\Omega_1)} &+ \|g_1 + i\mu g_0\|_{L^2(\Omega_2)} \|v_0\|_{L^2(\Omega_2)} \\ &+ \Big(\mu^2 \|v_0\|_{H^1(\Omega_2)} + \|g_1 + i\mu g_0\|_{L^2(\Omega_2)} \Big) \|g_0\|_{H^{\frac{1}{2}}(\gamma)}.\end{aligned}$$

Then

$$\begin{aligned}\|\nabla u_0\|^2_{L^2(\Omega_1)} \leq \frac{C}{\epsilon} \|f_0\|^2_{L^2(\Omega_1)} + \epsilon \|u_0\|^2_{L^2(\Omega_1)} &+ \frac{C}{\epsilon} \|g_1 + i\mu g_0\|^2_{L^2(\Omega_2)} + \epsilon \|v_0\|^2_{L^2(\Omega_2)} \\ &+ \Big(\mu^2 \|v_0\|_{H^1(\Omega_2)} + \|g_1 + i\mu g_0\|_{L^2(\Omega_2)} \Big) \|g_0\|_{H^{\frac{1}{2}}(\gamma)}. \end{aligned} \tag{2.8}$$

Now we use this result proved in Appendix A.

Lemma 2.1. *Let $\mathcal{O}$ be a bounded open set of $\mathbb{R}^n$. Then for all $\mu_0 > 0$, there exists $C > 0$ such that for u and f satisfying $(\triangle - \mathrm{i}\mu)u = f$ in $\mathcal{O}$, $|\mu| \geq \mu_0$, we have the following estimate*

$$\|u\|_{H^1(\mathcal{O})} \leq C \left(\|\nabla u\|_{L^2(\mathcal{O})} + \|f\|_{L^2(\mathcal{O})}\right). \tag{2.9}$$

Using this lemma and (2.8), we obtain, for ϵ small enough

$$\begin{aligned}\|u_0\|^2_{H^1(\Omega_1)} \leq C\|f_0\|^2_{L^2(\Omega_1)} + C_\epsilon \|g_1 + \mathrm{i}\mu g_0\|^2_{L^2(\Omega_2)} + \epsilon \|v_0\|^2_{L^2(\Omega_2)} \\ + \left(\mu^2 \|v_0\|_{H^1(\Omega_2)} + \|g_1 + \mathrm{i}\mu g_0\|_{L^2(\Omega_2)}\right) \|g_0\|_{H^{\frac{1}{2}}(\gamma)}.\end{aligned}$$

Then there exists $c_3 \gg c_1$ such that

$$\|u_0\|^2_{H^1(\Omega_1)} \leq C \left(\|f_0\|^2_{L^2(\Omega_1)} + \epsilon \mathrm{e}^{-c_3|\mu|} \|v_0\|^2_{H^1(\Omega_2)} + C_\epsilon \mathrm{e}^{-c_3|\mu|} \|g_1 + \mathrm{i}\mu g_0\|^2_{L^2(\Omega_2)} + \mathrm{e}^{c_3|\mu|} \|g_0\|^2_{H^1(\Omega_2)}\right). \tag{2.10}$$

Plugging (2.10) in (2.4), we obtain, for ϵ small enough

$$\|v_0\|^2_{H^1(\Omega_2)} \leq C\mathrm{e}^{c|\mu|} \left(\|f_0\|^2_{L^2(\Omega_1)} + \|g_0\|^2_{H^1(\Omega_2)} + \|g_1 + \mathrm{i}\mu g_0\|^2_{L^2(\Omega_2)}\right). \tag{2.11}$$

Combining (2.10) and (2.11), we obtain

$$\|u_0\|^2_{H^1(\Omega_1)} \leq C\mathrm{e}^{c|\mu|} \left(\|f_0\|^2_{L^2(\Omega_1)} + \|g_0\|^2_{H^1(\Omega_2)} + \|g_1 + \mathrm{i}\mu g_0\|^2_{L^2(\Omega_2)}\right). \tag{2.12}$$

Recalling that $v_1 = g_0 + \mathrm{i}\mu v_0$ and using (2.11), we obtain

$$\|v_1\|^2_{H^1(\Omega_2)} \leq C\mathrm{e}^{c|\mu|} \left(\|f_0\|^2_{L^2(\Omega_1)} + \|g_0\|^2_{H^1(\Omega_2)} + \|g_1 + \mathrm{i}\mu g_0\|^2_{L^2(\Omega_2)}\right). \tag{2.13}$$

Combining (2.11), (2.12) and (2.13), we obtain Theorem 1.1. □

Proof of Theorem 2.1. Estimate (2.4) is the consequence of two important results. The first one is a known result proved by Lebeau and Robbiano in [10] and the second one is given by Theorem 2.2 and proved in Section 3.

Let $0 < \epsilon_1 < \epsilon_2$ and V_{ϵ_j}, $j = 1, 2$, such that $V_{\epsilon_j} = \{x \in \Omega_2,\ d(x, \gamma) < \epsilon_j\}$.

Recalling that $(\triangle + \mu^2)v_0 = g_1 + \mathrm{i}\mu g_0$, then for all $D > 0$, there exists $C > 0$ and $\nu \in]0, 1[$ such that we have the following estimate (see [10])

$$\|v_0\|_{H^1(\Omega_2 \setminus V_{\epsilon_1})} \leq C\mathrm{e}^{D|\mu|} \|v_0\|^{1-\nu}_{H^1(\Omega_2)} \left(\|g_1 + \mathrm{i}\mu g_0\|_{L^2(\Omega_2)} + \|v_0\|_{H^1(V_{\epsilon_2})}\right)^\nu. \tag{2.14}$$

Moreover we have the following result proved in Section 3.

Theorem 2.2. *There exist $C > 0$, $\epsilon_2 > 0$ and $\mu_0 > 0$ such that for any $|\mu| \geq \mu_0$, for all $k_2 > 0$, there exists $k_1 > 0$ such that we have the following estimate*

$$\begin{aligned}\|v_0\|^2_{H^1(V_{\epsilon_2})} \quad \leq \quad & C\mathrm{e}^{k_1|\mu|} \left[\|f_0\|^2_{L^2(\Omega_1)} + \|g_1 + \mathrm{i}\mu g_0\|^2_{L^2(\Omega_2)} + \|g_0\|^2_{H^1(\Omega_2)} + \|u_0\|^2_{H^1(\Omega_1)}\right] \\ & + C\mathrm{e}^{-k_2|\mu|} \|v_0\|^2_{H^1(\Omega_2)}. \end{aligned} \tag{2.15}$$

Combining (2.14) and (2.15), we obtain

$$\|v_0\|^2_{H^1(\Omega_2\setminus V_{\epsilon_2})} \leq C\epsilon \|v_0\|^2_{H^1(\Omega_2)} + \frac{C}{\epsilon^{\frac{1-\nu}{\nu}}} \mathrm{e}^{2\frac{D}{\nu}|\mu|} \|g_1 + \mathrm{i}\mu g_0\|^2_{L^2(\Omega_2)} + \frac{C}{\epsilon^{\frac{1-\nu}{\nu}}} \mathrm{e}^{(2\frac{D}{\nu}-k_2)|\mu|} \|v_0\|^2_{H^1(\Omega_2)}$$
$$+ \frac{C}{\epsilon^{\frac{1-\nu}{\nu}}} \mathrm{e}^{(2\frac{D}{\nu}+k_1)|\mu|} \left[\|f_0\|^2_{L^2(\Omega_1)} + \|g_1 + \mathrm{i}\mu g_0\|^2_{L^2(\Omega_2)} + \|g_0\|^2_{H^1(\Omega_2)} + \|u_0\|^2_{H^1(\Omega_1)}\right]. \tag{2.16}$$

Adding (2.15) and (2.16), we obtain

$$\|v_0\|^2_{H^1(\Omega_2)} \leq C\epsilon \|v_0\|^2_{H^1(\Omega_2)} + C_\epsilon \mathrm{e}^{2\frac{D}{\nu}} \|g_1 + \mathrm{i}\mu g_0\|^2_{L^2(\Omega_2)} + C_\epsilon \mathrm{e}^{(2\frac{D}{\nu}-k_2)|\mu|} \|v_0\|^2_{H^1(\Omega_2)}$$
$$+ C_\epsilon \mathrm{e}^{(2\frac{D}{\nu}+k_1)|\mu|} \left[\|f_0\|^2_{L^2(\Omega_1)} + \|g_1 + \mathrm{i}\mu g_0\|^2_{L^2(\Omega_2)} + \|g_0\|^2_{H^1(\Omega_2)} + \|u_0\|^2_{H^1(\Omega_1)}\right].$$

We fix ϵ small enough and k_2 such that $2\frac{D}{\nu} \ll k_2$, then there exists $\mu_0 > 0$ such that for any $|\mu| \geq \mu_0$, we obtain (2.4). □

3. The Carleman estimate and its consequence

In this part, we prove the new Carleman estimate and Theorem 2.2 which is a consequence of this estimate.

3.1. Statement of the Carleman estimate

In this subsection we state the Carleman estimate which is the starting point of the proof of the main result. We begin by giving some notations and definitions used in the sequel.

Let τ be a positive real number such that $\tau \geq C_0 |\mu|$, $C_0 > 0$. We define the Sobolev spaces with a parameter τ, H^s_τ by

$$u(x,\tau) \in H^s_\tau \iff \langle \xi,\tau\rangle^s \widehat{u}(\xi,\tau) \in L^2, \quad \langle \xi,\tau\rangle^2 = |\xi|^2 + \tau^2,$$

where $\widehat{u}$ denoted the partial Fourier transform with respect to x.

For a differential operator

$$P(x,D,\tau,\mu) = \sum_{|\alpha|+k+j\leq m} a_{\alpha,k}(x)\mu^k \tau^j D^\alpha,$$

we denote the associated symbol by

$$p(x,\xi,\tau,\mu) = \sum_{|\alpha|+k+j\leq m} a_{\alpha,k}(x)\mu^k \tau^j \xi^\alpha.$$

The class of symbols of order m is defined by

$$S^m_\tau = \left\{p(x,\xi,\tau,\mu) \in C^\infty,\ \left|D^\alpha_x D^\beta_\xi p(x,\xi,\tau,\mu)\right| \leq C_{\alpha,\beta}\langle \xi,\tau\rangle^{m-|\beta|}\right\}$$

and the class of tangential symbols of order m by

$$\mathcal{T}S^m_\tau = \left\{p(x,\xi',\tau,\mu) \in C^\infty,\ \left|D^\alpha_x D^\beta_{\xi'} p(x,\xi',\tau,\mu)\right| \leq C_{\alpha,\beta}\langle \xi',\tau\rangle^{m-|\beta|}\right\}.$$

We denote by $\mathcal{O}^m$ (resp. $\mathcal{T}\mathcal{O}^m$) the set of pseudo-differential operators $P = \mathrm{op}(p)$, $p \in S^m_\tau$ (resp. $\mathcal{T}S^m_\tau$) and by $\sigma(P)$ the principal symbol of P.

We shall frequently use the symbol $\Lambda = \langle \xi',\tau\rangle = (|\xi'|^2 + \tau^2)^{\frac{1}{2}}$.

We use the following Gårding estimate: if $p \in \mathcal{T}S^2_\tau$ satisfies for $C_0 > 0$, $p(x,\xi',\tau) + \overline{p}(x,\xi',\tau) \geq C_0\Lambda^2$, then

$$\exists C_1 > 0,\ \exists \tau_0 > 0,\ \forall \tau > \tau_0,\ \forall u \in C^\infty_0(K),\ \mathrm{Re}(P(x,D',\tau,\mu)u,u) \geq C_1 \|\mathrm{op}(\Lambda)u\|^2_{L^2}. \tag{3.1}$$

Let $u = (u_0, v_0)$ satisfy the equation

$$\begin{cases} -(\triangle - \mathrm{i}\mu)u_0 = f_1 & \text{in } \Omega_1, \\ -(\triangle + \mu^2)v_0 = f_2 & \text{in } \Omega_2, \\ \mathrm{op}(B_1)u = u_0 - \mathrm{i}\mu v_0 = e_1 & \text{on } \gamma, \\ \mathrm{op}(B_2)u = \partial_n u_0 - \partial_n v_0 = e_2 & \text{on } \gamma. \end{cases} \tag{3.2}$$

We will proceed like Bellassoued in [3], we will reduce the problem of transmission to a diagonal system defined only on one side of the interface with boundary conditions.

Let $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. In a neighborhood $W \subset \mathbb{R}^n$ of $(0,0)$, we use normal geodesic coordinates (we can assume W symmetric with respect to $x_n \mapsto -x_n$). We denote

$$\Theta_2 = \{x \in \mathbb{R}^n,\ x_n > 0\} \cap W, \quad \text{and} \quad \Theta_1 = \{x \in \mathbb{R}^n,\ x_n < 0\} \cap W.$$

The Laplacian on Θ_2 is written in the form

$$\triangle = -A_2(x, D) = -\left(D_{x_n}^2 + R(+x_n, x', D_{x'})\right).$$

The Laplacian on Θ_1 can be identified locally to an operator in Θ_2 given by

$$\triangle = -A_1(x, D) = -\left(D_{x_n}^2 + R(-x_n, x', D_{x'})\right).$$

We denote the operator, with C^∞ coefficients defined in Θ_2, by

$$A(x, D) = \mathrm{diag}\Big(A_1(x, D_x), A_2(x, D_x)\Big)$$

and the tangential operator by

$$R(x, D_{x'}) = \mathrm{diag}\Big(R(-x_n, x', D_{x'}), R(+x_n, x', D_{x'})\Big) = \mathrm{diag}\Big(R_1(x, D_{x'}), R_2(x, D_{x'})\Big).$$

The principal symbol of the differential operator $A(x, D)$ satisfies $\sigma(A) = \xi_n^2 + r(x, \xi')$, where $r(x, \xi') = \mathrm{diag}\Big(r_1(x, \xi'), r_2(x, \xi')\Big) = \sigma(R(x, D_{x'}))$ and the quadratic form $r_j(x, \xi')$, $j = 1, 2$, satisfies

$$\exists\, C > 0, \quad \forall (x, \xi'), \quad r_j(x, \xi') \geq C\, |\xi'|^2, \quad j = 1, 2.$$

We denote $P(x, D)$ the matrix operator with C^∞ coefficients defined in Θ_2, by

$$P(x, D) = \mathrm{diag}(P_1(x, D), P_2(x, D)) = \begin{pmatrix} A_1(x, D) + \mathrm{i}\mu & 0 \\ 0 & A_2(x, D) - \mu^2 \end{pmatrix}.$$

Let $\varphi(x) = \mathrm{diag}(\varphi_1(x), \varphi_2(x))$, with φ_j, $j = 1, 2$, are C^∞ functions in Θ_2. For τ large enough, we define the operator

$$A(x, D, \tau) = \mathrm{e}^{\tau\varphi} A(x, D) \mathrm{e}^{-\tau\varphi}$$

where the principal symbol of $A(x, D, \tau)$ is given by

$$\sigma(A) = \left(\xi_n + \mathrm{i}\tau \frac{\partial \varphi}{\partial x_n}\right)^2 + r\left(x, \xi' + \mathrm{i}\frac{\partial \varphi}{\partial x'}\right) \in S_\tau^2.$$

Let

$$\tilde{Q}_{2,j} = \frac{1}{2}(A_j + A_j^*), \qquad \tilde{Q}_{1,j} = \frac{1}{2\mathrm{i}}(A_j - A_j^*), \qquad j = 1, 2$$

its real and imaginary part. Then we have

$$\begin{cases} A_j = \tilde{Q}_{2,j} + i\tilde{Q}_{1,j}, \\ \sigma(\tilde{Q}_{2,j}) = \xi_n^2 + q_{2,j}(x,\xi',\tau), \quad \sigma(\tilde{Q}_{1,j}) = 2\tau\frac{\partial\varphi_j}{\partial x_n}\xi_n + 2\tau q_{1,j}(x,\xi',\tau), \quad j=1,2, \end{cases} \tag{3.3}$$

where $q_{1,j} \in \mathcal{T}S^1_\tau$ and $q_{2,j} \in \mathcal{T}S^2_\tau$ are two tangential symbols given by

$$\begin{cases} q_{2,j}(x,\xi',\tau) = r_j(x,\xi') - (\tau\frac{\partial\varphi_j}{\partial x_n})^2 - \tau^2 r_j(x,\frac{\partial\varphi_j}{\partial x'}), \\ q_{1,j}(x,\xi',\tau) = \tilde{r}_j(x,\xi',\frac{\partial\varphi_j}{\partial x'}), \quad j=1,2, \end{cases} \tag{3.4}$$

where $\tilde{r}(x,\xi',\eta')$ is the bilinear form associated to the quadratic form $r(x,\xi')$.

In the sequel, $P(x,D,\tau,\mu)$ is the matrix operator with C^∞ coefficients defined in Θ_2 by

$$P(x,D,\tau,\mu) = \mathrm{diag}(P_1(x,D,\tau,\mu), P_2(x,D,\tau,\mu)) = \begin{pmatrix} A_1(x,D,\tau)+\mathrm{i}\mu & 0 \\ 0 & A_2(x,D,\tau)-\mu^2 \end{pmatrix} \tag{3.5}$$

and $u=(u_0,v_0)$ satisfies the equation

$$\begin{cases} Pu = f & \text{in } \{x_n>0\}\cap W, \\ \mathrm{op}(b_1)u = u_0|_{x_n=0} - \mathrm{i}\mu v_0|_{x_n=0} = e_1 & \text{on } \{x_n=0\}\cap W, \\ \mathrm{op}(b_2)u = \left(D_{x_n} + \mathrm{i}\tau\frac{\partial\varphi_1}{\partial x_n}\right)u_0|_{x_n=0} + \left(D_{x_n} + \mathrm{i}\tau\frac{\partial\varphi_2}{\partial x_n}\right)v_0|_{x_n=0} = e_2 & \text{on } \{x_n=0\}\cap W, \end{cases} \tag{3.6}$$

where $f=(f_1,f_2)$, $e=(e_1,e_2)$ and $B=(\mathrm{op}(b_1),\mathrm{op}(b_2))$. We note $p_j(x,\xi,\tau,\mu)$, $j=1,2$, the principal symbol associated to $P_j(x,D,\tau,\mu)$. We have

$$\begin{cases} p_1(x,\xi,\tau,\mu) = \xi_n^2 + q_{2,1}(x,\xi',\tau) + \mathrm{i}(2\tau\frac{\partial\varphi_1}{\partial x_n}\xi_n + 2\tau q_{1,1}(x,\xi',\tau)) \\ p_2(x,\xi,\tau,\mu) = \xi_n^2 + q_{2,2}(x,\xi',\tau) - \mu^2 + \mathrm{i}(2\tau\frac{\partial\varphi_2}{\partial x_n}\xi_n + 2\tau q_{1,2}(x,\xi',\tau)). \end{cases} \tag{3.7}$$

We assume that φ satisfies

$$\begin{cases} \varphi_1(x) = \varphi_2(x) & \text{on } \{x_n=0\}\cap W \\ \dfrac{\partial\varphi_1}{\partial x_n} > 0 & \text{on } \{x_n=0\}\cap W \\ \left(\dfrac{\partial\varphi_1}{\partial x_n}\right)^2 - \left(\dfrac{\partial\varphi_2}{\partial x_n}\right)^2 > 1 & \text{on } \{x_n=0\}\cap W \end{cases} \tag{3.8}$$

and the following hypoellipticity condition of Hörmander: $\exists C>0,\ \forall x\in K,\ \forall\xi\in\mathbb{R}^n\backslash\{0\}$,

$$\left(\mathrm{Re}p_j = 0 \quad \text{et} \quad \frac{1}{2\tau}\mathrm{Im}p_j = 0\right) \Rightarrow \left\{\mathrm{Re}p_j, \frac{1}{2\tau}\mathrm{Im}p_j\right\} \geq C\langle\xi,\tau\rangle^2, \tag{3.9}$$

where $\{f,g\}(x,\xi) = \sum\left(\frac{\partial f}{\partial\xi_j}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial\xi_j}\right)$ is the Poisson bracket of two functions $f(x,\xi)$ and $g(x,\xi)$ and K is a compact in $\{x_n\geq 0\}\cap W$.

We denote by

$$\|u\|_{L^2(\Theta_2)} = \|u\|, \quad \|u\|_{k,\tau}^2 = \sum_{j=0}^{k} \tau^{2(k-j)} \|u\|_{H^j(\Theta_2)}^2, \quad \|u\|_k^2 = \left\|\mathrm{op}(\Lambda^k)u\right\|^2,$$

$$|u|_{k,\tau}^2 = \|u|_{x_n=0}\|_{k,\tau}^2, \quad |u|_k^2 = \|u|_{x_n=0}\|_k^2, \ k \in \mathbb{R} \quad \text{and} \quad |u|_{1,0,\tau}^2 = |u|_1^2 + |D_{x_n}u|^2.$$

We are now ready to state our result.

Theorem 3.1. *Let φ satisfies (3.8) and (3.9). Then there exist constants $C > 0$, $\tau_0 > 0$ and $\mu_0 > 0$ such that for any $\tau \geq \tau_0$, $|\mu| \geq \mu_0$ such that $\tau \geq C_0 |\mu|$, $C_0 > 0$, we have the following estimate*

$$\tau \|w\|_{1,\tau}^2 + \tau^2 |w|_{\frac12}^2 + \tau^2 |D_{x_n}w|_{-\frac12}^2 \leq C\left(\|P(x,D,\tau)w\|^2 + \frac{\tau^2}{\mu^2}|\mathrm{op}(b_1)w|_{\frac12}^2 + \tau|\mathrm{op}(b_2)w|^2\right), \tag{3.10}$$

for any $w \in C_0^\infty(\{x_n \geq 0\} \cap W)$.

Corollary 3.1. *Let φ satisfies (3.8) and (3.9). Then there exist constants $C > 0$, $\tau_0 > 0$ and $\mu_0 > 0$ such that for any $\tau \geq \tau_0$, $|\mu| \geq \mu_0$ such that $\tau \geq C_0 |\mu|$, $C_0 > 0$, we have the following estimate*

$$\mu \|\mathrm{e}^{\mu\varphi}h\|_{H^1(\Theta_2)}^2 \leq C(\|\mathrm{e}^{\mu\varphi}P(x,D)h\|_{L^2(\Theta_2)}^2 + \frac{\tau^2}{\mu^2}\|\mathrm{e}^{\mu\varphi}\mathrm{op}(B_1)h\|_{H^{\frac12}(x_n=0)}^2 + \mu\|\mathrm{e}^{\mu\varphi}\mathrm{op}(B_2)h\|_{L^2(x_n=0)}^2), \tag{3.11}$$

for any $h \in C_0^\infty(\{x_n \geq 0\} \cap W)$.

Proof. Let $w = \mathrm{e}^{\tau\varphi}h$. Recalling that $P(x,D,\tau,\mu)w = \mathrm{e}^{\tau\varphi}P(x,D)\mathrm{e}^{-\tau\varphi}w$ and using (3.10), we obtain (3.11). □

3.2. **Proof of Theorem 2.2**

To apply Corollary 3.1, we have to choose φ_1 and φ_2 satisfying (3.8) and (3.9).
We denote $x = (x', x_n)$ a point in Ω. Let $x_0 = (0, -\delta)$, $\delta > 0$. We set

$$\psi(x) = |x - x_0|^2 - \delta^2 \qquad \text{and}$$

$$\varphi_1(x) = \mathrm{e}^{-\beta\psi(x',-x_n)}, \quad \varphi_2(x) = \mathrm{e}^{-\beta(\psi(x)-\alpha x_n)}, \quad \beta > 0, \quad \text{and} \quad \frac{\delta}{2} < \alpha < 2\delta.$$

The weight function $\varphi = \mathrm{diag}(\varphi_1, \varphi_2)$ has to satisfy (3.8) and (3.9). With these choices, we have $\varphi_1|_{x_n=0} = \varphi_2|_{x_n=0}$ and $\frac{\partial\varphi_1}{\partial x_n}|_{x_n=0} > 0$. It remains to verify

$$\left(\frac{\partial\varphi_1}{\partial x_n}\right)^2 - \left(\frac{\partial\varphi_2}{\partial x_n}\right)^2 > 1 \quad \text{on } \{x_n = 0\} \tag{3.12}$$

and the condition (3.9). We begin by condition (3.9) and we compute for φ_1 and p_1 (the computation for φ_2 and p_2 is made in the same way). Recalling that

$$\left\{\mathrm{Re}p_1, \frac{1}{2\tau}\mathrm{Im}p_1\right\}(x,\xi) = \frac{\mathrm{Im}}{2\tau}\left[\partial_\xi p_1(x,\xi - \mathrm{i}\tau\varphi_1'(x))\,\partial_x p_1(x,\xi + \mathrm{i}\tau\varphi_1'(x))\right]$$
$$+^t\left[\partial_\xi p_1(x,\xi - \mathrm{i}\tau\varphi_1'(x))\right]\varphi_1''(x)\left[\partial_\xi p_1(x,\xi - \mathrm{i}\tau\varphi_1'(x))\right].$$

We replace $\varphi_1(x)$ by $\varphi_1(x) = \mathrm{e}^{-\beta\psi(x',-x_n)}$, $\beta > 0$, we obtain, by noting $\xi = -\beta\varphi_1(x)\eta$

$$\left\{\mathrm{Re}p_1, \frac{1}{2\tau}\mathrm{Im}p_1\right\}(x,\xi) = (-\beta\varphi_1)^3 \left[\left\{\mathrm{Re}p_1(x,\eta - \mathrm{i}\tau\psi'), \frac{1}{2\tau}\mathrm{Im}p_1(x,\eta+\mathrm{i}\tau\psi')\right\}(x,\eta) - \beta\left|\psi'(x)\partial_\eta p_1(x,\eta+\mathrm{i}\tau\psi')\right|^2\right]$$

and

$$\left|\psi'(x)\partial_\eta p_1(x,\eta+\mathrm{i}\tau\psi')\right|^2 = 4\left[\tau^2\left|p_1(x,\psi')\right|^2 + \left|\tilde{p}_1(x,\eta,\psi')\right|^2\right]$$

where $\tilde{p}_1(x,\eta,\psi')$ is the bilinear form associated to the quadratic form $p_1(x,\eta)$. We have

$$\left(\mathrm{Re}p_1 = 0 \quad \text{et} \quad \frac{1}{2\tau}\mathrm{Im}p_1 = 0\right) \Longleftrightarrow p_1(x,\eta+\mathrm{i}\tau\psi') = 0.$$

- If $\tau = 0$, we have $p_1(x,\xi) = 0$ which is impossible. Indeed, we have $p_1(x,\xi) \geq C\left|\xi\right|^2$, $\forall(x,\xi) \in K \times \mathbb{R}^n$, K compact in $\{x_n \geq 0\} \cap W$.
- If $\tau \neq 0$, we have $\tilde{p}_1(x,\eta,\psi') = 0$.
  Then $\left|\psi'(x)\partial_\eta p_1(x,\eta+\mathrm{i}\tau\psi')\right|^2 = 4\tau^2\left|p_1(x,\psi')\right|^2 > 0$. On the other hand, we have

$$\left\{\mathrm{Re}p_1(x,\eta-\mathrm{i}\tau\psi'), \frac{1}{2\tau}\mathrm{Im}p_1(x,\eta+\mathrm{i}\tau\psi')\right\}(x,\eta) \leq C_1(\left|\eta\right|^2 + \tau^2\left|\psi'\right|^2)$$

where C_1 is a positive constant independent of ψ'. Then for $\beta \geq C_1$, the condition (3.9) is satisfied.

Now let us verify (3.12). We have, on $\{x_n = 0\}$,

$$\left(\frac{\partial\varphi_1}{\partial x_n}\right)^2 - \left(\frac{\partial\varphi_2}{\partial x_n}\right)^2 = \beta^2\alpha(4\delta-\alpha)\mathrm{e}^{-2\beta\psi}.$$

Then to satisfy (3.12), it suffices to choose $\beta = \frac{M}{\delta}$ where $M > 0$ such that $\frac{M}{\delta} \geq C_1$.

In the sequel, to respect the geometry we return in Θ_1 for the heat component u_0 (in this case φ_1 defined above becomes $\varphi_1(x) = \mathrm{e}^{-\beta\psi(x)}$).

Let us choose $r_1 < r_1' < r_2 < 0 = \psi(0) < r_2' < r_3 < r_3'$. We denote

$$w_j = \{x \in \Omega,\ r_j < \psi(x) < r_j'\} \quad \text{and} \quad T_{x_0} = w_2 \cap \Theta_2.$$

We set $R_j = \mathrm{e}^{-\beta r_j}$, $R_j' = \mathrm{e}^{-\beta r_j'}$, $j = 1,2,3$.

Then $R_3' < R_3 < R_2' < R_2 < R_1' < R_1$. We introduce a cut-off function $\tilde{\chi} \in C_0^\infty(\mathbb{R}^{n+1})$ such that

$$\tilde{\chi}(\rho) = \begin{cases} 0 & \text{if} \quad \rho \leq r_1, \quad \rho \geq r_3' \\ 1 & \text{if} \quad \rho \in [r_1', r_3]. \end{cases}$$

Let $\tilde{u} = (\tilde{u}_0, \tilde{v}_0) = \tilde{\chi}u = (\tilde{\chi}u_0, \tilde{\chi}v_0)$, we get the following system

$$\begin{cases} (\triangle - \mathrm{i}\mu)\tilde{u}_0 &= \tilde{\chi}f_0 + [\triangle - \mathrm{i}\mu, \tilde{\chi}]u_0 \\ (\triangle + \mu^2)\tilde{v}_0 &= \tilde{\chi}(g_1 + \mathrm{i}\mu g_0) + [\triangle + \mu^2, \tilde{\chi}]v_0, \\ \tilde{v}_1 &= g_0 + \mathrm{i}\mu\tilde{v}_0, \end{cases}$$

with the boundary conditions

$$\begin{cases} \tilde{u}_0|_{\Gamma_1} = \tilde{v}_0|_{\Gamma_2} & = \quad 0, \\ \mathrm{op}(B_1)\tilde{u} = \tilde{u}_0 - \mathrm{i}\mu\tilde{v}_0 & = \quad (\tilde{\chi}g_0)|_\gamma, \\ \mathrm{op}(B_2)\tilde{u} & = \quad ([\partial_n, \tilde{\chi}]u_0 - [\partial_n, \tilde{\chi}]v_0)|_\gamma. \end{cases}$$

From the Carleman estimate of Corollary 3.1 , we have

$$\tau \|\mathrm{e}^{\tau\varphi}\tilde{u}\|^2_{H^1} \leq C(\|\mathrm{e}^{\tau\varphi_1}(\triangle - \mathrm{i}\mu)\tilde{u}_0\|^2_{L^2(\Theta_1)} + \left\|\mathrm{e}^{\tau\varphi_2}(\triangle + \mu^2)\tilde{v}_0\right\|^2_{L^2(\Theta_2)}$$
$$+ \frac{\tau^2}{\mu^2} \|\mathrm{e}^{\tau\varphi}\mathrm{op}(b_1)\tilde{u}\|^2_{H^{\frac{1}{2}}(x_n=0)} + \tau \|\mathrm{e}^{\tau\varphi}\mathrm{op}(b_2)\tilde{u}\|^2_{L^2(x_n=0)}). \tag{3.13}$$

Using the fact that $[\triangle - \mathrm{i}\mu, \tilde{\chi}]$ is the first order operator supported in $(w_1 \cup w_3) \cap \Theta_1$, we have

$$\|\mathrm{e}^{\tau\varphi_1}(\triangle - \mathrm{i}\mu)\tilde{u}_0\|^2_{L^2(\Theta_1)} \leq C\left(\mathrm{e}^{2\tau R_1}\|f_0\|^2_{L^2(\Omega_1)} + \mathrm{e}^{2\tau R_1}\|u_0\|^2_{H^1(\Omega_1)}\right). \tag{3.14}$$

Recalling that $[\triangle + \mu^2, \tilde{\chi}]$ is the first order operator supported in $(w_1 \cup w_3) \cap \Theta_2$, we show

$$\left\|\mathrm{e}^{\tau\varphi_2}(\triangle + \mu^2)\tilde{v}_0\right\|^2_{L^2(\Theta_2)} \leq C\left(\mathrm{e}^{2\tau}\|g_1 + \mathrm{i}\mu g_0\|^2_{L^2(\Omega_2)} + \mathrm{e}^{2\tau R_3}\|v_0\|^2_{H^1(\Omega_2)}\right). \tag{3.15}$$

From the trace formula and recalling that $\mathrm{op}(b_2)\tilde{u}$ is an operator of order zero and supported in $\{x_n = 0\} \cap w_3$, we show

$$\tau \|\mathrm{e}^{\tau\varphi}\mathrm{op}(b_2)\tilde{u}\|^2_{L^2(x_n=0)} \leq C\mathrm{e}^{2\tau R_3}\|u\|^2_{H^1(\Omega)} \leq C\left(\mathrm{e}^{2\tau R_3}\|u_0\|^2_{H^1(\Omega_1)} + \mathrm{e}^{2\tau R_3}\|v_0\|^2_{H^1(\Omega_2)}\right). \tag{3.16}$$

Now we need to use this result shown in Appendix B.

Lemma 3.1. *There exists $C > 0$ such that for all $s \in \mathbb{R}$ and $u \in C_0^\infty(\Omega)$, we have*

$$\|\mathrm{op}(\Lambda^s)\mathrm{e}^{\tau\varphi}u\| \leq C\mathrm{e}^{\tau C}\|\mathrm{op}(\Lambda^s)u\|. \tag{3.17}$$

Following this lemma and recalling that $\tau \geq C_0|\mu|$, $C_0 > 0$, and $|\mu| \geq \mu_0$, we obtain

$$\frac{\tau^2}{\mu^2}\|\mathrm{e}^{\tau\varphi}\mathrm{op}(b_1)\tilde{u}\|^2_{H^{\frac{1}{2}}(x_n=0)} \leq C\tau^2\mathrm{e}^{2\tau c}|g_0|^2_{H^{\frac{1}{2}}} \leq C\tau^2\mathrm{e}^{2\tau c}\|g_0\|^2_{H^1(\Omega_2)}. \tag{3.18}$$

Combining (3.13)–(3.16) and (3.18), we obtain

$$C\tau\mathrm{e}^{2\tau R_2'}\|u_0\|^2_{H^1(w_2\cap\Theta_1)} + C\tau\mathrm{e}^{2\tau R_2'}\|v_0\|^2_{H^1(T_{x_0})} \leq C(\mathrm{e}^{2\tau R_1}\|f_0\|^2_{L^2(\Omega_1)} + \mathrm{e}^{2\tau R_1}\|u_0\|^2_{H^1(\Omega_1)}$$
$$+ \mathrm{e}^{2\tau}\|g_1 + \mathrm{i}\mu g_0\|^2_{L^2(\Omega_2)} + \mathrm{e}^{2\tau R_3}\|v_0\|^2_{H^1(\Omega_2)} + \mathrm{e}^{2\tau R_3}\|u_0\|^2_{H^1(\Omega_1)} + \mathrm{e}^{2\tau c}\|g_0\|^2_{H^1(\Omega_2)}).$$

Since $R_3 < R_2' < R_1$. Then there exist $c_1 > 0$ and $c_2 > 0$ such that

$$\|v_0\|^2_{H^1(T_{x_0})} \leq C\mathrm{e}^{c_1\tau}\left[\|f_0\|^2_{L^2(\Omega_1)} + \|g_1 + \mathrm{i}\mu g_0\|^2_{L^2(\Omega_2)} + \|g_0\|^2_{H^1(\Omega_2)} + \|u_0\|^2_{H^1(\Omega_1)}\right]$$
$$+ C\mathrm{e}^{-c_2\tau}\|v_0\|^2_{H^1(\Omega_2)}. \tag{3.19}$$

Now we must distinguish two cases:

- **Case 1:** $\|v_0\|^2_{H^1(\Omega_2)} \geq \left(\|f_0\|^2_{L^2(\Omega_1)} + \|g_1 + \mathrm{i}\mu g_0\|^2_{L^2(\Omega_2)} + \|g_0\|^2_{H^1(\Omega_2)} + \|u_0\|^2_{H^1(\Omega_1)}\right)$

  Minimizing the right-hand side of (3.19) with respect to τ, we get with $\delta_0 = c_2/(c_1 + c_2)$, the following estimate

$$\|v_0\|^2_{H^1(T_{x_0})} \leq C\left(\|f_0\|^2_{L^2(\Omega_1)} + \|g_1 + \mathrm{i}\mu g_0\|^2_{L^2(\Omega_2)} + \|g_0\|^2_{H^1(\Omega_2)} + \|u_0\|^2_{H^1(\Omega_1)}\right)^{\delta_0} \times \left(\|v_0\|^2_{H^1(\Omega_2)}\right)^{1-\delta_0}. \tag{3.20}$$

- **Case 2:** $\|v_0\|^2_{H^1(\Omega_2)} \leq \left(\|f_0\|^2_{L^2(\Omega_1)} + \|g_1 + \mathrm{i}\mu g_0\|^2_{L^2(\Omega_2)} + \|g_0\|^2_{H^1(\Omega_2)} + \|u_0\|^2_{H^1(\Omega_1)}\right)$

  In this case (3.20) is trivial.

  Then for all $k_2 > 0$, there exists $k_1 > 0$ such that we have

$$\|v_0\|^2_{H^1(T_{x_0})} \leq C\mathrm{e}^{k_1|\mu|}\left[\|f_0\|^2_{L^2(\Omega_1)} + \|g_1 + \mathrm{i}\mu g_0\|^2_{L^2(\Omega_2)} + \|g_0\|^2_{H^1(\Omega_2)} + \|u_0\|^2_{H^1(\Omega_1)}\right] + C\mathrm{e}^{-k_2|\mu|}\|v_0\|^2_{H^1(\Omega_2)}. \tag{3.21}$$

Since γ is compact, then there exists a finite number of T_{x_0} such that $\gamma \subset \cup\, T_{x_0}$ and if ϵ_2 small enough, we have $V_{\epsilon_2} \subset \cup\, T_{x_0}$. Then (2.15) follows from (3.21).

3.3. **Proof of the Carleman estimate (Thm. 3.1)**

In the next section we will prove the following theorem which is analogous to Theorem 3.1 with another scale of Sobolev spaces.

Theorem 3.2. *Let φ satisfy (3.8) and (3.9). Then there exist constants $C > 0$, $\tau_0 > 0$ and $\mu_0 > 0$ such that for any $\tau \geq \tau_0$, $|\mu| \geq \mu_0$ such that $\tau \geq C_0|\mu|$, $C_0 > 0$, we have the following estimate*

$$\tau\|u\|^2_{1,\tau} \leq C\left(\|P(x, D, \tau, \mu)u\|^2 + \tau|u|^2_{1,0,\tau}\right) \tag{3.22}$$

and

$$\tau\|u\|^2_{1,\tau} + \tau|u|^2_{1,0,\tau} \leq C\left(\|P(x, D, \tau, \mu)u\|^2 + \frac{\tau}{\mu^2}|\mathrm{op}(b_1)u|^2_1 + \tau|\mathrm{op}(b_2)u|^2\right), \tag{3.23}$$

for any $u \in C_0^\infty(\{x_n \geq 0\} \cap W)$.

To prove Theorem 3.1, we need the following lemma.

Lemma 3.2. *There exist constants $C > 0$ and $\tau_0 > 0$ such that for any $\tau \geq \tau_0$, $|\mu| \geq 1$ such that $\tau \geq C_0|\mu|$, $C_0 > 0$, we have the following estimate*

$$\left\|D^2_{x_n}\mathrm{op}(\Lambda^{-\frac{1}{2}})u\right\|^2 + \left\|D_{x_n}\mathrm{op}(\Lambda^{\frac{1}{2}})u\right\|^2 + \left\|\mathrm{op}(\Lambda^{\frac{3}{2}})u\right\|^2 + \tau|u|^2_{1,0,\tau} \leq C\Bigg(\|P(x, D, \tau, \mu)u\|^2 + \frac{\tau}{\mu^2}|\mathrm{op}(b_1)u|^2_1 + \tau|\mathrm{op}(b_2)u|^2\Bigg), \tag{3.24}$$

for any $u \in C_0^\infty(\{x_n \geq 0\} \cap W)$.

Proof. We have

$$P(x, D, \tau, \mu) = D^2_{x_n} + R + \tau C_1 + \tau^2 C_0,$$

where $R \in \mathcal{TO}^2$, $C_1 = c_1(x)D_{x_n} + T_1$, with $T_1 \in \mathcal{TO}^1$ and $C_0 \in \mathcal{TO}^0$. Then we have

$$\left\|(D_{x_n}^2 + R)\mathrm{op}(\Lambda^{-\frac{1}{2}})u\right\|^2 \le C\left(\left\|P\mathrm{op}(\Lambda^{-\frac{1}{2}})u\right\|^2 + \tau^2\left\|\mathrm{op}(\Lambda^{\frac{1}{2}})u\right\|^2 + \tau^2\left\|D_{x_n}\mathrm{op}(\Lambda^{-\frac{1}{2}})u\right\|^2 + \tau^4\left\|\mathrm{op}(\Lambda^{-\frac{1}{2}})u\right\|^2\right).$$

Since

$$\begin{aligned}
&\tau^4\left\|\mathrm{op}(\Lambda^{-\frac{1}{2}})u\right\|^2 \le C\tau^3\|u\|^2,\\
&\tau^2\left\|D_{x_n}\mathrm{op}(\Lambda^{-\frac{1}{2}})u\right\|^2 \le C\tau\|D_{x_n}u\|^2 \quad \text{and}\\
&\tau^2\left\|\mathrm{op}(\Lambda^{\frac{1}{2}})u\right\|^2 = \tau^2(\tfrac{1}{\sqrt{\tau}}\mathrm{op}(\Lambda)u, \sqrt{\tau}u) \le C\left(\tau\|\mathrm{op}(\Lambda)u\|^2 + \tau^3\|u\|^2\right).
\end{aligned}$$

Using the fact that $\|u\|_{1,\tau}^2 \simeq \|\mathrm{op}(\Lambda)u\|^2 + \|D_{x_n}u\|^2$, we obtain

$$\left\|(D_{x_n}^2 + R)\mathrm{op}(\Lambda^{-\frac{1}{2}})u\right\|^2 \le C\left(\left\|P\mathrm{op}(\Lambda^{-\frac{1}{2}})u\right\|^2 + \tau\|u\|_{1,\tau}^2\right).$$

Following (3.22), we have

$$\left\|(D_{x_n}^2 + R)\mathrm{op}(\Lambda^{-\frac{1}{2}})u\right\|^2 \le C\left(\left\|P\mathrm{op}(\Lambda^{-\frac{1}{2}})u\right\|^2 + \|Pu\|^2 + \tau|u|_{1,0,\tau}^2\right). \tag{3.25}$$

We can write

$$\begin{aligned}
P\mathrm{op}(\Lambda^{-\frac{1}{2}})u &= \mathrm{op}(\Lambda^{-\frac{1}{2}})Pu + [P, \mathrm{op}(\Lambda^{-\frac{1}{2}})]u\\
&= \mathrm{op}(\Lambda^{-\frac{1}{2}})Pu + [R, \mathrm{op}(\Lambda^{-\frac{1}{2}})]u\\
&\quad + \tau[C_1, \mathrm{op}(\Lambda^{-\frac{1}{2}})]u + \tau^2[C_0, \mathrm{op}(\Lambda^{-\frac{1}{2}})]u\\
&= \mathrm{op}(\Lambda^{-\frac{1}{2}})Pu + t_1 + t_2 + t_3.
\end{aligned} \tag{3.26}$$

Let us estimate t_1, t_2 and t_3. We have $[R, \mathrm{op}(\Lambda^{-\frac{1}{2}})] \in \mathcal{TO}^{\frac{1}{2}}$, then following (3.22), we have

$$\|t_1\|^2 \le C\left\|\mathrm{op}(\Lambda^{\frac{1}{2}})u\right\|^2 \le C\left(\|\mathrm{op}(\Lambda)u\|^2 + \|u\|^2\right) \le C\left(\|Pu\|^2 + \tau|u|_{1,0,\tau}^2\right). \tag{3.27}$$

We have $t_2 = \tau[C_1, \mathrm{op}(\Lambda^{-\frac{1}{2}})]u = \tau[c_1(x)D_{x_n}, \mathrm{op}(\Lambda^{-\frac{1}{2}})]u + \tau[T_1, \mathrm{op}(\Lambda^{-\frac{1}{2}})]u$. Then following (3.22), we obtain

$$\|t_2\|^2 \le C\left(\tau^{-1}\|D_{x_n}u\|^2 + \tau\|u\|^2\right) \le C\left(\|Pu\|^2 + \tau|u|_{1,0,\tau}^2\right). \tag{3.28}$$

We have $[C_0, \mathrm{op}(\Lambda^{-\frac{1}{2}})] \in \mathcal{TO}^{-\frac{3}{2}}$, then following (3.22), we obtain

$$\left\|\tau^2[C_0, \mathrm{op}(\Lambda^{-\frac{1}{2}})]u\right\|^2 \le C\tau\|u\|^2 \le C\left(\|Pu\|^2 + \tau|u|_{1,0,\tau}^2\right). \tag{3.29}$$

From (3.26)–(3.29), we have

$$\left\|P\mathrm{op}(\Lambda^{-\frac{1}{2}})u\right\|^2 \le C\left(\|Pu\|^2 + \tau|u|_{1,0,\tau}^2\right),$$

and from (3.25), we obtain

$$\left\|(D_{x_n}^2 + R)\mathrm{op}(\Lambda^{-\frac{1}{2}})u\right\|^2 \le C\left(\|Pu\|^2 + \tau|u|_{1,0,\tau}^2\right). \tag{3.30}$$

Moreover, we have

$$\left\|(D_{x_n}^2+R)\mathrm{op}(\Lambda^{-\frac{1}{2}})u\right\|^2=\left\|D_{x_n}^2\mathrm{op}(\Lambda^{-\frac{1}{2}})u\right\|^2+\left\|R\mathrm{op}(\Lambda^{-\frac{1}{2}})u\right\|^2+2\mathrm{Re}(D_{x_n}^2\mathrm{op}(\Lambda^{-\frac{1}{2}})u,R\mathrm{op}(\Lambda^{-\frac{1}{2}})u),$$

where $(.,.)$ denote the scalar product in L^2. By integration by parts, we find

$$\begin{aligned}\left\|(D_{x_n}^2+R)\mathrm{op}(\Lambda^{-\frac{1}{2}})u\right\|^2&=\left\|D_{x_n}^2\mathrm{op}(\Lambda^{-\frac{1}{2}})u\right\|^2+\left\|R\mathrm{op}(\Lambda^{-\frac{1}{2}})u\right\|^2\\&\quad+2\mathrm{Re}\left(\mathrm{i}(D_{x_n}u,R\mathrm{op}(\Lambda^{-1})u)_0+\mathrm{i}(D_{x_n}u,[\mathrm{op}(\Lambda^{-\frac{1}{2}}),R]\mathrm{op}(\Lambda^{-\frac{1}{2}})u)_0\right)\\&\quad+2\mathrm{Re}\left((RD_{x_n}\mathrm{op}(\Lambda^{-\frac{1}{2}})u,D_{x_n}\mathrm{op}(\Lambda^{-\frac{1}{2}})u)+(D_{x_n}\mathrm{op}(\Lambda^{-\frac{1}{2}})u,[D_{x_n},R]\mathrm{op}(\Lambda^{-\frac{1}{2}})u)\right).\end{aligned}\tag{3.31}$$

Since, we have

$$\left\|\mathrm{op}(\Lambda^{\frac{3}{2}})u\right\|^2=(\mathrm{op}(\Lambda^2)\mathrm{op}(\Lambda^{\frac{1}{2}})u,\mathrm{op}(\Lambda^{\frac{1}{2}})u)=\sum_{j\le n-1}(D_j^2\mathrm{op}(\Lambda^{\frac{1}{2}})u,\mathrm{op}(\Lambda^{\frac{1}{2}})u)+\tau^2(\mathrm{op}(\Lambda^{\frac{1}{2}})u,\mathrm{op}(\Lambda^{\frac{1}{2}})u).$$

By integration by parts, we find

$$\left\|\mathrm{op}(\Lambda^{\frac{3}{2}})u\right\|^2=\sum_{j\le n-1}(D_j\mathrm{op}(\Lambda^{\frac{1}{2}})u,D_j\mathrm{op}(\Lambda^{\frac{1}{2}})u)+\tau^2\left\|\mathrm{op}(\Lambda^{\frac{1}{2}})u\right\|^2=k+\tau^2\left\|\mathrm{op}(\Lambda^{\frac{1}{2}})u\right\|^2.\tag{3.32}$$

Let $\chi_0\in C_0^\infty(\mathbb{R}^{n+1})$ such that $\chi_0=1$ in the support of u. We have

$$k=\sum_{j\le n-1}(\chi_0D_j\mathrm{op}(\Lambda^{\frac{1}{2}})u,D_j\mathrm{op}(\Lambda^{\frac{1}{2}})u)+\sum_{j\le n-1}((1-\chi_0)D_j\mathrm{op}(\Lambda^{\frac{1}{2}})u,D_j\mathrm{op}(\Lambda^{\frac{1}{2}})u).$$

Recalling that $\chi_0u=u$, we obtain

$$k=\sum_{j\le n-1}(\chi_0D_j\mathrm{op}(\Lambda^{\frac{1}{2}})u,D_j\mathrm{op}(\Lambda^{\frac{1}{2}})u)+\sum_{j\le n-1}([(1-\chi_0),D_j\mathrm{op}(\Lambda^{\frac{1}{2}})]u,D_j\mathrm{op}(\Lambda^{\frac{1}{2}})u)=k'+k".\tag{3.33}$$

Using the fact that $[(1-\chi_0),D_j\mathrm{op}(\Lambda^{\frac{1}{2}})]\in\mathcal{TO}^{\frac{1}{2}}$ and $D_j\mathrm{op}(\Lambda^{\frac{1}{2}})\in\mathcal{TO}^{\frac{3}{2}}$, we show

$$k''\le C\left\|\mathrm{op}(\Lambda)u\right\|^2.\tag{3.34}$$

Using the fact that $\sum_{j,k\le n-1}\chi_0a_{j,k}D_jv\overline{D_kv}\ge\delta\chi_0\sum_{j\le n-1}|D_jv|^2$, $\delta>0$, we obtain

$$\begin{aligned}k'&\le C\sum_{j,k\le n-1}(\chi_0a_{jk}D_j\mathrm{op}(\Lambda^{\frac{1}{2}})u,D_k\mathrm{op}(\Lambda^{\frac{1}{2}})u)\\&\le C\sum_{j,k\le n-1}([\chi_0,a_{jk}D_j\mathrm{op}(\Lambda^{\frac{1}{2}})]u,D_k\mathrm{op}(\Lambda^{\frac{1}{2}})u)+\sum_{j,k\le n-1}(a_{jk}D_j\mathrm{op}(\Lambda^{\frac{1}{2}})u,D_k\mathrm{op}(\Lambda^{\frac{1}{2}})u).\end{aligned}$$

Using the fact that $[\chi_0,a_{jk}D_j\mathrm{op}(\Lambda^{\frac{1}{2}})]\in\mathcal{TO}^{\frac{1}{2}}$ and $D_k\mathrm{op}(\Lambda^{\frac{1}{2}})u\in\mathcal{TO}^{\frac{3}{2}}$, we obtain

$$k'\le C\left(\sum_{j,k\le n-1}(a_{jk}D_j\mathrm{op}(\Lambda^{\frac{1}{2}})u,D_k\mathrm{op}(\Lambda^{\frac{1}{2}})u)+\left\|\mathrm{op}(\Lambda)u\right\|^2\right).\tag{3.35}$$

By integration by parts and recalling that $R=\sum_{j,k\leq n-1} a_{j,k}D_jD_k$, we have

$$\sum_{j,k\leq n-1}(a_{jk}D_j\mathrm{op}(\Lambda^{\frac12})u,D_k\mathrm{op}(\Lambda^{\frac12})u)=(R\mathrm{op}(\Lambda^{\frac12})u,\mathrm{op}(\Lambda^{\frac12})u+\sum_{j,k\leq n-1}([D_k,a_{jk}]D_j\mathrm{op}(\Lambda^{\frac12})u,\mathrm{op}(\Lambda^{\frac12})u). \quad (3.36)$$

Since $[D_k,a_{jk}]D_j\mathrm{op}(\Lambda^{\frac12})\in\mathcal{T}\mathcal{O}^{\frac32}$, then

$$\sum_{j,k\leq n-1}([D_k,a_{jk}]D_j\mathrm{op}(\Lambda^{\frac12})u,\mathrm{op}(\Lambda^{\frac12})u)\leq C\left\|\mathrm{op}(\Lambda)u\right\|^2.$$

Following (3.36), we obtain

$$\sum_{j,k\leq n-1}(a_{jk}D_j\mathrm{op}(\Lambda^{\frac12})u,D_k\mathrm{op}(\Lambda^{\frac12})u)\leq C\left((R\mathrm{op}(\Lambda^{\frac12})u,\mathrm{op}(\Lambda^{\frac12})u)+\left\|\mathrm{op}(\Lambda)u\right\|^2\right). \quad (3.37)$$

Since

$$(R\mathrm{op}(\Lambda^{\frac12})u,\mathrm{op}(\Lambda^{\frac12})u)=(R\mathrm{op}(\Lambda^{-\frac12})u,\mathrm{op}(\Lambda^{\frac32})u)+([\mathrm{op}(\Lambda^{-1}),R]\mathrm{op}(\Lambda^{\frac12})u,\mathrm{op}(\Lambda^{\frac32})u).$$

Using the fact that $[\mathrm{op}(\Lambda^{-1}),R]\mathrm{op}(\Lambda^{\frac12})\in\mathcal{T}\mathcal{O}^{\frac12}$ and the Cauchy Schwartz inequality, we obtain

$$(R\mathrm{op}(\Lambda^{\frac12})u,\mathrm{op}(\Lambda^{\frac12})u)\leq\epsilon C\left\|\mathrm{op}(\Lambda^{\frac32})u\right\|^2+\frac{C}{\epsilon}\left\|R\mathrm{op}(\Lambda^{-\frac12})u\right\|^2+C\left\|\mathrm{op}(\Lambda)u\right\|^2. \quad (3.38)$$

Combining (3.32)–(3.35), (3.37) and (3.38), we obtain

$$\left\|\mathrm{op}(\Lambda^{\frac32})u\right\|^2\leq\epsilon C\left\|\mathrm{op}(\Lambda^{\frac32})u\right\|^2+\frac{C}{\epsilon}\left\|R\mathrm{op}(\Lambda^{-\frac12})u\right\|^2+C\left\|\mathrm{op}(\Lambda)u\right\|^2.$$

For ϵ small enough, we obtain

$$\left\|R\mathrm{op}(\Lambda^{-\frac12})u\right\|^2\geq C\left(\left\|\mathrm{op}(\Lambda^{\frac32})u\right\|^2-\tau^2\left\|\mathrm{op}(\Lambda^{\frac12})u\right\|^2\right). \quad (3.39)$$

Using the same computations, we show

$$(RD_{x_n}\mathrm{op}(\Lambda^{-\frac12})u,D_{x_n}\mathrm{op}(\Lambda^{-\frac12})u)\geq C\left(\left\|D_{x_n}\mathrm{op}(\Lambda^{\frac12})u\right\|^2-\tau\left\|D_{x_n}u\right\|^2\right). \quad (3.40)$$

Combining (3.31), (3.39) and (3.40), we obtain

$$\begin{aligned}\left\|(D_{x_n}^2+R)\mathrm{op}(\Lambda^{-\frac12})u\right\|^2 &+ \left|(D_{x_n}u,R\mathrm{op}(\Lambda^{-1})u)_0\right|+\left|(D_{x_n}u,[\mathrm{op}(\Lambda^{-\frac12}),R]\mathrm{op}(\Lambda^{-\frac12})u)_0\right|\\ &+ \left|(D_{x_n}\mathrm{op}(\Lambda^{-\frac12})u,[D_{x_n},R]\mathrm{op}(\Lambda^{-\frac12})u)\right|+\tau\left\|u\right\|_{1,\tau}^2 \quad (3.41)\\ &\geq C\left(\left\|D_{x_n}^2\mathrm{op}(\Lambda^{-\frac12})u\right\|^2+\left\|D_{x_n}\mathrm{op}(\Lambda^{\frac12})u\right\|^2+\left\|\mathrm{op}(\Lambda^{\frac32})u\right\|^2\right).\end{aligned}$$

Since

$$\left|(D_{x_n}u,R\mathrm{op}(\Lambda^{-1})u)_0\right|+\left|(D_{x_n}u,[\mathrm{op}(\Lambda^{-\frac12}),R]\mathrm{op}(\Lambda^{-\frac12})u)_0\right|\leq C\left(\left|D_{x_n}u\right|^2+\left|u\right|_1^2\right)=C\left|u\right|_{1,0,\tau}^2 \quad (3.42)$$

and

$$\left|(D_{x_n}\mathrm{op}(\Lambda^{-\frac{1}{2}})u, [D_{x_n}, R]\mathrm{op}(\Lambda^{-\frac{1}{2}})u)\right| \le C\tau \|u\|^2_{1,\tau}. \tag{3.43}$$

From (3.30), (3.41)–(3.43) and (3.22), we obtain

$$\left\|D^2_{x_n}\mathrm{op}(\Lambda^{-\frac{1}{2}})u\right\|^2 + \left\|D_{x_n}\mathrm{op}(\Lambda^{\frac{1}{2}})u\right\|^2 + \left\|\mathrm{op}(\Lambda^{\frac{3}{2}})u\right\|^2 \le C\left(\|P(x,D,\tau,\mu)u\|^2 + \tau\,|u|^2_{1,0,\tau}\right).$$

Following (3.23), we obtain (3.24). □

We are now ready to prove that Theorem 3.2 and Lemma 3.2 imply Theorem 3.1.

Let $\chi \in C_0^\infty(\mathbb{R}^{n+1})$ such that $\chi = 1$ in the support of w and $u = \chi\mathrm{op}(\Lambda^{-\frac{1}{2}})w$. Then

$$\begin{aligned}
Pu &= \mathrm{op}(\Lambda^{-\frac{1}{2}})Pw + [P, \mathrm{op}(\Lambda^{-\frac{1}{2}})]w + P[\chi, \mathrm{op}(\Lambda^{-\frac{1}{2}})]w\\
&= \mathrm{op}(\Lambda^{-\frac{1}{2}})Pw + [P, \mathrm{op}(\Lambda^{-\frac{1}{2}})]w + D^2_{x_n}[\chi, \mathrm{op}(\Lambda^{-\frac{1}{2}})]w\\
&\quad + R[\chi, \mathrm{op}(\Lambda^{-\frac{1}{2}})]w + \tau c_1(x) D_{x_n}[\chi, \mathrm{op}(\Lambda^{-\frac{1}{2}})]w\\
&\quad + \tau T_1[\chi, \mathrm{op}(\Lambda^{-\frac{1}{2}})]w + \tau^2 C_0[\chi, \mathrm{op}(\Lambda^{-\frac{1}{2}})]w\\
&= \mathrm{op}(\Lambda^{-\frac{1}{2}})Pw + [P, \mathrm{op}(\Lambda^{-\frac{1}{2}})]w + a_1 + a_2 + a_3 + a_4 + a_5.
\end{aligned} \tag{3.44}$$

Let us estimate a_1, a_2, a_3, a_4 and a_5. Recalling that $[\chi, \mathrm{op}(\Lambda^{-\frac{1}{2}})] \in \mathcal{TO}^{-\frac{3}{2}}$ and $\chi w = w$. Using the fact that $[D_{x_n}, T_k] \in \mathcal{TO}^k$ for all $T_k \in \mathcal{TO}^k$, we show

$$\|a_1\|^2 \le C\left(\left\|D^2_{x_n}\mathrm{op}(\Lambda^{-\frac{3}{2}})w\right\|^2 + \left\|D_{x_n}\mathrm{op}(\Lambda^{-\frac{3}{2}})w\right\|^2 + \left\|\mathrm{op}(\Lambda^{-\frac{3}{2}})w\right\|^2\right) \tag{3.45}$$

and

$$\|a_3\|^2 \le C\left(\tau^2\left\|D_{x_n}\mathrm{op}(\Lambda^{-\frac{3}{2}})w\right\|^2 + \tau^2\left\|\mathrm{op}(\Lambda^{-\frac{3}{2}})w\right\|^2\right). \tag{3.46}$$

We have $R[\chi, \mathrm{op}(\Lambda^{-\frac{1}{2}})] \in \mathcal{TO}^{\frac{1}{2}}$, $T_1[\chi, \mathrm{op}(\Lambda^{-\frac{1}{2}})] \in \mathcal{TO}^{-\frac{1}{2}}$ and $C_0[\chi, \mathrm{op}(\Lambda^{-\frac{1}{2}})] \in \mathcal{TO}^{-\frac{3}{2}}$. Then we obtain

$$\|a_2\|^2 + \|a_4\|^2 + \|a_5\|^2 \le C\left\|\mathrm{op}(\Lambda^{\frac{1}{2}})w\right\|^2. \tag{3.47}$$

Using the same computations made in the proof of Lemma 3.2 (*cf.* t_1, t_2 and t_3 of (3.26)), we show

$$\left\|[P, \mathrm{op}(\Lambda^{-\frac{1}{2}})]w\right\|^2 \le C\left(\left\|\mathrm{op}(\Lambda^{\frac{1}{2}})w\right\|^2 + \tau^{-1}\|D_{x_n}w\|^2\right). \tag{3.48}$$

Following (3.44)–(3.48), we obtain

$$\|Pu\|^2 \le C\left(\tau^{-1}\|Pw\|^2 + \left\|\mathrm{op}(\Lambda^{\frac{1}{2}})w\right\|^2 + \tau^{-1}\|D_{x_n}w\|^2 + \mu^{-1}\left\|D^2_{x_n}\mathrm{op}(\Lambda^{-1})w\right\|^2\right). \tag{3.49}$$

We have

$$\mathrm{op}(b_1)u = \mathrm{op}(b_1)\chi\mathrm{op}(\Lambda^{-\frac{1}{2}})w = \mathrm{op}(\Lambda^{-\frac{1}{2}})\mathrm{op}(b_1)w + \mathrm{op}(b_1)[\chi, \mathrm{op}(\Lambda^{-\frac{1}{2}})]w + [\mathrm{op}(b_1), \mathrm{op}(\Lambda^{-\frac{1}{2}})]w.$$

Recalling that $\mathrm{op}(b_1) \in \mathcal{TO}^1$, we obtain

$$\frac{\tau}{\mu^2}|\mathrm{op}(b_1)u|_1^2 = \frac{\tau}{\mu^2}|\mathrm{op}(\Lambda)\mathrm{op}(b_1)u|^2 \le C\left(\frac{\tau}{\mu^2}\left|\mathrm{op}(\Lambda^{\frac{1}{2}})\mathrm{op}(b_1)w\right|^2 + \frac{\tau}{\mu^2}\left|\mathrm{op}(\Lambda^{\frac{1}{2}})w\right|^2\right). \tag{3.50}$$

We have

$$\mathrm{op}(b_2)u = \mathrm{op}(b_2)\chi\mathrm{op}(\Lambda^{-\frac{1}{2}})w = \mathrm{op}(\Lambda^{-\frac{1}{2}})\mathrm{op}(b_2)w + \mathrm{op}(b_2)[\chi, \mathrm{op}(\Lambda^{-\frac{1}{2}})]w + [\mathrm{op}(b_2), \mathrm{op}(\Lambda^{-\frac{1}{2}})]w.$$

Recalling that $\mathrm{op}(b_2) \in D_{x_n} + \mathcal{T}\mathcal{O}^1$, we obtain

$$\tau \left|\mathrm{op}(b_2)u\right|^2 \leq C\left(\tau \left|\mathrm{op}(\Lambda^{-\frac{1}{2}})\mathrm{op}(b_2)w\right|^2 + \tau \left|\mathrm{op}(\Lambda^{-\frac{1}{2}})w\right|^2 + \tau \left|D_{x_n}\mathrm{op}(\Lambda^{-\frac{3}{2}})w\right|^2\right). \tag{3.51}$$

Moreover, we have

$$\tau \left|u\right|^2_{1,0,\tau} = \tau \left|u\right|^2_1 + \tau \left|D_{x_n}u\right|^2 = \tau \left|\mathrm{op}(\Lambda)u\right|^2 + \tau \left|D_{x_n}u\right|^2.$$

We can write

$$\mathrm{op}(\Lambda)u = \mathrm{op}(\Lambda)\chi\mathrm{op}(\Lambda^{-\frac{1}{2}})w = \mathrm{op}(\Lambda^{\frac{1}{2}})w + \mathrm{op}(\Lambda)[\chi, \mathrm{op}(\Lambda^{-\frac{1}{2}})]w.$$

Then

$$\tau \left|\mathrm{op}(\Lambda)u\right|^2 \geq \tau \left|\mathrm{op}(\Lambda^{\frac{1}{2}})w\right|^2 - C\tau \left|\mathrm{op}(\Lambda^{-\frac{1}{2}})w\right|^2 \geq \tau \left|\mathrm{op}(\Lambda^{\frac{1}{2}})w\right|^2 - C\tau^{-1} \left|\mathrm{op}(\Lambda^{\frac{1}{2}})w\right|^2.$$

For τ large enough, we obtain

$$\tau \left|\mathrm{op}(\Lambda)u\right|^2 \geq C\tau \left|\mathrm{op}(\Lambda^{\frac{1}{2}})w\right|^2. \tag{3.52}$$

By the same way, we prove, for τ large enough

$$\tau \left|D_{x_n}u\right|^2 \geq C\tau \left|D_{x_n}\mathrm{op}(\Lambda^{-\frac{1}{2}})w\right|^2. \tag{3.53}$$

Combining (3.52) and (3.53), we obtain

$$\tau \left|u\right|^2_{1,0,\tau} \geq C\left(\tau \left|\mathrm{op}(\Lambda^{\frac{1}{2}})w\right|^2 + \tau \left|D_{x_n}\mathrm{op}(\Lambda^{-\frac{1}{2}})w\right|^2\right). \tag{3.54}$$

By the same way, we prove

$$\left\|\mathrm{op}(\Lambda^{\frac{3}{2}})u\right\|^2 \geq \left\|\mathrm{op}(\Lambda)w\right\|^2 - C\left\|w\right\|^2, \tag{3.55}$$

$$\left\|D_{x_n}\mathrm{op}(\Lambda^{\frac{1}{2}})u\right\|^2 \geq \left\|D_{x_n}w\right\|^2 - C\left\|\mathrm{op}(\Lambda^{-1})D_{x_n}w\right\|^2 - C\left\|\mathrm{op}(\Lambda^{-1})w\right\|^2 \tag{3.56}$$

and

$$\left\|D^2_{x_n}\mathrm{op}(\Lambda^{-\frac{1}{2}})u\right\|^2 \geq \left\|D^2_{x_n}\mathrm{op}(\Lambda^{-1})w\right\|^2 - C\left\|D^2_{x_n}\mathrm{op}(\Lambda^{-2})w\right\|^2 - C\left\|D_{x_n}\mathrm{op}(\Lambda^{-2})w\right\|^2 - C\left\|\mathrm{op}(\Lambda^{-2})w\right\|^2. \tag{3.57}$$

Combining (3.55)–(3.57), we obtain for τ large enough

$$\left\|D^2_{x_n}\mathrm{op}(\Lambda^{-\frac{1}{2}})u\right\|^2 + \left\|D_{x_n}\mathrm{op}(\Lambda^{\frac{1}{2}})u\right\|^2 + \left\|\mathrm{op}(\Lambda^{\frac{3}{2}})u\right\|^2 \geq C\left(\left\|D^2_{x_n}\mathrm{op}(\Lambda^{-1})w\right\|^2 + \left\|D_{x_n}w\right\|^2 + \left\|\mathrm{op}(\Lambda)w\right\|^2\right). \tag{3.58}$$

Combining (3.24), (3.49)–(3.51), (3.54) and (3.58), we obtain (3.10), for τ large enough and $|\mu| \geq \mu_0$. □

4. Proof of Theorem 3.2

In this section, we use especially microlocal analysis and we recall and follow the notations used in [11]. The techniques used are the Calderon projector for the elliptic regions and Carleman estimates for non elliptic regions.

4.1. Study of the eigenvalues

The proof is based on a partition argument related to the nature of the roots of the polynomial $p_j(x,\xi',\xi_n,\tau,\mu)$, $j=1,2$, in ξ_n. On $x_n=0$, we note

$$q_1(x',\xi',\tau)=q_{1,1}(0,x',\xi',\tau)=q_{1,2}(0,x',\xi',\tau).$$

Let us introduce the following microlocal regions

$$\mathcal{E}_1^+=\left\{(x,\xi',\tau,\mu)\in K\times\mathbb{R}^{n-1}\times\mathbb{R}\times\mathbb{R},\quad q_{2,1}+\frac{q_1^2}{(\frac{\partial\varphi_1}{\partial x_n})^2}>0\right\},$$
$$\mathcal{Z}_1=\left\{(x,\xi',\tau,\mu)\in K\times\mathbb{R}^{n-1}\times\mathbb{R}\times\mathbb{R},\quad q_{2,1}+\frac{q_1^2}{(\frac{\partial\varphi_1}{\partial x_n})^2}=0\right\},$$
$$\mathcal{E}_1^-=\left\{(x,\xi',\tau,\mu)\in K\times\mathbb{R}^{n-1}\times\mathbb{R}\times\mathbb{R},\quad q_{2,1}+\frac{q_1^2}{(\frac{\partial\varphi_1}{\partial x_n})^2}<0\right\},$$
$$\mathcal{E}_2^+=\left\{(x,\xi',\tau,\mu)\in K\times\mathbb{R}^{n-1}\times\mathbb{R}\times\mathbb{R},\quad q_{2,2}-\mu^2+\frac{q_1^2}{(\frac{\partial\varphi_2}{\partial x_n})^2}>0\right\},$$
$$\mathcal{Z}_2=\left\{(x,\xi',\tau,\mu)\in K\times\mathbb{R}^{n-1}\times\mathbb{R}\times\mathbb{R},\quad q_{2,2}-\mu^2+\frac{q_1^2}{(\frac{\partial\varphi_2}{\partial x_n})^2}=0\right\},$$
$$\mathcal{E}_2^-=\left\{(x,\xi',\tau,\mu)\in K\times\mathbb{R}^{n-1}\times\mathbb{R}\times\mathbb{R},\quad q_{2,2}-\mu^2+\frac{q_1^2}{(\frac{\partial\varphi_2}{\partial x_n})^2}<0\right\}.$$

We consider $p_{1/2}(x,\xi,\tau,\mu)$ as a polynomial in ξ_n. Then we have the following lemma describing the root localization of $p_{1/2}$ (here and in the sequel the index $\frac{1}{2}$ $1/2$ means 1 or 2).

Lemma 4.1. *We have the following:*

1. *For $(x,\xi',\tau,\mu)\in\mathcal{E}_{1/2}^+$, the roots of $p_{1/2}$ denoted $z_{1/2}^{\pm}$ satisfy $\pm\operatorname{Im} z_{1/2}^{\pm}>0$.*
2. *For $(x,\xi',\tau,\mu)\in\mathcal{Z}_{1/2}$, one of the roots of $p_{1/2}$ is real.*
3. *For $(x,\xi',\tau,\mu)\in\mathcal{E}_{1/2}^-$, the roots of $p_{1/2}$ are in the half- plane $\operatorname{Im}\xi_n>0$ if $\frac{\partial\varphi_{1/2}}{\partial x_n}<0$ (resp. in the half-plane $\operatorname{Im}\xi_n<0$ if $\frac{\partial\varphi_{1/2}}{\partial x_n}>0$)·*

Proof. Using (3.3) and (3.4), we can write

$$\begin{cases} p_1(x',\xi,\tau,\mu)=\left(\xi_n+\mathrm{i}\tau\dfrac{\partial\varphi_1}{\partial x_n}-\mathrm{i}\alpha_1\right)\left(\xi_n+\mathrm{i}\tau\dfrac{\partial\varphi_1}{\partial x_n}+\mathrm{i}\alpha_1\right),\\[2ex] p_2(x',\xi,\tau,\mu)=\left(\xi_n+\mathrm{i}\dfrac{\partial\varphi_2}{\partial x_n}-\mathrm{i}\alpha_2\right)\left(\xi_n+\mathrm{i}\tau\dfrac{\partial\varphi_2}{\partial x_n}+\mathrm{i}\alpha_2\right),\end{cases}\tag{4.1}$$

where $\alpha_j\in\mathbb{C}$, $j=1,2$, defined by

$$\begin{cases}\alpha_1^2(x',\xi',\tau,\mu)=\left(\tau\dfrac{\partial\varphi_1}{\partial x_n}\right)^2+q_{2,1}+2\mathrm{i}\tau q_1,\\[2ex] \alpha_2^2(x',\xi',\tau,\mu)=\left(\tau\dfrac{\partial\varphi_2}{\partial x_n}\right)^2-\mu^2+q_{2,2}+2\mathrm{i}\tau q_1.\end{cases}\tag{4.2}$$

We set

$$z^{\pm}_{1/2} = -\mathrm{i}\tau \frac{\partial \varphi_{1/2}}{\partial x_n} \pm \mathrm{i}\alpha_{1/2}, \tag{4.3}$$

the roots of $p_{1/2}$. The imaginary parts of the roots of $p_{1/2}$ are

$$-\tau \frac{\partial \varphi_{1/2}}{\partial x_n} - \operatorname{Re} \alpha_{1/2}, \quad -\tau \frac{\partial \varphi_{1/2}}{\partial x_n} + \operatorname{Re} \alpha_{1/2}.$$

The signs of the imaginary parts are opposite if $\left|\partial \varphi_{1/2}/\partial x_n\right| < \left|\operatorname{Re} \alpha_{1/2}\right|$, equal to the sign of $-\partial \varphi_{1/2}/\partial x_n$ if $\left|\partial \varphi_{1/2}/\partial x_n\right| > \left|\operatorname{Re} \alpha_{1/2}\right|$ and one of the imaginary parts is null if $\left|\partial \varphi_{1/2}/\partial x_n\right| = \left|\operatorname{Re} \alpha_{1/2}\right|$. However the lines $\operatorname{Re} z = \pm \tau\, \partial \varphi_{1/2}/\partial x_n$ change by the application $z \mapsto z' = z^2$ into the parabolic curve $\operatorname{Re} z' = \left|\tau\, \partial \varphi_{1/2}/\partial x_n\right|^2 - \left|\operatorname{Im} z'\right|^2 / 4(\tau\, \partial \varphi_{1/2}/\partial x_n)^2$. Thus we obtain the lemma by replacing z' by $\alpha^2_{1/2}$. □

Lemma 4.2. *If we assume that the function φ satisfies the following condition*

$$\left(\frac{\partial \varphi_1}{\partial x_n}\right)^2 - \left(\frac{\partial \varphi_2}{\partial x_n}\right)^2 > 1, \tag{4.4}$$

then the following estimate holds

$$q_{2,2} - \mu^2 + \frac{q_1^2}{\left(\partial \varphi_2/\partial x_n\right)^2} > q_{2,1} + \frac{q_1^2}{\left(\partial \varphi_1/\partial x_n\right)^2}. \tag{4.5}$$

Proof. Following (3.4), on $\{x_n = 0\}$, we have

$$q_{2,2}(x,\xi',\tau) - q_{2,1}(x,\xi',\tau) = \left(\tau \frac{\partial \varphi_1}{\partial x_n}\right)^2 - \left(\tau \frac{\partial \varphi_2}{\partial x_n}\right)^2 \cdots \tag{4.6}$$

Using (4.4), we have (4.5). □

Remark 4.1. *The result of this lemma imply that $\mathcal{E}_1^+ \subset \mathcal{E}_2^+$.*

4.2. Estimate in $\mathcal{E}_1^+$

In this part we study the problem in the elliptic region $\mathcal{E}_1^+$. In this region we can inverse the operator and use the Calderon projectors. Let $\chi^+(x,\xi',\tau,\mu) \in \mathcal{T}S^0_\tau$ such that in the support of χ^+ we have $q_{2,1} + \frac{q_1^2}{(\partial\varphi_1/\partial x_n)^2} \geq \delta > 0$. Then we have the following estimate.

Proposition 4.1. *There exist constants $C > 0$, $\tau_0 > 0$ and $\mu_0 > 0$ such that for any $\tau \geq \tau_0$, $|\mu| \geq \mu_0$ such that $\tau \geq C_0 |\mu|$, $C_0 > 0$, we have*

$$\tau^2 \left\|\operatorname{op}(\chi^+)u\right\|^2_{1,\tau} \leq C\left(\|P(x,D,\tau,\mu)u\|^2 + \|u\|^2_{1,\tau} + \tau\,|u|^2_{1,0,\tau}\right), \tag{4.7}$$

for any $u \in C_0^\infty(\{x_n \geq 0\} \cap W)$.

If we suppose moreover that φ satisfies (4.4) then the following estimate holds

$$\tau\left|\operatorname{op}(\chi^+)u\right|^2_{1,0,\tau} \leq C\left(\|P(x,D,\tau,\mu)u\|^2 + \frac{\tau}{\mu^2}\left|\operatorname{op}(b_1)u\right|^2_1 + \tau\left|\operatorname{op}(b_2)u\right|^2 + \|u\|^2_{1,\tau} + \tau^{-2}\,|u|^2_{1,0,\tau}\right), \tag{4.8}$$

for any $u \in C_0^\infty(\{x_n \geq 0\} \cap W)$ and b_j, $j = 1, 2$, defined in (3.6).

Proof. Let $\tilde{u} = \mathrm{op}(\chi^+)u$. From (3.2), we get

$$\begin{cases} P\tilde{u} = \tilde{f} & \text{in } \{x_n > 0\} \cap W, \\ op(b_1)\tilde{u} = \tilde{u}_0|_{x_n=0} - \mathrm{i}\mu\tilde{v}_0|_{x_n=0} = \tilde{e}_1 & \text{on } \{x_n = 0\} \cap W, \\ \mathrm{op}(b_2)\tilde{u} = \left(D_{x_n} + \mathrm{i}\tau\frac{\partial\varphi_1}{\partial x_n}\right)\tilde{u}_0|_{x_n=0} + \left(D_{x_n} + \mathrm{i}\tau\frac{\partial\varphi_2}{\partial x_n}\right)\tilde{v}_0|_{x_n=0} = \tilde{e}_2 & \text{on } \{x_n = 0\} \cap W, \end{cases} \tag{4.9}$$

with $\tilde{f} = \mathrm{op}(\chi^+)f + [P, \mathrm{op}(\chi^+)]\, u$. Since $[P, \mathrm{op}(\chi^+)] \in (\mathcal{TO}^0)D_{x_n} + \mathcal{TO}^1$, we have

$$\|\tilde{f}\|^2_{L^2} \le C\left(\|P(x, D, \tau, \mu)u\|^2_{L^2} + \|u\|^2_{1,\tau}\right) \tag{4.10}$$

and $\tilde{e}_1 = \mathrm{op}(\chi^+)e_1$ satisfying

$$|\tilde{e}_1|^2_1 \le C\,|e_1|^2_1 \tag{4.11}$$

and

$$\tilde{e}_2 = \left[(D_{x_n} + \mathrm{i}\tau\frac{\partial\varphi_1}{\partial x_n}), \mathrm{op}(\chi^+)\right] u_0|_{x_n=0} + \left[(D_{x_n} + \mathrm{i}\tau\frac{\partial\varphi_2}{\partial x_n}), \mathrm{op}(\chi^+)\right] v_0|_{x_n=0} + \mathrm{op}(\chi^+)e_2.$$

Since $[D_{x_n}, \mathrm{op}(\chi^+)] \in \mathcal{TO}^0$, we have

$$|\tilde{e}_2|^2 \le C\left(|u|^2 + |e_2|^2\right). \tag{4.12}$$

Let $\underline{\tilde{u}}$ the extension of $\tilde{u}$ by 0 in $x_n < 0$. According to (3.3)–(3.5), we obtain, by noting $\partial\varphi/\partial x_n = \mathrm{diag}\,(\partial\varphi_1/\partial x_n, \partial\varphi_2/\partial x_n)$, $\gamma_j(\tilde{u}) = {}^t\left(D^j_{x_n}(\tilde{u}_0)\,|_{x_n=0^+}, D^j_{x_n}(\tilde{v}_0)\,|_{x_n=0^+}\right)$, $j = 0, 1$ and $\delta^{(j)} = (\mathrm{d}/\mathrm{d}x_n)^j\,(\delta_{x_n=0})$,

$$P\underline{\tilde{u}} = \underline{\tilde{f}} - \gamma_0(\tilde{u}) \otimes \delta' + \frac{1}{\mathrm{i}}\left(\gamma_1(\tilde{u}) + 2\mathrm{i}\tau\frac{\partial\varphi}{\partial x_n}\right) \otimes \delta \tag{4.13}$$

Let $\chi(x, \xi, \tau, \mu) \in S^0_\tau$ be equal to 1 for sufficiently large $|\xi| + \tau$ and in a neighborhood of $\mathrm{supp}(\chi^+)$. We assume p is elliptic in support of χ. These conditions are compatible due to the choice made for $\mathrm{supp}(\chi^+)$ and Remark 4.1. Let m large enough chosen later, by ellipticity of p on $\mathrm{supp}(\chi)$ there exists a parametric $E = \mathrm{op}(e)$ of P. We recall that $e \in S^{-2}_\tau$ and e has the following form $e(x, \xi, \tau, \mu) = \sum_{j=0}^m e_j(x, \xi, \tau, \mu)$, where $e_0 = \chi p^{-1}$ and $e_j = \mathrm{diag}(e_{j,1}, e_{j,2}) \in S^{-2-j}_\tau$ where $e_{j,1}$ and $e_{j,2}$ are rational functions with respect to ξ_n. Then we have

$$EP = \mathrm{op}(\chi) + R_m, \quad R_m \in \mathcal{O}^{-m-1}. \tag{4.14}$$

Following (4.13) and (4.14), we obtain

$$\begin{cases} \underline{\tilde{u}} = E\underline{\tilde{f}} + E\left[-h_1 \otimes \delta' + \frac{1}{\mathrm{i}}h_0 \otimes \delta\right] + w_1, \\ h_0 = \gamma_1(\tilde{u}) + 2\mathrm{i}\tau\frac{\partial\varphi}{\partial x_n}\gamma_0(\tilde{u}), \quad h_1 = \gamma_0(\tilde{u}), \\ w_1 = (\mathrm{Id} - \mathrm{op}(\chi))\,\underline{\tilde{u}} - R_m\underline{\tilde{u}}. \end{cases} \tag{4.15}$$

Using the fact that $\mathrm{supp}(1 - \chi) \cap \mathrm{supp}(\chi^+) = \varnothing$ and the symbolic calculus (see Lem. 2.10 in [8]), we have $(\mathrm{Id} - \mathrm{op}(\chi))\,\mathrm{op}(\chi^+) \in \mathcal{O}^{-m}$, we obtain

$$\|w_1\|^2_{2,\tau} \le C\tau^{-2}\,\|u\|^2_{L^2}\,. \tag{4.16}$$

Now, let us look at the term $E\left[-h_1\otimes\delta'+\frac{1}{\mathrm{i}}h_0\otimes\delta\right]$. For $x_n>0$, we get

$$\begin{cases} E\left[-h_1\otimes\delta'+\dfrac{1}{\mathrm{i}}h_0\otimes\delta\right]=\hat{T}_1h_1+\hat{T}_0h_0,\\ \hat{T}_j(h)=\left(\dfrac{1}{2\pi}\right)^{n-1}\displaystyle\int \mathrm{e}^{\mathrm{i}(x'-y')\xi'}\hat{t}_j(x,\xi',\tau,\mu)h(y')\mathrm{d}y'\mathrm{d}\xi'=\mathrm{op}(\hat{t}_j)h\\ \hat{t}_j=\dfrac{1}{2\pi\mathrm{i}}\displaystyle\int_\gamma \mathrm{e}^{\mathrm{i}x_n\xi_n}e(x,\xi,\tau,\mu)\xi_n^j\mathrm{d}\xi_n\end{cases}$$

where γ is the union of the segment $\{\xi_n\in\mathbb{R},\,|\xi_n|\le c_0\sqrt{|\xi'|^2+\tau^2}\}$ and the half circle $\{\xi_n\in\mathbb{C},\,|\xi_n|=c_0\sqrt{|\xi'|^2+\tau^2},\,\mathrm{Im}\xi_n>0\}$, where the constant c_0 is chosen large enough for the roots z_1^+ and z_2^+ to be enclosed by γ (if c_0 is large enough, the change of contour $\mathbb{R}\longrightarrow\gamma$ is possible because the symbol $e(x,\xi,\tau,\mu)$ is a holomorphic function for large $|\xi_n|$; $\xi_n\in\mathbb{C}$). In particular we have in $x_n\ge 0$

$$\left|\partial_{x_n}^k\partial_{x'}^\alpha\partial_{\xi'}^\beta\hat{t}_j\right|\le C_{\alpha,\beta,k}\langle\xi',\tau\rangle^{j-1-|\beta|+k},\quad j=0,1. \tag{4.17}$$

We now choose $\chi_1(x,\xi',\tau,\mu)\in\mathcal{T}S^0_\tau$, satisfying the same requirement as χ^+, equal to 1 in a neighborhood of $\mathrm{supp}(\chi^+)$ and such that the symbol χ be equal to 1 in a neighborhood of $\mathrm{supp}(\chi_1)$. We set $t_j=\chi_1\hat{t}_j$, $j=0,1$. Then we obtain

$$\underline{\tilde{u}}=E\underline{\tilde{f}}+\mathrm{op}(t_0)h_0+\mathrm{op}(t_1)h_1+w_1+w_2 \tag{4.18}$$

where $w_2=\mathrm{op}((1-\chi_1)\hat{t}_0)h_0+\mathrm{op}((1-\chi_1)\hat{t}_1)h_1$. By using the composition formula of tangential operator, estimate (4.17), the fact that $\mathrm{supp}(1-\chi_1)\cap\mathrm{supp}(\chi^+)=\varnothing$ and the following trace formula

$$|\gamma_0(u)|_j\le C\tau^{-\frac{1}{2}}\|u\|_{j+1,\tau},\quad j\in\mathbb{N}, \tag{4.19}$$

we obtain

$$\|w_2\|^2_{2,\tau}\le C\tau^{-2}\left(\|u\|^2_{1,\tau}+|u|^2_{1,0,\tau}\right). \tag{4.20}$$

Since $\chi=1$ in the support of χ_1, we have $e(x,\xi,\tau,\mu)$ is meromorphic w.r.t. ξ_n in the support of χ_1. The roots $z^+_{1/2}$ are in $\mathrm{Im}\xi_n\ge c_1\sqrt{|\xi'|^2+\tau^2}$ $(c_1>0)$. If c_1 is small enough we can choose fixed contours $\gamma_{1/2}$ in $\mathrm{Im}\xi_n\ge\frac{c_1}{2}\sqrt{|\xi'|^2+\tau^2}$ and we can write

$$t_j=\mathrm{diag}(t_{j,1},t_{j,2}),\quad t_{j,1/2}(x,\xi',\tau,\mu)=\chi_1(x,\xi',\tau,\mu)\frac{1}{2\pi\mathrm{i}}\int_{\gamma_{1/2}}\mathrm{e}^{\mathrm{i}x_n\xi_n}\mathrm{e}_{1/2}(x,\xi,\tau,\mu)\xi_n^j\mathrm{d}\xi_n,\quad j=0,1. \tag{4.21}$$

Then there exists $c_2>0$ such that in $x_n\ge0$, we obtain

$$\left|\partial_{x_n}^k\partial_{x'}^\alpha\partial_{\xi'}^\beta t_j\right|\le C_{\alpha,\beta,k}\mathrm{e}^{-c_2x_n\langle\xi',\tau\rangle}\langle\xi',\tau\rangle^{j-1-|\beta|+k}. \tag{4.22}$$

In particular, we have $\mathrm{e}^{c_2x_n\tau}(\partial^k_{x_n})t_j$ bounded in $\mathcal{T}S^{j-1+k}_\tau$ uniformly w.r.t. $x_n\ge0$. Then

$$\|\partial_{x'}\mathrm{op}(t_j)h_j\|^2_{L^2}+\|\mathrm{op}(t_j)h_j\|^2_{L^2}\le C\int_{x_n>0}\mathrm{e}^{-2c_2x_n\tau}\left|\mathrm{op}(\mathrm{e}^{c_2x_n\tau}t_j)h_j\right|^2_1(x_n)\mathrm{d}x_n\le C\tau^{-1}|h_j|^2_j$$

and

$$\|\partial_{x_n}\mathrm{op}(t_j)h_j\|^2_{L^2}\le C\int_{x_n>0}\mathrm{e}^{-2c_2x_n\tau}\left|\mathrm{op}(\mathrm{e}^{c_2x_n\tau}\partial_{x_n}t_j)h_j\right|^2_{L^2}(x_n)\mathrm{d}x_n\le C\tau^{-1}|h_j|^2_j.$$

Using the fact that $h_0 = \gamma_1(\tilde{u}) + 2\mathrm{i}\tau\frac{\partial\varphi}{\partial x_n}\gamma_0(\tilde{u})$ and $h_1 = \gamma_0(\tilde{u})$, we obtain

$$\|\mathrm{op}(t_j)h_j\|^2_{1,\tau} \le C\tau^{-1}|u|^2_{1,0,\tau}. \tag{4.23}$$

From (4.18) and estimates (4.10), (4.16), (4.20) and (4.23), we obtain (4.7).

It remains to proof (4.8). We recall that, in $\mathrm{supp}(\chi_1)$, we have

$$e_0 = \mathrm{diag}\,(e_{0,1}, e_{0,2}) = \mathrm{diag}\left(\frac{1}{p_1}, \frac{1}{p_2}\right) = \mathrm{diag}\left(\frac{1}{(\xi_n - z_1^+)(\xi_n - z_1^-)}, \frac{1}{(\xi_n - z_2^+)(\xi_n - z_2^-)}\right).$$

Using the residue formula, we obtain

$$\mathrm{e}^{-\mathrm{i}x_n z^+_{1/2}} t_{j,1/2} = \chi_1 \frac{(z^+_{1/2})^j}{z^+_{1/2} - z^-_{1/2}} + \lambda_{1/2}, \quad j = 0, 1, \quad \lambda_{1/2} \in \mathcal{T}S^{-2+j}_\tau. \tag{4.24}$$

Taking the traces of (4.18), we obtain

$$\gamma_0(\tilde{u}) = \mathrm{op}(c)\gamma_0(\tilde{u}) + \mathrm{op}(d)\gamma_1(\tilde{u}) + w_0, \tag{4.25}$$

where $w_0 = \gamma_0(E\underline{\tilde{f}} + w_1 + w_2)$ satisfies, according to the trace formula (4.19), the estimates (4.10), (4.16) and (4.20), the following estimate

$$\tau\,|w_0|_1^2 \le C\left(\|P(x, D, \tau, \mu)u\|^2 + \|u\|^2_{1,\tau} + \tau^{-2}\,|u|^2_{1,0,\tau}\right). \tag{4.26}$$

Following (4.23), c and d are two tangential symbols of order respectively 0 and -1 given by

$$c_0 = \mathrm{diag}(c_{0,1}, c_{0,2}) \qquad \text{with} \quad c_{0,1/2} = -\left(\chi_1 \frac{z^-_{1/2}}{z^+_{1/2} - z^-_{1/2}}\right),$$

$$d_{-1} = \mathrm{diag}(d_{-1,1}, d_{-1,2}) \quad \text{with} \quad d_{-1,1/2} = \left(\chi_1 \frac{1}{z^+_{1/2} - z^-_{1/2}}\right).$$

Following (4.9), the transmission conditions give

$$\begin{cases} \gamma_0(\tilde{u}_0) - \mathrm{i}\mu\gamma_0(\tilde{v}_0) = \tilde{e}_1 \\ \gamma_1(\tilde{u}_0) + \gamma_1(\tilde{v}_0) + \mathrm{i}\tau\frac{\partial\varphi_1}{\partial x_n}\gamma_0(\tilde{u}_0) + \mathrm{i}\tau\frac{\partial\varphi_2}{\partial x_n}\gamma_0(\tilde{v}_0) = \tilde{e}_2. \end{cases} \tag{4.27}$$

We recall that $\tilde{u} = (\tilde{u}_0, \tilde{v}_0)$, combining (4.25) and (4.27) we show that

$$\mathrm{op}(k)^{\,t}\left(\gamma_0(\tilde{u}_0), \gamma_0(\tilde{v}_0), \Lambda^{-1}\gamma_1(\tilde{u}_0), \Lambda^{-1}\gamma_1(\tilde{v}_0)\right) = w_0 + \frac{1}{\mu}\mathrm{op}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\tilde{e}_1 + \mathrm{op}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\Lambda^{-1}\tilde{e}_2, \tag{4.28}$$

where k is a 4×4 matrix, with principal symbol defined by

$$k_0+\frac{1}{\mu}r_0=\begin{pmatrix} 1-c_{0,1} & 0 & -\Lambda\, d_{-1,1} & 0\\ 0 & 1-c_{0,2} & 0 & -\Lambda\, d_{-1,2}\\ 0 & -\mathrm{i} & 0 & 0\\ \mathrm{i}\tau\Lambda^{-1}\dfrac{\partial\varphi_1}{\partial x_n} & \mathrm{i}\tau\Lambda^{-1}\dfrac{\partial\varphi_2}{\partial x_n} & 1 & 1\end{pmatrix}+\frac{1}{\mu}r_0,$$

where r_0 is a tangential symbol of order 0.

We now choose $\chi_2(x,\xi',\tau,\mu)\in\mathcal{T}S^0_\tau$, satisfying the same requirement as χ^+, equal to 1 in a neighborhood of $\mathrm{supp}(\chi^+)$ and such that the symbol χ_1 be equal to 1 in a neighborhood of $\mathrm{supp}(\chi_2)$. In $\mathrm{supp}(\chi_2)$, we obtain

$$k_0|_{\mathrm{supp}(\chi_2)}=\begin{pmatrix} \dfrac{z_1^+}{z_1^+-z_1^-} & 0 & -\dfrac{\Lambda}{z_1^+-z_1^-} & 0\\ 0 & \dfrac{z_2^+}{z_2^+-z_2^-} & 0 & -\dfrac{\Lambda}{z_2^+-z_2^-}\\ 0 & -\mathrm{i} & 0 & 0\\ \mathrm{i}\tau\Lambda^{-1}\dfrac{\partial\varphi_1}{\partial x_n} & \mathrm{i}\tau\Lambda^{-1}\dfrac{\partial\varphi_2}{\partial x_n} & 1 & 1\end{pmatrix}.$$

Then, following (4.3),

$$\det(k_0)|_{\mathrm{supp}(\chi_2)}=-\left(z_1^+-z_1^-\right)^{-1}\left(z_2^+-z_2^-\right)^{-1}\Lambda\,\alpha_1.$$

To prove that there exists $c>0$ such that $\left|\det(k_0)|_{\mathrm{supp}(\chi_2)}\right|\geq c$, by homogeneity it suffices to prove that $\det(k_0)|_{\mathrm{supp}(\chi_2)}\neq 0$ if $|\xi'|^2+\tau^2=1$.

If we suppose that $\det(k_0)|_{\mathrm{supp}(\chi_2)}=0$, we obtain $\alpha_1=0$ and then $\alpha_1^2=0$.

Following (4.2),we obtain

$$q_1=0\quad\text{and}\quad\left(\tau\frac{\partial\varphi_1}{\partial x_n}\right)^2+q_{2,1}=0. \tag{4.29}$$

But in $\mathcal{E}_1^+$, this implies $q_{2,1}>0$, then (4.29) is impossible.

Therefore $\det(k_0)|_{\mathrm{supp}(\chi_2)}\neq 0$. It follows that, for τ large enough, $k=k_0+\frac{1}{\mu}r_0$ is elliptic in $\mathrm{supp}(\chi_2)$. Then there exists $l\in\mathcal{T}S^0_\tau$, such that

$$\mathrm{op}(l)\mathrm{op}(k)=\mathrm{op}(\chi_2)+\tilde{R}_m,$$

with $\tilde{R}_m\in\mathcal{T}\mathcal{O}^{-m-1}$, for m large enough. This yields

$$\begin{aligned}{}^t\left(\gamma_0(\tilde u_0),\gamma_0(\tilde v_0),\Lambda^{-1}\gamma_1(\tilde u_0),\Lambda^{-1}\gamma_1(\tilde v_0)\right)&=\mathrm{op}(l)w_0+\tfrac{1}{\mu}\mathrm{op}(l)\mathrm{op}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\tilde e_1+\mathrm{op}(l)\mathrm{op}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\Lambda^{-1}\tilde e_2\\ &\quad+\left(\mathrm{op}(1-\chi_2)-\tilde R_m\right)^t\left(\gamma_0(\tilde u_0),\gamma_0(\tilde v_0),\Lambda^{-1}\gamma_1(\tilde u_0),\Lambda^{-1}\gamma_1(\tilde v_0)\right).\end{aligned}$$

Since $\mathrm{supp}(1-\chi_2)\cap\mathrm{supp}(\chi^+)=\varnothing$ and by using (4.26), we obtain

$$\tau|\tilde u|^2_{1,0,\tau}\le C\left(\frac{\tau}{\mu^2}|\tilde e_1|_1^2+\tau|\tilde e_2|^2+\|P(x,D,\tau,\mu)u\|^2_{L^2}+\|u\|^2_{1,\tau}+\tau^{-2}\,|u|^2_{1,0,\tau}\right).$$

From estimates (4.11) and (4.12) and the trace formula (4.19), we obtain (4.8). □

4.3. **Estimate in $\mathcal{Z}_1$**

The aim of this part is to prove the estimate in the region $\mathcal{Z}_1$. In this region, if φ satisfies (4.4), the symbol $p_1(x,\xi,\tau,\mu)$ admits a real root and $p_2(x,\xi,\tau,\mu)$ admits two roots $z_2^\pm$ satisfy $\pm\,\mathrm{Im}(z_2^\pm)>0$. Let $\chi^0(x,\xi',\tau,\mu)\in\mathcal{TS}^0_\tau$ equal to 1 in $\mathcal{Z}_1$ and such that in the support of χ^0 we have $q_{2,2}-\mu^2+\frac{q_1^2}{(\partial\varphi_2/\partial x_n)^2}\ge\delta>0$. Then we have the following estimate.

Proposition 4.2. *There exist constants $C>0$, $\tau_0>0$ and $\mu_0>0$ such that for any $\tau\ge\tau_0$, $|\mu|\ge\mu_0$ such that $\tau\ge C_0\,|\mu|$, $C_0>0$, we have the following estimate*

$$\tau\left\|\mathrm{op}(\chi^0)u\right\|^2_{1,\tau}\le C\left(\|P(x,D,\tau,\mu)u\|^2+\tau\,|u|^2_{1,0,\tau}+\|u\|^2_{1,\tau}\right),\tag{4.30}$$

for any $u\in C_0^\infty(\{x_n\ge0\}\cap W)$.

If we assume moreover that φ satisfies (4.4) then we have

$$\tau\left|\mathrm{op}(\chi^0)u\right|^2_{1,0,\tau}\le C\left(\|P(x,D,\tau,\mu)u\|^2+\frac{\tau}{\mu^2}\,|\mathrm{op}(b_1)u|_1^2+\tau\,|\mathrm{op}(b_2)u|^2+\|u\|^2_{1,\tau}+\tau^{-2}\,|u|^2_{1,0,\tau}\right),\tag{4.31}$$

for any $u\in C_0^\infty(\{x_n\ge0\}\cap W)$ and b_j, $j=1,2$, defined in (3.6).

4.3.1. *Preliminaries*

Let $u\in C_0^\infty(K)$, $\tilde u=\mathrm{op}(\chi^0)u$ and P the differential operator with principal symbol given by

$$p(x,\xi,\tau,\mu)=\mathrm{diag}(p_1,p_2)$$

with p_1 and p_2 defined in (3.7). Then we have the following system

$$\begin{cases}P\tilde u=\tilde f & \text{in }\{x_n>0\}\cap W,\\ B\tilde u=\tilde e=(\tilde e_1,\tilde e_2) & \text{on }\{x_n=0\}\cap W,\end{cases}\tag{4.32}$$

where $\tilde f=\mathrm{op}(\chi^0)f+\left[P,\mathrm{op}(\chi^0)\right]u$. Since $\left[P,\mathrm{op}(\chi^0)\right]\in(\mathcal{TO}^0)D_{x_n}+\mathcal{TO}^1$, we have

$$\|\tilde f\|^2_{L^2}\le C\left(\|P(x,D,\tau,\mu)u\|^2_{L^2}+\|u\|^2_{1,\tau}\right),\tag{4.33}$$

B defined in (3.6) and $\tilde e_1=\mathrm{op}(\chi^0)e_1$ satisfying

$$|\tilde e_1|_1^2\le C\,|e_1|_1^2\tag{4.34}$$

and

$$\tilde e_2=\left[(D_{x_n}+\mathrm{i}\tau\tfrac{\partial\varphi_1}{\partial x_n}),\mathrm{op}(\chi^0)\right]u_0|_{x_n=0}+\left[\left(D_{x_n}+\mathrm{i}\tau\frac{\partial\varphi_2}{\partial x_n}\right),\mathrm{op}(\chi^0)\right]v_0|_{x_n=0}+\mathrm{op}(\chi^0)e_2.$$

Since $[D_{x_n}, \mathrm{op}(\chi^+)] \in \mathcal{TO}^0$, we have

$$|\tilde{e}_2|^2 \leq C\left(|u|^2 + |e_2|^2\right). \tag{4.35}$$

Let us reduce the problem (4.32) to a first order system. Put $v = {}^t\,(\langle D', \tau\rangle \tilde{u}, D_{x_n}\tilde{u})$. Then we obtain the following system

$$\begin{cases} D_{x_n} v - \mathrm{op}(\mathcal{P})v = F & \text{in } \{x_n > 0\} \cap W, \\ \mathrm{op}(\mathcal{B})v = (\frac{1}{\mu}\Lambda\tilde{e}_1, \tilde{e}_2) & \text{on } \{x_n = 0\} \cap W, \end{cases} \tag{4.36}$$

where $\mathcal{P}$ is a 4×4 matrix, with principal symbol defined by

$$\mathcal{P}_0 = \begin{pmatrix} 0 & \Lambda\,\mathrm{Id}_2 \\ \Lambda^{-1} l_2 & -\mathrm{i} l_1 \end{pmatrix}, \quad l_1 = \begin{pmatrix} q_{1,1} & 0 \\ 0 & q_{1,2} \end{pmatrix}, \quad l_2 = \begin{pmatrix} q_{2,1} & 0 \\ 0 & q_{2,2} - \mu^2 \end{pmatrix},$$

$\mathcal{B}$ is a tangential symbol of order 0, with principal symbol given by

$$\mathcal{B}_0 + \frac{1}{\mu} r_0 = \begin{pmatrix} 0 & -\mathrm{i} & 0 & 0 \\ \mathrm{i}\Lambda^{-1}\frac{\partial\varphi_1}{\partial x_n} & \mathrm{i}\tau\Lambda^{-1}\frac{\partial\varphi_2}{\partial x_n} & 1 & 1 \end{pmatrix} + \frac{1}{\mu} r_0$$

(r_0 a tangential symbol of order 0), $F = {}^t(0, \tilde{f})$ and $\Lambda = \langle \xi', \tau\rangle = \left(|\xi'|^2 + \tau^2\right)^{\frac{1}{2}}$.

For a fixed $(x_0, \xi_0', \tau_0, \mu_0)$ in $\mathrm{supp}\chi_0$, the generalized eigenvalues of the matrix $\mathcal{P}$ are the zeroes in ξ_n of p_1 and p_2 *i.e.* $z_1^{\pm} = -\mathrm{i}\tau\frac{\partial\varphi_1}{\partial x_n} \pm \mathrm{i}\alpha_1$ and $z_2^{\pm} = -\mathrm{i}\tau\frac{\partial\varphi_2}{\partial x_n} \pm \mathrm{i}\alpha_2$ with $\pm\mathrm{Im}(z_2^{\pm}) > 0$ and $z_1^{+} \in \mathbb{R}$.

Let $s(x, \xi', \tau, \mu) = (s_1^-, s_2^-, s_1^+, s_2^+)$ a basis of the generalized eigenspace of $\mathcal{P}(x_0, \xi_0', \tau_0, \mu_0)$ corresponding to eigenvalues with positive or negative imaginary parts. The vectors $s_j^{\pm}(x, \xi', \tau, \mu)$, $j = 1, 2$ are C^∞ functions on a conic neighborhood of $(x_0, \xi_0', \tau_0, \mu_0)$ of degree zero in (ξ', τ, μ). We denote $\mathrm{op}(s)(x, D_{x'}, \tau, \mu)$ the pseudo-differential operator associated to the principal symbol

$$s(x, \xi', \tau, \mu) = \left(s_1^-(x, \xi', \tau, \mu), s_2^-(x, \xi', \tau, \mu), s_1^+(x, \xi', \tau, \mu), s_2^+(x, \xi', \tau, \mu)\right).$$

Let $\hat{\chi}(x, \xi', \tau, \mu) \in \mathcal{T}S_\tau^0$ equal to 1 in a conic neighborhood of $(x_0, \xi_0', \tau_0, \mu_0)$ and in a neighborhood of $\mathrm{supp}(\chi^0)$ and satisfies that in the support of $\hat{\chi}$, s is elliptic. Then there exists $n \in \mathcal{T}S_\tau^0$, such that

$$\mathrm{op}(s)\mathrm{op}(n) = \mathrm{op}(\hat{\chi}) + \hat{R}_m,$$

with $\hat{R}_m \in \mathcal{TO}^{-m-1}$, for m large.

Let $V = \mathrm{op}(n)v$. Then we have the following system

$$\begin{cases} D_{x_n} V = GV + AV + F_1 & \text{in } \{x_n > 0\} \cap W, \\ \mathrm{op}(\mathcal{B}_1)V = (\frac{1}{\mu}\Lambda\tilde{e}_1, \tilde{e}_2) + v_1 & \text{on } \{x_n = 0\} \cap W, \end{cases} \tag{4.37}$$

where $G = \mathrm{op}(n)\mathrm{op}(\mathcal{P})\mathrm{op}(s)$, $A = [D_{x_n}, \mathrm{op}(n)]\,\mathrm{op}(s)$, $F_1 = \mathrm{op}(n)F + \mathrm{op}(n)\mathrm{op}(\mathcal{P})(\mathrm{op}(1 - \hat{\chi}) - \hat{R}_m)v + [D_{x_n}, \mathrm{op}(n)]\,(\mathrm{op}(1 - \hat{\chi}) - \hat{R}_m)v$, $\mathrm{op}(\mathcal{B}_1) = \mathrm{op}(\mathcal{B})\mathrm{op}(s)$ and $v_1 = \mathrm{op}(\mathcal{B})(\mathrm{op}(\hat{\chi} - 1) + \hat{R}_m)v$.

Using the fact that $\mathrm{supp}(1 - \hat{\chi}) \cap \mathrm{supp}(\chi^0) = \varnothing$, $\hat{R}_m \in \mathcal{TO}^{-m-1}$, for m large and estimate (4.33), we show

$$\|F_1\|^2 \leq C\left(\|P(x, D, \tau, \mu)u\|_{L^2}^2 + \|u\|_{1,\tau}^2\right). \tag{4.38}$$

Using the fact that $\mathrm{supp}(1 - \hat{\chi}) \cap \mathrm{supp}(\chi^0) = \varnothing$, $\hat{R}_m \in \mathcal{TO}^{-m-1}$, for m large and the trace formula (4.19), we show

$$\tau\,|v_1|^2 \leq C\left(\tau^{-2}\,|u|_{1,0,\tau}^2 + \|u\|_{1,\tau}^2\right). \tag{4.39}$$

Here we recall an argument shown in Taylor [14] given by this lemma:

Lemma 4.3. *Let v solves the system*

$$\frac{\partial}{\partial y}v = Gv + Av$$

where $G = \begin{pmatrix} E & \\ & F \end{pmatrix}$ *and A are pseudo-differential operators of order 1 and 0, respectively. We suppose that the symbols of E and F are two square matrices and have disjoint sets of eigenvalues. Then there exists a pseudo-differential operator K of order -1 such that $w = (I + K)v$ satisfies*

$$\frac{\partial}{\partial y}w = Gw + \begin{pmatrix} \alpha_1 & \\ & \alpha_2 \end{pmatrix} w + R_1 w + R_2 v$$

where α_j and R_j, $j = 1, 2$ are pseudo-differential operators of order 0 and $-\infty$, respectively.

By this argument, there exists a pseudo-differential operator $K(x, D_{x'}, \tau, \mu)$ of order -1 such that the boundary problem (4.37) is reduced to the following

$$\begin{cases} D_{x_n} w - \mathrm{op}(\mathcal{H})w = \tilde{F} & \text{in } \{x_n > 0\} \cap W, \\ \mathrm{op}(\tilde{\mathcal{B}})w = (\frac{1}{\mu}\Lambda \tilde{e}_1, \tilde{e}_2) + v_1 + v_2 & \text{on } \{x_n = 0\} \cap W, \end{cases} \tag{4.40}$$

where $w = (I + K)V$, $\tilde{F} = (I + K)F_1$, $\mathrm{op}(\mathcal{H})$ is a tangential of order 1 with principal symbol $\mathcal{H} = \mathrm{diag}(\mathcal{H}^-, \mathcal{H}^+)$ and $-\mathrm{Im}(\mathcal{H}^-) \geq C\Lambda$, $\mathrm{op}(\tilde{\mathcal{B}}) = \mathrm{op}(\mathcal{B}_1)(I + K')$ with K' is such that $(I + K')(I + K) = Id + R'_m$ ($R'_m \in \mathcal{O}^{-m-1}$, for m large) and $v_2 = \mathrm{op}(\mathcal{B}_1)R'_m V$.

According to (4.38), we have

$$\|\tilde{F}\|^2 \leq C\left(\|P(x, D, \tau, \mu)u\|^2_{L^2} + \|u\|^2_{1,\tau}\right). \tag{4.41}$$

Using the fact that $R'_m \in \mathcal{O}^{-m-1}$, for m large, the trace formula (4.19) and estimates (4.34), (4.35) and (4.39), we show

$$\tau\left|\mathrm{op}(\tilde{\mathcal{B}})w\right|^2 \leq C\left(\frac{\tau}{\mu^2}|e_1|_1^2 + \tau|e_2|^2 + \tau^{-2}|u|^2_{1,0,\tau} + \|u\|^2_{1,\tau}\right). \tag{4.42}$$

Lemma 4.4. *Let $\mathcal{R} = \mathrm{diag}(-\rho Id_2, 0)$, $\rho > 0$. Then there exists $C > 0$ such that:*

(1) $\mathrm{Im}(\mathcal{RH}) = \mathrm{diag}\,(e(x, \xi', \tau, \mu), 0)$, *with* $e(x, \xi', \tau, \mu) = -\rho\mathrm{Im}(\mathcal{H}^-)$,
(2) $e(x, \xi', \tau, \mu) \geq C\Lambda$ *in* $\mathrm{supp}\,(\chi^0)$,
(3) $-\mathcal{R} + \tilde{\mathcal{B}}^\star\tilde{\mathcal{B}} \geq C.Id$ *on* $\{x_n = 0\} \cap W \cap \mathrm{supp}\,(\chi^0)$.

Proof. We have

$$\mathrm{Im}(\mathcal{RH}) = \mathrm{diag}\left(-\rho\,\mathrm{Im}(\mathcal{H}^-), 0\right) = \mathrm{diag}\left(e(x, \xi', \tau, \mu), 0\right), \tag{4.43}$$

where $e(x, \xi', \tau, \mu) = -\rho\,\mathrm{Im}(\mathcal{H}^-) \geq C\Lambda$, $C > 0$. It remains to prove (3).

We denote the principal symbol $\tilde{\mathcal{B}}$ of the boundary operator $\mathrm{op}(\tilde{\mathcal{B}})$ by $\left(\tilde{\mathcal{B}}^-, \tilde{\mathcal{B}}^+\right)$ where $\tilde{\mathcal{B}}^+$ is the restriction of $\tilde{\mathcal{B}}$ to subspace generated by $\left(s_1^+, s_2^+\right)$. We begin by proving that $\tilde{\mathcal{B}}^+$ is an isomorphism. Denote

$$w_1 =^t (1, 0) \quad \text{and} \quad w_2 =^t (0, 1).$$

Then

$$\begin{cases} s_1^+ = \left(w_1, z_1^+\Lambda^{-1}w_1\right) \\ s_2^+ = \left(w_2, z_2^+\Lambda^{-1}w_2\right) \end{cases}$$

are eigenvectors associated to z_1^+ and z_2^+. We have $\tilde{\mathcal{B}}^+ = (\mathcal{B}_0 + \frac{1}{\mu} r_0)(s_1^+ \; s_2^+) = \mathcal{B}_0^+ + \frac{1}{\mu} r_0^+$. To proof that $\tilde{\mathcal{B}}^+$ is an isomorphism it suffices, for τ large, to proof that $\mathcal{B}_0^+$ is an isomorphism. Following (4.3), we obtain

$$\mathcal{B}_0^+ = \begin{pmatrix} 0 & -\mathrm{i} \\ \Lambda^{-1}\mathrm{i}\alpha_1 & \Lambda^{-1}\mathrm{i}\alpha_2 \end{pmatrix}.$$

Then

$$\det(\mathcal{B}_0^+) = -\Lambda^{-1}\alpha_1.$$

If we suppose that $\det(\mathcal{B}_0^+) = 0$, we obtain $\alpha_1 = 0$ and then $\alpha_1^2 = 0$. Following (4.2), we obtain

$$q_1 = 0 \quad \text{and} \quad \left(\tau \frac{\partial \varphi_1}{\partial x_n}\right)^2 + q_{2,1} = 0.$$

Combining with the fact that $q_{2,1} + \frac{q_1^2}{(\partial\varphi_1/\partial x_n)^2} = 0$, we obtain $\left(\tau \frac{\partial \varphi_1}{\partial x_n}\right)^2 = 0$, that is impossible because following (4.4), we have $\left(\frac{\partial \varphi_1}{\partial x_n}\right)^2 \neq 0$ and following (3.4), we have $\tau \neq 0$. We deduce that $\tilde{\mathcal{B}}^+$ is an isomorphism.

Let $w = (w^-, w^+) \in \mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2$. Then we have $\tilde{\mathcal{B}} w = \tilde{\mathcal{B}}^- w^- + \tilde{\mathcal{B}}^+ w^+$. Since $\tilde{\mathcal{B}}^+$ is an isomorphism, then there exists a constant $C > 0$ such that

$$\left|\tilde{\mathcal{B}}^+ w^+\right|^2 \geq C \left|w^+\right|^2.$$

Therefore, we have

$$\left|w^+\right|^2 \leq C \left(\left|\tilde{\mathcal{B}} w\right|^2 + \left|w^-\right|^2\right).$$

We deduce

$$-(\mathcal{R}w, w) = \rho \left|w^-\right|^2 \geq \frac{1}{C}\left|w^+\right|^2 + (\rho - 1)\left|w^-\right|^2 - \left|\tilde{\mathcal{B}} w\right|^2.$$

Then, we obtain the result, if ρ is large enough. □

4.3.2. *Proof of Proposition* 4.2

We start by showing (4.30). We have

$$\begin{aligned} \|P_1(x, D, \tau, \mu) u_0\|^2 &= \|(\mathrm{Re} P_1) u_0\|^2 + \|(\mathrm{Im} P_1) u_0\|^2 \\ &\quad + \mathrm{i}\left[\Big((\mathrm{Im} P_1) u_0, (\mathrm{Re} P_1) u_0\Big) - \Big((\mathrm{Re} P_1) u_0, (\mathrm{Im} P_1) u_0\Big)\right]. \end{aligned}$$

By integration by parts we find

$$\|P_1(x, D, \tau, \mu) u_0\|^2 = \|(\mathrm{Re} P_1) u_0\|^2 + \|(\mathrm{Im} P_1) u_0\|^2 + \mathrm{i}\Big(\left[\mathrm{Re} P_1, \mathrm{Im} P_1\right] u_0, u_0\Big) + \tau Q_0(u_0),$$

where

$$\left\{ \begin{aligned} Q_0(u_0) &= \Big(-2\frac{\partial \varphi_1}{\partial x_n} D_{x_n} u_0, D_{x_n} u_0\Big)_0 + (\mathrm{op}(r_1) u_0, D_{x_n} u_0)_0 \\ &\quad + (\mathrm{op}(r_1') D_{x_n} u_0, u_0)_0 + (\mathrm{op}(r_2) u_0, u_0)_0 + \tau\Big(\frac{\partial \varphi_1}{\partial x_n} u_0, u_0\Big)_0, \\ &\quad r_1 = r_1' = 2 q_{1,1}, \qquad r_2 = -2\frac{\partial \varphi_1}{\partial x_n} q_{2,1}. \end{aligned} \right.$$

Then we have

$$|Q_0(u_0)|^2 \leq C\, |u_0|_{1,0,\tau}^2.$$

We obtain the same estimate on v_0 by the same method. In addition we know that the principal symbol of the operator $[\mathrm{Re}P_j, \mathrm{Im}P_j]$, $j = 1, 2$, is given by $\frac{1}{\mathrm{i}}\{\mathrm{Re}P_j, \mathrm{Im}P_j\}$. Proceeding like Lebeau and Robbiano in paragraph 3 in [10], we obtain (4.30).

It remains to prove (4.31). Following Lemma 4.4, let $G(x_n) = \mathrm{d}/\mathrm{d}x_n(\mathrm{op}(\mathcal{R})w, w)_{L^2(\mathbb{R}^{n-1})}$.

Using $D_{x_n}w - \mathrm{op}(\mathcal{H}) = \tilde{F}$, we obtain

$$G(x_n) = -2\,\mathrm{Im}(\mathrm{op}(\mathcal{R})\tilde{F}, w) - 2\,\mathrm{Im}(\mathrm{op}(\mathcal{R})\mathrm{op}(\mathcal{H})w, w).$$

The integration in the normal direction gives

$$(\mathrm{op}(\mathcal{R})w, w)_0 = \int_0^\infty \mathrm{Im}(\mathrm{op}(\mathcal{R})\mathrm{op}(\mathcal{H})w, w)\mathrm{d}x_n + 2\int_0^\infty \mathrm{Im}(\mathrm{op}(\mathcal{R})\tilde{F}, w)\mathrm{d}x_n. \tag{4.44}$$

From Lemma 4.4 and the Gårding inequality, we obtain, for τ large enough,

$$\mathrm{Im}(\mathrm{op}(\mathcal{R})\mathrm{op}(\mathcal{H})w, w) \geq C\left|w^-\right|^2_{\frac{1}{2}}, \tag{4.45}$$

moreover we have for all $\epsilon > 0$

$$\int_0^\infty \left|(\mathrm{op}(\mathcal{R})\tilde{F}, w)\right|\mathrm{d}x_n \leq \epsilon C\tau\left\|w^-\right\|^2 + \frac{C_\epsilon}{\tau}\|\tilde{F}\|^2. \tag{4.46}$$

Applying Lemma 4.4 and the Gårding inequality, we obtain, for τ large enough,

$$-(\mathrm{op}(\mathcal{R})w, w) + |\mathrm{op}(\tilde{\mathcal{B}})w|^2 \geq C\left|w\right|^2. \tag{4.47}$$

Combining (4.47), (4.46), (4.45) and (4.44), we get

$$C\left|w^-\right|^2_{\frac{1}{2}} + C\left|w\right|^2 \leq \frac{C}{\tau}\|\tilde{F}\|^2 + |\mathrm{op}(\tilde{\mathcal{B}})w|^2. \tag{4.48}$$

Then

$$\tau\left|w\right|^2 \leq C\|\tilde{F}\|^2 + \tau|\mathrm{op}(\tilde{\mathcal{B}})w|^2.$$

Recalling that $w = (I + K)V$, $V = \mathrm{op}(n)v$, $v = {}^t(\langle D', \tau\rangle\tilde{u}, D_{x_n}\tilde{u})$ and $\tilde{u} = \mathrm{op}(\chi^0)u$ and using estimates (4.41) and (4.42), we prove (4.31). □

4.4. **Estimate in $\mathcal{E}_1^-$**

Let $\chi^-(x, \xi', \tau, \mu) \in \mathcal{T}S^0_\tau$ equal to 1 in $\mathcal{E}_1^-$ and such that in the support of χ^- we have $q_{2,1} + \frac{q_1^2}{(\partial\varphi_1/\partial x_n)^2} \leq -\delta < 0$. Then we have the following estimate.

Proposition 4.3. *There exist constants $C > 0$, $\tau_0 > 0$ and $\mu_0 > 0$ such that for any $\tau \geq \tau_0$, $|\mu| \geq \mu_0$ such that $\tau \geq C_0|\mu|$, $C_0 > 0$, we have the following estimate*

$$\tau\left\|\mathrm{op}(\chi^-)u\right\|^2_{1,\tau} \leq C\left(\left\|P(x, D, \tau, \mu)u\right\|^2 + \tau\left|u\right|^2_{1,0,\tau} + \|u\|^2_{1,\tau}\right), \tag{4.49}$$

for any $u \in C_0^\infty(\{x_n \geq 0\} \cap W)$.

Moreover if we assume $\frac{\partial\varphi_1}{\partial x_n} > 0$, we have

$$\tau\left|\mathrm{op}(\chi^-)u_0\right|^2_{1,0,\tau} \leq C\left(\|P(x, D, \tau, \mu)u\|^2 + \tau^{-2}\left|u\right|^2_{1,0,\tau} + \|u\|^2_{1,\tau}\right) \tag{4.50}$$

for any $u = (u_0, v_0) \in C_0^\infty(\{x_n \geq 0\} \cap W)$.

Proof. Let $\tilde{u} = \mathrm{op}(\chi^-)u = (\mathrm{op}(\chi^-)u_0, \mathrm{op}(\chi^-)v_0) = (\tilde{u}_0, \tilde{v}_0)$.

In this region we have not *a priori* information for the roots of $p_2(x,\xi,\tau,\mu)$. Following the proof of (4.30), we obtain

$$\tau \left\|\mathrm{op}(\chi^-)v_0\right\|_{1,\tau}^2 \le C\left(\|P(x,D,\tau,\mu)v_0\|^2 + \tau\,|v_0|_{1,0,\tau}^2 + \|v_0\|_{1,\tau}^2\right). \tag{4.51}$$

In $\mathrm{supp}(\chi^-)$ the two roots $z_1^{\pm}$ of $p_1(x,\xi,\tau,\mu)$ are in the half-plane $\mathrm{Im}\xi_n < 0$. Then we can use the Calderon projector. By the same way that the proof of (4.7) and using the fact that the operators $t_{0,1}$ and $t_{1,1}$ vanish in $x_n > 0$ (because the roots are in $\mathrm{Im}\xi_n < 0$, see (4.21)), the counterpart of (4.18) is then

$$\tilde{u}_0 = E\underline{\tilde{f}}_1 + w_{1,1} + w_{2,1}, \quad \text{for } x_n > 0, \tag{4.52}$$

where $w_{1,1}$ and $w_{2,1}$ satisfy (4.16) and (4.20) respectively.

We then obtain (see proof of (4.7))

$$\tau^2 \left\|\mathrm{op}(\chi^-)u_0\right\|_{1,\tau}^2 \le C\left(\|P_1(x,D,\tau,\mu)u_0\|^2 + \tau\,|u_0|_{1,0,\tau}^2 + \|u_0\|_{1,\tau}^2\right). \tag{4.53}$$

Combining (4.51) and (4.53), we obtain (4.49).

It remains to proof (4.50). We take the trace at $x_n = 0^+$ of (4.52),

$$\gamma_0(\tilde{u}_0) = w_{0,1} = \gamma_0(E\underline{\tilde{f}}_1 + w_{1,1} + w_{2,1}),$$

which, by the counterpart of (4.26), gives

$$\tau\,|\gamma_0(\tilde{u}_0)|_1^2 \le C\left(\|P_1(x,D,\tau,\mu)u_0\|^2 + \|u_0\|_{1,\tau}^2 + \tau^{-2}\,|u_0|_{1,0,\tau}^2\right). \tag{4.54}$$

From (4.52) we also have

$$D_{x_n}\tilde{u}_0 = D_{x_n}E\underline{\tilde{f}}_1 + D_{x_n}w_{1,1} + D_{x_n}w_{2,1}, \quad \text{for } x_n > 0.$$

We take the trace at $x_n = 0^+$ and obtain

$$\gamma_1(\tilde{u}_0) = \gamma_0(D_{x_n}(E\underline{\tilde{f}}_1 + w_{1,1} + w_{2,1})).$$

Using the trace formula (4.19), we obtain

$$|\gamma_1(\tilde{u}_0)|^2 \le C\tau^{-1}\left\|D_{x_n}(E\underline{\tilde{f}}_1 + w_{1,1} + w_{2,1})\right\|_{1,\tau}^2 \le C\tau^{-1}\left\|E\underline{\tilde{f}}_1 + w_{1,1} + w_{2,1}\right\|_{2,\tau}^2$$

and, by the counterpart of (4.10), (4.16) and (4.20), this yields

$$\tau\,|\gamma_1(\tilde{u}_0)|^2 \le C\left(\|P_1(x,D,\tau,\mu)u_0\|^2 + \|u_0\|_{1,\tau}^2 + \tau^{-2}\,|u_0|_{1,0,\tau}^2\right). \tag{4.55}$$

Combining (4.54) and (4.55), we obtain (4.50). □

4.5. **End of the proof**

We can choose a partition of unity $\chi^+ + \chi^0 + \chi^- = 1$ such that χ^+, χ^0 and χ^- satisfy the properties listed in Propositions 4.1, 4.2 and 4.3 respectively. We have

$$\|u\|_{1,\tau}^2 \le \left\|\mathrm{op}(\chi^+)u\right\|_{1,\tau}^2 + \left\|\mathrm{op}(\chi^0)u\right\|_{1,\tau}^2 + \left\|\mathrm{op}(\chi^-)u\right\|_{1,\tau}^2.$$

Combining this inequality and (4.7), (4.30) and (4.49), we obtain, for τ large, the first estimate (3.22) of Theorem 3.2, *i.e.*

$$\tau \|u\|^2_{1,\tau} \leq C\left(\|P(x,D,\tau,\mu)u\|^2 + \tau |u|^2_{1,0,\tau}\right).$$

It remains to estimate $\tau |u|^2_{1,0,\tau}$. We begin by giving an estimate of $\tau |u_0|^2_{1,0,\tau}$.

We have

$$|u_0|^2_{1,0,\tau} \leq \left|\mathrm{op}(\chi^+)u_0\right|^2_{1,0,\tau} + \left|\mathrm{op}(\chi^0)u_0\right|^2_{1,0,\tau} + \left|\mathrm{op}(\chi^-)u_0\right|^2_{1,0,\tau},$$

$$\left|\mathrm{op}(\chi^+)u_0\right|^2_{1,0,\tau} \leq \left|\mathrm{op}(\chi^+)u\right|^2_{1,0,\tau}$$

and

$$\left|\mathrm{op}(\chi^0)u_0\right|^2_{1,0,\tau} \leq \left|\mathrm{op}(\chi^0)u\right|^2_{1,0,\tau}.$$

Combining these inequalities, (4.8), (4.31), (4.50) and the fact that $\tau^{-2}|u|^2_{1,0,\tau} = \tau^{-2}|u_0|^2_{1,0,\tau} + \tau^{-2}|v_0|^2_{1,0,\tau}$, we obtain, for τ large enough.

$$\tau |u_0|^2_{1,0,\tau} \leq C\left(\|P(x,D,\tau,\mu)u\|^2 + \frac{\tau}{\mu^2}|\mathrm{op}(b_1)u|^2_1 + \tau|\mathrm{op}(b_2)u|^2 + \tau^{-2}|v_0|^2_{1,0,\tau} + \|u\|^2_{1,\tau}\right). \tag{4.56}$$

For estimate $\tau |v_0|^2_{1,0,\tau}$, we use the transmission conditions given by (3.6). We have

$$\mathrm{op}(b_1)u = u_0|_{x_n=0} - \mathrm{i}\mu v_0|_{x_n=0} \quad \text{on } \{x_n = 0\} \cap W.$$

Then

$$\tau |v_0|^2_1 \leq C\left(\frac{\tau}{\mu^2}|u_0|^2_1 + \frac{\tau}{\mu^2}|\mathrm{op}(b_1)u|^2_1\right).$$

Since, for $|\mu| \geq \mu_0$, we have $\frac{\tau}{\mu^2}|u_0|^2_1 \leq C\tau |u_0|^2_{1,0,\tau}$. Then using (4.56), we obtain

$$\tau |v_0|^2_1 \leq C\left(\|P(x,D,\tau,\mu)u\|^2 + \frac{\tau}{\mu^2}|\mathrm{op}(b_1)u|^2_1 + \tau|\mathrm{op}(b_2)u|^2 + \tau^{-2}|v_0|^2_{1,0,\tau} + \|u\|^2_{1,\tau}\right). \tag{4.57}$$

We have also

$$\mathrm{op}(b_2)u = \left(D_{x_n} + \mathrm{i}\tau\frac{\partial\varphi_1}{\partial x_n}\right)u_0|_{x_n=0} + \left(D_{x_n} + \mathrm{i}\tau\frac{\partial\varphi_2}{\partial x_n}\right)v_0|_{x_n=0} \quad \text{on } \{x_n = 0\} \cap W.$$

Then

$$\tau |D_{x_n}v_0|^2 \leq C\left(\tau|\mathrm{op}(b_2)u|^2 + \tau|D_{x_n}u_0|^2 + \tau^3|u_0|^2 + \tau^3|v_0|^2\right).$$

Using the fact that $|u|_{k-1} \leq \tau^{-1}|u|_k$, we obtain

$$\tau |D_{x_n}v_0|^2 \leq C\left(\tau|\mathrm{op}(b_2)u|^2 + \tau|D_{x_n}u_0|^2 + \tau|u_0|^2_1 + \tau|v_0|^2_1\right).$$

Since we have $\tau |u_0|^2_{1,0,\tau} = \tau|D_{x_n}u_0|^2 + \tau|u_0|^2_1$. Then using (4.56) and (4.57), we obtain

$$\tau |D_{x_n}v_0|^2 \leq C\left(\|P(x,D,\tau,\mu)u\|^2 + \frac{\tau}{\mu^2}|\mathrm{op}(b_1)u|^2_1 + \tau|\mathrm{op}(b_2)u|^2 + \tau^{-2}|v_0|^2_{1,0,\tau} + \|u\|^2_{1,\tau}\right). \tag{4.58}$$

Combining (4.57) and (4.58), we have

$$\tau |v_0|^2_{1,0,\tau} \leq C\left(\|P(x,D,\tau,\mu)u\|^2 + \frac{\tau}{\mu^2}|\mathrm{op}(b_1)u|^2_1 + \tau|\mathrm{op}(b_2)u|^2 + \|u\|^2_{1,\tau}\right). \tag{4.59}$$

Combining (4.56) and (4.59), we obtain

$$\tau\,|u|^2_{1,0,\tau} \le C\left(\|P(x,D,\tau,\mu)u\|^2 + \frac{\tau}{\mu^2}\,|\mathrm{op}(b_1)u|^2_1 + \tau\,|\mathrm{op}(b_2)u|^2 + \|u\|^2_{1,\tau}\right). \tag{4.60}$$

Inserting (4.60) in (3.22) and for τ large enough, we obtain (3.23). □

Appendix A: Proof of Lemma 2.1

To prove Lemma 2.1, we need to distinguish two cases.

(1) **Inside $\mathcal{O}$**
To simplify the expressions, we note $\|u\|_{L^2(\mathcal{O})} = \|u\|$.
Let $\chi \in C_0^\infty(\mathcal{O})$. We have by integration by part

$$((\triangle - \mathrm{i}\,|\mu|)u, \chi^2 u) = (-\nabla u, \chi^2 \nabla u) - (\nabla u, \nabla(\chi^2)u) - \mathrm{i}\,|\mu|\,\|\chi u\|^2\,.$$

Then

$$|\mu|\,\|\chi u\|^2 \le C\left(\|f\|\,\|\chi^2 u\| + \|\nabla u\|^2 + \|\nabla u\|\,\|\chi u\|\right).$$

Then

$$|\mu|\,\|\chi u\|^2 \le C\left(\frac{1}{\epsilon}\,\|f\|^2 + \epsilon\,\|\chi^2 u\| + \|\nabla u\|^2 + \frac{1}{\epsilon}\,\|\nabla u\|^2 + \epsilon\,\|\chi u\|^2\right).$$

Recalling that $|\mu| \ge \mu_0$, we have for ϵ small enough

$$\|\chi u\|^2 \le C\left(\|\nabla u\|^2 + \|f\|^2\right). \tag{4.61}$$

Hence the result inside $\mathcal{O}$.

(2) **In the neighborhood of the boundary**
Let $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Then

$$\partial\mathcal{O} = \{x \in \mathbb{R}^n,\ x_n = 0\}.$$

Let $\epsilon > 0$ such that $0 < x_n < \epsilon$. Then we have

$$u(x', \epsilon) - u(x', x_n) = \int_{x_n}^{\epsilon} \partial_{x_n} u(x', \sigma)\mathrm{d}\sigma.$$

Then

$$|u(x', x_n)|^2 \le 2\,|u(x', \epsilon)|^2 + 2\left(\int_{x_n}^{\epsilon} |\partial_{x_n} u(x', \sigma)|\,\mathrm{d}\sigma\right)^2.$$

Using the Cauchy Schwartz inequality, we obtain

$$|u(x', x_n)|^2 \le 2\,|u(x', \epsilon)|^2 + 2\epsilon^2 \int_0^{\epsilon} |\partial_{x_n} u(x', x_n)|^2\,\mathrm{d}x_n.$$

Integrating with respect to x', we obtain

$$\int_{|x'|<\epsilon} |u(x', x_n)|^2\,\mathrm{d}x' \le 2\int_{|x'|<\epsilon} |u(x', \epsilon)|^2\,\mathrm{d}x' + 2\epsilon^2 \int_{|x'|<\epsilon,\,|x_n|<\epsilon} \left(|\partial_{x_n} u(x', x_n)|^2\,\mathrm{d}x_n\right)\mathrm{d}x'. \tag{4.62}$$

Using the trace theorem, we have

$$\int_{|x'|<\epsilon} |u(x',\epsilon)|^2 \,\mathrm{d}x' \leq C \int_{|x'|<2\epsilon,\, |x_n-\epsilon|<\frac{\epsilon}{2}} (|u(x)|^2 + |\nabla u(x)|^2)\mathrm{d}x. \tag{4.63}$$

Now we introduce the following cut-off functions

$$\chi_1(x) = \begin{cases} 1 & \text{if } 0 < x_n < \frac{\epsilon}{2}, \\ 0 & \text{if } x_n > \epsilon \end{cases}$$

and

$$\chi_2(x) = \begin{cases} 1 & \text{if } \frac{\epsilon}{2} < x_n < \frac{3\epsilon}{2}, \\ 0 & \text{if } x_n < \frac{\epsilon}{4},\ x_n > 2\epsilon. \end{cases}$$

Combining (4.62) and (4.63), we obtain for ϵ small enough

$$\|\chi_1 u\|^2 \leq C\left(\|\chi_2 u\|^2 + \|\nabla u\|^2\right). \tag{4.64}$$

Since following (4.61), we have

$$\|\chi_2 u\|^2 \leq C\left(\|f\|^2 + \|\nabla u\|^2\right).$$

Inserting in (4.64), we obtain

$$\|\chi_1 u\|^2 \leq C\left(\|f\|^2 + \|\nabla u\|^2\right). \tag{4.65}$$

Hence the result in the neighborhood of the boundary.

Following (4.61), we can write

$$\|(1-\chi_1)u\|^2 \leq C\left(\|f\|^2 + \|\nabla u\|^2\right). \tag{4.66}$$

Adding (4.65) and (4.66), we obtain (2.9).

Appendix B: Proof of Lemma 3.1

Let $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $\chi = 1$ in the support of u. It suffices to show that $\mathrm{op}(\Lambda^s)\mathrm{e}^{\tau\varphi}\chi\mathrm{op}(\Lambda^{-s})$ is bounded in L^2. Recalling that for all u and $v \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\mathcal{F}(uv)(\xi') = \left(\frac{1}{2\pi}\right)^{n-1} \mathcal{F}(u) * \mathcal{F}(v)(\xi'), \qquad \forall \xi' \in \mathbb{R}^{n-1}.$$

Then

$$\begin{aligned} \mathcal{F}(\mathrm{op}(\Lambda^s)\mathrm{e}^{\tau\varphi}\chi\mathrm{op}(\Lambda^{-s})v)(\xi',\tau) &= \langle \xi',\tau\rangle^s \mathcal{F}(\mathrm{e}^{\tau\varphi}\chi\mathrm{op}(\Lambda^{-s})v)(\xi',\tau) \\ &= \left(\frac{1}{2\pi}\right)^{n-1} \langle \xi',\tau\rangle^s (g(\xi',\tau) * \langle \xi',\tau\rangle^{-s}\mathcal{F}(v))(\xi',\tau), \end{aligned}$$

where $g(\xi',\tau)=\mathcal{F}(\mathrm{e}^{\tau\varphi}\chi)(\xi',\tau)$. Then we have

$$\mathcal{F}(\mathrm{op}(\Lambda^s)\mathrm{e}^{\tau\varphi}\chi\mathrm{op}(\Lambda^{-s})v)(\xi',\tau)=\int g(\xi'-\eta',\tau)\langle\xi',\tau\rangle^s\langle\eta',\tau\rangle^{-s}\mathcal{F}(v)(\eta',\tau)\mathrm{d}\eta'.$$

Let $k(\xi',\eta')=g(\xi'-\eta',\tau)\langle\xi',\tau\rangle^s\langle\eta',\tau\rangle^{-s}$. Our goal is to show that $\int K(\xi',\eta')\mathcal{F}(v)(\eta',\tau)\mathrm{d}\eta'$ is bounded in L^2. To do it, we will use Schur's Lemma. It suffices to prove that there exist $M>0$ and $N>0$ such that

$$\int|K(\xi',\eta')|\,\mathrm{d}\xi'\le M\qquad\text{and}\qquad\int|K(\xi',\eta')|\,\mathrm{d}\eta'\le N.$$

In the sequel, we suppose $s\ge 0$ (the case where $s<0$ is treated in the same way).

For $R>0$, we have

$$\begin{aligned}\langle\xi',\tau\rangle^{2R}g(\xi',\tau) &= \int\langle\xi',\tau\rangle^{2R}\mathrm{e}^{-\mathrm{i}x'\xi'}\chi(x)\mathrm{e}^{\tau\varphi(x)}\mathrm{d}x'\\ &= \int(1-\Delta+\tau^2)^R(\mathrm{e}^{-\mathrm{i}x'\xi'})\chi(x)\mathrm{e}^{\tau\varphi(x)}\mathrm{d}x'\\ &= \int\mathrm{e}^{-\mathrm{i}x'\xi'}(1-\Delta+\tau^2)^R(\chi(x)\mathrm{e}^{\tau\varphi(x)})\mathrm{d}x'.\end{aligned}$$

Then there exists $C>0$, such that

$$\left|\langle\xi',\tau\rangle^{2R}g(\xi',\tau)\right|\le C\mathrm{e}^{C\tau}.\tag{4.67}$$

Moreover, we can write

$$\int|K(\xi',\eta')|\,\mathrm{d}\xi'=\int\left|g(\xi'-\eta',\tau)\langle\xi'-\eta',\tau\rangle^{2R}\frac{\langle\xi',\tau\rangle^s\langle\eta',\tau\rangle^{-s}}{\langle\xi'-\eta',\tau\rangle^{2R}}\right|\mathrm{d}\xi'.$$

Using (4.67), we obtain

$$\int|K(\xi',\eta')|\,\mathrm{d}\xi'\le C\mathrm{e}^{C\tau}\int\frac{\langle\xi',\tau\rangle^s\langle\eta',\tau\rangle^{-s}}{\langle\xi'-\eta',\tau\rangle^{2R}}\mathrm{d}\xi'.$$

Since

$$\int\frac{\langle\xi',\tau\rangle^s\langle\eta',\tau\rangle^{-s}}{\langle\xi'-\eta',\tau\rangle^{2R}}\mathrm{d}\xi'=\int_{|\xi'|\le\frac{1}{\epsilon}|\eta'|}\frac{\langle\xi',\tau\rangle^s\langle\eta',\tau\rangle^{-s}}{\langle\xi'-\eta',\tau\rangle^{2R}}\mathrm{d}\xi'+\int_{|\eta'|\le\epsilon|\xi'|}\frac{\langle\xi',\tau\rangle^s\langle+\eta',\tau\rangle^{-s}}{\langle\xi'-\eta',\tau\rangle^{2R}}\mathrm{d}\xi',\quad\epsilon>0.$$

If $|\xi'|\le\frac{1}{\epsilon}|\eta'|$, we have

$$\frac{\langle\xi',\tau\rangle^s\langle\eta',\tau\rangle^{-s}}{\langle\xi'-\eta',\tau\rangle^{2R}}\le C\frac{\langle\eta',\tau\rangle^s\langle\eta',\tau\rangle^{-s}}{\langle\xi'-\eta',\tau\rangle^{2R}}\le\frac{C}{\langle\xi'-\eta',\tau\rangle^{2R}}\quad\in L^1\quad\text{if }2R>n-1.$$

If $|\eta'|\le\epsilon|\xi'|$, *i.e.* $\langle\xi'-\eta',\tau\rangle\ge\delta\langle\xi',\tau\rangle$, $\delta>0$, we have

$$\frac{\langle\xi',\tau\rangle^s\langle\eta',\tau\rangle^{-s}}{\langle\xi'-\eta',\tau\rangle^{2R}}\le\frac{C}{\langle\xi'-\eta',\tau\rangle^{2R-s}}\quad\in L^1\quad\text{if }2R-s>n-1.$$

Then there exists $M>0$, such that

$$\int|K(\xi',\eta')|\,\mathrm{d}\xi'\le M\mathrm{e}^{C\tau}.$$

By the same way, we show that there exists $N > 0$, such that

$$\int |K(\xi', \eta')| \, \mathrm{d}\eta' \leq N\mathrm{e}^{C\tau}.$$

Using Schur's Lemma, we have $(\mathrm{op}(\Lambda^s)\mathrm{e}^{\tau\varphi}\chi\mathrm{op}(\Lambda^{-s}))$ is bounded in L^2 and

$$\left\|\mathrm{op}(\Lambda^s)\mathrm{e}^{\tau\varphi}\chi\mathrm{op}(\Lambda^{-s})\right\|_{\mathcal{L}(L^2)} \leq C\mathrm{e}^{C\tau}.$$

Applying in $\mathrm{op}(\Lambda^s)u$, we obtain the result.

Appendix C: Proof of Theorem 1.1, for $\mu = 0$

Let $U = (u_0, v_0, v_1) \in D(\mathcal{A})$ and $F = (f_0, g_0, g_1) \in H$ such that $F = \mathcal{A}U$. Then we have the following system

$$\begin{cases} \triangle u_0 &= f_0 \quad \text{in } \Omega_1, \\ \triangle v_0 &= g_1 \quad \text{in } \Omega_2, \\ v_1 &= g_0 \quad \text{in } \Omega_2, \end{cases}$$

with the following boundary conditions

$$\begin{cases} u_0|_{\Gamma_1} &= 0, \qquad v_0|_{\Gamma_2} = 0 \\ u_0|_\gamma &= g_0|_\gamma, \\ (\partial_n u_0 - \partial_n v_0)|_\gamma &= 0|_\gamma. \end{cases}$$

From $\triangle u_0 = f_0$ in Ω_1, $u_0|_{\Gamma_1} = 0$ and $u_0 = g_0|_\gamma$, we have the following estimate

$$\|u_0\|^2_{H^1(\Omega_1)} \leq C\left(\|f_0\|^2_{H^{-1}(\Omega_1)} + \|g_0\|^2_{H^{\frac{1}{2}}(\gamma)}\right).$$

Then

$$\|u_0\|^2_{H^1(\Omega_1)} \leq C\left(\|f_0\|^2_{L^2(\Omega_1)} + \|g_0\|^2_{H^1(\Omega_2)}\right). \tag{4.68}$$

Moreover, from $\triangle v_0 = g_1$ in Ω_2, $v_0|_{\Gamma_2} = 0$, $\Gamma_2 \neq \emptyset$ and $\partial_n v_0 = \partial_n u_0|_\gamma$, we have the following estimate

$$\|v_0\|^2_{H^1(\Omega_2)} \leq C\left(\|g_1\|^2_{L^2(\Omega_2)} + \|\partial_n u_0\|^2_{H^{-\frac{1}{2}}(\gamma)}\right). \tag{4.69}$$

Recalling that $\triangle u_0 = f_0$ and using the trace lemma (Lem. 3.4 in [6]), we obtain

$$\|\partial_n u_0\|_{H^{-\frac{1}{2}}(\gamma)} \leq C\left(\|u_0\|_{H^1(\Omega_1)} + \|f_0\|_{L^2(\Omega_1)}\right).$$

Combining with (4.69), we obtain

$$\|v_0\|^2_{H^1(\Omega_2)} \leq C\left(\|g_1\|^2_{L^2(\Omega_2)} + \|u_0\|^2_{H^1(\Omega_1)} + \|f_0\|^2_{L^2(\Omega_1)}\right).$$

Combining with (4.68), we get

$$\|v_0\|^2_{H^1(\Omega_2)} \leq C\left(\|f_0\|^2_{L^2(\Omega_1)} + \|g_0\|^2_{H^1(\Omega_2)} + \|g_1\|^2_{L^2(\Omega_2)}\right). \tag{4.70}$$

Recalling that $v_1 = g_0$ and combining (4.68) and (4.70), we obtain Theorem 1.1, for $\mu = 0$.

Acknowledgements. Sincere thanks to professor Luc Robbiano for inspiring questions, his great contribution to this work and for careful reading of the manuscript. I want to thank also professor Mourad Bellassoued for his offer to work in this domain.

References

[1] C. Bardos, G. Lebeau and J. Rauch, Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. *SIAM J. Control Optim.* **30** (1992) 1024–1065.
[2] M. Bellassoued, Distribution of resonances and decay rate of the local energy for the elastic wave equation. *Comm. Math. Phys.* **215** (2000) 375–408.
[3] M. Bellassoued, Carleman estimates and distribution of resonances for the transparent obstacle and application to the stabilization. *Asymptot. Anal.* **35** (2003) 257–279.
[4] M. Bellassoued, Decay of solutions of the elastic wave equation with a localized dissipation. *Ann. Fac. Sci. Toulouse Math.* **12** (2003) 267–301.
[5] N. Burq, Décroissance de l'énergie locale de l'équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel. *Acta Math.* **180** (1998) 1–29.
[6] T. Duyckaerts, Optimal decay rates of the energy of a hyperbolic-parabolic system coupled by an interface. *Asymptot. Anal.* **51** (2007) 17–45.
[7] X Fu, Logarithmic decay of hyperbolic equations with arbitrary small boundary damping. *Commun. Partial Differ. Equ.* **34** (2009) 957–975.
[8] J. Le Rousseau and L. Robbiano, Carleman estimate for elliptic operators with coefficients with jumps at an interface in arbitrary dimension and application to the null controllability of linear parabolic equations. *Arch. Ration. Mech. Anal.* (to appear).
[9] G. Lebeau, Équation des ondes amorties, in *Algebraic and geometric methods in mathematical physics Kaciveli, 1993*, Kluwer Acad. Publ., Dordrecht, *Math. Phys. Stud.* **19** (1996) 73–109.
[10] G. Lebeau and L. Robbiano, Contrôle exact de l'équation de la chaleur. *Commun. Partial Differ. Equ.* **20** (1995) 335–356.
[11] G. Lebeau and L. Robbiano, Stabilisation de l'équation des ondes par le bord. *Duke Math. J.* **86** (1997) 465–491.
[12] J. Rauch, X. Zhang and E. Zuazua, Polynomial decay for a hyperbolic-parabolic coupled system. *J. Math. Pures Appl.* **84** (2005) 407–470.
[13] L. Robbiano, Fonction de coût et contrôle des solutions des équations hyperboliques. *Asymptot. Anal.* **10** (1995) 95–115.
[14] M.E. Taylor, Reflection of singularities of solutions to systems of differential equations. *Comm. Pure Appl. Math.* **28** (1975) 457–478.
[15] X. Zhang and E. Zuazua, Long-time behavior of a coupled heat-wave system arising in fluid-structure interaction. *Arch. Ration. Mech. Anal.* **184** (2007) 49–120.