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LIPSCHITZ REGULARITY FOR SOME ASYMPTOTICALLY CONVEX PROBLEMS *

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Abstract. We establish a local Lipschitz regularity result for local minimizers of asymptotically convex variational integrals.

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1. Introduction

We consider local minimizers of variational integrals of the type

$$\mathcal{F}(\mathbf{u}) = \int_\Omega f(\nabla \mathbf{u})\,\mathrm{d}x, \tag{1.1}$$

where Ω is a bounded, open subset of $\mathbb{R}^n$, $\mathbf{u} : \Omega \to \mathbb{R}^N$ is a vector valued function and $\nabla \mathbf{u}$ stands for the total derivative of u. A function $\mathbf{u} \in W^{1,p}(\Omega)$ is a local minimizer of $\mathcal{F}(\mathbf{u})$ if $\mathcal{F}(\mathbf{u}) \le \mathcal{F}(\mathbf{u}+\eta)$, for every test function $\eta \in W_0^{1,p}(\Omega)$ with compact support in Ω.

In 1977 Uhlenbeck (see [26]) proved everywhere $C^{1,\alpha}$ regularity for local minimizers of functional when the integrand $f \in C^2$ is assumed to behave like $|\xi|^p$, with $p \ge 2$; Acerbi and Fusco considered the case $1 < p < 2$. Later on a large number of generalizations have been made, see for example the survey [22].

For the (p, q) case and the general growth case, see the papers of Marcellini [18–21] and [6,7].

Another direction of research is the one arising in the model of electro-rheological fluids [2,3].

For the Lipschitz regularity, the results are available when $f \in C^2$ is asymptotically, in a C^2-sense, quadratic or super-quadratic at infinity (see [4] for the case $p = 2$ and [15,24] for the case $p > 2$; for the subquadratic case see [17]). For related results, see [11–14,23].

Keywords and phrases. Local minimizers, decay estimates, asymptotic behaviour.

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The argument of such results is the following: if the gradients of minimizers are very large, the problem becomes "regular" and so good estimates are known.

Moreover, Dolzmann and Kristensen [10] have proved local higher integrability with large exponents of minimizers when $f \in C^0$ approaches at infinity, in a C^0-sense, the p-Dirichlet integrand, for some arbitrary $p > 1$, see also [16].

In a recent paper Diening and Ettwein [8] considered fractional estimates for non-differentiable systems with φ-growth. Using some of their techniques, we were able to prove in [9] excess decay estimates for vectorial functionals with φ-growth. In this paper we extend the results found in [4,15,17,24] to the case of a convex function satisfying the Δ_2-condition with its conjugate ($\Delta_2(\{\varphi, \varphi^*\}) < \infty$), see Section 2 for the definitions. More precisely we have the following theorem:

Theorem 1.1. *Let φ be an N-function such that*

- (H1) $\varphi \in C^2((0,\infty)) \cap C([0,\infty))$ *and* $\varphi \in \Delta_2(\{\varphi, \varphi^*\})$;
- (H2) $\Delta_2(\{\varphi, \varphi^*\}) < \infty$;
- (H3) $\varphi'(t) \sim t\,\varphi''(t)$;
- (H4) *there exists* $\beta \in (0,1]$ *and* $c > 0$ *such that*

$$|\varphi''(s+t) - \varphi''(t)| \le c_1\, \varphi''(t) \left(\frac{|s|}{t}\right)^{\beta}$$

for all $t > 0$ *and* $s \in \mathbb{R}$ *with* $|s| < \frac{1}{2}t$.

Moreover let $f : \mathbb{R}^{n\times N} \to \mathbb{R}$ *be such that*

- (F1) $f \in C^2(\mathbb{R}^{n\times N})$;
- (F2) *there exists* $L > 0$ *such that for all* $\xi \in \mathbb{R}^{n\times N} \setminus \{0\}$

$$|\nabla^2 f(\xi)| \le L\, \varphi''(|\xi|); \tag{1.2}$$

- (F3) *there holds*[3]

$$\lim_{|\xi|\to\infty} \frac{|\nabla^2 f(\xi) - \nabla^2 \varphi(\xi)|}{\varphi''(|\xi|)} = 0. \tag{1.3}$$

If $\mathbf{u} \in W^{1,\varphi}(\Omega)$ *is a local minimizer of the functional* $\mathcal{F}$, *see* (1.1), *then* $\nabla \mathbf{u}$ *is locally bounded in* Ω. *Moreover, for every ball* $B \subset \Omega$ *we have*

$$\operatorname*{esssup}_{\frac{1}{2}B} \varphi(|\nabla \mathbf{u}|) \le c \left(1 + \fint_B \varphi(|\nabla \mathbf{u}|)\,\mathrm{d}x\right), \tag{1.4}$$

where c *depends only on* n, N, L, $\Delta_2(\{\varphi, \varphi^*\})$, c_1, β, *and the convergence in* (1.3).

Let us point out that in the power case, with $1 < p < 2$ [17], the authors considered the asymptotic behaviour like $(\mu + t^2)^{\frac{p}{2}}, \mu > 0$. Here we are able to recover also the case $\mu = 0$.

[3]We use that φ can also be interpreted as a function from $\mathbb{R}^{n\times N}$ to $\mathbb{R}^n$ by $\varphi(\xi) := \varphi(|\xi|)$.

2. Technical lemmas

In the sequel Ω will denote a bounded, open set of $\mathbb{R}^n$. To simplify the notation, the letter c will denote any positive constant, which may vary throughout the paper. For $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and a ball $B \subset \mathbb{R}^n$ we define

$$\langle w \rangle_B := \fint_B w(x)\,\mathrm{d}x := \frac{1}{|B|}\int_B w(x)\,\mathrm{d}x, \tag{2.1}$$

where $|B|$ is the n-dimensional Lebesgue measure of B. For $\lambda > 0$ we denote by λB the ball with the center as B but λ-times the radius. We write $B_r(x)$ for the ball with radius R and center x. For $U, \Omega \subset \mathbb{R}^n$ we write $U \Subset \Omega$ if the closure of U is a compact subset of Ω. We define $\delta_{i,j} := 0$ for $i \neq j$ and $\delta_{i,i} = 1$.

The following definitions and results are standard in the context of N-functions. A real function $\varphi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ is said to be an N-function if it satisfies the following conditions: there exists the derivative φ' of φ, it is right continuous, non-decreasing and satisfies $\varphi'(0) = 0$ and $\varphi'(t) > 0$ for $t > 0$. In addition, φ is convex.

We say that φ satisfies the Δ_2-condition, if there exists $K > 0$ such that for all $t \geq 0$ holds $\varphi(2t) \leq K\,\varphi(t)$. By $\Delta_2(\varphi)$ we denote the smallest constant K. Since $\varphi(t) \leq \varphi(2t)$ the Δ_2 condition is equivalent to $\varphi(2t) \sim \varphi(t)$. For a family $\{\varphi_\lambda\}_\lambda$ of N-functions we define $\Delta_2(\{\varphi_\lambda\}_\lambda) := \sup_\lambda \Delta_2(\varphi_\lambda)$.

By L^φ and $W^{1,\varphi}$ we denote the classical Orlicz and Sobolev-Orlicz spaces, *i.e.* $f \in L^\varphi$ iff $\int \varphi(|f|)\,\mathrm{d}x < \infty$ and $f \in W^{1,\varphi}$ iff $f, \nabla f \in L^\varphi$. The space L^φ equipped with the norm $\|f\|_\varphi := \inf\{\lambda > 0 \,:\, \int \varphi(|f/\lambda|)\,\mathrm{d}x \leq 1\}$ is a Banach space. By $W^{1,\varphi}_0(\Omega)$ we denote the closure of $C^\infty_0(\Omega)$ in $W^{1,\varphi}(\Omega)$, where $W^{1,\varphi}(\Omega)$ is equipped with the norm $\|f\|_\varphi + \|\nabla f\|_\varphi$ [5].

By $(\varphi')^{-1} : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ we denote the function

$$(\varphi')^{-1}(t) := \sup\{s \in \mathbb{R}^{\geq 0} \,:\, \varphi'(s) \leq t\}.$$

If φ' is strictly increasing then $(\varphi')^{-1}$ is the inverse function of φ'. Then $\varphi^* : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ with

$$\varphi^*(t) := \int_0^t (\varphi')^{-1}(s)\,\mathrm{d}s$$

is again an N-function and $(\varphi^*)'(t) = (\varphi')^{-1}(t)$ for $t > 0$. It is the complementary function of φ. Note that $\varphi^*(t) = \sup_{s \geq 0}(st - \varphi(s))$ and $(\varphi^*)^* = \varphi$. For all $\delta > 0$ there exists c_δ (only depending on $\Delta_2(\{\varphi, \varphi^*\})$) such that for all $t, s \geq 0$ holds

$$t\,s \leq \delta\,\varphi(t) + c_\delta\,\varphi^*(s). \tag{2.2}$$

For $\delta = 1$ we have $c_\delta = 1$. This inequality is called Young 's inequality. For all $t \geq 0$

$$\begin{aligned} \frac{t}{2}\varphi'\Big(\frac{t}{2}\Big) \leq \varphi(t) \leq t\,\varphi'(t),\\ \varphi\bigg(\frac{\varphi^*(t)}{t}\bigg) \leq \varphi^*(t) \leq \varphi\bigg(\frac{2\,\varphi^*(t)}{t}\bigg). \end{aligned} \tag{2.3}$$

Therefore, uniformly in $t \geq 0$

$$\varphi(t) \sim \varphi'(t)\,t, \qquad \varphi^*\big(\varphi'(t)\big) \sim \varphi(t), \tag{2.4}$$

where the constants only depend on $\Delta_2(\{\varphi, \varphi^*\})$.

We define the *shifted N-function* $\varphi_a : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ by

$$\varphi_a(t) = \int_0^t \varphi'_a(s)\mathrm{d}s \quad \text{where} \quad \varphi'_a(t) = \frac{\varphi'(a+t)}{a+t}t. \tag{2.5}$$

The shifted N-functions have been introduced in [8]. See [25] for a detailed study of the shifted N-functions. The function φ_a and its dual φ_a are again N-functions and satisfy the Δ_2-condition uniformly in $a \geq 0$. In particular, $\Delta_2(\{\varphi_a, (\varphi_a)^*\}_{a\geq 0}) < \infty$. For given φ we define the N-function ψ by

$$\frac{\psi'(t)}{t} := \left(\frac{\varphi'(t)}{t}\right)^{\frac{1}{2}}. \tag{2.6}$$

It is shown in [8] that ψ also satisfies (H2)–(H3) and uniformly in $t > 0$ holds $\psi''(t) \sim \sqrt{\varphi''(t)}$. We define the function $\mathbf{V}(\mathbf{Q})$:

$$\mathbf{V}(\mathbf{Q}) := \frac{\psi'(|\mathbf{Q}|)}{|\mathbf{Q}|}\mathbf{Q}.$$

The following lemma can be found in [1].

Lemma 2.1. *Let $\alpha > -1$ then uniformly in $\xi_0, \xi_1 \in \mathbb{R}^{n\times N}$ with $|\xi_0| + |\xi_1| > 0$ holds*

$$\big(|\xi_0| + |\xi_1|\big)^{\alpha} \sim \int_0^1 |\xi_\theta|^{\alpha}\,\mathrm{d}\theta, \tag{2.7}$$

where $\xi_\theta := (1-\theta)\xi_0 + \theta\xi_1$.

Moreover, we need the following generalization of Lemma 2.1.

Lemma 2.2 ([8], Lem. 20)**.** *Let φ be an N-function with $\Delta_2(\{\varphi, \varphi^*\}) < \infty$. Then uniformly for all $\xi_0, \xi_1 \in \mathbb{R}^{n\times N}$ with $|\xi_0| + |\xi_1| > 0$ holds*

$$\int_0^1 \frac{\varphi'(|\xi_\theta|)}{|\xi_\theta|}\,\mathrm{d}\theta \sim \frac{\varphi'(|\xi_0| + |\xi_1|)}{|\xi_0| + |\xi_1|}, \tag{2.8}$$

where $\xi_\theta := (1-\theta)\xi_0 + \theta\xi_1$. The constants only depend on $\Delta_2(\{\varphi, \varphi^*\})$.

Remark 2.3. Let φ be an N-function with $\Delta_2(\{\varphi, \varphi^*\}) < \infty$. Then it has been shown in [8], p. 546, and [25], Lemma 5.19, that there exists $0 < \gamma < 1$ and and N-function ρ with $\Delta_2(\{\rho, \rho^*\}) < \infty$ such that $(\varphi(t))^{\gamma} \sim \rho(t)$ uniformly in $t \geq 0$. It is important to remark that γ and $\Delta_2(\{\rho, \rho^*\})$ only depend on $\Delta_2(\{\varphi, \varphi^*\})$. Note that $\varphi(t) \sim t\,\varphi'(t)$, $\varphi(t) \sim (\rho(t))^{\frac{1}{\gamma}}$, and $\rho(t) \sim t\rho'(t)$ imply $\varphi'(t) \sim (\rho'(t))^{1/\gamma} t^{1/\gamma - 1}$.

The next Lemma contains useful properties of the function $\mathbf{V}$ (see [8], Lem. 3, or [9,25]).

Lemma 2.4. *For every $\xi_0, \xi_1 \in \mathbb{R}^{n\times N}$ with $|\xi_0| + |\xi_1| > 0$ holds*

$$\begin{aligned} |\mathbf{V}(\xi_0) - \mathbf{V}(\xi_1)|^2 &\sim |\xi_0 - \xi_1|^2 \varphi''(|\xi_0| + |\xi_1|) \\ |\mathbf{V}(\xi_0)|^2 &\sim \varphi(|\xi_0|). \end{aligned} \tag{2.9}$$

3. Proof of the main result

We need two lemmas that measures the differences of f and φ in a C^2 sense. The first lemma is a rough estimate using only the upper estimates for $\nabla^2 f$ and $\nabla^2\varphi$. The second lemma is more subtle using that $\nabla^2 f$ and $\nabla^2\varphi$ are close for large arguments. It is the analogue of Lemma 5.1 in [15] and Lemma 2.4 in [17].

Lemma 3.1. *Let φ satisfy* (H1)–(H4) *and f satisfy* (F1)–(F3)*. Then there exists $c > 0$ such that for all $\xi_0, \xi_1 \in \mathbb{R}^{n\times N}$ holds*

$$\int_0^1 \big|[\nabla^2 f(\xi_\theta) - \nabla^2\varphi(\xi_\theta)]\big|\mathrm{d}\theta\,|\xi_1 - \xi_0|^2 \leq c\,\big|\mathbf{V}(\xi_1) - \mathbf{V}(\xi_0)\big|^2, \tag{3.1}$$

where $\xi_\theta = (1-\theta)\xi_0 + \theta\xi_1$. Note that c depends only on $n, N, L,$ and $\Delta_2(\{\varphi, \varphi^*\})$.

Proof. Due to (1.2), Lemmas. 2.2 and 2.4 we estimate

$$\begin{aligned}\int_0^1 \big|[\nabla^2 f(\xi_\theta) - \nabla^2\varphi(\xi_\theta)]\big| \mathrm{d}\theta |\xi_1 - \xi_0|^2 &\le (L+1)\int_0^1 \varphi''(\xi_\theta)\,\mathrm{d}\theta |\xi_1-\xi_0|^2 \\ &\le c\,(L+1)\varphi''(|\xi_0|+|\xi_1|)\,|\xi_1-\xi_0|^2 \\ &\le c\,(L+1)|\mathbf{V}(\xi_1)-\mathbf{V}(\xi_0)|^2.\end{aligned}$$

This proves the assertion. □

Lemma 3.2. *Let φ satisfy* (H1)–(H4) *and f satisfy* (F1)–(F3). *Then for every $\varepsilon > 0$ there exist $\sigma(\varepsilon) > 0$ such that for all $\xi_0, \xi_1 \in \mathbb{R}^{n\times N}$ with* $\max\{|\xi_0|, |\xi_1|\} \ge \sigma(\varepsilon)$ *holds*

$$\int_0^1 \big|[\nabla^2 f(\xi_\theta) - \nabla^2\varphi(\xi_\theta)]\big| \mathrm{d}\theta\, |\xi_1 - \xi_0|^2 \le \varepsilon \big|\mathbf{V}(\xi_1)-\mathbf{V}(\xi_0)\big|^2, \tag{3.2}$$

where $\xi_\theta = (1-\theta)\xi_0 + \theta\xi_1$. Note that σ depends only on $\varepsilon, n, N, L, \Delta_2(\{\varphi,\varphi^*\})$, and the limit in (1.3).

Proof. Fix $\varepsilon > 0$. In the following let $\delta = \delta(\varepsilon) > 0$. The precise value of δ will be chosen later. Due to (1.3) there exists $\Lambda(\delta) > 0$ such that

$$|\nabla^2 f(\xi) - \nabla^2\varphi(\xi)| \le \delta\,\varphi''(|\xi|) \tag{3.3}$$

for all $\xi \in \mathbb{R}^{n\times N}$ with $|\xi| \ge \Lambda(\delta)$.

Let $\sigma(\varepsilon) := K\,\Lambda(\delta)$ with $K \ge 2$, where the precise value of K will be chosen later. Let $\xi_0, \xi_1 \in \mathbb{R}^{n\times N}$ with $\max\{|\xi_0|, |\xi_1|\} \ge \sigma(\varepsilon)$. By symmetry we can assume without loss of generality $|\xi_1| \ge \sigma(\varepsilon)$. For $\theta \in (0,1)$ define $\xi_\theta := (1-\theta)\xi_0 + \theta\xi_1$. We split the domain of integration on the left hand side of (3.2) into $I^{\le} = \{\theta \in [0,1]:\ |\xi_\theta| \le \Lambda(\delta)\}$ and $I^{>} = \{\theta \in [0,1]:\ |\xi_\theta| > \Lambda(\delta)\}$. Thanks to (3.3), Lemmas 2.2 and 2.4 we get

$$\begin{aligned}(I) &:= \int_{I^>} \big|[\nabla^2 f(\xi_\theta) - \nabla^2\varphi(\xi_\theta)]\big| \mathrm{d}\theta\, |\xi_1-\xi_0|^2 \\ &\le \delta\,\varphi''(|\xi_0|+|\xi_1|)\,|\xi_1-\xi_0|^2 \\ &\le c\,\delta\,|\mathbf{V}(\xi_1)-\mathbf{V}(\xi_0)|^2.\end{aligned}$$

If we choose $\delta > 0$ small enough, then

$$(I) \le \frac{\varepsilon}{2}|\mathbf{V}(\xi_1)-\mathbf{V}(\xi_0)|^2.$$

Assumptions (F2) and (H3) yield

$$\begin{aligned}(II) &:= \int_{I^\le} \big|[\nabla^2 f(\xi_\theta) - \nabla^2\varphi(\xi_\theta)]\big| \mathrm{d}\theta\, |\xi_1-\xi_0|^2 \\ &\le c\,(L+1)\int_{I^\le} \frac{\varphi'(|\xi_\theta|)}{|\xi_\theta|}\,\mathrm{d}\theta\, |\xi_1-\xi_0|^2.\end{aligned}$$

Due to Remark 2.3 there exists $0 < \gamma < 1$ and an N-function ρ with $\Delta_2(\{\rho,\rho^*\}) < \infty$ such that $(\varphi(t))^\gamma \sim \rho(t)$ uniformly in $t \ge 0$. Since $1/\gamma - 2 > -1$ we can find $\alpha > 1$ such that $\alpha'(1/\gamma - 2) > -1$, where $1 = \frac{1}{\alpha} + \frac{1}{\alpha'}$.

With the previous estimate, Hölder's inequality, and $\varphi'(t) \sim (\rho'(t))^{1/\gamma}\, t^{1/\gamma-1}$ we get

$$
\begin{aligned}
(II) &\le c\,(L+1)|I^{\le}|^{\frac{1}{\alpha}} \left(\int_0^1 \frac{\varphi'(|\xi_\theta|)^{\alpha'}}{|\xi_\theta|^{\alpha'}}\mathrm{d}\theta\right)^{\frac{1}{\alpha'}} |\xi_1-\xi_0|^2 \\
&\le c\,(L+1)|I^{\le}|^{\frac{1}{\alpha}} \left(\int_0^1 \frac{(\rho'(|\xi_\theta|))^{\alpha'/\gamma}}{|\xi_\theta|^{\alpha'(1/\gamma-2)}}\mathrm{d}\theta\right)^{\frac{1}{\alpha'}} |\xi_1-\xi_0|^2.
\end{aligned}
$$

Using that $\rho'(|\xi_\theta|) \le \rho'(|\xi_0|+|\xi_1|)$, Lemma 2.1, $\varphi'(t) \sim (\rho'(t))^{1/\gamma}\, t^{1/\gamma-1}$, and Lemma 2.4 we get

$$
\begin{aligned}
(II) &\le c\,(L+1)|I^{\le}|^{\frac{1}{\alpha}} \frac{(\rho'(|\xi_0|+|\xi_1|))^{1/\gamma}}{(|\xi_0|+|\xi_1|)^{1/\gamma-2}}|\xi_1-\xi_0|^2 \\
&\le c\,(L+1)|I^{\le}|^{\frac{1}{\alpha}}|\mathbf{V}(\xi_1)-\mathbf{V}(\xi_0)|^2.
\end{aligned}
$$

Let us now estimate $|I^{\le}|$. Recall that $|\xi_0| \ge \sigma(\varepsilon) = K\Lambda(\delta)$. If $|\xi_1-\xi_0| \ge (K-1)\Lambda(\delta)$, then

$$
|I^{\le}| \le \frac{2\Lambda(\delta)}{|\xi_1-\xi_0|} \le \frac{2}{K-1}\cdot \tag{3.4}
$$

If on the other hand $|\xi_1-\xi_0| < (K-1)\Lambda(\delta)$, then $|I^{\le}| = 0$. Thus, (3.4) holds in both cases. It follows that

$$
(II) \le c\,(L+1)\left(\frac{2}{K-1}\right)^{\frac{1}{\alpha}}|\mathbf{V}(\xi_1)-\mathbf{V}(\xi_0)|^2.
$$

If we choose $K \ge 2$ large enough, then

$$
(II) \le \frac{\varepsilon}{2}|\mathbf{V}(\xi_1)-\mathbf{V}(\xi_0)|^2.
$$

Combining the estimates for (I) and (II) we get the claim. □

We define the functional $\mathcal{F}_\varphi : W^{1,\varphi}(\Omega) \to \mathbb{R}$ by

$$
\mathcal{F}_\varphi(\mathbf{u}) := \int_\Omega \varphi(|\nabla \mathbf{u}|)\,\mathrm{d}x. \tag{3.5}
$$

Lemma 3.3 (comparison estimate)**.** *Let φ, f, and $\mathbf{u}$ be as in Theorem 1.1. Then for every $\varepsilon > 0$ there exists $\kappa(\varepsilon) > 0$ such that the following holds: If B be a ball with $B \subset \Omega$ and $\mathbf{v}$ is the local minimizer of the functional $\mathcal{F}_\varphi$, see (3.5), satisfying $\mathbf{v}-\mathbf{u} \in W_0^{1,\varphi}(B)$, then*

$$
\fint_B \left|\mathbf{V}(\nabla \mathbf{u})\right|^2 \mathrm{d}x \le \kappa(\varepsilon) \tag{3.6}
$$

or

$$
\fint_B \left|\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\nabla \mathbf{v})\right|^2 \mathrm{d}x \le \varepsilon \fint_B \left|\mathbf{V}(\nabla \mathbf{u}) - \langle \mathbf{V}(\nabla \mathbf{u})\rangle_B\right|^2 \mathrm{d}x. \tag{3.7}
$$

Note that $\kappa(\varepsilon)$ and $\gamma(\varepsilon)$ depend only on ε, n, N, L, $\Delta_2(\{\varphi,\varphi^*\})$, and the convergence in (1.3).

Proof. In the following let B always be a ball and let $\mathbf{v}$ be the local minimizer of the functional $\mathcal{F}_\varphi$, see (3.5), satisfying $\mathbf{v} - \mathbf{u} \in W_0^{1,\varphi}(B)$. Since V is surjective we can choose $\xi_0 \in \mathbb{R}^{n\times N}$ such that $\mathbf{V}(\xi_0) = \langle \mathbf{V}(\nabla\mathbf{u})\rangle_B$. Let σ be as in Lemma 3.2.

We start the proof with an auxiliary result.

Claim. There holds

$$\fint_B \big|\mathbf{V}(\nabla\mathbf{u}) - \mathbf{V}(\nabla\mathbf{v})\big|^2 \,\mathrm{d}x \le \frac{\varepsilon}{2} \fint_B \big|\mathbf{V}(\nabla\mathbf{u}) - \langle\mathbf{V}(\nabla\mathbf{u})\rangle_B\big|^2 \,\mathrm{d}x + \Gamma_B, \tag{3.8}$$

where $\Gamma_B := 0$ if $|\xi_0| \ge \sigma(\varepsilon/16)$ and $\Gamma_B := 2c\,\varphi(c\,\sigma(\varepsilon/16))$ if $|\xi_0| < \sigma(\varepsilon/16)$. The constant c depends only on n, N, L, and $\Delta_2(\varphi,\varphi^*)$.

Define $g : \mathbb{R}^{n\times N} \to \mathbb{R}$ by

$$g(\xi) = \varphi(\xi) + f(\xi_0) - \varphi(\xi_0) + [\nabla f(\xi_0) - \nabla\varphi(\xi_0)](\xi - \xi_0).$$

It is easy to see that $\mathbf{v}$ is also a local minimizer of

$$\int_B g(\nabla\mathbf{v})\,\mathrm{d}x. \tag{3.9}$$

such that $\mathbf{v} - \mathbf{u} \in W_0^{1,\varphi}(B)$. The Euler equation for (3.9) and the ellipticity of g yield

$$\begin{aligned}\int_B [g(\nabla\mathbf{u}) - g(\nabla\mathbf{v})]\,\mathrm{d}x &= \int_B \left\langle \int_0^1 (1-\theta)\nabla^2 g((1-\theta)\nabla\mathbf{v} + \theta\nabla\mathbf{u})\,\mathrm{d}\theta\,(\nabla\mathbf{u} - \nabla\mathbf{v}), (\nabla\mathbf{u} - \nabla\mathbf{v})\right\rangle \mathrm{d}x \\ &\ge c\int_B\int_0^1 (1-\theta)\varphi''(|(1-\theta)\nabla\mathbf{v} + \theta\nabla\mathbf{u}|)\,\mathrm{d}\theta\,|\nabla\mathbf{u} - \nabla\mathbf{v}|^2\,\mathrm{d}x.\end{aligned}$$

Now with $\varphi''(t)\,t \sim \varphi'(t)$, Lemmas 2.2 and 2.4 it follows

$$\begin{aligned}\int_B [g(\nabla\mathbf{u}) - g(\nabla\mathbf{v})]\mathrm{d}x &\ge c\int_B \varphi''(|\nabla\mathbf{u}| + |\nabla\mathbf{v}|)|\nabla\mathbf{u} - \nabla\mathbf{v}|^2\,\mathrm{d}x \\ &\ge c\int_B |\mathbf{V}(\nabla\mathbf{u}) - \mathbf{V}(\nabla\mathbf{v})|^2\,\mathrm{d}x.\end{aligned} \tag{3.10}$$

Now, since $\mathbf{u}$ is a local minimizer for $\mathcal{F}$, $\mathbf{u} - \mathbf{v} \in W_0^{1,\varphi}(B)$, and $B \subset \Omega$ it follows that

$$\begin{aligned}\int_B [g(\nabla\mathbf{u}) - g(\nabla\mathbf{v})]\,\mathrm{d}x &= \int_B [g(\nabla\mathbf{u}) - f(\nabla\mathbf{u})]\,\mathrm{d}x + \int_B [f(\nabla\mathbf{u}) - f(\nabla\mathbf{v})]\,\mathrm{d}x \\ &\quad + \int_B [f(\nabla\mathbf{v}) - g(\nabla\mathbf{v})]\mathrm{d}x \\ &\le \int_B [g(\nabla\mathbf{u}) - f(\nabla\mathbf{u})]\,\mathrm{d}x + \int_B [f(\nabla\mathbf{v}) - g(\nabla\mathbf{v})]\,\mathrm{d}x =: (I).\end{aligned}$$

Observe that for every $\xi_1 \in \mathbb{R}^{n\times N}$ holds

$$f(\xi_1) - g(\xi_1) = \left\langle \int_0^1 (1-\theta)[\nabla^2 f(\xi_\theta) - \nabla^2 g(\xi_\theta)]\mathrm{d}\theta\,(\xi_1 - \xi_0), (\xi_1 - \xi_0)\right\rangle, \tag{3.11}$$

where $\xi_\theta := (1-\theta)\xi_0 + \theta\xi_1$.

If $|\xi_0| \geq \sigma(\varepsilon/16)$, then it follows from (3.11) and Lemma 3.2 that

$$(I) \leq \frac{\varepsilon}{16}\bigg(\int_B |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\xi_0)|^2 \,\mathrm{d}x + \int_B |\mathbf{V}(\nabla \mathbf{v}) - \mathbf{V}(\xi_0)|^2 \,\mathrm{d}x\bigg).$$

If on the other hand $|\xi_0| \leq \sigma(\varepsilon/16)$, then it follows from (3.11), Lemmas 3.2 and 3.1, and (2.9) that

$$\begin{aligned}(I) &\leq \frac{\varepsilon}{16}\bigg(\int_B \chi_{\{|\nabla \mathbf{u}|\geq\sigma(\varepsilon/16)\}}|\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\xi_0)|^2 \,\mathrm{d}x + \int_B \chi_{\{|\nabla \mathbf{v}|\geq\sigma(\varepsilon/16)\}}|\mathbf{V}(\nabla \mathbf{v}) - \mathbf{V}(\xi_0)|^2 \,\mathrm{d}x\bigg) \\ &\quad + c\,\bigg(\int_B \chi_{\{|\nabla \mathbf{u}|<\sigma(\varepsilon/16)\}}|\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\xi_0)|^2 \,\mathrm{d}x + \int_B \chi_{\{|\nabla \mathbf{v}|<\sigma(\varepsilon/16)\}}|\mathbf{V}(\nabla \mathbf{v}) - \mathbf{V}(\xi_0)|^2 \,\mathrm{d}x\bigg) \\ &\leq \frac{\varepsilon}{16}\bigg(\int_B |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\xi_0)|^2 \,\mathrm{d}x + \int_B |\mathbf{V}(\nabla \mathbf{v}) - \mathbf{V}(\xi_0)|^2 \,\mathrm{d}x\bigg) + c\,\varphi\big(c\,\sigma(\varepsilon/16)\big).\end{aligned}$$

This and the previous estimate prove

$$\fint_B \big|\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\nabla \mathbf{v})\big|^2 \,\mathrm{d}x \leq \frac{\varepsilon}{16}\bigg(\int_B |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\xi_0)|^2 \,\mathrm{d}x + \int_B |\mathbf{V}(\nabla \mathbf{v}) - \mathbf{V}(\xi_0)|^2 \,\mathrm{d}x\bigg) + \tfrac{1}{2}\Gamma_B,$$

where $\Gamma_B := 0$ if $|\xi_0| \geq \sigma(\varepsilon/16)$ and $\Gamma_B := 2c\,\varphi(c\,\sigma(\varepsilon/16))$ if $|\xi_0| < \sigma(\varepsilon/16)$. We estimate by adding and subtracting $\mathbf{V}(\nabla \mathbf{v})$ in the second integrand

$$\begin{aligned}&\frac{\varepsilon}{16}\bigg(\int_B |\mathbf{V}(\nabla \mathbf{u}) - \langle \mathbf{V}(\nabla \mathbf{u})\rangle_B|^2 \,\mathrm{d}x + \int_B |\mathbf{V}(\nabla \mathbf{v}) - \langle \mathbf{V}(\nabla \mathbf{u})\rangle_B|^2 \,\mathrm{d}x\bigg) \\ &\qquad\leq \frac{\varepsilon}{4}\bigg(\int_B |\mathbf{V}(\nabla \mathbf{u}) - \langle \mathbf{V}(\nabla \mathbf{u})\rangle_B|^2 \,\mathrm{d}x + \int_B |\mathbf{V}(\nabla \mathbf{v}) - \mathbf{V}(\nabla \mathbf{u})|^2 \,\mathrm{d}x\bigg).\end{aligned}$$

This and the previous estimate shows

$$\fint_B \big|\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\nabla \mathbf{v})\big|^2 \,\mathrm{d}x \leq \frac{\varepsilon}{2}\,\bigg(\int_B |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\xi_0)|^2 \,\mathrm{d}x\bigg) + \Gamma_B,$$

which proves the auxiliary result (3.8).

Let us now prove the claim of the lemma. If $|\xi_0| \geq \sigma(\varepsilon/16)$, then the claim follows from (3.8), since in this case $\Gamma_B = 0$. So let us assume in the following that $|\xi_0| < \sigma(\varepsilon/16)$, which implies $\Gamma_B = 2c\,\varphi(c\,\sigma(\varepsilon/16))$. If

$$\Gamma_B \leq \frac{\varepsilon}{2}\fint_B \big|\mathbf{V}(\nabla \mathbf{u}) - \langle \mathbf{V}(\nabla \mathbf{u})\rangle_B\big|^2 \,\mathrm{d}x,$$

then the claim follows again from (3.8). So we can assume in the following that

$$\fint_B \big|\mathbf{V}(\nabla \mathbf{u}) - \langle \mathbf{V}(\nabla \mathbf{u})\rangle_B\big|^2 \,\mathrm{d}x \leq \frac{2\Gamma_B}{\varepsilon}.$$

This, $|\xi| \le \sigma(\varepsilon/16)$, and (2.9) imply

$$\begin{aligned}\fint_B \left|\mathbf{V}(\nabla\mathbf{u})\right|^2 \mathrm{d}x &\le 2\fint_B \left|\mathbf{V}(\nabla\mathbf{u}) - \langle\mathbf{V}(\nabla\mathbf{u})\rangle_B\right|^2 \mathrm{d}x + 2\left|\mathbf{V}(\xi_0)\right|^2\\ &\le \frac{4\Gamma_B}{\varepsilon} + c\,\varphi\big(c\,\sigma(\varepsilon/16)\big) =: \kappa(\varepsilon).\end{aligned}$$

This proves the lemma. □

The following result on the decay of the excess functional for local minimizers can be found in [9], Theorem 6.4.

Proposition 3.4 (decay estimate for $\mathbf{v}$)**.** *Let φ satisfy* (H1)–(H4)*, let $B \subset \Omega$ be a ball, and let $\mathbf{v}$ be the local minimizer of the functional $\mathcal{F}_\varphi$, see* (3.5)*, satisfying $\mathbf{v} - \mathbf{u} \in W_0^{1,\varphi}(B)$. Then there exists $\beta > 0$ and $c > 0$ such that for every ball $B \subset \Omega$ and every $\lambda \in (0,1)$ holds*

$$\fint_{\lambda B} |\mathbf{V}(\nabla\mathbf{v}) - \langle\mathbf{V}(\nabla\mathbf{v})\rangle_B|^2 \,\mathrm{d}x \le c\,\lambda^\sigma \fint_B |\mathbf{V}(\nabla\mathbf{v}) - \langle\mathbf{V}(\nabla\mathbf{v})\rangle_B|^2 \,\mathrm{d}x.$$

Note that c and β depend only on n, N, L, $\Delta_2(\{\varphi,\varphi^*\})$, and c_1.

We will now combine Proposition 3.4 and Lemma 3.3 to derive a decay estimate for the excess functional of $\mathbf{u}$.

Lemma 3.5 (decay estimate for $\mathbf{u}$)**.** *Let φ, f, and $\mathbf{u}$ be as in Theorem 1.1. Then exists $\kappa_0 > 0$ and λ_0 such that the following holds: if B is a ball with $B \subset \Omega$, then*

$$\fint_B |\mathbf{V}(\nabla\mathbf{u})|^2 \le \kappa_0 \tag{3.12}$$

or

$$\fint_{\lambda_0 B} |\mathbf{V}(\nabla\mathbf{u}) - \langle\mathbf{V}(\nabla\mathbf{u})\rangle_{\lambda_0 B}|^2 \,\mathrm{d}x \le \frac{1}{2}\fint_B |\mathbf{V}(\nabla\mathbf{u}) - \langle\mathbf{V}(\nabla\mathbf{u})\rangle_B|^2 \,\mathrm{d}x. \tag{3.13}$$

Note that κ_0 and λ_0 depend only on n, N, L, $\Delta_2(\{\varphi,\varphi^*\})$, c_1, β, and the limit in (1.3)*.*

Proof. Let B be a ball with $B \subset \Omega$. With Proposition 3.4 we estimate for any $\lambda \in (0,1)$

$$\begin{aligned}\fint_{\lambda B} |\mathbf{V}(\nabla\mathbf{u}) - \langle\mathbf{V}(\nabla\mathbf{u})\rangle_{\lambda B}|^2 \mathrm{d}x &\le 2\fint_{\lambda B} |\mathbf{V}(\nabla\mathbf{u}) - \mathbf{V}(\nabla\mathbf{v})|^2 \,\mathrm{d}x + \fint_{\lambda B} |\mathbf{V}(\nabla\mathbf{v}) - \langle\mathbf{V}(\nabla\mathbf{v})\rangle_{\lambda B}|^2 \,\mathrm{d}x\\ &\le 2\,\lambda^{-n}\fint_B |\mathbf{V}(\nabla\mathbf{u}) - \mathbf{V}(\nabla\mathbf{v})|^2 \,\mathrm{d}x + c\,\lambda^\sigma \fint_B |\mathbf{V}(\nabla\mathbf{v}) - \langle\mathbf{V}(\nabla\mathbf{v})\rangle_B|^2 \mathrm{d}x\\ &\le c\,\lambda^{-n}\fint_B |\mathbf{V}(\nabla\mathbf{u}) - \mathbf{V}(\nabla\mathbf{v})|^2 \,\mathrm{d}x + c\,\lambda^\sigma \fint_B |\mathbf{V}(\nabla\mathbf{u}) - \langle\mathbf{V}(\nabla\mathbf{u})\rangle_B|^2 \mathrm{d}x.\end{aligned}$$

In the following we fix $\lambda \in (0,1)$ such that $c\,\lambda^\sigma \le \frac{1}{4}$, which implies

$$\fint_{\lambda B} |\mathbf{V}(\nabla\mathbf{u}) - \langle\mathbf{V}(\nabla\mathbf{u})\rangle_{\lambda B}|^2 \,\mathrm{d}x \le c\,\lambda^{-n}\fint_B |\mathbf{V}(\nabla\mathbf{u}) - \mathbf{V}(\nabla\mathbf{v})|^2 \,\mathrm{d}x + \frac{1}{4}\fint_B |\mathbf{V}(\nabla\mathbf{u}) - \langle\mathbf{V}(\nabla\mathbf{u})\rangle_B|^2 \,\mathrm{d}x. \tag{3.14}$$

Due to Lemma 3.3 there exists $\kappa_0 > 0$ such that

$$\fint_B \big|\mathbf{V}(\nabla\mathbf{u})\big|^2\,\mathrm{d}x \le \kappa_0 \tag{3.15}$$

or

$$c\,\lambda^{-n}\fint_B \big|\mathbf{V}(\nabla\mathbf{u}) - \mathbf{V}(\nabla\mathbf{v})\big|^2\,\mathrm{d}x \le \frac{1}{4}\fint_B \big|\mathbf{V}(\nabla\mathbf{u}) - \langle\mathbf{V}(\nabla\mathbf{u})\rangle_B\big|^2\,\mathrm{d}x. \tag{3.16}$$

In combination with (3.14) we get that (3.15) holds or

$$\fint_{\lambda B} |\mathbf{V}(\nabla\mathbf{u}) - \langle\mathbf{V}(\nabla\mathbf{u})\rangle_{\lambda B}|^2\,\mathrm{d}x \le \frac{1}{2}\fint_B |\mathbf{V}(\nabla\mathbf{u}) - \langle\mathbf{V}(\nabla\mathbf{u})\rangle_B|^2\,\mathrm{d}x.$$

This proves the claim. □

We are now in position to prove our main result.

Proof of Theorem 1.1. Let $B \subset \Omega$ and let R denote the radius of B. Due to (2.9) it suffices to show that

$$|\mathbf{V}(\nabla\mathbf{u}(z))|^2 \le c\left(1 + \fint_B |\mathbf{V}(\nabla\mathbf{u})|^2\,\mathrm{d}x\right) \tag{3.17}$$

for almost all $z \in \frac{1}{2}B$. Since $\mathbf{u} \in W^{1,\varphi}(\Omega)$, it follows by Lemma 2.4 that $\mathbf{V}(\nabla\mathbf{u}) \in L^2(\Omega)$. Thus for almost every $z \in \frac{1}{2}B$ holds

$$\lim_{r\to 0}\fint_{B_r(z)} |\mathbf{V}(\nabla\mathbf{u}(z)) - \langle\mathbf{V}(\nabla\mathbf{u})\rangle_{B_r(z)}|^2\,\mathrm{d}x = 0. \tag{3.18}$$

Let E denote the set of $z \in \frac{1}{2}B$ such (3.18) holds. To prove the theorem it suffices to show that

$$\big|\mathbf{V}\big(\nabla\mathbf{u}(z)\big)\big|^2 \le c\left(1 + \fint_{B_R(z)} |\mathbf{V}(\nabla\mathbf{u})|^2\,\mathrm{d}x\right) \tag{3.19}$$

for every $z \in E$.

Fix $z \in E$. Then due to Lemma 3.5 there exists $\kappa_0 > 0$ and $\lambda_0 \in (0,1)$ such that for every $r \in (0, R/2)$ holds

$$\fint_{B_r(z)} \big|\mathbf{V}(\nabla\mathbf{u})\big|^2\,\mathrm{d}x \le \kappa_0 \tag{3.20}$$

or

$$\fint_{B_{\lambda_0 r}(z)} |\mathbf{V}(\nabla\mathbf{u}) - \langle\mathbf{V}(\nabla\mathbf{u})\rangle_{B_{\lambda_0 r}(z)}|^2\,\mathrm{d}x \le \frac{1}{2}\fint_{B_r(z)} |\mathbf{V}(\nabla\mathbf{u}) - \langle\mathbf{V}(\nabla\mathbf{u})\rangle_{B_r(z)}|^2\,\mathrm{d}x. \tag{3.21}$$

This allows us to distinguish two cases:

(i) There exists a sequence of radii $r_j \to 0$ such that (3.20) holds for every r_j.

(ii) There exists $R_0 > 0$ such that (3.20) holds for all $r \le R_0$.

In the case (i) it follows with (3.18) that

$$|\mathbf{V}(\nabla\mathbf{u})(z)|^2 \le \kappa_0.$$

Let us now consider the case (i). Let $r_0 := \sup\{s \in (0, R/2) : (3.21) \text{ holds for all } r \le s\}$, then $r_0 \ge R_0 > 0$. By continuity of the expressions in (3.21) with respect to $r \in (0, R)$, it follows that also r_0 satisfies (3.21). Let $r_k := \lambda_0^{-k} r_0$ for $k \in \mathbb{N}_0$. Repeated use of (3.21) shows

$$\fint_{B_{r_k}(z)} |\mathbf{V}(\nabla\mathbf{u}) - \langle\mathbf{V}(\nabla\mathbf{u})\rangle_{B_{r_k}(z)}|^2 \,\mathrm{d}x \le 2^{-k} \fint_{B_{r_0}(z)} |\mathbf{V}(\nabla\mathbf{u}) - \langle\mathbf{V}(\nabla\mathbf{u})\rangle_{B_{r_0}(z)}|^2 \,\mathrm{d}x$$

for every $k \in \mathbb{N}$. But then, since

$$\big|\langle\mathbf{V}(\nabla\mathbf{u})\rangle_{B_{r_k}(z)} - \langle\mathbf{V}(\nabla\mathbf{u})\rangle_{B_{r_{k+1}}(z)}\big| \le \lambda_0^{-n} \left(\fint_{B_{r_k}(z)} \big|\mathbf{V}(\nabla\mathbf{u}) - \langle\mathbf{V}(\nabla\mathbf{u})\rangle_{B_{r_k}(z)}\big|^2 \,\mathrm{d}x \right)^{\frac{1}{2}},$$

we get using (3.18)

$$\begin{aligned} |\mathbf{V}(\nabla\mathbf{u})(z)| &\le \sum_{k=0}^{\infty} \big|\langle\mathbf{V}(\nabla\mathbf{u})\rangle_{B_{r_{k+1}}(z)} - \langle\mathbf{V}(\nabla\mathbf{u})\rangle_{B_{r_k}(z)}\big| + |\langle\mathbf{V}(\nabla\mathbf{u})\rangle_{B_{r_0}(z)}| \\ &\le \lambda_0^{-n} \sum_{k=0}^{\infty} 2^{-k} \Big(\fint_{B_{r_0}(z)} |\mathbf{V}(\nabla\mathbf{u}) - \langle\mathbf{V}(\nabla\mathbf{u})\rangle_{B_{r_0}(z)}|^2 \,\mathrm{d}x \Big)^{\frac{1}{2}} + |\langle\mathbf{V}(\nabla\mathbf{u})\rangle_{B_{r_0}(z)}| \\ &\le \big(2\,\lambda_0^{-n}\big) \Big(\fint_{B_{r_0}(z)} |\mathbf{V}(\nabla\mathbf{u}) - \langle\mathbf{V}(\nabla\mathbf{u})\rangle_{B_{r_0}(z)}|^2 \,\mathrm{d}x \Big)^{\frac{1}{2}} + |\langle\mathbf{V}(\nabla\mathbf{u})\rangle_{B_{r_0}(z)}| \\ &\le \big(4\,\lambda_0^{-n} + 1\big) \Big(\fint_{B_{r_0}(z)} |\mathbf{V}(\nabla\mathbf{u})|^2 \,\mathrm{d}x \Big)^{\frac{1}{2}}. \end{aligned} \tag{3.22}$$

If $r_0 = R/2$, then we estimate with (3.22), $B_{R/2}(z) \subset B$, and $|B_{R/2}(z)| = 2^{-n}|B|$

$$|\mathbf{V}(\nabla\mathbf{u})(z)|^2 \le 2^n \big(4\,\lambda_0^{-n} + 1\big)^2 \fint_B |\mathbf{V}(\nabla\mathbf{u})|^2 \,\mathrm{d}x,$$

which proves (3.19). So we can continue under the assumption that $0 < r_0 < R/2$. We will show in the following that in this case r_0 satisfies (3.20). The definition of r_0 and $r_0 < R/2$ imply that for every $j \in \mathbb{N}$ there exists $r_j \in [r_0, \min\{r_0 + \frac{1}{j}, R/2\})$ such that (3.20) holds. Since $r_j \to r_0$ for $j \to \infty$, we conclude by continuity of $r \mapsto \fint_{B_r(z)} |\mathbf{V}(\nabla\mathbf{u})|^2 \,\mathrm{d}x$ on $(0, R)$ that also r_0 satisfies (3.20). This and (3.22) imply

$$|\mathbf{V}(\nabla\mathbf{u})(z)|^2 \le \big(4\,\lambda_0^{-n} + 1\big)^2 \kappa_0.$$

This concludes the proof of Theorem 1.1. $\square$

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