ESAIM: COCV 16 (2010) 247–274
DOI: 10.1051/cocv/2008077

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

UNIQUE CONTINUATION PRINCIPLE FOR SYSTEMS OF PARABOLIC EQUATIONS

Otared Kavian[1] and Luz de Teresa[2]

Abstract. In this paper we prove a unique continuation result for a cascade system of parabolic equations, in which the solution of the first equation is (partially) used as a forcing term for the second equation. As a consequence we prove the existence of ε-insensitizing controls for some parabolic equations when the control region and the observability region do not intersect.

Mathematics Subject Classification. 35B37, 35B60, 93C20.

Received December 10, 2007. Revised October 10, 2008.
Published online February 10, 2009.

1. Statement of the problem and main results

This paper is devoted to the study of unique continuation properties for cascade systems of parabolic equations. This kind of problems has been studied in particular by Bodart and Fabre [1] in the context of the so called ε-insensitizing control problems for the heat equation, and has been solved only in the particular case in which the control domain and the observability domain have non empty intersection (see Sect. 5 or [1] for a complete description of the problem).

To begin with a simple example, as far as the unique continuation property is concerned, let $\Omega \subset \mathbb{R}^N$ be a Lipschitz bounded domain and, for $p_0 \in L^2(\Omega)$, let p be the solution of

$$\begin{cases} \partial_t p - \operatorname{div}(a\nabla p) = 0 & \text{in } (0,T)\times\Omega \\ p(0,x) = p_0(x) & \text{in } \Omega \\ p(t,\sigma) = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{1.1}$$

Here $a := (a_{ij})_{1\le i,j\le N}$ is a self-adjoint matrix such that for some positive constant $c_0 > 0$ and all $\xi \in \mathbb{R}^N$, and all $x \in \Omega$

$$\sum_{1\le i,j\le N} a_{ij}(x)\xi_i\xi_j \ge c_0|\xi|^2, \qquad a_{ij} \in W^{1,\infty}(\Omega). \tag{1.2}$$

The first kind of *unique continuation result* which we are interested in, can be illustrated with the following:

Keywords and phrases. Unique continuation, approximate controllability, cascade systems of parabolic equations.

[1] Département de Mathématiques, Université de Versailles Saint Quentin, 45 avenue des États-Unis, 78035 Versailles Cedex, France. kavian@math.uvsq.fr

[2] Instituto de Matemáticas, Universidad Nacional Autónoma de México, Circuito Exterior, C.U. 04510, D.F., México. deteresa@matem.unam.mx

Article published by EDP Sciences

Theorem 1.1. *Let Ω be a bounded Lipschitz domain and let the matrix a satisfy* (1.2)*. For $p_0, u_0 \in L^2(\Omega)$ denote by p the solution of* (1.1)*, and for an open $\omega_0 \subset \Omega$ let u satisfy the equation*

$$\begin{cases} \partial_t u - \operatorname{div}(a\nabla u) = p 1_{\omega_0} & \text{in } (0,T)\times\Omega \\ u(0,x) = u_0(x) & \text{in } \Omega \\ u(t,\sigma) = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{1.3}$$

Assume that $\omega_1 \subset \Omega$ is an open subdomain and for some $T_2 > T_1 > 0$ and an infinite sequence $(t_j)_{j\geq 1}$ with $t_j \in [T_1, T_2]$ we have $u(t_j, x) \equiv 0$ in ω_1. Then we have $p_0 \equiv u_0 \equiv 0$ in Ω (hence $p \equiv 0$ and $u \equiv 0$).

Another variant of this *unique continuation principle* concerns a cascade system of parabolic equations in which the second equation is a backward evolution equation.

Theorem 1.2. *Let Ω be a bounded Lipschitz domain and let the matrix a satisfy* (1.2)*. For $p_0 \in L^2(\Omega)$ denote by p the solution of* (1.1)*, and for an open $\omega_0 \subset \Omega$ let z be the solution of the backward heat equation*

$$\begin{cases} -\partial_t z - \operatorname{div}(a\nabla z) = p(t,x) 1_{\omega_0} & \text{in } (0,T)\times\Omega \\ z(T,x) = 0 & \text{in } \Omega \\ z(t,\sigma) = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{1.4}$$

Assume that $\omega_1 \subset \Omega$ is an open subdomain and for some $T > T_2 > T_1 > 0$ and an infinite sequence $(t_j)_{j\geq 1}$ with $t_j \in [T_1, T_2]$ we have $z(t_j, x) \equiv 0$ in ω_1. Then we have $p_0 \equiv 0$ in Ω (and hence $p \equiv 0$ and $z \equiv 0$).

Note that when $\omega := \omega_0 \cap \omega_1 \neq \emptyset$, the above assumptions on u (or on z) imply easily that $p(t,x) \equiv 0$ on $(0,T)\times\omega$, and hence the classical unique continuation principle for the heat equation (see for instance Saut and Scheurer [9]) implies that $p(t,x) \equiv 0$ on $(0,T)\times\Omega$, and consequently $p_0 \equiv 0$. However, as we shall see, when $\omega_0 \cap \omega_1 = \emptyset$ the result is not obvious.

Actually the main ingredients of the proof of the above results consist in two properties, shared by a large class of evolution equations associated to a self-adjoint operator A (for instance $Au := -\operatorname{div}(a\nabla u)$ with Dirichlet boundary conditions): the first ingredient is the fact that the semi-group $S(t) := \exp(-tA)$ generated by such operators on a Hilbert space have the unique continuation property: if $S(t)f = 0$ on $(0,T)\times\omega$ with $\omega \subset \Omega$ an open subset (for instance), then $f \equiv 0$. The second ingredient is that the semi-group $S(t)$ satisfies the so-called backward unique continuation property, that is if for some $T > 0$ one has $S(T)f = 0$, then $f = 0$. Indeed we do not claim that we can prove a unique continuation result for a cascade system of evolution equations with such general operators, since our arguments need some more technical assumptions. However, the assumptions we make are weak enough to include a large class of parabolic systems.

In Sections 2 and 3 we prove our main results, in an abstract setting, for a cascade system of equations. More precisely we consider equations such as

$$\begin{cases} \partial_t p + Ap = 0 & \text{for } t > 0 \\ \partial_t u + Au = B_0 p(t) & \text{for } t > 0 \\ p(0) = p_0 \\ u(0) = u_0 \end{cases} \qquad \text{or} \qquad \begin{cases} \partial_t p + Ap = 0 & \text{for } t > 0 \\ -\partial_t z + Az = B_0 p(t) & \text{for } 0 < t < T \\ p(0) = p_0 \\ z(T) = 0. \end{cases}$$

We denote by H a Hilbert space of functions defined on Ω, where $\Omega \subset \mathbb{R}^N$ is an open set (bounded or not); as a typical example one can think of H as being $(L^2(\Omega))^m$, for some integer $m \geq 1$. The norm of H is denoted by $\|\cdot\|$ and its scalar product by $(\cdot|\cdot)$. We consider $(A, D(A))$ an unbounded self-adjoint operator acting on H, that is $D(A) \subset H$ and $A : D(A) \longrightarrow H$. We assume that

$$A \text{ is self-adjoint and has a compact resolvent,} \tag{1.5}$$

and that the sequence of eigenvalues of $(A, D(A))$, denoted by $(\lambda_k)_{k\geq 1}$ satisfies

$$\exists\,\beta > 0, \quad \exists c_0 > 0, \quad \exists\, k_0 \geq 1, \quad \forall\, k \geq k_0, \quad \lambda_k \geq c_0\, k^{\beta}. \tag{1.6}$$

We denote by

$$\mathbb{P}_k : H \longrightarrow N(A - \lambda_k I), \tag{1.7}$$

the orthogonal projection on the eigenspace $N(A - \lambda_k I)$ associated to λ_k. Thus for $f \in H$ we have $f = \sum_{k\geq 1} \mathbb{P}_k f$. The operator A being as above, for each real number $\gamma \geq 0$ we assume that the domain $D(A^\gamma)$ is endowed with its natural norm, that is

$$u \in D(A^\gamma) \iff \|u\|^2_{D(A^\gamma)} := \|u\|^2 + \sum_{k\geq 1} \lambda_k^{2\gamma}\, \|\mathbb{P}_k(u)\|^2 < \infty,$$

or, in the simpler case in which the least eigenvalue λ_1 is positive,

$$u \in D(A^\gamma) \iff \|u\|^2_{D(A^\gamma)} := \sum_{k\geq 1} \lambda_k^{2\gamma}\, \|\mathbb{P}_k(u)\|^2 < \infty.$$

As a matter of fact, as we shall see below, we can always assume that the condition $\lambda_1 > 0$ is satisfied. Also, by an abuse of notation, when $\gamma < 0$, denoting by $n_0 \geq 1$ a (possible) integer such that $\lambda_{n_0} = 0$, we shall denote again $D(A^\gamma)$ as being the closure of H for the norm

$$u \mapsto \left(\|\mathbb{P}_{n_0}(u)\|^2 + \sum_{k\neq n_0} |\lambda_k|^{2\gamma} \|\mathbb{P}_k(u)\|^2 \right)^{1/2}$$

and for $f \in D(A^\gamma)$, $g \in D(A^\beta)$ with $\alpha + \beta \geq 0$ we write

$$(f|g) := \sum_{k\geq 1} (\mathbb{P}_k(f)|\mathbb{P}_k(g)).$$

This amounts to identifying H with its dual H', and then the dual of $D(A^\gamma)$ equipped with the above norm is identified with $D(A^{-\gamma})$.

With these conventions in mind, we shall need bounded linear operators noted B which satisfy certain properties.

Definition 1.3. The operator $(A, D(A))$ being as in (1.5)–(1.6), we shall say that $B \in SAP(A^\gamma, H)$ if

$$\begin{cases} B : D(A^\gamma) \longrightarrow H \ \text{ is a bounded linear operator for some } \gamma \geq 0, \\ \forall\, f, g \in D(A^\gamma), \quad (Bf|g) = (f|Bg), \quad (Bf|f) \geq 0. \end{cases} \tag{1.8}$$

Another useful class of operators consists in those which satisfy a certain *abstract unique continuation property* for A with respect to B. To be more precise, we introduce the following definition:

Definition 1.4. The operator $(A, D(A))$ being as in (1.5)–(1.6), and X being a Banach space, we shall say that $B \in UCP(A^\gamma, X)$ if

$$\begin{cases} B : D(A^\gamma) \longrightarrow X \ \text{ is a linear bounded operator for some } \gamma \geq 0, \\ \varphi \in N(A - \lambda_k I) \ \text{ and } \ B\varphi = 0 \quad \Longrightarrow \quad \varphi \equiv 0 \ \text{ on } \ \Omega. \end{cases} \tag{1.9}$$

For $p_0 \in H$ we denote by p the solution of the evolution equation

$$\begin{cases} \partial_t p + Ap = 0 & \text{on } (0,\infty)\times\Omega \\ p(0) = p_0 & \text{in } \Omega \\ p(t) \in D(A) & \text{for } t>0. \end{cases} \tag{1.10}$$

With p solution to (1.10), for $B_0 \in SAP(A^\gamma, H) \cap UCP(A^\gamma, H)$, we consider u the solution of

$$\begin{cases} \partial_t u + Au = B_0 p(t) & \text{on } (0,\infty) \\ u(0) = u_0 & \\ u(t) \in D(A) & \text{for a.e. } t>0, \end{cases} \tag{1.11}$$

where $u_0 \in H$ is a given initial data, and, for a given positive $T > 0$, we denote by z the solution of the backward evolution equation

$$\begin{cases} -\partial_t z + Az = B_0 p(t) & \text{on } (0,T)\times\Omega \\ z(T) = 0 & \text{in } \Omega \\ z(t) \in D(A) & \text{for a.e. } t>0. \end{cases} \tag{1.12}$$

Our first main result concerns the system of forward-forward equations (1.10)–(1.11):

Theorem 1.5 (forward-forward)**.** *Let the operator A satisfy conditions (1.5), (1.6), and let $B_0 \in SAP(A^\gamma, H) \cap UCP(A^\gamma, H)$. Assume that $u_0 \in H$, and $p_0 \in D(A^\beta)$ for some $\beta > \gamma_0 - 1$ are given and denote by p the solution of (1.10) and by u the solution of (1.11). If $B_1 \in UCP(A^\gamma, H)$ is such that $B_1 u(t_j) = 0$ for an infinite sequence $t_j \in [T_1, T_2]$ for some $0 < T_1 < T_2$, then $u_0 = p_0 = 0$, that is $p = u \equiv 0$.*

The second result concerns the system of forward-backward equations (1.10)–(1.12):

Theorem 1.6 (forward-backward)**.** *Assume that A satisfies (1.5), (1.6), and let $B_0 \in SAP(A^\gamma, H) \cap UCP(A^\gamma, H)$. Let $B_1 \in UCP(A^\gamma, H)$. For a given $p_0 \in H$ let p be the solution of (1.10) and z be the solution of equation (1.12). If $B_1 z(t_j) = 0$ for an infinite sequence $t_j \in [T_1, T_2]$ for some $0 < T_1 < T_2$, then we have $p_0 = 0$ and $p \equiv z \equiv 0$.*

To illustrate the above assumptions on A, B, assume that $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, set $D(A) := H^2(\Omega) \cap H_0^1(\Omega)$ and $Au := -\Delta u$. Then if $\omega \subset \Omega$ is an open subset and $Bu := u\,1_\omega$, we may choose $\gamma = 0$ and $X := L^2(\Omega)$: in this case property (1.9) boils down to the classical unique continuation property for the Laplacian: if $-\Delta\varphi = \lambda\varphi$ and $u \in H_0^1(\Omega)$, then the assumption $\varphi = 0$ in ω, implies $\varphi \equiv 0$ (see Sect. 4). Note also that in this situation the property (1.8) is also satisfied. Another example, with A as previously, is the following: choose $\gamma := 1$, set $X := L^2(\partial\Omega)$ and $\Gamma \subset \partial\Omega$ being a relatively open subset of the boundary, consider

$$Bu := 1_\Gamma \frac{\partial u}{\partial \mathbf{n}}\cdot$$

In this case A, B satisfy again (1.9) since

$$\varphi \in H_0^1(\Omega), \quad -\Delta\varphi = \lambda\varphi, \quad \frac{\partial\varphi}{\partial\mathbf{n}} = 0 \text{ on } \Gamma,$$

implies again that $\varphi \equiv 0$ in Ω.

The remainder of this paper is organized as follows. In Section 2 we establish a representation formula for u and z and we prove Theorem 1.5. In Section 3 we prove Theorem 1.6, while in Section 4 we show how our abstract result can be applied to some heat equations, such as those considered in Theorems 1.1 and 1.2. In Section 5 we consider a system of cascade Stokes equations and in Section 6 we give a few applications of Theorem 1.6 in control theory.

2. Preliminary results for the representation of solutions and proof of Theorem 1.5

Observe that in (1.6) there is no need to assume that $\lambda_1 > 0$. As a matter of fact, once the eigenvalues are assumed to be distinct, indeed each having its own multiplicity greater or equal than one, condition (1.6) can be replaced by the condition

$$0 < \lambda_k < \lambda_{k+1}, \qquad \exists\, \beta > 0, \quad \exists\, c_0 > 0, \quad \forall\, k \geq 1, \quad \lambda_k \geq c_0\, k^{\beta}. \tag{2.1}$$

Indeed we choose $\lambda_0 > -\lambda_1$ so that upon setting $A_0 := A + \lambda_0 I$, and

$$\widetilde{p} := \mathrm{e}^{-\lambda_0 t} p(t), \qquad \widetilde{u}(t) := \mathrm{e}^{-\lambda_0 t} u(t),$$

we have

$$\partial_t \widetilde{p} + A_0 \widetilde{p} = 0, \qquad \partial_t \widetilde{u} + A_0 \widetilde{u} = B\widetilde{p}$$

and therefore replacing the operator A by A_0, which satisfies conditions (2.1), if a unique continuation result is proved for A_0, $\widetilde{p}$ and $\widetilde{u}$, then clearly our main theorems apply to A, p and u. A similar modification can be applied to the cascade system involving p and z (see Sect. 3). From now on we assume that $\lambda_1 > 0$, that is that A satisfies (2.1).

We recall here that if B satisfies condition (1.8), it is an elementary exercise (which consists in expanding the scalar product $(B(f + tg)|f + tg)$ for $g \in D(A^\gamma)$ and $t > 0$, and then letting t converge to zero) to observe that the semi-positivity assumption in (1.8) on B, together with the fact that $D(A^\gamma)$ is dense in H, yield

$$f \in D(A^\gamma), \qquad (Bf|f) = 0 \Longrightarrow Bf = 0. \tag{2.2}$$

We denote by $S(t) := \exp(-tA)$ the semigroup generated by A on H. Thus, for $p_0 \in H$ if $p(t)$ is given by (1.10) we have ($\mathbb{P}_k$ being the orthogonal projection on $N(A - \lambda_k I)$, see (1.7))

$$S(t)p_0 = p(t) = \sum_{k\geq 1} \mathrm{e}^{-\lambda_k t} \mathbb{P}_k p_0. \tag{2.3}$$

Since A is a self-adjoint operator on H (with $\lambda_1 > 0$), the semi-group $S(t)$ is holomorphic, contractive, and in particular for $t > 0$ and $p_0 \in H$ we have $S(t)p_0 \in D(A^\gamma)$ for all $\gamma \geq 0$ (see for instance Yosida [10], Chap. IX), and more generally for $p_0 \in D(A^\alpha)$ where $\alpha \leq \gamma$, we have for a constant c depending on $(\gamma - \alpha)$

$$\|A^\gamma p(t)\| \leq c(\gamma - \alpha)\, t^{-(\gamma-\alpha)}\, \|p_0\|_{D(A^\alpha)}. \tag{2.4}$$

A straightforward consequence of the unique continuation assumption (1.9) is the following unique continuation principle for solutions of equation (1.10).

Theorem 2.1. *Let A satisfy (1.5), (2.1), $p_0 \in H$ and let p be the solution of equation (1.10). If $B \in UCP(A^\gamma, X)$, and $T_2 > T_1 > 0$ are such that $Bp(t_j) = 0$ for an infinite sequence $t_j \in [T_1, T_2]$, then we have $p_0 = 0$ and $p(t) \equiv 0$ on $(0, \infty)$.*

Proof. Recall that $t \mapsto p(t)$ is analytic on $(0, \infty) \longrightarrow D(A^\gamma)$. Denoting by $\langle \cdot, \cdot \rangle$ the duality between the X' and X, for $g \in X'$ setting $F(t) := \langle g, Bp(t) \rangle$, we have an analytic function F on $(0, \infty) \longrightarrow \mathbb{R}$. Since we have $F(t_j) = 0$ for an infinite sequence $t_j \in [T_1, T_2]$, it follows that $F(t) \equiv 0$ for all $t \in (0, \infty)$. Therefore for all $t \in (0, \infty)$, and all $g \in X'$ we have

$$0 = \langle g, Bp(t) \rangle = \sum_{k\geq 1} \mathrm{e}^{-\lambda_k t}\, \langle g, B\mathbb{P}_k(p_0) \rangle.$$

Since the numbers λ_k are all distinct, from this and the classical well known result concerning the topological independence of the exponentials $\left(e^{-\lambda_k t}\right)_{k\geq 1}$, we conclude that $\langle g, B\mathbb{P}_k(p_0)\rangle = 0$ for all $g \in X'$ and all $k \geq 1$, that is $B\mathbb{P}_k(p_0) = 0$. Thus by (1.9) we have $\mathbb{P}_k(p_0) = 0$ for all $k \geq 1$. Therefore $p_0 \equiv 0$. □

Another result which shall be needed, is the so called backward uniqueness result for the semi-group $S(t)$:

Theorem 2.2. *Let $p_0 \in H$. If p satisfies (1.10) and for some $T > 0$ one has $p(T) = 0$ then $p(t) = 0$ for all $t \in (0, \infty)$ and $p_0 = 0$. More precisely for $0 < t < T$, setting $\theta := t/T$ we have*

$$\|p(t)\| \leq \|p_0\|^{1-\theta} \|p(T)\|^{\theta}.$$

This is a classical result concerning semi-groups generated by self-adjoint operators A such that $(Af|f) \geq 0$. The proof, in a general setting is based on the fact that the function $t \mapsto h(t) := \log \|p(t)\|^2$ is convex. Another proof, more elementary but applying to our case, consists in writing

$$\|p(t)\|^2 = \sum_{k\geq 1} e^{-2\lambda_k t} \|\mathbb{P}_k(p_0)\|^2 = \sum_{k\geq 1} \left(\|\mathbb{P}_k(p_0)\|^2\right)^{1-\theta} \left(e^{-2\lambda_k T} \|\mathbb{P}_k(p_0)\|^2\right)^{\theta}$$
$$\leq \left(\sum_{k\geq 1} \|\mathbb{P}_k(p_0)\|^2\right)^{1-\theta} \left(\sum_{k\geq 1} e^{-2\lambda_k T} \|\mathbb{P}_k(p_0)\|^2\right)^{\theta} = \|p_0\|^{2(1-\theta)} \|p(T)\|^{2\theta},$$

where we use Hölder's inequality in $\ell^q(\mathbb{N}^*)$ and $\ell^{q'}(\mathbb{N}^*)$ with $q := (1-\theta)^{-1}$ and $q' := \theta^{-1}$.

Remark 2.3. Note that the above argument is also valid for $p_0 \in D(A^\alpha)$, that is

$$\|p(t)\|_{D(A^\alpha)} \leq \|p_0\|_{D(A^\alpha)}^{1-\theta} \|p(T)\|_{D(A^\alpha)}^{\theta}$$

for any $\alpha \in \mathbb{R}$.

Our aim is to show that if $B_1 \in UCP(A^\gamma, X)$ is such that $B_1 u(t_j) = 0$, or $B_1 z(t_j) = 0$, for an infinite sequence $t_j \in [T_1, T_2]$ with $T_1, T_2 \in (0, T)$, then we have $p_0 \equiv 0$. To this end we begin by representing the solutions of the system of equations (1.10)–(1.11), or (1.10)–(1.12), in terms of the initial data p_0. For $p_0 \in H$ and $p_0 \neq 0$, we consider the subset $\mathbb{K} \subset \mathbb{N}$

$$\mathbb{K} := \mathbb{K}(p_0) := \{n \geq 1;\ \mathbb{P}_n p_0 \neq 0\} \tag{2.5}$$

and we denote

$$\alpha_n := \|\mathbb{P}_n p_0\|, \qquad \text{and for } n \in \mathbb{K}, \quad \varphi_n := \frac{1}{\alpha_n} \mathbb{P}_n p_0, \tag{2.6}$$

so that for $n \in \mathbb{K}$ we have $A\varphi_n = \lambda_n \varphi_n$, with $\|\varphi_n\| = 1$, and we may write

$$p_0 = \sum_{n\in\mathbb{K}} \alpha_n \varphi_n.$$

Even though we are defining the eigenfunction φ_n when $n \notin \mathbb{K}$, it is sometimes convenient, and harmless, to use the (abuse of) notation $p_0 = \sum_{n\geq 1} \alpha_n \varphi_n$, since for $n \notin \mathbb{K}$ by definition we have $\alpha_n = 0$. It follows that $p(t)$ is given by

$$p(t) = S(t)p_0 = \sum_{n\in\mathbb{K}} \alpha_n e^{-\lambda_n t} \varphi_n = \sum_{n\geq 1} \alpha_n e^{-\lambda_n t} \varphi_n. \tag{2.7}$$

Note also that since for any $n \in \mathbb{K}$ we have $B_0\varphi_n = \sum_{j\geq 1} \mathbb{P}_j(B_0\varphi_n)$, we can express $B_0 p(t)$ as

$$B_0 p(t) = \sum_{n\in\mathbb{K}} e^{-\lambda_n t} B_0\varphi_n = \sum_{n\in\mathbb{K}} \sum_{j\geq 1} \alpha_n e^{-\lambda_n t} \mathbb{P}_j(B_0\varphi_n) = \sum_{j\geq 1} \sum_{n\in\mathbb{K}} \alpha_n e^{-\lambda_n t} \mathbb{P}_j(B_0\varphi_n).$$

The solution of the equation

$$\begin{cases} \partial_t v + Av = \widetilde{f}(t) & \text{in } (0,T) \\ v(0) = v_0 & \\ v(t) \in D(A) & \text{a.e. on } (0,T), \end{cases} \tag{2.8}$$

is given by

$$v(t) = S(t)v_0 + \int_0^t S(t-\tau)\widetilde{f}(\tau)\mathrm{d}\tau, \tag{2.9}$$

provided that, for instance, $\widetilde{f} \in C([0,T];H)$ (actually it is enough to have $\widetilde{f} \in L^q((0,T);H)$ for some $q > 1$ but, as we shall see below, at this point it is not necessary to enter into this kind of subtleties). Since for $t > 0$ we have

$$S(t)g = \sum_{k\geq 1} \exp(-\lambda_k t)\, \mathbb{P}_k g, \qquad \mathbb{P}_k S(t-\tau)\widetilde{f}(\tau) = \exp(-\lambda_k(t-\tau))\mathbb{P}_k(\widetilde{f}(\tau)),$$

we can write (with a convergence in $C([0,T];H)$ of the series involved)

$$v(t) = \sum_{k\geq 1} \mathrm{e}^{-\lambda_k t}\mathbb{P}_k(v_0) + \sum_{k\geq 1}\int_0^t \exp(-\lambda_k(t-\tau))\, \mathbb{P}_k(\widetilde{f}(\tau))\mathrm{d}\tau. \tag{2.10}$$

For the remainder of this section we assume that for some $\gamma_0 \geq 0$ the operator B_0 satisfies

$$B_0 : D(A_0^{\gamma}) \longrightarrow H \ \text{ is a linear bounded operator.} \tag{2.11}$$

First consider

$$v_0 := u_0, \qquad \widetilde{f}(t) := B_0 p(t), \qquad p_0 \in D(A^{\gamma_0}),$$

so that $p \in C([0,T];D(A^{\gamma_0}))$ and v is in fact the function u solution of (1.11). Then by (2.4) we have

$$\|B_0 p(t)\| \leq \|B_0\|\, \|p(t)\|_{D(A^{\gamma_0})} \leq \|B_0\|\, \|p_0\|_{D(A^{\gamma_0})}$$

and we know that $t \mapsto B_0 p(t)$ belongs to $C([0,T];H)$. Since

$$B_0 p(t) = \sum_{n\in\mathbb{K}} \alpha_n \mathrm{e}^{-\lambda_n t}\, B_0\varphi_n, \qquad \text{and for } n \in \mathbb{K}, \quad B_0\varphi_n = \sum_{k\geq 1}\mathbb{P}_k(B_0\varphi_n),$$

one can check quite easily that the convergence of the series is uniform in $t \in [\varepsilon, T]$ for any $0 < \varepsilon < T$. From the very expression of the mapping $t \mapsto B_0 p(t)$, one sees that this function has a natural holomorphic extension to the right half-plane of $\mathbb{C}$, that is to the set $[\Re(t) > 0]$. Due to the fact that we need the analyticity of the mapping $t \mapsto u(t)$, we detail somewhat this aspect of the convergence of the series involved at various levels. Let $N \geq 1$ be an integer and $p_{0N} := \sum_{n=1}^{n=N} \alpha_n\varphi_n$, and denote by p_N, u_N the corresponding solutions constructed through the above approach: we have indeed

$$\widetilde{f}_N(t) = B_0 p_N(t) = \sum_{n=1}^{N} \alpha_n \mathrm{e}^{-\lambda_n t}\sum_{k\geq 1}\mathbb{P}_k(B_0\varphi_n) = \sum_{k\geq 1}\sum_{n=1}^{N}\alpha_n\mathrm{e}^{-\lambda_n t}\mathbb{P}_k(B_0\varphi_n).$$

It is clear that the function $\widetilde{f}_N$ is analytic and has a natural holomorphic extension to the right half of the complex plane, that is $[\Re(t) > 0] := \{t = \tau + \mathrm{i}s \in \mathbb{C}\ ;\ \tau > 0\}$. If we fix $0 < \varepsilon < T$, for any $j \geq 1$, and any $t \in \mathbb{C}$

with $\Re(t) \in [\varepsilon, T]$ we have the estimate

$$\|\widetilde{f}_{N+j}(t) - \widetilde{f}_N(t)\| \leq \sum_{n=N+1}^{N+j} \alpha_n \mathrm{e}^{-\varepsilon\lambda_n} \left\| \sum_{k\geq 1} \mathbb{P}_k(B_0\varphi_n) \right\| \leq \|B_0\| \sum_{n=N+1}^{N+j} \alpha_n \mathrm{e}^{-\varepsilon\lambda_n}$$
$$\leq \|B_0\| \left(\sum_{n=N+1}^{\infty} \alpha_n^2 \right)^{1/2} \left(\sum_{n=N+1}^{\infty} \mathrm{e}^{-2\varepsilon\lambda_n} \right)^{1/2}.$$

This means that $(\widetilde{f}_N)_{N\geq 1}$ is a sequence of holomorphic functions on the right hand half-plane of $\mathbb{C}$ which is a uniformly convergent Cauchy sequence on any strip of the type $[\varepsilon \leq \Re(t) \leq T]$ and thus converges uniformly to a holomorphic function $\widetilde{f}$ which can be written, for any t with $\Re(t) > 0$ in the form

$$\widetilde{f}(t) = B_0 p(t) = \sum_{n\in\mathbb{K}} \sum_{k\geq 1} \alpha_n \mathrm{e}^{-\lambda_n\tau} \mathbb{P}_k(B_0\varphi_n) = \sum_{k\geq 1} \sum_{n\geq 1} \alpha_n \mathrm{e}^{-\lambda_n\tau} \mathbb{P}_k(B_0\varphi_n). \tag{2.12}$$

In the same manner we are going to show that $t \mapsto u(t)$ has a holomorphic extension to the half plane $\Re(t) > 0$:

Lemma 2.4. *Let A satisfy (1.5), (2.1), and let $B_0 \in UCP(A^{\gamma_0}, H)$. For $p_0 \in D(A^{\gamma_0})$ and $p_0 \neq 0$, the solution u of (1.11) is given by the series*

$$u(t) = \sum_{k\geq 1} \mathrm{e}^{-\lambda_k t} \mathbb{P}_k(u_0) + \sum_{k\geq 1} \sum_{n\neq k} \alpha_n \frac{\mathrm{e}^{-\lambda_k t} - \mathrm{e}^{-\lambda_n t}}{\lambda_n - \lambda_k} \mathbb{P}_k(B_0\varphi_n) + t \sum_{k\geq 1} \alpha_n \, \mathrm{e}^{-\lambda_n t} \, \mathbb{P}_n(B_0\varphi_n), \tag{2.13}$$

for any $t \in \mathbb{C}$ with $\Re(t) > 0$, and $t \mapsto u(t)$ is holomorphic.

Proof. With p_{0N}, p_N as above, we can write

$$u_N(t) = S(t)u_0 + \int_0^t S(t-\tau)\widetilde{f}_N(\tau)\mathrm{d}\tau = S(t)u_0 + \sum_{k\geq 1} \sum_{n=1}^{N} \alpha_n \int_0^t \mathrm{e}^{-\lambda_k(t-\tau)} \, \mathrm{e}^{-\lambda_n\tau} \mathrm{d}\tau \mathbb{P}_k(B_0\varphi_n).$$

Step 1. We show first that u_N is a holomorphic function on $[\Re(t) > 0]$. Upon calculating the integrals, according to whether $k \neq n$ or $k = n$, we find that

$$u_N(t) = S(t)u_0 + \sum_{k\geq 1} \sum_{\substack{1\leq n\leq N \\ n\neq k}} \alpha_n \frac{\mathrm{e}^{-\lambda_n t} - \mathrm{e}^{-\lambda_k t}}{\lambda_k - \lambda_n} \mathbb{P}_k(B_0\varphi_n) + t \sum_{n=1}^{N} \alpha_n \, \mathrm{e}^{-\lambda_n t} \, \mathbb{P}_n(B_0\varphi_n). \tag{2.14}$$

Observe that

$$\sum_{k\geq 1} \sum_{\substack{1\leq n\leq N \\ n\neq k}} \alpha_n \frac{\mathrm{e}^{-\lambda_n t} - \mathrm{e}^{-\lambda_k t}}{\lambda_k - \lambda_n} \mathbb{P}_k(B_0\varphi_n) = \sum_{n=1}^{N} \sum_{\substack{1\leq k\leq N \\ k\neq n}} \alpha_n \frac{\mathrm{e}^{-\lambda_n t} - \mathrm{e}^{-\lambda_k t}}{\lambda_k - \lambda_n} \mathbb{P}_k(B_0\varphi_n)$$
$$+ \sum_{n=1}^{N} \sum_{k\geq N+1} \alpha_n \frac{\mathrm{e}^{-\lambda_n t} - \mathrm{e}^{-\lambda_k t}}{\lambda_k - \lambda_n} \mathbb{P}_k(B_0\varphi_n),$$

so that if we show that for N fixed, the mapping

$$t \mapsto F(t) := \sum_{n=1}^{N} \sum_{k\geq N+1} \alpha_n \frac{\mathrm{e}^{-\lambda_n t} - \mathrm{e}^{-\lambda_k t}}{\lambda_k - \lambda_n} \mathbb{P}_k(B_0\varphi_n) \tag{2.15}$$

is analytic and has a holomorphic extension to $\Re(t) > 0$, then $t \mapsto u_N(t)$ is analytic on $\Re(t) > 0$ (recall that we already know that $t \mapsto S(t)u_0$ has an analytic extension to this half plane). Now, $N \geq 1$ being fixed, for any integer $m \geq 1$ consider the holomorphic function F_m defined for $t \in [\Re(t) > 0]$ by

$$F_m(t) := \sum_{n=1}^{N} \sum_{k=N+1}^{N+m} \alpha_n \frac{\mathrm{e}^{-\lambda_n t} - \mathrm{e}^{-\lambda_k t}}{\lambda_k - \lambda_n} \mathbb{P}_k(B_0\varphi_n).$$

For $a, b \geq \lambda_1$ and $t \in \mathbb{C}$ with $\Re(t) \in [\varepsilon, T]$, consider the function

$$g(a,b,t) := \begin{cases} \dfrac{\mathrm{e}^{-at} - \mathrm{e}^{-bt}}{b-a} & \text{if } b \neq a \\ t\mathrm{e}^{-at} & \text{if } a = b. \end{cases} \tag{2.16}$$

Since we have

$$\mathrm{e}^{-at} - \mathrm{e}^{-bt} = t(b-a)\int_0^1 \exp(-(a+(b-a)\sigma)t)\mathrm{d}\sigma,$$

for $0 < a < b$ and $\Re(t) \in [\varepsilon, T]$, we have the bound

$$|g(a,b,t)| \leq T\,\mathrm{e}^{-a\varepsilon},$$

and therefore for $n < k$ the mapping $t \mapsto g(\lambda_n, \lambda_k, t)$ is holomorphic on the strip $\Re(t) \in [\varepsilon, T]$ and we have the estimate

$$|g(\lambda_n, \lambda_k, t)| = \left| \frac{\mathrm{e}^{-\lambda_n t} - \mathrm{e}^{-\lambda_k t}}{\lambda_k - \lambda_n} \right| \leq T\,\mathrm{e}^{-\varepsilon\lambda_n}.$$

Thus we can write, proceeding as above,

$$F_{m+j}(t) - F_m(t) = \sum_{n=1}^{N} \alpha_n \sum_{k=N+m+1}^{N+m+j} g(\lambda_n, \lambda_k, t)\,\mathbb{P}_k(B_0\varphi_n), \tag{2.17}$$

and since

$$\begin{aligned}\left\| \sum_{k=N+m+1}^{N+m+j} g(\lambda_n, \lambda_k, t)\,\mathbb{P}_k(B_0\varphi_n) \right\|^2 &= \sum_{k=N+m+1}^{N+m+j} |g(\lambda_n, \lambda_k, t)|^2\,\|\mathbb{P}_k(B_0\varphi_n)\|^2 \\ &\leq T^2\,\mathrm{e}^{-2\varepsilon\lambda_n} \sum_{k=N+m+1}^{N+m+j} \|\mathbb{P}_k(B_0\varphi_n)\|^2 \\ &\leq T^2\,\mathrm{e}^{-\varepsilon\lambda_n} \sum_{k\geq N+m+1} \|\mathbb{P}_k(B_0\varphi_n)\|^2,\end{aligned}$$

we can conclude from (2.17)

$$\|F_{m+j}(t) - F_m(t)\| \leq T \left(\sum_{n=1}^{N} \alpha_n\,\mathrm{e}^{-\varepsilon\lambda_n} \right) \left(\sum_{k\geq N+m+1} \|\mathbb{P}_k(B_0\varphi_n)\|^2 \right)^{1/2}.$$

This shows that the sequence of holomorphic functions $(F_m)_m$ converges uniformly on any strip $[\varepsilon \leq \Re(t) \leq T]$ to the function F defined in (2.15), and thus finally we can induce that u_N is holomorphic on $[\Re(t) > 0]$.

Step 2. In order to finish the proof of our lemma, we have to show that $(u_N)_{N\geq 1}$ is a Cauchy sequence of holomorphic functions on any strip $[\varepsilon \leq \Re(t) \leq T]$ of the complex plane. As we know already that $t \mapsto S(t)u_0$

is holomorphic on this half plane, we can assume without loss of generality that $u_0 := 0$ and thus using the function $g(\lambda_n, \lambda_k, t)$ we can write

$$u_N := \sum_{n=1}^{N} \sum_{k\geq 1} \alpha_n g(\lambda_n, \lambda_k, t)\, \mathbb{P}_k(B_0\varphi_n), \tag{2.18}$$

so that for $j \geq 1$ we have

$$u_{N+j}(t) - u_N(t) = \sum_{n=N+1}^{N+j} \sum_{k\geq 1} \alpha_n g(\lambda_n, \lambda_k, t)\, \mathbb{P}_k(B_0\varphi_n) =: E_1(t) + E_2(t) + E_3(t) \tag{2.19}$$

where for convenience we have set

$$\begin{aligned}
E_1(t) &:= \sum_{n=N+1}^{N+j} \sum_{k=1}^{N} \alpha_n\, g(\lambda_n, \lambda_k, t)\, \mathbb{P}_k(B_0\varphi_n) = \sum_{k=1}^{N} \mathbb{P}_k\left(B_0 \sum_{n=N+1}^{N+j} \alpha_n\, g(\lambda_n, \lambda_k, t)\varphi_n\right)\\
E_2(t) &:= \sum_{n=N+1}^{N+j} \sum_{k=N+1}^{n} \alpha_n\, g(\lambda_n, \lambda_k, t)\, \mathbb{P}_k(B_0\varphi_n) = \sum_{k=N+1}^{N+j} \mathbb{P}_k\left(B_0 \sum_{n=k}^{N+j} \alpha_n\, g(\lambda_n, \lambda_k, t)\varphi_n\right)\\
E_3(t) &:= \sum_{n=N+1}^{N+j} \sum_{k=n+1}^{\infty} \alpha_n\, g(\lambda_n, \lambda_k, t)\, \mathbb{P}_k(B_0\varphi\varphi_n).
\end{aligned}$$

With a little bit patience, using the same arguments as when we established the estimates for $(F_m)_m$, one checks easily that for $t \in \mathbb{C}$ and $\varepsilon \leq \Re(t) \leq T$ we have

$$\begin{aligned}
\|E_1(t)\|^2 &= \sum_{k=1}^{N} \left\| \mathbb{P}_k \left(B_0 \sum_{n=N+1}^{N+j} \alpha_n g(\lambda_n, \lambda_k, t)\varphi_n\right)\right\|^2\\
&\leq \sum_{k=1}^{N} \left\| B_0 \sum_{n=N+1}^{N+j} \alpha_n g(\lambda_n, \lambda_k, t)\varphi_n \right\|^2\\
&\leq \sum_{k=1}^{N} \|B_0\|^2 \sum_{n=N+1}^{N+j} \alpha_n^2\, |g(\lambda_n, \lambda_k, t)|^2\\
&\leq \|B_0\|^2\, T^2 \left(\sum_{k=1}^{N} \mathrm{e}^{-2\varepsilon\lambda_k}\right) \left(\sum_{n=N+1}^{N+j} \alpha_n^2\right),
\end{aligned}$$

so that finally for some constant $c(\varepsilon, T)$ depending only on ε, T

$$\|E_1(t)\|^2 \leq c(\varepsilon, T)\|B_0\|^2 \sum_{n=N+1}^{\infty} \alpha_n^2. \tag{2.20}$$

In order to estimate $\|E_2(t)\|$ first we write E_2 in the form

$$E_2(t) = \sum_{k=N+1}^{N+j} \mathbb{P}_k \left(B_0 \sum_{n=k}^{N+j} \alpha_n g(\lambda_n, \lambda_k, t)\varphi_n\right),$$

so that

$$\begin{aligned}\|E_2(t)\|^2 &= \sum_{k=N+1}^{N+j}\left\|\mathbb{P}_k\left(B_0\sum_{n=k}^{N+j}\alpha_n g(\lambda_n,\lambda_k,t)\varphi_n\right)\right\|^2\\ &\le \|B_0\|^2\sum_{k=N+1}^{N+j}\left\|\sum_{n=k}^{N+j}\alpha_n g(\lambda_n,\lambda_k,t)\varphi_n\right\|^2\\ &\le T^2\,\|B_0\|^2\sum_{k=N+1}^{N+j}\mathrm{e}^{-2\varepsilon\lambda_k}\sum_{n=k}^{N+j}\alpha_n^2\end{aligned}$$

and finally we get the following estimate on $\|E_2(t)\|$

$$\|E_2(t)\|^2 \le T^2\,\|B_0\|^2\left(\sum_{n=N+1}^{\infty}\alpha_n^2\right)\sum_{k=N+1}^{\infty}\mathrm{e}^{-2\varepsilon\lambda_k}. \tag{2.21}$$

The estimate on E_3 is straightforward: indeed

$$\|E_3(t)\| \le \sum_{n=N+1}^{N+j}\alpha_n\left\|\sum_{k=n}^{\infty}g(\lambda_n,\lambda_k,t)\mathbb{P}_k(B_0\varphi_n)\right\|$$

and since

$$\begin{aligned}\left\|\sum_{k=n}^{\infty}g(\lambda_n,\lambda_k,t)\mathbb{P}_k(B_0\varphi_n)\right\|^2 &= \sum_{k=n}^{\infty}|g(\lambda_n,\lambda_k,t)|^2\|\mathbb{P}_k(B_0\varphi_n)\|^2\\ &\le T^2\,\mathrm{e}^{-2\varepsilon\lambda_n}\sum_{k=n}^{\infty}\|\mathbb{P}_k(B_0\varphi_n)\|^2 = T^2\,\mathrm{e}^{-2\varepsilon\lambda_n}\,\|B_0\varphi_n\|^2,\\ &\le T^2\,\mathrm{e}^{-2\varepsilon\lambda_n}\,\|B_0\|^2,\end{aligned}$$

we get finally

$$\|E_3(t)\| \le T\,\|B_0\|\sum_{n=N+1}^{\infty}\alpha_n\,\mathrm{e}^{-\varepsilon\lambda_n}. \tag{2.22}$$

Therefore, thanks to the assumption (2.1) on the growth of the eigenvalues, using (2.20), (2.21), (2.22) we obtain that $(u_N)_N$ is a Cauchy sequence of holomorphic functions converging uniformly to u on any strip $[\varepsilon \le \Re(t) \le T]$, and we have

$$\begin{aligned}u(t) &= \sum_{k\ge1}\mathrm{e}^{-\lambda_k t}\mathbb{P}_k(u_0) + \sum_{k\ge1}\sum_{n\in\mathbb{K}}\alpha_n g(\lambda_n,\lambda_k,t)\mathbb{P}_k(B_0\varphi_n)\\ &= \sum_{k\ge1}\mathrm{e}^{-\lambda_k t}\mathbb{P}_k(u_0) + \sum_{k,n\ge1}g(\lambda_n,\lambda_k,t)\mathbb{P}_k(B_0\varphi_n),\end{aligned}$$

which, upon using the explicit expression for $g(\lambda_n,\lambda_k,t)$, that is (2.16), yields the representation formula for u solution of equation (1.11), and the lemma is proved. □

Once we have the above representation formulas for u, we can consider the unique continuation questions mentioned in Section 1. First from the representation formula for the solution u of (1.11), that is from Lemma 2.4 we conclude the following, which establishes in fact Theorem 1.5:

Lemma 2.5. *Under the assumptions of Theorem 1.5 if $B_1u(t_j) = 0$ for an infinite sequence $t_j \in [T_1, T_2]$ for some $0 < T_1 < T_2$, then for all $k \geq 1$ we have $\alpha_k = 0$, that is $p_0 \equiv 0$, and $p = u \equiv 0$.*

Proof. We proceed in two steps.
Step 1. In a first approach assume that $p_0 \in D(A^{\gamma_0})$. Then for any $g \in X'$, if $\langle g, B_1u(t_j)\rangle = 0$ for an infinite sequence $t_j \in [T_1, T_2]$, since $t \mapsto \langle g, B_1u(t)\rangle$ is holomorphic in t, we may conclude that $\langle g, B_1u(t)\rangle = 0$ on $(0, \infty)$. Therefore we have that $B_1u(t) = 0$ for all $t \in (0, \infty)$.

If $p_0 \not\equiv 0$, then the set $\mathbb{K}$ defined in (2.5) is non empty and we know that u is given by (2.13). Let $n_0 := \min\{n\ ;\ n \in \mathbb{K}\}$. Then multiplying the representation formula (2.13) by $t^{-1}\exp(\lambda_{n_0}t)$, we get for all $t > 0$

$$E_1(t) - \alpha_{n_0} B_1\mathbb{P}_k(B_0\varphi_{n_0}) = E_2(t), \tag{2.23}$$

where for convenience we have set

$$\begin{aligned} E_1(t) &:= \sum_{k<n_0} \frac{\mathrm{e}^{(\lambda_{n_0}-\lambda_k)t}}{t} B_1\mathbb{P}_k(u_0) + \sum_{k<n_0}\sum_{n\in\mathbb{K}} \alpha_n \frac{\mathrm{e}^{(\lambda_{n_0}-\lambda_k)t} - \mathrm{e}^{-(\lambda_n-\lambda_{n_0})t}}{t(\lambda_k-\lambda_n)} B_1\mathbb{P}_k(B_0\varphi_n) \\ E_2(t) &:= E_{21}(t) + E_{22}(t) + E_{23}(t) \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} E_{21}(t) &:= \sum_{k\geq n_0} \frac{\mathrm{e}^{-(\lambda_k-\lambda_{n_0})t}}{t} B_1\mathbb{P}_k(u_0) \\ E_{22}(t) &:= \sum_{k\geq n_0}\sum_{n\neq k} \alpha_n \frac{\mathrm{e}^{-(\lambda_k-\lambda_{n_0})t} - \mathrm{e}^{-(\lambda_n-\lambda_{n_0})t}}{t(\lambda_n-\lambda_k)} B_1\mathbb{P}_k(B_0\varphi_n) \\ E_{23}(t) &:= \sum_{n>n_0} \alpha_n \mathrm{e}^{-(\lambda_n-\lambda_{n_0})t} B_1\mathbb{P}_n(B_0\varphi_n). \end{aligned} \tag{2.25}$$

First one observes easily that $E_2(t) \to 0$, for instance weakly in X, as $t \to +\infty$. Next, we shall show that this implies that

$$\alpha_{n_0} B_1\mathbb{P}_{n_0}(B_0\varphi_{n_0}) = 0. \tag{2.26}$$

Assume for a moment that (2.26) is proved. Then since $\alpha_{n_0} > 0$, this implies that $B_1\mathbb{P}_{n_0}(B_0\varphi_{n_0}) = 0$: hence by the unique continuation assumption for the operators A, B_1 we conclude that $\mathbb{P}_{n_0}(B_0\varphi_{n_0}) = 0$. However this implies in particular that

$$(\mathbb{P}_{n_0}(B_0\varphi_{n_0})|\varphi_{n_0}) = (B_0\varphi_{n_0}|\varphi_{n_0}) = 0.$$

Since $B_0 \in SAP(A^\gamma, H)$, thanks to our observation (2.2), we conclude that $B_0\varphi_{n_0} = 0$. At this point, since $B_0 \in UCP(A^\gamma, H)$, we conclude that $\varphi_{n_0} = 0$, a contradiction with the fact that by definition we have $\|\varphi_{n_0}\| = 1$. This contradiction shows that we have $\mathbb{K} = \emptyset$, that is $p_0 \equiv 0$.

So, in order to finish the proof of the lemma in this first step (that is when $p_0 \in D(A^{\gamma_0})$), we have to prove (2.26). To this end we are going to look more closely at the behavior of $E_1(t)$, appearing in the left hand side of (2.23) as $t \to +\infty$, according to whether $n_0 = 1$ or $n_0 \geq 2$. It is clear that if $n_0 = 1$, then the left hand side of (2.23) is reduced to $-\alpha_{n_0}\mathbb{P}_{n_0}(B_0\varphi_{n_0})$, and thus after passing to the limit as $t \to +\infty$, we obtain (2.26).

If $n_0 \geq 2$, then the series in $E_1(t)$ can be written, for t large,

$$E_1(t) = \sum_{k<n_0} \frac{\mathrm{e}^{(\lambda_{n_0}-\lambda_k)t}}{t} B_1\mathbb{P}_k\left(u_0 + \sum_{n\in\mathbb{K}} \frac{\alpha_n}{\lambda_k-\lambda_n} B_0\varphi_n\right) + O\left(\frac{1}{t}\right).$$

Since for $k < n_0$ we have $E_2(t) \to 0$ and

$$\lim_{t\to+\infty} \frac{\mathrm{e}^{(\lambda_{n_0}-\lambda_k)t}}{t} = +\infty,$$

this means that (2.23) is possible only if for any $k < n_0$ we have

$$B_1\mathbb{P}_k\left(u_0 + \sum_{n\in\mathbb{K}} \frac{\alpha_n}{\lambda_k - \lambda_n} B_0\varphi_n\right) = 0.$$

Finally, we have $E_1(t) = O(1/t)$ as $t \to +\infty$, and thus (2.23) yields again (2.26).

Now to see that $u_0 = 0$ as well, we observe that since by the above argument we have $p_0 = 0$, then $u(t) = S(t)u_0$ satisfies $B_1u(t_j) = 0$ for an infinite sequence, therefore by Theorem 2.1 we have $u_0 \equiv u \equiv 0$. The proof of Theorem 1.5 is done when $p_0 \in D(A^{\gamma_0})$.

Step 2. Consider now the general case $p_0 \in D(A^\beta)$ for some $\beta > \gamma_0 - 1$. If $\beta \geq \gamma_0$ then we are again in the situation of the above first step. If $\gamma_0 - 1 < \beta < \gamma_0$, then

$$\|B_0p(t)\| \leq \|B_0\|\,\|p(t)\|_{D(A^{\gamma_0})} \leq c\,\|B_0\|\,t^{\beta-\gamma_0}\,\|p_0\|_{D(A^\beta)}.$$

In this case, choosing $1 < q < 1/(\gamma_0 - \beta)$, we see that the mapping $t \mapsto B_0p(t)$ belongs to $L^q(0,T;H)$ and thus by the maximum regularity results for inhomogeneous evolution equations (see for instance Pazy [8], or Coulhon and Duong [2]) we can assert that the solution of

$$\begin{cases} \partial_t\widetilde{u} + A\widetilde{u} = B_0p(t) & \text{on } (0,\infty) \\ \widetilde{u}(0) = 0 & \\ \widetilde{u}(t) \in D(A) & \text{for a.e.}\, t > 0, \end{cases} \tag{2.27}$$

exists and is unique in $L^q(0,T;D(A))$, for any $T > 0$. We can choose $0 < t_* < T_1$ such that $p(t_*) \in D(A^{\gamma_0})$ and $\widetilde{u}(t_*) \in D(A)$. Now if we set

$$p_{0*} := p(t_*), \quad u_{0*} := S(t_*)u_0 + \widetilde{u}(t_*), \qquad \text{and} \quad p_*(t) := p(t+t_*), \quad u_*(t) := u(t+t_*), \qquad t_{*j} := t_j - t_*$$

the function u_* satisfies the evolution equation

$$\begin{cases} \partial_t u_* + Au_* = B_0p_*(t) & \text{for } t > 0 \\ u_*(0) = u_{0*} & \\ u_*(t) \in D(A) & \text{for a.e. } t > 0, \end{cases}$$

and we have $B_1u_*(t_{*j}) = 0$ for an infinite sequence $t_{*j} \in [T_1 - t_*, T_2 - t_*]$. Since in this case we know that $p_{0*} \in D(A^{\gamma_0})$ we can apply the result of the previous step to u_*, p_* and conclude that $p_{0*} = 0$. At this point the backward uniqueness Theorem 2.2 implies that $p(t) \equiv 0$, that is $p_0 \equiv 0$, and using again the unique continuation theorem for $u(t) = S(t)u_0$ we conclude that $u(t) \equiv 0$ and thus $u_0 \equiv 0$. □

The proof of a unique continuation result for a cascade system of forward-backward evolution equations such as (1.10)–(1.12) is somewhat more delicate and is dealt with in the next section.

3. Unique continuation result for a cascade of forward-backward evolution equations

Using the approach of the previous section together with the notations thereof, we consider the solution of equations (2.8), with the particular choice:

$$v_0 := 0, \quad p_0 \in H, \quad f(t) := B_0 p(t) \text{ for } t > 0, \quad \text{and} \quad \widetilde{f}(t) := f(T-t) = B_0 p(T-t) \text{ for } 0 \leq t < T.$$

Since for any $\varepsilon > 0$ we know that $t \mapsto p(t)$ is analytic on $[\varepsilon, \infty)$, and also belongs to $C([\varepsilon, \infty); D(A^\gamma))$ for any $\gamma \geq 0$, it is clear that for $0 < \varepsilon < T$ we have $\widetilde{f} \in C([0, T-\varepsilon]; D(A^{\gamma_0}))$, whenever B_0 satisfies (2.11). It is moreover clear that $t \mapsto \widetilde{f}$ has a natural holomorphic extension to the strip of the complex plane $[0 < \Re(t) < T - \varepsilon]$. Therefore v is given by (2.9), and on the other hand the solution of (1.12) is $z(t) = v(T-t)$ where v satisfies (2.8), with the above choice of the right hand side $\widetilde{f}$. So, since $\varepsilon > 0$ is arbitrary, for $0 < t < T$ we have

$$\begin{aligned} z(t) = v(T-t) &= \int_0^{T-t} S(T-t-\tau)\widetilde{f}(\tau)\mathrm{d}\tau \\ &= \int_t^T S(\tau - t) f(\tau)\mathrm{d}\tau = \int_t^T S(\tau-t)B_0 p(\tau)\mathrm{d}\tau. \\ &= \int_t^T \sum_{k\geq 1} \mathrm{e}^{-\lambda_k(\tau - t)}\, \mathbb{P}_k(B_0 p(\tau))\mathrm{d}\tau. \end{aligned}$$

Finally, using (2.12) we obtain

$$z(t) = \sum_{k\geq 1}\sum_{n\geq 1} \alpha_n \left(\int_t^T \mathrm{e}^{-\lambda_k(\tau - t)}\, \mathrm{e}^{-\lambda_n \tau}\mathrm{d}\tau \right) \mathbb{P}_k(B_0\varphi_n),$$

and therefore, according to whether $\lambda_n + \lambda_k = 0$ or not, we have:

Lemma 3.1. *Let A satisfy (1.5), (1.6), and let $B_0 \in UCP(A^{\gamma_0}, H)$. If $p_0 \in H$, with $p_0 \neq 0$, then for $0 < t < T$, the solution z of (1.12) may be represented in the form of the series*

$$z(t) = (T-t)\sum_{\lambda_k+\lambda_n=0} \alpha_n \mathrm{e}^{\lambda_k t}\, \mathbb{P}_k(B_0\varphi_n) + \sum_{\lambda_k+\lambda_n\neq 0} \alpha_n\, \mathrm{e}^{\lambda_k t}\, \frac{\mathrm{e}^{-(\lambda_k+\lambda_n)t} - \mathrm{e}^{-(\lambda_k+\lambda_n)T}}{\lambda_k+\lambda_n}\, \mathbb{P}_k(B_0\varphi_n). \tag{3.1}$$

Moreover the mapping $t \mapsto z(t)$ has a holomorphic extension to the strip $[0 < \Re(t) < T]$.

Proof. Note that since $\lambda_j > 0$ for $j \geq k_0$ (given by (1.6)), the first sum in the right hand side of (3.1) is a finite sum of holomorphic functions in $t \in [0 < \Re(t) < T]$ (which may be reduced to zero if $\lambda_k + \lambda_n \neq 0$ for all $k, n \geq 1$), and so the main point is to show that the function defined by the second term, that is

$$t \mapsto \sum_{\lambda_k+\lambda_n\neq 0} \alpha_n\, \mathrm{e}^{\lambda_k t}\, \frac{\mathrm{e}^{-(\lambda_k+\lambda_n)t} - \mathrm{e}^{-(\lambda_k+\lambda_n)T}}{\lambda_k+\lambda_n}\, \mathbb{P}_k(B_0\varphi_n)$$

is holomorphic on $[0 < \Re(t) < T]$. Therefore there is no loss in generality to assume that

$$\text{for some } \delta > 0, \quad \text{for all } n, k \geq 1, \qquad |\lambda_n + \lambda_k| \geq \delta. \tag{3.2}$$

Assuming this we have to show that

$$z(t) = \sum_{n\geq 1}\sum_{k\geq 1} \alpha_n \,\mathrm{e}^{\lambda_k t}\, \frac{\mathrm{e}^{-(\lambda_k+\lambda_n)t} - \mathrm{e}^{-(\lambda_k+\lambda_n)T}}{\lambda_k+\lambda_n}\, \mathbb{P}_k(B_0\varphi_n). \tag{3.3}$$

Now if we set $p_{0N} := \sum_{n=1}^N \alpha_n\varphi_n$ and if we denote by p_N, z_N the corresponding solutions of (1.10) and (1.12), then it is easy to see that z_N is given by

$$z_N = \sum_{n=1}^{N}\sum_{k\geq 1} \alpha_n \frac{\mathrm{e}^{-\lambda_n t} - \mathrm{e}^{-\lambda_k(T-t)}\,\mathrm{e}^{-\lambda_n T}}{\lambda_k+\lambda_n}\, \mathbb{P}_k(B_0\varphi_n).$$

One may check easily that, for $N\geq 1$ fixed, the sequence of holomorphic functions

$$F_k(t) := \sum_{n=1}^{N}\sum_{k=1}^{N} \alpha_n \frac{\mathrm{e}^{-\lambda_n t} - \mathrm{e}^{-\lambda_k(T-t)}\,\mathrm{e}^{-\lambda_n T}}{\lambda_k+\lambda_n}\, \mathbb{P}_k(B_0\varphi_n),$$

defined on the strip $[0 < \Re(t) < T]$ converges uniformly to z_N on $[\varepsilon \leq \Re(t) \leq T-\varepsilon]$ for any $\varepsilon > 0$ small enough, implying that z_N is a holomorphic function defined on $[0 < \Re(t) < T]$.

Now for $j \geq 1$ we have

$$z_{N+j}(t) - z_N(t) = \sum_{n=N+1}^{N+j}\sum_{k\geq 1} \alpha_n \frac{\mathrm{e}^{-\lambda_n t} - \mathrm{e}^{-\lambda_k(T-t)}\,\mathrm{e}^{-\lambda_n T}}{\lambda_k+\lambda_n}\, \mathbb{P}_k(B_0\varphi_n),$$

and so

$$\|z_{N+j}(t) - z_N(t)\| \leq \sum_{n=N+1}^{N+j} \alpha_n \left\| \sum_{k\geq 1} \frac{\mathrm{e}^{-\lambda_n t} - \mathrm{e}^{-\lambda_k(T-t)}\,\mathrm{e}^{-\lambda_n T}}{\lambda_k+\lambda_n}\, \mathbb{P}_k(B_0\varphi_n)\right\|. \tag{3.4}$$

Since for $\varepsilon \leq \Re(t) \leq T-\varepsilon$ we have

$$\left|\frac{\mathrm{e}^{-\lambda_n t} - \mathrm{e}^{-\lambda_k(T-t)}\,\mathrm{e}^{-\lambda_n T}}{\lambda_k+\lambda_n}\right| \leq \frac{\mathrm{e}^{-\lambda_n\varepsilon} + \mathrm{e}^{-\lambda_k\varepsilon}\,\mathrm{e}^{-\lambda_n T}}{\lambda_n+\lambda_k} \leq \frac{\mathrm{e}^{-\lambda_n\varepsilon} + \mathrm{e}^{-\lambda_n T}}{\lambda_n+\lambda_k},$$

for n fixed we conclude that

$$\begin{aligned}
\left\| \sum_{k\geq 1} \frac{\mathrm{e}^{-\lambda_n t} - \mathrm{e}^{-\lambda_k(T-t)}\,\mathrm{e}^{-\lambda_n T}}{\lambda_k+\lambda_n}\, \mathbb{P}_k(B_0\varphi_n)\right\|^2 &= \sum_{k\geq 1} \left|\frac{\mathrm{e}^{-\lambda_n t} - \mathrm{e}^{-\lambda_k(T-t)}\,\mathrm{e}^{-\lambda_n T}}{\lambda_k+\lambda_n}\right|^2 \|\mathbb{P}_k(B_0\varphi_n)\|^2\\
&\leq \left(\frac{\mathrm{e}^{-\lambda_n\varepsilon} + \mathrm{e}^{-\lambda_n T}}{\lambda_n+\lambda_k}\right)^2 \sum_{k\geq 1} \|\mathbb{P}_k(B_0\varphi_n)\|^2\\
&\leq \left(\frac{\mathrm{e}^{-\lambda_n\varepsilon} + \mathrm{e}^{-\lambda_n T}}{\lambda_n+\lambda_k}\right)^2 \|(B_0\varphi_n)\|^2\\
&\leq \|B_0\|^2 \left(\frac{\mathrm{e}^{-\lambda_n\varepsilon} + \mathrm{e}^{-\lambda_n T}}{\lambda_n+\lambda_k}\right)^2.
\end{aligned}$$

Therefore, using (3.2) and reporting this into (3.4), we obtain

$$\|z_{N+j}(t) - z_N(t)\| \leq \|B_0\| \sum_{n=N+1}^{N+j} \alpha_n \left(\frac{\mathrm{e}^{-\lambda_n \varepsilon} + \mathrm{e}^{-\lambda_n T}}{|\lambda_n + \lambda_k|} \right) \leq \frac{2\|B_0\|}{\delta} \sum_{n \geq N+1} \alpha_n \, \mathrm{e}^{-\lambda_n \varepsilon}$$

for any $t \in \mathbb{C}$ such that $[\varepsilon \leq \Re(t) \leq T - \varepsilon]$. Therefore the sequence of holomorphic functions $(z_N)_N$ converges uniformly on the strip $[\varepsilon \leq \Re(t) \leq T - \varepsilon]$ to z, and thus z is given by the series stated in the lemma and $t \mapsto z(t)$ is holomorphic on $[0 < \Re(t) < T]$. □

It is easily seen that we may derive some consequences of the representation formula obtained in Lemma 3.1: indeed using the fact that $t \mapsto z(t)$ is holomorphic on $[0 < \Re(t) < T]$ we have:

Corollary 3.2. *Under the assumptions of Theorem 1.6, if $p_0 \in H$ with $p_0 \neq 0$, and if $B_1 z(t_j) = 0$ for an infinite sequence of $t_j \in [T_1, T_2]$ for some $0 < T_1 < T_2$, then $B_1 z(t) = 0$ for $t \in (0, T)$ and we have*

$$(T - t) \sum_{\lambda_n + \lambda_k = 0} \alpha_n \mathrm{e}^{\lambda_k t} B_1 \mathbb{P}_k(B_0 \varphi_n) \; + \sum_{\lambda_k + \lambda_n \neq 0} \alpha_n \, \frac{\mathrm{e}^{-\lambda_n t}}{\lambda_k + \lambda_n} \, B_1 \mathbb{P}_k(B_0 \varphi_n) = \sum_{\lambda_k + \lambda_n \neq 0} \alpha_n \, \mathrm{e}^{\lambda_k t} \, \frac{\mathrm{e}^{-(\lambda_k + \lambda_n)T}}{\lambda_k + \lambda_n} \, B_1 \mathbb{P}_k(B_0 \varphi_n).$$

At this point we recall the following result on Dirichlet series:

Lemma 3.3. *Let $(b_n)_{n \geq 1}$ be a sequence of H such that $\sum_{n \geq 1} \|b_n\| < \infty$, and $(\lambda_n)_{n \geq 1}$ a sequence of distinct real numbers. If*

$$\forall\, s \in \mathbb{R}, \qquad \sum_{n \geq 1} \mathrm{e}^{\mathrm{i}\lambda_n s} \, b_n = 0,$$

then for all $n \geq 1$ we have $b_n = 0$.

Proof. If this were not the case, let n_0 be the least integer $n \geq 1$ such that $b_n \neq 0$. Multiplying the series $\sum_{n \geq j} b_n \mathrm{e}^{\mathrm{i}\lambda_n s}$ by $\mathrm{e}^{-\mathrm{i}\lambda_{n_0} s}$ and integrating over $[-\ell, \ell]$ for some $\ell > 0$, we have

$$0 = b_{n_0} + \sum_{n \geq n_0 + 1} \left(\frac{1}{2L} \int_{-\ell}^{+\ell} \mathrm{e}^{\mathrm{i}(\lambda_n - \lambda_{n_0})s} \mathrm{d}s \right) b_n = b_{n_0} + \sum_{n \geq n_0 + 1} \frac{\sin((\lambda_n - \lambda_{n_0})\ell)}{(\lambda_n - \lambda_{n_0})\ell} \, b_n.$$

It is clear that letting $\ell \longrightarrow +\infty$ yields $b_{n_0} = 0$, which is in contradiction with the definition of b_{n_0}. □

Now returning to the result of Corollary 3.2, we see that the series on each side converge uniformly and define a holomorphic function on the strip $[\varepsilon \leq \Re(t) \leq T - \varepsilon]$ (for any ε such that $0 < \varepsilon < T$), and therefore in particular choosing $t := \frac{T}{2} + \mathrm{i}s$, with $s \in \mathbb{R}$, we can conclude that for all $s \in \mathbb{R}$

$$\left(\frac{T}{2} - \mathrm{i}s \right) \sum_{\lambda_n + \lambda_k = 0} \alpha_n \mathrm{e}^{\lambda_k T/2} \, \mathrm{e}^{\mathrm{i}\lambda_k s} B_1 \mathbb{P}_k(B_0 \varphi_n) + \sum_{\lambda_k + \lambda_n \neq 0} \frac{\alpha_n}{\lambda_k + \lambda_n} \, \mathrm{e}^{-\lambda_n T/2} \mathrm{e}^{-\mathrm{i}\lambda_n s} \, B_1 \mathbb{P}_k(B_0 \varphi_n) = \sum_{\lambda_k + \lambda_n \neq 0} \frac{\alpha_n}{\lambda_k + \lambda_n} \, \mathrm{e}^{-(\lambda_k + 2\lambda_n)T/2} \, \mathrm{e}^{\mathrm{i}\lambda_k s} \, B_1 \mathbb{P}_k(B_0 \varphi_n). \quad (3.5)$$

This can be written in the form

$$\forall\, s \in \mathbb{R}, \qquad \left(\frac{T}{2} - \mathrm{i}s \right) \sum_{\lambda_n + \lambda_k = 0} \alpha_n \mathrm{e}^{\lambda_k T/2} \, \mathrm{e}^{\mathrm{i}\lambda_k s} B_1 \mathbb{P}_k(B_0 \varphi_n) + \sum_{n \geq 1} \mathrm{e}^{-\mathrm{i}\lambda_n s} b_{1n} = \sum_{n \geq 1} \mathrm{e}^{\mathrm{i}\lambda_n s} b_{2n} \qquad (3.6)$$

where we have set (for $k, n \geq 1$):

$$b_{1n} := \sum_{\substack{k\geq 1\\ \lambda_k+\lambda_n\neq 0}} \frac{\alpha_n}{\lambda_k+\lambda_n}\,\mathrm{e}^{-\lambda_n T/2}\, B_1\mathbb{P}_k(B_0\varphi_n) = \alpha_n\,\mathrm{e}^{-\lambda_n T/2} \sum_{\substack{k\geq 1\\ \lambda_k+\lambda_n\neq 0}} \frac{1}{\lambda_k+\lambda_n}\, B_1\mathbb{P}_k(B_0\varphi_n), \tag{3.7}$$

and

$$b_{2k} := \sum_{\substack{n\geq 1\\ \lambda_k+\lambda_n\neq 0}} \frac{\alpha_n}{\lambda_k+\lambda_n}\,\mathrm{e}^{-(\lambda_k+2\lambda_n)T/2}\, B_1\mathbb{P}_k(B_0\varphi_n) = \mathrm{e}^{-\lambda_k T/2}\, B_1 \left(\sum_{\substack{n\geq 1\\ \lambda_k+\lambda_n\neq 0}} \frac{\alpha_n}{\lambda_k+\lambda_n}\,\mathrm{e}^{-\lambda_n T}\,\mathbb{P}_k(B_0\varphi_n)\right). \tag{3.8}$$

Now using Lemma 3.3 we can state

Proposition 3.4. *Let $p_0 \in H$ and $p_0 \neq 0$. Assume that $B_1 z(t) = 0$ for $t \in (0, T)$. Then b_{1n} and b_{2k} being defined in* (3.7) *and* (3.8)*, for all $n, k \geq 1$ we have:*

$$\mathbb{P}_k B_0 \left(\sum_{\lambda_k+\lambda_n=0} \alpha_n\varphi_n \right) = 0, \qquad b_{1n} = b_{2k} \equiv 0. \tag{3.9}$$

Proof. In order to apply Lemma 3.3, we verify first that for $j = 1, 2$ we have

$$\sum_{n\geq 1} \|b_{1n}\| < \infty, \qquad\qquad \sum_{n\geq 1} \|b_{2n}\| < \infty. \tag{3.10}$$

Let $\delta := \min\{|\lambda_n + \lambda_k|;\ n, k \geq 1,\ \lambda_n + \lambda_k \neq 0\}$. It is clear that thanks to assumption (1.6) we have $\delta > 0$. Now for $n \in \mathbb{K}$ define the function

$$\psi_n := \sum_{\substack{k\geq 1\\ \lambda_n+\lambda_k\neq 0}} \frac{1}{\lambda_k+\lambda_n}\,\mathbb{P}_k(B_0\varphi_n). \tag{3.11}$$

One can see that $\psi_n \in D(A)$ satisfies

$$\|\psi_n\|^2 = \sum_{\substack{k\geq 1\\ \lambda_n+\lambda_k\neq 0}} \left(\frac{1}{\lambda_k+\lambda_n}\right)^2 \|\mathbb{P}_k(B_0\varphi_n)\|^2 \leq \frac{1}{\delta^2} \sum_{\substack{k\geq 1\\ \lambda_n+\lambda_k\neq 0}} \|\mathbb{P}_k(B_0\varphi_n)\|^2 = \frac{1}{\delta^2}\|B_0\varphi_n\|^2 \tag{3.12}$$

and thus we have the estimate

$$\|\psi_n\| \leq \frac{\|B_0\|}{\delta}. \tag{3.13}$$

Therefore, noting that as a matter of fact b_{1n} can be expressed as

$$b_{1n} = \alpha_n\,\mathrm{e}^{-\lambda_n T/2}\, B_1(\psi_n), \tag{3.14}$$

we have

$$\|b_{1n}\| \leq \alpha_n\,\mathrm{e}^{-\lambda_n T/2}\|B_1\|\,\|\psi_n\| \leq \frac{\alpha_n}{\delta}\,\|B_1\|\,\|B_0\|\,\mathrm{e}^{-\lambda_n T/2}.$$

Therefore we have $\sum_{n\geq 1}\|b_{1n}\| < \infty$.

In order to see that $\sum_{k\geq 1}\|b_{2k}\| < \infty$, we observe that upon setting

$$F_k := \sum_{\substack{n\geq 1\\ \lambda_k+\lambda_n\neq 0}} \frac{\alpha_n}{\lambda_k+\lambda_n}\,\mathrm{e}^{-\lambda_n T}\varphi_n, \tag{3.15}$$

we have clearly $\|F_k\| \leq \delta^{-1}\, e^{-\lambda_1 T}\|p_0\|$ and

$$b_{2k} = e^{-\lambda_k T/2}\, B_1\left(\mathbb{P}_k(B_0 F_k)\right). \tag{3.16}$$

Therefore, since

$$\begin{aligned}\sum_{k\geq 1}\|b_{2k}\| &\leq \|B_1\| \sum_{k\geq 1} e^{-\lambda_k T/2}\, \|\mathbb{P}_k(B_0F_k)\| \\ &\leq \delta^{-1}\, e^{-\lambda_1 T}\, \|B_1\| \sum_{k\geq 1} e^{-\lambda_k T/2} < \infty.\end{aligned}$$

Thus knowing that now $\sum_{n\geq 1}\|b_1 n\| < \infty$ and $\sum_{k\geq 1}\|b_{2k}\| < \infty$ we may apply Lemma 3.3 in the following way. First for $k_0 \geq 1$ fixed such that $\lambda_n + \lambda_{k_0} = 0$ for at least some $n \geq 1$, we multiply (3.5) by $\exp(-i\lambda_{k_0} s)$, we integrate in s on the interval $[-\ell, \ell]$ and we let $\ell \to +\infty$ we conclude that

$$e^{\lambda_{k_0} T/2}\, B_1 \mathbb{P}_{k_0}\left(\sum_{\substack{n\geq 1\\ \lambda_n+\lambda_{k_0}=0}} \alpha_n B_0 \varphi_n\right) = 0.$$

Therefore, since $B_1 \in UCP(A^{\gamma_0}, H)$, for all $k \geq 1$ such that for some $n \geq 1$ one has $\lambda_n + \lambda_k = 0$ we have

$$\mathbb{P}_k\left(\sum_{\substack{n\geq 1\\ \lambda_n+\lambda_k=0}} \alpha_n B_0\varphi_n\right) = 0$$

and thus (3.6) reduces to

$$\forall\, s \in \mathbb{R}, \qquad \sum_{n\geq 1} e^{-i\lambda_n s} b_{1n} = \sum_{n\geq 1} e^{i\lambda_n s} b_{2n}.$$

At this point it is clear that Lemma 3.3 implies that $b_{1n} = b_{2k} = 0$, and the proposition is proved. □

Using (3.14) and (3.16), we may conclude the following: if $B_1 z(t) = 0$ on $(0, T)$, then by Proposition 3.4 we know that $b_{1n} \equiv b_{2k} \equiv 0$ for all $n, k \geq 1$. So by (3.14) we induce that $\alpha_n B_1 \psi_n = 0$ for all $n \geq 1$. On the other hand by relation (3.16) we conclude that $B_1 \mathbb{P}_k(B_0 F_k) = 0$. Therefore by (1.9), that is the unique continuation property for the operator $B_1 \in UCP(A^\gamma, H)$, we have that $\mathbb{P}_k(B_0F_k) \equiv 0$ on Ω for all $k \geq 1$. These observations can be gathered in the following corollary:

Corollary 3.5. *Under the assumptions of Theorem 1.6 on A, B_0, B_1 if $p_0 \in H$ and $p_0 \neq 0$, define ψ_n as in (3.11), b_{1n}, b_{2k} being defined by (3.7)–(3.8), and F_k being as in (3.15). If $B_1 z(t) \equiv 0$ for $t \in (0, T)$ then $\alpha_n B_1 \psi_n \equiv 0$ for all $n \in \mathbb{K}$ and*

$$\forall\, k \geq 1, \qquad \mathbb{P}_k(B_0 F_k) = 0. \tag{3.17}$$

From (3.17) we conclude in particular that for all $k \in \mathbb{K}$ we have

$$(\mathbb{P}_k(B_0F_k)|\varphi_k) = \sum_{\substack{n\in\mathbb{K}\\ \lambda_n+\lambda_k\neq 0}} \frac{\alpha_n}{\lambda_k+\lambda_n}\, e^{-\lambda_n T}\, (B_0\varphi_n|\varphi_k) = 0,$$

a result which can be noted in the following corollary:

Corollary 3.6. *Under the assumptions of Corollary 3.5, if* $B_1 z(t) = 0$ *on* $(0,T)$ *then we have*

$$\forall\, k \in \mathbb{K}, \qquad \sum_{\substack{n\in\mathbb{K} \\ \lambda_n+\lambda_k\neq 0}} \frac{c_{kn}}{\lambda_k+\lambda_n}\,\alpha_n\,\mathrm{e}^{-\lambda_n T} = 0, \tag{3.18}$$

where we have set $c_{nk} = c_{kn} := (B_0\varphi_n|\varphi_k)$.

In order to prove our unique continuation result for z, we are going to show that as a matter of fact the relations (3.18) imply that $p(T) = 0$, and thus $p_0 = 0$, yielding a contradiction.

Assuming $p_0 \neq 0$, we define H_0 as the span of the eigenfunctions φ_n for $n \in \mathbb{K}$ and we denote by $(A_0, D(A_0))$ the restriction of the operator $(A, D(A))$ to the space H_0, that is

$$H_0 := \bigoplus_{n\in\mathbb{K}} \mathbb{R}\varphi_n, \qquad D(A_0) := D(A)\cap H_0, \quad A_0 u := Au \ \text{ for } \ u \in D(A_0).$$

Note that for $n \in \mathbb{K}$, each λ_n is a simple eigenvalue of A_0 and φ_n is its eigenfunction. We denote by $\mathbb{P}_0$ the orthogonal projection of H into H_0 (this amounts to setting $\mathbb{P}_0 := \sum_{n\in\mathbb{K}} \mathbb{P}_n$). For $n \in \mathbb{K}$

$$\psi_n = \sum_{\substack{k\in\mathbb{K} \\ \lambda_n+\lambda_k\neq 0}} \frac{1}{\lambda_n+\lambda_k}\,\mathbb{P}_k(B_0\varphi_n).$$

This means that ψ_n is the solution of the equation

$$A_0\psi_n + \lambda_n\psi_n = \mathbb{P}_{0,n}(B_0\varphi_n), \qquad \psi_n \in D(A_0), \qquad \text{with } \mathbb{P}_{0,n} := \sum_{\substack{k\geq 1 \\ \lambda_n+\lambda_k\neq 0}} \mathbb{P}_k.$$

Then we can define a linear operator $L : D(A_0^{m_0}) \longrightarrow H_0$ (with $m_0 \geq 0$ a sufficiently large integer, or even real number, chosen below), by setting $L\varphi_n := \psi_n$ and more generally for $f = \sum_{n\in\mathbb{K}} (f|\varphi_n)\varphi_n \in D(A_0^{m_0})$

$$Lf := \sum_{n\in\mathbb{K}} (f|\varphi_n) L\varphi_n = \sum_{n\in\mathbb{K}} (f|\varphi_n)\psi_n.$$

Since by (3.13) we have $\|L\varphi_n\| = \|\psi_n\| \leq \delta^{-1}\,\|B_0\|$, the integer m_0 needs to be large enough to ensure that Lf is well defined for any $f \in D(A_0^{m_0})$. Observe that

$$\begin{aligned}\|Lf\| &\leq \delta^{-1}\,\|B_0\| \sum_{n\in\mathbb{K}} |(f|\varphi_n)| = \delta^{-1}\,\|B_0\| \sum_{n\in\mathbb{K}} (1+|\lambda_n|^2)^{m_0/2} |(f|\varphi_n)|\,(1+|\lambda_n|^2)^{-m_0/2} \\ &\leq \delta^{-1}\|B_0\| \left(\sum_{n\in\mathbb{K}} (1+|\lambda_n|^2)^{m_0}\,|(f|\varphi_n)|^2\right)^{1/2} \left(\sum_{n\in\mathbb{K}} (1+|\lambda_n|^2)^{-m_0}\right)^{1/2} < \infty,\end{aligned}$$

provided that $2m_0\beta > 1$, since by Assumption (2.1) we have $\lambda_n \geq c_0\, n^\beta$: therefore L is well defined on $D(A_0^{m_0})$ for such a choice of m_0.

We shall need the following representation result regarding the relationship between the operator L and the semi-group $S_0(t) := \exp(-tA_0)$ acting on H_0.

Lemma 3.7. *Assume that A satisfies (1.5), (2.1), and let $B_0 \in SAP(A^{\gamma_0}, H) \cap UCP(A^{\gamma_0}, H)$ (that is satisfying (1.8) and (1.9)). Let m_0 be an integer (or a real number) such that $2(m_0+1)\beta > 1$. For $f \in D(A_0^{m_0})$, denoting by $S_0(t) = \exp(-tA_0)$ the semi-group generated by the operator A_0 on H_0 we have*

$$(Lf|f) = \int_0^\infty (B_0 S_0(t) f | S(t) f) \mathrm{d}t.$$

In particular this implies that if $Lf = 0$, then $f = 0$.

Proof. For $t > 0$ define $F(t) := S_0(t) f = \sum_{n \geq \mathbb{K}} \mathrm{e}^{-\lambda_n t} (f|\varphi_n) \varphi_n$. Note that for $g \in D(A^{\gamma_0})$ one has

$$(B_0 g|g) = \sum_{n,k \in \mathbb{K}} (g|\varphi_n)(g|\varphi_k)\,(B_0 \varphi_n|\varphi_k) = \sum_{n,k \geq 1} c_{nk}\,(g|\varphi_n)(g|\varphi_k).$$

So we have

$$\begin{aligned}
\int_0^\infty (B_0 S_0(t) f | S_0(t) f) \mathrm{d}t &= \int_0^\infty \left[\sum_{n,k \in \mathbb{K}} c_{nk}\, \mathrm{e}^{-\lambda_n t} (f|\varphi_n)\, \mathrm{e}^{-\lambda_k t} (f|\varphi_k) \right] \mathrm{d}t \\
&= \sum_{n,k \in \mathbb{K}} c_{nk}\, (f|\varphi_n)(f|\varphi_k) \int_0^\infty \mathrm{e}^{-(\lambda_n + \lambda_k)t} \mathrm{d}t \\
&= \sum_{n,k \in \mathbb{K}} \frac{c_{nk}}{\lambda_n + \lambda_k} (f|\varphi_n)(f|\varphi_k) = (Lf|f).
\end{aligned}$$

In order to finish the proof of the lemma, observe that if $Lf = 0$ then

$$\int_0^\infty (B_0 S_0(t) f | S_0(t) f) \mathrm{d}t = 0,$$

which means $(B_0 S_0(t) f | S_0(t) f) = 0$ for all $t \in (0, \infty)$, since B_0 satisfies (1.8). According to (2.2) this implies that $B_0 F(t) = B_0 S(t) f \equiv 0$ in $(0, \infty)$. However since $\partial_t F + A_0 F = 0$ in $(0, \infty)$, the unique continuation principle for this evolution equation, that is Theorem 2.1 applied to A_0 and F, implies that $F \equiv 0$ on $(0, \infty)$, that is $f = 0$. ☐

Now we are in a position to show our unique continuation principle for a cascade of forward-backward evolution equations:

Main Theorem 3.8 (forward-backward)**.** *Assume that A satisfy (1.5), (2.1), and $B_0 \in SAP(A^\gamma, H) \cap UCP(A^\gamma, H)$. Let $B_1 \in UCP(A^\gamma, H)$. For a given $p_0 \in H$ let p be the solution of (1.10). If z is the solution of equation (1.12) and satisfies $B_1 z(t_j) = 0$ for an infinite sequence $(t_j)_j$ with $t_j \in [T_1, T_2]$ for some $0 < T_1 < T_2 < T$, then we have $p \equiv z \equiv 0$ and $p_0 = 0$.*

Proof. We know that the assumption $B_1 z(t_j) = 0$ for an infinite sequence $(t_j)_j$ yields that $B_1 z(t) = 0$ for all $t \in (0, T)$. If we had $p_0 \neq 0$, setting $f := p(T) = \sum_{n \in \mathbb{K}} \alpha_n\, \mathrm{e}^{-\lambda_n T} \varphi_n$, then one sees that for any choice of m_0 as above, we have $f \in D(A_0^{m_0})$, so Lf is well defined and is given by

$$Lf = \sum_{n,k \in \mathbb{K}} \frac{c_{nk}}{\lambda_n + \lambda_k} (f|\varphi_n) \varphi_k = \sum_{n,k \in \mathbb{K}} \frac{c_{nk}}{\lambda_n + \lambda_k} \alpha_n\, \mathrm{e}^{-\lambda_n T} \varphi_k.$$

In particular for any $k \in \mathbb{K}$ we have

$$(Lf|\varphi_k) = \sum_{n \in \mathbb{K}} \frac{c_{nk}}{\lambda_n + \lambda_k} \alpha_n\, \mathrm{e}^{-\lambda_n T} = \sum_{n \in \mathbb{K}} \frac{c_{kn}}{\lambda_n + \lambda_k} \alpha_n\, \mathrm{e}^{-\lambda_n T} = 0,$$

where we use the fact that $c_{nk} = c_{kn}$ and Corollary 3.6. Therefore we have $Lf = 0$, and by Lemma 3.7 we conclude that $f = 0$, that is $p(T) = 0$. At this point, using the backward uniqueness result recalled in Theorem 2.2, we conclude that $p_0 = 0$, which is a contradiction. $\square$

Remark 3.9. It is noteworthy that the proof of Lemma 3.7 yields also the following result which seems interesting in its own right:

Lemma 3.10. *Let H be a separable Hilbert space, $(\varphi_n)_{n\geq 1}$ a Hilbert basis of H and $B : H \longrightarrow H$ a self-adjoint bounded operator such that $(Bf|f) \geq 0$ for all $f \in H$. If $(\lambda_n)_{n\geq 1}$ is a sequence of positive numbers such that for some positive number $c_0 > 0$ one has $\lambda_n \geq c_0$ for all $n \geq 1$, then upon setting*

$$Lf := \sum_{n,j\geq 1} \frac{1}{\lambda_n + \lambda_j}(B\varphi_n|\varphi_j)(f|\varphi_n)\varphi_j,$$

L is non-negative that is $(Lf|f) \geq 0$, more precisely for $t \geq 0$ defining the semi-group $S(t)f := \sum_{n\geq 1} \mathrm{e}^{-\lambda_n t}(f|\varphi_n)\varphi_n$ we have

$$(Lf|f) = \int_0^\infty (BS(t)f|S(t)f)\mathrm{d}t \geq 0.$$

When H is finite dimensional, for instance when $H = \mathbb{C}^n$ or $H = \mathbb{R}^n$, this shows that if B is a non-negative self-adjoint matrix, then the matrix L defined by

$$L_{ij} := \frac{B_{ij}}{\lambda_i + \lambda_j}$$

is also non-negative, and if B is positive definite so is L. However L can be positive definite even if B is only non-negative. For instance, this particular example is of interest: if $\lambda_i \neq \lambda_j$ for $i \neq j$, taking $B_{ij} := 1$ for $1 \leq i, j \leq n$, one may verify that $(Bf|f) \geq 0$ and in fact the matrix

$$L := \left(\frac{1}{\lambda_i + \lambda_j}\right)_{1\leq i,j\leq n}$$

is positive definite: for we know already that $(Lf|f) \geq 0$ and therefore if it happens that $(Lf|f) = 0$, for all $t > 0$ we must have $(BS(t)f|S(t)f) = 0$. However B being self-adjoint, $(BS(t)f|S(t)f) = 0$ means that $BS(t)f = 0$, which in turn means that for all $t > 0$ we have $\sum_{1\leq k\leq n} f_k \mathrm{e}^{-\lambda_k t} = 0$, that is $f_k = 0$ for $1 \leq k \leq n$. Note that in this example the matrix B is not positive definite (being a matrix of rank one). As a matter of fact this observation is a generalization of the well known result asserting that the Hilbert matrix $L := (L_{ij})_{1\leq i,j\leq n}$ defined by $L_{ij} = 1/(i + j)$ is positive definite.

4. The case of Stokes equations

Consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$, and

$$\mathbb{H} := \left\{u \in (L^2(\Omega))^N;\ \mathrm{div}(u) = 0\right\}\cdot$$

Consider the Stokes operator on Ω, that is the unbounded operator defined on $\mathbb{H}$ by

$$Au := -\Delta u, \qquad D(A) := \left\{u \in (H_0^1(\Omega))^N;\ \Delta u \in \mathbb{H}\right\}\cdot$$

In order to apply our unique continuation result of Sections 2 and 3 to this case, among other things we have to check that the eigenvalues λ_k for the Stokes eigenvalue problem, that is

$$\begin{cases} -\Delta\varphi_k + \nabla\pi_k = \lambda_k\varphi_k & \text{in } \Omega \\ \operatorname{div}(\varphi_k) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

have a lower bound such as $\lambda_k \geq c_0\, k^{2/N}$. Indeed by the variational characterization of the eigenvalues, we have

$$\lambda_k = \inf_{A\in\mathcal{A}_k^0} \max_{u\in A} \int_\Omega |\nabla u(x)|^2 \mathrm{d}x,$$

where the class $\mathcal{A}_k^0$ is defined as the set of subsets of genus k on the sphere

$$S_0 := \left\{ u \in (H_0^1(\Omega))^N;\ \operatorname{div}(u) = 0,\ \int_\Omega |u(x)|^2 \mathrm{d}x = 1 \right\}$$

(endowed with the topology of $H_0^1(\Omega)$) that is precisely

$$\mathcal{A}_k^0 := \left\{ f(S^{k-1});\ f : S^{k-1} \longrightarrow S_0 \ \text{ is odd and continuous} \right\}\cdot$$

Now denoting by λ_k^{D} the eigenvalues for the Dirichlet problem (here $\varphi_k^{\mathrm{D}} \in (H_0^1(\Omega))^N$)

$$\begin{cases} -\Delta\varphi_k^{\mathrm{D}} = \lambda_k^{\mathrm{D}}\varphi_k^{\mathrm{D}} & \text{in } \Omega \\ \varphi_k^{\mathrm{D}} = 0 & \text{on } \partial\Omega, \end{cases}$$

we have

$$\lambda_k^{\mathrm{D}} = \inf_{A\in\mathcal{A}_k} \max_{u\in A} \int_\Omega |\nabla u(x)|^2 \mathrm{d}x,$$

where denoting by

$$S := \left\{ u \in (H_0^1(\Omega))^N;\ \int_\Omega |u(x)|^2 \mathrm{d}x = 1 \right\}$$

the class $\mathcal{A}_k$ is defined as being

$$\mathcal{A}_k := \left\{ f(S^{k-1});\ f : S^{k-1} \longrightarrow S \ \text{ is odd and continuous} \right\}\cdot$$

Since clearly we have $\mathcal{A}_k^0 \subset \mathcal{A}_k$, it follows that $\lambda_k \geq \lambda_k^{\mathrm{D}}$. As a consequence, the eigenvalues of the Stokes operator satisfy the growth condition mentioned in (1.6) with $k_0 = 1$ and $\beta = 2/N$.

Next observe that the classical unique continuation principle for the Laplacian yields easily a unique continuation principle for the Stokes operator. Namely, if $\omega \subset \Omega$ is an open ball such that $\varphi_k \in (H_0^1(\Omega))^N$ satisfies (4.1) and $\varphi_k = 0$ on ω, then according to the first equation we have $\nabla\pi_k = 0$ in ω, so that π_k is equal to constant in ω, which can be taken to be zero. However taking the divergence of this first equation, and using the fact that $\operatorname{div}(\varphi_{\mathrm{k}}) = 0$, we have also that $\Delta\pi_k = 0$ in Ω while $\pi_k = 0$ in ω: therefore the classical unique continuation principle for the Laplacian implies that $\pi_k \equiv 0$ in Ω. It follows that $-\Delta\varphi_k - \lambda_k\varphi_k = 0$ in Ω and $\varphi_k = 0$ in ω: applying again the unique continuation principle for the Laplacian, we conclude that $\varphi_k \equiv 0$. This means that if for $u \in \mathbb{H}$ we set

$$Bu := 1_\omega\, u$$

then B satisfies both properties (1.8) and (1.9) with $\gamma = 0$.

Therefore we can consider a coupled system of Stokes equations such as:

$$\begin{cases} \partial_t u^1 - \Delta u^1 + \nabla \pi^1 = 0 & \text{in } (0,\infty)\times\Omega \\ \partial_t u^2 - \Delta u^2 + \nabla \pi^2 = 1_{\omega_0}\, u^1(t) & \text{in } (0,\infty)\times\Omega \\ \operatorname{div}(u^1) = \operatorname{div}(u^2) = 0 & \text{in } (0,\infty)\times\Omega \\ u^1(t,\sigma) = u^2(t,\sigma) = 0 & \text{on } (0,\infty)\times\partial\partial\Omega \\ u^1(0,x) = u^1_0(x) & \text{in } \Omega \\ u^2(0,x) = u^2_0(x) & \text{in } \Omega \end{cases} \tag{4.2}$$

and state the following unique continuation result:

Theorem 4.1 (forward-forward coupled Stokes systems)**.** *Let $\omega_0, \omega_1 \subset \Omega$ be two open subsets of Ω, and for given $u^1_0, u^2_0 \in \mathbb{H}$, let u^1, u^2 be solution to (4.2). Then if for an infinite sequence $(t_j)_j \geq 1$ with $t_j \in [T_1, T_2]$ and $0 < T_1 < T_2$ we have $u^2(t_j, x) = 0$ for $x \in \omega_1$, then we have $u^1_0 \equiv u^2_0 \equiv 0$.*

Indeed we have also an analogous result for the following forward-backward Stokes system:

$$\begin{cases} \partial_t u^1 - \Delta u^1 + \nabla \pi^1 = 0 & \text{in } (0,T)\times\Omega \\ -\partial_t u^2 - \Delta u^2 + \nabla \pi^2 = 1_{\omega_0}\, u^1(t) & \text{in } (0,T\infty)\times\Omega \\ \operatorname{div}(u^1) = \operatorname{div}(u^2) = 0 & \text{in } (0,T)\times\Omega \\ u^1(t,\sigma) = u^2(t,\sigma) = 0 & \text{on } (0,T)\times\partial\Omega \\ u^1(0,x) = u^1_0(x) & \text{in } \Omega \\ u^2(T,x) = 0 & \text{in } \Omega \end{cases} \tag{4.3}$$

and we can state the following unique continuation result:

Theorem 4.2 (forward-backward coupled Stokes systems)**.** *Let $\omega_0, \omega_1 \subset \Omega$ be two open subsets of Ω, and for given $u^1_0 \in \mathbb{H}$, let u^1, u^2 be solution to (4.3). Then if for an infinite sequence $(t_j)_j \geq 1$ with $t_j \in [T_1, T_2]$ and $0 < T_1 < T_2 < T$ we have $u^2(t_j, x) = 0$ for $x \in \omega_1$, then we have $u^1_0 \equiv 0$.*

The reader may consider other types of operators B, as the ones given in the previous section, since using the above unique continuation property for the Stokes operator one may easily see that if φ_k satisfies (4.1) and if for a relatively open subset $\Gamma \subset \partial\Omega$ we have

$$\frac{\partial \varphi_k}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma,$$

then $\varphi_k \equiv 0$ in Ω.

Let us denote $\sigma(v,p) := -\pi I + (\nabla v + {}^t\nabla v)$ for $v : \Omega \longrightarrow \mathbb{R}^N$ and $\pi : \Omega \longrightarrow \mathbb{R}$. Then since Stokes equations (4.1) may be written as $\operatorname{div}(\sigma(\varphi_k, \pi_k)) = \lambda_k \varphi_k$ with $\operatorname{div}(\varphi_k) = 0$ and $\varphi_k \in (H^1_0(\Omega))^N$, instead of the normal derivative $\partial\varphi_k/\partial\mathbf{n}$ one may consider a more physical boundary information, such as

$$\sigma(\varphi_k, \pi_k)\mathbf{n} = 0 \quad \text{on } \Gamma$$

where $\sigma(\varphi_k, \pi_k)\mathbf{n}$ corresponds to the so-called Cauchy forces on the boundary. Then the unique continuation property mentioned above for the Stokes equation yields that $\varphi_k \equiv 0$ in Ω. For instance as another example of cascade equations we may consider (4.3) which can be written as (here $B_0 v := 1_{\omega_0} v$)

$$\begin{cases} \partial_t u^1 - \operatorname{div}(\sigma(u^1,\pi^1)) = 0 & \text{in } (0,T)\times\Omega \\ -\partial_t u^2 - \operatorname{div}(\sigma(u^2,\pi^2)) = B_0 u^1(t) & \text{in } (0,T)\times\Omega \\ \operatorname{div}(u^1) = \operatorname{div}(u^2) = 0 & \text{in } (0,T)\times\Omega \\ u^1(t,\sigma) = u^2(t,\sigma) = 0 & \text{on } (0,T)\times\partial\Omega \\ u^1(0,x) = u^1_0(x) & \text{in } \Omega \\ u^2(T,x) = 0 & \text{in } \Omega. \end{cases}$$

Now proceeding as in the previous section with $B_1(v) := 1_\Gamma\, \sigma(v,\pi)\mathbf{n}$, we have that if $\sigma(u^2(t_j),\pi^2(t_j)) = 0$ on Γ for an infinite sequence $(t_j)_j \in [T_1,T_2] \subset (0,T)$, then we have $u^1_0 \equiv 0$. There are numerous other such remarks, and we leave them to the reader's interest.

5. Application to control problems and examples

The concept of *insensitizing control* for the heat equation was introduced by Lions in [7] and is in relation with the following heat equation

$$\begin{cases} \partial_t y - \operatorname{div}(a\nabla y) = \xi + h 1_{\omega_1} & \text{in } Q := (0,T)\times\Omega \\ y(0,x) = y_0(x) + \tau\widetilde{y}_0 & \text{in } \Omega \\ y(t,\sigma) = 0 & \text{on } (0,T)\times\partial\Omega \end{cases} \tag{5.1}$$

in which the data are *incomplete* in the following sense: ξ, y^0 are given in $L^2(Q)$ and $L^2(\Omega)$ respectively, while $h = h(x,t)$ is a control term in $L^2(Q)$ to be determined, but

- $\widetilde{y}_0 \in L^2(\Omega)$ is unknown and $\|\widetilde{y}_0\|_{L^2(\Omega)} = 1$ and represents in some sense the *uncertainty* on the initial data;
- $\tau \in \mathbb{R}$ is unknown and small enough.

An observation functional

$$\Phi(y) := \frac{1}{2}\int_0^T \int_{\omega_0} y^2(x,t)\mathrm{d}x\,\mathrm{d}t \tag{5.2}$$

being given, the question is whether there exists a control $h \in L^2((0,T)\times\omega_1)$ such that

$$\frac{\partial\Phi(y(h,\tau))}{\partial\tau}\Big|_{\tau=0} = 0. \tag{5.3}$$

Here $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $\omega_1 \subset \Omega$ is a small control region, while $\omega_0 \subset \Omega$ is a small observation set. We denote by $y(h,\tau) := y(t,x;h,\tau)$ the solution of (5.1). Bodart and Fabre [1] relaxed the notion to the ε-insensitivity in the following way: for every $\varepsilon > 0$ find a control $h \in L^2(\omega_1 \times (0,T))$ such that

$$\left|\frac{\partial\Phi(y(h,\tau))}{\partial\tau}\Big|_{\tau=0}\right| \le \varepsilon. \tag{5.4}$$

It is not difficult to see that condition (5.4) (resp. (5.3)) is equivalent to the partial approximate (resp. null) controllability of the cascade system:

$$\begin{cases} \partial_t y - \operatorname{div}(a\nabla y) = \xi + h_{\omega_1} & \text{in } Q \\ y(0,x) = y_0(x) & \text{in } \Omega \\ y(t,\sigma) = 0 & \text{on } (0,T)\times\partial\Omega \end{cases} \tag{5.5}$$

$$\begin{cases} -\partial_t q - \operatorname{div}(a\nabla q) = y 1_{\omega_0} & \text{in } Q \\ q(T,x) = 0 & \text{in } \Omega \\ q(t,\sigma) = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{5.6}$$

That is, the ε-insensitivity (resp. insensitivity) condition (5.4) (resp. (5.3)) is equivalent to

$$\|q(0)\|_{L^2(\Omega)} \le \varepsilon, \qquad (\text{resp. } q(0) = 0).$$

When $\omega_0 \cap \omega_1 \neq \emptyset$, the problem was completely solved, even in the semilinear case, in Bodart and Fabre [1] for the approximate framework, and partially solved (*i.e.* for $y^0 = 0$) by the second author (de Teresa [3]) in the insensitizing context. The results of the previous sections allow us to solve the ε-insensitizing control problem when $\omega_0 \cap \omega_1 = \emptyset$. So the main result in this section is the following:

Theorem 5.1. *Assume that the matrix a satisfies (1.2) and that the domain Ω is bounded and Lipschitz. For any given initial data $y_0 \in L^2(\Omega)$, $\xi \in L^2(Q)$ and $\varepsilon > 0$, if $\omega_0, \omega_1 \subset \Omega$ are respectively any observation and control subdomains, there exists an ε-insensitizing control $h = h(\varepsilon, y_0, \xi)$ for the functional given by (5.2), where y is the solution to (5.1).*

The proof of this result is a direct consequence of the unique continuation property proved in Theorem 1.2 and is by now classical (see *e.g.* [4] or [1]), nevertheless for the sake of completeness we give a sketch of the proof. To this aim we need to introduce a new functional: given $y_0 \in L^2(\Omega)$, $\xi \in L^2(Q)$ and $\varepsilon > 0$, we consider for $p_0 \in L^2(\Omega)$

$$J(p_0) := \frac{1}{2}\int_0^T\int_{\omega_1} z^2 \,\mathrm{d}x\,\mathrm{d}t + \varepsilon\|p_0\|_{L^2} + \int_\Omega y_0 z(0)\mathrm{d}x + \int_0^T\int_\Omega \xi z \,\mathrm{d}x\,\mathrm{d}t \tag{5.7}$$

where z is the solution to (1.4) when p is solution to (1.1). We have the following result.

Lemma 5.2. *J is continuous, strictly convex and coercive, that is*

$$\lim_{\|p_0\|_{L^2}\to\infty} J(p_0) = +\infty.$$

More precisely we have

$$\liminf_{\|p_0\|_{L^2}\to\infty} \frac{J(p_0)}{\|p_0\|_{L^2}} \ge \varepsilon. \tag{5.8}$$

In particular, J achieves its minimum at a unique $\widehat{p}_0$. When $\widehat{p}_0 \neq 0$, it satisfies the following optimality condition:

$$\int_0^T\int_{\omega_1} \widehat{z} z \,\mathrm{d}x\,\mathrm{d}t + \frac{\varepsilon}{\|\widehat{p}_0\|_{L^2}}\int_\Omega \widehat{p}_0 p_0 \,\mathrm{d}x + \int_\Omega y_0 z(0)\mathrm{d}x + \int_0^T\int_\Omega \xi z\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{5.9}$$

for any $p_0 \in L^2(\Omega)$, where z is the corresponding solution to (1.4).

Proof. The continuity and convexity are straightforward as is the optimality condition, once the existence of the minimum is proved. We concentrate in proving (5.8). Suppose that this is not so; then there would exist a sequence of initial data p_0^n such that

$$\lim_{n\to\infty} \frac{J(p_0^n)}{\|p_0^n\|_{L^2}} < \varepsilon \qquad \text{and} \qquad \lim_{n\to\infty} \|p_0^n\|_{L^2} = \infty. \tag{5.10}$$

Defining the normalized data $\widetilde{p}_0^n := p_0^n/\|p_0^n\|_{L^2}$, we denote by $\widetilde{p}^n$ and $\widetilde{z}^n$ be the corresponding solutions to (1.1) and (1.4). It is easily seen (refer *e.g.* Bodart and Fabre [1]) that for a subsequence (still denoted by n) we have:

$$\widetilde{p}_0^n \rightharpoonup p_0 \quad \text{weakly in } L^2(\Omega), \qquad \widetilde{p}^n \to p \quad \text{and} \quad \widetilde{z}^n \to z \quad \text{strongly in } L^2(Q), \qquad \widetilde{z}^n(0) \to z(0) \text{ in } L^2(\Omega),$$

with p, z satisfying equations (1.1), (1.4) corresponding to p_0. Observe that since

$$\frac{J(p_0^n)}{\|p_0^n\|_{L^2}} = \frac{\|p_0^n\|_{L^2}}{2} \int_0^T \int_{\omega_1} |\widetilde{z}^n|^2 \mathrm{d}x\,\mathrm{d}t + \varepsilon + \int_0^T \int_\Omega \xi \widetilde{z}^n \mathrm{d}x\,\mathrm{d}t + \int_\Omega y^0 \widetilde{z}^n(0)\mathrm{d}x, \tag{5.11}$$

if we have $\int_0^T \int_{\omega_1} z^2 \mathrm{d}x\,\mathrm{d}t > 0$, then

$$\liminf_{n\to\infty} \frac{J(p_0^n)}{\|p_0^n\|} \geq \varepsilon,$$

contrary to (5.10). Therefore assumption (5.10) implies necessarily that $\int_0^T \int_{\omega_1} z^2 \mathrm{d}x\,\mathrm{d}t = 0$, that is $z \equiv 0$ on $(0,T) \times \omega_1$, and consequently according to the unique continuation principle proved in Theorem 1.2, we have that $z \equiv 0 \equiv p$ on $(0,T)\times\Omega$ and that $p_0 \equiv 0$. Passing to the limit in (5.11) we obtain a contradiction with (5.10). This contradiction shows that our claim (5.8) holds. □

Proof of Theorem 5.1. The claim of Theorem 5.1 follows now easily. Indeed, observe that the system (1.4), (1.1) is the adjoint of system (5.5), (5.6). Therefore taking $h := \widehat{z}$, the solution of (1.4) corresponding to the minimizer $\widehat{p}_0$ of J, thanks to the optimality condition (5.9), we obtain a partial ε-control for the cascade system (5.5), (5.6). That is, we get $\|q(0)\|_{L^2(\Omega)} \leq \varepsilon$. □

Observe that our main result Theorem 3.8 allows also to control equation (5.5) from a non empty subset $\Gamma_1 \subset \partial\Omega$ of the boundary or to use a coupling in (5.6) on a subset $\Gamma_0 \subset \partial\Omega$ on the boundary.

For the case of two forward-forward equations simultaneous approximate controllability of both equations can be performed. As an example of the kind of situations we can expect to hold, we have the following result:

Proposition 5.3. *Let* $a := (a_{ij})_{1\leq i,j\leq N}$ *satisfy* (1.2)*. For any* $\varepsilon > 0$ *and data* (y_0, w_0), $(y_1, w_1) \in H^{-1}(\Omega)$ *given, and for every domain* $\omega \subset \Omega$ *and every nonempty* $\Gamma \subset \partial\Omega$ *subset of the boundary, there exists* $h \in L^2((0,T)\times\Gamma)$ *such that if* y *is the solution to*

$$\begin{cases} \partial_t y - \operatorname{div}(a\nabla y) = 0 & \text{in } (0,T)\times\Omega \\ y(0,x) = y_0 & \text{in } \Omega \\ y(t,\sigma) = 0 & \text{on } (0,T)\times(\partial\Omega\backslash\Gamma) \\ y(t,\sigma) = h & \text{on } (0,T)\times\Gamma, \end{cases} \tag{5.12}$$

and if w *satisfies*

$$\begin{cases} \partial_t w - \operatorname{div}(a\nabla w) = y 1_\omega & \text{in } (0,T)\times\Omega \\ w(0,x) = w_0 & \text{in } \Omega \\ w(t,\sigma) = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{5.13}$$

then we have

$$\|y(T) - y_1\|_{H^{-1}} + \|w(T) - w_1\|_{H^{-1}} \leq \varepsilon.$$

The results in the previous section give also ε-insensitizing results for the Stokes system (we thank an anonymous referee of this paper for having kindly informed us that the problem of (null) insensitizing control, in the easier case in which $\omega_1 \cap \omega_0 \neq \emptyset$, has been treated by Guerrero in [6]).

6. Open problems

First of all we notice that in the latter controllability result, we are only obtaining a partial approximate control for the cascade system in the sense that we are not *simultaneously* obtaining an approximate control for the state y. An interesting problem is to control simultaneously (5.5), (5.6), that is, to get a control h such that the corresponding solution satisfies

$$\|q(0)\|_{L^2} \leq \varepsilon, \quad \|y(T)\|_{L^2} \leq \varepsilon.$$

In fact a controllability result in this direction is equivalent to the unique continuation property of system (1.1)–(1.4) but with the additional assumption that $z(T) = z_0 \in L^2(\Omega)$, instead of $z(T) = 0$ as in this paper. Observe that the proof of Theorem 1.2 uses the fact that $z(T) = 0$.

The techniques used along this paper cannot be applied in the case of a linear system with potentials, *i.e.*, we do not know if Theorem 1.2 holds true for p, z solution of

$$\begin{cases} \partial_t p - \Delta p + b(x,t)p = 0 & \text{in } Q \\ p(0,x) = p_0(x) & \text{in } \Omega \\ p(t,\sigma) = 0 & \text{on } (0,T) \times \partial\Omega \end{cases} \tag{6.1}$$

$$\begin{cases} -\partial_t z - \Delta z + a(x,t)z = p1_{\omega_0} & \text{in } Q \\ z(T,x) = 0 & \text{in } \Omega \\ z(t,\sigma) = 0 & \text{on } (0,T) \times \partial\Omega \end{cases} \tag{6.2}$$

with $a, b \in L^\infty(Q)$ and then the semilinear ε-insensitizing control problem remains open. See [1] for a description and solution of the problem when $\omega_0 \cap \omega_1 \neq \emptyset$.

Observe also that the linear operators involved in the cascade systems are assumed to be the same: indeed, as far as the control problem is concerned this is not an annoyance, but from a mathematical point of view it would be interesting to consider situations in which two different linear operators are involved in the cascade systems. In a forthcoming study [5], Fernández-Cara *et al.* show a counterexample to the unique continuation in the following case: assuming that p satisfies

$$\begin{cases} \partial_t p - \mu p_{xx} = 0 & \text{in } (0,T) \times (0,1) \\ p(0,x) = p_0(x) & \text{in } (0,1) \\ p(t,0) = p(t,1) = 0 & \text{for } t \in (0,T), \end{cases} \tag{6.3}$$

and z is solution to

$$\begin{cases} \partial_t z - z_{xx} = p & \text{in } (0,T) \times (0,1) \\ z(0,x) = z_0(x) & \text{in } (0,1) \\ z(t,0) = z(t,1) = 0 & \text{for } t \in (0,T). \end{cases} \tag{6.4}$$

More precisely, when $\mu \neq 1$ and $\sqrt{\mu} \in \mathbb{Q}$ there exist non zero solutions to (6.3), (6.4) such that $z_x|_{x=0} = 0$, that is, the unique continuation principle is not any more true.

Acknowledgements. The authors thank Enrique Fernández-Cara for fruitful discussions about the problems presented in this paper, and are grateful to Enrique Zuazua for having invited the authors to the *Centro de Ciencias de Benasque "Pedro Pascual"* where the final version of this work was written. The second author was partially supported by PAPIIT IN106006 of UNAM, Mexico.

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