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POLYNOMIAL DIFFEOMORPHISMS OF $\mathbf{C}^2$: VII. HYPERBOLICITY AND EXTERNAL RAYS

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ABSTRACT. – For a polynomial automorphism we study the topology of J, the analogue of the Julia set, in the case where J is connected and hyperbolic. © Elsevier, Paris

RÉSUMÉ. – À partir d'un automorphisme polynomial de $\mathbf{C}^2$, nous étudions l'analogue de l'ensemble de Julia, noté J, dans le cas où J est hyperbolique et connexe. © Elsevier, Paris

0. Introduction

In this paper we consider the dynamics of polynomial diffeomorphisms of $\mathbf{C}^2$. The family of polynomial diffeomorphisms $f : \mathbf{C}^2 \to \mathbf{C}^2$ contains the simplest invertible holomorphic transformations with interesting dynamical behavior. Given a dynamical system to investigate, a first problem is the identification of sets of dynamical interest, and when such a set is identified, a second problem is then the description, up to topological equivalence, of the dynamics on this set.

For this class of diffeomorphisms there is a natural candidate for a dynamically significant set, namely the analogue of the Julia set $J = J_f \subset \mathbf{C}^2$ defined below. We are interested in attacking the problem of finding a topological description of the restriction of f to J. A paradigm for what we would like to achieve is the Douady-Hubbard theory of external rays for polynomial maps of $\mathbf{C}$, which we recall briefly. Douady and Hubbard show that when the Julia set J of a polynomial map f is connected, and f is expanding on J, then the map $f|_J$ is semiconjugate to the map $z \mapsto z^d$ on the unit circle. Furthermore this semiconjugacy can be realized concretely by external rays. An external ray is a gradient curve for the Green function of the complement of the filled Julia set. Douady and Hubbard show that each external ray limits on a well-defined point in the Julia set. It is this "landing map" which provides the semiconjugacy from the space of external rays (which is the circle) to J. This explicit description of maps with expanding connected Julia sets provides a first step in the combinatorial description of parameter space.

Our aim in this paper is to lay the groundwork for a theory of polynomial diffeomorphisms of $\mathbf{C}^2$ analogous to the Douady-Hubbard theory for 1-variable polynomial expanding maps with connected Julia sets. We showed in [BS6] that external rays for

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polynomial diffeomorphisms can be defined when J is connected. In this paper we make the additional assumption† that f is hyperbolic on J. This condition is analogous to the expanding condition for maps in one variable. In §3 we establish a range of topological conditions equivalent to connectivity. Assuming that J is connected we describe the topology of the space of rays in §4. In §5 we show that the landing map from the space of rays to J is defined, continuous, surjective and bounded-to-one. This allows us to describe J as a quotient of the solenoid under an equivalence relation. In §6 we describe topological conditions that this equivalence relation must satisfy. We also relate the properties of the equivalence relation to the local topology of J and in §7 we identify a finite set of periodic points that play a distinguished role.

To describe the results of this paper more precisely, we will recall some standard terminology and known results. We denote by K^+/K^- the sets of points in $\mathbf{C}^2$ with bounded forward/backward orbits under f. Let $J^{\pm} = \partial K^{\pm}$ and let $J = J^+ \cap J^-$. We refer to J as the Julia set of f. Let $U^{\pm} = \mathbf{C}^2 - K^{\pm}$. There are real valued functions $G^{\pm}$ defined on $\mathbf{C}^2$ which play the role of Green functions. The functions $G^{\pm}$ are continuous, non-negative, plurisubharmonic, equal to zero on $K^{\pm}$ and pluriharmonic on $U^{\pm}$.

The properties of stable and unstable connectivity were defined in [BS6]. Let p be a periodic saddle point, and let $W^u(p)$ be its unstable manifold. We say f is *unstably connected* if $W^u(p) \cap K^+$ has no compact components. It is shown in [BS6] that this definition does not depend on which saddle point p is chosen. Stable connectivity is defined analogously. Let $J^-_+ = J^- - K^+$. When f is unstably connected, J^-_+ has a lamination $\mathcal{M}^-$ by Riemann surfaces. Furthermore the restriction of G^+ to a leaf of this lamination is a harmonic function without critical points. An (unstable) external ray is then a gradient curve of the restriction of G^+ to a leaf (with respect to some conformal metric on the leaf). We denote by $\mathcal{E}$ the set of external rays. This set inherits a topology from the space J^-_+.

The function $\det \mathrm{D}f$ is constant on $\mathbf{C}^2$. We say that f increases, preserves, or decreases volume, depending on whether $|\det \mathrm{D}f|$ is greater than, equal to, or less than, one. By replacing f by f^{-1} if necessary we may assume that f is not volume increasing. This will be a constant assumption throughout this paper. By Corollary 6.3, if J is connected (and hyperbolic as we also constantly suppose), then f cannot preserve volume. In particular, since f does not increase volume, it follows from [BS6, Corollary 7.6] that unstable rays are defined exactly when J is connected, and stable rays are never defined.

We will review some results from [BS1] and [BS2] on hyperbolicity. When f restricted to J is hyperbolic, then f is Axiom A. In this case Smale's spectral theorem gives a decomposition of the nonwandering set into basic sets. The basic sets are: J, which is the unique (complex) index one basic set, and a finite set of periodic sinks $S = \{s_1, \dots s_k\}$. The stable set of J, $W^s(J)$, is $J^+ = \partial K^+$ and the interior of K^+ consists of the basins of the sinks. The unstable set of J, $W^u(J)$, is $J^- - S$ and the interior of K^- is empty. The sets $W^{s/u}(J)$ have dynamically defined Riemann surface laminations, $\mathcal{W}^{s/u}$, whose leaves consist of stable/unstable manifolds of points. Each leaf of this lamination is conformally equivalent to $\mathbf{C}$. When f is unstably connected, the restriction of the lamination $\mathcal{W}^u$ to J^-_+ is the same as the lamination $\mathcal{M}^-$ given in [BS6].

† Generally speaking, the map f is assumed hyperbolic, and J is assumed connected thoughout this paper. In parts of §3 and §5, J is not assumed to be connected.

As was observed in [H] the function G^+ is pluriharmonic on U^+, and the holomorphic 1-form ∂G^+ defines a holomorphic foliation $\mathcal{G}^+$ on U^+. In §2 we prove the useful technical result that if f is hyperbolic and unstably connected, then the laminations $\mathcal{G}^+$ and $\mathcal{W}^s$ fit together continuously, or, more precisely, $\mathcal{G}^+ \cup \mathcal{W}^s$ is a Riemann surface lamination of $J^+ \cup U^+$ (Proposition 2.7). We note in Appendix A, however, that $\mathcal{G}^-$ and $\mathcal{W}^u$ (the corresponding objects for the inverse diffeomorphism) do not fit together continuously in this case.

When f is hyperbolic, we will sharpen some of the topological criteria for unstable connectivity given in [BS6]. The property of unstable connectivity is defined in terms of slices of K^+ by unstable manifolds. We say f is unstably connected when, for any saddle point p, $K^+ \cap W^u(p)$ has no compact components. In §3, we show that in the hyperbolic case, the property of unstable connectivity is characterized by slices by more general "transversals" (The use of the term transversal is meant to be suggestive. In fact our transversals are very general complex one dimensional submanifolds). A typical result from §3 is that, when f is hyperbolic, f is unstably connected if and only if a transversal slices K^+ (locally) into only finitely many components.

We give a second topological characterization of unstable connectivity in the hyperbolic case, we recall the observation of [HO1] that the homology of the set U^+ is independent of the mapping f and is given by $\mathrm{H}_1(U^+;\mathbf{Z}) \cong \mathbf{Z}[\frac{1}{d}]$, where d denotes the degree of f (see §1). Note that this homology group is not finitely generated. The condition that the map f be unstably connected can be characterized by the finite generation of the homology of a bounded portion of U^+. Precisely, f is unstably connected if and only if for every bounded set $B \subset \mathbf{C}^2$ the image of the inclusion map

$$\iota_* : \mathrm{H}_1(B \cap U^+;\mathbf{Z}) \to \mathrm{H}_1(U^+;\mathbf{Z})$$

is finitely generated (Proposition 3.2 and Theorem 3.4).

In [BS6] it was shown that the property of unstable connectivity has consequences for the topology of J^-_+. In particular we can construct an abstract "model" for the space J^-_+. We recall some notation. Let Σ be the complex solenoid, the inverse limit of $\mathbf{C}^*$ under the map $\sigma : z \mapsto z^d$. Let $\Sigma_+ \subset \Sigma$ denote the inverse limit of the set $\{z : |z| > 1\}$ under σ, and let $\Sigma_0 \subset \Sigma$ denote the inverse limit of the set $\{z : |z| = 1\}$ under σ, which is the (real) solenoid. The shift map σ acts on each of these spaces (these are discussed further in §1). We showed in [BS6] that when f is unstably connected there is a semiconjugacy Φ between the action of σ on Σ_+ and the action of f on J^-_+. In §4 we show that when f is hyperbolic and unstably connected there is a map Ψ from Σ_+ to J^-_+ which is in fact a conjugacy. The conjugacy Ψ induces a conjugacy between Σ_0 and the space of external rays $\mathcal{E}$.

In §5 we derive geometric information about unstable manifolds. For every point $p \in J$ the stable/unstable manifolds $W^{s/u}(p)$ are uniformized by $\mathbf{C}$. This endows the manifolds $W^{s/u}(p)$ with affine structures. In §5 we use the work of Ghys [G] to show that for any hyperbolic mapping these affine structures vary continuously with p. The affine structure gives us a tool for proving a John-type condition for the sets $W^u(p) \cap U^+$ inside $W^u(p)$. This implies (Proposition 5.6 and Corollary 5.13) that when f is unstably connected the sets $K^+ \cap W^u(p)$ and $J^+ \cap W^u(p)$ are connected and locally connected. (For a general unstably connected map we know only that $K^+ \cap W^u(p)$ has no compact components.)

We can use the function G^+ to parametrize external rays. For an external ray $\gamma \in \mathcal{E}$ and a real number $r > 1$, let $e_r(\gamma)$ be the point on γ for which $G^+ = r$. We say that γ lands if $\lim_{r\to 0} e_r(\gamma)$ exists; when γ lands we denote the limit by $e(\gamma)$. We prove that external rays land at well-defined points of J, and we prove that the landing map $\gamma \mapsto e(\gamma)$ is a continuous map from $\mathcal{E}$ to J. We use the John property to show that the landing map is a surjection and finite-to-one, with the number of preimages of a point being uniformly bounded. The existence of the landing map allows us to represent J as a quotient of Σ_0 by an equivalence relation.

At this stage it is natural to ask what general sort of equivalence relations can arise to give the restriction $f|_J$. In §6 we show that the quotient map $\psi : \Sigma_0 \to J$ respects local product structures. We give some additional conditions which this quotient must satisfy; these arise from the topological condition that certain subset of J must be contained in sets $W^u(p)$ and thus must be planar.

In §7 we consider a special kind of cut point of $K \cap W^u$, which we call pinch points. It is also shown (Theorem 7.1) that all pinch points of the slices $W^u \cap K$ of the Julia set lie on the stable manifolds of a finite set of "primary" periodic points. In Appendix B we present an example and show how a computer picture may be used to illustrate the results of this paper.

This paper is not the first to make a connection between polynomial diffeomorphisms of $\mathbf{C}^2$ and solenoids. Such a connection first appears in the paper of Hubbard [H]. More recently, Hubbard and Oberste-Vorth [HOV1] have established the existence of a "solenoid at infinity" in connection with a certain compactification of $\mathbf{C}^2$. By contrast our use of the solenoid is more closely connected with the topology of J. While the solenoid at infinity exists for all parameter values, our "solenoid of external rays" exists only when J is connected. Yet when the solenoid of external rays does exist it seems more directly related to the dynamics of J than does the solenoid at infinity.

1. Notation and Preliminaries

We consider mappings of the form $f = f_1 \circ \cdots \circ f_m$, where

$$f_j(x, y) = (y, p_j(y) - a_j x), \tag{1.1}$$

with $p_j(y) = y^{d_j} + O(y^{d_j-2})$, and $d_j \geq 2$. It is a result of [FM] that every polynomial diffeomorphism of $\mathbf{C}^2$ which is not conjugate to a linear map or a shear is conjugate to a map of this form. We let $d = d_1 \cdots d_n$ denote the degree of f.

The iterates of f have the form

$$f^k(x, y) = (y^{d^k/d_1}(1 + \ldots), y^{d^k}(1 + \ldots)). \tag{1.2}$$

The rate of escape to infinity in forward/backward time is given by

$$G^{\pm} = \lim_{n\to\infty} \frac{1}{d^n} \log^+ \left|f^{\pm n}(x, y)\right|. \tag{1.3}$$

The function $G^{\pm}$ is continuous and pluri-subharmonic on $\mathbf{C}^2$. Further, if we let $U^{\pm}$ denote the points that escape to infinity in forward/backward time, then $U^{\pm} = \{G^{\pm} > 0\}$, and $G^{\pm}$ is pluri-harmonic on $U^{\pm}$. Thus the complex 1-form $\partial G^{\pm}$ is holomorphic on $U^{\pm}$ and defines a plane field which determines a holomorphic foliation $\mathcal{G}^{\pm}$ on $U^{\pm}$.

We define the sets

$$V^+ = \{|y| \geq |x|, G^+ \geq \log R\}, \quad V^- = \{|y| \leq |x|, G^- \geq \log R\}, \tag{1.4}$$

and

$$V = \{G^+, G^- \leq \log R\}. \tag{1.5}$$

For R sufficiently large, $fV^+ \subset V^+$, $fV \subset V \cup V^+$, and for any point $(x,y) \in V^-$, the orbit $f^n(x,y)$ can remain in V^- only for finitely many $n > 0$. If $K^\pm$ denotes the points with bounded forward/backward orbits, then

$$U^+ = \mathbf{C}^2 - K^+ = \bigcup_{n=0}^{\infty} f^{-n}V^+, \quad \text{and} \quad U^- = \mathbf{C}^2 - K^- = \bigcup_{n=0}^{\infty} f^n V^-. \tag{1.6}$$

Laminations of sets occur at various points in our work. Loosely speaking, a lamination $\mathcal{L}$ of a set X is a partition of X into subsets which are manifolds, such that the partition is locally trivial. This means that each point has a neighborhood U such that the partition of $\mathcal{L}|U$ into components of leaves, or "plaques" is homeomorphic to a product lamination. For further details, see [C] or [MS].

We collect here some facts which will be useful in §3. Let us set $\pi_2(x,y) = y$, and then as in [HOV1], let us define an analytic function φ^+ on V^+ by the formula

$$\varphi^+(x,y) = \lim_{n\to\infty} (\pi_2 \circ f^n(x,y))^{\frac{1}{d^n}}, \tag{1.7}$$

where we take the d^n-th root so that $\varphi^+(x,y) = y + o(1)$ holds on V^+. It is immediate that $\varphi^+ \circ f = (\varphi^+)^d$ and $\log|\varphi^+| = G^+$ hold on V^+. In particular, any analytic continuation of φ^+ is locally constant on the leaves of $\mathcal{G}^+$. Indeed locally the plaques of the foliation $\mathcal{G}^+$ are just the level sets of φ^+. In [BS6] we showed that if f is unstably connected, then φ^+ has an extension to J^-_+.

We collect some background which will be useful in §4. The complex solenoid Σ is the inverse limit of $\mathbf{C}^*$ under the map $z \mapsto z^d$ (A more detailed discussion of the complex solenoid is given in [BS]). We can realize this explicitly as the set of bi-infinite sequences $z = (z_j)$ such that $z_j \in \mathbf{C}^*$ and $z_j^d = z_{j+1}$. The space Σ has the topology of a closed subset of the bi-infinite product $(\mathbf{C}^*)^{\mathbf{Z}}$. The shift mapping $\sigma : \Sigma \to \Sigma$ is defined by setting $\sigma(z) = w$, where $w_j = z_{j+1} = z_j^d$. The map σ induces a homeomorphism of Σ.

We define the projection $\pi : \Sigma \to \mathbf{C}^*$ by $\pi(z) = z_0$. The set $\pi^{-1}(w)$ has the topology of a Cantor set. We define a modulus function on Σ by $|z| = |\pi(z)|$. We have $|\sigma(z)| = |z|^d$. We set $\Sigma_0 = \{z \in \Sigma : |z| = 1\}$ and $\Sigma_+ = \{z \in \Sigma : |z| > 1\}$. The map σ induces homeomorphisms on Σ_+ and on Σ_0.

The complex solenoid is a topological group under coordinatewise multiplication. The unit element "1" of this group is the element with all coordinates equal to 1. The subset Σ_0 is a subgroup. We let $m_s : \Sigma \to \Sigma$ denote the operation of multiplication by s. If $s \in \Sigma_0$, then m_s maps Σ_+ to itself. For $z \in \mathbf{C}$ we define $\exp(z) \in \Sigma$ by defining the n-th coordinate to be $[\exp(z)]_n = e^{zd^n}$. This gives a $\mathbf{C}$-action on the solenoid by $\mathbf{C} \ni z \mapsto m_{\exp(z)}$. For $t \in \mathbf{R}$ we have $\exp(it) \in \Sigma_0$ This gives an $\mathbf{R}$-action on the solenoid by $\mathbf{R} \ni t \mapsto m_{\exp(it)}$.

The orbits of the $\mathbf{C}$ action on Σ are the leaves of a lamination of Σ. This lamination has a dynamical interpretation as the lamination by unstable manifolds of σ. The map from

$\mathbf{C}$ to the orbit of a point w given by $z \mapsto \exp(z) \cdot w$ induces an affine structure on each leaf. These affine structures on leaves come from an affine lamination on Σ. The subspace Σ_+ inherits the affine lamination structure from Σ even though it does not inherit the $\mathbf{C}$ action. Each leaf of the lamination of Σ_+ is affinely equivalent to a half-plane.

We use the extension of φ^+ to construct a semi-conjugacy from J_+^- to Σ_+ (cf. [BS6, Theorem 3.2]). Specifically the map $\Phi : J_+^- \to \Sigma_+$ given by $[\Phi(p)]_n = \varphi^+(f^n p)$ is continuous and satisfies $\sigma \circ \Phi = \Phi \circ f$.

The extension φ^+ has the property that $\varphi^+|_M : M \to \mathbf{C} - \overline{\Delta}$ is a holomorphic covering, for every leaf M of $\mathcal{M}^-$. As in Proposition 2.2 of [BS6], this may be lifted to a conformal equivalence $\alpha : M \to H$ to the right half plane H. For each $\theta \in \mathbf{R}$, we let $\tau_\theta : H \to H$ be the translation $H \ni z \mapsto z + i\theta$. We may now define an $\mathbf{R}$-action on J_+^- by pulling the translation τ_θ back to the leaf M. We denote this action by $(\theta, z) \mapsto \exp(i\theta) \cdot z$. We have $\exp(i\theta) \circ \Phi = \Phi \circ \exp(i\theta)$.

2. Continuity of the Stable Lamination

In this section we show that when f is a hyperbolic and unstably connected map, the leaves of the foliation $\mathcal{G}^+$ of U^+ and the stable lamination $\mathcal{W}^s$ of J^+ fit together to make a locally trivial lamination of $U^+ \cup J^+$ (Theorem 2.7). In the process of proving this result, we show that the external rays converge to well defined points of J and that this "landing map" is continuous. We also obtain some results on the interplay between the landing map and the stable manifolds.

We begin with a preliminary result about the landing map. We will investigate the properties of the landing map more fully in §3.

PROPOSITION 2.1. – *Let f be unstably connected and hyperbolic. Then the mapping $e : \mathcal{E} \to J$ is defined on all of $\mathcal{E}$ and is continuous and equivariant.*

Proof. – Since f is hyperbolic, there exist constants $C < \infty$ and $\lambda < 1$ such that

$$||\mathrm{D}f^{-n}(x)|_{\mathcal{W}^u}|| \leq C\lambda^n \tag{2.1}$$

for x in a neighborhood of J in J^-. By applying the appropriate iterate of the map f and changing the constant we may assume that this holds on $J^- \cap \{0 \leq G^+ \leq 1\}$.

Let $\mathcal{G}$ denote the family of arcs γ obtained by starting at points of $J^- \cap \{G^+ = 1\}$ and following the gradient line of $G^+|W^u(p)$ in the direction of decreasing G^+, so that γ ends at a point of $J^- \cap \{G^+ = d^{-1}\}$ (note that these arcs are pieces of external rays, as defined in §3 of [BS6]). The family $\mathcal{G}$ has the properties:

(i) For every $p \in J^- \cap \{G^+ = 1\}$ there is a curve $\gamma \in \mathcal{G}$ starting at p.

(ii) Every $\gamma \in \mathcal{G}$ lies inside $W^u(x)$ for some $x \in J$.

The curves of $f^{-n}\mathcal{G}$ connect $J^- \cap \{G^+ = d^{-n}\}$ to $J^- \cap \{G^+ = d^{-n-1}\}$ inside $\mathcal{W}^u$. Let $\tilde{\mathcal{G}}$ denote the paths of the form

$$\tilde{\gamma} = \gamma \cup f^{-1}\gamma_1 \cup f^{-2}\gamma_2 \cup \ldots,$$

that is, γ starts at a point $p \in J^- \cap \{G^+ = 1\}$ and ends at a point $p_1 \in J^- \cap \{G^+ = d^{-1}\}$. This is followed by $f^{-1}\gamma_1$, where $\gamma_1 \in \mathcal{G}$ starts at $fp_1 \in J^- \cap \{G^+ = 1\}$ and ends at a point $p_2 \in J^- \cap \{G^+ = d^{-1}\}$.

The endpoint map will be defined as $e(s) = \lim_{n\to\infty} e_n(s)$, where $e_n(s)$ is obtained by following the ray γ_s to the level $\{G^+ = 1\}$ and then following the curve $\tilde{\gamma}$ in to the level $\{G^+ = d^{-n}\}$. The map $e_n : \mathcal{E} \to J^-$ is continuous.

If we set

$$M = \max_{\gamma \in \mathcal{G}} \text{Length}(\gamma), \tag{2.2}$$

then the curves in the family $f^{-n}\mathcal{G}$ all have length no greater than $MC\lambda^n$. Thus for $n < m$

$$\max_{s\in\mathcal{E}} |e_n(s) - e_m(s)| \le CM(\lambda^{n+1} + \ldots + \lambda^m) \le CM(1-\lambda)^{-1}\lambda^{n+1},$$

so the e_n converge uniformly, and the limit $e : \mathcal{E} \to J$ exists and is continuous. □

Further properties of the endpoint map will be discussed in §5. Our next objective is to show that the foliation $\mathcal{G}^+$ of U^+ is compatible with the lamination of J^+ by stable manifolds.

It will be useful to set up some of the machinery from the graph transform proof of the stable manifold theorem. At each point $p \in J$ we can split the tangent space $T_p\mathbf{C}^2 = E_p^u \oplus E_p^s$. This splitting depends continuously on the point p. We can also choose a continuous adapted metric on the tangent spaces E_p^u and E_p^s so that $\mathrm{D}f : E_p^u \to E_{f(p)}^u$ uniformly expands distances and $\mathrm{D}f : E_p^s \to E_{f(p)}^s$ uniformly contracts distances. At each point $p \in J$ we choose a coordinate map $\theta_p : \mathbf{C}^2 \to \mathbf{C}^2$. We choose θ_p to be an affine map which takes 0 to p, takes the x-axis to E_p^u and takes the y-axis to E_p^s. We also require that θ_p take the standard metrics on the axes to the adapted metrics on the stable and unstable tangent spaces.

Let $B_\epsilon(p)$ be the image of the set $\{(x,y) \in \mathbf{C}^2 : |x| < \epsilon,\ |y| < \epsilon\}$ under the map θ_p. A *vertical* (resp. *horizontal*) disk in $B_\epsilon(p)$ is a set which can be written as $\{(\Gamma(y), y) : |y| < \epsilon\}$ (resp. $\{(x, \Gamma(x)) : |x| < \epsilon\}$) for some holomorphic map $\Gamma : \{|z| < \epsilon\} \to \{|z| < \epsilon\}$. For $p \in J$ we define the *local stable/unstable manifold* through p, written $W_\epsilon^{s/u}(p)$ to be the component of $B_\epsilon(p) \cap W^{s/u}(p)$ containing p.

It is sometimes useful to consider the local unstable manifold of p relative to a nearby point p': For $p \in J$ we define $W_\epsilon^{s/u}(p,p')$ to be the component of $B_\epsilon(p') \cap W^{s/u}(p)$ containing p.

We may assume that ϵ is chosen sufficiently small for the following properties to hold:

VERTICAL OVERFLOWING PROPERTY. – *If Δ is a vertical disk in $B_\epsilon(p)$, then*

$$f^{-1}(\Delta) \cap B_\epsilon(f^{-1}(p))$$

is a vertical disk in $B_\epsilon(f^{-1}(p))$.

STABLE MANIFOLD THEOREM. – *$W_\epsilon^{s/u}(p)$ is a vertical/horizontal disk in $B_\epsilon(p)$.*

SMOOTH LAMBDA LEMMA. – *If M is a smooth manifold which intersects $W^u(p)$ transversally at a point q, then there is an n such that the component of $f^{-n}(M) \cap B_\epsilon(f^n(p))$ containing $f^{-n}(p)$ is a vertical disk in $B_\epsilon(f^n(p))$.*

The first of these properties is easily verified. The second two properties are consequences of the graph-transform proof of the Stable Manifold Theorem.

For $p \in J$ and $q \in B_\epsilon(p) - K^+$ we denote by $\mathcal{G}^+_\epsilon(q,p)$ the component of $\mathcal{G}^+(q) \cap B_\epsilon(p)$ containing q. We say that $\mathcal{G}^+_\epsilon(q,p)$ is *vertical* if it is a vertical disk in $B_\epsilon(p)$. Using the compactness of J it is easy to show that there is a $\delta > 0$ such that for $p \in J$ and $p' \in J$ with $d(p,p') < \delta$, $W^s_\epsilon(p,p')$ is vertical and $W^u_\epsilon(p,p')$ is horizontal. The next Proposition shows an analogous result for local leaves of the $\mathcal{G}^+$ foliation.

PROPOSITION 2.2. – *There is a $\delta > 0$ such that for $p \in J$ and $q \in U^+$ with $d(p,q) < \delta$, $\mathcal{G}^+_\epsilon(q,p)$ is vertical.*

The proof will be given after a sequence of lemmas.

LEMMA 2.3. – *If $\mathcal{G}^+_\epsilon(q,p)$ is vertical then $\mathcal{G}^+_\epsilon(f^{-1}(q), f^{-1}(p))$ is vertical.*

Proof. – If $\mathcal{G}^+_\epsilon(q,p)$ is vertical then the component of $\mathcal{G}^+(q) \cap B_\epsilon(p)$ containing q is a vertical disk Δ in $B_\epsilon(p)$. The vertical overflowing property of f^{-1} implies that $f^{-1}(\Delta) \cap B_\epsilon(f^{-1}(p))$ is a vertical disk in $B_\epsilon(f^{-1}(p))$. The invariance of the foliation $\mathcal{G}^+$ implies that $f^{-1}(\mathcal{G}^+(q)) = \mathcal{G}^+(f^{-1}(q))$. Thus $f^{-1}(\Delta) \cap B_\epsilon(f^{-1}(p))$ is the component of $\mathcal{G}^+(f^{-1}(q)) \cap B_\epsilon(f^{-1}(p))$ containing $f^{-1}(q)$. This is the definition of $\mathcal{G}^+_\epsilon(f^{-1}(q), f^{-1}(p))$. $\square$

LEMMA 2.4. – *There is an $\alpha > 0$ so that if $\mathcal{G}^+_\epsilon(q,p)$ is vertical and $d(p', f^{-1}(p)) < \alpha$ then $\mathcal{G}^+_\epsilon(f^{-1}(q), p')$ is vertical.*

Proof. – The Vertical Overflowing Property implies that $f^{-1}(B_\epsilon(p))$ overflows the box $B_\epsilon(f^{-1}(p))$. If the box $B_\epsilon(f^{-1}(p))$ is changed slightly then the overflowing condition still holds. Since the stable and unstable tangent spaces and the adapted metric vary continuously with the point, a small change in the point causes a small change in the box. Thus there is some α_p such that $d(p', f^{-1}(p)) < \alpha_p$ implies that $\mathcal{G}^+_\epsilon(f^{-1}(q), p')$ is vertical. The compactness of J allows us to find a positive lower bound α independent of $p \in J$. $\square$

LEMMA 2.5. – *There is a $\beta > 0$ so that if $\mathcal{G}^+_\epsilon(q,p)$ is vertical for some p with $d(q,p) < \beta$ then $\mathcal{G}^+_\epsilon(f^{-1}(q), p')$ is vertical for all p' with $d(f^{-1}(q), p') < \beta$.*

Proof. – Choose $\beta < \alpha/2$ such that $d(p,q) < \beta$ implies that $d(f^{-1}(p), f^{-1}(q)) < \alpha/2$. If $d(p,q) < \beta$ and $d(p', f^{-1}(q)) < \beta$ then $d(p', f^{-1}(p)) \le d(p', f^{-1}(q)) + d(f^{-1}(q), f^{-1}(p)) < \alpha$ so by Lemma 2.4, $\mathcal{G}^+_\epsilon(f^{-1}q, p')$ is vertical. $\square$

We will say that a point $q \in U^+$ is *good with respect to* $p \in J$ if either $d(p,q) > \beta$ or $\mathcal{G}^+_\epsilon(q,p)$ is vertical. We will say that a point q is *good* if it is good with respect to any $p \in J$ with $d(p,q) < \beta$. Lemma 2.5 says that if q is good with respect to some p then $f^{-1}(q)$ is good. Note that a point q can be good "vacuously" if its distance from J is greater than β. In this case $f^{-1}(q)$ need not be good. On the other hand if q is good and $d(q,J) < \beta$ then, by Lemma 2.5, $f^{-1}(q)$ is also good.

LEMMA 2.6. – *There is an N_1 such that every q with $0 < G^+(q) \le 1/d^{N_1}$ and $G^-(q) \le 1/d^{N_1}$ satisfies $d(q,J) < \beta$.*

Proof. – Let V be the set of points within distance β of J. The set $V \cup \operatorname{int} K^+$ is an open set containing the compact set K, and K is the set of common zeros of the functions G^+ and G^-; so there is some N_1 such that the set

$$\{q : G^+(q) \le 1/d^{N_1},\ G^-(q) \le 1/d^{N_1}\} \subset V \cup \operatorname{int} K^+.$$

If we remove the set $int\ K^+ = \{q : G^+(q) = 0\}$ from the left hand side of the equation, then we have

$$\{q : 0 < G^+(q) \leq 1/d^{N_1},\ G^-(q) \leq 1/d^{N_1}\} \subset V. \qquad \square$$

Proof of Proposition 2.2. – Let $S(N_1) = \{q \in J^- : 1/d^{N_1+1} \leq G^+(q) \leq 1/d^{N_1}\}$. This set is compact and serves as a fundamental domain for the action of f on J^-_+. Fix a point $q \in S(N_1)$. Since f is hyperbolic each point of J^-_+ is in the stable manifold of some point of J. Choose a p so that $q \in W^u(p)$. Since G^+ has no critical points on $W^u(p) \cap U^+$ the leaf $\mathcal{G}^+(q)$ is transverse to $W^u(p)$ at q. The Smooth Lambda Lemma implies that there is an n so that $\mathcal{G}^+(f^{-n}(q), f^{-n}(p))$ is vertical. If this condition holds for a fixed value of n, then Lemma 2.3 implies that it holds for all larger values of n. Choose an n_q large enough for $\mathcal{G}^+(f^{-n}(q), f^{-n}(p))$ to be vertical and $d(f^{-n}(q), f^{-n}(p)) < \beta$. In fact there is a neighborhood U_q of q consisting of points q' so that $\mathcal{G}^+(f^{-n}(q'), f^{-n}(p))$ is vertical and $d(f^{-n}(q'), f^{-n}(p)) < \beta$. The set $S(N_1)$ is compact, so we can cover it by a finite number of such open sets, say $U_{q_1}, \ldots, U_{q_k}$. Let U be the union of these neighborhoods, and let N_2 be the maximum of the n_{q_j}'s. Thus for each $q' \in U$ there is an $n \leq N_2$ and a $p \in J$ so that $\mathcal{G}^+(f^{-n}(q), f^{-n}(p))$ is vertical and $d(f^{-n}(q'), f^{-n}(p)) < \beta$. Choose N_3 so that $1/d^{N_1+1} \leq G^+(q) \leq 1/d^{N_1}$ and $G^-(q) < 1/d^{N_3}$ implies that $q \in U$.

We complete the proof by showing that any q satisfying $0 < G^+(q) \leq 1/d^{N_1+N_2+2}$ and $G^-(q) \leq 1/d^{N_3}$ is good. Assume that q satisfies the above conditions. We can write q as $f^{-n}(q')$ for a unique q' with $1/d^{N_1+1} < G^+(q') \leq 1/d^{N_1}$. Now

$$\frac{1}{d^{N_1+N_2+2}} \geq G^+(q) = G^+(f^{-n}(q')) = \frac{1}{d^n} G^+(q') \geq \frac{1}{d^{N_1+n+1}}$$

so that $n \geq N_2 + 1$. Since $G^-(q') = (1/d^n)G^-(q) \leq G^-(q) \leq 1/d^{N_3}$ we see that $q' \in U$. Since $q' \in U$ there is an integer ℓ and a point $p' \in J$ so that $f^{-\ell}(q')$ is good with respect to $f^{-\ell}(p')$ and $d(f^{-\ell}(q'), f^{-\ell}(p')) < \beta$. It follows from Lemma 2.5 that $f^{-(\ell+1)}(q')$ is good.

Now $0 \leq \ell \leq N_2$ and $n \geq N_2 + 1$, so $\ell + 1$ satisfies $0 \leq \ell + 1 \leq n$. We observe that for each m with $0 \leq m \leq n$ we have

$$G^+(f^{-m}(q')) \leq G^+(q') \leq \frac{1}{d^{N_1}}$$

and

$$G^-(f^{-m}(q')) \leq G^-(q') \leq \frac{1}{d^{N_3}} \leq \frac{1}{d^{N_1}}.$$

Thus by Lemma 2.6 $d(f^{-m}(q'), J) \leq \beta$.

In particular, $d(f^{-\ell+1}(q'), J) < \beta$. It follows that $f^{-(\ell+2)}(q')$ is good. By induction on the size of the exponent m for which $f^{-m}(q')$ is good we see that $q = f^{-n}(q')$ is good as was to be proved. $\square$

Let $\mathcal{W}^s$ be the lamination of J^+ by stable manifolds. Let $\mathcal{L}^s$ be the partition of $J^+ \cup U^+$ whose leaves are the leaves of $\mathcal{W}^s$ and $\mathcal{G}^+$.

PROPOSITION 2.7. – *If f is hyperbolic and unstably connected, then partition $\mathcal{L}^s$ is a (locally trivial) lamination.*

Remark. – In Corollary A2 in the Appendix, we show that under these hypotheses, there are points of J^-_+ where $\mathcal{L}^u$ is not locally trivial.

Proof. – We must show that $\mathcal{L}^s$ is locally trivial. For points $p \in U^+$ the local triviality of $\mathcal{L}^s$ is a consequence of the local triviality of the foliation $\mathcal{G}^+$. Let $p \in J^+$. Consider the restriction of the lamination $\mathcal{L}^s$ to the box $B_\epsilon(p)$. Let $V \subset B_\epsilon(p)$ be the union of leaves which intersect the δ-ball around p. The leaves of the restriction of $\mathcal{W}^s$ to V are local stable manifolds of the form $W^s_\epsilon(p', p)$. It follows from the remarks prior to Proposition 2.2 that these disks are vertical in $B_\epsilon(p)$. The leaves of the restriction of $\mathcal{G}^+$ to V are sets of the form $\mathcal{G}^+_\epsilon(p', p)$. It follows from Proposition 2.2 that these sets are vertical disks in $B_\epsilon(p)$. If we view these vertical disks as graphs of functions in the box $B_\epsilon(p)$ they constitute a holomorphic motion. It follows from the complex λ-Lemma (see [MSS]) that the restriction of $\mathcal{L}^s$ to V is homeomorphic to a product lamination. □

Remarks. – Related results, which give the conclusion of Proposition 2.7, have been obtained by Pixton [P] and Buzzard [B]. They work, however, under the additional hypothesis that the slice of $\mathcal{W}^u$ by a transversal is totally disconnected. In fact, it will be shown in §4 (using Proposition 2.7) that this slice property also holds for our mappings.

PROPOSITION 2.8. – *If $p \in J$ and $q \in U^+$ and $d(p,q) < \delta$ then $W^u_\epsilon(p) \cap \mathcal{G}^+_\epsilon(q,p)$ consists of precisely one point.*

Proof. – It follows from Proposition 2.2 that $\mathcal{G}^+_\epsilon(q,p)$ is vertical in $B_\epsilon(p)$. The constant ϵ was chosen so that $W^u_\epsilon(p)$ is horizontal in $B_\epsilon(p)$. It follows from topology that vertical and horizontal disks intersect in precisely one point. □

Choose a reference point $p \in J$. Let q and q' be points in U^+ so that q and q' are within distance δ of p. Assume also that $e(q)$ and $e(q')$ and the tails of external rays through q and q' lie completely within the ball of radius δ around p. For the next two propositions we will use the term "local leaves" to mean local with respect to the box $B_\epsilon(p)$.

PROPOSITION 2.9. – *If q and q' lie on the same local $\mathcal{G}^+$ leaf then $e(q)$ and $e(q')$ lie on the same local $\mathcal{W}^s$ leaf.*

Proof. – Let V be the union of local $\mathcal{L}^s$ leaves which intersect the δ-ball around p. We have an extension of the function φ^+ to the set $V \cap U^+$. φ^+ is constant on local leaves so $\varphi^+(q) = \varphi^+(q')$. The construction of external rays implies that for $t > 0$ $\varphi^+(e_t(q)) = \varphi^+(e_t(q'))$ so that $e_t(q)$ and $e_t(q')$ are on the same local leaf. Now $\mathcal{L}^s|V_p$ is a product lamination so $e(q) = \lim_{t\to 0} e_t(q)$ and $e(q') = \lim_{t\to 0} e_t(q')$ lie on the same local $\mathcal{W}^s$ leaf. □

PROPOSITION 2.10. – *Let p, q and q' be chosen as above. If q and q' lie on the same local $\mathcal{G}^+$ leaf and $e(q) = e(q')$ then $q = q'$.*

Proof. – The external ray through q, $e_t(q)$, lies on a single unstable manifold. Since the tail of this ray is contained in $B_\epsilon(p)$, the external ray lies on a single *local* unstable manifold. It follows that $e(q)$ lies on the same local unstable manifold. We conclude that $q \in W^u_\epsilon(e(q))$. Similarly $q' \in W^u_\epsilon(e(q'))$. Our hypothesis implies $W^u_\epsilon(e(q)) = W^u_\epsilon(e(q'))$ so that q and q' both lie on the same local unstable manifold. On the other hand q and q' both belong to the same local leaf of $\mathcal{G}^+$. It follows from Proposition 2.8 on the uniqueness of intersection points that $q = q'$. □

The following proposition describes one property of the landing map. We will collect other properties in §6.

PROPOSITION 2.11. – *If $x, y \in U^+$ are sufficiently close and $z \in U^+$ is sufficiently close to $e(x)$ then there is a unique w close to z in $\mathcal{G}^+_\epsilon(z)$ with $e(y) = e(w)$.*

Proof. – Define $w = W^u_\epsilon(e(y)) \cap \mathcal{G}^+_\epsilon(z)$. Then $e(w) \in W^u_\epsilon(e(y))$ and $e(w) \in \mathcal{G}^+_\epsilon(e(z)) = \mathcal{G}^+_\epsilon(e(x)) = \mathcal{G}^+_\epsilon(e(y))$. But $e(y) \in W^u_\epsilon(e(y)) \cap \mathcal{G}^+_\epsilon(e(y))$ so the uniqueness of the intersection implies that $e(y) = e(w)$. □

For a point $p \in J^-_+$, we let $R(p)$ denote the external ray containing p. We let $R_0(p)$ denote the bounded component of $(R(p) \cup \{e(p)\}) - \{p\}$.

LEMMA 2.12. – *For any $c > 0$, there is an $\epsilon > 0$ such that, for any $\delta > 0$, there exists $\eta > 0$ such that if $p_1, p_2 \in J^- \cap \{G^+ = c\}$, $\epsilon \geq \operatorname{dist}(p_1, p_2) \geq \delta$, and*

$$N_j = \bigcup_{q \in R_0(p_j)} B_\eta(q) \cap \mathcal{G}^+_\epsilon(q) \tag{2.3}$$

for $j = 1, 2$, then $N_1 \cap N_2 = \emptyset$.

Proof. – By Proposition 2.1, the mapping $J^-_+ \ni p \mapsto R_0(p)$ is continuous. Thus it is uniformly continuous on the set $\{G^+ = c\}$. Since e is locally injective (Proposition 2.10), it follows that $R_0(p_1) \cap R_0(p_2) = \emptyset$ if $p_1 \neq p_2$ and $\operatorname{dist}(p_1, p_2)) \leq \epsilon$. If $\epsilon \geq \operatorname{dist}(p_1, p_2) \geq \delta$, then $R_0(p_1)$ and $R_0(p_2)$ are uniformly separated by a distance $\eta > 0$. □

3. Topological Characterizations

In this Section we give two characterizations of unstable connectivity for mappings which are hyperbolic. The first condition involves slices by transversals, and the second is closely related to the function φ^+. In [BS6] it was shown that the property of being unstably connected is characterized by the property that slices of K^+ by unstable manifolds have only noncompact components. Here, in Theorems 3.1 and 3.6 we give conditions for unstable connectivity in terms of slices by more general transversals. Then in Theorem 3.4 we show that for hyperbolic mappings the property of unstable connectivity is characterized by the topology of the intersection of $\mathbf{C}^2 - K^+$ with large balls.

Let us define a *transversal* T to be the image of the closed unit disk under a continuous map $h : D \to \mathbf{C}^2$ which is holomorphic on the interior. Further, to avoid trivialities assume that T meets J^+ and that T is not contained in K^+.

THEOREM 3.1. – *Let f be hyperbolic and unstably disconnected. Then for any transversal T, $K^+ \cap T$ contains uncountably many point components in its interior.*

Proof. – We prove this first in a special case. Say $T = W^u_\epsilon(p)$ with $p \in J$. Let $\mathcal{T}^u$ denote the set of points $p \in J$ such that the component of p in $W^u(p) \cap K^+$ is exactly p. According to Theorem 7.1 of [BS6] for μ-almost every point p' in J the set $\mathcal{T}^u$ has full μ^+ measure in $W^u(p')$. The point p is in the support of μ so we can find a p' close to p with this property. The point p' is in the support of $\mu^+|W^u(p')$ so the set $W^u_\epsilon(p')$ has positive μ^+ measure. The measure μ^+ has no atoms so any set of positive measure is uncountable. Thus $K^+ \cap W^u_\epsilon(p')$ has uncountably many point components. Using the hyperbolic structure of f the set $W^u_\epsilon(p')$ is homeomorphic to a neighborhood of p in $W^u(p)$ by means of a homeomorphism that takes $K^+ \cap W^u_\epsilon(p')$ to $K^+ \cap W^u(p)$. Thus there are uncountably many point components of $K^+ \cap W^u(p)$ near p.

Now we consider a general transversal. Since T meets K^+, but also meets the complement of K^+, and since T is connected, T meets J^+. The set J^+ has a lamination by stable manifolds. For $q \in J^+ \cap T$ let L_q be the local stable manifold through q. Lemma 6.4 of [BLS] implies that the set of points q for which T meets L_q non-transversally is discrete. On the other hand, the potential theory of $G^+|T$ shows that $T \cap J^+$ has no isolated points, so we can find a q for which T meets L_q transversally. The condition that T meets L_q transversally is an open condition. Since stable manifolds of periodic saddle points are dense in J^+ we may find a periodic point p in J so that $W^s(p)$ meets T transversally at $q \in T$. For n sufficiently large the dynamical Lambda Lemma gives the existence of a disk D around q in T so that $f^n(D)$ is C^1-close to $W^u(p)$. In particular using the hyperbolicity of f there is a neighborhood of $f^n(q)$ in $f^n(D)$ which is homeomorphic to a neighborhood of p in $W^u(p)$ by a homeomorphism that preserves K^+. We conclude that $f^n(D) \cap K^+$ contains uncountably many point components. Since K^+ is invariant, $D \cap K^+$ contains uncountably many point components. □

In [BS6] it is shown that the property of unstable connectivity can be determined from the topology of the intersections of K^+ with the unstable manifolds of periodic points. According to Theorems 3.1 and 3.6, when f is hyperbolic we can determine the unstable connectivity of f from the topology of $T \cap K^+$ for any one-dimensional complex manifold T which meets K^+.

Now we discuss the analytic continuation of φ^+. Since $\log|\varphi^+| = G^+$ is pluriharmonic on V^+, we may continue φ^+ analytically along any path in U^+ starting in V^+. The problem of finding a continuation to a larger set is thus the question of whether this continuation is single-valued. In fact this is a topological problem. The connection between the cohomology class η and φ^+ comes from the fact that *if γ is a closed curve in U^+, then any analytic continuation $\tilde{\varphi}^+$ of φ^+ along γ satisfies*

$$\delta_\gamma \log \tilde{\varphi}^+ = 2\pi i \eta(\gamma), \tag{3.1}$$

where $\delta_\gamma \log \tilde{\varphi}^+$ denotes the difference between the values of $\log \tilde{\varphi}^+$ at the beginning and end of γ.

There is a relation between holomorphic extensions of φ^+ and homology. The form $(2\pi)^{-1} d^c G^+$ is closed on $U^+ = \mathbf{C}^2 - K^+$, so it defines a cohomology class $\eta \in \mathrm{H}^1(U^+; \mathbf{R})$. This class can be evaluated on $\gamma \in \mathrm{H}_1(U^+; \mathbf{Z})$. We have:

$$\eta(\gamma) = \frac{1}{2\pi} \int_\gamma d^c G^+.$$

If γ is a 1-cycle supported on U^+, then we may choose n sufficiently large so that $f^n\gamma \subset V^+$. If γ is an integral 1-cycle, $f^n\gamma$ is homologous in V^+ (and thus in U^+) to $k\,[\partial D]$, where $D = \{0\} \times \{|y| < 2R\}$. Clearly $k = 0$ if and only if $\gamma \sim 0$. Since $\int_{\partial D} d^c G^+ = 2\pi$, we have $\eta(f^n(\gamma)) = k$. Thus $\eta(\gamma) = kd^{-n}$, and so

$$\eta : \mathrm{H}_1(U^+; \mathbf{Z}) \to \mathbf{Z}[d^{-1}] \tag{3.2}$$

is an isomorphism.

This topological condition for analytic continuation may also be stated as follows: *Let $U \subset U^+$ be a connected open set. The equation*

$$\log|\psi| = G^+ \tag{3.3}$$

has a solution ψ which is holomorphic on U if and only if η is an integral class, i.e., $\eta \in \mathrm{H}^1(U;\mathbf{Z})$. Furthermore, any two solutions ψ_1 and ψ_2 satisfy $\psi_1 = c\psi_2$ where c is a complex constant with $|c| = 1$.

When f is unstably connected then [BS6, Theorem 6.3] shows that the function φ^+ has an analytic extension on a neighborhood of J_+^-. We say that φ^+ *extends uniformly* if there exists an open set U containing K such that φ^+ has an analytic continuation to $U - K^+$. We note that by the functional equation, $\varphi^+ = (\varphi^+ \circ f^{-n})^{d^n}$. Thus, if φ^+ extends uniformly, then it extends to $\bigcup_{n\geq 0} f^n U - K^+$, which is a neighborhood of J_+^-.

PROPOSITION 3.2. – *If f is hyperbolic and unstably connected, then φ^+ extends uniformly.*

Proof. – Let us choose $\epsilon > 0$ small enough to have good coordinate boxes $B_\epsilon(p)$ as in §2, and let $c > 0$ be small enough for $\{G^+ = c\} \cap J^-$ to be contained in an ϵ-neighborhood of J. Since f is unstably connected, φ^+ has an analytic continuation to a neighborhood of J_+^-. And since φ^+ is locally constant on the leaves of $\mathcal{G}^+$, there is a $\delta > 0$ such that for all $p \in J^- \cap \{G^+ = c\}$, φ^+ is constant on $\mathcal{G}_\delta^+(p)$, i.e. on the component of $B_\delta(p) \cap \mathcal{G}^+$ containing p. Now choose $\eta < \delta$ according to Lemma 2.12, and let

$$\mathcal{N} := \bigcup_{p \in J^- \cap \{G^+ = c\}} \; \bigcup_{q \in R_0(p)} B_\eta(q) \cap \mathcal{G}_\epsilon^+(q).$$

We claim that we may extend φ^+ to $\mathcal{N}$ by making it constant on the sets $B_\eta(q) \cap \mathcal{G}_\epsilon^+(q)$.

For this definition to be consistent, we need to show that if $p_1, p_2 \in J^- \cap \{G^+ = c\}$, and if $\mathcal{G}_\eta^+(q_1) \cap \mathcal{G}_\eta^+(q_2) \neq \emptyset$ for some $q_j \in R_0(p_j)$, $j = 1,2$, then $\varphi^+(q_1) = \varphi^+(q_2)$. The function defined this way will be holomorphic, since it is an extension of φ^+, and it is clear that it contains an η-neighborhood of $J^- \cap \{G^+ \leq c\}$ inside $\mathbf{C}^2 - K^+$.

Let us fix $p_1, p_2 \in J^- \cap \{G^+ = c\}$, and points $q_j \in R_0(p_j)$, $j = 1,2$, with $\mathcal{G}_\eta^+(q_1) \cap \mathcal{G}_\eta^+(q_2) \neq \emptyset$. Now each ray $R_0(p_j)$, $j = 1,2$, is contained in a local unstable manifold $W_\epsilon^u(r_j)$. All of the local unstable manifolds near $W_\epsilon^u(r_1)$ are locally graphs over it. Since the intersection above is nonempty, $W_\epsilon^u(r_1)$ and $W_\epsilon^u(r_2)$ come within η of each other over some point of $W_\epsilon^u(r_1) \cap \{G^+ \leq c\}$. We apply the Lambda Lemma [MSS] to conclude that if η is sufficiently small, then the portion of the unstable manifold $W_\epsilon^u(r_2)$ lying over $W_\epsilon^u(r_1) \cap \{G^+ \leq c\}$ is in fact contained within a δ-neighborhood. It follows that $R_0(p_2)$ is contained in $\bigcup_{q \in R_0(p_1)} B_\delta(q) \cap \mathcal{G}_\epsilon^+(q)$. By the choice of δ, it follows that $\varphi^+(p_1) = \varphi^+(p_2)$. Finally, since φ^+ is a covering, and since the rays are defined by lifting radial lines, it follows that $\varphi^+(q_j) = \varphi^+(p_j)e^{G^+(q_j) - G^+(p_j)}$. Since the function G^+ is constant on overlapping disks, the values of φ^+ agree on the overlapping disks, and the extension of φ^+ is well defined. □

The condition that φ^+ extends uniformly seems to be an interesting condition in its own right. For the rest of this section we will explore some of its consequences. In particular we will not assume that f is hyperbolic.

THEOREM 3.3. – *The following are equivalent:*

(1) *φ^+ extends uniformly.*

(2) *There exists a neighborhood U of K such that $(\varphi^+)^{d^n}$ extends analytically to $U - K^+$ for some n.*

Proof. – It is obvious that (1) $\Rightarrow$ (2). For the converse, we not that if U is a neighborhood of K, then so is $f^n U$. If ψ is an analytic continuation of $(\varphi^+)^{d^n}$ along a path γ from V^+

to U, then $\psi \circ f^{-n}$ has an analytic continuation along $f^n\gamma$ from f^nV^+ to f^nU. It follows from the functional equation that $\psi \circ f^{-n} = \epsilon\varphi^+$ on f^nV^+, where ϵ is a d^nth root of unity. Thus $\epsilon^{-1}\psi \circ f^n$ is an analytic continuation of φ^+ along the path $f^n\gamma$, so (2) $\Rightarrow$ (1). □

For a complex disk D in $\mathbf{C}^2$ with smooth boundary and $\partial D \cap K^+ = \emptyset$, we may define the degree of the boundary curve about K^+ as

$$\deg(\partial D, K^+) = \frac{1}{2\pi}\int_{\partial D} d^cG^+.$$

If $D = \{x = x_0, |y| = R\}$ for R large, then $\deg(\partial D, K^+) = 1$, and in general, the degree measures the extent to which the curve ∂D goes around K^+, as measured by μ^+. We can also interpret this quantity as the total mass of the restriction of μ^+ to D.

THEOREM 3.4. – *Let f be given. Then the following are equivalent:*

(1) *φ^+ extends uniformly.*

(2) *For any bounded open set B, the image of the inclusion map*

$$i_* : \mathrm{H}_1(B - K^+; \mathbf{Z}) \to \mathrm{H}_1(\mathbf{C}^2 - K^+; \mathbf{Z})$$

is finitely generated.

(3) *For any bounded set $B \subset \mathbf{C}^2$ there is a constant C such that for any complex disk $D \subset B$ with $\partial D \cap K^+ = \emptyset$,*

$$\dim \mathrm{H}_1(D - K^+; \mathbf{R}) \le C\deg(\partial D, K^+).$$

Note that $\dim \mathrm{H}_1(D - K^+; \mathbf{R})$ is just the number of components of $K^+ \cap D$ which do not meet the boundary of D.

(4) *For any holomorphic mapping $h : \Delta^2 \to \mathbf{C}^2$ of the unit polydisk Δ^2 and any $\epsilon > 0$, there exists $\delta > 0$ such that if $|x| < 1 - \epsilon$ and $\omega \subset \{|y| < 1\}$ is a relatively compact domain with $h(\{y\} \times \omega) \cap K^+ \ne \emptyset$ and $h(\{y\} \times \partial\omega) \cap K^+ = \emptyset$, then*

$$\frac{1}{2\pi}\int_{h(\{x\}\times\partial\omega)} d^cG^+ \ge \delta.$$

(5) *For $R_1, R_2 > 0$ such that $K \subset \{|x| < R_1, |y| < R_2\}$ and $K^+ \cap \{|x| \le R_1, |y| = R_2\} = \emptyset$, there is a $\delta > 0$ such that for each disk $\omega \subset \{x = x_0, |y| < R_2\}$ with $\partial\omega \cap K^+ = \emptyset$*

$$\deg(\partial\omega, K^+) \ge \delta.$$

Proof. – Let us suppose that φ^+ extends analytically to $\{G^- < c\} - K^+$. Let n be such that $cd^n > \max_B G^-$. Thus $\varphi^+ \circ f^n$ is analytic on $\{G^- < cd^n\} - K^+ \supset B - K^+$. It follows that, in the notation of (3.1),

$$\delta_\gamma(\log\varphi^+ \circ f^n) = d^n\delta_\gamma \log\varphi^+ \in 2\pi\mathbf{Z}$$

for any path $\gamma \subset B - K^+$. Using the isomorphism (3.2), we see that $\eta i_*\mathrm{H}_1(B - K^+; \mathbf{Z}) \subset d^{-n}\mathbf{Z}$, and so the image of the homology group is generated by d^{-n}. Thus (1) $\Rightarrow$ (2).

Now let us suppose that (2) holds so that the image homology group in (2) is given by $d^{-n}\mathbf{Z}$ under the isomorphism (3.2). Let D be a complex disk in $\mathbf{C}^2$ with $\partial D \cap K^+ = \emptyset$. For $\epsilon := \min_{\partial D} G^+ > 0$, the components $S_1, \ldots, S_r$ of $D \cap \{G^+ < \epsilon\}$ are relatively compact, and we denote $\gamma_j = \partial S_j$. Thus there is an integer $k \geq 1$ such that

$$\eta(\gamma_j) = kd^{-n} = \frac{1}{2\pi} \int_{S_j} dd^c G^+.$$

It follows that

$$r \leq \frac{d^n}{2\pi} \int_{\gamma_1 + \ldots + \gamma_r} d^c G^+ = \frac{d^n}{2\pi} \int_{S_1 \cup \ldots \cup S_r} dd^c G^+ \leq \frac{d^n}{2\pi} \int_D dd^c G^+ \leq d^n \deg(\partial D, K^+)$$

Since $r = \dim \mathrm{H}_1(D - K^+; \mathbf{R})$, we have (2) $\Rightarrow$ (3).

For (3) $\Rightarrow$ (4), we may assume without loss of generality that $B = h(\Delta^2)$ is bounded, and let C denote the constant in condition (3). Let $g^+(y) := G^+(h(x_0, y))$ be the function G^+ pulled back to the disk $\{x_0\} \times \Delta$. Since $g \geq 0$ and $\{g^+ = 0\} = \Delta \cap h^{-1}K^+$, the homology group $\mathrm{H}_1(h(\{x_0\} \times \Delta) - K^+; \mathbf{Z})$ is generated by the boundaries of the relatively compact components of $\{g^+ < \epsilon\}$. By the maximum principle, these components are disks. Let ω denote such a disk. By (3) $\mathrm{H}_1(\omega - K^+; \mathbf{R})$ is finite dimensional, so taking ϵ sufficiently small, we have $\dim \mathrm{H}_1(\omega - K^+; \mathbf{R}) = 1$, and so by condition (3), we may take $\delta = C^{-1}$.

The implication (4) $\Rightarrow$ (5) is clear; it remains therefore to prove (5) $\Rightarrow$ (1). Let us choose m such that $f^m\{|x| \leq R_1, |y| = R_2\} \subset V^+$, and choose n such that $d^{-n} \leq \delta$. Let $\mathcal{H}$ denote the subgroup of $\mathrm{H}_1(\mathbf{C}^2 - K^+; \mathbf{Z})$ generated by integral 1-cycles in $\{x = x_0, |y| < R_2\} - K^+$ for $|x_0| < R_1$. By condition (4) and the isomorphism (3.2), we have $\mathcal{H} \subset d^{-n}\mathbf{Z}$. For $\gamma \in \mathrm{H}_1(f^m\{x = x_0, |y| < R_2\} - K^+; \mathbf{Z})$, we have $f_*^{-m}\gamma \in \mathcal{H}$, so $\eta(\gamma) \in d^{-n-m}\mathbf{Z}$. It follows from the topological condition (3.3) for extension that $(\varphi^+)^{d^{n+m}} = \varphi^+ \circ f^{n+m}$ can be analytically from V^+ along the Riemann surface $f^m\{x = x_0, |y| < R_2\}$ for each $|x_0| < R_1$. It follows, then, that $(\varphi^+)^{d^{n+m}}$ has an analytic continuation to $f^m\{|x| < R_1, |y| < R_2\}$. By Theorem 3.2, φ^+ extends uniformly. □

Remark. – If f is hyperbolic, then if follows from Theorem 3.1 that the equivalent conditions of Theorem 3.4 are equivalent to f being unstably connected. In connection with condition (2), we recall that Hubbard and Oberste-Vorth [HO1] have shown that the topological type of $\mathbf{C}^2 - K^+$ is the same for all mappings f of the same degree. By Theorem 3.4, the bounded part of the topology of $\mathbf{C}^2 - K^+$ determines whether φ^+ extends uniformly. The extendability of φ^+ can change as parameter values change.

THEOREM 3.5. – *Let f be hyperbolic, let T be a transversal, and let $T_0 \subset T$ be relatively compact. Then the number of components of $T \cap K^+$ that are contained in T_0 is finite or uncountable. f is unstably connected in the first case and unstably disconnected in the second case.*

Proof. – If f is unstably disconnected then by Theorem 3.1, $K^+ \cap \operatorname{int} T$ contains uncountably many point components. If f is unstably connected then by Proposition 3.2 the function φ^+ extends uniformly so by Theorem 3.4 (3) there are only finitely many components of K^+ that do not meet the boundary of T. □

4. Conjugacy with the Solenoid

In this section we relate the dynamics of f on J_+^- to the complex solenoid when f is unstably connected.

THEOREM 4.1. – *Let f be hyperbolic and unstably connected. There is a homeomorphism $\Psi : \Sigma_+ \to J_+^-$ which is holomorphic on leaves and yields a conjugacy between the action of f on J_+^- and the action of the shift map on Σ_+.*

COROLLARY 4.2. – *If f is hyperbolic and unstably connected, the space of external rays $\mathcal{E}$ is topologically conjugate to the solenoid Σ_0.*

For general hyperbolic unstably connected diffeomorphisms we do not know whether the semiconjugacy Φ defined in §3 of [BS6] is in fact a homeomorphism. When Φ is a homeomorphism, then Ψ of Theorem 4.1 coincides with Φ^{-1}.

We will use the map $\Theta_k : \Sigma \to \Sigma$, $[\Theta_k s]_n = (s_n)^k$ which raises each coordinate to the k-th power. If k and d are relatively prime then Θ_k is an finite covering that commutes with the shift. The following result contains Theorem 4.1.

THEOREM 4.3. – *If f is hyperbolic and unstably connected, then*

(1) *$\Phi : J_+^- \to \Sigma_+$ is a covering map of degree k, with $(k, d) = 1$.*

(2) *There exists a homeomorphism $\Psi : \Sigma_+ \to J_+^-$ which is holomorphic on leaves and which has the following properties*

  (*a*) $\Phi \circ \Psi = m_s \circ \Theta_k$ for some $s \in \Sigma_0$*;*

  (*b*) $\Psi \circ \sigma = f \circ \Psi$.

The first part of this section will be devoted to proving that the Φ is a covering map. The rest of the section is devoted to showing the existence of the function Ψ of Theorem 4.3.

PROPOSITION 4.4. – *The mapping Φ is locally injective.*

Proof. – For p and q in J_+^- we say $p \sim q$ if $\Phi(p) = \Phi(q)$. We wish to prove that if p and q are sufficiently close, and if $p \sim q$, then $p = q$. To show this, we define a second equivalence relation. If p and q are points in J_+^-, we say $p \approx q$ if $p \sim q$ and $e(p)$ and $e(q)$ are in the same stable manifold. Suppose that p and q are close and $p \sim q$. Since $\Phi(p) = \Phi(q)$ we have $\varphi^+(p) = \varphi^+(q)$ so p and q are on the same local $\mathcal{G}^+$ leaf. Now by Proposition 2.9 $e(p)$ and $e(q)$ lie on the same stable manifold so $p \approx q$. Thus to prove Proposition 4.4, it suffices to show that if p and q are sufficiently close, and if $p \approx q$, then $p = q$.

The set of points q such that $q \sim p$ is closed because it is the inverse image of a point under the continuous map Φ. It is also bounded because φ^+ is constant on the equivalence class. Thus it is compact. We can partition this set into $\approx$ equivalence classes. Let C_p denote the set of q such that $q \approx p$. By Proposition 2.9 each of these sets C_p is open in the corresponding $\sim$ equivalence class. In particular there are only finitely many such classes. It follows that each class is also closed and hence compact. Since f contracts stable manifolds we can find an $n \geq 0$ such that $f^n(e(C_p))$ is contained in a single box B_ϵ. In fact for p' sufficiently close to p, $f^n(e(C_{p'}))$ will be contained in B_ϵ. It follows from the compactness of $\mathcal{E}$ that the diameters of the images of equivalence classes (measured with respect to the standard metric on $\mathbf{C}^2$) is bounded above. Let M denote the supremum of these diameters. Now f^n decreases Euclidean distances between points in stable manifolds by a factor of at least c/λ^n where $\lambda > 1$. Thus

$$\operatorname{diam}(f^n(e(C_p))) \leq \frac{c}{\lambda^n}\operatorname{diam}(e(C_p)).$$

On the other hand the equivalence relation $\approx$ is invariant under f so that $f^n(e(C_p)) = e(C_{f^n p})$ so that:

$$\begin{aligned} M &= \sup_p \operatorname{diam}(e(C_p)) \\ &= \sup_p \operatorname{diam}(e(C_{f^n p})) \\ &= \sup_p \operatorname{diam}(f^n(e(C_p))) \\ &\leq \sup_p \frac{c}{\lambda^n}\operatorname{diam}(e(C_p)) = \frac{c}{\lambda^n} M \end{aligned}$$

Choosing n large enough so that $c/\lambda^n < 1$ we see that the only possibility is that $M = 0$, so the image of each equivalence class under e consists of a single point. Now e is locally injective on equivalence classes, by Proposition 2.10, so the $\approx$ equivalence classes are discrete. □

For any $\zeta \in \mathbf{C}$ with $|\zeta| > 1$, the slice $\pi^{-1}\{\zeta\} \subset \Sigma_+$ is a Cantor set, and all slices are homeomorphic via the holonomy map. Fix ζ_0 with $|\zeta_0| > 1$. Let $S_0 = \pi^{-1}\{\zeta_0\} \subset \Sigma_+$ consisting of points with 0-th coordinate equal ζ_0. Let $T_0 := \Phi^{-1}(S_0) \subset J_+^-$ so that T_0 is also equal to $(\varphi^+)^{-1}(\zeta_0)$. Let

$$\Phi_0 : T_0 \to S_0$$

be the restriction of Φ to T_0.

The map $\exp(2\pi i)$ is a homeomorphism of Σ which preserves S_0. Let $\chi : S_0 \to S_0$ be the restriction map. The corresponding homeomorphism of J_+^-, also denoted by $\exp(2\pi i)$, stablizes T_0 and we denote the restriction to T_0 as $\tilde{\chi}$. The space S_0 together with the action of χ is a well known dynamical system called the d-adic adding machine. The action of χ on S_0 is minimal, which means that the only closed invariant sets are the empty set and all of S_0.

PROPOSITION 4.5. – *The map Φ_0 is a covering map and the covering degree is a finite constant m.*

Proof. – Recall that Φ_0 is a local homeomorphism at p if there is a neighborhood U of p in T_0 so that the restriction of Φ_0 to U is a homeomorphism onto an open set in S_0. Since Φ_0 is locally injective, to show that Φ_0 is a local homeomorphism at a point p it suffices to check the local homeomorphism property on a neighborhood V of p on which Φ_0 is injective.

The set of points p where Φ_0 is a local homeomorphism is open. It is also invariant under $\tilde{\chi}$, where $\tilde{\chi}$ is the mapping induced on J_+^-. To see this note that if $\Phi_0 = \chi \circ \Phi_0 \circ \tilde{\chi}^{-1}$ is a local homeomorphism at p, then since χ and $\tilde{\chi}$ are homeomorphisms, Φ_0 is a local homeomorphism at $\tilde{\chi}(p)$.

Let T' be the subset of T_0 on which Φ_0 is *not* a local homeomorphism. The set T' is closed and invariant. Let S' be the set $\Phi_0(T')$. The set S' is closed and invariant.

We will show that S' is a proper subset of S_0. Cover T_0 by compact sets $K_1, \dots, K_n$ on which Φ_0 is injective. Then $S' = \bigcup_{j=1}^{n} \Phi_0(T' \cap K_j)$. Each set $\Phi_0(T' \cap K_j)$ is closed. Since Φ_0 is not a local homeomorphism on T' the set $\Phi_0(T' \cap K_j)$ does not contain a neighborhood of any of its points; thus $\Phi_0(T' \cap K_j)$ is nowhere dense. Since S' is a finite union of closed nowhere dense sets it is meager in the sense of Baire category. Now S_0 is a compact metric space, so by the Baire category theorem, S' is not all of S_0.

Since S' is a proper closed χ-invariant subset of S_0, and χ is minimal, we conclude that $S' = \emptyset$. Thus T' is empty and Φ_0 is a local homeomorphism at every point. The map Φ_0 is proper and a local homeomorphism, so it is a covering map.

Since T_0 is compact, the covering degree at each point is finite. Viewed as a function of p the covering degree is continuous. Since this covering degree is invariant under χ it is constant. □

Proof of Theorem 4.3, *part* (1). – The fact that Φ is a covering map follows from the previous proposition and the fact that it is a bijection when restricted to leaves. Let $U \subset \mathbf{C} - \overline{\Delta}$ be simply connected and contain ζ_0. Then $(\varphi^+)^{-1}(U)$ has the structure of a product $S_0 \times U$. If $V \subset S_0$ is evenly covered under Φ_0, then the set $V \times U$ is evenly covered by Φ. The space Σ_+ is covered by open sets of this form. □

If the space Σ_+ were topologically well behaved it would be possible to appeal to the Galois correspondence between covering spaces and subgroups of the fundamental group to analyze topological type of J_+^-. In fact Σ_+ is neither locally connected nor path connected. Nevertheless we will analyze the possible finite degree covering spaces of Σ_+. We will see in fact that Σ_+ has the property that every connected finite degree covering space is homeomorphic to Σ_+. In this respect Σ_+ is like the circle.

For $n \geq 0$ we let $\mathcal{P}^n$ be the partition of S_0 into classes so that s and t are in the same class if $s_{-n} = t_{-n}$. We note that $\mathcal{P}^n$ consists of the d^n sets of the form $\{s : s_{-n} = \zeta_0^{1/d^n} e^{2\pi i j/d^n}\}$ for $1 \leq j \leq d^n$.

LEMMA 4.6. – *If P denotes a class in the partition $\mathcal{P}^n$ of S_0 then the following are equivalent:*

(1) $\exp(2\pi i r)(P) \cap P \neq \emptyset$.

(2) $\exp(2\pi i r)(P) = P$.

(3) $r \in d^n\mathbf{Z}$.

Proof. – If (1) holds there is some point $s \in P$ so that $\exp(2\pi i r)s \in P$. The $\mathcal{P}$ equivalence class of a point is determined by the $(-n)$-th coordinate of the point, so $s \in P$ and $\exp(2\pi i r)s \in P$ have the same $(-n)$-th component. Now $\exp(2\pi i r)$ acts on the $(-n)$-th coordinate of points in the solenoid by multiplication by $e^{2\pi i r/d^n}$. We conclude that $e^{2\pi i r/d^n} = 1$. So $r \in d^n\mathbf{Z}$. Conversely if $r \in d^n\mathbf{Z}$ then (2) and (3) follow. □

Let $\tilde{\mathcal{P}}^n$ be the pullback of the partition $\mathcal{P}^n$ to T_0. That is to say two elements of T_0 are in the same $\tilde{\mathcal{P}}^n$ class if their images under Φ_0 are in the same $\mathcal{P}^n$ class.

To say that Φ_0 is a covering map of degree m says that for any $p \in \Sigma_0$ there is a neighborhood U_p such that $\Phi_0^{-1}(U_p)$ consists of m sets $V_1, \dots, V_m$, each of which maps homeomorphically to U_p. If the base space were locally connected the set U_p could be chosen to be connected and the sets $V_1, \ldots, V_m$ would be just the connected components of $\Phi_0^{-1}(U_p)$. In this case the decomposition of $\Phi_0^{-1}(U_p)$ into sets $V_1, \ldots, V_m$ is unique. Since S_0 is not locally connected there is some choice of the partition of $\Phi_0^{-1}(U_p)$ into sets which map homeomorphically to U_p. The next Lemma gives a specific choice about how to divide $\Phi_0^{-1}(P)$ into subsets mapping homeomorphically to P when P is an equivalence class in some partition $\mathcal{P}^n$.

LEMMA 4.7. – *For some value of n there is a partition $\tilde{\mathcal{Q}}^n$ of T_0 such that:*

(1) *Each class $\tilde{P}$ of $\tilde{\mathcal{P}}^n$ is the union of m classes $\tilde{Q}_1, \ldots, \tilde{Q}_m$ of $\tilde{\mathcal{Q}}^n$.*

(2) *Each $\tilde{\mathcal{Q}}^n$ equivalence class maps bijectively to a $\mathcal{P}^n$ equivalence class under Φ_0.*

(3) *$\tilde{\chi}$ takes $\tilde{\mathcal{Q}}^n$ equivalence classes to $\tilde{\mathcal{Q}}^n$ equivalence classes.*

Proof. – Let D be the minimum distance between points p and p' such that $\Phi_0(p) = \Phi_0(p')$. Choose $\delta < D/3$ so that if $d(p,p') < \delta$ then $d(\tilde{\chi}(p), \tilde{\chi}(p')) < D/3$. Choose n sufficiently large so that (a) each $\mathcal{P}^n$ equivalence class P in S_0 is evenly covered by m homeomorphic sets $Q_1, \ldots, Q_m$ and (b) each Q_j has diameter less than δ. Let $\tilde{\mathcal{Q}}^n$ be the partition of T_0 into the md^n sets Q_j.

Assertions (1) and (2) are true by construction. We will show (3). Consider a $\tilde{\mathcal{Q}}^n$ equivalence class $\tilde{Q}$. $\Phi_0(\tilde{Q})$ is a $\mathcal{P}^n$ equivalence class which we call P. If p and q are points in $\tilde{Q}$ and $\tilde{\chi}(p)$ and $\tilde{\chi}(q)$ lie in distinct $\tilde{\mathcal{Q}}^n$ equivalence classes then there is a point p', in the same $\tilde{\mathcal{Q}}^n$ equivalence class as $\tilde{\chi}(q)$, having the property that $\Phi_0(\tilde{\chi}(p)) = \Phi_0(p')$. Now $d(\tilde{\chi}(p), \tilde{\chi}(q)) \geq d(\tilde{\chi}(p), p') - d(p', \tilde{\chi}(q)) \geq D - \delta \geq (2/3)D$. On the other hand since $d(p,q) < \delta$ we have $d(\tilde{\chi}(p), \tilde{\chi}(q)) < D/3$. We conclude from this contradiction that $\tilde{\chi}(p)$ and $\tilde{\chi}(q)$ are in the same $\tilde{\mathcal{Q}}^n$ equivalence class. □

We fix this value of n for the rest of the section, and we write $\mathcal{P}^n$, $\tilde{\mathcal{P}}^n$ and $\tilde{\mathcal{Q}}^n$ simply as $\mathcal{P}$, $\tilde{\mathcal{P}}$, and $\tilde{\mathcal{Q}}$. Now χ cyclically permutes the $\mathcal{P}$ equivalence classes, and χ^{d^n} stabilizes each $\mathcal{P}$ equivalence class. So $\tilde{\chi}^{d^n}$ stabilizes each $\tilde{\mathcal{P}}$ equivalence classes. Each $\tilde{\mathcal{P}}$ equivalence class $\tilde{P}$ consists of m $\tilde{\mathcal{Q}}$ equivalence classes $\tilde{Q}_1, \ldots, \tilde{Q}_m$, which are permuted by $\tilde{\chi}^{d^n}$.

LEMMA 4.8. – *The action of $\tilde{\chi}^{d^n}$ on $\tilde{P} = \tilde{Q}_1 \cup \ldots \cup \tilde{Q}_m$ is minimal.*

Proof. – It follows from [BS6, Theorem 2.1 (5)] that each leaf of the unstable lamination of J_+^- is dense in J_+^- (That is to say the unstable lamination is minimal). Say $p \in J_+^-$ then the intersection of the leaf through p with T_0 is the $\tilde{\chi}$ orbit of p. Since each leaf is dense for any point $q \in \tilde{P}$ there is a sequence of points $p_j \to q$ with p_j in the leaf of p. The local triviality of the lamination implies that there is a sequence $p_j' \to q$ with $p_j \in \tilde{P}$. Thus every $\tilde{\chi}^{d^n}$ orbit is dense. □

COROLLARY 4.9. – *$\tilde{\chi}^{d^n}$ cyclically permutes the $\tilde{\mathcal{Q}}$ equivalence classes $\tilde{Q}_1, \ldots, \tilde{Q}_m$ which compose $\tilde{P}$. The map $\tilde{\chi}^{md^n}|\tilde{Q}_1$ is minimal.*

Proof. – The equivalence classes $\tilde{Q}_1, \ldots, \tilde{Q}_m$ are disjoint closed and open sets which are permuted by $\tilde{\chi}^{d^n}$. In order for $\tilde{\chi}^{d^n}$ to act minimally it must permute the equivalence classes transitively. This proves the first assertion of the corollary. It follows that $\tilde{\chi}^{md^n}$ stabilizes each of these classes. If $p \in \tilde{Q}_1$ then the intersection of the $\tilde{\chi}^{d^n}$ orbit of p with $\tilde{Q}_1$ is just the $\tilde{\chi}^{md^n}|\tilde{Q}_1$ orbit of p. It follows that the $\tilde{\chi}^{md^n}|\tilde{Q}_1$ orbit of p is dense in $\tilde{Q}_1$ for every $p \in \tilde{Q}_1$. So $\tilde{\chi}^{md^n}|\tilde{Q}_1$ is minimal. □

COROLLARY 4.10. – *The covering degree m of the map Φ_0 is relatively prime to d.*

Proof. – The map Φ_0 gives a conjugacy between the action of $\tilde{\chi}^{md^n}|\tilde{Q}_1$ and the action of χ^{md^n} on P. Now χ^{md^n} acts on the $-(n+1)$-st coordinate of a point in P by multiplication by ζ^m where $\zeta = e^{2\pi i/d}$ is a primitive d-th root of unity. As in the previous Corollary the minimality of $\tilde{\chi}^m|\tilde{Q}_1$ forces this action to be transitive. In order for multiplication by ζ^m to transitively permute the d-th roots of unity the element ζ^m must have order d. If $s = (m, d) > 1$ then $\zeta^{m(d/s)} = 1$, so the order of ζ^m divides d/s and hence is strictly less than d. □

LEMMA 4.11. – *If $\tilde{Q}$ denotes a class in the partition $\tilde{\mathcal{Q}}$ of T_0, the following are equivalent:*
(1) $\exp(2\pi i r)(\tilde{Q}) \cap \tilde{Q} \neq \emptyset$, (2) $\exp(2\pi i r)(\tilde{Q}) = \tilde{Q}$, (3) $r \in md^n\mathbf{Z}$.

Proof. – If $\exp(2\pi i r)(Q) \cap Q \neq \emptyset$ then $\exp(2\pi i r)(P) \cap P \neq \emptyset$, so by Lemma 4.7, $r \in d^n\mathbf{Z}$. Write r as ad^n for $a \in \mathbf{Z}$. Now the map $\exp(2\pi i d^n)|\tilde{P} = \tilde{\chi}^{d^n}$, so $\exp(2\pi i r) = \exp(2\pi i a d^n) = \tilde{\chi}^{ad^n}$. By Corollary 4.9, $\tilde{\chi}^{d^n}$ transitively permutes the $\tilde{\mathcal{Q}}$ classes $\tilde{Q}_1, \ldots, \tilde{Q}_m$. If some power of $\tilde{\chi}^{d^n}$ fixes an equivalence class, then that power must be a multiple of m. We conclude that a is a multiple of m. So $r \in md^n\mathbf{Z}$. Conversely if $r \in md^n\mathbf{Z}$ then (1) and (2) hold. □

LEMMA 4.12. – *Let ℓ be relatively prime to d. Let t be an element of Σ_+. Fix an index k. Let s_k be a complex number satisfying $s_k^\ell = t_k$. Then there is a unique element $s \in \Sigma_+$ with k-th coordinate equal to s_k such that $\Theta_\ell(s) = t$.*

Proof. – For $j > k$ define $s_j = s_k^{d^{j-k}}$. Since $t_j = t_k^{d^{j-k}}$ it follows that $s_j^\ell = t_j$. Now assume that s_j is defined. In order to define s_{j-1} we need to find a simultaneous solution to $s_{j-1}^d = s_j$ and $s_{j-1}^\ell = t_{j-1}$, given that $s_j^\ell = t_j = t_{j-1}^d$.

Choose a root w of the equation $Z^{d\ell} = t_j$. If ζ is a $d\ell$-th root of unity then ζw is another solution of $Z^{d\ell} = t_j$. In this way we can identify the set of solutions of the equation $Z^{d\ell} = t_j$ with the group of $d\ell$-th roots of unity. Now ζ^d is a solution of the equation $Z^\ell = t_j$. Using this particular solution we can identify the set of solutions to $Z^\ell = t_j$ with the group of ℓ-th roots of unity.

Similarly ζ^ℓ is a solution of $Z^d = t_j$. Using this particular solution we can identify the set of solutions of $Z^d = t_j$ with the group of d-th roots of unity.

To solve our original equations we need to find a $d\ell$-th root of unity with appropriate images under the d-th power map and the ℓ-th power map. If we write our groups additively then the fact that our original equation can be solved uniquely is now equivalent to the fact that the map from $\mathbf{Z}_{d\ell} \to \mathbf{Z}_d \oplus \mathbf{Z}_\ell$ whose first coordinate is multiplication by ℓ and whose second coordinate is multiplication by d is a bijection when d and ℓ are relatively prime. This assertion is the Chinese Remainder Theorem from elementary number theory. □

Let Σ'_+ be a copy of the solenoid Σ_+, and define $\exp' : \Sigma'_+ \to \Sigma'_+$ for $s \in \Sigma'_+$ by $\exp'(it) = \exp(it/m)$ so that $[\exp'(2\pi i r)(s)]_n = e^{2\pi i r d^n/m} s_n$. It follows that

$$\Theta_m \circ \exp' = \exp \circ \Theta_m.$$

Let $T'_0 \subset \Sigma'_+$ be the set $(\Theta_m)^{-1}(S_0) = \{s : (s_0)^m = \zeta_0\}$. Let $\Theta'_m : T'_0 \to S_0$ be the restriction of Θ_m to T'_0. Let $\mathcal{P}'$ be the partition of T'_0 which is the pullback of the partition $\mathcal{P}$ of S_0. Thus two points are in the same equivalence class of the partition $\mathcal{P}'$ if their n-th coordinates have equal m-th powers. Let $\mathcal{Q}'$ be the partition of T_0' such that two points are in the same equivalence class if their n-th coordinates are equal.

LEMMA 4.13. – *Each equivalence class P' of $\mathcal{P}'$ may be partitioned into m subclasses $Q_1', \ldots, Q_m'$ under $\mathcal{Q}'$ such that Θ_m' is bijective on each Q_j'.*

Proof. – This is similar to Lemma 4.7. □

LEMMA 4.14. – *If Q' denotes a class in the partition $\mathcal{Q}'$ of T_0' then the following are equivalent:*

(1) $\exp'(2\pi ir)(Q') \cap Q' \neq \emptyset$

(2) $\exp'(2\pi ir)(Q') = Q'$

(3) $r \in md^n\mathbf{Z}$.

Proof. – This is similar to Lemma 4.6. □

Proof of Theorem 4.3, *part* (2a). – We define a function $\Psi : \Sigma_+' \to J_+^-$ satisfying (a) of part (2) of Theorem 4.3. Choose a class P of the partition $\mathcal{P}$ of S_0. Choose a class Q of the partition $\tilde{\mathcal{Q}}$ of T_0 which maps bijectively to P under Φ_0. And choose a class Q' of the partition $\mathcal{Q}'$ of T_0' which maps to P under Θ_m. The maps $(\Phi_0|Q)^{-1}$ and $\Theta_m|Q'$ are bijections. Define Ψ on Q' by setting $\Psi_0 = (\Phi_0)^{-1} \circ (\Theta_m|Q')$. We will extend this map to all of Σ_0 in two stages. First we define it by setting $\Psi(\exp'(2\pi it)s) = \exp(2\pi it)\Psi(s)$ for all $t \in \mathbf{R}$. To show that this definition is consistent, we must show that if $p, q \in Q'$ are such that $\exp(2\pi ia)p = \exp(2\pi ib)q$, then $\exp(2\pi ia)\Psi(p) = \exp(2\pi ib)\Psi(q)$. This follows because Φ commutes with exp, $\Theta_m \circ \exp' = \exp \circ \Theta_m$, and by the application of Lemmas 4.11–13.

Thus we have defined the map Ψ on one point on each ray. Each ray is mapped bijectively to a ray under Φ and Θ_m. Thus there is a unique extension of Ψ to entire rays so that (2a) holds. □

The map Ψ defined so far does not necessarily conjugate the shift σ to the action of f on J_+^-. We will show Ψ can be modified so that it does preserve the dynamics.

Define the norm of an element $s \in \Sigma$ to be $|s_0|$. The following Lemma characterizes the action of the subgroup Σ_0 by multiplication on Σ_+.

LEMMA 4.15. – *A homeomorphism of the solenoid Σ_+ which commutes with the action of* exp, *which preserves rays and norms of elements, is multiplication by a group element $s \in \Sigma_0$.*

Proof. – Let g be such a homeomorphism. Choose a $t \in \Sigma_+$. Let $s = g(t)/t$. Since g preserves norms, s has norm 1. For $s \in \Sigma_0$, if $m_s : \Sigma_+ \to \Sigma_+$ is the multiplication by s, then $m_s^{-1}g$ fixes t and commutes with the action of exp. In particular it fixes the path component of t. The path component of the identity is dense in the set of elements with fixed norm so $m_s^{-1} \circ g$ is the identity. Thus $m_s^{-1}g = 1$ and $g = m_s$. □

Proof of Theorem 4.3, *part* (2b). – We want the map Ψ to conjugate the function σ to the function f. That is, we would like to have $f' := \Psi^{-1} \circ f \circ \Psi$ to be equal to σ. The maps σ and f' both expand the exponential parametrization by a factor of d, which is to say that $\sigma(\exp(2\pi it)s) = \exp(2\pi idt)\sigma(s)$ and $f'(\exp(2\pi it)s) = \exp(2\pi idt)f'(s)$. Thus the composition $f' \circ \sigma^{-1}$ preserves the exponential parametrization. The map $f' \circ \sigma^{-1}$ also preserves norms of elements and rays. By the Lemma 4.15, $f'\sigma^{-1}$ has the form m_s for some s in the solenoid Σ_0 so $f' = m_s\sigma$.

We show next that there is a $t \in \Sigma_0$ so that $f' = m_t^{-1} \circ \sigma \circ m_t$. In order to do this we need to solve the following equation for t:

$$m_s\sigma = m_t^{-1}\sigma m_t \tag{4.1}$$

It is easy to check that $\sigma m_t = m_{\sigma(t)}\sigma$. So that equation (4.1) is equivalent to the equation

$$m_s\sigma = {m_t}^{-1} m_{\sigma(t)}\sigma. \tag{4.2}$$

Thus it suffices to find a t such that

$$s = t^{-1}\sigma(t). \tag{4.3}$$

Now t^{-1} is exactly $\Theta_{-1}(t)$ and $\sigma(t)$ is $\Theta_d(t)$. So that (4.3) is equivalent to the equation

$$s = \Theta_{-1}(t) \cdot \Theta_d(t) = \Theta_{d-1}(t). \tag{4.4}$$

Since d and $d-1$ are relatively prime, there is a solution t_0 to equation (4.4) by Lemma 4.12.

Thus we have $f' = \Psi^{-1} f \Psi = {m_{t_0}}^{-1}\sigma m_{t_0}$. So $f = (\Psi {m_{t_0}}^{-1})\sigma(m_{t_0}\Psi^{-1})$. Now if we replace Ψ by the product $\Psi \cdot {m_{t_0}}^{-1}$, then this new map conjugates the dynamics of f on J_+^- to the dynamics of the shift map σ on Σ_+. □

5. Affine Structure and Landing Map: Dynamics on J

We want to study more closely the intersection $J \cap W^u(x)$; for this we need more information on the structure of the unstable lamination $\mathcal{W}^u$. If f is hyperbolic, then for each $p \in J$, there is a uniformization $\psi_p : \mathbf{C} \to \mathbf{C}^2$ such that $\psi_p(0) = p$ and $\psi_p(\mathbf{C}) = W^u(p)$. This uniformization is not unique: the derivative at the origin is not determined. Indeed if we choose a point y so that $W^u(x) = W^u(y)$, we produce uniformizations which differ by an affine transformation. It is useful to introduce the notion of an affine structure as a way of dealing with this ambiguity.

A *complex affine structure* on a holomorphic curve L is given by an atlas consisting of holomorphic diffeomorphisms χ_j from open sets U_j of L to open sets of $\mathbf{C}$ such that the U_j cover L and the $\chi_j \circ \chi_i^{-1}$ are restrictions of affine diffeomorphisms $z \mapsto az + b$ of $\mathbf{C}$. Two atlases give the same affine structure if their union satisfies the compatibility condition. The identity map gives $\mathbf{C}$ an affine structure which we call the canonical affine structure on $\mathbf{C}$.

For three distinct points x, y, and z in $\mathbf{C}$ the ratio $\frac{x-y}{x-z}$ is invariant under the group of affine motions of $\mathbf{C}$. If x, y, and z are distinct nearby points of $U_j \subset \mathbf{C}$, the ratio $\frac{\chi_j(x)-\chi_j(y)}{\chi_j(x)-\chi_j(z)}$ depends only on the points $\chi_j(x)$, $\chi_j(y)$ and $\chi_j(z)$ and not on the particular coordinate chart χ_j. Note that this function on nearby triples in L actually determines the affine structure in L. One says that an affine structure is complete if it is isomorphic to $\mathbf{C}$ with its canonical affine structure. Note that the canonical affine structure on $\mathbf{C}$ is the unique complete affine structure on $\mathbf{C}$.

The affine structure is defined on the whole unstable lamination $\mathcal{W}^u$ and is f-invariant. Let us fix $x_0 \in J^-$ and disjoint transversals T_1, T_2, T_3 to the local unstable manifold

$W^u_\epsilon(x_0)$. For $x \in J^-$ near x_0, there are three points $p_j := T_j \cap W^u_\epsilon(x)$, $j = 1, 2, 3$. The ratio $\frac{\zeta(p_1)-\zeta(p_2)}{\zeta(p_1)-\zeta(p_3)}$ is well-defined, independently of any particular choice of complex affine coordinate ζ on $W^u(x)$. To say that the affine structure is continuous is to say that this ratio varies continuously with x.

The following, an adaptation of Proposition 3.1 of Ghys [G], states that this affine structure varies continuously from leaf to leaf.

THEOREM 5.1. – *If f is hyperbolic then there is a unique way to equip each leaf of $\mathcal{W}^u$ with an affine structure so that:*

(i) *f acts affinely on the leaves of $\mathcal{W}^u$ and*

(ii) *The ratio function on nearby triples of points is continuous.*

Moreover, for this structure, each leaf of $\mathcal{W}^u$ is complete.

Proof. – The proof follows the line of argument of Proposition 3.1 of [G] with the following modifications. We choose a compact neighborhood N of J inside J^- so that $f(N) \supset N$. As in [Gh, Lemma 3.2] we can identify affine structures on $N \cap W^u$ with certain sections of bundles of jets. It is necessary to modify slightly the definition of the action of f on the space of sections to deal with the fact that N is not invariant. If σ is a section we can define a new section $\overline{f}(\sigma)$ by $\overline{f}(\sigma)(x) = \mathrm{D}f^*(\sigma(f^{-1}(x)))$. This map on sections is a contraction of a metric space. Thus it has a unique fixed point. This fixed section produces an affine structure on N which is continuous and backward invariant. The pushforward of this affine structure on $f(N)$ agrees with the affine structure on N, and so we have a natural way to extend the affine structure to $\bigcup_n f^n(N) = \mathcal{W}^u$. Now the argument in [G] shows that each leaf of this affine structure is complete. □

The uniqueness of the complete affine structure on the complex line shows that the affine structure constructed in Theorem 5.1 agrees with the affine structure obtained by parametrizing unstable manifolds. Thus we can read the proposition as saying that the affine structure obtained by parametrizing unstable manifolds by copies of $\mathbf{C}$ is transversely continuous.

For any point p in J we write U^+_p for $W^u(p) - K^+ = W^u(p) \cap U^+$. If $\mathcal{O}^+_p \subset U^+_p$ is a connected component, then by [BS6, Proposition 2.2] there is a conformal equivalence $\alpha_p : \mathcal{O}^+_p \to H$, where H is the right half plane in $\mathbf{C}$, and which satisfies $\Re\alpha_p = G^+$. For fixed $a \in \mathbf{R}$, the curves $R_p(a) := \alpha_p^{-1}(ia, ia + \infty)$ are gradient lines of $G^+|\mathcal{O}^+_p$ and hence external rays. Thus $\mathcal{O}^+_p = \bigcup_{a\in\mathbf{R}} R(a)$ is the union of external rays. We identify $\mathbf{R}$ and the external rays contained in $\mathcal{O}^+_p$, and this identification is unique modulo translation, since α_p is uniquely defined modulo an additive constant ib. Thus it is natural to define the distance between two external rays $R(a)$ and $R(a')$ to be $|a - a'|$. As was noted in the remark after [BS6, Proposition 2.7], the maps α_p serve to define another affine structure, $\mathcal{A}'$, on the Riemann surface lamination $\mathcal{W}^u|U^+$. This affine structure is not to be confused with the affine structure of $\mathcal{W}^u$.

Let a and b be points in $\mathbf{C}$. Let E be a path connecting them. Let $c > 0$, and define

$$\operatorname{car}(E, c) = \{z \in \mathbf{C} : |z - x| < c|x - a| \text{ for some } x \in E\}$$

$$\operatorname{cig}(E, c) = \{z \in \mathbf{C} : |z - x| < c\min(|x - a|, |x - b|) \text{ for some } x \in E\}.$$

The carrot $\operatorname{car}(E, c)$ has its small end at a. In the cigar $\operatorname{cig}(E, c)$ the ends are treated symmetrically. Let D be a connected open set in $\mathbf{C}$. We say that the domain D *satisfies*

c-cig if each pair of points in D can be joined by a path E so that $\operatorname{cig}(E,c) \subset D$. Note that this property is scale invariant. If D satisfies c-cig then any translate of D satisfies c-cig, and λD satisfies c-cig for any $\lambda \neq 0$ in $\mathbf{C}$. In other words, this property depends only on the complex affine structure of $\mathbf{C}$.

THEOREM 5.2. – *Let f be hyperbolic and unstably connected, and give $\mathcal{W}^u$ the natural affine structure. Then there is a constant $c > 0$ so that for every $p \in J$ each component $\mathcal{O}_p^+$ of U_p^+ satisfies the condition c-*cig*. In addition, if R is any external ray then* $\operatorname{car}(R,c) \subset U^+$.

Proof. – First we deal with carrots centered about external rays. Choose ϵ small enough so that for every $p \in J$ the set $B(p,\epsilon) \cap J$ is contained in a subset of J with local product structure. Let N be the set of points in J^- within distance ϵ of J. The set of pairs of points $M = \{(x,y) \in N \times N : d(x,y) \leq \epsilon/2 \text{ and } x \in W_\epsilon^u(y)\}$ is compact. For $(x,y) \in M$, $y \notin J$, consider the function

$$\lambda(x,y) = \left| \frac{x-y}{e(y)-y} \right|$$

where the ratio is computed with respect to the affine structure on $W^u(x) = W^u(y)$. The continuity of the affine structure implies that this function is continuous. Choose n sufficiently large so that if $x \in J^-$ and $G^+(x) \leq 1/d^n$, then we have $d(x, e(x)) \leq \epsilon/2$. We restrict our attention to the compact set of pairs $(x,y) \in M$ such that $x \in K$ and $1/d^{n+1} \leq G^+(y) \leq 1/d^n$. The function λ is defined and continuous on this compact set, so it assumes a minimum value $c > 0$ there.

Since the affine structure is invariant under f, and the map e is equivariant with respect to f, the function λ is also invariant under f. Since any point in $\mathcal{W}^u - K$ is equivalent under f^k, for some $k \in \mathbf{Z}$, to a point in the set $\{q : 1/d^{n+1} \leq G^+(q) \leq 1/d^n\}$, it follows that c is a lower bound for λ on the set $(x,y) \in K \times (\mathcal{W}^u - K)$.

Let R denote an external ray in some unstable manifold $W^u(p)$ with p_0 as its endpoint. It follows from the definition of c that for any $z \in R$ the open disk around z of radius $c|z - p_0|$ is contained in $W^u(p) - K$. In other words $\operatorname{car}(R,c) \subset W^u(p) - K \subset U^+$.

The next step is to construct cigars. Let us choose a component $\mathcal{O}_p^+$ of U_p^+ and recall the conformal mapping to the right half plane, $\alpha_p : \mathcal{O}_p^+ \to H$, defined above. For $q \in \mathcal{O}_p^+$ and $t \in \mathbf{R}$, we define $q_t := \alpha_p^{-1}(\alpha_p(q) + it)$ and

$$\lambda'(q) = \min_{t \in [-1,1]} \min_{z \in K^+ \cap W^u(q)} \frac{|z - q_t|}{|e(q) - q|}.$$

Since $[-1,1] \ni t \mapsto q_t$ lies in a compact subset of $\mathcal{O}_p^+$, this minimum is attained. By the continuity of the affine structure $\mathcal{A}'$, $\lambda'(q)$ is continuous for $q \in J_+^-$. Thus

$$c' := \min_{q \in \{G^+=1\} \cap J^-} \lambda'(q) > 0$$

since $\lambda' > 0$ on J_+^-.

To finish the proof, we must show that, given any two points $q, q' \in \mathcal{O}_p^+$, there is a cigar connecting them. Since k-cigars are mapped to k-cigars under f, we may replace q and q' by $f^n q$ and $f^n q'$. Thus we may assume that $G^+(q), G^+(q') < 1$. Let us choose $a, a' \in \mathbf{R}$ such that the corresponding external rays satisfy $q \in R(a)$ and $q' \in R(a')$. Let

γ (resp. γ') denote the path inside $R(a)$ (resp. $R(a')$) running from q (resp. q') up to the level set $\{G^+ = 1\}$. Define the path $\sigma = \alpha_p^{-1}([1+ia, 1+ia'])$, which runs inside the set $\{G^+ = 1\}$ and connects γ to γ'. Let us consider the case $|a - a'| \leq 1$. By the definition of c' it follows that if $w \in \sigma$, then, in the affine structure of $W^u(p)$, the disk centered about w with radius $c' \min(|e(q) - w|, |e(q') - w|)$ is contained in $\mathcal{O}_p^+$. Thus the curve $\gamma \cup \sigma \cup \gamma'$ is the center of a $\min(c, c')$-cigar contained in $\mathcal{O}_p^+$ and connecting q to q'.

Finally, when $|a - a'| > 1$, we observe that f^{-n} acts on the affine structure $\mathcal{A}'$ as $\zeta \mapsto d^{-n}\zeta + c$. Thus $R_p(a)$ and $R_p(a')$ are taken to $R_{f^{-n}p}(b)$ and $R_{f^{-n}p}(b')$, where $|b - b'| = d^{-n}|a - a'|$. Thus for n large, the procedure above constructs a $\min(c, c')$-carrot connecting $f^{-n}q$ and $f^{-n}q'$. □

In [NV] a topological disk in $\mathbf{R}^2$ which satisfies the c-cig condition is called a *c-John disk.* According to the previous theorem each component $\mathcal{O}_p^+$ is a c-John disk. The following proposition summarizes some topological properties of John disks.

PROPOSITION 5.3. – *Let $\mathcal{O}_p^+$ be a connected component of U_p^+. The boundary of $\mathcal{O}_p^+$ is locally connected, and $\mathcal{O}_p^+$ is locally connected at ∞.*

Proof. – The local connectivity of a subset of R^2 is equivalent to the local connectivity of its boundary. A strong form of local connectivity for the complement of a John disk is proved in Theorem 4.5 (6) of [NV]. The assertion that $\mathcal{O}_p^+$ is locally connected at ∞ is proved in Theorem 2.23 of [NV]. □

PROPOSITION 5.4. – *For each component $\mathcal{O}_p^+$ the landing map gives a proper continuous map from the space of rays in $\mathcal{O}_p^+$ to the topological boundary of $\mathcal{O}_p^+$.*

Proof. – Following Theorem 5.1 we described a Riemann map $\alpha_p : \mathcal{O}_p^+ \to H$. Let β be the inverse Riemann map: $\beta_p : H \to \mathcal{O}_p^+ \subset W_p^u \cup \{\infty\}$. We give W_p^u the leaf topology so that $W_p^u \cup \{\infty\}$ is just the Riemann sphere. Since the boundary of $\mathcal{O}_p^+$ is locally connected, Caratheodory's Theorem states that β extends to a continuous map β^* from $H \cup i\mathbf{R} \cup \{\infty\}$ to the closure of $\mathcal{O}_p^+$ in $W_p^u \cup \{\infty\}$. The fact that $\mathcal{O}_p^+$ is locally connected at ∞ implies that $(\beta^*)^{-1}(\infty) = \{\infty\}$. So the restriction of β^* to the finite points gives a proper map from $H \cup i\mathbf{R}$ to the closure of $\mathcal{O}_p^+$ in W_p^u. The landing point of the ray $R(a)$ is the point $\beta_p(a)$ so we see that the landing map is continuous and proper. □

PROPOSITION 5.5. – *There is a number N, depending only on the John constant c, so that each point $\partial(W_p^u \cap K^+)$ is the image of at least one and at most N external rays. Furthermore each set U_p^+ has at most N components.*

Proof. – Each point in $\partial(W_p^u \cap K^+)$ is in the boundary of some component $\mathcal{O}_p^+$, and hence is the landing point of some ray in $\mathcal{O}_p^+$ by the previous result. Let $N = N(c)$ be the maximum number of disjoint open disks of radius 1 which can have their centers on the circle of radius c. The proof that N is an upper bound for the number of rays that land at a point follows from the proof of [NV, Thm. 2.18 part (1)]. The proof that N is an upper bound for the number of components of U_p^+ follows from the proof of [NV, Thm. 2.18 part (2)] and the observation that each component of U_p^+ is unbounded. □

PROPOSITION 5.6. – *The sets $\partial(W_p^u \cap K^+)$ and $W_p^u \cap K^+$ are locally connected.*

Proof. – By Propositions 5.4 and 5.5, $\partial(K^+ \cap W^u)$ is the proper image of $k \leq N$ copies of $\mathbf{R}$. Let X_k be the one point compactification of the union of k disjoint copies of $\mathbf{R}$. The set X_k is the wedge of k circles. Since the parametrization of $\partial(K^+ \cap W^u)$ is proper it gives a continuous map from X_k to the one point compactification of W^u. The image of a compact locally connected set is locally connected. Thus $\partial(K^+ \cap W^u) \cup \infty$ is locally compact hence $\partial(K^+ \cap W^u)$ is locally connected.

The local connectivity of $W^u_p \cap K^+$ follows from the local connectivity of its boundary. □

Both Σ_0 and J have local product structure. That is, there exist $\delta, \epsilon > 0$ such that if p and q are points with $\operatorname{dist}(p,q) < \delta$, then there is a point $[p,q]$ such that

$$W^s_\epsilon(p) \cap W^u_\epsilon(q) = \{[p,q]\}. \tag{5.1}$$

The bracket is a continuous function of both variables. By Proposition 2.7, the bracket on J has a continuous extension to $(J^-_+ \cap U) \times J$ for some neighborhood U of J, by setting

$$[p,q] = \mathcal{L}^s_\epsilon(p) \cap W^u_\epsilon(q)$$

for $p \in J^-_+ \cap U$ and $q \in J$ with $\operatorname{dist}(p,q) < \delta$.

A set is called a rectangle if it is closed under application of the bracket. If R is a rectangle, and $r \in R$, we set $D^{s/u}(r) = W^{s/u}_\epsilon(r) \cap R$. It follows that the map $[\cdot,\cdot] : D^s(r) \times D^u(r) \to R$ is a homeomorphism. For any two points $r', r'' \in R$, the bracket gives a natural homeomorphism between $D^{s/u}(r')$ and $D^{s/u}(r'')$; when no confusion may result we suppress r and write $R = D^s \times D^u$.

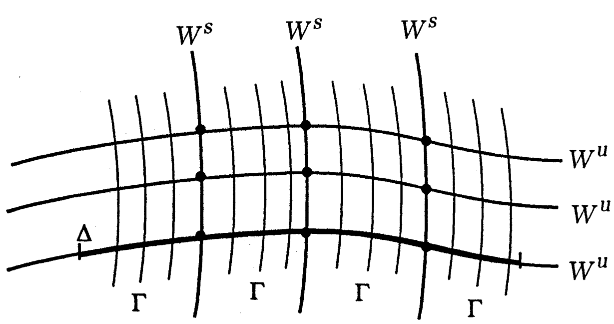
We may cover J by finitely many such rectangles R_j. We now enlarge R_j to a set $\mathcal{R}_j$ with local product structure and such that $\bigcup \mathcal{R}_j$ is a neighborhood of J inside J^-. Let $R = D^s(r) \times D^u(r)$ be a rectangle as above, and set $\Delta = W^u_\epsilon(r)$. We may consider

$$\Delta \cap (J^- \cap U^+) \ni q \mapsto \mathcal{L}^s_\epsilon(q)$$

as a holomorphic motion. By [MSS] this may be extended to a neighborhood Δ' of r in Δ. That is, there is a continuous map $\Delta' \ni q' \mapsto \Gamma_{q'}$, where $\Gamma_{q'}$ is a complex manifold, and $\Gamma_{q'_1} \cap \Gamma_{q'_2} = \emptyset$ if $q'_1 \neq q'_2$. Thus, shrinking ϵ if necessary, we have a bracket

$$[\cdot,\cdot] : D^s(r) \times \Delta' \to \mathcal{R} \subset J^-$$

given by $[p,q] = W^s_\epsilon(p) \cap \Gamma_q$, and where we set $\mathcal{R} = [D^s(r), \Delta']$. In the illustration, the local pieces of $\mathcal{W}^s$ appear as darker vertical curves, and the manifolds Γ_q are thinner vertical curves.



Extension of Local Product Structure

We note that this enlargement of the local product structure is not f-invariant. But from it we get useful topological information such as the following:

PROPOSITION 5.7. – *If f is hyperbolic, then $\partial(K^+ \cap W^u) = J \cap W^u$, where the boundary operation is with respect to the leaf topology on W^u.*

Proof. – For $p \in J$, the set $\bigcup_{q \in D^u(p)} \Gamma_q$ contains a neighborhood of p inside $\mathbf{C}^2$. It follows by the enlarged local product structure that $\bigcup_{q \in D^u(p) \cap K^+} \Gamma_q$ coincides with K^+ in a neighborhood of p. Thus for a product neighborhood $\mathcal{N} = [D^s(p), D^u(p)]$ of p, we have $(\partial K^+) \cap \mathcal{N} = \bigcup_{q \in D^u(p) \cap \partial K^+} \Gamma_q \cap \mathcal{N} = \bigcup_{r \in D^s(p)} \partial(D^u(r) \cap K^+)$. □

This allows us to restate Proposition 5.6:

COROLLARY 5.8. – *If f is hyperbolic and unstably connected, then for each leaf W^u the set $J \cap W^u$ is locally connected.*

The following is a corollary of Theorem 4.1.

COROLLARY 5.9. – *If f is hyperbolic and unstably connected, then $J \cap W^s(p)$ is totally disconnected. In particular, f is not stably connected.*

Proof. – In the extended local product structure, the intersection $W^s_\epsilon(p) \cap J$ is locally homeomorphic to $\Gamma_q \cap J^-$. If we choose $q \in W^u_\epsilon(p) - K$, then $\Gamma_q \subset U^+$. By Theorem 4.1 $J^- \cap U^+$ is homeomorphic to the complex solenoid. Since Γ_q is transversal to J^-, it follows that $\Gamma_q \cap J^-$ is totally disconnected. □

Recall from Section 2.1 that there is a continuous and equivariant landing map e from the space $\mathcal{E}$ of all external rays, to J.

THEOREM 5.10. – *If f is hyperbolic and unstably connected then the map $e : \mathcal{E} \to J$ is surjective, and each point has at most N preimages.*

Proof. – Each point $p \in J$ lies in some unstable manifold W^u_p. With respect to the leaf topology on W^u_p, the point p is in $\partial(K^+ \cap W^u)$ by Propostion 5.7. By Proposition 5.5 the point p is the landing point of at least one and at most N external rays in W^u_p. □

We will make additional use of the enlargement of the local product structure. The manifolds $\mathcal{L}^s_\epsilon$ provide a family of transversals with respect to which the affine structure of $\mathcal{W}^u$ varies continuously. If $p \in J$, and $q \in W^s_\epsilon(p) \cap J$, then the map $W^u_{\rm loc}(p) \ni r \mapsto \mathcal{L}^s_\epsilon(r) \cap W^u_{\rm loc}(q)$ is a local homeomorphism. By the continuity of the affine structure, this homeomorphism takes cigars to cigars and carrots to carrots, where we define carrots and cigars with respect to the affine structure of $\mathcal{W}^u$, and the carrots and cigars must be small enough to lie inside the domain of definition of the local homeomorphism. This gives us a device for transporting carrots and cigars from one unstable manifold to another within a neighborhood of J (inside J^-). By [MSS] the local homeomorphism is quasiconformal. Thus, locally, it preserves carrots and cigars with respect to the induced euclidean metric.

In the following Proposition, dist will denote the distance function on an unstable manifold W^u induced by the Euclidean metric on $\mathbf{C}^2$.

PROPOSITION 5.11. – *There exists $M < \infty$ with the property that if $p, q \in W^u - K$ satisfy* $\mathrm{dist}(p, q) \le 1$, $\mathrm{dist}(p, J) \le 1$, *then one of the following holds:*

(i) *There exists a cigar $\gamma \subset W^u - K$ connecting p and q, and* $\mathrm{diam}(\gamma) \le M$.

(ii) *There exists* $r \in J \cap W^u$ *and cigars* $\gamma_1, \gamma_2 \subset (W^u - K) \cup \{r\}$ *such that* γ_1 *connects* p *to* r, γ_2 *connects* r *to* q, *and* $\mathrm{diam}(\gamma_j) \le M$ *for* $j = 1, 2$.

Proof. – We may cover J by a finite number of triples of local product neighborhoods $\mathcal{R}'_j \cong D_j \times \Delta'_j$, $\mathcal{R}''_j \cong D_j \times \Delta''_j$, and $\mathcal{R}'''_j \cong D_j \times \Delta'''_j$ with the properties that Δ'_j is a relatively compact subset of Δ''_j, and Δ''_j is relatively compact in Δ'''_j. Let us fix j and drop the subscript j from our notation.

If p is a point of $\{r\} \times \Delta' - K$ then by Theorem 5.2 there is a carrot Γ with vertex at p connecting p to a point $p'' \in \{r\} \times \partial\Delta''$. If δr denotes the distance between $\overline{\Delta}'$ and $\partial\Delta''$, then the carrot contains a disk of radius $c\,\delta r$, centered at p''. Thus, possibly after shrinking c and Δ', it follows that if p, q lie in the same connected component of $\{r\} \times \Delta'$, there is a c-cigar contained in $\{r\} \times \Delta'' - K$ which connects p to q. It is evident that by the uniformity of the c-carrot condition in Theorem 5.2, this holds for all $r \in D_j$.

Next we observe that there exists $\delta_0 > 0$ such that if $\mathcal{O}_k$ and $\mathcal{O}_m$ are components of Δ''' which intersect $\partial\Delta''$, and if $\overline{\mathcal{O}}_k \cap \overline{\mathcal{O}}_m \cap \overline{\Delta}'' = \emptyset$, then $\mathrm{dist}\,(\mathcal{O}_k \cap \Delta'', \mathcal{O}_m \cap \Delta'') \ge \delta_0$. By the local product structure on $\mathcal{R}$, we may shrink δ_0 if necessary so that this holds for all unstable disks $\{r\} \times \Delta'''$. Further, we may assume that $\mathrm{dist}(\partial\Delta'_j, \partial\Delta''_j) > \delta_0$. Since there are only finitely many $\mathcal{R}_j$, we may assume that δ_0 has this property for all $\mathcal{R}_j$.

Let $\mathcal{N} = \{z \in \mathbf{C}^2 : \mathrm{dist}(z, J) \le 1\}$. Without loss of generality (applying f^{-k} if necessary), we may assume that $J^- \cap \mathcal{N} \subset \bigcup \mathcal{R}_j$. Now for n sufficiently large we have $\mathrm{dist}(f^{-n}p, f^{-n}q) < \delta_0$. Choose j such that $f^{-n}p \in \{r\} \times \Delta'_j \subset \mathcal{R}'_j$, and thus $f^{-n}q \in \{r\} \times \Delta''_j$. By the definition of δ_0, either there is a connected open set $\mathcal{O} \subset \{r\} \times \Delta'''$ containing both p and q, or there are connected open sets $\mathcal{O}_1, \mathcal{O}_2 \subset \{r\} \times \Delta'''$ with $\partial\mathcal{O}_1 \cap \partial\mathcal{O}_2 \cap \overline{\Delta}'' \ne \emptyset$. In the first case, we may connect p and q by a cigar contained in $\mathcal{O}$, and in the second case, there is a point $r \in \partial\mathcal{O}_1 \cap \partial\mathcal{O}_2 \cap \overline{\Delta}''$, and a pair of cigars $\gamma_j \subset \mathcal{O}_k$, $k = 1, 2$, such that γ_1 connects p to r, and γ_2 connects r to q.

Thus $f^{-n}p$ and $f^{-n}q$ are connected according to either (i) or (ii). If C and λ are as in (2.1), then it is sufficient to take $n = n_0$ to satisfy $C\lambda^{n_0} < \delta_0$. The closures of $\mathcal{R}'''_j$ are compact, so it follows that the closures of $f^{n_0}\mathcal{R}'''_j$ are compact. Thus there exists $M < \infty$ such that the diameters of $f^{n_0}(\{r\} \times \Delta''')$ are bounded by M for all r and j. Thus the diameters of $f^{n_0}\gamma_k$ are bounded by M for $k = 1, 2$. □

THEOREM 5.12. – *If* f *is hyperbolic, then there exists* M *such that the following holds: If* W^u *is an unstable manifold, if* d *is a distance function compatible with the affine structure of* W^u, *and if* p, q *are two points of* W^u, *then one of the following holds:*

(i) *There exists a cigar* $\gamma \subset W^u - K$ *connecting* p *and* q, *and* $\mathrm{diam}(\gamma) \le Md(p, q)$.

(ii) *There exists* $r \in J \cap W^u$ *and cigars* $\gamma_1, \gamma_2 \subset (W^u - K) \cup \{r\}$ *such that* γ_1 *connects* p *to* r, γ_2 *connects* r *to* q, *and* $\mathrm{diam}(\gamma_j) \le Md(p, q)$ *for* $j = 1, 2$.

In either case, if $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *denote the components of* $W^u - K$ *containing* p *and* q, *then* $\partial\mathcal{O}_1 \cap \partial\mathcal{O}_2 \ne \emptyset$.

Proof. – Let $\mathcal{R}'_j, \mathcal{R}''_j, \mathcal{R}'''_j$ denote triples of local product neighborhoods as used in the proof of Proposition 5.11. For $\delta_0 > 0$, let $\mathcal{N} = \{p \in J^- : \mathrm{dist}(p, J) \le 1$. We may choose δ_0 sufficiently small for $\{\mathcal{R}'_j\}$ to cover $\mathcal{N}$ and (2.1) to hold on $\mathcal{N}$. Choose M_0 such that $|\mathrm{D}f| \le M_0$ on $J^- \cap \mathcal{N}$, and set $\mathcal{N}_0 = \{p \in J^- : \mathrm{dist}(p, J) \le \delta_0/M_0\}$.

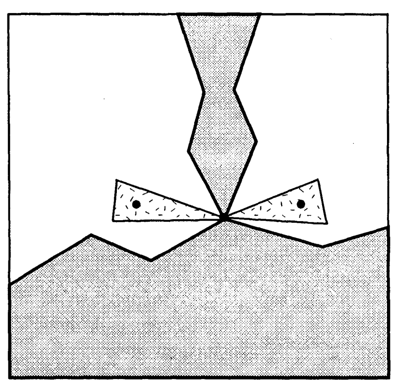
For $p, q \in \mathcal{N}$ in the same unstable manifold W^u, we have $\mathrm{dist}\,((f^k p, f^k q) \to 0$ as $k \to -\infty$. Further, since $p, q \in J^-$, we have $\mathrm{dist}\,((f^k p, J) \to 0$ and $\mathrm{dist}\,((f^k q, J) \to 0$

as $k \to -\infty$. Thus we have $\text{dist}(f^k p, f^k q) < \delta_0$ and $f^k p, f^k q \in \mathcal{N}_0$ for $k \ll 0$. As long as $f^k p$ and $f^k q$ remain in $\mathcal{N}$, it follows from (2.1) that dist $((f^k p, f^k q)$ will grow as k increases. Now there are two possibilities. The first is that there exists k such that dist $((f^k p, f^k q) \leq \delta_0/M_0$, $f^k p, f^k q \in \mathcal{N}_0$, and $f^{k+1} p \notin \mathcal{N}_0$. In this case, $f^k p$ and $f^k q$ lie in a disk of radius δ_0/M_0 (inside W^u) which is disjoint from J. Thus $f^k p$ may be connected to $f^k q$ by a cigar of diameter bounded by δ_0/M_0. Thus we have (i) in this case with the constant $M = 1$, except that the distance involved is dist instead of d.

The other case is that there exists k such that $f^k p, f^k q \in \mathcal{N}_0$ and that $\delta_0/M_0 \leq$ dist $((f^{k+1} p, f^{k+1} q) \leq \delta_0$. As in the proof of Proposition 5.11, we have p connected to q, either by one or two cigars (corresponding to cases (i) or (ii)), with the constant $M = DM_0/\delta_0$, where D is an upper bound on the diameters of the disks $\{r\} \times \Delta_j'''$, except that the distance involved is dist instead of d.

Since each disk $\{r\} \times \Delta_j''' \subset \mathcal{R}_j'''$ is contained in a leaf of $\mathcal{W}^u$, we may give it the induced affine structure. Thus each (unstable) lamina of each local product neighborhood $\mathcal{R}_j'''$ has a complex affine structure. By Theorem 5.1, the affine structure of $\{r\} \times \Delta_j'''$ varies continuously with r. We assign a metric δ on these disks as follows: we choose two points $a_j, b_j \in \Delta_j'''$, and we set $\delta(a_j, b_j) = 1$. This determines a unique distance function on Δ_j'' which is compatible with the induced affine structure. Although we have no a priori bound on the ratio between the distance function on $\{r\} \times \Delta_j'''$ and the Euclidean distance, i.e., dist/δ, this ratio varies continuously throughout $\mathcal{R}_j'''$. Thus there is a constant κ with the property that $\kappa^{-1} \leq \text{dist}/\delta \leq \kappa$ holds on the unstable leaves of $\bigcup \mathcal{R}_j'''$.

Now let $p, q \in W^u - K$ be given. It follows that for some $k \in \mathbf{Z}$, $f^k p$ and $f^k q$ are contained in $\{r\} \times \Delta_j''$ and are connected by one or two c-cigars γ_1, γ_2, inside $\{r\} \times \Delta_j'''$. Now if we switch from the distance dist to δ, we change scales by at most κ. Thus, with respect to δ, we have $f^k p$ and $f^k q$ connected by c'-cigars of diameter $\leq D\kappa$, and $\delta(f^k p, f^k q) \geq \delta_0/(M_0\kappa)$. Thus the alternatives (i) and (ii) continue to hold for c'-cigars, and a new constant M'. Recall that the affine structure of $\{r\} \times \Delta_j'''$ was induced from the affine structure of W^u. Thus the restrictions of d and δ to $\{r\} \times \Delta_j'''$ are constant multiples of each other. Thus (i) and (ii) hold for d, with the same constants M' and c'. Finally, the conditions (i) and (ii) are complex affine invariant, so they hold for p, q, with cigars $f^{-k}\gamma_j$, $j = 1, 2$. □



A short bow tie

Remark. – In general the diameter (and thus length) of the shortest path inside $W^u - K$ connecting two points will be much larger than the Euclidean distance between the two points. Thus we cannot hope to connect two points by "short cigars," whose diameter (or length) is comparable to euclidean distance. Theorem 5.12, however, shows that points may always be connected by short bow ties.

COROLLARY 5.13. – *If f is hyperbolic and unstably connected, then $J \cap W^u_p$ and $K \cap W^u_p$ are path connected.*

Proof. – By Propositions 5.4 and 5.5, $\partial K \cap W^u_p$ is the union of the continuous images of a finite number of copies of $\mathcal{R}$. By Theorem 5.12 these images are connected, so $\partial K \cap W^u_p$ is path connected. Thus $K \cap W^u_p$ is also path connected. □

COROLLARY 5.14. – *If f is hyperbolic and unstably connected, the path components of J are precisely the sets $J \cap W^u_p$.*

Proof. – By Corollary 5.13 the sets $J \cap W^u_p$ are path connected. The result now follows from the local product structure, since (Corollary 5.9) $J \cap W^s_p$ is totally disconnected. □

THEOREM 5.15. – *If f is hyperbolic, then there exists M such that the following holds: If W^u is an unstable manifold, if d is a distance function compatible with the affine structure of W^u, and if $p_1, \ldots, p_n \in W^u - K$, then there exists a point $r \in J \cap W^u$ and cigars $\sigma_1, \ldots, \sigma_n$ in $(W^u - K) \cup \{r\}$ such that σ_j connects p_j to r, and* $\operatorname{diam}(\sigma_j) \leq M \operatorname{diam}(\{p_1, \ldots, p_n\})$. *In particular, if $\mathcal{O}_1, \ldots, \mathcal{O}_n$ are connected components of $W^u - K$, and if $n \geq 3$, then $\partial\mathcal{O}_1 \cap \ldots \cap \partial\mathcal{O}_n = \{r\}$ is a single point.*

Proof. – Theorem 5.12 is the statement of this Theorem in the case $n = 2$. The case $n \geq 3$ is proved along similar lines, with the following modification. We must show that if $\mathcal{O}_m$, $m = 1, 2, 3, \ldots, n$ are components of $W^u - K$, then $\bigcap \partial\mathcal{O}_m \neq \emptyset$. For this we introduce triples of local product neighboroods as before, and we define the number $\delta_0 > 0$ by the property: If $\mathcal{W}_m$, $m = 1, 2, 3, \ldots, n$, are connected components of $\{r\} \times \Delta_j''' - K$ such that $\bigcap_{m=1}^n \overline{\mathcal{W}}_m \cap (\{r\} \times \overline{\Delta}_j'') = \emptyset$, then every n-tuple of points $p_m \in \mathcal{W}_m \cap (\{r\} \times \overline{\Delta}_j'')$, $m = 1, 2, 3, \ldots, n$, satisfies $\operatorname{diam}(p_1, \ldots, p_n) \geq \delta_0$. The fact that $\delta_0 > 0$ follows from the local product structure. Repeating the arguments of Proposition 5.11, we have $\bigcap \overline{\mathcal{O}}_m \supset \{r\} \neq \emptyset$. By applying f^k for some $k \in \mathbf{Z}$, we may assume that $p_1, \ldots, p_n \in \{r_j\} \times \Delta_j'$, and that the diameter of $\{p_1, \ldots, p_n\}$ is proportional to the diameter of $\{r_j\} \times \Delta_j'$. Further, by the John property we may assume that r lies in the boundary of the component of $\{r_j\} \times \Delta''' - K$ containing p_i for each $1 \leq i \leq n$. Thus we may connect r to p_i by a cigar σ_i inside $(\{r_j\} \times \Delta''' - K) \cup \{r\}$. The uniformity of the constant M follows from the use of the continuity of the affine structure of $\mathcal{W}^u$, as in Theorem 5.12.

It follows from the first part of this Theorem that if $\mathcal{O}_1, \ldots, \mathcal{O}_n$ are components of $W^u - K$, then $r \in \mathcal{O}_1 \cap \ldots \cap \mathcal{O}_n$. Now let us show that r is the unique point of intersection. By Proposition 5.8, we know that $\partial\mathcal{O}_m$, $1 \leq m \leq n$, contains the point of infinity in its closure. Thus each $\overline{\mathcal{O}}_m$ connects r to ∞ inside the 2-sphere. It follows that if s is a point of the 2-sphere which lies in the boundaries of ≥ 3 components $\mathcal{O}_m$, then $s = r$. □

6. Quotients of the Solenoid

The set $\mathcal{E}$ of external rays is topologically conjugate to Σ_0 via a map Ψ (Corollary 4.2). We let $\psi : \Sigma_0 \to J$ denote the composition of this map with the endpoint mapping. By Theorem 5.10, ψ is bounded-to-one. Here we take the point of view that ψ represents J as a finite quotient $\Sigma_0/\sim$, where the equivalence relation $\sim$ is given by $p \sim q$ iff $\psi(p) = \psi(q)$. The topology of J is determined by this equivalence relation, and we give in this section some general properties that such equivalence relations must satisfy. The first (Proposition 6.1) is that no identifications can occur within stable manifolds. A general consequence of this is Proposition 6.2, which states that $\sim$ respects a local product structure in which the quotient is taken in the unstable factor. Then we give some "planarity" conditions, which arise from the fact that the unstable identifications occur as the endpoints of disjoint rays lying in the same plane. We call these "non-linking," "orientation," and "isolation."

PROPOSITION 6.1 (Injectivity on Stable Manifolds). – *For any $t \in \Sigma_0$, the restriction of ψ to $W^s(t)$ is injective.*

Proof. – Suppose that $t', t'' \in W^s(t, \Sigma_0)$ satisfy $\psi(t') = \psi(t'')$. We let $\gamma_t, \gamma_{t'}, \gamma_{t''} \subset J_+^-$ denote the external rays corresponding to t, t' and t'', respectively. We set $p = \psi(t)$ and use the notation $\mathcal{G}_\epsilon^+(q,p)$ introduced before Proposition 2.2. By taking forward iterates, we may suppose that $\psi(t') = \psi(t'')$ is within ϵ of p. If $q \in \gamma_t$ is close to p, then by the local triviality of the lamination $\mathcal{L}^s = \mathcal{W}^s \cup \mathcal{G}^+$, there are points $q' = \gamma_{t'} \cap \mathcal{G}_\epsilon^+(q,p)$ and $q'' = \gamma_{t''} \cap \mathcal{G}_\epsilon^+(q,p)$. Since t' and t'' are distinct points on the stable manifold, it follows that $\gamma_{t'}$ and $\gamma_{t''}$ are disjoint, so $q' \neq q''$. By definition, $\psi(t') = e(q')$ and $\psi(t'') = e(q'')$ are the endpoints the external rays $\gamma_{t'}$ and $\gamma_{t''}$, respectively. Since q' and q'' are on the same local leaf of $\mathcal{G}^+$, we conclude by Proposition 2.10 that the endpoints must be distinct. This contradiction proves the Proposition. □

PROPOSITION 6.2 (Local Product Structure). – *Each point of J is contained in the interior of a rectangle R such that $\psi^{-1}(R) = R_1 \cup \cdots \cup R_N$, where each $R_j = R_j^s \times R_j^u$ is a rectangle in Σ_0 such that $\psi : R_j \to R$ preserves the product structure, that is to say that with respect to the natural product structures on R_j and R we can write $\psi(x,y) = [\psi^s(x), \psi^u(y)]$. Further, this may be arranged such that ψ^s is bijective, R_j^s is totally disconnected, and R_j^u is an interval.*

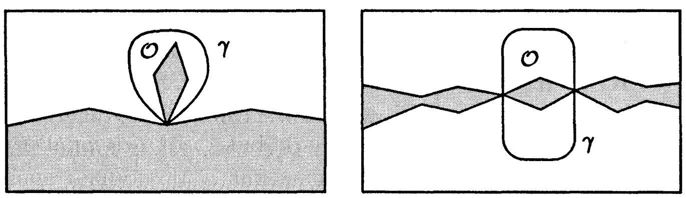
Proof. – This Proposition follows from rather general properties of ψ. Since ψ is a semi-conjugacy, it preserves the local product structure. Since the number of preimages of a point is bounded, it follows from continuity that for a sufficiently small rectangle R, the preimage $\psi^{-1}R$ is contained in a finite number of δ-balls B_j, and each set $B_j \cap \psi^{-1}R$ must be a rectangle. The statement that the stable factor of the map ψ is an isomorphism follows from Proposition 6.1. R_j^s is totally disconnected by Corollary 5.9. Finally, by Proposition 5.6 each $p \in \partial W_p^u \cap K^+$ has a neighborhood which is the image, under the landing map, of finitely many intervals of external rays. Since ψ^u is essentially the endpoint map of the external rays, each R_j^u may be taken to be an interval. □

Every path component H_0 of Σ_0 is homeomorphic to $\mathbf{R}$, and we will give H_0 the orientation for which the projection $\pi : H_0 \to \partial(\mathbf{C} - \overline{\Delta})$ preserves orientation. We use this particular orientation here because in our planarity conditions we exploit the fact that Σ_0 is the boundary of Σ_+. In Appendix B, we use the opposite orientation, which is induced by the imbedding $t \mapsto \exp(it)$.

For each point $p \in J$ and each $t \in \Sigma_0$ with $\psi(t) = p$, we let $H_t = W^u(t, \Sigma_+)$ denote the component of Σ_+ containing the ray R_t. Then $\Psi(H_t)$ is a connected component of $W^u(p) - K^+$. If R_j^u is an interval from Proposition 6.2, and if $\tilde{R}_j^u \subset \Sigma_+$ corresponds to the set of rays with endpoints in R_j^u, then there is an injection $\tilde{\psi}^u : \tilde{R}_j^u \to \mathbf{C}$, which has a continuous extension to $\tilde{R}_j^u \cup R_j^u$, and $\tilde{\psi}^u|_{R_j^u} = \psi^u$. The planarity conditions below arise in various ways from the injection of these rays into $\mathbf{C}$.

PROPOSITION 6.3 (Non-Linking). – *Suppose that $a, b, c, d \in H_0$ are points with $a < b$ and $\psi(a) = \psi(b) \neq \psi(c) = \psi(d)$. If $c \in (a, b)$, then $d \in (a, b)$; and if $c \notin [a, b]$, then $d \notin [a, b]$.*

Proof. – Assume that $\psi(a) = \psi(b)$. Let $\rho : [0, 1] \to H$ parametrize the half circle in H centered on the real axis with endpoints at a and b. The composition $\psi \circ \rho : [0, 1] \to W^u$ has the property that $\psi \circ \rho(0) = \psi \circ \rho(1)$ so $\psi \circ \rho([0, 1])$ is a simple closed curve in the plane W^u which we call γ. By the Jordan Curve Theorem, γ divides the plane into two components, one bounded and one unbounded. Let $c \in (a, b)$. We claim that $\psi(c)$ is in the bounded component. Now let $\rho_c(t) = c + ti$ for $t \geq 0$ parametrize the vertical line ending at c. In the upper halfplane this line has a unique intersection with the half-circle with endpoints at a and b. This intersection is transverse. The images of the line and the half-circle have a unique intersection point in W^u because ψ is injective on $H - H_0$ and the images of the endpoints are distinct by hypothesis. Thus for T sufficiently large and ϵ sufficiently small the points $\psi(\rho_c(T))$ and $\psi(\rho_c(\epsilon))$ are in different components of the complement of γ. Now $G^+(\psi(\rho_c(t))) = \Im(c+ti) = kt$ so as $t \to \infty$, $G^+(\psi(\rho_c(t))) \to \infty$. Since G^+ is a continuous function on W^u we see that $\psi(\rho_c(t))$ eventually leaves any bounded set in W^u. So for $t > t_0$, $\psi(\rho_c(t))$ is in the unbounded component of the complement of γ. Since $\psi \circ \rho_c$ crosses γ at t_0 we see that for $t < t_0$, $\psi(\rho_c(t))$ is in the bounded component of the complement of γ.

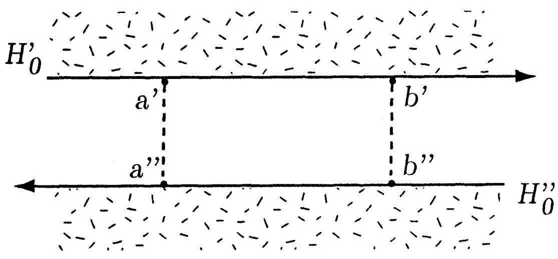


Examples of Curves γ in Propositions 6.3 and 6.4

Now consider a point $d \in H_0$ but not in $[a, b]$. We see that the vertical line based at d does not cross γ. As before the image of this vertical line in W^u contains points going to infinity so we conclude that $\psi(d)$ is in the unbounded component of the complement of γ. If d lies in some other path component but $\psi(d) \in W^u$ the same construction shows that $\psi(d)$ is in the unbounded component of the complement of γ. Thus the only way that $\psi(c) = \psi(d)$ is possible is that $d \in (a, b)$. □

PROPOSITION 6.4 (Orientation). – *Let H_0' and H_0'' be distinct path components of Σ_0. Suppose that there are points $a', b' \in H_0'$ and $a'', b'' \in H_0''$ with $\psi(a') = \psi(a'') \neq \psi(b') = \psi(b'')$. If $a' < b'$, then $a'' > b''$.*

Proof. – Let H' (respectively, H'') denote the component of Σ_+ corresponding to H_0' (respectively H_0''). It follows that $\Psi(H') \cup \Psi(H'') \subset W^u(\psi(a'))$. Thus the orientations of



Orientation

H' and H'' agree with the orientation of $W^u(\psi(a'))$. Let $g' : [0,1] \to H'$ parametrize the half-circle with $g'(0) = a'$ and $g'(1) = b'$. Let $g'' : [0,1] \to H''$ parametrize the half-circle with $g''(0) = b''$ and $g''(1) = a''$. The union of the images $\psi \circ g'$ and $\psi \circ g''$ is an embedded circle which we call γ. The maps g' and g'' induce a consistent orientation on the circle γ. Choose $c' \in (a', b')$ and $c'' \in (a'', b'')$. Let $\rho_{c'} : [0,\infty] \to H'_0$ and $\rho_{c''} : [0,\infty] \to H''_0$ be parametrizations of the vertical lines ending at c' and c''. As before we see that $\psi(c')$ and $\psi(c'')$ are in the bounded component of the complement of γ and for T large $\psi(\rho_{c'}(T))$ and $\psi(\rho_{c''}(T))$ are in the unbounded component. It follows that the algebraic intersection numbers of $\psi \circ \rho_{c'}[0,\infty]$ and $\psi \circ \rho_{c''}[0,\infty]$ with γ are the same. Since ψ, being holomorphic, is orientation preserving, it suffices to calculate the intersection numbers of $\rho_{c'}[0,\infty]$ and $\rho_{c''}[0,\infty]$ with $g'[0,1]$ and $g''[0,1]$. These intersection numbers are $\operatorname{sgn}(a' - b')$ and $\operatorname{sgn}(b'' - a'')$. So $\operatorname{sgn}(a'' - b'') = -\operatorname{sgn}(b'' - a'') = -\operatorname{sgn}(a' - b') = -1$ and $b'' < a''$. □

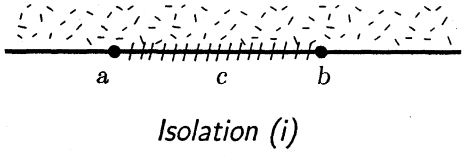
The next condition says that in the presence of certain identifications, the points that can be identified with c must lie in the hatched portions of the intervals indicated. In particular, these identifications are isolated from other path components of the solenoid.

PROPOSITION 6.5 (Isolation)

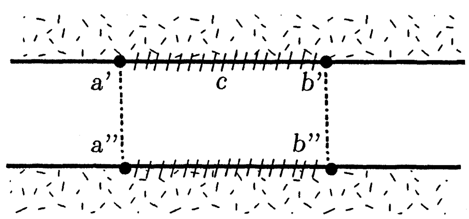
(i) *Suppose that $a, b, c \in H_0$, and $a < c < b$ with $\psi(a) = \psi(b) \neq \psi(c)$. If $d \in \Sigma_0$ satisfies $\psi(d) = \psi(c)$, then $d \in H_0$.*

(ii) *Let H'_0 and H''_0 be distinct path components of Σ_0 with $a', b' \in H'_0$, $a'', b'' \in H''_0$ and $\psi(a') = \psi(a'')$, $\psi(b') = \psi(b'')$. If $a' < c' < b'$ and $c'' \in \Sigma_0$ satisfy $\psi(c') = \psi(c'')$, $\psi(c') \neq \psi(a')$, $\psi(c') \neq \psi(b')$, then $c'' \in H'_0 \cup H''_0$.*

Remark. – It follows from Proposition 6.5 (Non-Linking) that in case (i) we have $d \in (a, b)$; and in case (ii), if $c'' \in H'_0$, then $c'' \in (a', b')$. If $c'' \in H''_0$, then it follows from Proposition 6.4 (Orientation) that $c'' \in (a'', b'')$.



Isolation (i)



Isolation (ii)

Proof. – To prove (i), let H denote the path component of Σ_+ corresponding to H_0, and let $\gamma \subset W^u(\psi(a))$ be the Jordan curve constructed in the proof of Proposition 6.3 Let $\mathcal{O}$ denote the bounded component of the complement of γ. Let H_d denote the path component of Σ_+ containing the ray R_d. We claim that $H_d = H$. Otherwise, H and H_c are disjoint, so $\Psi(H_d)$ is disjoint from γ. Further, since $\Psi(H)$ contains a neighborhood of $\gamma - \{\psi(a), \psi(b)\}$, and since $\psi(d) \neq \psi(a) = \psi(b)$, it follows that $\psi(d) \in \mathcal{O}$. Thus we must have $\Psi(H_d) \subset \mathcal{O}$, forcing $\Psi(H_d)$ to be bounded, contradicting Corollary 5.8.

For the proof of (ii), we let γ be the curve constructed in the proof of Proposition 6.4. It follows that c' lies in the bounded component $\mathcal{O}$ of the complement of γ. If $H_{c''}$ denotes the path component of Σ_+ corresponding to c'', then $\Psi(H_{c''})$ is a domain in $W^u(\psi(a'))$. Let H' and H'' denote the path components of Σ_+ corresponding to H'_0 and H''_0. If $c'' \notin H'_0 \cup H''_0$, then $H_{c''}$ is disjoint from $H' \cup H''$, and $\Psi(H_{c''})$ is disjoint from γ. Since $\psi(c') \in \mathcal{O}$ is in the closure of $\Psi(H_{c''})$, it follows that $H_{c''} \subset \mathcal{O}$. But $\mathcal{O}$ is bounded, so this contradicts Corollary 5.8, and so $H_{c''}$ must be equal either to H' or H''. This contradiction shows that $c'' \in H'_0 \cup H''_0$. □

We have been considering how points inside various path components are identified. We could also consider an equivalence relation on path components: two path components H_0 and H_1 satisfy $H_0 \approx \mathrm{H}_1$ if $\psi(H_0)$ and $\psi(\mathrm{H}_1)$ are contained in the same unstable manifold $W^u(p)$. By Theorem 5.12, $H_0 \approx \mathrm{H}_1$ implies that $a_0 \sim a_1$ for some $a_0 \in H_0$ and $a_1 \in \mathrm{H}_1$. Suppose that $\mathrm{H}'_1 \approx \ldots \approx H'_k$ are path components of Σ_0, so there exist $p_j \in H'_j$ and $q_j \in H'_{j+1}$ such that $p_j \sim q_j$. By Theorem 5.15 there exist points $a_j \in H'_j$, $1 \leq j \leq k$ such that $a_1 \sim \ldots \sim a_k$. If $k \geq 3$, there is a unique class $[a]$ such that $H'_j \cap [a] \neq \emptyset$ for $1 \leq j \leq k$. We note that, by Proposition 5.6, $k \leq N$.

7. Pinch points and Primary Pinch Points

In this section we deal with the local topology of the sets $K \cap W^u(p)$. Let us say that the slices at points p and p' in J are locally homeomorphic if there is a homeomorphism of a neighborhood N of p in $W^u(p)$ to a neighborhood of N' of p' in $W^u(p')$ which takes $K \cap N$ to $K \cap N'$.

There is one local homeomorphism invariant which is closely related to the equivalence relation E. By construction the equivalence classes of the equivalence relation E are in one-to-one correspondence with the elements of J. If $p \in J$ corresponds to an equivalence class $a_1 \sim a_2 \sim \ldots \sim a_k$ with k elements then we say that the *valence* of p is k and we denote it by $v(p)$. By the remark preceding Corollary 5.4 the valence of p is equal to the number of local components of $W^u(p) - K$ at p. In particular, it is a local homeomorphism invariant of p. Since $K \cap W^u$ is connected and simply connected, the number of local components of $W^u \cap K - \{p\}$ at p is the same as the number of components of $W^u \cap K - \{p\}$.

A point p is a *cut point* for $K \cap W^u(p)$ if its removal disconnects $W^u(p) \cap K$. This occurs exactly when $v(p) \geq 2$. Thus the study of cut points of $K \cap W^u$ is equivalent to the study of equivalence classes of E with cardinality greater than or equal to two.

We define a second local homeomorphism invariant. Let $a(p)$ be the number of local components of $W^u - J$ which contain p in their boundaries. We have seen that the local components of $W^u - K$ at p have this property, so $v(p) \leq a(p)$. As will be shown in Lemma 7.5, any two local components U_1 and U_2 of $W^u - J$ at p with $p \in \overline{U}_1 \cap \overline{U}_2$ are separated by a component of $W^u - K$. Thus $a(p) \le 2v(p)$. We define p to be a *pinch point* if $a(p) \ge 3$. It follows that $v(p) \ge 2$, so every pinch point is a cut point.

We obtain a different notion of valence if we count only the unbounded components of $W^u(p) \cap K - \{p\}$. This quantity is not a local homeomorphism invariant. The number of unbounded components of $W^u \cap K - \{p\}$ is the same as the number of path components of the solenoid which are identified at p. This corresponds to the equivalence relation $\approx$ discussed at the end of §6. It is also the number of components of $W^u(p) - K$ which have p in their closures. Let us define $V(p)$ to be this number. We define $A(p)$ to be the number of unbounded components of $W^u(p) - J$ which have p in their closures. A point p is a *primary pinch point* if $A(p) \ge 3$.

In this section we prove the following.

THEOREM 7.1. – *The set of primary pinch points is finite. If $p \in J$ is any pinch point, then there exists a primary pinch point $q \in J$ such that $p \in W^s(q)$.*

If points p and p' are on the same stable manifold it follows from the local product structure that p and p' are locally homeomorphic. In particular the points p and q in Theorem 7.1 are locally homeomorphic (as are all points on $W^s(q) \cap J$).

If the set of pinch points is nonempty, it is dense in J. Theorem 7.1 tells us that under the equivalence relation of local homeomorphism these pinch points fall into a finite number of classes. Since the set of primary pinch points is finite, these points are periodic. We note two consequences of this fact.

COROLLARY 7.2. – *If p is a primary pinch point, then each component of $(W^u(p) \cap K) - \{p\}$ is noncompact.*

Proposition 6.4 described certain restrictions on how a pair of path components of the solenoid can be identified. Additional information from Theorem 7.1 provides additional restrictions in this case.

COROLLARY 7.3. – *Suppose that H_0' and H_0'' are distinct path components of Σ_0, and suppose that $a', b' \in H_0'$, $a' < b'$ and $a'', b'' \in H_0''$ with $a' \sim a''$ and $b' \sim b''$. If there are no identifications between points of (b', ∞) and $(-\infty, b'')$, then $\psi(b')$ is periodic, and there are infinitely many identifications between (a', b') and (b'', a''), as well as an infinite number of identifications between $(-\infty, a')$ and (a'', ∞) (A corresponding conclusion holds if there are no identifications between $(-\infty, a')$ and (a'', ∞)).*

Let us fix an unstable manifold W^u, which we identify with $\mathbf{C}$. Let $D_1 \subset D_2$ be open disks in $\mathbf{C}$ with the same center and radii $r_1 < r_2$. For a connected subset C of $\mathbf{C} - J$, we let $\tilde{C}$ denote the connected component of $\mathbf{C} - J$ which contains C. We define $\mathcal{C}(D_2)$ as the set of connected components C of $D_2 - J$ such that $\tilde{C} \cap \partial C \ne \emptyset$ (It follows that $\tilde{C} \cap \partial C \subset \partial D_2$). Similarly, $\mathcal{C}(D_1, D_2)$ is defined to be the set of connected components C of $D_2 - \overline{D}_1$ such that $\tilde{C} \cap \partial C$ intersects both ∂D_1 and ∂D_2. For $p \in D_2$ we set

$$\mathcal{C}_p = \mathcal{C}_p(D_2) = \{c \in \mathcal{C}(D_2) : p \in \partial C\}.$$

We say that $p \in D_2$ is *primary with respect to* D_2 if $\#\mathcal{C}_p(D_2) \ge 3$. Let $\mathcal{P} = \mathcal{P}(D_1, D_2)$ denote the set of points of D_1 which are primary with respect to D_2. We write

$$\mathcal{C}_{\mathcal{P}} = \bigcup_{p \in \mathcal{P}} \mathcal{C}_p(D_2).$$

Since each component $C \in \mathcal{C}_p$ intersects $D_2 - \overline{D}_1$ in at least one element of $\mathcal{C}(D_1, D_2)$, it is immediate that

$$\#\mathcal{C}_{\mathcal{P}} \leq \#\mathcal{C}(D_1, D_2).$$

The circular ordering on ∂D_2 induces a circular ordering on the set $\mathcal{C}(D_1, D_2)$. To see this, we note that we may assign to each $C \in \mathcal{C}(D_1, D_2)$ an open arc σ_C whose boundary points lie in $\tilde{C} \cap (\partial D_1 \cup \partial D_2)$. Since for distinct components, the corresponding arcs will have disjoint closures, the intersections of the arcs with ∂D_2 yield a well-defined ordering on $\mathcal{C}(D_1, D_2)$. Given $C', C'' \in \mathcal{C}(D_1, D_2)$, there is a well-defined component of $(D_2 - \overline{D}_1) - (\sigma_{C'} \cup \sigma_{C''})$ corresponding to the interval $(\sigma_{C'}, \sigma_{C''})$, in the sense of increasing angle. Thus we may define the portion of the complement $(D_2 - \overline{D}_1) - (C' \cup C'')$ between C' and C'' in the sense of increasing angle, as the portion of the complement which lies between $\sigma_{C'}$ and $\sigma_{C''}$.

For each $C \in \mathcal{C}(D_2)$ there are two cases: the first is when $C \cap K = \emptyset$, and the second is when $C \subset \operatorname{int}(K)$. We refer to these two cases as components of the first and second kind. In §5 we showed that a component of the first kind satisfies a John condition and thus has a well-behaved boundary.

The remainder of this section will be devoted to proving Theorem 7.1. The main step (Lemma 7.7) will be to show that $\mathcal{P}$ is finite.

LEMMA 7.4. – $\#\mathcal{C}(D_1, D_2) < \infty$.

Proof. – Let $\{C_1, \ldots, C_N\} \subset \mathcal{C}(D_1, D_2)$ be a finite set of components of the first kind, i.e. $C_j \cap K = \emptyset$ for $1 \leq j \leq N$. Since C_j is connected and meets both the inner and outer boundaries of the annulus, there is a point $z_j \in C_j$ with distance $(r_1 + r_2)/2$ from the center. There is a c-carrot γ_j connecting z_j to infinity inside $W^u - K$. This carrot must exit U_j through an interval inside $\partial C_j \cap (\partial D_1 \cup \partial D_2)$. Since this interval must have length at least $c(r_2 - r_1)/2$, and since these intervals have disjoint interiors, it follows that there can be no more than $4\pi(r_1 + r_2)/(c(r_2 - r_1))$ intervals of the first kind.

Now let us suppose that $\{C_1, \ldots, C_N\} \subset \mathcal{C}(D_1, D_2)$ is a finite set of components of the second kind, i.e. $U_j \subset \operatorname{int}(K)$. Since each C_j is a component of $int(K \cap W^u)$, $\partial C_j \cap \partial C_{j+1}$ is nowhere dense in ∂C_j (and in ∂C_{j+1}). For $\epsilon > 0$, let $A \subset D_2 - \overline{D}_1$ denote the annulus with inner radius $\frac{1}{2}(r_1 + r_2) - \epsilon$ and outer radius $\frac{1}{2}(r_1 + r_2) + \epsilon$. There exist points $z_j \in A \cap (W^u - K)$ such that z_j is between C_j and C_{j+1} in the sense of increasing angle with respect to the circular ordering. As before, there is a c-carrot γ_j connecting z_j to infinity inside $W^u - K$. This c-carrot must exit the annulus in an open interval of $(\partial D_1 \cup \partial D_2) - K$ between $\overline{C}_j$ and $\overline{C}_{j+1}$. Thus the intervals are pairwise disjoint. Since $\epsilon > 0$ may be taken arbitrarily small, we have the same lower bound on the length of the intervals, and thus the same upper bound on the number of components of the second kind. □

LEMMA 7.5. – *Let $p \in \mathcal{P}$ be given, and let $\mathcal{O}', \mathcal{O}'' \in \mathcal{C}_p$ be two components of the second kind. Then there are components $C_1, C_2 \in \mathcal{C}_p$ which are of the first kind and such that C_1 lies between $\mathcal{O}'$ and $\mathcal{O}''$ in the sense of increasing angle, and C_2 lies between $\mathcal{O}''$ and $\mathcal{O}'$.*

Proof. – By Lemma 5.6, $\mathbf{C} - K$ has finitely may components at p. Thus there exist $\epsilon > 0$ and a finite number of components $C_1, \ldots, C_N$ of the first kind such that

$$D(\epsilon) - K = D(\epsilon) \cap \bigcup_{j=1}^{N} C_j$$

where $D(\epsilon) := \{z \in \mathbf{C} : |z - p| < \epsilon\}$. Now suppose that no component C_j, $1 \le j \le N$, lies between $\mathcal{O}'$ and $\mathcal{O}''$. Then, relabeling the components if necessary, we have

$$\mathcal{O}' < \mathcal{O}'' < C_1 < \cdots < C_N < \mathcal{O}'$$

in the sense of the circular ordering. There is an arc $\gamma \subset (\partial C_1 \cup \partial C_N) \cap D(\epsilon)$ such that $\bigcup C_j \cap D(\epsilon)$ lies on one side of γ. Given the circular ordering above, however, $\mathcal{O}'$ and $\mathcal{O}''$ must lie on the other side of γ. In other words, γ divides $D(\epsilon)$ into two components $D(\epsilon) - \gamma = D'(\epsilon) \cup D''(\epsilon)$, where $\bigcup C_j \cap D(\epsilon) = D(\epsilon) - K \subset D'(\epsilon)$, and $D(\epsilon) \cap (\mathcal{O}' \cup \mathcal{O}'') \subset D''(\epsilon)$. But this is not possible, since the first containment means that $D''(\epsilon) \subset K$, so

$$J \cap D''(\epsilon) = D'' \cap \partial(K \cap D(\epsilon)) = \emptyset.$$

This contradicts the fact that there must be a nonempty boundary between $\mathcal{O}'$ and $\mathcal{O}''$ in $D''(\epsilon)$, which implies that $\emptyset \ne \partial\mathcal{O}' \cap D''(\epsilon) \subset J \cap D''(\epsilon)$. □

COROLLARY 7.6. – *If $p \in \mathcal{P}$, then there are at least two components of the first kind in $\mathcal{C}_p$.*

LEMMA 7.7. – $\#\mathcal{P} < \infty$.

Proof. – For each $p \in \mathcal{P}$, we have $\#\mathcal{C}_p \ge 3$, so there are at least three components $C_1, C_2, C_3 \in \mathcal{C}_\mathcal{P}$ with $p \in \partial C_1 \cap \partial C_2 \cap \partial C_3$. By Corollary 7.6, we may assume that C_1 and C_2 are of the first kind, that is $(C_1 \cup C_2) \cap K = \emptyset$. Here we will show that, conversely, given components $C_1, C_2, C_3 \in \mathcal{C}_\mathcal{P}$ with C_1 and C_2 of the first kind, the intersection $\partial C_1 \cap \partial C_2 \cap \partial C_3$ can contain at most one point. Thus it will follow that the cardinality $\#\mathcal{P}$ is bounded by the number of triples of elements of $\mathcal{C}_\mathcal{P}$. Since by Lemma 7.4 we know that $\#\mathcal{C}_\mathcal{P} < \infty$, it will follow that $\#\mathcal{P} < \infty$.

Let p and q be distinct points of $\partial C_1 \cap \partial C_2$. Since C_1 and C_2 are of the first kind, we have arcs $\gamma_{1,p}, \gamma_{1,q} \subset C_1$ and $\gamma_{2,p}, \gamma_{2,q} \subset C_2$ such that $\gamma_{j,p}$ connects a point of $\tilde{C}_j \cap \overline{C}_j \subset \partial D_2$ to p, for $j = 1, 2$, and $\gamma_{j,q}$ connects a point of $\tilde{C}_j \cap \overline{C}_j \subset \partial D_2$ to q, for $j = 1, 2$. Thus $\gamma_p := \gamma_{1,p} \cup \gamma_{2,p}$ and $\gamma_q := \gamma_{1,q} \cup \gamma_{2,q}$ are disjoint arcs in $\overline{D}_2$, each of which disconnects D_2 into two components. Let $e_{j,p}, e_{j,q} \in \partial D_2$ denote the enpoints of $\gamma_{j,p}$ and $\gamma_{j,q}$. Without loss of generality, we may assume that $e_{1,q} < e_{1,p} < e_{2,p} < e_{2,q}$ in the sense of the circular ordering. Now suppose that $C_3 \in \mathcal{C}_\mathcal{P}$ contains p in its closure. Note that C_3 is disjoint from γ_p and γ_q, and we must have $C_1 < C_3 < C_2$ in the sense of the circular ordering. It follows that C_3 is contained in the connected component of $D_2 - \gamma_p$ that does not contain γ_q. Thus $q \notin \partial C_3$, so $\partial C_1 \cap \partial C_2 \cap \partial C_3$ consists of only one point. □

We will use the following variant of the Contraction Mapping Principle.

LEMMA 7.8. – *Suppose that $X_1, \ldots, X_N$ are compact metric spaces, that $f : \bigcup X_j \to \bigcup X_j$ is a homeomorphism, and that $f|_{X_j}$ is a contraction for each $1 \le j \le N$. Then $\bigcup X_j$ is finite.*

Proof of Theorem 7.1. – Let us cover J by a finite number of pairs $\mathcal{R}'_j, \mathcal{R}''_j$ of local product neighborhoods as in §5. For each j and for $z_1, z_2 \in D^s_j$, $(\{z_1\} \times \Delta'_j) \cap J$ is homeomorphic via the local product structure to $(\{z_2\} \times \Delta'_j) \cap J$. For each z, we set $D_1 = \{z\} \times \Delta'_j$ and $D_2 = \{z\} \times \Delta''_j$. For each j and z we let $\mathcal{P}_j(z) = \mathcal{P}(D_1, D_2)$ denote the points in D_1 which are primary with respect to D_2. By Lemma 7.7 the sets $\mathcal{P}_j(z)$ are finite. The homeomorphism between $(\{z_1\} \times \Delta'_j) \cap J$ and $(\{z_2\} \times \Delta'_j) \cap J$ carries $\mathcal{P}(z_1)$ to $\mathcal{P}(z_2)$. Thus $\bigcup_{z \in D^s_j} \mathcal{P}_j(z)$ is a finite union of local stable disks. Let S be the union of these sets as j varies. It is clear from the construction that f is a local contraction on S. We may assume that S is a closed set.

Now q is a primary pinch point if and only if q is a primary point with respect to D for all disks $D \subset W^u(q)$ containing q, and it follows that if q is a primary pinch point then $f^n(q) \in S$ for all n. Let S^∞ be the set of points with this property. The set S^∞ is invariant under f hence the restriction of f to S^∞ is a locally contracting homeomorphism. Lemma 7.8 then shows that S^∞, the set of primary pinch points, is finite.

For the second assertion in Theorem 7.1, let p be a pinch point. We may choose a small disk D around p such that p is primary with respect to D. There is an n_0 with the property that if $n \geq n_0$ and $f^n(p) \in \mathcal{R}'_j$, then $f^n(D)$ overflows the disk $\{z\} \times \Delta''_j$ containing $f^n(p)$. It follows that $f^n(p) \in S$ for $n \geq n_0$. Write $\omega(p)$ for the ω-limit set of p. Since S is closed, $\omega(p) \subset S$. Since ω-limit sets are invariant under f^{-1}, $\omega(p) \subset S^\infty$. It follows that for n large $f^n(p)$ is in the same stable disk as an element q of S^∞ thus $p \in W^s(f^{-n}(q))$. □

Appendix A: Critical Points

In [BS5] we defined certain critical points for a polynomial diffeomorphism f. On the set $U^+ \cap U^-$, the critical points are defined as the heteroclinic tangencies, i.e. the tangencies between $\mathcal{G}^+$ and $\mathcal{G}^-$. Thus on $U^+ \cap U^-$, the critical locus is given by $\mathcal{C} = \{\zeta \in U^+ \cap U^- : \partial G^+(\zeta) \wedge \partial G^-(\zeta) = 0\}$. The critical points of $U^+ \cap J^-$ (resp. $U^- \cap J^+$) are the heteroclinic tangencies of $\mathcal{G}^+$ and $\mathcal{W}^u$ (resp. $\mathcal{G}^-$ and $\mathcal{W}^s$) and are denoted by $\mathcal{C}^u$ (resp. $\mathcal{C}^s$). If we let τ^- denote a holomorphic vector field defining the kernel of ∂G^-, then τ^- is tangent to $\mathcal{G}^-$, and so $\mathcal{C}$ is the set where τ^- is tangent to $\mathcal{G}^+$, and $\mathcal{C}^s$ is the set where τ^- is tangent to $\mathcal{W}^s$.

It is natural to define unstable critical measures $\mu_c^\pm$ on $\mathcal{C}^{s/u}$. Let U be an open set where the lamination $\mathcal{W}^s$ is trivial so that $\mathcal{W}^s \cap U$ consists of $\{\Gamma_e : e \in E\}$ for some compact E. We let μ_τ^+ denote any transversal measure on $U \cap \mathcal{W}^s$. For a stratum Γ_e we let $[\mathrm{Crit}(G^-, \Gamma_e)]$ denote the current of integration over the critical points of the restriction of G^- to Γ_e, which is the sum of point masses at the critical points (with multiplicity) of $G^-|\Gamma_e$. The restriction of the critical measure to U is given by

$$\mu_c^+ \llcorner U = \int_{e \in E} \mu_\tau^+(e)\, [\mathrm{Crit}(G^-, \Gamma_e)].$$

For $\epsilon > 0$, we have currents

$$\mu_\epsilon^\pm := \frac{1}{2\pi} dd^c \max(G^\pm, \epsilon),$$

which approximate $\mu^\pm$ in the sense that $\lim_{\epsilon \to 0} \mu_\epsilon^\pm = \mu^\pm$ as currents on $\mathbf{C}^2$. These currents give critical slice measures on $U^+ \cap U^-$:

$$\mu_{c,\epsilon}^\pm := [\mathcal{C}] \wedge \mu_\epsilon^\pm.$$

As a consequence of the local triviality of the lamination $\mathcal{L}^s = \mathcal{G}^+ \cup \mathcal{W}^s$, we have the following:

PROPOSITION A1. – *If f is hyperbolic, and $\mathcal{L}^s = \mathcal{G}^+ \cup \mathcal{W}^s$ is a (locally trivial) lamination, then $\mathcal{C}^s = (J^+ \cap U^-) \cap \overline{\mathcal{C}}$, and*

$$\mu_c^+ = \lim_{\epsilon \to 0} \mu_{c,\epsilon}^+.$$

Remark. – By Proposition 2.7, this holds whenever f is hyperbolic and unstably connected.

Proof. – Let us fix a point $p_0 \in J^+ \cap U^-$ and let T be a complex 1-dimensional manifold which intersects $\mathcal{W}^s$ transversally at p_0. Shrinking T, if necessary, we have a neighborhood U of p_0 on which $\mathcal{L}^s$ is trivial: $\mathcal{L}^s \cap U$ is homeomorphic to a product lamination $\{\Gamma_e : e \in E\}$, with $E = T \cap (J^+ \cup U^+)$. If we choose a holomorphic vector field τ^- on U, then $\tau^-(G^-)$ is a holomorphic function on U. Thus

$$\mathrm{Crit}(G^-, \Gamma_e) = \{\tau^-(G^-)|_{\Gamma_e} = 0\}$$

varies continuously in $e \in E$.

If $p_0 \in \mathcal{C}^s$, and thus if $\tau^-(G^-)(p_0) = 0$, then it follows that the set $\{\tau^-(G^-) = 0\} \cap \Gamma_e \neq \emptyset$ contains points close to p_0 if e is close to e_0. Conversely, if $p_0 \notin \mathcal{C}^s$, then $\{\tau^-(G^-) = 0\} \cap \Gamma_e$ contains no points near p_0 if e is close to e_0. This shows that $\mathcal{C}^s = \overline{\mathcal{C}} \cap (J^+ \cap U^-)$.

Now we note that

$$[\mathcal{C}] \wedge \mu_\epsilon^+ = \frac{1}{2\pi} dd^c \max(G^+|_{\mathcal{C}}, \epsilon),$$

since dd^c is intrinsic to the complex structure of $\mathcal{C}$. We may also write $[\mathcal{C}] \wedge \mu_\epsilon^+ = \mu_\epsilon^+|_{\mathcal{C}}$ as the (transversal) measure induced by μ_ϵ^+ on $\mathcal{C}$.

Let χ denote the holonomy map taking $U \cap \mathcal{C}$ to T by sliding inside the strata Γ_e of the product lamination. It follows that χ preserves $dd^c G^+$ on the sets $\{G^+ = \epsilon\}$. That is, χ preserves the transversal measures in the sense that if $\mathcal{C}$ has multiplicity m in a neighborhood V of $q_0 \in U \cap \mathcal{C} \cap \{G^+ = \epsilon\}$, then

$$\chi_*(\mu_\epsilon^+|_{\mathcal{C}\cap V}) = m\, \mu_\epsilon^+|_{T \cap \chi(V)}.$$

It follows that the restriction of $\mu_{c,\epsilon}^+$ to the open set U is given by

$$[\mathcal{C}] \wedge \mu_\epsilon^+ \llcorner U = \int_{e \in E} \mu_\epsilon^+|_T(e)\, [\mathrm{Crit}(G^-, \Gamma_e)].$$

Finally, we observe that as $\epsilon \to 0$, the transversal measure $\mu_\epsilon^+|_T$ converges to the transversal measure $\mu^+|_T$; and by Proposition 2.7, $[\mathrm{Crit}(G^-, \Gamma_e)]$ varies continuously as $e \in E$ approaches J^+. Thus the integral expression for the restriction to U of $[\mathcal{C}] \wedge \mu_\epsilon^+$ converges to μ_c^+, restricted to U. □

In general $\overline{\mathcal{C}} \cap (J^\pm \cap U^\mp) \neq \emptyset$ (see [BS5, Prop. 1.7]), but we do not know the relationship between $\mathcal{C}$ and $\mathcal{C}^{s/u}$, nor do we know the relationship between $[\mathcal{C}]$ and $\mu_c^\pm$. For instance, if f is unstably connected, then $\overline{\mathcal{C}} \cap (J_+^-) \neq \emptyset$, but $\mu_c^- = 0$, and $\mathcal{C}^u = \emptyset$.

COROLLARY A2. – *If f is hyperbolic and unstably connected, then the partition $\mathcal{L}^u = \mathcal{G}^- \cup \mathcal{W}^u$ is not a (locally trivial) lamination in a neighborhood of any point of $(J_+^-) \cap \overline{\mathcal{C}}$.*

Proof. – Let us suppose that $\mathcal{L}^u$ is a lamination in a neighborhood of a point $p_0 \in (J_+^-) \cap \overline{\mathcal{C}}$. Then by the proof of the Proposition above, it follows that $\mathcal{C}^u = (J_+^-) \cap \overline{\mathcal{C}}$ in a neighborhood of p_0, and thus the set is nonempty. This contradicts the fact that $\mathcal{C}^u = \emptyset$ when f is unstably connected. □

COROLLARY A3. – *If f is hyperbolic and J is connected, then f is not volume preserving.*

Proof. – Let f be hyperbolic and volume preserving with J connected. It follows from [BS6] that f is both stably and unstably connected. If f were stably connected then applying Proposition 2.7 to f^{-1} would show that $\mathcal{L}^u$ is a locally trivial lamination. This contradicts Corollary A.2. □

Appendix B: An Illustration

One goal of this paper has been to represent J as a quotient of the solenoid. In §6 and §7 we discussed this quotient in terms of topological properties of the slices $W^u \cap J$. Another approach is to use symbol sequences. We write the equivalence relation as $E = \{(t_1, t_2) \in \Sigma_0 \times \Sigma_0 : \psi(t_1) = \psi(t_2)\}$. We represent the solenoid as a quotient of the bilateral shift on d symbols $\mathcal{S}_d = \{0, 1, \ldots, d-1\}^{\mathbf{Z}}$, with the quotient map being written as $\theta : \mathcal{S}_d \to \Sigma_0$. This equivalence relation may then be lifted to the space of sequences: $\theta^{-1}E = \{(t_1, t_2) \in \mathcal{S}_d \times \mathcal{S}_d : \psi \circ \theta(t_1) = \psi \circ \theta(t_2)\}$. By Fried [F], $\theta^{-1}E$ is a subshift of finite type. The object is to know what symbolic relations $\theta^{-1}E$ can occur, as well as their relationship to the topological conditions discussed in §6 and §7.

A: $^{\infty}0101.001(01)^{\infty}$
B: $^{\infty}0100.110(10)^{\infty}$
C: $^{\infty}0100.101(01)^{\infty}$
D: $^{\infty}0100.100(10)^{\infty}$
E: $^{\infty}0100.011(01)^{\infty}$
F: $^{\infty}0011.100(10)^{\infty}$
G: $^{\infty}0011.011(01)^{\infty}$
H: $^{\infty}0011.010(10)^{\infty}$
I: $^{\infty}0011.001(01)^{\infty}$
J: $^{\infty}0010.110(10)^{\infty}$
K: $^{\infty}0010.101(01)^{\infty}$

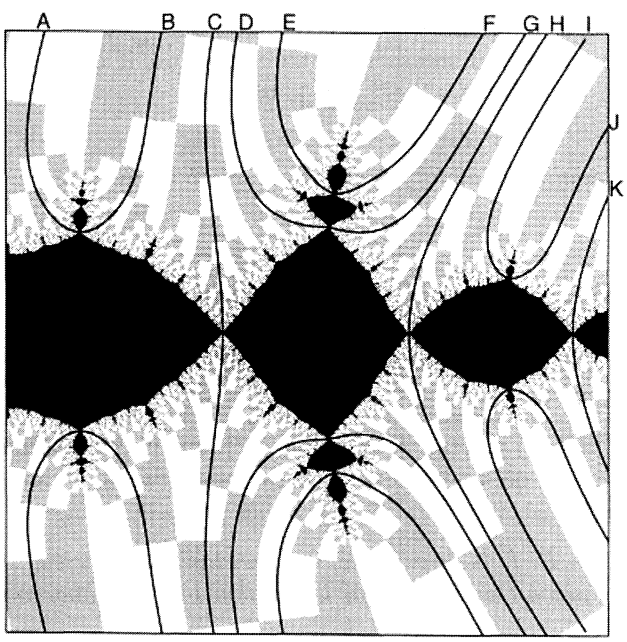


Image by Ricardo Oliva, Cornell University.

External rays in the unstable manifold of the fixed point with positive eigenvalues, for the Hénon map, $f(x, y) = (x^2 + c - ay, x)$, with $a = 0.124$, $c = -1.250$.

Ricardo Oliva [O] has carried out work in this direction, calculating combinatorial models for equivalence relations for certain mappings and their bifurcations. We are grateful to Oliva for allowing us to include a computer graphic from [O]. Let f be mapping defined by $f(x,y) = (x^2 + c - ay, x)$ with $a = 0.124$ and $c = -1.25$. For these values, f has an attracting 2-cycle and two saddle fixed points. We may write the saddle points as $p^{\pm}$, with the choice of $\pm$ determined by the condition that the eigenvalues of $\mathrm{D}f(p^+)$ are positive, while the eigenvalues of $\mathrm{D}f(p^-)$ are negative. We choose the saddle point p^+ and let $\psi : \mathbf{C} \to W^u(p^+)$ be an analytic uniformization which takes real values on $\mathbf{R}$ and such that $\psi(0) = p^+$. The Figure illustrates the n-th level of the Green function restricted to the unstable manifold $W^u(p^+)$, i.e. these are the sets $\{2^{-n-1} < G^+ \circ \psi < 2^{-n}\}$ inside $\mathbf{C}$. Within the n-th level of the Green function, the colors alternate to reveal the sets $\frac{j}{2^{n+1}} < \operatorname{Arg}(\varphi^+ \circ \psi) < \frac{j}{2^n}$. In other words, within the n-th level, the region is colored white if the n-th binary digit of $\operatorname{Arg}(\varphi^+ \circ \psi)$ is $\mathtt{0}$ and gray if it is $\mathtt{1}$.

Certain external rays γ (orthogonal to the level sets of $G^+ \circ \psi$) are indicated in the picture as dark curves. The set $\mathcal{E}$ of external rays is parametrized by the solenoid Σ_0 (via the map Ψ). We use a point of the solenoid to refer both to an external ray γ and to its endpoint $e(\gamma) \in J$. By the quotient map $\theta : \mathcal{S}_2 \to \Sigma_0$, a point of the solenoid may be represented by a sequence of $\mathtt{0}$'s and $\mathtt{1}$'s; this is determined in the Figure according to the white or gray patches that γ traverses. A ray γ corresponds to the sequence $(s_n)_{n\in\mathbf{Z}}$ (sometimes written $\ldots s_{-1}s_0.s_1s_2\ldots$) where s_n corresponds to the color at level n.

The condition that the parameters a and c are real means that f is the complexification of a real mapping, so f is invariant under complex conjugation $(x,y) \mapsto (\overline{x},\overline{y})$. Thus the family of all codings and identifications are invariant under the substitution $\mathtt{0}\leftrightarrow\mathtt{1}$. If $A \in \mathcal{S}_2$, we let $\overline{A}$ denote the sequence obtained from A under this substitution, so that the rays corresponding to A and $\overline{A}$ are complex conjugate to each other.

The fixed point p^+ corresponds to ${}^\infty\mathtt{0}.\mathtt{0}^\infty \sim {}^\infty\mathtt{1}.\mathtt{1}^\infty$. The notations ${}^\infty\alpha$ and α^∞ apply to a string α, which is repeated infinitely often, to the left or to the right. With this notation, p^- corresponds to ${}^\infty(\mathtt{01}).(\mathtt{01})^\infty \sim {}^\infty(\mathtt{10}).(\mathtt{10})^\infty$. The Figure represents a subset of $W^u(p^+)$, so the codings of the external rays have the form ${}^\infty\mathtt{0}\cdots$ or ${}^\infty\mathtt{1}\cdots$, where a ray starts with ${}^\infty\mathtt{0}$ if it lies in the upper half plane and ${}^\infty\mathtt{1}$ otherwise. The real axis is the axis of symmetry of the picture, and the origin (corresponding to p^+) is to the right, out of the picture.

Perhaps the most apparent external rays are those which make their final approach J entirely through the boundaries between gray and white regions. (None of these is marked explicitly as a curve in the Figure.) Since they lie in the unstable manifold of p^+, the ones that lie in the upper half plane have the form ${}^\infty\mathtt{0}w\mathtt{01}^\infty \sim {}^\infty\mathtt{0}w\mathtt{10}^\infty$ for some finite word w. The ambiguity of their coding illustrates the identification arising from the equivalence of binary expansions, i.e., these are mapped to the same point under the mapping $\theta : \mathcal{S}_2 \to \Sigma_0$. The fact that the codings end on the right in $\mathtt{0}^\infty$ or $\mathtt{1}^\infty$ indicates that the landing points of these rays belong to $W^s(p^+)$.

The choice of which level set $\{2^{-n-1} < G^+ < 2^{-n}\}$ corresponds to the 0-th generation is somewhat arbitrary. Any level $n = n_0$ may be used, as long as it is used consistently to determine the other generations. In the figure, the placement of the point "." is chosen to be approximately where the rays enter the picture. We note that the solenoidal addresses of the rays show the solenoidal ordering $A > B > C > \cdots > K$. The identifications $A \sim B$, $D \sim G$ $E \sim F$ illustrate the non-linking property. The addresses of the conjugate rays satisfy the reverse inequality $\overline{A} < \overline{B} < \cdots < \overline{K}$. The identifications $C \sim \overline{C}$, $H \sim \overline{H}$,

$K \sim \overline{K}$, together with the inequalities $C > H > K$ and $\overline{C} < \overline{H} < \overline{K}$ correspond to a local version of the orientation principle.

The fact that the codings of these rays all end on the right in $(\mathtt{01})^\infty$ indicates that the landing points belong to $W^s(p^-)$. Thus the shape of $W^u(p^+) \cap K$ in a neighborhood of these landing points gives a picture of $W^u(p^-) \cap K$ in a neighborhood of p^-. By Theorem 7.1, p^- is a primary pinch point.

Each component U of the interior of $K \cap W^u(p)$ inside $W^u(p)$ corresponds to the intersection of $W^u(p)$ with one of the two basins of attraction. These components "alternate" in the sense that if U_1 and U_2 are components of the interior of $K \cap W^u(p)$, and if $\partial U_1 \cap \partial U_2 \neq \emptyset$, then U_1 and U_2 belong to different basins. Further, f acts on the components of the interior $W^u(p^+) \cap K$ which intersect the real axis by moving each of them three components to the left. This is consistent with the observation that the shift σK of K satisfies $\sigma K > A$.

A detailed discussion of diffeomorphisms of $\mathbf{C}^2$ which are perturbations of one-dimensional mappings is given in [HO2]; the mapping illustrated here is not equivalent to a mapping of that form (see Oliva [O]).

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