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J.-B. BOST

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POTENTIAL THEORY AND LEFSCHETZ THEOREMS FOR ARITHMETIC SURFACES

BY J.-B. BOST

ABSTRACT. – We prove an arithmetic analogue of the so-called Lefschetz theorem which asserts that, if D is an effective divisor in a projective normal surface X which is nef and big, then the inclusion map from the support $|D|$ of D in X induces a surjection from the (algebraic) fondamental group of $|D|$ onto the one of X. In the arithmetic setting, X is a normal arithmetic surface, quasi-projective over Spec $\mathbf{Z}$, D is an effective divisor in X, proper over Spec $\mathbf{Z}$, and furthermore one is given an open neighbourhood Ω of $|D|(\mathbf{C})$ on the Riemann surface $X(\mathbf{C})$ such that the inclusion map $|D|(\mathbf{C}) \hookrightarrow \Omega$ is a homotopy equivalence. Then we may consider the equilibrium potential $g_{D,\Omega}$ of the divisor $D(\mathbf{C})$ in Ω and the Arakelov divisor $(D, g_{D,\Omega})$, and we show that if the latter is nef and big in the sense of Arakelov geometry, then the fundamental group of $|D|$ still surjects onto the one of X. This results extends an earlier theorem of Ihara, and is proved by using a generalization of Arakelov intersection theory on arithmetic surfaces, based on the use of Green functions which, up to logarithmic singularities, belong to the Sobolev space L^2_1. © Elsevier, Paris

RÉSUMÉ. – Nous établissons un analogue arithmétique du classique « théorème de Lefschetz », affirmant que si D est un diviseur effectif nef et big d'une surface projective normale X, alors le morphisme d'inclusion du support $|D|$ de D dans X induit une surjection du groupe fondamental (algébrique) de $|D|$ vers celui de X. Dans le cadre arithmétique, X est une surface arithmétique normale, quasiprojective sur Spec $\mathbf{Z}$, D est un diviseur effectif sur X, de support $|D|$ propre sur Spec $\mathbf{Z}$, et l'on se donne un voisinage ouvert Ω de $|D|(\mathbf{C})$ sur la surface de Riemann $X(\mathbf{C})$ tel que l'inclusion $|D|(\mathbf{C}) \hookrightarrow \Omega$ soit une équivalence d'homotopie. On peut alors considérer le potentiel d'équilibre $g_{D,\Omega}$ du diviseur $D(\mathbf{C})$ dans Ω et le diviseur d'Arakelov $(D, g_{D,\Omega})$, puis montrer que si ce dernier est nef et big au sens de la géométrie d'Arakelov, alors le groupe fondamental de $|D|$ se surjecte encore vers celui de X. Ce résultat, qui étend un théorème antérieur d'Ihara, se démontre au moyen d'une généralisation de la théorie d'intersection d'Arakelov sur les surfaces arithmétiques utilisant des fonctions de Green qui, à des singularités logarithmiques près, appartiennent à l'espace de Sobolev L^2_1. © Elsevier, Paris

1. Introduction

1.1. In this paper, we prove arithmetic analogues of the following classical theorem in algebraic geometry, which is a particular instance of the so-called Lefschetz theorems (see section 2 *infra* for a discussion and references):

Theorem 1.1. (Lefschetz theorem for projective surfaces over a field). *Let X be a smooth projective connected surface over an algebraically closed field, and D an effective divisor on X. If D is ample, then its support $|D|$ is connected, and for any geometric point η of $|D|$, the map between algebraic fundamental groups*

$$i_* : \pi_1(|D|, \eta) \to \pi_1(X, \eta), \tag{1.1}$$

induced by the inclusion i of $|D|$ in X, is surjective.

Our arithmetic analogues are theorems concerning arithmetic surfaces, whose statement and proof rely on Arakelov geometry and potential theory on Riemann surfaces. Let us state a special case of them, the formulation of which requires only the most basic definition in Arakelov geometry, namely the one of the Arakelov degree of an hermitian line bundle over the (spectrum of the) ring of integers of a number field.

Let us denote by K a number field and by $\mathcal{O}_K$ its ring of integers. Recall that an *hermitian line bundle* $\overline{\mathcal{L}}$ over Spec $\mathcal{O}_K$ is defined as a pair $(\mathcal{L}, \|\ \|)$ formed by an invertible sheaf $\mathcal{L}$ over Spec $\mathcal{O}_K$ and a family $\|\cdot\| = (\|\cdot\|_\sigma)_{\sigma:K\hookrightarrow\mathbb{C}}$, where for any field embedding $\sigma : K \hookrightarrow \mathbb{C}$, $\|\cdot\|_\sigma$ denotes an hermitian metric on the complex line[1] $\mathcal{L}_\sigma := \mathcal{L}\otimes_\sigma \mathbb{C}$. Any non-zero rational section s of $\mathcal{L}$ over Spec $\mathcal{O}_K$ (*i.e.*, any non-zero element of $\mathcal{L}_K\backslash\{0\}$) has a well-defined $\wp$-adic valuation $v_\wp(s) \in \mathbb{Z}$ for each non-zero prime ideal $\wp$ in $\mathcal{O}_K$. It vanishes for almost every $\wp$, and the real number

$$\sum_{\substack{\wp \text{ prime} \\ \neq 0}} v_\wp(s) \log N\wp - \sum_{\sigma:K\hookrightarrow\mathbb{C}} \log \|s\|_\sigma$$

does not depend on the choice of s by the product formula, and defines the *Arakelov degree* $\widehat{\deg}\,\overline{\mathcal{L}}$ of $\overline{\mathcal{L}}$.

Let $\pi : \mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$ be an integral normal quasi-projective arithmetic surface over Spec $\mathcal{O}_K$[2] and $P \in \mathcal{X}(\mathcal{O}_K)$ a section of π (therefore $\mathcal{X}_K$ is a smooth geometrically connected curve over K). Assume that π is smooth along the image of P and that, for every embedding $\sigma : K \hookrightarrow \mathbb{C}$, we are given an open holomorphic immersion

$$\varphi_\sigma : \overset{\circ}{D}(0;1) := \{z \in \mathbb{C} \mid |z| < 1\} \hookrightarrow \mathcal{X}_\sigma(\mathbb{C})$$

such that

$$\varphi_\sigma(0) = P_\sigma .$$

[1] Usually this family is required to be invariant under complex conjugation; it is convenient to omit this condition, to make the statement of Theorem 1.2 simpler.

[2] *i.e.*, an integral normal quasi-projective flat scheme over Spec $\mathcal{O}_K$ of Krull dimension 2.

Then the relative tangent bundle T_π is well-defined over an open neighbourhood of the image of P, and its inverse image $P^* T_\pi$ is a well-defined line bundle over Spec $\mathcal{O}_K$. Moreover, the φ_σ's determine an hermitian structure on $P^* T_\pi$; namely, for any embedding $\sigma : K \hookrightarrow \mathbb{C}$, the complex line $(P^* T_\pi)_\sigma \simeq T_{P_\sigma} \mathcal{X}_\sigma(\mathbb{C})$ may be equipped with the (hermitian) norm $\| \cdot \|_{\varphi_\sigma}$ defined by

$$\left\| D\varphi_\sigma(0) \left(\frac{\partial}{\partial z} \right) \right\|_{\varphi_\sigma} = 1 .$$

Using the notation we have just introduced, we may state the special case of the main result of this paper (Theorems 4.3 and 4.3′ below) alluded to above:

Theorem 1.2. *If*

$$\widehat{\deg}\, (P^* T_\pi, (\| \cdot \|_{\varphi_\sigma})_{\sigma : K \hookrightarrow \mathbb{C}}) > 0 ,$$

or if

$$\widehat{\deg}\, (P^* T_\pi, (\| \cdot \|_{\varphi_\sigma})_{\sigma : K \hookrightarrow \mathbb{C}}) = 0$$

and $\coprod_{\sigma : K \hookrightarrow \mathbb{C}} \mathcal{X}_\sigma(\mathbb{C}) \backslash \varphi_\sigma(\overset{\circ}{D}\,(0,1))$ *has a non-empty interior, then, for any geometric point* η *of* Spec $\mathcal{O}_K$, *the maps between algebraic fundamental groups*

$$P_* : \pi_1(\text{Spec } \mathcal{O}_K, \eta) \to \pi_1(\mathcal{X}, P(\eta))$$

and

$$\pi_* : \pi_1(\mathcal{X}, P(\eta)) \to \pi(\text{Spec } \mathcal{O}_K, \eta) ,$$

determined by the section P : Spec $\mathcal{O}_K \to \mathcal{X}$ *and the morphism* $\pi : \mathcal{X} \to$ Spec $\mathcal{O}_K$, *are isomorphisms inverse of each other.*

1.2. The first results in the direction of Theorem 1.2 and of its generalizations Theorems 4.3 and 4.3′ are due to Ihara ([I]). In **7.1** below, we discuss how to recover Ihara's statements from ours. It is worth noting that his proof relies on techniques quite different from the ones used in the present paper. Namely, he uses special cases of a theorem of Harbater ([H]) concerning the rationality of power series with coefficients in number fields under suitable hypotheses on their algebraicity and their radii of convergence. However, I would like to emphasize that the paper of Ihara has been the starting point of the present work, and that the question of understanding it by means of Arakelov geometry is indeed asked by Ihara himself (see [I], introduction). We refer the reader to [I] for striking applications of Ihara's theorem (= Theorem 7.1 below) to modular curves and to curves equipped with Belyi functions, and for a discussion of the "ramification theoretic character" of this theorem. Proving such results concerning ramification—at least implicitly—appears as an application of Arakelov geometry of a rather unexpected kind.

1.3. Besides, Theorem 1.2 admits consequences which are not covered by Ihara's paper, for instance applications to elliptic curves, which we now describe briefly.

Let E be an elliptic curve over a number field K, and let $\pi : \mathcal{E} \to S := \operatorname{Spec} \mathcal{O}_K$ be the connected Néron model of E and $\varepsilon : \operatorname{Spec} \mathcal{O}_K \to \mathcal{E}$ its zero section. The line bundle $\omega_{\mathcal{E}/S} := \varepsilon^* \Omega^1_{\mathcal{E}/S}$ on S may be equipped with the natural hermitian structure $\| \ \|_{\mathrm{L}^2} = (\| \ \|_{\mathrm{L}^2,\sigma})_{\sigma : K \hookrightarrow \mathbb{C}}$ defined by

$$\|\alpha\|^2_{\mathrm{L}^2,\sigma} = \frac{i}{2\pi} \int_{E_\sigma(\mathbb{C})} \alpha \wedge \overline{\alpha}$$

for any embedding $\sigma : K \hookrightarrow \mathbb{C}$ and any $\alpha \in \omega_\pi \otimes_\sigma \mathbb{C} \simeq \Omega^1(E_\sigma)$. The Faltings height of E is defined as

$$h_F(E) := \frac{1}{[K:\mathbb{Q}]} \widehat{\deg}\,(\omega_{\mathcal{E}/S}, \| \ \|_{\mathrm{L}^2}) . \tag{1.2}$$

On the other hand, for any embedding $\sigma : K \hookrightarrow \mathbb{C}$, we may choose τ_σ in the usual fundamental domain

$$\left\{ \tau \in \mathbb{C} \mid \operatorname{Im} \tau > 0 \, , \, -\frac{1}{2} \leq \operatorname{Re} \tau \leq \frac{1}{2} \, , \text{ and } |\tau| \geq 1 \right\} \tag{1.3}$$

for the action of $SL_2(\mathbb{Z})$ on the upper half-plane, such that the Riemann surface $E_\sigma(\mathbb{C})$ is isomorphic to $\mathbb{C}/\mathbb{Z} + \tau_\sigma \mathbb{Z}$.

If we apply Theorem 1.2 to some integral normal quasi-projective arithmetic surface $\mathcal{X}$ over $\operatorname{Spec} \mathcal{O}_K$ containing $\mathcal{E}$ as an open $\operatorname{Spec} \mathcal{O}_K$-subscheme (*e.g.* to the Néron model of E, or to the minimal regular model, or – when E has semi-stable reduction over K – to the stable model of E) and to $\pi = \varepsilon$, we get (see § 7.2 below for details and numerical examples):

Corollary 1.3. *If*

$$h_F(E) \leq \frac{1}{2[K:\mathbb{Q}]} \sum_{\sigma : K \hookrightarrow \mathbb{C}} \log \frac{\pi}{4 \operatorname{Im} \tau_\sigma} , \tag{1.4}$$

then, for any geometric point η of $\operatorname{Spec} \mathcal{O}_K$, *we have isomorphisms:*

$$\pi_1(\operatorname{Spec} \mathcal{O}_K, \eta) \underset{\varepsilon_*}{\overset{\pi_*}{\leftrightarrows}} \pi_1(\mathcal{X}, \varepsilon(\eta)) .$$

1.4. The conclusion of Theorem 1.2 and Corollary 1.3 becomes especially simple when $K = \mathbb{Q}$. Indeed, then $S = \operatorname{Spec} \mathbb{Z}$ has a trivial fundamental group according to a classical theorem of Minkowski; therefore when the hypothesis of Theorem 1.2 or Corollary 1.3 is satisfied, the arithmetic surface $\mathcal{X}$ has a trivial fundamental group. There are particularly simple examples of such simply connected arithmetic surfaces, for instance, for any $n \in \mathbb{N}^*$, the subscheme $\mathcal{X}$ of $\mathbb{A}^2_{\mathbb{Z}}$ defined by the equation

$$y^2 + y = x^n ,$$

or more generally, any open subscheme of this one containing the section $(0,0) \in \mathbb{A}^2_{\mathbb{Z}}(\mathbb{Z})$ (see [I], § 4, and **7.1.4** below).

1.5. Our approach to "arithmetic Lefschetz theorems" via Arakelov geometry relies on two main observations, which we would like to discuss now:

i) The geometric Lefschetz Theorem 1.1 may be proved in various ways (see *infra*). One of these, maybe not so widely known, relies on the notion of *numerical connectedness* for an effective divisor on a projective surface, which was introduced in the 40's by Franchetta ([F1]) and reintroduced in "modern" algebraic geometry by C.P. Ramanujam, in a famous paper [R] where he proves – as an auxiliary result – the connectedness assertion in Theorem 1.1 by a simple argument based on the Hodge index inequality. Once the connectedness assertion is proved, the surjectivity of i_* in Theorem 1.1 formally follows by applying this assertion to connected étale covers of $\mathcal{X}$.

It turns out to be possible to transfer this proof in the arithmetic setting, by using the Hodge index inequality of Faltings-Hriljac for the Arakelov intersection pairing on arithmetic surfaces.

ii) To achieve this, a crucial technical point is the use of an extended definition of the Arakelov-Chow group attached to a projective arithmetic surface, based on the use of Green functions which, up to a logarithmic singularity, belong to the Sobolev space L^2_1 of functions in L^2 whose first order derivatives belong to L^2. This is the weakest natural choice of regularity for these Green functions which leads to a well defined intersection pairing on the Arakelov-Chow groups.

These generalized Green functions contain as noteworthy instances the equilibrium potentials associated to non-polar compact subsets in compact Riemann surfaces. The arithmetic significance of these equilibrium potentials and of the associated *capacities* has been already made clearly visible by the remarkable work of Rumely [Ru1], which relates integral points and capacity theory on algebraic curves, and indeed has a long history, which goes back to the contribution of Fekete and Szegö (see [Ru1], § 0.1 for a discussion and references; see also [C]). Our main theorem (Theorem 4.2 *infra*) is phrased in term of capacities[3], and provide another illustration of the role of capacity theory in the investigation of arithmetic surfaces. Our proof of this theorem shows how capacity theory on Riemann surfaces naturally fits with the version of Arakelov intersection theory on arithmetic surfaces first introduced by Deligne ([D3]; see also [G-S1]), which involves Green functions which are not necessarily admissible in the sense of Arakelov. After this paper was submitted, the author learnt from R. Rumely of his paper [Ru2], where he develops a related, but different, extension of Arakelov intersection theory.

Let us finally indicate that the use of these L^2_1-Green functions in the Arakelov geometry of arithmetic surfaces is a natural and convenient tool in other contexts, such as the Arakelov geometry of correspondences between curves, and of modular curves. We plan to return latter on these matters.

1.6. This article is organized as follows.

Section 2 is devoted to a discussion of the geometric Lefschetz theorem and its variants, and to the proof of a stronger form of Theorem 1.1 which will provide a motivation for our

[3] More precisely, in terms of the hermitian metrics which provide an intrinsic interpretation of capacities on Riemann surfaces.

arithmetic version and a model of its proof. In section 3, we introduce the basic concepts of potential theory and of Arakelov geometry needed for the statement of our results. Our presentation emphasizes the role of the Sobolev space L^2_1 and is somewhat different from the classical presentations of potential theory on Riemann surfaces; to remedy the lack of suitable references, we include a proof of the results we need in Appendix A at the end of the paper. Our main results are described in section 4. In section 5, we introduce a variant of the Arakelov-Chow group of [D3], [G-S1], attached to any integral normal projective arithmetic surface, which is based on the use of L^2_1-Green currents, and we discuss the existence and the properties of the Arakelov intersection pairing on these generalized Arakelov-Chow groups. In section 6, we analyze the notion of numerical effectivity for elements of these groups, and we complete the proof of our main theorem. Finally, section 7 is devoted to a discussion of Ihara's results and to examples.

1.7. The content of this paper was presented at the Arithmetic Algebraic Geometry Conference, Oberwolfach, July 1996, and at the Number Theory Seminar, Oxford, October 1996. I am grateful to A. Abbes, J.-M. Bony, G. Courtois, C. Margerin and S. Semmes for helpful discussions, and to the referee for his careful reading of the manuscript.

2. Geometric Lefschetz theorems for projective surfaces

2.1. Let us begin by reviewing various proofs of Theorem 1.1 and of variants of it. We shall restrict ourselves to the connectedness assertion; indeed, as already pointed out, the surjectivity of (1.1) formally follows from the connectedness assertion applied to étale covers of X (see **2.3.1** *infra*).

2.1.1. An easy modern proof (see [Ha], III, Corollary 7.9) is to consider the Cartier divisors nD ($n \in \mathbb{N}^*$) and the associated exact sequence of coherent sheaves on X

$$0 \to \mathcal{O}_X(-nD) \to \mathcal{O}_X \to \mathcal{O}_{nD} \to 0\,. \tag{2.1}$$

The so-called Lemma of Enriques-Severi-Zariski – now a straightforward consequence of Serre duality and vanishing theorems – asserts that $\mathrm{H}^1(X; \mathcal{O}_X\ (-nD))$ vanishes if n is large enough. Then the long exact cohomology sequence associated to (2.1) shows that $\mathrm{H}^0(X; \mathcal{O}_X) = k$ maps onto $\mathrm{H}^0(X; \mathcal{O}_{nD})$ which is therefore one-dimensional; hence nD and D are connected.

This proof leads to a stronger version of Theorem 1.1. Indeed, the vanishing of $\mathrm{H}^1(X; \mathcal{O}_X(-nD))$ when $n \gg 0$ holds for any ample Cartier divisor on a projective *normal* connected surface X over the algebraically closed field k. Accordingly, the smoothness assumption on X in Theorem 1.1 may be weakened to mere normality.

2.1.2. It is worth recalling, however, that Theorem 1.1 is a classical result in algebraic geometry, which was known long before the advent of cohomological techniques. Over the complex numbers, it goes back to italian geometers: the connectedness assertion appears in the work of Bertini, and the surjectivity assertion – at least for first homology groups instead of fundamental groups – in the classical papers of Castelnuovo and Enriques. Historically, the main contribution of Lefschetz to these questions has been to extend their results to algebraic varieties of dimension > 2. For a survey of the modern developments of Lefschetz types theorems over $\mathbb{C}$ and their relation to the topology of algebraic varieties, we refer the reader to [Fu].

To the knowledge of the author, a rigorous proof of Theorem 1.1, valid over a field of arbitrary characteristic, first appears – at least implicitly – in the work of Zariski (see also [Gr1], § X.2.8). It turns out to lead to a proof of the connectedness assertion in Theorem 1.1 under a weaker assumption, namely the ampleness condition on D may be replaced by the following one, where $(D_i)_{i\in I}$ denotes the family of irreducible components of D:

$$\begin{cases} D^2 > 0 \\ \forall i \in I\,,\ D_i^2 \geq 0\,. \end{cases} \tag{2.2}$$

Indeed, (2.2) implies that for n sufficiently large, the linear system $|nD|$ has no base point and is not composite with a pencil ([Z3], § 6); in turn, this implies that the generic element of $|nD|$ has an absolutely irreducible support (by Zariski's version of Bertini's theorem; see [Z2], § I.6). Finally, by Zariski's connectedness theorem ([Z1]), all divisors in $|nD|$ – hence nD and D – are connected.

2.1.3. In the mid-sixties, Lefschetz type theorems were revisited using the tools of modern algebraic geometry: the general strategy was to relate the geometry of an ample divisor D in a scheme X to the geometry of X *via* the formal completion $\widehat{D}$ of D in X.

In particular, Grothendieck's seminar SGA 2 ([Gr2]) is devoted to this circle of ideas. The following relative version is proved there (see also [Ray] for variants and further developments):

Theorem 2.1. *Let $f: X \longrightarrow S$ be a projective flat morphism of connected nœtherian schemes, and let D be an effective Cartier divisor in X, flat and relatively ample with respect to f.*

1) If, for any $s \in S$, the depth of X_s at any of its closed points is ≥ 2, then D is connected and, for any open subscheme U of X containing D and any geometric point η in D, the map between fundamental groups

$$i_{U*} : \pi_1(|D|, \eta) \longrightarrow \pi_1(U, \eta)\,, \tag{2.3}$$

induced by the inclusion i_U of $|D|$ in U, is onto.

2) If, moreover, the depth of X_s at any of the closed points of D_s is ≥ 3, and if the local ring $\mathcal{O}_{X,x}$ of X at any closed point x of X is pure[4]*, then, for any geometric point η in D, the map i_{X*} is an isomorphism*[5].

When $S = \operatorname{Spec} k$, k a field, the depth condition in 1) is satisfied by any normal scheme over k, of dimension ≥ 2 at every point. In particular, Theorem 2.1 implies a stronger version of Theorem 1.1, valid for an effective ample divisor on a projective normal integral surface over an arbitrary field.

[4] *e.g.*, if $\mathcal{O}_{X,x}$ is regular or, more generally, a complete intersection; see [Gr2] X.3.4.

[5] As this theorem is not explicitly stated in [Gr2], we indicate how to recover it from the main results in *loc. cit.* To prove 1), observe that, using the terminology of *loc. cit.* X.2, the ampleness of D and the lower bound on depths implies that the condition Lef (X, D) holds, by *loc. cit.* XII.2.4; the conclusion of 1) follows by *loc. cit.* X.2.6. Similarly, the assumption on depths in 2) imply that the condition Leff (X, D) holds by *loc. cit.* XII.3.4; the conclusion of 2) follows by *loc. cit.* X.2.6 and X.3.3.

When S is regular and X is "nice-enough" (*e.g.* regular, or locally a complete intersection) and has relative dimension d over S, the hypothesis of 1) (resp. 2)) is satisfied as soon as $d \geq 2$ (resp. $d \geq 3$).

Observe that the arithmetic situation investigated in the present paper is an instance of this relative setting, with $S = \text{Spec } \mathbb{Z}$ and $d = 1$, which however is not covered by these results, but appears as a natural complement to them.

2.1.4. In a spirit close to [Gr2], Hironaka and Matsumura investigated the ring of formal rational functions along some subvariety of a variety ([H-M]). As a byproduct, they showed ([H-M], Prop. 3.6) that if D is an effective divisor in a smooth projective connected surface X over an algebraically closed field k which satisfies condition (2.2), then D is connected (indeed they proved more, namely that the ring of formal rational functions along D coincides with the function field of X).

2.2. These various approaches to the Lefschetz theorem do not seem so easy to transpose into the framework of Arakelov geometry (known results [So], [Ga] concerning "vanishing theorems" for ample hermitian line bundles on arithmetic surfaces apparently do not lead to connectedness theorems with a simple form; on the other hand, one should observe that Ihara's argument in [I] is somewhat in the spirit of the approach **2.1.3**). Fortunately, there is another algebraic approach, which also goes back to the Italian tradition of algebraic geometry, based on the notion of numerical connectedness for an effective divisor on a surface. This notion is apparently due to Franchetta (see for instance [F1] and [F2]), and has been revived in the work of C.P. Ramanujam ([R]). It plays an important role in the study of pluricanonical systems on complex surfaces (Franchetta, Bombieri), and also in the study of curves over function fields of positive characteristic (Szpiro; see [Sz], [LM]).

Recall that an effective divisor D on a smooth projective surface X over an algebraically closed field k is called *numerically connected* if, for any two divisors D_1 and D_2 on X such that

$$D_1 > 0\,,\ D_2 > 0\,,\ \text{and } D = D_1 + D_2\,, \tag{2.4}$$

we have:

$$D_1 \cdot D_2 > 0\,.$$

Clearly, this condition implies the connectedness of D, and conversely when D is reduced (but not in general).

In his paper [R], C.P. Ramanujam implicitly proves the following statement:

Proposition 2.2. *Let X be a smooth projective connected surface over an algebraically closed field, and let D be an effective Cartier divisor on X. If D is nef* [6] *and if $D^2 > 0$, then D is numerically connected, and therefore $|D|$ is connected.*

This is a straightforward (but clever!) consequence of Hodge index inequality. As the argument is short and plays a central role in this paper, we reproduce it.

[6] Recall that D is called *numerically effective* or *nef* if, for any effective divisor D' on X, $D \cdot D' \geq 0$.

Let D_1 and D_2 be two divisors on X which satisfy (2.4). Then, as D is nef, we have:

$$D \cdot D_1 \geq 0 \qquad \text{and} \qquad D \cdot D_2 \geq 0\,,$$

hence:

$$D_1^2 \geq -D_1 \cdot D_2 \qquad \text{and} \qquad D_2^2 \geq -D_1 \cdot D_2\,.$$

Therefore, if $D_1 \cdot D_2 \leq 0$, by multiplying these two inequalities, we get:

$$D_1^2 \cdot D_2^2 \geq (D_1 \cdot D_2)^2\,. \tag{2.5}$$

On the other hand, since $(D_1 + D_2)^2 > 0$, the Hodge index theorem implies that

$$\begin{vmatrix} D_1^2 & D_1 \cdot D_2 \\ D_1 \cdot D_2 & D_2^2 \end{vmatrix} \leq 0\,.$$

Comparing with (2.5), we get that this determinant vanishes, and by Hodge index theorem again, that D_1 and D_2 are linearly dependent in the group $\mathrm{Num}\,(X)_{\mathbb{Q}}$ of $\mathbb{Q}$-divisors on X modulo numerical equivalence. As D_1 and D_2 are non zero effective cycles, they have positive intersection numbers with any ample line bundle. Therefore their classes in $\mathrm{Num}\,(X)_{\mathbb{Q}}$ do not vanish, and there exists $\lambda \in \mathbb{Q}_+^*$ such that

$$D_2 = \lambda\, D_1 \qquad \text{in} \quad \mathrm{Num}\,(X)_{\mathbb{Q}}\,.$$

Finally the positivity of

$$D^2 = (D_1 + D_2)^2 = (1+\lambda)^2\, D_1^2$$

contradicts the non-positivity of

$$D_1 \cdot D_2 = \lambda\, D_1^2\,.$$

2.3. We proceed now with a few comments concerning Proposition 2.2 and its proof which will be useful when we investigate its arithmetic analogue. We denote by D an effective divisor on a smooth projective surface X over an algebraically closed field k.

2.3.1. Let $D = \sum_{i \in I} n_i\, D_i$ be the decomposition of D into irreducible components. As D is effective, it is nef iff

$$\forall i \in I\,, \quad D \cdot D_i \geq 0\,.$$

This condition is implied by the second part of condition (2.2), namely

$$\forall i \in I\,, \quad D_i^2 \geq 0\,.$$

Therefore Proposition 2.2 is stronger than the version of the connectedness assertion in Theorem 1.1 alluded to in **2.1.2** and **2.1.4**, and in particular than the connectedness assertion in Theorem 1.1.

Moreover, the condition

$$\text{(2.6)} \qquad D \textit{ is nef and } D^2 > 0$$

is preserved if X is replaced by another smooth projective connected surface X' over k which maps to X by a dominant k-morphism $\varphi : X' \to X$, and D by its inverse image $D' := \varphi^*(D)$. Therefore, any such D' has a connected support.

Applied to étale coverings $\varphi : X' \to X$, this observation shows, by the very definition of the algebraic fundamental group ([Gr1]), that for any geometric point η in $|D|$, the map

$$i_* : \pi_1(|D|, \eta) \to \pi_1(X, \eta)$$

induced by the injection $i : |D| \hookrightarrow X$ is onto.

More generally, if U is any open subscheme of X containing D, and $\varphi_0 : U' \to U$ is any étale covering, we may consider the normalization $\widetilde{X}$ of X in the function field of U', and a resolution X' of $\widetilde{X}$. The surface U' may be identified with an open subscheme of X', and the map φ_0 extends to a map $\varphi : X' \to X$. Applied to this map, the observation above shows that $\varphi^{-1}(D)$, hence $\varphi_0^{-1}(D)$, has a connected support. This implies the following refinement of Theorem 1.1:

Proposition 2.3. *If* (2.6) *holds, then, for any open neighbourhood U of $|D|$ in X and any geometric point η of $|D|$, the map*

$$i_{U^*} : \pi_1(|D|, \eta) \to \pi_1(U, \eta),$$

induced by the injection $i_U : |D| \hookrightarrow U$, is onto.

2.3.2. Recall that the numerical effectivity of D already implies that $D^2 \geq 0$. (Indeed, the nef cone in $\mathrm{Num}\,(X)_{\mathbb{Q}}$ is the closure of the ample cone, as follows for instance from Seshadri's ampleness criterion.) Therefore we get, by considering the "equality case" $D^2 = 0$ in the argument used to prove Proposition 2.2, that, if D is nef, then

- either $|D|$ is connected;
- or $|D|$ is not connected, and then $D^2 = 0$, and for any decomposition $D = D_1 + D_2$ of D as a sum of effective divisors with disjoint supports, the classes of D_1 and D_2 in $\mathrm{Num}\,(X)_{\mathbb{Q}}$ are colinear.

2.4. Finally, Proposition 2.2 and its variants described in the previous subsection may be extended to *normal*, non-necessarily smooth, surfaces.

To achieve this, one relies on Mumford's intersection theory on normal complete surfaces ([M] II.b). Let us briefly recall the main features of this theory.

In this subsection, X will denote a normal complete integral surface over an algebraically closed field k. Let $\nu : \widetilde{X} \to X$ be a resolution of X, and let $(E_i)_{i\in I}$ be the (finite) family of one-dimensional components of the fibers of ν: these are projective curves, sent by ν onto the non-smooth points of X. The intersection matrix $(E_i \cdot E_j)_{(i,j)\in I^2}$ is negative definitive, hence invertible over $\mathbb{Q}$ ([M] p. 17 or [D1], Corollaire 1.9). Therefore, there is a well defined linear map between spaces of $\mathbb{Q}$-Weil divisors

$$\mu : Z_1\,(X)_{\mathbb{Q}} \to Z_1\,(\widetilde{X})_{\mathbb{Q}},$$

which sends a $\mathbb{Q}$-divisor D on X onto the $\mathbb{Q}$-divisor D' on X given by

$$D' = \widetilde{D} + \sum_{i \in I} r_i \, E_i \,, \tag{2.7}$$

where $\widetilde{D}$ denotes the proper transform of D by ν, and where $(r_i)_{i \in I} \in \mathbb{Q}^I$ is chosen such that

$$\forall \, i \in I \,, \quad D' \cdot E_i = 0 \,. \tag{2.8}$$

Indeed, this condition may be also written

$$\forall \, i \in I \,, \quad \sum_{j \in I} r_j \, (E_i \cdot E_j) = -\widetilde{D} \cdot E_i \,.$$

Let

$$\nu_* : Z_1 \, (\widetilde{X})_{\mathbb{Q}} \to Z_1 \, (X)_{\mathbb{Q}}$$

be the push-forward of $\mathbb{Q}$-cycles by ν.

The following properties hold:

i) Clearly we have:

$$\nu_* \circ \mu = \mathrm{id}_{Z_1(X)_{\mathbb{Q}}} \,,$$

and

$$\mathrm{id}_{Z_1 \, (\widetilde{X})_{\mathbb{Q}}} - \mu \circ \nu_*$$

is an idempotent of image $\bigoplus_{i \in I} \mathbb{Q} \, E_i$ and of kernel the space of $\mathbb{Q}$-divisor on $\widetilde{X}$ whose intersection products with the E_i's vanish.

ii) If D is a Cartier divisor, $\mu(D)$ coincides with its inverse image $\nu^*(D)$. In particular, for any $f \in k(X)^* = k(\widetilde{X})^*$, the divisor $\mathrm{div}_X \, f$ of f in X is mapped onto its divisor $\mathrm{div}_{\widetilde{X}} \, f$ in $\widetilde{X}$. Consequently, μ defines a morphism

$$\mathrm{CH}_1 \, (X)_{\mathbb{Q}} \to \mathrm{CH}_1 \, (\widetilde{X})_{\mathbb{Q}}$$

between rational homological Chow groups, which we shall still denote by μ.

iii) If D_1 and D_2 belong to $Z_1 \, (X)_{\mathbb{Q}}$ or to $\mathrm{CH}_1 \, (X)_{\mathbb{Q}}$, their *intersection number* is defined as

$$D_1 \cdot D_2 := \mu \, (D_1) \cdot \mu \, (D_2) \in \mathbb{Q} \,. \tag{2.9}$$

The so-defined intersection pairing turns out not to depend on the choice of the resolution $\nu : \widetilde{X} \to X$, and to coincide with the usual $\mathbb{Z}$-valued intersection pairing if D_1 and D_2 belong to $Z_1(X)$ and one of them is Cartier. Moreover, for any $D \in Z_1(X)_{\mathbb{Q}}$ and any $\widetilde{D} \in Z_1(\widetilde{X})_{\mathbb{Q}}$, the following projection formula holds

$$\mu \, (D) \cdot \widetilde{D} = D \cdot \nu_* \widetilde{\mathrm{D}}. \tag{2.10}$$

iv) For any effective $\mathbb{Q}$-Weil divisor D on X, $D' := \mu \, (D)$ also is effective. Indeed, in the defining equation (2.7), when D is effective, $\widetilde{D}$ also is effective, and $r_i > 0$ (resp. $r_i = 0$) if $\nu \, (E_i) \in |D|$ (resp. if $\nu \, (E_i) \notin |D|$), as follows from [M] II.b, Property (ii).

v) If D_1 and D_2 are effective $\mathbb{Q}$-Weil divisors on X, without a common irreducible component, then

$$D_1 \cdot D_2 \geq 0$$

and

$$D_1 \cdot D_2 > 0 \quad \text{if} \quad |D_1| \cap |D_2| \neq \emptyset \,.$$

Indeed, if $\widetilde{D}_2$ denotes the proper transform of D_2 in $\widetilde{X}$, we have by (2.10):

$$D_1 \cdot D_2 = D_1 \cdot \nu_* (\widetilde{D}_2) = \mu (D_1) \cdot \widetilde{D}_2 \,,$$

and $\mu (D_1)$ and $\widetilde{D}_2$ are effective $\mathbb{Q}$-divisors on $\widetilde{X}$, without a common irreducible component, and their supports meet iff the ones of D_1 and D_2 do.

vi) We shall say that $D \in Z_1(X)_\mathbb{Q}$ is *numerically effective* or *nef* if, for any effective $D' \in Z_1(X)_\mathbb{Q}$, we have

$$D \cdot D' \geq 0 \,.$$

Using (2.9), (2.10), and **iv)**, we get:

D is nef iff $\mu(D)$ is nef.

We shall define a $\mathbb{Q}$-vector space $\mathrm{Num}\,(X)_\mathbb{Q}$ as the quotient of $Z_1(X)_\mathbb{Q}$ by the relation of numerical equivalence $\sim_{\mathrm{num}}$ defined by:

$$D_1 \sim_{\mathrm{num}} D_2 \Leftrightarrow \forall\, D \in Z_1(X)_\mathbb{Q} \,,\; D \cdot D_1 = D \cdot D_2 \,.$$

By construction, the intersection product defines a non-degenerate bilinear form on $\mathrm{Num}\,(X)_\mathbb{Q}$. Moreover, the adjonction formula (2.10) shows that the maps μ and ν_* are compatible with numerical equivalence, and therefore define $\mathbb{Q}$-linear maps:

$$\mu : \mathrm{Num}\,(X)_\mathbb{Q} \to \mathrm{Num}\,(\widetilde{X})_\mathbb{Q}$$

and

$$\nu_* : \mathrm{Num}\,(\widetilde{X})_\mathbb{Q} \to \mathrm{Num}\,(X)_\mathbb{Q} \,.$$

The classes $[E_i]$ in $\mathrm{Num}\,(\widetilde{X})_\mathbb{Q}$ of the curves E_i, $i \in I$, are linearly independant and the intersection form on $\bigoplus_{i\in I} \mathbb{Q}\,[E_i]$ is negative definite, since $(E_i \cdot E_j)_{(i,j)\in I^2}$ is negative definite; moreover, μ maps bijectively $\mathrm{Num}\,(X)_\mathbb{Q}$ onto the orthogonal complement $\left(\bigoplus_{i\in I} \mathbb{Q}\,[E_i]\right)^\perp$, and preserves the intersection pairings; its adjoint with respect to these pairings coincides with ν_*. In particular, the finite dimensionality of $\mathrm{Num}\,(\widetilde{X})_\mathbb{Q}$ and the Hodge index theorem for $\mathrm{Num}\,(\widetilde{X})_\mathbb{Q}$ imply that $\mathrm{Num}\,(X)_\mathbb{Q}$ also is finite dimensional and satisfies the Hodge index theorem (namely, the intersection pairing on $\mathrm{Num}\,(X)_\mathbb{Q}$ has signature $(+, -, \ldots, -)$).

Using this formalism, we finally get our most general version for the connectedness theorem in the geometric case:

Theorem 2.4. *Let X be an integral normal complete surface over an algebraically closed field, and let D be an effective Cartier divisor on X. If D is nef then:*

- *either $|D|$ is connected;*

- *or $|D|$ is not connected, and then $D^2 = 0$, and for any decomposition $D = D_1 + D_2$ of D has a sum of effective divisors with disjoint supports, the classes of D_1 and D_2 in $\mathrm{Num}(X)_{\mathbb{Q}}$ are collinear.*

This follows, either by using the same argument based on the Hodge index theorem as the one used to prove Proposition 2.3 and its variant in **2.3.2** or by reducing to this variant applied to $\tilde{X}$ by means of the following observation: the connexity of the fibers of ν (consequence of Zariski's main theorem) and the property **iv)** above imply that the support $|D|$ of an effective $\mathbb{Q}$-divisor D on X is connected iff $|\mu(D)|$ is.

3. Equilibrium potentials and Arakelov divisors on arithmetic surfaces

3.1. Potential theory on Riemann surfaces

3.1.1. Some classical definitions

Let M be a Riemann surface (*i.e.*, a one-dimensional complex manifold). A function f from M into $[-\infty, +\infty[$ is called *subharmonic* if:

(i) f is upper semi-continuous;

(ii) f is not identically $-\infty$ on any connected component of M;

(iii) f satisfies the "local submean inequality", namely, for any holomorphic embedding φ of an open neighbourhood V of the closed disk $\{z \in \mathbb{C} \mid |z| \leq 1\}$ into M, we have

$$f(\varphi(0)) \leq \int_0^1 f\left(\varphi\left(e^{2\pi i t}\right)\right) \mathrm{d}t.$$

(As f is locally bounded from above by i), the last integral is well defined and not equal to $+\infty$).

Any subharmonic function f on M is locally L^1, and the distribution (still denoted f) it defines satisfies

$$\mathrm{dd}^c f := -\frac{1}{2\pi i}\, \partial\overline{\partial} f \geq 0; \tag{3.1}$$

in other words, the current $\mathrm{dd}^c f$ on M is a positive measure on M. Conversely, for any locally L^1 function real f on M which satisfies (3.1), there exists a unique subharmonic function on M which coincides almost everywhere with f.

A subset E of M is called *polar* if, locally on M, there exists a subharmonic function such that

$$f_{|E} = -\infty.$$

Polar subsets of M are the negligible sets of potential theory. They turn out to be "very small". For instance, they are Lebesgue negligible, and compact polar sets are totally disconnected. Countable subsets of M, and more generally countable unions of polar subsets of M, are polar.

For more details on these basic notions, we refer the reader to the monographs [T], [Ru1] and [Ra].

3.1.2. The space $L^2_1(M)$

We now assume that M is compact. Then we may consider the vector space $L^2(M)$ of L^2 complex functions on M. For any positive continuous 2-form μ on M, the hermitian form which maps $f \in L^2(M)$ to $\int_M |f|^2 \mu$ defines a Hilbert space structure on $L^2(M)$, whose underlying topology does not depend on μ.

We may also consider the *Dirichlet* or *Sobolev space*

$$L^2_1(M) = \{f \in L^2(M) \mid \text{ the current } \partial f \text{ is } L^2\}.$$

It is equipped with the *Dirichlet* hermitian *form*, which maps $f \in L^2_1(M)$ to

$$\|f\|^2_{Dir} := \frac{i}{2\pi}\int_M \partial f \wedge \overline{\partial f} \in \mathbb{R}_+ . \tag{3.2}$$

For any μ as before the hermitian form which maps f to

$$\int_M |f|^2 \mu + \frac{i}{2\pi}\int_M \partial f \wedge \overline{\partial f} \tag{3.3}$$

defines a Hilbert space structure on $L^2_1(M)$, the underlying topology of which does not depend on μ; equipped with this topology, $L^2_1(M)$ contains $C^\infty(M)$ as a dense subspace[7]. Moreover, for any $f \in C^2(M)$, we have

$$\begin{aligned}\frac{i}{2\pi}\int_M \partial f \wedge \overline{\partial f} &= \frac{i}{2\pi}\int_M [\mathrm{d}(f \wedge \overline{\partial f}) - f\,\partial\overline{\partial f}] \\ &= -\int_M f\, \mathrm{dd}^c \overline{f}\end{aligned}$$

and, by a similar computation:

$$\frac{i}{2\pi}\int_M \partial f \wedge \overline{\partial f} = -\int_M \overline{f}\, \mathrm{dd}^c f .$$

In particular, the Dirichlet form (3.2) is invariant by complex conjugation (on $C^2(M)$, hence on $L^2_1(M)$), and $L^2_1(M)$ is conjugation invariant. In particular, for any $f \in L^2_1(M)$, $\overline{\partial} f = \overline{\partial \overline{f}}$ is L^2.

When M is connected, the quotient $L^2_1(M)/\mathbb{C}$ of $L^2_1(M)$ by the closed subspace of constant functions has a canonical Hilbert space structure defined by the Dirichlet form. In

[7] As $L^2_1(M)$ is stable by multiplication by functions in $C^\infty(M)$, by using partitions of unity, one has only to prove that elements of $L^2_1(M)$ with "small supports" may be approximated in $L^2_1(M)$ by functions in $C^\infty(M)$. This follows from a standard convolution argument.

general, the topology of $L^2_1(M)$ is defined by the hermitian form (3.3) for any non-negative continuous real 2-form μ on M, the restriction of which to any connected component of M does not vanish identically.

For any open subset Ω of M, we define $L^2_1(\Omega)_0$ as the closure in $L^2_1(M)$ of the space $C^\infty_c(\Omega)$ of C^∞ function with compact supports from Ω to $\mathbb{C}$. The restriction map with values in the generalized functions on Ω

$$L^2_1(\Omega)_0 \to C^{-\infty}(\Omega)$$

is injective, as easily follows from the definition of $L^2_1(\Omega)_0$. Observe also that the notation $L^2_1(\Omega)_0$, which omits to specify the ambiant compact surface M is legitimate. Indeed, if Ω' is an open subset of some other compact Riemann surface, and if $\varphi : \Omega' \to \Omega$ is a biholomorphic map, the isomorphism

$$\varphi^* : C^\infty_c(\Omega) \to C^\infty_c(\Omega')$$

extends to a continuous isomorphism

$$\varphi^* : L^2_1(\Omega)_0 \to L^2_1(\Omega')_0 .$$

This immediately follows from the description in the preceding paragraph of the topology on $L^2_1(M)$ and $L^2_1(M')$ by means of the hermitian forms (3.3).

Also observe that, if Ω_1 and Ω_2 are disjoint open subsets of M and if $f_1 \in L^2_1(\Omega_1)_0$ and $f_2 \in L^2_1(\Omega_2)_0$, then

$$\int_M \partial f_1 \wedge \overline{\partial} f_2 = 0 . \tag{3.4}$$

This is clear if $f_1 \in C^\infty_c(\Omega_1)$ and $f_2 \in C^\infty_c(\Omega_2)$, and follows in general by density.

Finally, for any open subset U of M (or more generally, for any Riemann surface U), we shall define $L^2_1(U)_{\text{loc}}$ as the vector space of generalized functions f on U such that f and df are locally L^2 on U, and $L^2_{-1}(U)_{\text{loc}}$ as the vector space of currents α of degree 2 on U which locally on U may be written

$$\alpha = \partial \beta$$

for some locally L^2 1-form β; these are also the currents which may be locally written

$$\alpha = \overline{\partial} \gamma$$

for some locally L^2 1-form γ, or the currents which, over any coordinate chart in U, locally coincide with an element of the usual Sobolev space $L^2_{-1}(U)$. The spaces $L^2_1(U)_{\text{loc}}$ and $L^2_{-1}(U)_{\text{loc}}$ are stable by multiplications by functions in $C^\infty(U)$, and for any $f \in L^2_1(U)_{\text{loc}}$ and $\alpha \in L^2_{-1}(U)_{\text{loc}}$, the current (of degree 2) $f\alpha$ is well defined; namely, on any open subset of U where $\alpha = \partial\beta$ for some locally L^2 1-form β, we have:

$$f\alpha = f\partial\beta := \partial(f\beta) - \partial f \wedge \beta$$

(observe that $f\beta$ and $\partial f \wedge \beta$ are well defined locally L^1 currents, since f, ∂f and β are locally L^2).

3.1.3. Green functions and equilibrium potentials

For any divisor $D = \sum_{i \in I} n_i P_i$ on the compact Riemann surface M, a *Green function* for D is, by definition, a real generalized function (*i.e.* a current of degree 0) g on M such that

$$\omega := \mathrm{dd}^c g + \delta_D$$

is C^∞. The ellipticity of dd^c and the equation of currents over $\mathbb{C}$

$$\mathrm{dd}^c \log |z|^2 = \delta_0$$

show that a generalized function g on M is a Green function for D iff for any open subset U of M and any meromorphic function f on U such that div f is defined and coincides with D on U, we have

$$g_{|U} = \varphi + \log |f|^{-2} \tag{3.5}$$

for some real valued φ in $\mathrm{C}^\infty(U)$.

We shall define a L_1^2-*Green function* g for D as a real generalized function g on M such that for any U and f as above, (3.5) holds, as an equality of distributions (or, equivalently, as an equality almost everywhere on U), for some φ in $\mathrm{L}_1^2(U)_{\mathrm{loc}}$. To emphasize the distinction between these L_1^2-Green functions and the previously defined Green functions, we shall sometimes call the latter *Green functions with* C^∞ *regularity.*

It turns out to be convenient to extend this definition to divisors D with real coefficients: if $D = \sum_{i \in I} n_i P_i$, one asks that, for any local holomorphic coordinate z on some open subset U of M, we have, almost everywhere on U:

$$g(z) = \varphi(z) + \sum_{\substack{i \in I \\ P_i \in U}} n_i \log |z - z(P_i)|^{-2}$$

for some $\varphi \in \mathrm{L}_1^2(U)_{\mathrm{loc}}$.

The main result in potential theory we shall rely on in this paper is the existence of equilibrium potentials for compact non-polar subsets of M:

Theorem 3.1. *Let M be a compact connected Riemann surface. For any open subset Ω of M such that $M \backslash \Omega$ is not polar, and any point P in Ω, there exists a unique generalized function $g_{P,\Omega}$ on M which satisfies the following two conditions:*

(i) *there exists a compact subset K of Ω and a function $f \in \mathrm{L}_1^2(\Omega)_0$ such that*

$$g_{P,\Omega} = f \quad \text{on} \quad M \backslash K\,;$$

(ii) *on $\Omega \backslash \{P\}$, $g_{P,\Omega}$ is harmonic, and if z denotes a local holomorphic coordinate near P, we have on some open neighbourhood of P:*

$$g_{P,\Omega} = \log |z - z(P)|^{-2} + h\,,$$

where h is harmonic.

Moreover the following properties holds:

(iii) *on* $M\backslash\{P\}$, $g_{P,\Omega}$ *is defined by a non-negative subharmonic function and the current of type* $(1,1)$ *on* M

$$\mu := \mathrm{dd}^c\, g_{P,\Omega} + \delta_P$$

is a probability measure supported on $\partial\Omega = \overline{\Omega}\backslash\Omega$*;*

(iv) *the subset* E *of* $M\backslash\Omega$ *where the subharmonic representative of* $g_{P,\Omega}$ *over* $M\backslash\{P\}$ *does not vanish is a polar subset of* $\partial\Omega$.

Roughly speaking, $g_{P,\Omega}$ is the unique function which is harmonic on $\Omega\backslash\{P\}$, has a logarithmic singularity at P (condition (ii)), and "vanishes on $M\backslash\Omega$" (condition (i)). It represents the electric field of a unit charge placed at the point P in the two-dimensional world modelled by M, where Ω (resp. $M\backslash\Omega$) is made of an insulating material (resp. of a conducting material wired to the earth).

In the sequel, it will be convenient to denote by $g_{P,\Omega}$ the subharmonic representative of the distribution $g_{P,\Omega}$ on $M\backslash\{P\}$. It is a well defined upper semi-continuous function from $M\backslash\{P\}$ to $\mathbb{R}_+$, and not only a function defined almost everywhere.

Observe that, if condition (ii) holds, then condition (i) holds for any compact neighbourhood K of P in Ω as soon as it holds for one compact subset K of Ω.

Also observe that condition (ii) may be rephrased as the equality of currents:

$$\text{(ii)}' \qquad \mathrm{dd}^c\, g_{P,\Omega} + \delta_P = 0 \qquad \text{on} \quad \Omega\,,$$

and that (i) and (ii) show that $g_{P,\Omega}$ is a L^2_1-Green function for P. More generally, for any divisor $D = \sum_{i\in I} n_i\, P_i$ on M such that the P_i's belong to some open subset Ω with non-polar complement, we let:

$$g_{D,\Omega} := \sum_{i\in I} n_i\, g_{P_i,\Omega}\,.$$

It is a L^2_1-Green function for D, which satisfies (i) and (ii)′ (with D instead of P); moreover, as for $g_{P,\Omega}$, these properties characterize $g_{D,\Omega}$ (*cf.* Appendix **A.4** below). This definition immediately extends to a possibly disconnected compact Riemann surface: to define $g_{D,\Omega}$, we require that, for any connected component M_i of M, $M_i\backslash\Omega$ is not polar, and we let

$$g_{D,\Omega|M_i} := g_{D_{|M_i},\Omega\cap M_i}\,.$$

All assertions in Theorem 3.1 are presumably well-known to specialists. However, as we could not find an explicit description of the equilibrium potentials $g_{P,\Omega}$ in terms of the spaces $\mathrm{L}^2_1(\Omega)_0$ in the literature, we give some details on its proof in Appendix A.

3.1.4. Remarks.

i) Let us keep the notation of Theorem 3.1, and consider the connected component Ω_0 of Ω containing P.

First, the maximum principle shows that $g_{P,\Omega}$ is positive on $\Omega_0\backslash\{P\}$.

Moreover, the complement $M\backslash\Omega_0$ contains $M\backslash\Omega$ and therefore is not polar. So the equilibrium potential g_{P,Ω_0} is well defined. Moreover, properties i), ii), and iv) in Theorem 3.1 with Ω_0 instead of Ω show that it satisfies properties i) and ii) (for Ω). This shows that

$$g_{P,\Omega_0} = g_{P,\Omega} \, .$$

In particular, $g_{P,\Omega}$ vanishes outside $\overline{\Omega}_0$, and $E \subset \partial\Omega_0$.

This result immediately implies that, more generally, if D is any divisor on M supported by Ω and if Ω' is an open subset of Ω containing every connected component of Ω which meets $|D|(\mathbb{C})$, then

$$g_{D,\Omega'} = g_{D,\Omega} \, .$$

ii) As $g_{P,\Omega}$ is non-negative and upper semi-continuous, it is continuous at every point of M where it vanishes. Therefore $g_{P,\Omega}$ is continuous outside P and the exceptional subset E of $\partial\Omega$ in condition iv).

It is indeed possible to characterize the continuity of $g_{P,\Omega}$ at some point Q of $\partial\Omega$ in terms of the so-called *regularity* and *thinness* of the connected component Ω_0 of P in Ω and of its complement. See Appendix **A.8**, for a detailed discussion.

iii) In Appendix **A.6**, we shall see that *any non-negative subharmonic function on* $M\backslash\{P\}$ *which satisfies conditions* (ii) *and* (iv) *in Theorem* 3.1 *coincides with* $g_{P,\Omega}$. This characterization provides another possible definition of $g_{P,\Omega}$, which avoids the introduction of the space $\mathrm{L}^2_1(\Omega)_0$. Moreover, it shows how our definition of the Green function $g_{P,\Omega}$ relates to the construction of Rumely, in [Ru1], § 3: with Rumely's notations in *loc. cit.*, if

$$M = \mathcal{C}(\mathbb{C})$$

$$P = \xi$$

and

$$M\backslash\Omega = E \, ,$$

then

$$g_{P,\Omega} = 2(V_\xi(E) - u_E(\cdot, \xi))\, ;$$

indeed this function satisfies condition (ii) and is subharmonic on $M\backslash\{P\}$ by *loc. cit.* Lemma 3.1.2, is non-negative and vanishes on the complement in $M\backslash\Omega$ of some polar subset by *loc. cit.* Theorem 3.1.7 (Frostman's Theorem). Outside the set of non-regular points of $\partial\Omega_0$, it coincides with twice the function $G(\cdot, \xi; E)$ introduced in *loc. cit.* Definition 3.2.1.

More generally, we shall see that for any *effective* divisor $D = \sum_{i \in I} n_i P_i$ on M, supported by Ω, $g_{D,\Omega}$ *is the unique non-negative subharmonic function on* $M\backslash\{P_i, i \in I\}$ *such that:*

- $g_{D,\Omega}$ *is harmonic on* $\Omega\backslash\{P_i, i \in I\}$, *and for any local holomorphic coordinate* z *on some open subset* U *of* Ω,

$$g_{D,\Omega} = h + \sum_{\substack{i \in I \\ P_i \in U}} n_i \log |z - z(P_i)|^{-2} \qquad on \quad U$$

for some harmonic function h *on* U;

• $g_{D,\Omega}$ *vanishes on the complement in* $M \backslash \Omega$ *of some polar subset of* $\partial \Omega$.

iv) The simplest example of equilibrium potential arises when $M = \mathbb{P}^1(\mathbb{C})$, $\Omega = \overset{\circ}{D}(0;1)$, and $P = 0$. Then we have

$$\begin{aligned} g_{0,\overset{\circ}{D}(0,1)}(z) &= \log |z|^{-2} \quad \text{if} \quad z \in \overset{\circ}{D}(0;1) \\ &= 0 \qquad\qquad \text{if} \quad z \in \mathbb{P}^1(\mathbb{C}) \backslash \overset{\circ}{D}(0;1)\,. \end{aligned}$$

Indeed, this function is non-negative, subharmonic on $\mathbb{P}^1(\mathbb{C}) \backslash \{0\}$ and satisfies (ii) and (iv). (One may also check condition (i) directly.)

v) The value $h(P)$ of the function h in condition (ii) may be used to define the *capacity* of the compact set $M \backslash \Omega$ with respect to P. Of course, the value $h(P)$ depends on the choice of local coordinate z. Intrinsically, we may define a "capacitary norm" $\| \ \|_{P,\Omega}$ on the complex line $T_P M = \mathbb{C}\, \frac{\partial}{\partial z}_{|P}$ by the equality

$$\left\| \frac{\partial}{\partial z}_{|P} \right\|_{P,\Omega} := e^{-\frac{1}{2} h(P)} = \lim_{Q \to P} |z(Q) - z(P)|^{-1} e^{-\frac{1}{2} g_{P,\Omega}(Q)}\,. \tag{3.6}$$

It is possible to show that

$$-\log \| \ \|_{P,\Omega}\,,$$

as a function of Ω, is increasing and strongly subadditive (see [Ch] and [Do], 1.XIII.18 when $M = \mathbb{P}^1(\mathbb{C})$; Doob's proof extends to the present setting).

When $M = \mathbb{P}^1(\mathbb{C})$, $P = \infty$, and z is the local coordinate X^{-1}, vanishing at ∞, then $\|\frac{\partial}{\partial z}\|_{P,\Omega}$ coincides with the classical *logarithmic capacity*[8] $c(\mathbb{P}^1(\mathbb{C}) \backslash \Omega)$ of $\mathbb{P}^1(\mathbb{C}) \backslash \Omega$, and $-\log \|\frac{\partial}{\partial z}\|_{P,\Omega}$ with its *Robin constant* $r(\mathbb{P}^1(\mathbb{C}) \backslash \Omega)$ (*cf.* [Do], 1.XIII.18 and [Ra], Theorem 5.2.1).

3.1.5. Functoriality properties

The construction of equilibrium potentials satisfy the following compatibility with biholomorphic maps:

Lemma 3.2. *Let* M *(resp.* M'*) be a compact connected Riemann surface,* Ω *(resp.* Ω'*) an open subset of* M *(resp. of* M'*), and* P *(resp.* P'*) a point in* M *(resp.* M'*). Moreover, let* $\varphi : \Omega \to \Omega'$ *be a biholomorphic map such that* $\varphi(P') = P$*. If* $M \backslash \Omega$ *is not polar, then* $M' \backslash \Omega'$ *is not polar and*

$$g_{P',\Omega'} = \varphi^* g_{P,\Omega} \; (:= g_{P,\Omega} \circ \varphi) \qquad on \quad \Omega\,. \tag{3.7}$$

[8] This terminology is somewhat misleading: as a function of $M \backslash \Omega$, $\log \|\frac{\partial}{\partial z}\|_{P,\Omega}$ shares the usual properties of capacities of being increasing and strongly subadditive, while $\|\frac{\partial}{\partial z}\|_{P,\Omega}$ itself does not in general.

We shall see the non-polarity of $M'\backslash\Omega'$ in Appendix **A.3**. Taking it for granted, (3.7) follows from the characterization by conditions (i) and (ii) of equilibrium potentials and from the fact that φ^* defines an isomorphism from $\mathrm{L}^2_1(\Omega')_0$ to $\mathrm{L}^2_1(\Omega)_0$.

Together with the computation of the equilibrium potential $g_{0,\overset{\circ}{D}(0,1)}$ on $\mathbb{P}^1(\mathbb{C})$ (see Remarks 3.1.4, **iv)** above), this lemma leads to the following:

Proposition 3.3. *Let M be a compact connected Riemann surface, and $\varphi : \overset{\circ}{D}(0;1) \to M$ a holomorphic embedding of the unit disk, and let*

$$\Omega := \varphi(\overset{\circ}{D}(0;1)) \qquad \textit{and} \quad P = \varphi(0)\,.$$

Then $M\backslash\Omega$ is not polar, and

$$(3.8)\qquad \begin{aligned} g_{P,\Omega}(Q) &= \log|\varphi^{-1}(Q)|^{-2} \quad \textit{if} \quad Q \in \Omega\backslash\{P\} \\ &= 0 \qquad\qquad\qquad\quad\ \textit{if} \quad Q \in M\backslash\Omega\,. \end{aligned}$$

Moreover, the capacitary norm $\|\cdot\|_{P,\Omega}$ is given by the equality

$$(3.9)\qquad \left\| \mathrm{D}\varphi(0)\left(\frac{\partial}{\partial z}\right)\right\|_{P,\Omega} = 1\,.$$

Indeed, $M\backslash\Omega$ is not polar since $\mathbb{P}^1(\mathbb{C})\backslash\overset{\circ}{D}(0,1)$ is not. Moreover, the function defined by the right-hand side of (3.8) coincides with $g_{P,\Omega}$ almost everywhere on $M\backslash\{P\}$. As it is continuous (hence upper semicontinuous) and $g_{P,\Omega}$ is subharmonic, they coincide everywhere on $M\backslash\{P\}$. If we let Q go to P in (3.8), we get (3.9).

Example 3.4. Let Ω be a connected and *simply connected* open subset of $\mathbb{P}^1(\mathbb{C})$ such that

$$0 \in \Omega \qquad \text{and} \qquad |\mathbb{P}^1(\mathbb{C})\backslash\Omega| \geq 2\,.$$

Then there exists a biholomorphic mapping

$$\varphi : \overset{\circ}{D}(0;1) \xrightarrow{\sim} \Omega$$

such that $\varphi(0) = 0$. Moreover, φ is unique up to multiplication by a complex number of modulus 1. In particular,

$$(3.10)\qquad \rho(\Omega) := |\varphi'(0)|$$

is a well defined positive number. For instance, for any $r \in \mathbb{R}^*_+$,

$$\rho(\overset{\circ}{D}(0;r)) = r\,.$$

From (3.9), we get

$$(3.11)\qquad \left\|\frac{\partial}{\partial z}\right\|_{0,\Omega} = \rho(\Omega)^{-1}\,.$$

The construction of equilibrium potentials is also compatible with pull-backs by finite holomorphic maps:

Proposition 3.5. *Let $f : M' \to M$ be a finite holomorphic mapping between compact Riemann surfaces, let Ω be an open subset of M the complement of which in each connected component of M is not polar. Then the complement of $f^{-1}(\Omega)$ in each connected component of M' is not polar, and for any divisor D in M supported by Ω, we have*

$$g_{f^*(D),f^{-1}(\Omega)} = f^* g_{D,\Omega} \, (:= g_{D,\Omega} \circ f) \, .$$

Proof. It is enough to prove the proposition when D is effective. Then $f^*(D)$ also is effective, and we may use the characterization of equilibrium potentials attached to effective divisors given at the end of Remarks 3.1.4, **iii)**. Proposition 3.5 therefore follows from the following elementary observation: let $f : X' \to X$ be some holomorphic map between connected Riemann surfaces; if f is finite, then, for any subset E of X, E is polar iff $f^{-1}(E)$ is, and, for any harmonic (resp. subharmonic) function on X, $f^* \varphi$ is harmonic (resp. subharmonic) on X'.

q.e.d.

3.2. Arakelov divisors on arithmetic surfaces attached to equilibrium potentials

3.2.1. The Arakelov divisor $\widehat{D}_\Omega$

Let $\mathcal{X}$ be a projective integral normal arithmetic surface, namely, a projective integral normal scheme over Spec $\mathbb{Z}$ of Krull dimension 2. The ring of regular function on $\mathcal{X}$ is the ring of integers $\mathcal{O}_K$ of some number field K, and the canonical morphism

$$\pi : \mathcal{X} \to \text{Spec } \mathcal{O}_K \, ,$$

is projective, flat, with geometrically connected fibers (it is indeed the "Stein factorization" of the map $\mathcal{X} \to$ Spec $\mathbb{Z}$). For any embedding $\sigma : K \to \mathbb{C}$, the complex curve

$$\mathcal{X}_\sigma := \mathcal{X} \otimes_\sigma \mathbb{C}$$

is projective, smooth, and connected. Moreover, we have the decomposition into connected components:

$$\mathcal{X}(\mathbb{C}) = \coprod_{\sigma : K \to \mathbb{C}} \mathcal{X}_\sigma(\mathbb{C}) \, .$$

Let Ω be an open subset of $\mathcal{X}(\mathbb{C})$, invariant by complex conjugation, such that, for any embedding $\sigma : K \to \mathbb{C}$, the complement $\mathcal{X}_\sigma(\mathbb{C}) \backslash \Omega$ is not polar.

For any Weil divisor D on $\mathcal{X}$ such that

$$|D|(\mathbb{C}) \subset \Omega \, ,$$

we may consider the L^2_1-Green function $g_{D,\Omega}$ for D, defined as $g_{D_{\mathbb{C}},\Omega}$ or, equivalently, as the generalized function on $\mathcal{X}(\mathbb{C})$ such that

$$g_{D,\Omega} := g_{D_\sigma, \Omega \cap \mathcal{X}_\sigma(\mathbb{C})} \qquad \text{on} \quad \mathcal{X}_\sigma(\mathbb{C}) \, .$$

We may also introduce the "Arakelov" or "compactified" divisor

$$\widehat{D}_{\Omega} := (D, g_{D,\Omega}).$$

It is not an Arakelov divisor in the usual sense – since $g_{D,\Omega}$ is not, in general, a Green function with C^{∞} regularity for D – but in the generalized sense we introduce in § **5.2** below.

3.2.2. The height $h_{\widehat{D}_{\Omega}}$

Recall that, if $\overline{\mathcal{L}} = (\mathcal{L}, \| \ \|)$ is an hermitian line bundle and Z a Weil divisor on $\mathcal{X}$, we may define the height $h_{\overline{\mathcal{L}}}(Z)$ of Z with respect to $\overline{\mathcal{L}}$ as the Arakelov degree $\widehat{\deg}\,(\widehat{c}_1(\overline{\mathcal{L}}) \mid Z)$ of $\overline{\mathcal{L}}$ along Z (*cf.* [B-G-S] for instance). Consider now an Arakelov divisor $\widehat{E} = (E, g)$, defined by a Cartier divisor E on $\mathcal{X}$ and a Green function g for $E_{\mathbb{C}}$ (*cf.* 3.1.3). We may define the height $h_{\widehat{E}}(Z)$ as $h_{\overline{\mathcal{O}(E)}}(Z)$, where $\overline{\mathcal{O}(E)}$ is the line bundle $\mathcal{O}(E)$ equipped with the metric $\| \cdot \|$ such that, if $\mathbf{1}$ denotes the canonical section of $\mathcal{O}(E)$ with divisor E,

$$g = \log \|\mathbf{1}\|^{-2}.$$

More generally the definition of $h_{\overline{\mathcal{L}}}(Z)$ (resp. $h_{\widehat{E}}(Z)$) still makes sense as soon as $\mathcal{L}$ is equipped with a C^{∞} (or even continuous) metric on a neighbourhood of $|Z|(\mathbb{C})$ (resp., if g is defined and if $dd^c g + \delta_{Z_{\mathbb{C}}}$ is C^{∞} on an open neighbourhood of $|Z|(\mathbb{C})$). Let us recall how it is defined. The height $h_{\overline{\mathcal{L}}}(Z)$ is additive in Z, and is therefore specified by its values on cycles Z defined by integral subschemes; let Z be such a subscheme, and $\nu : \widetilde{Z} \to Z$ its normalization; then $\widetilde{Z}$ is either a projective smooth, geometrically irreducible curve over some finite field $\mathbb{F}_q$, or isomorphic to Spec $\mathcal{O}_K$ for some number field K. By definition, we have

$$\begin{aligned} h_{\overline{\mathcal{L}}}(Z) &:= \deg_{\mathbb{F}_q} \nu^* \mathcal{L} \cdot \log q \quad \text{in the first case} \\ &:= \widehat{\deg}\, \nu^* \overline{\mathcal{L}} \quad \text{in the second one.} \end{aligned}$$

The height $h_{\widehat{E}}(Z)$ is additive in $\widehat{E}$ also. In particular, for any $n \in \mathbb{N}^*$,

$$h_{\widehat{E}}(Z) = \frac{1}{n} h_{n\widehat{E}}(Z). \tag{3.12}$$

This equality leads to a definition of $h_{\widehat{E}}(Z)$ when E is a Weil divisor, not necessarily Cartier. Indeed, as shown by Moret-Bailly ([MB2]), the local Picard groups of $\mathcal{X}$ at non smooth points are finite, and any Weil divisor E on $\mathcal{X}$ has a multiple $n E$, $n \in \mathbb{N}^*$, which is Cartier; then $h_{\widehat{E}}(Z)$ may be defined by (3.12).

This discussion applies to the Arakelov divisor $\widehat{E} = \widehat{D}_{\Omega} := (D, g_{D,\Omega})$ defined above and shows that, for any Weil divisor Z on $\mathcal{X}$ such that $|Z|(\mathbb{C}) \subset \Omega$, the height $h_{\widehat{D}_{\Omega}(Z)}$ is well defined.

The height $h_{\widehat{D}_{\Omega}}$ satisfies the following projection formula:

Proposition 3.6. *Let $\mathcal{X}$ and $\mathcal{X}'$ be two projective integral normal arithmetic surfaces, and $f : \mathcal{X}' \to \mathcal{X}$ a generically finite morphism. Let D be a Cartier divisor on $\mathcal{X}$, and let Ω be an open neighbourhood of $|D|(\mathbb{C})$ in $\mathcal{X}(\mathbb{C})$, invariant under complex conjugation such* *that, for any connected component M of $\mathcal{X}(\mathbb{C})$, $M\backslash\Omega$ is not polar. Let $D' := f^* D$ and $\Omega' := f_{\mathbb{C}}^{-1}(\Omega)$, and let Z be a Weil divisor in $\mathcal{X}'$ such that $|Z|(\mathbb{C}) \subset \Omega'$.*

Then, for any connected component M' of $\mathcal{X}'(\mathbb{C})$, $M'\backslash\Omega'$ is not polar, and $\widehat{D}'_{\Omega'} := (D', g_{D',\Omega'})$ is therefore well defined. Moreover,

$$h_{\widehat{D}'_{\Omega'}}(Z) = h_{\widehat{D}_{\Omega}}(\nu_* Z). \tag{3.13}$$

This follows from Proposition 3.5 and the definitions of $h_{\widehat{D}'_{\Omega'}}$ and $h_{\widehat{D}_{\Omega}}$.

3.2.3. Examples

Let P be a section of $\pi : \mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$ the image of which lies in the smooth locus of π, and let Ω be an open neighbourhood of $P(\mathbb{C})$ in $\mathcal{X}(\mathbb{C})$, invariant under complex conjugation such that, for any $\sigma : K \hookrightarrow \mathbb{C}$, the complement of $\Omega_\sigma := \Omega \cap \mathcal{X}_\sigma(\mathbb{C})$ in $\mathcal{X}_\sigma(\mathbb{C})$ is not polar. We may apply the discussion above to the divisor on $\mathcal{X}$ defined by P; thus we get an Arakelov divisor

$$\widehat{P}_\Omega := (P, g_{P,\Omega})$$

where

$$g_{P,\Omega} := g_{P_\sigma,\Omega\cap\mathcal{X}_\sigma(\mathbb{C})} \qquad \text{on} \quad \mathcal{X}_\sigma(\mathbb{C}).$$

We shall denote $\overline{\mathcal{O}(P)}_\Omega$ the associated hermitian line bundle.

We have a canonical adjunction isomorphism of K-lines:

$$\mathcal{O}(P_K)_{|P_K} \simeq T_{P_K}\mathcal{X}_K;$$

for any uniformizing parameter t at P_K on $\mathcal{X}_K$, it maps $t^{-1}_{|P_K}$ to $\frac{\partial}{\partial t}_{|P_K}$. The smoothness of π along P and the definition of the capacitary metrics $\| \ \|_{P_\sigma,\Omega_\sigma}$ show that this isomorphism defines an isometric isomorphism

$$P^*\overline{\mathcal{O}(P)}_\Omega \simeq (P^* T_\pi \mathcal{X}, (\| \ \|_{P_\sigma,\Omega_\sigma})).$$

Therefore

$$\begin{aligned} h_{\widehat{P}_\Omega}(P) &:= \widehat{\deg}\, P^*\overline{\mathcal{O}(P)}_\Omega \\ &= \widehat{\deg}\,(P^* T_\pi \mathcal{X}, (\| \ \|_{P_\sigma,\Omega_\sigma})_{\sigma:K\hookrightarrow\mathbb{C}}). \end{aligned}$$

This relates the height $h_{\widehat{P}_\Omega}(P)$ and the capacitary metrics $\| \cdot \|_{P_\sigma,\Omega_\sigma}$.

In particular, when $\mathcal{X} = \mathbb{P}^1_{\mathbb{Z}}$ and $P = \infty$, $h_{\widehat{\infty}_\Omega}(\infty)$ coincides with the Robin constant $r(\mathbb{P}^1(\mathbb{C})\backslash\Omega)$, and the projection formula of Proposition 3.6, for $\mathcal{X} = \mathcal{X}' = \mathbb{P}^1_{\mathbb{Z}}$ translates into relations between Robin constants of various compact subsets of $\mathbb{C}$. For instance, consider the map

$$\begin{aligned} f : \mathbb{P}^1_{\mathbb{Z}} &\to \mathbb{P}^1_{\mathbb{Z}}, \\ x &\mapsto y = \tfrac{(1-x)^2}{x}; \end{aligned}$$

(up to the automorphisms $v = -y^{-1}$ and $u = \frac{x}{1-x}$ of $\mathbb{P}^1_{\mathbb{Z}}$, it coincides with the Artin-Schreier map $u \mapsto v = u^2 + u$), and let $\Omega = \mathbb{P}^1(\mathbb{C})\backslash[0,4]$, $D = D_\infty$, and $Z = D_0$, where for any $a \in \mathbb{P}^1(\mathbb{Q}) = \mathbb{P}^1(\mathbb{Z})$, D_a denotes the image of the section $a : \operatorname{Spec}\mathbb{Z} \to \mathbb{P}^1_{\mathbb{Z}}$. Then

$\Omega' = \mathbb{P}^1(\mathbb{C})\backslash\{z \in \mathbb{C} \mid |z| = 1\}$, $D' = D_0 + D_\infty$, $\nu_* Z = D_\infty$, $h_{\widehat{D'}_{\Omega'}}(Z) = 0$, and the projection formula (3.13) becomes the classical formula:

$$r([0,4]) = 0 .$$

Finally, if, as in **1.1** and Theorem 1.2, each Ω_σ is the image of some holomorphic embedding

$$\varphi_\sigma : \overset{\circ}{D}(0;1) \hookrightarrow \mathcal{X}_\sigma(\mathbb{C})$$

such that

$$\varphi_\sigma(0) = P_\sigma ,$$

then Proposition 3.3 shows that, with the notation of **1.1**, we have an equality of metrics on $T_{P_\sigma}\mathcal{X}_\sigma$ for any $\sigma : K \hookrightarrow \mathbb{C}$:

$$\|\cdot\|_{\varphi_\sigma} = \|\cdot\|_{P_\sigma,\Omega_\sigma} ,$$

and therefore

$$h_{\widehat{P}_\Omega}(P) = \widehat{\deg}\,(P^* T_\pi \mathcal{X}, (\|\cdot\|_{\varphi_\sigma})_{\sigma:K\hookrightarrow\mathbb{C}}) . \tag{3.14}$$

4. Nef divisors, connectedness, and fundamental groups of arithmetic surfaces

4.1. The following definition is motivated by the fact that an effective Weil divisor on a projective normal surface over a field is nef iff it intersects non-negatively all its irreducible components (*cf.* **2.4**, **v)** and **vi)**) (see **6.1** *infra* for further discussion):

Definition 4.1. *Let $\mathcal{X}$ be a projective integral normal arithmetic surface, D an* effective *Weil divisor on $\mathcal{X}$, and Ω an open neighbourhood of D in $\mathcal{X}(\mathbb{C})$, invariant under complex conjugation, and such that, for any connected component M of $\mathcal{X}(\mathbb{C})$, $M\backslash\Omega$ is not polar.*

The Arakelov divisor $\widehat{D}_\Omega := (D, g_{D,\Omega})$ is called numerically effective *or* nef *if, for any irreducible component D_i of D,*

$$h_{\widehat{D}_\Omega}(D_i) \geq 0 .$$

Using this definition, we may formulate our theorems of Lefschetz type for arithmetic surfaces.

Technically, the main result of this paper is the following connectedness theorem, which we will prove in section 6.

Theorem 4.2. *Let $\mathcal{X}$ be a projective integral normal arithmetic surface, let D be an effective Weil divisor on $\mathcal{X}$ which is not vertical*[9]*, and let Ω be an open neighbourhood of $|D|(\mathbb{C})$ in $\mathcal{X}(\mathbb{C})$, invariant under complex conjugation, such that*

[9] In other words, if $\pi : \mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$ is as in **3.2**, the divisor D_K on $\mathcal{X}_K$ is not zero.

$(4.1)_{\mathcal{X},\Omega}$ *for any connected component M of $\mathcal{X}(\mathbb{C})$, $M\backslash\Omega$ is not polar.*

If each connected component of Ω contains at most one point in $|D|(\mathbb{C})$, if $\widehat{D}_\Omega := (D, g_{D,\Omega})$ is nef and if

$(4.2)_{\mathcal{X},\Omega}$ *the interior of $\mathcal{X}(\mathbb{C})\backslash\Omega$ is not empty,*

then $|D|$ is connected.

This statement is an arithmetic counterpart to Theorem 2.4. Indeed, it admits the following informal interpretation: the divisor $\widehat{D}_\Omega$ on the "Arakelov compactification" of $\mathcal{X}$ is effective and nef and therefore, by Theorem 2.4, is expected to be connected (whatever it means; the analogue of the second case in Theorem 2.4 is excluded by condition $(4.2)_{\mathcal{X},\Omega}$); moreover, its various components do not meet over archimedean places (since distinct points in $|D|(\mathbb{C})$ are disconnected by $\mathcal{X}(\mathbb{C})\backslash\Omega$); hence D itself must be connected.

Also observe that Theorem 4.1 does not hold anymore when condition $(4.2)_{\mathcal{X},\Omega}$ is omitted. For instance, all the hypotheses of Theorem 4.1 except $(4.2)_{\mathcal{X},\Omega}$ are satisfied by

$$\mathcal{X} = \mathbb{P}^1_{\mathbb{Z}} \quad , \quad D = D_0 + D_\infty$$

where, as before, for any $a \in \mathbb{P}^1(\mathbb{Q}) = \mathbb{P}^1(\mathbb{Z})$, D_a denotes the image of the section $a : \operatorname{Spec} \mathbb{Z} \to \mathbb{P}^1_{\mathbb{Z}}$, and

$$\Omega = \{(z_0 : z_1) \in \mathbb{P}^1(\mathbb{C}) \mid |z_0| \neq |z_1|\},$$

but $|D|$ is not connected. However, it follows from the proof of Theorem 4.1 that condition $(4.2)_{\mathcal{X},\Omega}$ may be replaced by weaker conditions, for instance by the following one:

$(4.3)_{\mathcal{X},\Omega}$ *There exists $\sigma : K \hookrightarrow \mathbb{C}$ and $P \in F := \mathcal{X}_\sigma(\mathbb{C})\backslash\Omega$ and an open neighbourhood U of P in $\mathcal{X}_\sigma(\mathbb{C})\backslash|D|(\mathbb{C})$ such that any harmonic function on U which vanishes nearly everywhere*[10] *on $U \cap F$ vanishes on U.*

This condition is satisfied as soon as there exists $P \in F$ on a neighbourhood of which F is a C^1 curve (*i.e.* a C^1-subvariety of dimension 1) which is not real analytic.

Another useful variant of Theorem 4.1 involves the *self-intersection* $\widehat{D}_\Omega \cdot \widehat{D}_\Omega$ of the Arakelov divisor $\widehat{D}_\Omega$. It is a real number which will be defined in section 5.3, and will be shown to be given by the following formula, where $(D_i)_{i\in I}$ is the family of irreducible components of $|D|$ and n_i the multiplicity of D_i in D:

$$\widehat{D}_\Omega \cdot \widehat{D}_\Omega = h_{\widehat{D}_\Omega}(D) = \sum_{i\in I} n_i\, h_{\widehat{D}_\Omega}(D_i)\,. \tag{4.4}$$

In particular, when $\widehat{D}_\Omega$ is nef, the self intersection $\widehat{D}_\Omega \cdot \widehat{D}_\Omega$ is non-negative, and positive iff, for some $i \in I$, $h_{\widehat{D}_\Omega}(D_i) > 0$. The following statement is an arithmetic analogue of Proposition 2.2:

Theorem 4.2′. *Consider $\mathcal{X}$, D, and Ω as in the first paragraph of Theorem 4.2, such that condition $(4.1)_{\mathcal{X},\Omega}$ holds.*

If each connected component of Ω contains at most one point of $|D|(\mathbb{C})$, if $\widehat{D}_\Omega$ is nef, and if $\widehat{D}_\Omega \cdot \widehat{D}_\Omega > 0$, then $|D|$ is connected.

[10] *i.e.* on the complement of a polar subset

4.2. We shall amplify Theorem 4.1 by means of the following observations:

Lemma 4.3. *Let $\mathcal{X}$ and $\mathcal{X}'$ be two projective integral normal arithmetic surfaces, and $f : \mathcal{X}' \to \mathcal{X}$ a generically finite morphism. Let D be a non vertical effective Cartier divisor on $\mathcal{X}$, and let Ω be an open neighbourhhod of $|D|(\mathbb{C})$ in $\mathcal{X}(\mathbb{C})$, invariant under complex conjugation. Let $D' := f^* D$ and $\Omega' := f_{\mathbb{C}}^{-1}(\Omega)$.*

1) *If $\mathcal{X}$ and Ω satisfy $(4.1)_{\mathcal{X},\Omega}$ (resp. $(4.2)_{\mathcal{X},\Omega}$), then $\mathcal{X}'$ and Ω' satisfy $(4.1)_{\mathcal{X}',\Omega'}$ (resp. $(4.2)_{\mathcal{X}',\Omega'}$).*

2) *If the inclusion*

$$|D|(\mathbb{C}) \hookrightarrow \Omega$$

is a homotopy equivalence (i.e., if each connected component of Ω is biholomorphic to $\overset{\circ}{D}(0;1)$ or to $\mathbb{C}$ and contains exactly one point of $|D|(\mathbb{C})$), and if $f_{\mathbb{C}} : \mathcal{X}'(\mathbb{C}) \to \mathcal{X}(\mathbb{C})$ is unramified over $\Omega \backslash |D|(\mathbb{C})$, then the inclusion $|D'|(\mathbb{C}) \hookrightarrow \Omega'$ is also a homotopy equivalence.

3) *If $\mathcal{X}$ and Ω satisfy $(4.1)_{\mathcal{X},\Omega}$ and $\widehat{D}_\Omega = (D, g_{D,\Omega})$ is nef, then $\widehat{D}'_{\Omega'} := (D', g_{D',\Omega'})$ is nef.*

4) *If $\mathcal{X}$ and Ω satisfy $(4.1)_{\mathcal{X},\Omega}$ and if $\widehat{D}_\Omega \cdot \widehat{D}_\Omega > 0$, then $\widehat{D}'_{\Omega'} \cdot \widehat{D}'_{\Omega'} > 0$.*

Proof. 1) holds since the inverse image of a non-empty open subset (resp. of a non-polar compact subset) by a surjective holomorphic map between compact Riemann surface is non-empty and open (resp. compact and non-polar; *cf.* Prop. 3.5).

To prove 2), let assume that $|D|(\mathbb{C}) \hookrightarrow \Omega$ is a homotopy equivalence. For any $P \in |D|(\mathbb{C})$, let us denote by Ω_P the connected component of Ω containing P, and let

$$\Omega'_P := f_{\mathbb{C}}^{-1}(\Omega_P).$$

Then $\Omega' = \coprod_{P \in |D|(\mathbb{C})} \Omega'_P$, and

$$f_{\mathbb{C}} : \Omega'_P \to \Omega_P$$

is a finite holomorphic map, ramified only over P; therefore, for any connected component $\Omega'_{P,i}$ of Ω'_P,

$$f_{\mathbb{C}} : \Omega'_{P,i} \to \Omega_P \tag{4.5}$$

is also a finite holomorphic map, ramified only over P. There exists a holomorphic map

$$\varphi_P : \Omega_P \to \mathbb{C}$$

which maps Ω_P biholomorphically onto $U = \overset{\circ}{D}(0,1)$ or $\mathbb{C}$, and P to 0, and an elementary topological argument shows that, if $n_{P,i}$ denotes the degree of (4.5), there exists a biholomorphic map $\varphi'_{P,i}$ from Ω'_P onto U, such that the following diagram commutes

$$\begin{array}{ccc} \Omega'_{P_i} & \xrightarrow{f_{\mathbb{C}}} & \Omega_P \\ \downarrow \varphi'_{P,i} & & \downarrow \varphi_P \\ U & \longrightarrow & U \\ & & \\ z & \mapsto & z^{n_{P,i}} \end{array}$$

In particular, each connected component $\Omega'_{P,i}$ is biholomorphic to $\overset{\circ}{D}(0,1)$ or $\mathbb{C}$, and contains exactly one point of $|D'|(C) := f_{\mathbb{C}}^{-1}(|D|(\mathbb{C}))$.

Assertions 3) and 4) follow from the projection formula of Proposition 3.6. Indeed, this formula shows that, for any component C' of $|D'|$

$$h_{\widehat{D}'_{\Omega'}}(C') = h_{\widehat{D}_\Omega}(\nu_* C');$$

moreover the cycle $\nu_* C'$ is either 0 or a positive multiple of component C of $|D|$, and any component of $|D|$ appears at least once in this way.

q.e.d.

4.3. From Theorem 4.2 and Lemma 4.3, we easily deduce an arithmetic analogue of the geometric Lefschetz theorem 1.1.

Theorem 4.4. *Let $\mathcal{X}$ be a projective integral normal arithmetic surface, D an effective Weil divisor on $\mathcal{X}$ which is not vertical, Ω an open neighbourhood of $|D|(\mathbb{C})$, invariant under complex conjugation, which satisfies $(4.1)_{\mathcal{X},\Omega}$ and $(4.2)_{\mathcal{X},\Omega}$, and F a closed subset of $\mathcal{X}$ such that*

$$F \cap |D| = \emptyset$$

and

$$F(\mathbb{C}) \cap \Omega = \emptyset .$$

If $\widehat{D}_\Omega := (D, g_{D,\Omega})$ is nef and if the inclusion

$$|D|(\mathbb{C}) \hookrightarrow \Omega$$

is a homotopy equivalence, then $|D|$ is connected, and for any geometric point η of $|D|$, the inclusion map $i : |D| \hookrightarrow \mathcal{X}\backslash F$ induces a surjection

$$i_* : \pi_1(D, \eta) \to \pi_1(\mathcal{X}\backslash F, i(\eta)) .$$

Indeed, the connexity of $|D|$ is asserted by Theorem 4.2. By the very definition of the algebraic fundamental group, the surjectivity of i_* is equivalent to the following assertion:

For any finite étale connected covering

$$\nu : \mathcal{Y} \to \mathcal{X}\backslash F ,$$

the inverse image $\nu^{-1}(|D|)$ of $|D|$ is connected.

To prove it, let us consider the integral closure

$$\widetilde{\nu} : \widetilde{\mathcal{Y}} \to \mathcal{X}$$

of $\mathcal{X}$ in the function field of $\mathcal{Y}$. It is a finite morphism, $\widetilde{\mathcal{Y}}$ is a projective normal arithmetic surface, and $\mathcal{Y}$ may be embedded in $\widetilde{\mathcal{Y}}$ as an open subscheme, in such a way that the diagram

$$\begin{array}{ccc} \widetilde{\mathcal{Y}} & \xrightarrow{\widetilde{\nu}} & \mathcal{X} \\ \cup & & \cup \\ \mathcal{Y} & \xrightarrow{\nu} & \mathcal{X}\backslash F \end{array}$$

commutes. Lemma 4.3, applied to $\mathcal{X}' = \widetilde{\mathcal{Y}}$ and $f = \widetilde{\nu}$, shows that the arithmetic surface $\widetilde{\mathcal{Y}}$, the divisor $\widetilde{\nu}^*(D)$ on $\mathcal{Y}$, and the open subset $\widetilde{\nu}_{\mathbb{C}}^{-1}(\Omega)$ of $\widetilde{\mathcal{Y}}(\mathbb{C})$ satisfy the hypotheses of Theorem 4.2. Therefore

$$\nu^{-1}(|D|) = |\widetilde{\nu}^*(D)|$$

is connected, as claimed.

By using Theorem 4.2′ instead of Theorem 4.2, the same reasoning based on Lemma 4.3 leads to:

Theorem 4.4′. *The conclusion of Theorem 4.3 still holds when condition* $(4.2)_{\mathcal{X},\Omega}$ *is not assumed, provided* $\widehat{D}_\Omega \cdot \widehat{D}_\Omega > 0$.

4.4. From Theorems 4.4 and 4.4′, Theorem 1.2 follows easily.

Observe indeed that any integral normal quasi-projective surface $\mathcal{X}_0$ may be written as $\mathcal{X}_0 = \mathcal{X}\backslash F$, where $\mathcal{X}$ and $F \subset \mathcal{X}$ are as in Theorem 4.4. Moreover, divisors D in $\mathcal{X}$ such that $|D| \cap F = \emptyset$ correspond to divisors in $\mathcal{X}_0$ with proper supports. This remark shows that the surjectivity of P_* in Theorem 1.2 follows from Theorems 4.4 and 4.4′ and formula (3.14), when $\coprod_{\sigma:K\hookrightarrow\mathbb{C}} \varphi_\sigma(\overset{\circ}{D}(0,1))$ is invariant under complex conjugation. As

$$\pi_* \circ P_* = (\pi \circ P)_* = \mathrm{id}_* = \mathrm{id},$$

this implies that π_* and P_* are inverse of each other, under this additional hypothesis. To conclude, it is enough to note that, by replacing some of the φ_σ's by

$$\begin{array}{rcl} \widetilde{\varphi_{\overline{\sigma}}} : \overset{\circ}{D}(0,1) & \rightarrow & \mathcal{X}_\sigma(\mathbb{C}) \\ z & \mapsto & \overline{\varphi_{\overline{\sigma}}(z)}, \end{array}$$

it is possible to achieve it and simultaneously not to decrease $\widehat{\deg}\,(P^* T_\pi, (\| \, \|_{\varphi_\sigma}))$.

5. Arakelov intersection theory on arithmetic surfaces and L^2_1-Green functions

In this section, we introduce the version of the arithmetic Chow group $\widehat{\mathrm{CH}}^1(\mathcal{X})$ and of the intersection pairing that we shall need for the proof of Theorems 4.2 and 4.2′.

5.1. Integrals of star-products of L^2_1-Green functions

Let M be a compact Riemann surface, and D_1 and D_2 divisors with disjoint support on M. For any two Green functions g_1 and g_2 for D_1 and D_2 respectively, their star-product is defined as the current of degree 2 on M:

$$g_1 * g_2 := g_1\,\omega_2 + g_2\,\delta_{D_1}, \tag{5.1}$$

where ω_2 is the C^∞ form of degree 2 on M defined by:

$$\omega_i := \mathrm{dd}^c\, g_i + \delta_{D_i}.$$

From (5.1), we get:

$$g_1 * g_2 = g_1 \,\mathrm{dd}^c g_2 + g_1\,\delta_{D_2} + g_2\,\delta_{D_1}\,;$$

therefore

$$\begin{aligned} g_1 * g_2 - g_2 * g_1 &= -\frac{1}{2\pi i}\, g_1\,\partial\overline{\partial}\, g_2 + \frac{1}{2\pi i}\, g_2\,\partial\overline{\partial}\, g_1 \\ &= \frac{1}{2\pi i}\,\mathrm{d}(g_2\,\overline{\partial}\, g_1 + g_1\,\partial\, g_2)\,, \end{aligned}$$

and

$$\int_M g_1 * g_2 = \int_M g_2 * g_1\,. \tag{5.2}$$

From (5.1) and (5.2), it follows that, for any two functions φ_1 and φ_2 in $\mathrm{C}^\infty(M)$, if we define new Green functions $\widetilde{g}_1$ and $\widetilde{g}_2$ for D_1 and D_2 by

$$\widetilde{g}_1 = g_1 + \varphi_1 \qquad \text{and} \qquad \widetilde{g}_2 = g_2 + \varphi_2\,, \tag{5.3}$$

then we have

$$\begin{aligned} \int_M \widetilde{g}_1 * \widetilde{g}_2 &= \int_M g_1 * g_2 + \int_M \varphi_1\,\omega_2 + \int_M \varphi_2\,\omega_1 + \int_M \mathrm{dd}^c\varphi_1 \cdot \varphi_2 \\ &= \int_M g_1 * g_2 + \int_M \varphi_1\,\omega_2 + \int_M \varphi_2\,\omega_1 + \frac{1}{2\pi i}\int_M \partial\varphi_1 \wedge \overline{\partial}\varphi_2\,. \end{aligned} \tag{5.4}$$

Together with the fact that

$$\operatorname{supp} g_1 \cap \operatorname{supp} g_2 = \emptyset \qquad \Longrightarrow \qquad \int_M g_1 * g_2 = 0\,, \tag{5.5}$$

this property completely determines the pairing $(g_1, g_2) \mapsto \int_M g_1 * g_2$. Indeed, if $\widetilde{g}_i$ and g_i are any Green functions for D_i ($i = 1, 2$), there exist φ_1 and φ_2 in $\mathrm{C}^\infty(M)$ such that (5.3) holds.

More generally, the right hand side of (5.4) is well-defined as soon as φ_1 and φ_2 belong to $\mathrm{L}^2_1(M)$. As any two L^2_1-Green functions $\widetilde{g}_1$ and $\widetilde{g}_2$ for D_1 and D_2 may be written as (5.3), where φ_1 and φ_2 belong to $\mathrm{L}^2_1(M)$, we may take (5.4) as a definition of $\int_M \widetilde{g}_1 * \widetilde{g}_2$ when $\widetilde{g}_1$ and $\widetilde{g}_2$ are L^2_1-Green functions (in which case $\widetilde{g}_1 * \widetilde{g}_2$ itself is not necessarily well defined). One readily checks that it does not depend on the choice of the Green functions g_1 and g_2, that it is symmetric in $\widetilde{g}_1$ and $\widetilde{g}_2$, and that (5.4) holds more generally when $g_1, g_2, \widetilde{g}_1$ and $\widetilde{g}_2$ are arbitrary L^2_1-Green functions (for D_1 or D_2) (then φ_1 and φ_2 belong to $\mathrm{L}^2_1(M)$, ω_1 and ω_2 are L^2_{-1}-currents of degree 2, and the integrals $\int_M \varphi_1\,\omega_2$ and $\int_M \varphi_2\,\omega_1$ are therefore well defined; *cf.* **3.1.2**). This generalized form of (5.4) shows that, for any sequence $(g_1^n)_{n\in\mathbb{N}}$ (resp. $(g_2^n)_{n\in\mathbb{N}}$) of L^2_1-Green functions for D_1 (resp. for D_2) such that

$$\|g_1^n - g_1\|_{\mathrm{L}^2_1(M)} \to 0 \quad \text{and} \quad \|g_2^n - g_2\|_{\mathrm{L}^2_1(M)} \to 0\,,$$

we have

$$\int_M g_1^n * g_2^n \to \int_M g_1 * g_2\,.$$

Observe also that (5.5) still holds when g_1 and g_2 are L^2_1-Green functions.

These properties allow us to prove the following simple but useful lemmas:

Lemma 5.1. *Let M be a compact connected Riemann surface, Ω_1 and Ω_2 two open subsets of M such that $M\backslash\Omega_1$ and $M\backslash\Omega_2$ are not polar, and D_1 (resp. D_2) a divisor on M supported by Ω_1 (resp. Ω_2). If*

$$\Omega_1 \cap \Omega_2 = \emptyset\,,$$

then

$$\int_M g_{D_1,\Omega_1} * g_{D_2,\Omega_2} = 0\,.$$

Proof. This follows from (5.5) when $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \emptyset$. To handle the general case, we argue as follows. Let ρ_1 (resp. ρ_2) be a function in $C_c^\infty(\Omega_1)$ (resp. $C_c^\infty(\Omega_2)$) such that $\rho_1 \equiv 1$ (resp. $\rho_2 \equiv 1$) on a neighbourhood of $|D_1|$ (resp. of $|D_2|$), and let, for $i = 1, 2$:

$$g_i := \rho_i\, g_{D_i,\Omega_i} \quad \text{and} \quad \varphi_i = g_{D_i,\Omega_i} - g_i\,.$$

Then g_i is a Green function for D_i, the support of which lies in Ω_i, φ_i belongs to $L_1^2(\Omega_i)_0$, and

$$\omega_i := dd^c\, g_i + \delta_{D_i}$$

is a C^∞ 2-form the support of which lies in $\Omega_i\backslash|D_i|$. In particular, we may use formula (5.4), and we get:

$$\begin{aligned}\int_M g_{D_1,\Omega_1} * g_{D_2,\Omega_2} &= \int_M (g_1+\varphi_1) * (g_2+\varphi_2)\\ &= \int_M g_1 * g_2 + \int_M \omega_1\,\varphi_2 + \int_M \varphi_1\,\omega_2\\ &+ \frac{1}{2\pi i}\int_M \partial\varphi_1 \wedge \overline{\partial}\varphi_2\,.\end{aligned}$$

The integral $\int_M g_1 * g_2$ (resp. $\int_M \omega_1\,\varphi_2$, resp. $\int_M \varphi_1\,\omega_2$) vanishes since the Green functions g_1 and g_2 (resp. ω_1 and φ_2, resp. φ_1 and ω_2) have disjoint supports, and the integral $\int_M \partial\varphi_1 \wedge \overline{\partial}\varphi_2$ vanishes as was observed in **3.1.2**, (3.4).

q.e.d.

Lemma 5.2. *Let D be an effective divisor on M, and g a L_1^2-Green function for D on M, such that $\omega := dd^c\, g + \delta_D$ is locally*[11] *L^∞ on some open subset U of M.*

Then, for any open subset U' of U and any meromorphic function f on U' such that $\operatorname{div} f = D_{|U'}$, *there exists a continuous function φ on U' such that:*

$$g = \varphi + \log|f|^{-2} \qquad \text{on} \quad U'\,.$$

Moreover, for any divisor D' on M such that $|D| \cap |D'| = \emptyset$ and $|D'| \subset U$, and any L_1^2-Green function g' for D' on M, we have:

$$\int_M g' * g = \int_M (g'\,\omega + g\,\delta_{D'}). \tag{5.6}$$

[11] The condition "locally L^∞" could be weakened to "locally L^p", for any $p > 1$.

In particular, for any divisor D' on M such that $|D| \cap |D'| = \emptyset$, (5.6) holds for any Green function g for D (with C^∞ regularity) and any L^2_1-Green function g' for D'.

Observe that the right hand side of (5.6) is indeed well defined:

• $g\,\delta_{D'}$ is well defined, since g is continuous on a neighbourhood of $|D'|$, which lies in U;

• $\omega\, g'$ is well defined on $M \backslash |D'|$, since $\omega \in L^2_{-1}(M)$ and $g'_{|M\backslash|D'|} \in L^2_1(M\backslash|D'|)_{\mathrm{loc}}$; it is also well defined on an open neighbourhood of $|D'|$ in M, since ω is locally L^∞ and g' locally L^1 near $|D'|$.

Proof. To prove the first assertion, observe that, if we let

$$\varphi := g - \log |f|^{-2} \qquad \text{on} \quad U',$$

then $dd^c\,\varphi$ is locally L^∞ on U'. But any generalized function on a Riemann surface, the image of which by dd^c is locally L^∞, is continuous: this is a local statement, which has to be checked on open domains in $\mathbb{C}$; then it follows by a standard convolution argument from the fact that a fundamental solution for dd^c over $\mathbb{C}$ is $\log|z|^2$, which is locally L^1, and C^∞ outside 0.

To prove (5.6), let us write

$$g' = g'_0 + \varphi'$$

where g'_0 is a Green function (with C^∞ regularity) for D' and $\varphi' \in L^2_1(M)$. According to (5.4), we have:

$$\int_M g' * g = \int_M g'_0 * g + \int_M \varphi'\,\omega\,.$$

This reduces the proof of (5.6) to the case where g' has C^∞ regularity. Then, we may write

$$g = g_0 + \varphi\,,$$

where g_0 is a Green function with C^∞ regularity for D, and where $dd^c\,\varphi$ is L^∞ and φ is continuous on some neighbourhood of $|D'|$. By (5.4) again, we have

$$\int_M g' * g = \int_M g' * g_0 + \int_M \varphi\,\omega'$$

where $\omega' := dd^c\,g' + \delta_{D'}$. On the other hand, we have:

$$\int_M g' * g_0 = \int_M (g'\,\omega_0 + g_0\,\delta_{D'})\,,$$

and:

$$\int_M (\omega\, g' + g\,\delta_{D'}) = \int_M [g'\,(\omega_0 + dd^c\,\varphi) + (g_0 + \varphi)\,\delta_{D'}]$$

where $\omega_0 := dd^c\,g_0 + \delta_D$. Using these three equalities, to complete the proof of (5.6), we need to prove the identity:

$$\int_M g'\,dd^c\,\varphi = \int_M \varphi\,dd^c\,g'\,.$$

This will follow from the equality of currents on M:

$$g'\,\mathrm{dd}^c\varphi - \varphi\,\mathrm{dd}^c g' = -\frac{1}{2\pi i}\, g'\,\partial\overline{\partial}\,\varphi + \frac{1}{2\pi i}\,\varphi\,\partial\overline{\partial}\,g'$$
$$= \frac{1}{2\pi i}\,\mathrm{d}(\varphi\,\overline{\partial}\,g' + g'\,\partial\,\varphi)\,.$$

This equality indeed holds on $M\backslash|D'|$, where g' is C^∞. To prove its validity on some neighbourhood of $|D'|$, we may introduce local coordinates near points of $|D'|$, and are reduced to prove it when φ is a continuous function on some open neighbourhood of 0 in $\mathbb{C}$ such that $\partial\overline{\partial}\,\varphi$ is bounded, and g' is a Green function for 0; then a standard approximation argument, which replaces φ by convolution products, shows that it follows from the case where φ is C^∞ – in which case it clearly holds.

q.e.d.

5.2. The arithmetic Chow group $\widehat{\mathrm{CH}}^1(\mathcal{X})$

From now on, and until the end of section **6**, *we will denote by* $\mathcal{X}$ *a* projective integral normal arithmetic surface, *and by*

$$\pi : \mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$$

the "Stein factorization" of the map $\mathcal{X} \to \operatorname{Spec} \mathbb{Z}$ (*cf.* **3.2.1**).

Let $Z^1(\mathcal{X})$ be the abelian group of (Weil) divisors on $\mathcal{X}$. We define an *Arakelov divisor* on $\mathcal{X}$ as a pair (D, g), where $D \in Z^1(\mathcal{X})$ and where g is a real L^2_1-Green function on $\mathcal{X}(\mathbb{C})$ for the divisor $Z_{\mathbb{C}}$ (*cf.* **3.1.3**), which is invariant by the complex conjugation on $\mathcal{X}(\mathbb{C})$. Equipped with the sum defined by

$$(D, g) + (D', g') = (D + D', g + g')\,,$$

the Arakelov divisors on $\mathcal{X}$ form an abelian group, that we shall denote $\widehat{Z}^1(\mathcal{X})$. To any non-zero rational function $f \in K(\mathcal{X})^*$ is attached its Arakelov divisor

$$\widehat{\mathrm{div}}\, f := (\mathrm{div}\ f, -\log|f_{\mathbb{C}}|^2)\,.$$

Indeed, we have:

$$\mathrm{dd}^c(-\log|f_{\mathbb{C}}|) + \delta_{\mathrm{div}\ f_{\mathbb{C}}} = 0\,.$$

The map

$$\widehat{\mathrm{div}} : K(\mathcal{X})^* \to \widehat{Z}^1(\mathcal{X})$$

is a homomorphism of abelian groups, and, by definition its cokernel is the *arithmetic Chow group* $\widehat{\mathrm{CH}}^1(\mathcal{X})$.

Observe that the group of Arakelov divisors $\widehat{Z}^1(\mathcal{X})$ we consider here is strictly larger than the ones used by previous authors: Arakelov [A], Faltings [F1] and Moret-Bailly [MB1] use admissible Green functions, and Deligne [D3] and Gillet-Soulé [G-S1], Green functions with C^∞ regularity; moreover, these authors consider only Cartier divisors on $\mathcal{X}$. The arithmetic (or Arakelov-) Chow groups they define by means of these more restrictive definitions of Arakelov cycles inject (when defined) in our $\widehat{\mathrm{CH}}^1(\mathcal{X})$.

For any Arakelov divisor $\alpha = (D, g)$ in $\widehat{Z}^1(\mathcal{X})$, we shall denote, as usual:

$$\omega(\alpha) := \mathrm{dd}^c g + \delta_{D_{\mathbb{C}}}.$$

This is an element of $L^2_{-1}(\mathcal{X}(\mathbb{C}))$, real and invariant under complex conjugation. Moreover, the morphism of abelian groups

$$\omega : \widehat{Z}^1(\mathcal{X}) \to L^2_{-1}(\mathcal{X}(\mathbb{C}))$$

vanishes on $\widehat{\mathrm{div}}\, K(\mathcal{X})^*$, and defines a morphism of abelian groups from $\widehat{\mathrm{CH}}^1(\mathcal{X})$ to $L^2_{-1}(\mathcal{X}(\mathbb{C}))$, which we shall still denote ω.

5.3. Arakelov intersection pairing

If $Z = \sum_{i\in I} n_i P_i$ is any 0-cycle on $\mathcal{X}$ with $\mathbb{Q}$-coefficients (*i.e.*, the P_i are closed points in $\mathcal{X}$ and the n_i belong to $\mathbb{Q}$), we define its Arakelov degree as

$$\widehat{\deg}\, Z := \sum_{i\in I} n_i \log N(P_i),$$

where the norm $N(P_i)$ of P_i is the cardinality of its residue field. The Arakelov degree of Z depends only on its class modulo $\mathbb{Q}$-linear equivalence supported by vertical fibers. Namely, if C is any integral (necessarily proper) curve contained in a closed fiber of $\pi : \mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$, and if f is any non-zero rational function on C, we have

$$\widehat{\deg}\ \mathrm{div}\ f = 0, \tag{5.7}$$

and "$\mathbb{Q}$-linear equivalence supported by vertical fibers" is defined by equality modulo elements of the $\mathbb{Q}$-vector space generated by such $\mathrm{div}\ f$'s. To prove (5.7) it is enough to observe that, if k denotes the field of constants of C and q its cardinality, we have more generally, for any 0-cycle Z on C:

$$\widehat{\deg}\, C = \deg_k C \cdot \log q.$$

Consider now two divisors D_1 and D_2 on $\mathcal{X}$, such that $|D_1|$ and $|D_2|$ do not meet on $\mathcal{X}_K$.

If D_1 and D_2 are Cartier, then their intersection 0-cycle $D_1 \cdot D_2$ is well defined up to linear equivalence supported by $|D_1| \cap |D_2|$, therefore up to linear equivalence supported by vertical fibers. Moreover, up to such equivalence,

$$D_1 \cdot D_2 = D_2 \cdot D_1.$$

In general, D_1 and D_2 are not necessarily Cartier. However, some multiples $n_1 D_1$ and $n_2 D_2$ (with $n_1, n_2 \in \mathbb{N}^*$) are, according to Moret-Bailly ([MB1]), and we may define $D_1 \cdot D_2$ as the 0-cycle with coefficients in $\mathbb{Q}$

$$D_1 \cdot D_2 := (n_1 n_2)^{-1} (n_1 D_1) \cdot (n_2 D_2),$$

which is well defined and symmetric in D_1, D_2, up to $\mathbb{Q}$-linear equivalence supported by vertical fibers.

Another way to define the 0-cycle with $\mathbb{Q}$-coefficients $D_1 \cdot D_2$ is to introduce a resolution $\nu : \widetilde{\mathcal{X}} \to \mathcal{X}$ and to use a variant of Mumford's construction we recalled in **2.4**; namely, we may define a linear map

$$\mu : Z^1(\mathcal{X})_{\mathbb{Q}} \to Z^1(\mathcal{X}')$$

as in **2.4**, by imposing (2.7) and (2.8) (now in (2.8), $D' \cdot E_i$ has to be seen as a local intersection number, as in [D1], 1, especially 1.1 and 1.5), and then define:

$$D_1 \cdot D_2 := \nu_* \left(\mu(D_1) \cdot \mu(D_2)\right),$$

where $\mu(D_1) \cdot \mu(D_2)$ is a well defined 0-cycle with $\mathbb{Q}$-coefficients on $\widetilde{\mathcal{X}}$, up to $\mathbb{Q}$-linear equivalence supported by vertical fibers, since $\widetilde{\mathcal{X}}$ is regular. As for any Cartier divisor D on $\mathcal{X}$,

$$\mu(D) = \nu^*(D),$$

this second definition of $D_1 \cdot D_2$ coincides with the previous one.

Finally, if $\widehat{D}_1 = (D_1, g_1)$ and $\widehat{D}_2 = (D_2, g_2)$ are two Arakelov divisors on $\mathcal{X}$ such that $|D_1|$ and $|D_2|$ do not meet on $\mathcal{X}_K$, we define their *Arakelov intersection product*:

$$\widehat{D}_1 \cdot \widehat{D}_2 := \widehat{\deg}\, D_1 \cdot D_2 + \frac{1}{2} \int_{\mathcal{X}(\mathbb{C})} g_1 * g_2 . \tag{5.8}$$

We claim that this intersection product depends only on the classes $[\widehat{D}_1]$ and $[\widehat{D}_2]$ of $\widehat{D}_1$ and $\widehat{D}_2$ in $\widehat{\mathrm{CH}}^1(\mathcal{X})$; as any two classes x_1 and x_2 in $\widehat{\mathrm{CH}}^1(\mathcal{X})$ may be written

$$x_1 = [\widehat{D}_1] \quad \text{and} \quad x_2 = [\widehat{D}_2]$$

where $|D_1|$ and $|D_2|$ do not meet on $\mathcal{X}_K$, this will show that the intersection product defined by (5.8) factorizes through a symmetric *intersection pairing*:

$$\begin{array}{rcl} \widehat{\mathrm{CH}}^1(\mathcal{X}) \times \widehat{\mathrm{CH}}^1(\mathcal{X}) & \to & \mathbb{R} \\ (x_1, x_2) & \mapsto & x_1 \cdot x_2 . \end{array}$$

Our claim is equivalent to the identity:

$$\widehat{\mathrm{div}}\, f \cdot \widehat{D} = 0 \tag{5.9}$$

for any Arakelov divisor $\widehat{D} = (D, g)$ on $\mathcal{X}$, and any non zero rational function f on $\mathcal{X}$ such that $|D|$ and $|\mathrm{div}\, f|$ do not meet on $\mathcal{X}_K$. This is well known when $\mathcal{X}$ is regular and g is a Green function with C^∞ regularity. In the present setting, we argue as follows. As a special case of (5.6), we get:

$$\int_{\mathcal{X}(\mathbb{C})} g * \log |f_{\mathbb{C}}|^{-2} = \int_{\mathcal{X}(\mathbb{C})} \log |f_{\mathbb{C}}|^{-2} \, \delta_{D_{\mathbb{C}}} ,$$

and (5.9) boils down to:

$$\widehat{\deg}\,(\mathrm{div}\, f \cdot D) - \int_{\mathcal{X}(\mathbb{C})} \log |f_{\mathbb{C}}| \, \delta_{D_{\mathbb{C}}} = 0 .$$

To prove this identity, we may assume that D is an integral one-dimensional subscheme of $\mathcal{X}$; when D is vertical (resp. horizontal) it follows from the vanishing of the degree of the divisor of a rational function (resp. from the product formula) on the normalization of D.

5.4. Heights and Arakelov intersection pairing

Let $\widehat{D}_1 = (D_1, g_1)$ and $\widehat{D}_2 = (D_2, g_2)$ be two Arakelov divisors in $\widehat{Z}^1(\mathcal{X})$. As explained in **3.2.2**, if g_1 has C^∞-regularity, the height $h_{\widehat{D}_1}(D_2)$ is well-defined. If moreover $|D_1|_K$ and $|D_2|_K$ do not meet, it follows from the definitions that

$$h_{\widehat{D}_1}(D_2) = \widehat{\deg}\, D_1 \cdot D_2 + \frac{1}{2}\int_{\mathcal{X}(\mathbb{C})} g_1 \cdot \delta_{D_2}\,. \tag{5.10}$$

Therefore, if we use the definitions (5.8) of $\widehat{D}_1 \cdot \widehat{D}_2$ and (5.1) of the star-product $g_2 * g_1$, we get, when g_1 and g_2 have C^∞ regularity:

$$\widehat{D}_1 \cdot \widehat{D}_2 = h_{\widehat{D}_1}(D_2) + \frac{1}{2}\int_{\mathcal{X}(\mathbb{C})} \omega_1\, g_2 \tag{5.11}$$

where $\omega_1 := \mathrm{dd}^c g_1 + \delta_{D_1}$. As the various terms in (5.11) are unchanged if we add an Arakelov divisor of the form $\widehat{\mathrm{div}}\, f$, $f \in K(\mathcal{X})^*$, to $\widehat{D}_1$, it remains true even when $|D_1|_K$ and $|D_2|_K$ are not disjoint.

Extending (5.10) and (5.11) to more general Green functions requires some regularity assumption on g_1 near $|D_2|(\mathbb{C})$. For instance, using Lemma 5.2, we get by the same reasoning:

Proposition 5.3. *Let $\widehat{D}_1 = (D_1, g_1)$ and $\widehat{D}_2 = (D_1, g_2)$ be two Arakelov divisors in $\widehat{Z}^1(\mathcal{X})$, such that $\omega_1 := \mathrm{dd}^c g_1 + \delta_{D_1}$ is locally L^∞ on a neighbourhood of $|D_2|(\mathbb{C})$.*

For any small enough open neighbourhood U of $|D_2|(\mathbb{C})$, and for any meromorphic function f on U such that div $f = D_{|U}$, *there exists a continuous function φ on U such that:*

$$g_1 = \varphi + \log|f|^{-2} \qquad \text{on } U\,.$$

Therefore, over U, there is a continuous metric $\|\cdot\|_{g_1}$ on the line bundle $\mathcal{O}(D_{\mathbb{C}})$ such that

$$\log \|\mathbf{1}\|_{g_1}^{-2} = g_1\,,$$

and $h_{\widehat{D}_1}(D_2)$ is well defined.

Moreover, we still have:

$$\widehat{D}_1 \cdot \widehat{D}_2 = h_{\widehat{D}_1}(D_2) + \frac{1}{2}\int_{\mathcal{X}(\mathbb{C})} \omega_1\, g_2\,. \tag{5.11}$$

Corollary 5.4. *Let D_1 and D_2 be two Weil divisors on $\mathcal{X}$, and Ω_1 (resp. Ω_2) be an open neighbourhood of $|D_1|(\mathbb{C})$ (resp. of $|D_2|(\mathbb{C})$) in $\mathcal{X}(\mathbb{C})$. Assume that $\Omega_2 \subset \Omega_1$ and that, for any $\sigma : K \hookrightarrow \mathbb{C}$, $\mathcal{X}_\sigma(\mathbb{C})\backslash\Omega_1$ is not polar. Then*

$$\widehat{D}_{1,\Omega_1} \cdot \widehat{D}_{2,\Omega_2} = h_{\widehat{D}_{1,\Omega_1}}(D_2)\,.$$

As a special case, we get the expression (4.4) for $\widehat{D}_\Omega \cdot \widehat{D}_\Omega$.

Proof. We need to prove that

$$\int_{\mathcal{X}(\mathbb{C})} \omega_1 \, g_{D_2,\Omega_2} = 0 \tag{5.12}$$

where $\omega_1 := \mathrm{dd}^c g_{D_2,\Omega_2} + \delta_{D_2}$. We already know that ω_1 is supported by $\mathcal{X}(\mathbb{C})\backslash\Omega_1$; therefore for any $\varphi \in \mathrm{C}_c^\infty(\Omega_1)$,

$$\int_{\mathcal{X}(\mathbb{C})} \omega_1 \, \varphi = 0\,. \tag{5.13}$$

Moreover $\omega_1 \in \mathrm{L}^2_{-1}(\mathcal{X}(\mathbb{C}))$; therefore, by continuity, (5.13) still holds for any φ in $\mathrm{L}^2_1(\Omega_1)_0$. This entails (5.12), since for any $\rho \in \mathrm{C}_c^\infty(\Omega_1)$ which takes the value 1 on some neighbourhood of $|D_2|$, we have

$$(1-\rho)\,\omega_1 = \omega_1$$

hence

$$\int_{\mathcal{X}(\mathbb{C})} \omega_1 \, g_{D_2,\Omega_2} = \int_{\mathcal{X}(\mathbb{C})} \omega_1 \, (1-\rho) \, g_{D_2,\Omega_2}\,,$$

and

$$(1-\rho)\, g_{D_2,\Omega_2} \in \mathrm{L}^2_1(\Omega_2)_0 \subset \mathrm{L}^2_1(\Omega_1)_0\,.$$

q.e.d.

5.5. The Hodge index theorem

To formulate the Hodge index theorem on the arithmetic surface $\mathcal{X}$, it is convenient to follow the presentation of [G-S2] and to introduce some variant "with real coefficients" of the Arakelov-Chow group $\widehat{\mathrm{CH}}^1(\mathcal{X})$.

Let $\widehat{Z}^1_{\mathbb{R}}(\mathcal{X})$ be the real vector spaces formed by the pairs (Z,g), where Z belongs to $Z^1(\mathcal{X})_{\mathbb{R}}$ (*i.e.*, Z is a divisor on $\mathcal{X}$ with real coefficients) and where $g \in \mathrm{C}^{-\infty}(\mathcal{X}_{\mathbb{R}})$ is a L^2_1-Green function for $Z_{\mathbb{C}}$, and let[12] $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X})$ be the quotient of $\widehat{Z}^1_{\mathbb{R}}(\mathcal{X})$ by its vector subspace over $\mathbb{R}$ generated by the arithmetic cycles $\widehat{\mathrm{div}}(f)$, where $f \in K(\mathcal{X})^*$. The inclusion

$$\widehat{Z}^1(\mathcal{X}) \hookrightarrow \widehat{Z}^1_{\mathbb{R}}(\mathcal{X})$$

defines a natural map

$$\rho : \widehat{\mathrm{CH}}^1(\mathcal{X}) \to \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X})\,.$$

The following statement follows from the Hodge index theorem for arithmetic surfaces of Faltings-Hriljac and from its proof (*cf.* [Fa], § 5 and [Hr]; see also [MB2], 6.16, [G-S2], and [Zh] Theorem 7.1; these references assume $\mathcal{X}$ stable or regular, and work with the arithmetic Chow groups defined by means of Green functions which are either admissible, or have C^∞ regularity; however the proofs given there still work, with simple modifications, in the present setting).

[12] We prefer to avoid the notation $\widehat{\mathrm{CH}}^1(\mathcal{X})_{\mathbb{R}}$ introduced in [G-S2], since $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X})$ cannot be identified with $\widehat{\mathrm{CH}}^1(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{R}$ (indeed, $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R} \not\simeq \mathbb{R}$).

Theorem 5.5. 1) *The image of* ρ *spans* $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X})$. *For any class* α *in* $\widehat{\mathrm{CH}}^1(\mathcal{X})$ *the following three conditions are equivalent:*

(i) α *belongs to the kernel of* ρ*;*

(ii) *there exists* $n \in \mathbb{N}^*$ *and* $\beta \in \widehat{\mathrm{CH}}^1(\mathrm{Spec}\, \mathcal{O}_K)$ *such that*

$$\widehat{\deg}\, \beta = 0 \quad \textit{and} \quad n\, \alpha = \pi^* \beta\, ;$$

(iii) *there exists* $n \in \mathbb{N}^*$ *and a family* $(r_\sigma)_{\sigma : K \hookrightarrow \mathbb{C}}$ *of real numbers such that*

$$r_\sigma = r_{\overline{\sigma}} \quad , \quad \sum_{\sigma : K \hookrightarrow \mathbb{C}} r_\sigma = 0\, , \tag{5.8}$$

and

$$n\, \alpha = [(0, g)] \quad \textit{where} \quad g := r_\sigma \quad \textit{on} \quad \mathcal{X}_\sigma(\mathbb{C})\, .$$

2) *The Arakelov pairing factorizes through* ρ *and defines a* $\mathbb{R}$*-bilinear pairing*

$$\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}) \times \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X}) \to \mathbb{R}$$

$$(\alpha, \beta) \mapsto \alpha \cdot \beta$$

which is non-degenerate and has signature $(+, -, -, \ldots)$.

6. Proof of the main theorems

6.1. Non-negative and nef classes

6.1.1. The following definitions will play a crucial role in the proof of our Lefschetz theorems.

Definitions 6.1. i) *An Arakelov divisor* $\widehat{Z} = (Z, g)$ *in* $\widehat{Z}^1_{\mathbb{R}}(\mathcal{X})$ *is called* non-negative *(notation:* $\widehat{Z} \geq 0$*) if the cycle* Z *is effective or zero, and the current* g *is non-negative. A class in* $\widehat{\mathrm{CH}}^1(\mathcal{X})$ *(resp.* $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X})$*) is called* non-negative *if it is the class of a non-negative element of* $\widehat{Z}^1(\mathcal{X})$ *(resp.* $\widehat{Z}^1_{\mathbb{R}}(\mathcal{X})$*).*

ii) *A class* α *in* $\widehat{\mathrm{CH}}^1(\mathcal{X})$ *or* $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X})$, *or an Arakelov divisor* α *in* $\widehat{Z}^1(\mathcal{X})$ *or* $\widehat{Z}^1_{\mathbb{R}}(\mathcal{X})$, *is called* numerically effective *or* nef *if*

$$\widehat{Z} \cdot \alpha \geq 0$$

for any non-negative $\widehat{Z} \in \widehat{Z}^1(\mathcal{X})$, *or equivalently, for any non-negative* $\widehat{Z} \in \widehat{Z}^1_{\mathbb{R}}(\mathcal{X})$.

We shall see that for the Arakelov divisors $\widehat{D}_\Omega$ considered in **3.2**, this notion of numerical effectivity is equivalent with the one in Definition 4.1 (*cf.* Proposition 6.10, *infra*). Observe also that a definition of effective arithmetic cycles, similar to i), has been considered, in a slightly different context, in [BGS1], 6.1.

6.1.2. To analyze properties of nef Arakelov divisors, we need a few preliminary facts concerning non-negative L^2_1-Green functions and their regularization.

Let us choose a function $\rho \in \mathrm{C}^\infty_c(\mathbb{C})$ such that the following conditions hold:

- $\rho \geq 0$;
- for any $z \in \mathbb{C}$, $\rho(z) = \rho(|z|)$;
- $\int_{\mathbb{R}^2} \rho(x + iy)\, \mathrm{d}x\, \mathrm{d}y = 1$.

For any $\varepsilon > 0$, let

$$\rho_\varepsilon := \varepsilon^{-2}\,\rho\,(\varepsilon^{-1}\cdot)\,,$$

and, for any current T on $\mathbb{C}$, let

$$T_\varepsilon := T * \rho_\varepsilon\,.$$

Finally, let E be the superharmonic[13] function on $\mathbb{C}$ defined by

$$E(z) = \log|z|^{-2}\,.$$

Lemma 6.2. *Let T be a generalized function (*i.e. *a current of degree* 0*) on* $\mathbb{C}$.

1) *If T belongs to $\mathrm{L}^2_1(\mathbb{C})_{\mathrm{loc}}$, then, when ε goes to zero in $\mathbb{R}^*_+$, T^ε converges to T in the Fréchet space $\mathrm{L}^2_1(\mathbb{C})_{\mathrm{loc}}$.*

2) *If $T \geq 0$, then for any $\varepsilon \in \mathbb{R}^*_+$, $T^\varepsilon \geq 0$ and $E + (T-E)^\varepsilon \geq 0$.*

3) *If $\mathrm{dd}^c T + \delta_0 \geq 0$, then, for any $\varepsilon \in \mathbb{R}^*_+$,*

$$\mathrm{dd}^c\left[E + (T-E)^\varepsilon\right] + \delta_0 = (\mathrm{dd}^c T + \delta_0)^\varepsilon \geq 0\,.$$

Proof. The only assertion which is possibly not well known is the fact that $E+(T-E)^\varepsilon \geq 0$ if $T \geq 0$. This follows from the equality

$$E + (T-E)^\varepsilon = E - E^\varepsilon + T^\varepsilon\,,$$

and from the fact that $T^\varepsilon \geq 0$ if $T \geq 0$, and from the superharmonicity of E, which implies that $E - E^\varepsilon \geq 0$.

q.e.d.

Lemma 6.2 shows how to approximate locally L^2_1-functions on $\mathbb{C}$ (resp. $\mathrm{L}^2_{1\,\mathrm{loc}}$-Green functions for the origin on $\mathbb{C}$) by C^∞ functions (resp. by Green functions with C^∞ regularity) in a way compatible with positivity. By using partitions of unity, and working on coordinate charts, we deduce:

Proposition 6.3. *Let M be a compact Riemann surface and D an effective divisor on M. For any non-negative L^2_1-Green function g for D on M, there exists a sequence $(g_n)_{n\in\mathbb{N}}$ of non-negative Green functions for D with C^∞ regularity such that*

$$\|g_n - g\|_{\mathrm{L}^2_1(M)} \to 0\,.$$

Moreover, if K is any compact subset of M contained in a (finite) disjoint union of holomorphic coordinate charts on M, each containing at most one point of $|D|$, and if $\omega := \mathrm{dd}^c g + \delta_D$ is ≥ 0 on some neighbourhood of K, the Green currents g_n may be chosen in such a way that, for any $n \in \mathbb{N}$, $\omega_n := \mathrm{dd}^c g_n + \delta_D$ is ≥ 0 on some neighbourhood of K.

Corollary 6.4. *Let M be a compact Riemann surface and D and D' two effective divisors with disjoint supports on M. For any two L^2_1-Green functions for D and D' respectively, such that*

$$g \geq 0 \quad , \quad g' \geq 0$$

[13] *i.e.* the opposite of a subharmonic function.

and

$$\omega' := \mathrm{dd}^c g' + \delta_D \geq 0\,,$$

we have:

$$\int_M g * g' \geq 0\,. \tag{6.1}$$

Proof. When g and g' are Green functions with C^∞ regularity this follows immediately from the definition

$$g * g' := g\,\omega' + g'\,\delta_D$$

of the star product. In general, we argue as follows.

First observe that the equality, for any $\varphi \in \mathrm{L}^2_1(M)$,

$$\int_M (g+\varphi) * g' - \int_M g * g' = \int_M \varphi\,\omega'$$

shows that $\int_M g * g'$ is an increasing function of g. Therefore, to prove (6.1), we may "truncate" g by multiplying it by some function χ in $\mathrm{C}^\infty_c(M)$ with values in $[0,1]$, supported on some arbitrarily small neighbourhood of $|D|$, and taking the value 1 on some smaller neighbourhood of $|D|$. In particular, we may assume that g is supported by a disjoint union of holomorphic coordinate charts on M, each containing at most one point of $|D|$. Then, according to Proposition 6.3, we can find a sequence $(g'_n)_{n\in\mathbb{N}}$ of Green functions for D', with C^∞ regularity, such that

$$\|g'_n - g'\|_{\mathrm{L}^2_1(M)} \to 0\,, \tag{6.2}$$

$$g'_n \geq 0\,,$$

and

$$\omega'_n := \mathrm{dd}^c g'_n + \delta_{D'} \geq 0 \text{ on a neighbourhood of } \operatorname{supp} g\,.$$

Then $\int_M g * g'_n$ coincides with

$$\int_M (g\,\omega'_n + g'_n\,\delta_D)$$

(*cf.* Lemma 5.2) and is therefore non-negative. Since (6.2) implies that it converges to $\int_M g * g'$ when n goes to infinity, this last integral also is non-negative.

q.e.d.

Remark 6.5. Under the assumption of Corollary 5.5, if moreover g and g' have C^∞ regularity, then

$$\int_M g * g' = \int_M (g\,\omega' + g'\,\delta_D)$$

vanishes iff

(6.3) *g vanishes almost everywhere with respect to the measure class defined by the positive current ω'*

(6.4) $g'_{| |D|} = 0.$

When g and g' are general L^2_1-Green functions satisfying the hypotheses of Corollary 5.5, the vanishing of $\int_M g * g'$ still implies that (6.3) holds. Indeed, for any χ as in the preceding proof, we have

$$0 \leq \int_M (\chi g) * g' \leq \int g * g'$$

(the first inequality follows from Corollary 6.4 applied to χg and g'; the second one from its proof). Therefore, the vanishing of $\int_M g * g'$ implies that for any such χ,

$$\int_M (1-\chi)\, g\, \omega' = \int_M g * g' - \int_M (\chi g) * g' \\ = 0\,.$$

Therefore g vanishes ω'-almost everywhere on $M \backslash |D|$, hence on M (indeed the positive measure defined by ω' cannot be atomic because it belongs to $L^2_{-1}(M)$; *cf.* Appendix, **A.3**, (A.3.1)).

6.1.3. We shall now apply the analytic results of the preceding subsection to establish the equivalence of various notions of numerical effectivity for Arakelov divisors.

Lemma 6.6. *A class α in $\widehat{\mathrm{CH}}^1(\mathcal{X})$, or an Arakelov divisor α in $\widehat{Z}^1(\mathcal{X})$, is nef iff*

$$\widehat{Z} \cdot \alpha \geq 0$$

for any non-negative $\widehat{Z} = (Z, g) \in \widehat{Z}^1(\mathcal{X})$, where g is a Green function with C^∞ regularity.

Proof. This condition is clearly weaker than the one in the definition 6.1, ii) of "nef". Conversely, assume that α satisfies this condition. To see that indeed this implies that α is nef, observe that for any non-negative $\widehat{Z} = (Z, g) \in \widehat{Z}^1(\mathcal{X})$ the first part of Proposition 6.3 implies the existence of a sequence $(g_n)_{n \in \mathbb{N}}$ of non-negative real valued Green functions with C^∞ regularity for $Z_{\mathbb{C}}$, invariant under complex conjugation, such that $g_n - g$ goes to zero in $L^2_1(\mathcal{X}(\mathbb{C}))$. Then, by hypothesis, $(Z, g_n) \cdot \alpha \geq 0$ for any $n \in \mathbb{N}$. Since

$$(Z, g_n) \cdot \alpha - (Z, g) \cdot \alpha = \frac{1}{2} \int_{\mathcal{X}(\mathbb{C})} (g_n - g)\, \omega(\alpha)$$

goes to zero when n goes to infinity, this implies that $(Z \cdot g) \cdot \alpha \geq 0$.

q.e.d.

Lemma 6.7. *Let $\widehat{D} = (D, g)$ and $\widehat{D}' = (D', g')$ be two non-negative Arakelov divisors in $\widehat{Z}^1(\mathcal{X})$. If D and D' meet properly, and*

$$\omega' := dd^c g' + \delta_{D'} \geq 0\,,$$

then

$$\widehat{D} \cdot \widehat{D}' \geq 0\,.$$

Proof. As D and D' meet properly, the intersection cycle $D \cdot D'$ is a well defined 0-cycle on $\mathcal{X}$. Moreover, as D and D' are effective, $D \cdot D'$ is effective (this is clear if D or D' is Cartier; one reduces to this case, either by considering multiples of D and D', or by using a resolution of $\mathcal{X}$ and by reasoning as in **2.4**, **v**)). Therefore

$$\widehat{\deg}\, D \cdot D' \geq 0\,.$$

On the other hand, Corollary 6.4 shows that

$$\int_{\mathcal{X}(\mathbb{C})} g * g' \geq 0\,.$$

Finally,

$$\widehat{D} \cdot \widehat{D}' := \widehat{\deg}\, D \cdot D' + \frac{1}{2} \int_{\mathcal{X}(\mathbb{C})} g * g' \geq 0\,.$$

q.e.d.

Remark 6.8. Under the assumptions of Lemma 6.7, if $\widehat{D} \cdot \widehat{D}' = 0$, then D and D' necessarily have disjoint supports and condition (6.3) holds. This follows from the proof above and Remark 6.5.

The next Proposition shows that for the Arakelov divisors $\widehat{D}_\Omega = (D, g_{D,\Omega})$ associated to equilibrium potentials introduced in **3.2**, the general definition of numerical effectivity coincides with the one introduced in **4.1**.

Proposition 6.9. *Let $\widehat{D} = (D, g)$ be a non-negative element of $\widehat{Z}_1(\mathcal{X})$ such that*

$$\omega(\widehat{D}) := \mathrm{dd}^c g + \delta_D$$

is locally L^∞ near $|D|(\mathbb{C})$. Then $\widehat{D}$ is nef iff

$$\omega(\widehat{D}) \geq 0\,,$$

and for any irreducible component E of D,

$$h_{\widehat{D}}(E) \geq 0\,.$$

Proof. 1) Let us assume that $\widehat{D}$ is nef.

For any $\varphi \in \mathrm{C}^\infty(\mathcal{X}(\mathbb{C}), \mathbb{R})$, invariant by complex conjugation, the pair $(0, \varphi)$ defines an element of $\widehat{Z}^1(\mathcal{X})$ such that

$$\widehat{D} \cdot (0, \varphi) = \frac{1}{2} \int_{\mathcal{X}(\mathbb{C})} g * \varphi = \frac{1}{2} \int_{\mathcal{X}(\mathbb{C})} \varphi\, \omega(\widehat{D})\,.$$

Moreover, if φ takes its values in $\mathbb{R}_+$, $(0, \varphi)$ is non-negative, and therefore

$$\int_{\mathcal{X}(\mathbb{C})} \varphi\, \omega(\widehat{D}) \geq 0\,.$$

As $\omega(\widehat{D})$ is invariant by complex conjugation, this holds more generally for any non-negative $\varphi \in \mathrm{C}^\infty(\mathcal{X}(\mathbb{C}), \mathbb{R})$, and this shows that $\omega(\widehat{D}) \geq 0$.

Let E be any irreducible component of $|D|$. For any real valued, invariant under complex conjugation, Green function g for $E_{\mathbb{C}}$, we have by Proposition 5.3:

$$\widehat{D}\cdot(E,g) = h_{\widehat{D}}(E) + \frac{1}{2}\int_{\mathcal{X}(\mathbb{C})} g\,\omega\,(\widehat{D})\,. \tag{6.5}$$

Moreover, it is possible to find a sequence g_n of such Green functions for $E_{\mathbb{C}}$ which are non-negative, and such that

$$\lim_{n\to+\infty}\int_{\mathcal{X}(\mathbb{C})} g_n\,\omega\,(\widehat{D}) = 0\,. \tag{6.6}$$

(For instance, define g_n as $\chi_n\,g$, where g is as before and where $\chi_n : \mathcal{X}(\mathbb{C}) \to [0,1]$ is C^∞, invariant under complex conjugation, takes the value 1 on some neighbourhood of $E(\mathbb{C})$, and has a support which shrinks to $E(\mathbb{C})$ when n goes to infinity; (6.6) holds by dominated convergence.) As (E,g_n) is non-negative, we get from (6.5):

$$h_{\widehat{D}}(E) + \frac{1}{2}\int_{\mathcal{X}(\mathbb{C})} g_n\,\omega\,(\widehat{D}) \geq 0\,,$$

and finally, by letting n go to infinity:

$$h_{\widehat{D}}(E) \geq 0\,.$$

2) Conversely, suppose that $\widehat{D}$ satisfies the two conditions. To prove that it is nef, observe that any non-negative $\widehat{Z}$ in $\widehat{Z}^1(\mathcal{X})$ may be decomposed as:

$$\widehat{Z} = (Z',g') + \sum_{\substack{E \text{ component}\\ \text{of } D}} n_E\,(E,g_E)\,,$$

where (Z',g') is non-negative, Z' meets D properly, $n_E \in \mathbb{N}$ and $g_E \geq 0$ (use a suitable partition of unity to define g' and the g_E's). Then we have:

$$\widehat{D}\cdot\widehat{Z} = \widehat{D}\cdot(Z',g') + \sum_E n_E\cdot\widehat{D}\cdot(E,g_E)\,.$$

By Lemma 6.7, $\widehat{D}\cdot(Z',g')$ is non-negative. Moreover, for every irreducible component E of D, Proposition 5.3 gives:

$$\widehat{D}\cdot(E,g_E) = h_{\widehat{D}}(E) + \frac{1}{2}\int_{\mathcal{X}(\mathbb{C})} g_E\,\omega\,(\widehat{D})\,.$$

This is non-negative by our hypotheses.

q.e.d.

6.2. Hyperplane sections and non-negative Arakelov divisors

Let $\mathcal{L}$ be an ample line bundle on $\mathcal{X}$, and let $\|\cdot\|$ a C^∞-hermitian metric on the holomorphic line bundle $\mathcal{L}_{\mathbb{C}}$ on $\mathcal{X}(\mathbb{C})$, invariant by complex conjugation, such that:

i) the first Chern form[14] $c_1(\mathcal{L}, \|\ \|)$ is > 0 on $\mathcal{X}(\mathbb{C})$;

ii) there exists a finite family $(s_i)_{i\in I}$ of regular, non identically zero, sections of $\mathcal{L}$ on $\mathcal{X}$ such that

$$\bigcap_{i\in I} |\operatorname{div} s_i| = \emptyset$$

and

$$\forall x \in \mathcal{X}(\mathbb{C}),\ \|s(x)\| < 1 .$$

(Such pairs $(\mathcal{L}, \|\ \|)$ do indeed exist; for instance if $i : \mathcal{X} \hookrightarrow \mathbb{P}^N_{\mathbb{Z}}$ is a closed embedding, the line bundle $\mathcal{L} = i^*\mathcal{O}(1)$, equipped with the restriction $\|\ \|$ of the canonical quotient metric on the holomorphic line bundle $\mathcal{O}(1)_{\mathbb{C}}$ over $\mathbb{P}^N(\mathbb{C})$, scaled by a factor $\lambda \in]0,1[$, satisfies these conditions.) Let us denote by h the first Chern class (in $\widehat{\mathrm{CH}}^1(\mathcal{X})$) of $(\mathcal{L}, \|\ \|)$; by definition, for any non zero rational section s of $\mathcal{L}$ over $\mathcal{X}$, it is:

$$h = [(\operatorname{div} s, -\log \|s\|^2)] .$$

Such a class h turns out to be nef (compare with [Zh]). More precisely, we have:

Proposition 6.10. *For any $(D,g) \in Z^1_{\mathbb{R}}(\mathcal{X})\backslash\{0\}$ such that $(D,g) \geq 0$, the class $x = [(D,g)]$ in $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X})$ satisfies:*

$$h \cdot x > 0 .$$

In particular $x \neq 0$.

Proof. Any such (D,g) may be written $\sum_{i\in F} \lambda_i (D_i, g_i)$ where F is finite and non-empty, and for any $i \in F$, $\lambda_i \in \mathbb{R}^*_+$, $(D_i, g_i) \geq 0$, $(D_i, g_i) \neq 0$, and D_i is either 0 or an integral one dimensional subscheme of $\mathcal{X}$ (use a suitable partition of unity to decompose g). Therefore, we may assume that either $D = 0$, or D is an integral one-dimensional subscheme.

In the first case,

$$h \cdot x = \int_{\mathcal{X}(\mathbb{C})} g \cdot c_1(\mathcal{L}, \|\ \|) > 0$$

since $c_1(\mathcal{L}, \|\ \|) > 0$ and g is not almost everywhere 0.

In the second case, there exist $i \in I$ such that $|\operatorname{div} s_i|$ does not contain D. Then $\operatorname{div} s_i$ and D meet properly, and

$$h \cdot x = (D,g) \cdot (\operatorname{div} s_i,\ -\log \|s_i\|^2)$$

[14] Recall that, for any non-zero rational section s of $\mathcal{L}$, $c_1(\mathcal{L}, \|\ \|) = dd^c \log \|s\|^{-2} + \delta_{\operatorname{div} s}$.

is non-negative by Lemma 6.7. If it would vanish, g would be zero by Remark 6.8, and therefore D would be vertical; so we would have:

$$(h \cdot x) = \deg_{\mathbb{F}_q} \mathcal{L}_{|D} \cdot \log q\,,$$

where $\mathbb{F}_q$ denotes the field of constants of D. However

$$\deg_{\mathbb{F}_q} \mathcal{L}_{|D} \neq 0\,,$$

since $\mathcal{L}$ is ample on D.

q.e.d.

6.3. Proof of the main theorems

In this section, we finally prove Theorems 4.2 and 4.2′.

We shall denote by $\mathcal{X}$ a projective integral normal arithmetic surface, and by

$$\pi : \mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$$

the "Stein factorization" of the map $\mathcal{X} \to \operatorname{Spec} \mathbb{Z}$.

First, an argument similar to the one leading to Proposition 2.2 and 2.4 in the geometric case will lead to:

Lemma 6.11. *For any two elements x and y of $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X})$ such that*

$$x \geq 0 \quad , \quad y \geq 0 \quad \text{and} \quad x + y \text{ is nef,} \tag{6.7}$$

we have

$$x \cdot y \geq 0\,.$$

Moreover the equality

$$x \cdot y = 0$$

holds iff either $x = 0$, or $y = 0$, or there exists $\lambda \in \mathbb{R}^*_+$ such that $y = \lambda x$ and $x \cdot x$, and therefore $x \cdot y$ and $y \cdot y$, vanish.

Proof. The assumptions on x and y imply that

$$x \cdot (x + y) \geq 0 \quad \text{and} \quad y \cdot (x + y) \geq 0\,, \tag{6.8}$$

or equivalently, that

$$x \cdot x \geq -x \cdot y \quad \text{and} \quad y \cdot y \geq -x \cdot y\,. \tag{6.9}$$

If $x \cdot y \leq 0$, we get from (6.9):

$$(x \cdot x)\,(y \cdot y) \geq (x \cdot y)^2\,,$$

that is, that the quadratic form

$$\begin{aligned} \mathbb{R}^2 &\to \mathbb{R} \\ (a,b) &\mapsto (ax+by)\cdot(ax+by) \end{aligned}$$

has a non-negative discriminant. Moreover it is not negative definite, since $(x+y)\cdot(x+y) \geq 0$. The Hodge index theorem (Theorem 5.5, 2)) therefore implies that the subspace $\mathbb{R}\,x + \mathbb{R}y$ of $\widehat{\mathrm{CH}}^1(\mathcal{X})_{\mathbb{R}}$ has dimension at most 1.

This already proves that if x and y satisfy (6.7) and are not colinear in $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X})_{\mathbb{R}}$, then $x \cdot y > 0$.

Suppose now that x and y are colinear and both not zero. Then there exists $\lambda \in \mathbb{R}^*$ such that $y = \lambda\, x$. Let $h \in \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X})$ be as in **6.2**. Then, by Proposition 6.10, we have

$$h \cdot x > 0 \quad \text{and} \quad h \cdot y > 0\,.$$

The equality

$$h \cdot y = \lambda\,(h \cdot x)$$

therefore implies that $\lambda > 0$. Now we get

$$x + y = (1+\lambda)\cdot x\,,$$

hence, according to (6.8):

$$x \cdot y = (1+\lambda)^{-1}\,(x+y)\cdot y \geq 0\,.$$

Finally, if $x \cdot y$ vanishes, then so do $x \cdot x = \lambda^{-1}\, x \cdot y$ and $y \cdot y = \lambda\, x \cdot y$.

q.e.d.

Consider now a non-vertical effective divisor D on $\mathcal{X}$, and Ω satisfying $(4.1)_{\mathcal{X},\Omega}$ as in Theorem 4.2. Let us assume that each connected component of Ω contains at most one point in $|D|\,(\mathbb{C})$, that $\widehat{D}_\Omega := (D, g_{D,\Omega})$ is nef, and that one of the conditions $(4.2)_{\mathcal{X},\Omega}$, or $(4.3)_{\mathcal{X},\Omega}$, or $\widehat{D}_\Omega \cdot \widehat{D}_\Omega > 0$ is satisfied. To complete the proof of Theorems 4.2 and 4.2′, we need to prove that $|D|$ is connected.

By contradiction, let us assume that $|D|$ is not connected. Then we may decompose D as

$$D = D_1 + D_2$$

where $D_1 \geq 0$, $D_2 \geq 0$, and $|D_1| \neq \emptyset$, $|D_2| \neq \emptyset$ and $|D_1| \cap |D_2| = \emptyset$. Then the Arakelov divisors $\widehat{D}_{1,\Omega}$ and $\widehat{D}_{2,\Omega}$ are ≥ 0, and their sum

$$\widehat{D}_\Omega = \widehat{D}_{1,\Omega} + \widehat{D}_{2,\Omega}$$

is nef. Moreover, their classes in $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X})$ are not zero, by Proposition 6.10.

On the other hand, we have:

$$\widehat{D}_{1,\Omega} \cdot \widehat{D}_{2,\Omega} = 0\,.$$

Indeed, since the supports of D_1 and D_2 do not meet, we get:

$$\widehat{D}_{1,\Omega} \cdot \widehat{D}_{2,\Omega} = \frac{1}{2} \int_{\mathcal{X}(\mathbb{C})} g_{D_1,\Omega} * g_{D_2,\Omega}\,. \tag{6.10}$$

Let Ω_1 (resp. Ω_2) the union of the connected components of Ω containing some point in $|D_1|(\mathbb{C})$ (resp. in $|D_2|(\mathbb{C})$). Then we have:

$$g_{D_1,\Omega} = g_{D_1,\Omega_1} \quad \text{and} \quad g_{D_2,\Omega} = g_{D_2,\Omega_2}$$

(*cf.* **3.1.4, i)**). Moreover, since each connected component of Ω contains at most one point in $|D|(\mathbb{C})$, the open subsets Ω_1 and Ω_2 are disjoint. Therefore, the right hand side of (6.10) vanishes according to Lemma 5.1.

We now apply Lemma 6.11 to $x = \widehat{D}_{1,\Omega}$ and $y = \widehat{D}_{2,\Omega}$, and we get that there exists $\lambda \in \mathbb{R}_+^*$ such that the classes $[\widehat{D}_{1,\Omega}]$ and $[\widehat{D}_{2,\Omega}]$ of $\widehat{D}_{1,\Omega}$ and $\widehat{D}_{2,\Omega}$ in $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{X})$ satisfy

$$[\widehat{D}_{2,\Omega}] = \lambda\,[\widehat{D}_{1,\Omega}]\,, \tag{6.11}$$

and that

$$\begin{aligned}\widehat{D}_\Omega \cdot \widehat{D}_\Omega &= (\widehat{D}_{1,\Omega} + \widehat{D}_{2,\Omega}) \cdot (\widehat{D}_{1,\Omega} + \widehat{D}_{2,\Omega})\\ &= 0\,.\end{aligned}$$

So, when $\widehat{D}_\Omega \cdot \widehat{D}_\Omega > 0$, we get a contradiction, which concludes the proof of Theorem 4.2′.

To get a contradiction when $(4.2)_{\mathcal{X},\Omega}$ or more generally $(4.3)_{\mathcal{X},\Omega}$ holds, observe that, according to Theorem 5.5, 1), the equality (6.11) is equivalent to the existence of a family $(r_\sigma)_{\sigma:K\hookrightarrow\mathbb{C}}$ of real numbers and of $f \in K(\mathcal{X})^*$ such that (5.8) is satisfied, and such that

$$n\,(D_2 - \lambda\,D_1) = \operatorname{div} f$$

and

$$n\,(g_{D_2,\Omega} - \lambda\,g_{D_1,\Omega}) = \log|f|^{-2} + r_\sigma \quad \text{on } \mathcal{X}_\sigma(\mathbb{C})\backslash|D|_\sigma(\mathbb{C})\,.$$

(This identity *a priori* holds almost everywhere on $\mathcal{X}_\sigma(\mathbb{C})$. As $n\,g_{D_2,\Omega}$ and $n\,\lambda\,g_{D_1,\Omega} + \log|f|^{-2} + r_\sigma$ are subharmonic on $\mathcal{X}_\sigma(\mathbb{C})\backslash|D|_\sigma(\mathbb{C})$, they coincide there.) In particular, for every embedding $\sigma : K \hookrightarrow \mathbb{C}$, the function $\log|f|^{-2} + r_\sigma$, vanishes nearly everywhere on $\mathcal{X}_\sigma(\mathbb{C})\backslash\Omega$. On the other hand, it is harmonic on $\mathcal{X}_\sigma(\mathbb{C})\backslash|D|_\sigma(\mathbb{C})$. If now σ, P and U are as in $(4.3)_{\mathcal{X},\Omega}$, it follows that $\log|f|^{-2} + r_\sigma$ vanishes on the open neighbourhood U of P_σ in $\mathcal{X}_\sigma(\mathbb{C})$, hence on the whole of $\mathcal{X}_\sigma(\mathbb{C})$ by analytic continuation. Therefore

$$n\,D_{2,\sigma} - n\,\lambda\,D_{1,\sigma} = 0$$

and the divisor with real coefficients $n\,D_2 - n\,\lambda\,D_1$ on $\mathcal{X}$ is vertical. As D_1 and D_2 have disjoint supports, this implies that D_1 and D_2 are vertical. This contradicts the assumption that $D = D_1 + D_2$ is not vertical, and concludes the proof of Theorem 4.2′.

7. Examples

7.1. A theorem of Ihara and some generalizations

7.1.1. As explained in the introduction, the present paper was motivated by Ihara's paper [I], where, using a slightly different formulation, he proves the following statement.

Theorem 7.1. *Let K be a number field, $\mathcal{O}_K$ its ring of integers, $\mathcal{X}$ an integral normal projective arithmetic surface over* Spec $\mathcal{O}_K$*, and*

$$f : \mathcal{X} \to \mathbb{P}^1_{\mathcal{O}_K}$$

a finite morphism. For every embedding $\sigma : K \to \mathbb{C}$, let $r_\sigma \in \mathbb{R}^*_+$ be such that the finite cover

$$f_\sigma : \mathcal{X}_\sigma(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$$

is unramified over the punctured disk

$$\dot{\mathrm{D}}(0, r_\sigma) := \{z \in \mathbb{C} \mid 0 < |z| < r_\sigma\}.$$

If $\prod_{\sigma:K\hookrightarrow\mathbb{C}} r_\sigma \geq 1$, then the inverse image $D = f^{-1}(D_0)$ of the section 0 of $\mathbb{P}^1_{\mathcal{O}_K} \to$ Spec $\mathcal{O}_K$ *is connected.*

By replacing r_σ by $\max(r_\sigma, r_{\overline{\sigma}})$, we reduce to the case where moreover, $r_\sigma = r_{\overline{\sigma}}$. Then it is a special case of Theorem 4.2, applied to $\mathcal{X}$, D, and to $\Omega = f_{\mathbb{C}}^{-1}(\Omega_0)$, where

$$\Omega_0 = \coprod_{\sigma:K\hookrightarrow\mathbb{C}} \{z \in \mathbb{C} \mid |z| < r_\sigma\} \subset \coprod_{\sigma:K\hookrightarrow\mathbb{C}} \mathbb{P}^1(\mathbb{C}) \simeq \mathbb{P}_{\mathcal{O}_K}(\mathbb{C}).$$

Indeed, if D_0 is the divisor on $\mathbb{P}^1_{\mathcal{O}_K}$ defined by the 0-section, we have by (3.14):

$$h_{\widehat{D}_{0_{\Omega_0}}}(D_0) = \sum_{\sigma:K\hookrightarrow\mathbb{C}} \log r_\sigma > 0;$$

therefore $\mathcal{X}$, D, and Ω satisfy the hypotheses of Theorem 4.2, by Lemma 4.3.

From Theorem 7.1, Ihara deduces statements concerning fundamental groups, by the argument used to prove Theorem 4.4. Here is a variant of his results, which may be obtained either as a consequence of Theorem 7.1, or as a special case of Theorem 4.4 (since the previous proof of Theorem 7.1 actually shows that the hypotheses of Theorem 4.4 are satisfied by $\mathcal{X}$, D, and Ω):

Corollary 7.2. *Let us keep the notations of Theorem 7.1. If $\prod_{\sigma:K\hookrightarrow\mathbb{C}} r_\sigma \geq 1$, then for any geometric point η of $|D|$ and any closed subset F of $\mathcal{X}$ such that*

$$F \cap |D| = \emptyset$$

and, for any embedding $\sigma : K \hookrightarrow \mathbb{C}$,

$$F_\sigma(\mathbb{C}) \cap f_\sigma^{-1}(\{z \in \mathbb{C} \mid |z| < r_\sigma\}) = \emptyset,$$

the inclusion $i : D \to \mathcal{X}\backslash F$ induces a surjection

$$i_* : \pi_1(|D|, \eta) \to \pi_1(\mathcal{X}\backslash F, i(\eta)).$$

7.1.2. A natural variation on Ihara's theorem, based on our general results in section 4, is as follows.

Consider the following data:

• *a number field K and $\pi : \mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$ an integral normal quasi-projective arithmetic surface over* $\operatorname{Spec} \mathcal{O}_K$*;*
• *a non-zero rational function R on $\mathcal{X}$;*
• *a section $P \in \mathcal{X}(\mathcal{O}_K)$ of π, such that R vanishes at its generic point P_K, at some order $n \in \mathbb{N}^*$;*
• *for any embedding $\sigma : K \hookrightarrow \mathbb{C}$, a connected open neighbourhood Ω_σ of P_σ in $\mathcal{X}_\sigma(\mathbb{C})$ such that:*

– *the open neighbourhood $R_\sigma(\Omega_\sigma)$ of 0 in $\mathbb{P}^1(\mathbb{C})$ is biholomorphic to an open disk,*
– $R_\sigma^{-1}(0) \cap \Omega_\sigma = P_\sigma$,
– *the map $R_\sigma : \Omega_s \backslash \{P_\sigma\} \to R_\sigma(\Omega_\sigma)\backslash\{0\}$ is a finite unramified covering (necessarily of degree n),*
– *the open subset $\coprod_{\sigma:K\hookrightarrow\mathbb{C}} \Omega_\sigma$ of $\mathcal{X}(\mathbb{C})$ is invariant under complex conjugation (equivalently, for every $\sigma : K \hookrightarrow \mathbb{C}$, $\Omega_{\overline{\sigma}} = \overline{\Omega}_\sigma$).*

Then the divisor of R may be written

$$\operatorname{div} R = n\,P + D,$$

where D meets P properly. Moreover, as D is $\mathbb{Q}$-Cartier, $P^* D$ is a well defined $\mathbb{Q}$-divisor on $\operatorname{Spec} \mathcal{O}_K$:

$$P^* D = \sum_{\wp} n_\wp\, \wp\,,$$

(the sum runs over the non-zero primes of $\mathcal{O}_K$, the $n_\wp$'s belong to $\mathbb{Q}$, and almost all of them vanish).

In this situation, the numerical effectivity of $\widehat{P}_\Omega$ may be checked by means of the following lemma:

Lemma 7.3. *Using the notation above, we have:*

$$n\, h_{\widehat{P}_\Omega}(P) = \sum_{\sigma:K\hookrightarrow\mathbb{C}} \log \rho(R_\sigma(\Omega_\sigma)) - \sum_{\wp} n_\wp \log N_\wp\,. \tag{7.1}$$

Recall that the invariant $\rho(R_\sigma(\Omega_\sigma))$ of the "holomorphic disk" $R_\sigma(\Omega_\sigma)$ has been introduced in **3.1.5**, Example 3.4.

Proof. For any embedding $\sigma : K \hookrightarrow \mathbb{C}$, let

$$\varphi_\sigma : \overset{\circ}{D}(0;1) \to R_\sigma(\Omega_\sigma)$$

be a biholomorphic map such that $\varphi_\sigma(0) = 0$. It follows from the hypotheses on Ω_σ and R_σ that there exists a biholomorphic map

$$\widetilde{\varphi}_\sigma : \overset{\circ}{D}\ (0;1) \to \Omega_\sigma$$

such that $\widetilde{\varphi}_\sigma(0) = P_\sigma$ and the following diagram commutes:

$$\begin{array}{cccc} z & \overset{\circ}{D}(0;1) & \xrightarrow{\widetilde{\varphi}_\sigma} & \Omega_\sigma \\ \updownarrow & \downarrow & & \downarrow R_\sigma \\ z^n & \overset{\circ}{D}(0;1) & \xrightarrow{\varphi_\sigma} & R_\sigma(\Omega_\sigma). \end{array}$$

It follows that on $\Omega_\sigma \backslash \{P_\sigma\}$, we have

$$\begin{aligned} g_{P_\sigma,\Omega_\sigma} &= -\log|\widetilde{\varphi}_\sigma^{-1}|^2 \\ &= -\frac{1}{n}\log|\varphi_\sigma^{-1}\circ R_\sigma|^2, \end{aligned}$$

and the function

$$\psi_\sigma := n\, g_{P_\sigma,\Omega_\sigma} + \log|R_\sigma|^2$$

in $L^2_{1,\mathrm{loc}}(\mathcal{X}_\sigma(\mathbb{C})\backslash|\operatorname{div} R_\sigma|)$ extends to a C^∞ function on Ω_σ, such that

$$\begin{aligned} \psi_\sigma(P_\sigma) &= -2\log|(\varphi_\sigma^{-1})'(0)| \\ &= 2\log\rho(R_\sigma(\Omega_\sigma)). \end{aligned} \tag{7.2}$$

On the other hand,

$$\begin{aligned} n\,h_{\widehat{P_\Omega}}(P) &= n\,h_{\widehat{P_\Omega}}(P) - h_{\widehat{\operatorname{div} R}}(P) \\ &= h_{(nP-\operatorname{div} R,(\psi_\sigma)_{\sigma:K\hookrightarrow\mathbb{C}})}(P) \\ &= -\sum_{\wp} n_\wp \log N_\wp + \frac{1}{2}\sum_{\sigma:K\hookrightarrow\mathbb{C}}\psi_\sigma(P_\sigma); \end{aligned} \tag{7.3}$$

indeed P meets $nP - \operatorname{div} R$ properly and

$$P^*(nP - \operatorname{div} R) = -\sum_\wp n_\wp\,\wp.$$

Formula (7.1) follows from (7.2) and (7.3).

q.e.d.

Taking Lemma 7.3 into account, we now get from Theorems 4.4 and 4.4′:

Theorem 7.4. *With the same notation as above, if*

$$\sum_{\sigma:K\hookrightarrow\mathbb{C}} \log \rho\,(R_\sigma\,(\Omega_\sigma)) > \sum_{\wp} n_\wp \log N_\wp\,,$$

or if

$$\sum_{\sigma:K\hookrightarrow\mathbb{C}} \log \rho\,(R_\sigma\,(\Omega_\sigma)) = \sum_{\wp} n_\wp \log N_\wp$$

and $\coprod_{\sigma:K\hookrightarrow\mathbb{C}} \mathcal{X}_\sigma\,(\mathbb{C})\backslash\Omega_\sigma$ *has a nonempty interior, then, for any geometric point* η *of* Spec $\mathcal{O}_K$, *we have isomorphisms*

$$\pi_1\,(\mathrm{Spec}\;\mathcal{O}_K,\eta) \underset{P_*}{\overset{\pi_*}{\leftrightarrows}} \pi_1\,(\mathcal{X},P(\eta))\,.$$

7.1.3. When applying Theorem 7.4 to explicit examples, the following observations are useful.

Let $\nu:\widetilde{\mathcal{X}}\to\mathcal{X}$ be a birational projective map of schemes, with $\widetilde{\mathcal{X}}$ integral and normal (hence a quasi-projective arithmetic surface over Spec $\mathcal{O}_K$) and let $\widetilde{P}\in\widetilde{\mathcal{X}}\,(\mathcal{O}_K)$ be the section of $\widetilde{\mathcal{X}}\to$ Spec $\mathcal{O}_K$ mapped onto P by ν. The rational function R on $\mathcal{X}$ may be seen as a rational function on $\widetilde{\mathcal{X}}$, which we shall denote $\widetilde{R}$; moreover for any $\sigma:K\hookrightarrow\mathbb{C}$, $\mathcal{X}_\sigma$ (resp. P_σ, resp. Ω_σ) may be identified with $\widetilde{\mathcal{X}}_\sigma$ (resp. with $\widetilde{P}_\sigma$, resp. with an open neighbourhood $\widetilde{\Omega}_\sigma$ of $\widetilde{P}_\sigma$).

Clearly, $\widetilde{R}_K$ vanishes at order n at $\widetilde{P}_K$, and we may write

$$\mathrm{div}\;\widetilde{R} = n\,\widetilde{P}+\widetilde{D},$$

where $\widetilde{D}$ meets $\widetilde{P}$-properly. On the other hand, if V is the effective vertical $\mathbb{Q}$-divisor on $\widetilde{\mathcal{X}}$ such that

$$\nu^*\,P=\widetilde{P}+V\,,$$

we have:

$$\mathrm{div}\;\widetilde{R}=\nu^*\,(\mathrm{div}\;R)=n\,\nu^*\,P+\nu^*\,D=n\,\widetilde{P}+n\,V+\nu^*\,D.$$

Therefore

$$\widetilde{D}=n\,V+\nu^*\,D,$$

and

$$\begin{aligned}\widetilde{P}^*\,\widetilde{D}&=n\,\widetilde{P}^*\,V+\widetilde{P}^*\,\nu^*\,D\\&=n\,\widetilde{P}^*\,V+P^*\,D.\end{aligned}$$

Finally, if we let

$$\widetilde{P}^*\,\widetilde{D}=\sum_{\wp}\widetilde{n}_\wp\,\wp\,,$$

we get that *for any* $\wp$,

$$\widetilde{n}_{\wp} \geq n_{\wp}\,; \tag{7.4}$$

moreover equality holds in (7.4) if ν *is an isomorphism on a neighbourhood of* $\widetilde{P}$.

Since

$$\begin{aligned} h_{\widehat{\widetilde{P}}_{\widetilde{\Omega}}}(\widetilde{P}) &:= \sum_{\sigma:K\hookrightarrow\mathbb{C}} \log \rho(\widetilde{R}_{\sigma}(\widetilde{\Omega}_{\sigma})) - \sum_{\wp} \widetilde{n}_{\wp} \log N_{\wp} \\ &= \sum_{\sigma:K\hookrightarrow\mathbb{C}} \log \rho(R_{\sigma}(\Omega_{\sigma})) - \sum_{\wp} \widetilde{n}_{\wp} \log N_{\wp} \\ &= h_{\widehat{P}_{\Omega}}(P) - \sum_{\wp} (\widetilde{n}_{\wp} - n_{\wp}) \log N_{\wp}\,, \end{aligned}$$

this shows that *if* $\widetilde{\mathcal{X}}$, $\widetilde{P}$, $\widetilde{\Omega}_{\sigma}$ *satisfy the conditions in Theorem 7.4, then* $\mathcal{X}$, P, Ω_{σ} *satisfy them too, and conversely when* ν *is an isomorphism on a neighbourhood of* $\widetilde{P}$ (for instance, if the image of P belongs to the smooth locus of $\pi : \mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$ and if $\nu : \widetilde{\mathcal{X}} \to \mathcal{X}$ is a resolution of the singularities of $\mathcal{X}$).

7.1.4. An explicit illustration of the preceding discussion is provided by the integral closed subscheme of $\mathbb{A}^2_{\mathbb{Z}}$ defined by the equation

$$Y^2 + Y = X^{2g+1}\,,$$

where $g \in \mathbb{N}^*$ (compare [I], § 4, Corollary 3). For any number field K, the hypersurface $\mathcal{H}_{g,\mathcal{O}_K}$ is integral, and the map $\mathcal{H}_{g,\mathcal{O}_K} \to \operatorname{Spec} \mathcal{O}_K$ is smooth on the complement of the 0-dimensional subscheme Σ of $\mathcal{H}_{g,\mathcal{O}_K}$ defined by

$$\begin{cases} 2g+1 = 0 \\ X^{2g+1} = -\frac{1}{4} \\ Y = -\frac{1}{2}\,. \end{cases}$$

In particular, $\mathcal{H}_{g,\mathcal{O}_K}$ is regular in codimension 1 (hence normal, since it is an hypersurface in $\mathbb{A}^2_{\mathcal{O}_K}$ and therefore Cohen-Macaulay).

Let F be any horizontal subscheme of $\mathcal{H}_{g,\mathcal{O}_K}$ defined as the closure in $\mathcal{H}_{g,\mathcal{O}_K}$ of a finite subset of points in $\mathcal{H}_g(\overline{K})$ whose first coordinate is a root of unity, let $O = (0,0)$ be the zero section of $\mathcal{H}_{g,\mathcal{O}_K} \to \operatorname{Spec} \mathcal{O}_K$, and let $R = X$. The divisor of R is the disjoint union of the reduced divisors ($X = 0$, $Y = 0$) and ($X = 0$, $Y = -1$), and the map $R : \mathcal{H}_g(\mathbb{C}) \to \mathbb{A}^1(\mathbb{C})$ is a finite covering, ramified only over 0. Therefore, the following data are instances of the ones considered in 7.1.1:

$$\mathcal{X} := \mathcal{H}_{g,\mathcal{O}_K} \backslash F\,,$$

$$P = O\,,$$

$$\Omega_{\sigma} = \text{ connected component of } (0,0) \text{ in } \{(x,y) \in \mathbb{C}^2 \mid y^2 + y = x^{2g+1} \text{ and } |x| < 1\}\,.$$

Moreover the hypothesis of Theorem 7.4 are satisfied; indeed

$$R_\sigma(\Omega_\sigma) = \overset{\circ}{D}(0,1), \text{and } \rho(\overset{\circ}{D}(0,1)) = 1\,,$$

$$n = 1 \text{ and } \widetilde{n}_\wp = 0 \text{ for any } \wp\,,$$

and $\mathcal{X}_\sigma \backslash \Omega_\sigma$ contains the non-empty open subset

$$\{(x,y) \in \mathbb{C}^2 \mid y^2 + y = x^{2g+1} \text{ and } |x| > 1\}\,.$$

Finally we get that, for any geometric point η of Spec $\mathcal{O}_K$,

$$\pi_1(\mathcal{H}_{g,\mathcal{O}_K} \backslash F, O(\eta)) \simeq \pi_1(\operatorname{Spec} \mathcal{O}_K, \eta)\,.$$

Using the observation at the end of 7.1.3, we also get that we have

$$\pi_1(\mathcal{X}, \eta) \overset{\pi_*}{\xrightarrow{\sim}} \pi_1(\operatorname{Spec} \mathcal{O}_K, \pi(\eta)) \tag{7.5}$$

for any resolution $\mathcal{X}$ of a projective completion of $\mathcal{H}_{g,\mathcal{O}_K}$ or, more generally, for any normal projective surface $\mathcal{X}$ obtained by contracting vertical curves in such a resolution—in particular for the minimal proper regular model of $\mathcal{H}_{g,K}$ over $\mathcal{O}_K$—and for any geometric point η of $\mathcal{X}$.

The semi-stable reduction of the curves $\mathcal{H}_{g,K}$, and more generally of "primitive Fermat curves", has been studied in detail by Coleman and McCallum in [C-MC]. In particular they prove that, if $2g+1$ is prime, then $\mathcal{H}_g$ has potentially good reduction, and therefore that there exists a number field K_0 such that for any finite extension K of K_0, the minimal proper regular model of $\mathcal{H}_{g,K}$ over $\mathcal{O}_K$ is smooth. This may also be established by a simple direct computation: the case $2g+1 = 5$ is treated in [B-M-MB], and, with obvious modifications, the same computation shows, when $p = 2g+1$ is prime, that the curve $\mathcal{H}_g$ has good reduction over $K_0 = \mathbb{Q}\left(\zeta_p, \sqrt[p]{2}, \sqrt{1-\zeta_p}\right)$, where ζ_p is a primitive p-th root of unity. In this way, for any odd prime p, we get a proper smooth arithmetic surface $\mathcal{X}$ of genus $g = \frac{p-1}{2}$ such that (7.5) holds.

7.2. Elliptic curves

This last section is devoted to the proof of Corollary 1.3 and to some explicit examples.

7.2.1. To prove Corollary 1.3, let us go back to the notation of **1.3**, and, for any embedding $\sigma : K \hookrightarrow \mathbb{C}$, let

$$I_\sigma : \mathbb{C}/\mathbb{Z} + \tau_\sigma \mathbb{Z} \xrightarrow{\sim} E_\sigma(\mathbb{C})$$

be an isomorphism of complex analytic elliptic curves. For any such σ, consider the composition $I_\sigma \circ p_\sigma$ of I_σ with the canonical quotient map $p_\sigma : \mathbb{C} \to \mathbb{C}/\mathbb{Z} + \tau_\sigma \mathbb{Z}$. Its restriction to the open disk $\overset{\circ}{D}\left(0; \frac{1}{2}\right) = \left\{z \in \mathbb{C} \mid |z| < \frac{1}{2}\right\}$ is injective, since τ_σ belongs to the fundamental domain (1.3). Therefore, the map

$$\begin{aligned} \varphi_\sigma : \overset{\circ}{D}(0;1) &\to E_\sigma(\mathbb{C}) \\ z &\mapsto I_\sigma \circ p_\sigma(z/2) \end{aligned}$$

is an open holomorphic immersion.

The construction of **1.1** attaches hermitian norms $\| \|_{\varphi_\sigma}$ on the complex lines $T_0 E_\sigma(\mathbb{C})$ to these maps. Corollary 1.3 now follows from Theorem 1.2 and the following simple lemma:

Lemma 7.5. *The following equality holds:*

$$\widehat{\deg}(\varepsilon^* T_\pi, (\| \|_{\varphi_\sigma})_{\sigma:K\hookrightarrow\mathbb{C}}) = -[K:\mathbb{Q}]\, h_F(E) + \frac{1}{2}\sum_{\sigma:K\hookrightarrow\mathbb{C}} \log \frac{\pi}{4\,\mathrm{Im}\,\tau_\sigma}. \tag{7.6}$$

Proof. Let us consider the hermitian norm $\| \|^\vee_{\varphi_\sigma}$ on

$$\omega_{\mathcal{E}/S,\sigma} \simeq T_0 E_\sigma(\mathbb{C})^\vee$$

dual to the norm $\| \|_{\varphi_\sigma}$, and the ratio

$$\lambda_\sigma := \frac{\|\cdot\|_{\mathrm{L}^2,\sigma}}{\|\cdot\|^\vee_{\varphi_\sigma}}.$$

To compute λ_σ, let us introduce the holomorphic differential form α_σ on $E_\sigma(\mathbb{C})$ such that

$$(I_\sigma \circ p_\sigma)^* \alpha_\sigma = \mathrm{d}z.$$

Then we have:

$$\|\alpha\|^2_{\mathrm{L}^2,\sigma} = \frac{i}{2\pi}\int_{\mathbb{C}/\mathbb{Z}+\tau_\sigma\mathbb{Z}} \mathrm{d}z\wedge \mathrm{d}\overline{z} = \frac{1}{\pi}\,\mathrm{Im}\,\tau_\sigma.$$

Moreover,

$$\varphi_\sigma^*\,\alpha_\sigma = \frac{1}{2}\,\mathrm{d}z,$$

and therefore

$$\|\alpha_\sigma\|^\vee_{\varphi_\sigma} = \frac{1}{2}.$$

Thus we get:

$$\lambda_\sigma = \sqrt{\frac{4\,\mathrm{Im}\,\tau_\sigma}{\pi}}. \tag{7.7}$$

On the other hand, we have:

$$\begin{aligned}\widehat{\deg}(\varepsilon^* T_\pi, (\| \|_{\varphi_\sigma})_{\sigma:K\hookrightarrow\mathbb{C}}) &= -\,\widehat{\deg}(\omega_{\mathcal{E}/S}, (\| \|^\vee_{\varphi_\sigma})_{\sigma:K\hookrightarrow\mathbb{C}})\\ &= -\,\widehat{\deg}(\omega_{\mathcal{E}/S}, \| \|_{\mathrm{L}^2}) - \sum_{\sigma:K\hookrightarrow\mathbb{C}} \log\lambda_\sigma.\end{aligned} \tag{7.8}$$

Finally, (7.6) follows from (7.7), (7.8), and the definition (1.2) of $h_F(E)$.

q.e.d.

7.2.2. Examples. Let us give a few explicit examples of elliptic curves E over a number field K such that the numerical condition

$$h_F(E) \leq \frac{1}{2[K:\mathbb{Q}]} \sum_{\sigma:K\hookrightarrow\mathbb{C}} \log \frac{\pi}{4 \operatorname{Im} \tau_\sigma}, \tag{1.4}$$

holds.

(i) Let E be an elliptic curve over a number field K, whose j-invariant is 0 and which has good reduction over K. Its Faltings height is given by the Chowla-Selberg formula (see for instance [D2]):

$$h_F(E) = -\frac{1}{2} \log \left[\frac{1}{\sqrt{3}} \left(\frac{\Gamma(1/3)}{\Gamma(2/3)} \right)^3 \right] = -0,749\ldots. \tag{7.9}$$

Moreover, for any $\sigma : K \hookrightarrow \mathbb{C}$, we may choose

$$\tau_\sigma = \frac{1+i\sqrt{3}}{2}$$

and therefore

$$\frac{1}{2} \log \frac{\pi}{4 \operatorname{Im} \tau_\sigma} = \frac{1}{2} \log \frac{\pi}{2\sqrt{3}} = -0,05\ldots.$$

(ii) Let E be the elliptic curve over $\mathbb{Q}$ defined by the plane affine curve

$$y^2 + y = x^3.$$

Its j-invariant is 0, and its minimal discriminant is

$$\Delta_E = 3^3.$$

Its Faltings height is the sum of its stable Faltings height, given by (7.9), and of the contribution

$$\frac{1}{12} \log \Delta_E = \frac{1}{4} \log 3 = 0,275\ldots$$

of its unique prime of bad reduction (namely 3). Therefore

$$h_F(E) = -0,749\ldots + 0,275\ldots = -0,474\ldots.$$

Moreover, as in **(i)**, we have

$$\tau = \frac{1+i\sqrt{3}}{2} \quad \text{and} \quad \frac{1}{2} \log \frac{\pi}{4 \operatorname{Im} \tau} = -0,05\ldots.$$

Observe that this example coincides with the special case $g = 1$ of the curve $\mathcal{H}_g$ studied in 6.1.2.

(iii) Let E be an elliptic curve over a number field K, whose j-invariant is 1728 and which has good reduction over K. Here again, the Faltings height of E is given by the Chowla-Selberg formula

$$h_F(E) = -\frac{1}{2} \log \left[\frac{1}{2} \left(\frac{\Gamma(1/4)}{\Gamma(1/2)\,\Gamma(3/4)} \right)^2 \right] = -0,166\ldots.$$

Moreover, for any $\sigma : K \hookrightarrow \mathbb{C}$,

$$\tau_\sigma = i$$

and therefore

$$\frac{1}{2} \log \frac{\pi}{4 \operatorname{Im} \tau_\sigma} = \frac{1}{2} \log \frac{\pi}{4} = -0,12\ldots.$$

Appendix: Potential theory on compact Riemann surfaces

This appendix is devoted to the proofs of various non-trivial facts concerning the equilibrium potentials $g_{P,\Omega}$ stated in section 3, notably of Theorem 3.1 and of the characterization of equilibrium potentials in Remarks 3.1.4, iii) (this characterization is used in the proof of the functoriality property in Proposition 3.5).

These proofs are similar to the ones in the classical works of Beurling, Brelot, Deny and Lions – see especially [Br], [B-De], [De1], [De2], [De-L] – and indeed rely crucially on deep results in these papers. However, we believe necessary to give them in some detail, since the present setting of potential theory on arbitrary compact surfaces is not considered by these authors and sometimes requires non-obvious modifications (see for instance section **A.5** below). We also refer the reader to the monographs [T], [Ru1] and [R] for additional material concerning potential theory on Riemann surfaces.

A.1. Capacity - Fine topology

Let K be a compact subset of $\mathbb{C}$, contained in some open disk of radius 1. Then, if K is not polar, its Robin constant (see **3.1.4, v)**), is positive, and we shall define the *capacity* of K as

$$\begin{aligned} \operatorname{cap}(K) &:= r(K)^{-1} && \text{if } K \text{ is not polar,} \\ &:= 0 && \text{if } K \text{ is polar.} \end{aligned}$$

This definition of capacity coincides with the one used in [De-L], II.1. If $c(K)$ denotes the logarithmic capacity of K defined as in [Ra], 5.1, we have:

$$\operatorname{cap}(K) = -\frac{1}{\log c(K)}\,.$$

The capacity of an open subset U of a disk of radius 1 in $\mathbb{C}$ is defined as

$$\operatorname{cap}(\Omega) := \sup_{\substack{K \text{ compact} \\ K \subset \Omega}} \operatorname{cap}(K)\,,$$

and finally, the (outer) capacity of an arbitrary subset X of a disk $\overset{\circ}{D}$ of radius 1 in $\mathbb{C}$ is defined as

$$\operatorname{cap}(X) := \inf_{\substack{\Omega \text{ open in } \overset{\circ}{D} \\ X \subset \Omega}} \operatorname{cap}(\Omega)\,.$$

It belongs to $[0, +\infty]$, and, as a function of X, is countably subadditive when restricted to subsets of a fixed subset of $\mathbb{C}$ of diameter ≤ 1. Moreover, a subset E of $\mathbb{C}$ is polar, iff for any $a \in \mathbb{C}$

$$\operatorname{cap}\left(\overset{\circ}{D}\left(a, \frac{1}{2}\right) \cap E\right) = 0\,.$$

This implies that a subset E of an arbitrary Riemann surface M is polar iff, for any holomorphic embedding $\varphi : \overset{\circ}{D}\,(0; 1/2) \to M$, one has:

$$\operatorname{cap}(\varphi^{-1}(E)) = 0\,.$$

It is indeed enough to assume that this condition holds for some family of holomorphic embeddings $\varphi : \overset{\circ}{D}(0;1/2) \to M$ the images of which cover M.

When dealing with the domain of validity of some property of points in some Riemann surface, or with the domain of definition of a function, we will use the expression *nearly everywhere* to mean "outside a polar subset".

The *fine topology* on a Riemann surface M is the coarsest topology on M, which is finer than the usual topology, and such that, for any subset U of M open in the usual topology, the subharmonic functions on U are continuous. Topological notions considered with respect to this topology will be preceded by the qualificative "fine" (*e.g.*, fine continuous, fine lim sup, ...). We refer to [Do], 1.XI, and to [Br] and [De1] for informations and references concerning the fine topology. Let us only recall that a function F defined nearly everywhere on some open set Ω is fine continuous at some point P of Ω iff there exist an open subset U of Ω, thin at P (*cf.* **A.8** *infra*), such that $F(P) = \lim_{\substack{Q \to P \\ Q \in \Omega \setminus U}} F(Q)$.

A.2. BLD functions

We shall say that a complex valued function f, defined nearly everywhere on some Riemann surface M satisfies *Property (P)* if, for any holomorphic imbedding $\varphi : \overset{\circ}{D}(0;1/2) \to M$ and any $\varepsilon \in \mathbb{R}_+^*$, there exists an open subset ω of $\overset{\circ}{D}(0;1/2)$ such that

$$\operatorname{cap}(\varphi^{-1}(\omega)) < \varepsilon$$

and $\varphi^* f := f \circ \varphi$, restricted to $\overset{\circ}{D}(0;1/2) \setminus \omega$, is continuous.

One easily sees that such a function f on M satisfies Property (P) iff, for any open subset U of $\mathbb{C}$ and any holomorphic chart $\varphi : U \to M$, the function $\varphi^* f$, defined nearly everywhere on U, satisfies Property (P) in [De-L], II.3. Accordingly, various results concerning Property (P) established in *loc. cit.* immediately extend to the present setting. First, we have, for any Riemann surface M:

Theorem A.2.1. *Any function in* $\mathrm{L}_1^2(M)_{\mathrm{loc}}$ *has a representative which satisfies Property* (P), *and any two such representatives coincide nearly everywhere on* M.

A function satisfying Property (P) and defining an element of $\mathrm{L}_1^2(M)_{\mathrm{loc}}$ will be called a *BLD function on* M (BLD is for "Beppo-Levy-Deny": we follow the terminology of [Do], 1.XIII.7, and our BLD functions are "fonctions précisées" in the sense of [De-L]; we refer to the introduction of this paper for historical references).

Then, we immediately get from [De-L], II, Théorème 3.2:

Theorem A.2.2. *A function on* M, *which defines an element of* $\mathrm{L}_1^2(M)_{\mathrm{loc}}$, *is a* BLD-*function iff it is fine continuous quasi-everywhere on* M.

Moreover, an easy variation on the proof of [De-L], II, Théorème 4.1 leads to the following:

Theorem A.2.3. *Let M be a compact Riemann surface, and (u_n) a sequence of* BLD *functions on M which converges in $\mathrm{L}_1^2(M)$ to some* BLD *function u. Then there exists some subsequence $(u_{n(k)})$ of (u_n) which converges quasi-everywhere to u.*

Indeed one even has uniform convergence outside an open subset of M "of arbitrary small capacity".

As a first illustration of the usefulness of BLD functions for potential theory, we now prove the last assertion in Theorem 3.1, taking the previous ones for granted. Indeed, if $\chi \in \mathrm{C}^\infty(M)$ takes the value 0 (resp. 1) on some neighbourhood of P (resp. $M \backslash \Omega$), then the first two assertions show that $\chi\, g_{P,\Omega}$ belongs to $\mathrm{L}_1^2(\Omega)_0$. The third one allows to assume that $g_{P,\Omega}$ is a subharmonic function on $M \backslash \{P\}$; then $\chi\, g_{P,\Omega}$ is fine continuous on M. It is therefore a BLD function (Theorem A.2.2), which defines an element in $\mathrm{L}_1^2(\Omega)_0$. Any such BLD function is the pointwise limit quasi-everywhere of a sequence in $\mathrm{C}_c^\infty(\Omega)$ (Theorem A.2.3); so it vanishes nearly everywhere on $M \backslash \Omega$.

A.3. A characterization of polar compact subsets

The following characterization of polar sets in terms of spaces of L_1^2 functions plays a key role in our study of equilibrium potentials.

Theorem A.3.1. *Let M be a connected compact Riemann surface. For any open subset Ω of M, the following conditions are equivalent:*

(i) *Ω is non-empty and there exists $\varphi \in \mathrm{L}_1^2(\Omega)_0$ such that $\varphi_{|\Omega} = 1$;*

(ii) *the constant function $1 : X \to \mathbb{C}$ belongs to $\mathrm{L}_1^2(\Omega)_0$;*

(iii) *$\mathrm{L}_1^2(X) = \mathrm{L}_1^2(\Omega)_0$;*

(iv) *$M \backslash \Omega$ is polar.*

Proof. The implications iii) $\Rightarrow$ ii) $\Rightarrow$ i) are clear. The implication (i) $\Rightarrow$ (ii) is a consequence of the following:

Lemma A.3.2. *With the notation of Theorem A.3.1, if Ω is non empty, for any $\varphi \in \mathrm{L}_1^2(\Omega)_0$, we have*

$$\varphi = 1 \Leftrightarrow \varphi_{|\Omega} = 1 .$$

Proof of Lemma A.3.2. The direct implication is clear. Conversely, if $\varphi_{|\Omega} = 1$, then for any $\rho \in \mathrm{C}_c^\infty(\Omega)$, we have

$$\langle \varphi, \rho \rangle_{\mathrm{Dir}} := \frac{i}{2\pi} \int_M \partial\varphi \wedge \overline{\partial\rho} = 0 .$$

By the density of $\mathrm{C}_c^\infty(\Omega)$ in $\mathrm{L}_1^2(\Omega)_0$, this still holds for any ρ in $\mathrm{L}_1^2(\Omega)_0$, in particular for $\rho = \varphi$. As $\| \ \|_{\mathrm{Dir}}$ is a norm on $\mathrm{L}_1^2(M)/\mathbb{C}$, this shows that φ is a constant function, and therefore coincides with the constant function 1, since it takes the value 1 on the non-empty open subset Ω.

q.e.d.

The implication (ii) $\Rightarrow$ (iv) follows from the basic results on BLD functions described in the preceding section. Indeed, if (ii) holds, then the constant function 1 on X is the limit in $\mathrm{L}^2_1(X)$ of a sequence of functions in $\mathrm{C}^\infty_c(\Omega)$; by Theorem A.2.3, and is therefore quasi-everywhere on M the pointwise limit of such a sequence. In particular, we have $1 = 0$ quasi-everywhere on $M\backslash\Omega$; this exactly means that $M\backslash\Omega$ is polar.

To complete the proof of Theorem A,3.1, it is now enough to prove the implication (iv) $\Rightarrow$ (iii). To achieve this, recall that the bilinear pairing

$$\mathrm{L}^2_1(M) \times \mathrm{L}^2_{-1}(M) \to \mathbb{C}$$

$$(\varphi, \omega) \mapsto \int_M \varphi\,\omega$$

identifies the topological dual of $\mathrm{L}^2_1(M)$ with $\mathrm{L}^2_{-1}(M)$, and observe that, for any open subset Ω of M and any $\omega \in \mathrm{L}^2_{-1}(M)$, the linear form

$$\varphi \mapsto \int_M \varphi\,\omega$$

vanishes on $\mathrm{C}^\infty_c(\Omega)$ iff ω is supported by $M\backslash\Omega$. Therefore, the Hahn-Banach theorem shows that $\mathrm{C}^\infty_c(\Omega)$ is dense in $\mathrm{L}^2_1(M)$ iff every $\omega \in \mathrm{L}^2_{-1}(M)$ such that supp $\omega \subset M\backslash\Omega$ vanishes. In particular the implication (iv) $\Rightarrow$ (iii) for Ω arbitrary may be rephrased as the following claim:

(A.3.1) *any $\omega \in \mathrm{L}^2_{-1}(M)$ supported by some polar compact subset of M vanishes.*

It is clearly a local statement, and it is enough to prove it when $M = \mathbb{P}^1(\mathbb{C})$. Finally, we are reduced to the following assertion:

(A.3.2) *for any open subset Ω of $\mathbb{P}^1(\mathbb{C})$ such that $\mathbb{P}^1(\mathbb{C})\backslash\Omega$ is polar, $\mathrm{C}^\infty_c(\Omega)$ is dense in $\mathrm{C}^\infty(\mathbb{P}^1(\mathbb{C}))$ for the L^2_1-topology.*

This follows from [De-L], Chapter II, Théorème 2.2. Indeed, this theorem and its proof show that, when $\infty \notin \Omega$, for any φ in $\mathrm{C}^\infty_c(\mathbb{P}^1(\mathbb{C})\backslash\{\infty\})$ there exists a sequence (φ_n) of functions in $\mathrm{C}^\infty_c(\Omega)$ such that

$$\lim_{n\to+\infty} \|\varphi_n - \varphi\|_{\mathrm{Dir}} = 0\,,$$

and which are supported by a fixed compact subset of $\mathbb{P}^1(\mathbb{C})\backslash\{\infty\}$; by the well known "Poincaré inequality", this implies that φ_n converges to φ in $\mathrm{L}^2_1(\mathbb{P}^1(\mathbb{C}))$. Consequently, for any $P \in \mathbb{P}^1(\mathbb{C})\backslash\Omega$, $\mathrm{C}^\infty_c(\Omega)$ is dense in $\mathrm{C}^\infty_c(\mathbb{P}^1(\mathbb{C})\ \backslash P)$ for the L^2_1-topology. This implies (A.3.2).

q.e.d.

Corollary A.3.3. *Let M and M' be two connected compact Riemann surfaces, and let Ω (resp. Ω') an open subset of M (resp. M'). If there is some biholomorphic map $f : \Omega \xrightarrow{\sim} \Omega'$, then $M\backslash\Omega$ is polar iff $M'\backslash\Omega'$ is.*

This follows from the equivalence of conditions (i) and (iv) in Theorem A.3.1, and from the fact that, for any biholomorphic map $f : \Omega \xrightarrow{\sim} \Omega'$, the map

$$f^* : \mathrm{C}^\infty_c(\Omega') \xrightarrow{\sim} \mathrm{C}^\infty_c(\Omega)$$

extends to an isomorphism

$$\mathrm{L}_1^2(\Omega')_0 \xrightarrow{\sim} \mathrm{L}_1^2(\Omega)_0$$

(*cf.* **3.1.2**).

Corollary A.3.4. *With the notation of Theorem A.3.1, if $M \backslash \Omega$ is not polar, then the Dirichlet form restricted to $\mathrm{L}_1^2(\Omega)_0$ is an hermitian scalar product which defines the natural topology of* $\mathrm{L}_1^2(\Omega)_0$ (which is defined as a closed vector subspace of $\mathrm{L}_1^2(X)$).

Indeed, when $M \backslash \Omega$ is non polar, any constant function in $\mathrm{L}_1^2(\Omega)_0$ vanishes, and therefore the composition of the inclusion of a closed subspace $\mathrm{L}_1^2(\Omega)_0 \hookrightarrow \mathrm{L}_1^2(X)$ and of the quotient map with finite dimensional kernel $\mathrm{L}_1^2(X) \to \mathrm{L}_1^2(X)/\mathbb{C}$ is a linear homeomorphism from $\mathrm{L}_1^2(\Omega)_0$ onto its image. Moreover, the quotient topology on $\mathrm{L}_1^2(X)/\mathbb{C}$ is defined by the Dirichlet norm.

Remark A.3.5. With the notation of Theorem 3.1, the current $\mu := dd^c g_{P,\Omega} + \delta_P$ is non zero, is supported by $\partial\Omega_0$, where Ω_0 is the connected component of Ω containing P, and belongs to $\mathrm{L}_{-1}^2(M)$. According to (A.3.1), this implies that $\partial\Omega_0$ is not polar. Therefore, when we shall have completed the proof of Theorem 3.1, we shall have shown that, for any compact Riemann surface M, and any open subset Ω of M, *if $M \backslash \Omega$ is not polar, the boundaries of the components of Ω also are not polar.*

A.4. Resolution of the Laplace equation and construction of $g_{P,\Omega}$

Thanks to the results in the preceding section, we may solve the Laplace equation on Ω by Hilbert space methods:

Theorem A.4.1. *Let M be a compact connected Riemann surface, and Ω an open subset of M such that $M \backslash \Omega$ is non polar. Then, for any α in $\mathrm{L}_{-1}^2(X)$, there exists a unique f in $\mathrm{L}_1^2(\Omega)_0$ such that*

$$dd^c f = \alpha \quad \text{on} \quad \Omega. \tag{A.4.1}$$

Proof. Let α be an element of $\mathrm{L}_{-1}^2(X)$, and let us consider the continuous linear form on $\mathrm{L}_1^2(X)$:

$$\lambda_\alpha : \varphi \mapsto -\int_M \varphi\,\overline{\alpha}.$$

Then, for any $f \in \mathrm{L}_1^2(X)$, the following assertions are clearly equivalent:

- $dd^c f = \alpha$ on Ω;
- for any $\varphi \in \mathrm{C}_c^\infty(\Omega)$, $-\int_M \varphi\, dd^c \overline{f} = -\int_M \varphi\,\overline{\alpha}$
- for any $\varphi \in \mathrm{C}_c^\infty(\Omega)$, $\langle \varphi, f\rangle_{\mathrm{Dir}} = -\int_M \varphi \cdot \overline{\alpha}$
- for any $\varphi \in \mathrm{L}_1^2(\Omega)_0$, $\langle \varphi, f\rangle_{\mathrm{Dir}} = \lambda_\alpha(\varphi)$.

Theorem A.4.1 therefore follows from the fact that the Dirichlet form $\langle\ ,\ \rangle_{\mathrm{Dir}}$ defines a *hilbertian* scalar product on $\mathrm{L}_1^2(\Omega)_0$ since $M \backslash \Omega$ is not polar (Corollary A.3.4) and from Riesz theorem: the unique solution f of (A.4.1) in $\mathrm{L}_1^2(\Omega)_0$ is the image of the linear form λ_α on $\mathrm{L}^2_1(\Omega)_0$ by the antilinear isomorphism of $\mathrm{L}^2_1(\Omega)_0$ onto its topological dual defined by $\langle\cdot,\cdot\rangle_{\mathrm{Dir}}$.

q.e.d.

Using Theorem A.4.1, it is now easy to complete the proof of the first part of Theorem 3.1.

Proof of the existence and unicity of equilibrium potentials. Let M be a compact connected Riemann surface, Ω an open subset of M such that $M\backslash\Omega$ is not polar, and P a point of Ω. Let us choose a holomorphic coordinate z on some open neighbourhood U of P in Ω, and $\rho \in \mathrm{C}^\infty_c(U,\mathbb{R})$ taking the value 1 on some neighbourhood of P, and let us introduce the function g_0 on X defined by:

$$\begin{aligned} g_0 &:= \rho \log |z - z(P)|^{-2} && \text{on} \quad U\backslash\{P\} \\ &:= 0 && \text{on} \quad M\backslash U\,. \end{aligned}$$

Then g_0 is C^∞ on $X\backslash\{P\}$, and summable over M. It is indeed a Green function for the divisor P on M: the distribution

$$\alpha := \mathrm{dd}^c\, g_0 + \delta_P$$

is C^∞ (with compact support in U).

According to Theorem A.4.1, there exists $u \in \mathrm{L}^2_1(\Omega)_0$ such that

$$\mathrm{dd}^c\, u = -\alpha \quad \text{on} \quad \Omega\,.$$

Let us check that

$$g_{P,\Omega} := g_0 + u$$

satisfies conditions (i) and (ii) in Theorem 3.1:

- outside $K := \operatorname{supp} u\,(\subset \Omega)$, we have $g_0 = 0$; therefore

$$g_{P,\Omega} = u \quad \text{on} \quad M\backslash K\,,$$

while $u \in \mathrm{L}^2_1(\Omega)_0$ by construction;

- we have:

$$\begin{aligned} \mathrm{dd}^c\, g_{P,\Omega} + \delta_P &= \mathrm{dd}^c(g_0 + u) + \delta_P \\ &= \alpha + \mathrm{dd}^c\, u \\ &= 0 \qquad \text{on} \quad \Omega\,. \end{aligned}$$

Let us finally check the unicity of the distribution $g_{P,\Omega}$ satisfying these conditions (i) and (ii). So, let us consider another distribution $\widetilde{g}_{P,\Omega}$ satisfying them, and let

$$\delta := \widetilde{g}_{P,\Omega} - g_{P,\Omega}\,.$$

Then

$$\mathrm{dd}^c\,\delta = 0 \quad \text{on} \quad \Omega\,; \tag{A.4.2}$$

therefore δ is harmonic, hence C^∞, on Ω; moreover, outside some compact subset K of Ω, δ coincides with some element of $\mathrm{L}^2_1(\Omega)_0$; this shows that δ belongs to $\mathrm{L}^2_1(\Omega)_0$. The equation (A.4.2), and the unicity assertion in Theorem A.4.1 imply that $\delta = 0$.

q.e.d.

Remark A.4.2. Observe that the proof of Theorem A.4.1 essentially relies on the fact that, as $M\backslash\Omega$ is non polar, the constant function 1 does not belong to $\mathrm{L}_1^2(\Omega)_0$. If we make the stronger hypothesis that $M\backslash\Omega$ has a non empty interior, this is immediate, and the proofs of Theorem A.4.1, and consequently of the existence and unicity of $g_{P,\Omega}$, become quite elementary.

A.5. The method of Beurling-Deny

This section is devoted to proving that the current

$$\mu := \mathrm{dd}^c g_{P,\Omega} + \delta_P$$

is a probability measure supported by $\partial\Omega$. As

$$\int_M \mu = \int_M (\mathrm{dd}^c g_{P,\Omega} + \delta_P) = \int_M \delta_P = 1\,,$$

and $g_{P,\Omega}$ and δ_P are supported by $\overline{\Omega}$, this follows from the apparently weaker statement:

(A.5.1) *the current μ is non-negative and supported by $M\backslash\Omega$.*

We shall also prove that *$g_{P,\Omega}$ is non-negative*. Together with (A.5.1), this will complete the proof of assertion (iii) in Theorem 3.1, and therefore the proof of this Theorem (see **A.4** for (i) and (ii), and the end of **A.2** for (iv)).

We shall prove these facts by using some variant of the *method of Beurling-Deny* (see [B-De], [De2]). This is a general method for proving subharmonicity properties of equilibrium potentials constructed by hilbertian techniques of the kind used in the previous section, which relies on the fact that *the composition $T\circ f$ of a function f in L_1^2 by a contraction T* (*i.e.*, a distance decreasing map from $\mathbb{C}$ to $\mathbb{C}$) *is again* L_1^2; more precisely, *the Dirichlet norm of $T\circ f$ is not larger than the one of f.*

Here, we shall make use of the following special case of this fact:

Proposition A.5.1. *For any Riemann surface M and any function $u : M \to \mathbb{R}$ in $\mathrm{L}_1^2(M)_{\mathrm{loc}}$, the function $u_+ := \max(0,u)$ belongs to $\mathrm{L}_1^2(M)_{\mathrm{loc}}$ and, for any compact subset K in M,*

(A.5.2) $$i\int_K \partial u_+ \wedge \overline{\partial} u_+ \leq i\int_K \partial u \wedge \overline{\partial} u\,.$$

This follows for instance from [De-L], I, Théorème 3.2[15].

We shall also need latter the following related result:

[15] It may also be proved by considering for any $\varepsilon \in \mathbb{R}_+$ the function $u_\varepsilon = \frac{1}{2}\left[u + \sqrt{u^2+\varepsilon}\right]$, by showing that u_ε belongs to $\mathrm{L}_1^2(M)_{\mathrm{loc}}$ and satisfies (A.5.2) with u_ε instead of u_+ (treat first the case where u is C^∞, then use an approximation argument), and by letting ε go to 0.

Proposition A.5.2. *Let M be a compact Riemann surface and let $f : M \to \mathbb{R}$ be a function which defines an element of $\mathrm{L}^2_1(X)$. For any $R \in \mathbb{R}_+$, let $f_R : M \to [-R, R]$ be the function defined by*

$$\begin{aligned} f_R(x) &= f(x) \quad \text{if} \quad f(x) \in [-R, R] \\ &= R \quad \text{if} \quad f(x) \geq R \\ &= -R \quad \text{if} \quad f(x) \leq -R. \end{aligned}$$

Then f_R defines an element of $\mathrm{L}^2_1(X)$, which converges to f in $\mathrm{L}^2_1(X)$ when R goes to $+\infty$.

This follows for instance from [De-L], I, Proposition 3.1.

We start by giving a variational characterization of the equilibrium potential $g_{P,\Omega}$ constructed in the preceding section.

We consider as before a compact connected Riemann surface M, an open subset Ω of M such that $M \backslash \Omega$ is not polar, and P a point of Ω. We shall denote by $V(P)$ the set of L^2_1-Green functions for P on M, and by $V(P, \Omega)$ the subset of $V(P)$ formed by L^2_1-Green functions which satisfy condition (i) of Theorem 3.1 (namely, which coincide with a function in $\mathrm{L}^2_1(\Omega)_0$ outside a neighbourhood of P). These sets $V(P)$ and $V(P, \Omega)$ are affine spaces, with the vector spaces

$$\mathrm{L}^2_1(M, \mathbb{R}) := \{\text{real valued functions in } \mathrm{L}^2_1(M)\}$$

and

$$\mathrm{L}^2_1(\Omega, \mathbb{R})_0 := \mathrm{L}^2_1(\Omega)_0 \cap \mathrm{L}^2_1(M, \mathbb{R})$$

as underlying vector spaces. In particular, the L^2_1-topology on these spaces induces a topology on $V(P)$ and $V(P, \Omega)$.

For any $g_0 \in V(P)$, we define a functional

$$F_{g_0} : V(P) \to \mathbb{R}$$

by the following formula, where φ is any element of $\mathrm{L}^2_1(\Omega, \mathbb{R})_0$ and $\omega_0 := \mathrm{dd}^c g_0 + \delta_P$:

$$F_{g_0}(g_0 + \varphi) = \frac{1}{2} \langle \varphi, \varphi \rangle_{\mathrm{Dir}} - \int_M \varphi\, \omega_0\,.$$

One easily checks that, if g_1 is another element of $V(P)$, the functionals F_{g_1} and F_{g_0} differ by the constant $F_{g_1}(g_0) = -F_{g_0}(g_1)$. These functionals are strictly convex and differentiable on $V(P)$, and there differential is easily checked to be given by the formula

$$\mathrm{D}F_{g_0}\,(g)\,(h) = -\int_M h\, \omega\,,$$

where g (resp. h) is any element of $V(P)$ (resp. $\mathrm{L}^2_1(X, \mathbb{R})$), and $\omega := \mathrm{dd}^c g + \delta_P$. In particular $\mathrm{D}F_{g_0}(g)$ vanishes on $\mathrm{L}^2_1(\Omega)_0$, or equivalently on $\mathrm{C}^\infty_c(\Omega)$, iff ω vanishes on Ω. This shows that an element $g \in V(P, \Omega)$ satisfies condition (ii) of Theorem 3.1 iff $\mathrm{D}F_{g_0}(g)$ vanishes on $\mathrm{L}^2_1(\Omega, \mathbb{R})_0$, and therefore that F_{g_0} *admits a minimal value on $V(P, \Omega)$, which is attained exactly at $g_{P,\Omega}$.*

A crucial technical ingredient in the proof of (A.5.1) is the following statement:

Proposition A.5.3. *For any g in $V(P)$, the function $g_+ := \max(0, g)$ also belongs to $V(P)$ and*

$$F_{g_0}(g_+) \leq F_{g_0}(g)\,. \tag{A.5.3}$$

Proof. We first prove the lemma when g is a Green function for P (with C^∞ regularity) supported in Ω. Then g is positive on some neighbourhood of P, on which clearly $g = g_+$; moreover, outside any compact neighbourhood of P, g is locally L_1^2, and therefore g_+ also, by Proposition A.5.1. This already implies that g_+ belong to $V(P)$.

The validity of (A.5.3) clearly does not depend on the choice of g_0. Therefore, we may assume that the support of g_0 lies in some open neighbourhood U of P on which g is positive, and accordingly, we have

$$g_+ = g \quad \text{on} \quad U \tag{A.5.4}$$

and

$$\int_M g\,\omega_0 = \int_M g_+\,\omega_0\,. \tag{A.5.5}$$

This implies:

$$\begin{aligned}
F_{g_0}(g) - F_{g_0}(g_+) &= \frac{1}{2}\,\|g - g_0\|_{\mathrm{Dir}}^2 - \int_M (g - g_0)\,\omega_0 \\
&\quad - \frac{1}{2}\,\|g_+ - g_0\|_{\mathrm{Dir}}^2 + \int_M (g_+ - g_0)\,\omega_0 \\
&= \frac{i}{4\pi}\int_M \partial(g - g_0)\wedge\overline{\partial}(g - g_0) \\
&\quad - \frac{i}{4\pi}\int_M \partial(g_+ - g_0)\wedge\overline{\partial}(g_+ - g_0) \quad \text{by (A.5.5)} \\
&= \frac{i}{4\pi}\int_{M\setminus U} \partial(g - g_0)\wedge\overline{\partial}(g - g_0) \\
&\quad - \frac{i}{4\pi}\int_{M\setminus U} \partial(g_+ - g_0)\wedge\overline{\partial}(g_+ - g_0) \quad \text{by (A.5.4)} \\
&= \frac{i}{4\pi}\int_{M\setminus U} \partial\, g\wedge\overline{\partial}\, g - \frac{i}{4\pi}\int_{M\setminus U} \overline{\partial}\, g_+\wedge\partial\, g_+ \\
&\qquad\qquad \text{since } \operatorname{supp} g_0 \subset U\,.
\end{aligned}$$

This is indeed non-negative by Proposition A.5.1.

We now suppose that g is an arbitrary element of $V(P)$. Then there exists a sequence (g^n) of Green functions (with C^∞ regularity) for P such that

$$\lim_{n\to\infty} \|g^n - g\|_{L_1^2(M)} = 0\,. \tag{A.5.6}$$

By the first part of the proof, for any n, we have:

$$g_+^n \in V(P) \qquad \text{and} \qquad F_{g_0}(g_+^n) \leq F_{g_0}(g^n)\,.$$

From (A.5.6), we get:

$$\lim_{n\to\infty} F_{g_0}(g^n) = F_{g_0}(g).$$

Therefore:

$$\limsup_{n\to\infty} F_{g_0}(g_+^n) \le F_{g_0}(g). \tag{A.5.7}$$

Moreover, as

$$|g_+^n - g_+| \le |g^n - g|,$$

(g_+^n) converges to g_+ in $\mathrm{L}^2(M)$, and consequently

$$\lim_{n\to\infty} \int_M g_+^n\, \omega_0 = \int_M g_+\, \omega_0 .$$

Together with (A.5.7), this implies:

$$\limsup_{n\to\infty} \|g_+^n - g_0\|_{\mathrm{Dir}} \le \|g - g_0\|_{\mathrm{Dir}} . \tag{A.5.8}$$

In particular, the sequence $(g_+^n - g_0)$, which converges to $g_+ - g_0$ in $\mathrm{L}^2(M)$, is bounded in $\mathrm{L}_1^2(M)$. This shows that $g_+ - g_0$ belongs to $\mathrm{L}_1^2(M)$, and is the weak limit of $(g_+^n - g_0)$ in $\mathrm{L}_1^2(M)$. This implies that

$$\|g_+ - g_0\|_{\mathrm{Dir}} \le \limsup_{n\to\infty} \|g_+^n - g_0\|_{\mathrm{Dir}} , \tag{A.5.9}$$

and therefore, by (A.5.8):

$$\|g_+ - g_0\|_{\mathrm{Dir}} \le \|g - g_0\|_{\mathrm{Dir}} .$$

Finally, using successively (A.5.9) and (A.5.7), we get:

$$F_{g_0}(g_+) \le \limsup_{n\to\infty} F_{g_0}(g_+^n) \le F_{g_0}(g).$$

q.e.d.

Let us observe that the argument in the second part of the proof shows that, for any g^n, $n \in \mathbb{N}$, and g in $V(P)$,

(A.5.10) if $\lim_{n\to\infty} \|g^n - g\|_{\mathrm{L}_1^2(M)} = 0$, then $g_+^n - g_+$ converges weakly to 0 in $\mathrm{L}_1^2(M)$.

Let $V_-(P,\Omega)$ be the (normic or weak) closure in $V(P)$ of the convex set

$$\{g \in V(P) \mid g \le 0 \text{ on some neighbourhood of } M\backslash\Omega\}. \tag{A.5.11}$$

It is a convex subset of $V(P)$, and contains $V(P,\Omega)$, which is indeed the (normic or weak) closure of the subspace

$$\{g \in V(P) \mid g = 0 \text{ on some neighbourhood of } M\backslash\Omega\}. \tag{A.5.12}$$

It is also stable under translations by functions in $\mathrm{C}^\infty(M,\mathbb{R})$ which are non-negative on some neighbourhood of $M\backslash\Omega$.

Observe that, from Proposition A.5.3 and from the continuity statement (A.5.10), and the definitions of $V_-(P,\Omega)$ and $V(P,\Omega)$ as the closures of (A.5.11) and (A.5.12), we get:

Corollary A.5.4. *The map* $(g \mapsto g_+)$ *sends* $V_-(P,\Omega)$ *into* $V(P,\Omega)$.

In particular $g_{P,\Omega+}$ belongs to $V(P,\Omega)$. Moreover, by (A.5.3), we have $F(g_{P,\Omega+}) \leq F(g_{P,\Omega})$. The variational characterization of $g_{P,\Omega}$ therefore implies that

$$g_{P,\Omega+} = g_{P,\Omega}\,,$$

or, in other words, that $g_{P,\Omega}$ *is almost everywhere non-negative.*

Moreover, for any $\varphi \in \mathrm{C}^\infty(M)$ non-negative on some neighbourhood of $M\backslash\Omega$ and any $t \in \mathbb{R}_+$, $(g_{P,\Omega} - t\,\varphi)_+$ belongs to $V(P,\Omega)$, and we get:

$$F_{g_0}(g_{P,\Omega} - t\,\varphi) \geq F_{g_0}((g_{P,\Omega} - t\,\varphi)_+) \geq F_{g_0}(g_{P,\Omega})\,.$$

Since we also have:

$$F_{g_0}(g_{P,\Omega} - t\,\varphi) - F_{g_0}(g_{P,\Omega}) = -t\,\mathrm{D}F_{g_0}(g_{P,\Omega}) + O(t^2) = t\int_M \varphi\,\mu + O(t^2)\,,$$

where $\mu := \mathrm{dd}^c\, g_{P,\Omega} + \delta_P$, this shows that, for any such φ,

$$\int_M \varphi\,\mu \geq 0\,.$$

This exactly establishes (A.5.1).

q.e.d.

A.6. A refined maximum principle

In this section, we shall prove the characterization of equilibrium potentials as subharmonic functions on $M\backslash\{P\}$ stated in Remarks 3.1.4, iii). For this, we shall use the following result of independent interest:

Theorem A.6.1. *Let M be a compact connected Riemann surface, and let Ω be an open subset of M such that $M\backslash\Omega$ is not polar. Let E be a polar subset of $\partial\,\Omega$ and $u : \Omega \to [-\infty, +\infty[$ a bounded above subharmonic function.*

If, for every point $P \in \partial\,\Omega\backslash E$, we have:

(A.6.1) $$\text{fine}\limsup_{\substack{M\in\Omega\\ M\to P}} u(M) \leq 0\,,$$

then $u \leq 0$ on Ω.

When $M = \mathbb{P}^1(\mathbb{C})$, this is proved by Brelot ([Br], Lemme 1, p. 301). Let us explain how to extend his result to the present setting. Observe that we get a weaker version of Theorem A.6.1 by replacing condition (A.6.1) by

(A.6.1′) $$\limsup_{\substack{M\in\Omega\\ M\to P}} u(M) \leq 0\,.$$

Let us call it Theorem A.6.1′. When E is empty, Theorem A.6.1′ is nothing else than the well known maximum principle for subharmonic function (which holds as soon as M is connected and $M\backslash\Omega$ not empty). The proof of Theorem A.6.1 proceeds in two steps:

- deduce Theorem A.6.1′ for an arbitrary E from Theorem A.6.1′ for $E = \emptyset$;
- deduce Theorem A.6.1 from Theorem A.6.1′.

The second step is a consequence of [Br], Théorème 1, which shows that for any regular point Q of $\partial\,\Omega$, and any polar subset E in $\partial\,\Omega$ containing the thin points of $\partial\,\Omega$ in some neighbourhood of Q:

$$\limsup_{\substack{M\to Q\\ M\in\Omega}} u(M) = \limsup_{\substack{P\to Q\\ P\in\partial\Omega\backslash E}} \left(\text{fine}\limsup_{\substack{M\to P\\ M\in\Omega}} u(M)\right).$$

The first step is classical, once one has enough "global" subharmonic functions taking the value $-\infty$ on E (see for instance [Do], 1.V.7; a special case of Theorem A.6.1′ appears as Theorem III.2.8 in [T]; however the proof seems incomplete). Here we shall use the following proposition:

Proposition A.6.2. *Let M be a compact connected Riemann surface, and E a polar subset of M. Then, for any compact subset K of M with non-empty interior, there exists a subharmonic function*

$$\varphi : M\backslash K \to [-\infty, 0]$$

such that

$$\varphi = -\infty \qquad \textit{on} \qquad E\backslash K. \tag{A.6.2}$$

Let us take it for granted for a while and let us prove Theorem A.6.1′. Let Q be an arbitrary point in Ω, and ε a positive real number. The boundary of Ω is not polar (*cf.* Remark A.3.5), therefore we may find some point P in $\partial\,\Omega\backslash E$; according to (A.6.1′), there exists some neighbourhood K of P in M such that

$$u \leq \varepsilon \qquad \text{on} \qquad K\cap\Omega.$$

Clearly, we may also assume that K is closed and does not contain Q. Consider now a function φ satisfying the conditions in Proposition A.6.2, and for any $t\in\mathbb{R}_+^*$, let

$$u_t := u + t\,\varphi : \Omega\backslash K \to \mathbb{R}.$$

It is a subharmonic function, and by construction, for any $P\in\partial(\Omega\backslash K)$,

$$\limsup_{\substack{M\to P\\ M\in\Omega\backslash K}} u_t(M) \leq \varepsilon.$$

Therefore, by the maximum principle

$$u_t \leq \varepsilon \qquad \text{on} \qquad \Omega\backslash K.$$

If we let t go to zero, we obtain that $u\leq\varepsilon$ nearly everywhere on $\Omega\backslash K$, hence everywhere. This shows that $u(Q)\leq\varepsilon$, and finally that $u(Q)\leq 0$ since ε is arbitrary.

Proof of Proposition A.6.2. When $M = \mathbb{P}^1(\mathbb{C})$, we may assume that ∞ lies in the interior of K. Then $M\backslash K$ is a bounded open subset of $\mathbb{C}$, and the Proposition is classical (see for instance [Do], 1.V.2).

In general, there exists some non constant meromorphic function f on X, whose divisor is supported by K (this follows for instance from Abel's theorem). Then $f : M \to \mathbb{P}^1(\mathbb{C})$ maps $M\backslash K$ onto the complement of some closed neighbourhood K' of ∞ in $\mathbb{P}^1(\mathbb{C})$, and E onto some polar subset E' of $\mathbb{P}^1(\mathbb{C})$. The first part of the proof shows the existence of a subharmonic function

$$\varphi' : \mathbb{P}^1(\mathbb{C})\backslash K' \to [-\infty, 0]$$

such that

$$\varphi' = -\infty \qquad \text{on} \qquad E'\backslash K',$$

and the composite function

$$\varphi := \varphi' \circ f : M\backslash K \to [-\infty, 0]$$

is subharmonic and satisfies (A.6.2).

q.e.d.

Corollary A.6.3. *Let M, Ω, and E be as in Theorem A.6.1. If $h : \Omega \to \mathbb{C}$ is a bounded harmonic function such that, for every point $P \in \partial\Omega\backslash E$, we have*

$$\text{(A.6.3)} \qquad \text{fine} \lim_{\substack{M\in\Omega \\ M\to P}} h(M) = 0,$$

then $h = 0$ on Ω.

Proof. Apply Theorem A.6.1 to $u = \pm h$.

q.e.d.

We may now prove the characterization of equilibrium potentials $g_{D,\Omega}$ of effective divisors by the conditions stated at the end of Remarks 3.1.4, iii). Indeed, if $g_{D,\Omega}$ and $\widetilde{g}_{D,\Omega}$ satisfy these conditions, then $h := \widetilde{g}_{D,\Omega} - g_{D,\Omega}$, *a priori* defined on $M\backslash|D|$, extends by continuity to M, is harmonic on Ω, bounded and fine continuous on M, and vanishes nearly everywhere on $M\backslash\Omega$. In particular, it satisfies (A.6.3) for nearly every P in $\partial\Omega$, and therefore, by Corollary A.6.3, vanishes on Ω. Thus, h vanishes nearly everywhere on M, and therefore everywhere as it is fine continuous.

A.7. Further results about $\mathrm{L}^2_1(\Omega)_0$

This section is devoted to the proofs of two theorems, which have not been used in the article, but are noteworthy consequences of the techniques developed so far.

Theorem A.7.1. *Let M be a compact connected Riemann surface, and Ω an open subset of M such that $M\backslash\Omega$ is not polar. Then any function $f \in \mathrm{L}^2_1(X)$ may be uniquely decomposed as*

$$\text{(A.7.1)} \qquad f = u + h,$$

where $u \in \mathrm{L}^2_1(\Omega)_0$, and where $h \in \mathrm{L}^2_1(X)$ is harmonic on Ω.

Moreover, if f takes its values in $\mathbb{R}_+$ (resp. in the disk $D(0;R)$), then so does h.

Roughly speaking, h is the solution of the Laplace equation on Ω, with boundary values $f_{|\partial\Omega}$ on $\partial\Omega$. Theorem A.7.1 therefore asserts the solvability of (some version of) the Dirichlet problem on Ω.

Proof. Observe that, as shown by the proof of Theorem A.4.1 with $\alpha = 0$, the following two assertions are equivalent, for any function $h \in \mathrm{L}^2_1(X)$:

(A.7.2) *h is harmonic on Ω;*

(A.7.3) *the class of h in the Hilbert space $\mathrm{L}^2_1(X)/\mathbb{C}$ equipped with the Dirichlet form $\langle\cdot,\cdot\rangle_{\mathrm{Dir}}$ is orthogonal to the image of $\mathrm{L}^2_1(\Omega)_0$.*

Therefore, by orthogonal decomposition, any element in $\mathrm{L}^2_1(X)/\mathbb{C}$ may be uniquely written as the sum of the class of some h satisfying (A.7.2) and of some element in the closure of the image of $\mathrm{L}^2_1(\Omega)_0$ in $\mathrm{L}^2_1(X)/\mathbb{C}$. Moreover, as $M\backslash\Omega$ is not polar, the map $\mathrm{L}^2_1(\Omega)_0 \to \mathrm{L}^2_1(X)/\mathbb{C}$ is a homeomorphism onto its image, which is closed (*cf.* Theorem A.3.1 and Corollary A.3.4). This already establishes the existence and the uniqueness of the decomposition (A.7.1).

The interpretation of this decomposition in terms of orthogonal decomposition with respect to the Dirichlet form also shows that, with the notation of Theorem A.7.1, the functional

$$\begin{aligned} \mathrm{L}^2_1(\Omega)_0 &\to \mathbb{R}_+ \\ v &\mapsto \|v - f\|_{\mathrm{Dir}}\,, \end{aligned} \tag{A.7.4}$$

possesses a unique minimum, attained by $v = u$.

If f is real valued, then, by unicity, u and h also are real valued, and u may be characterized as the element of $\mathrm{L}^2_1(\Omega,\mathbb{R})_0$ which minimizes the functional (A.7.4). Together with this variational characterization, the next lemma will show that, if f is non negative, so is h:

Lemma A.7.2. 1) *For any (f,v) in $\mathrm{L}^2_1(X,\mathbb{R})$, $\widetilde{v} := f - (f-v)_+$ belongs to $\mathrm{L}^2_1(X,\mathbb{R})$, and*

$$\|\widetilde{v} - f\|_{\mathrm{Dir}} \leq \|v - f\|_{\mathrm{Dir}}\,. \tag{A.7.5}$$

2) *If moreover $f \geq 0$ and $v \in \mathrm{L}^2_1(\Omega,\mathbb{R})_0$, then $\widetilde{v} \in \mathrm{L}^2_1(\Omega,\mathbb{R})_0$.*

Indeed, this shows that $\widetilde{u} \in \mathrm{L}^2_1(\Omega,\mathbb{R})$ and $\|\widetilde{u} - f\|_{\mathrm{Dir}} \leq \|u - f\|_{\mathrm{Dir}}$, and therefore that u coincides with $\widetilde{u}$, that is $h = f - u$ is non negative. This argument is another instance of the method of Beurling-Deny.

To prove Lemma A.7.2, observe that Proposition A.5.1 already implies assertion 1). In particular, if a sequence (v_n) converges to v in $\mathrm{L}^2_1(X,\mathbb{R})$, then $(\widetilde{v}_n)$ is bounded in $\mathrm{L}^2_1(X,\mathbb{R})$, and therefore, as it converges to $\widetilde{v}$ in $\mathrm{L}^2(M)$, converges weakly to v in $\mathrm{L}^2_1(X,\mathbb{R})$. Accordingly, to prove assertion 2), we may assume that v belongs to $\mathrm{C}^\infty_c(\Omega,\mathbb{R})$; then, on some neighbourhood of $M\backslash\Omega$,

$$(f-v)_+ = f_+ = f\,,$$

and therefore $\widetilde{v}$ has its support in Ω, and *a fortiori* belongs to $\mathrm{L}^2_1(\Omega,\mathbb{R})_0$.

Finally, the fact that the map $f \mapsto h$ sends functions with values in $D(0;R)$ to themselves formally follows from its linearity and from the fact that it sends real valued (resp. non negative) functions to themselves, and preserves constant functions.

q.e.d.

Our last theorem provides a description of the subspace $\mathrm{L}^2_1(\Omega)_0$ of $\mathrm{L}^2_1(X)$ in terms of BLD functions (compare with [De-L], II, Théorème 5.1).

Theorem A.7.3. *Let M be a compact Riemann surface, and Ω any open subset of M. For any* BLD *function f on M, the following conditions are equivalent:*

(i) *f defines an element of $\mathrm{L}^2_1(\Omega)_0$;*

(ii) *f vanishes nearly everywhere on $M\backslash\Omega$.*

Proof. (i) $\Rightarrow$ (ii). If f belongs to $\mathrm{L}^2_1(\Omega)_0$, then by Theorem A.2.3, it is nearly everywhere the pointwise limit of a sequence of function in $\mathrm{C}^\infty_c(\Omega)$, and therefore vanishes nearly everywhere on $M\backslash\Omega$.

(ii) $\Rightarrow$ (i). Let f be as in (ii). To prove (i), we may assume that M connected, that $M\backslash\Omega$ is not polar (otherwise, (i) holds by Theorem A.3.1), and moreover that f is real valued and bounded (by Proposition A.5.2). Let us consider its decomposition

$$f = u + h$$

provided by Theorem A.7.1, in which we may assume that u and h are BLD functions, and $h_{|\Omega}$ is a harmonic function. Then, by the first part of the proof, u vanishes quasi-everywhere on $M\backslash\Omega$. Therefore, the BLD function $h = f - u$ vanishes nearly everywhere on $M\backslash\Omega$; as it is fine continuous nearly everywhere on M, this implies that it satisfies condition (A.6.3) for nearly every $P \in \partial\Omega$. As h is harmonic and bounded on Ω, Corollary A.6.3 shows that $h_{|\Omega} = 0$. This proves that f coincides with u nearly everywhere on M, and therefore defines an element of $\mathrm{L}^2_1(\Omega)_0$.

q.e.d.

Remark A.7.4. A similar argument shows that, in the decomposition (A.7.1), the restriction of h to any connected component Ω_0 of Ω depends only the restriction of f to $\partial\Omega_0$.

A.8. Continuity properties of equilibrium potentials

We conclude this Appendix with a discussion of the continuity properties of the equilibrium potentials. Let us use the notation of Theorem 3.1. As we already observed in 3.1.4, ii), the equilibrium potential $g_{P,\Omega}$ is continuous outside P and the exceptional subset E of $\partial\Omega$.

A characterization of the continuity of $g_{P,\Omega}$ at some point Q of $\partial\Omega$ unavoidably involves the concepts of *regularity* and *thinness*. We refer for instance to [Ra], Chapters 3 and 4, for basic results about these (Ransford works over $\mathbb{P}^1(\mathbb{C})$; however regularity and thinness are local notions, and most results immediately extend to the present setting). Let us only recall that, if U is an open subset of M, a point Q of ∂U is called a *regular boundary point of U* if there exists a *barrier for U at Q*, namely a subharmonic function b defined on $U \cap N$, where N is an open neighbourhood of P, satisfying $b < 0$ on $U \cap N$ and $\lim_{z \to P} b(z) = 0$. A subset S of M is *non-thin* at some $Q \in M$ if $Q \in \overline{S\backslash\{Q\}}$, and if, for every subharmonic function u defined on a neighbourhood of Q, $\lim_{\substack{z \in S\backslash\{Q\} \\ z \to Q}} \sup u(z) = u(Q)$.

As before, let us denote by Ω_0 the connected component of Ω containing P. If $Q \in \partial\Omega \backslash \partial\Omega_0$, then $g_{P,\Omega}$ vanishes on some neighbourhood of Q, and *a fortiori* is continuous at Q. Moreover, if $Q \in \partial\Omega_0$, the following conditions are equivalent:

R1. $g_{P,\Omega}(Q) = 0$;

R2. $g_{P,\Omega}$ *is continuous and vanishes at* Q;

R3. $\lim_{\substack{Q \in \Omega_0 \\ P \to Q}} g_{P,\Omega}(Q) = 0$;

R4. Q *is a regular boundary point of* Ω_0;

R5. $M \backslash \Omega_0$ *is non-thin at* Q;

and so are the following ones:

S1. $g_{P,\Omega}$ *is continuous and does not vanish at* Q*;*

S2. *There exists an open neighbourhood* U *of* Q *in* M *such that* $U \backslash \Omega$ *(or equivalently* $U \cap \partial\Omega_0$*) is polar, in which case* $g_{P,\Omega} = g_{P,\Omega \cup U}$.

Indeed, we have already observed that **R1** $\Rightarrow$ **R2**. The implication **R2** $\Rightarrow$ **R3** is clear. When **R3** holds, $g_{P,\Omega}$ is a barrier for Ω_0 at Q, which therefore is a regular boundary point of Ω_0. The implication **R4** $\Rightarrow$ **R5** is classical (see for instance, [Ra], Theorem 4.2.4). Finally, when $M \backslash \Omega_0$ is non-thin at Q, then $M \backslash (\Omega_0 \cup E)$ is non-thin at Q (indeed E is a F_σ polar set, hence thin at Q – see [Ra], Theorem 3.8.2 – and the union of two sets thin at Q is also thin at Q); as $g_{P,\Omega}$ is subharmonic on some neighbourhood of Q and vanishes on $M \backslash (\Omega_0 \cup E)$, this implies **R1**.

Moreover if **S2** holds, then $g_{P,\Omega}$ is bounded on some neighbourhood of the polar subset $F := U \backslash (\Omega \cup \{P\})$ in $U \backslash \{P\}$, and harmonic on its complement $(U \backslash \{P\}) \cap \Omega$ in $U \backslash \{P\}$, and therefore $g_{P,\Omega|(U \backslash \{P\}) \cap \Omega}$ extends to some harmonic function on $U \backslash \{P\}$ (indeed bounded harmonic functions "do not see" polar subsets; *cf.* for instance [Ra], Corollary 3.6.2); as $U \backslash \Omega$ is negligible, this shows that $g_{P,\Omega}$ is harmonic on $U \backslash \{P\}$, and therefore coincides with $g_{P,\Omega \cup U}$. In particular, it is continuous at Q. Moreover since Q belongs to the connected component of P in $\Omega \cup U$, this implies that

$$g_{P,\Omega}(Q) = g_{P,\Omega \cup U}(Q) > 0 .$$

Conversely, we have **S1** $\Rightarrow$ **S2** by property (iv) in Theorem 3.1. Finally, the fact that in **S2** we may replace "$U \backslash \Omega$ is polar" by "$U \cap \partial\Omega_0$ is polar" follows from the equality

$$g_{P,\Omega} = g_{P,M \backslash \partial\Omega_0} ;$$

indeed both of these Green functions coincides with g_{P,Ω_0} by the **3.1.4**, i).

In "concrete" geometric situations, conditions **R1-5** are often satisfied by all points Q in $\partial\Omega_0$, in which case $g_{P,\Omega}$ is continuous on $M \backslash \{P\}$ and vanishes on $M \backslash \Omega_0$. This happens for instance when one of the following conditions hold:

R′1. Ω_0 *is simply connected*;

R′2. *The compact* $M \backslash \Omega_0$ *has no isolated point and is locally connected.*

Indeed, when **R′1** holds, Ω_0 is biholomorphic to the unit disk, and the continuity of $g_{P,\Omega} = g_{P,\Omega_0}$ follows from Proposition 3.3 . Moreover, condition **R′2** implies that $M \backslash \Omega_0$ is non-thin at any of its points, by [Ra], Theorem 3.8.3.

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J.B. BOST
Département de Mathématiques,
Bâtiment 425, Université Paris-Sud,
91405 Orsay Cedex, France.