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TYPES AND HECKE ALGEBRAS FOR PRINCIPAL SERIES REPRESENTATIONS OF SPLIT REDUCTIVE P-ADIC GROUPS

BY ALAN ROCHE

ABSTRACT. – We construct types in the sense of Bushnell and Kutzko for principal series representations of split connected reductive p-adic groups (with mild restrictions on the residual characteristic) and describe the resulting Hecke algebras. We discuss their interpretation as Iwahori Hecke algebras of related reductive groups (in general disconnected). In addition, we describe how (parabolic) induction and (Jacquet) restriction functors and questions about square-integrability can be transferred to this context. © Elsevier, Paris

RÉSUMÉ. – On construit des types au sens de Bushnell et Kutzko pour les représentations des séries principales des groupes déployés connexes réductifs p-adiques (avec de légères restrictions sur la caractéristique résiduelle) et on décrit les algèbres de Hecke qui en résultent. On discute comment ces algèbres peuvent être interprétées comme des algèbres de Iwahori-Hecke relatives à des groupes réductifs reliés (en général non-connexes). De plus, on décrit comment les foncteurs d'induction et de restriction et les questions relatives aux représentations de carré intégrable peuvent être considérés dans ce contexte. © Elsevier, Paris

Contents

1. Introduction

Let F be a non-Archimedean local field and G the group of F-rational points of a connected reductive algebraic group $\mathbb{G}$ defined over F. Let $\mathfrak{R}(G)$ denote the category of smooth complex representations of G. The Bernstein decomposition expresses the abelian category $\mathfrak{R}(G)$ as a direct product

$$\mathfrak{R}(G) = \prod_{\mathfrak{s} \in \mathfrak{B}(G)} \mathfrak{R}_{\mathfrak{s}}(G)$$

of full subcategories $\mathfrak{R}_{\mathfrak{s}}(G)$. The objects of $\mathfrak{R}_{\mathfrak{s}}(G)$ are all $(\pi, \mathcal{V}) \in \mathfrak{R}(G)$ such that all irreducible subquotients of π have fixed supercuspidal support modulo unramified twists. Thus the indexing set $\mathfrak{B}(G)$ consists of irreducible supercuspidal representations of Levi subgroups of G (up to equivalence) modulo G-conjugation and twisting by unramified characters. Following [13], we call $\mathfrak{B}(G)$ the Bernstein spectrum of G and refer to the subcategories $\mathfrak{R}_{\mathfrak{s}}(G)$ for $\mathfrak{s} \in \mathfrak{B}(G)$ as the components of the Bernstein decomposition.

If $\mathbb{G}$ is quasi-split (over F), then a minimal Levi subgroup of G is (the group of F-rational points of) a torus. Thus an irreducible supercuspidal representation of a minimal Levi subgroup is in this case a smooth character. We refer to the resulting subcategories $\mathfrak{R}_{\mathfrak{s}}(G)$ of $\mathfrak{R}(G)$ as the principal series components of the Bernstein decomposition (and the corresponding elements of the Bernstein spectrum as principal series elements).

In this paper we study the principal series components of $\mathfrak{R}(G)$ via compact open data under the assumption that $\mathbb{G}$ is split (over F). Fix such a component $\mathfrak{R}_{\mathfrak{s}}(G)$. When F has characteristic zero and certain mild restrictions are placed on the residual characteristic, we explicitly describe an $\mathfrak{s}$-type in G in the sense of Bushnell and Kutzko [13]. We also describe the structure of the resulting Hecke algebras. (See the list preceding Theorem 4.15 and Remark 4.14 for a precise statement of the restrictions on residual characteristic and related comments. If each irreducible factor of $\mathbb{G}$ is of type B, C or D, the only excluded residual characteristic is two.) By very slightly modifying the methods of section 4, the results of this paper may also be seen to hold when F has positive characteristic while keeping the same restrictions on residual characteristic (see [2]).

These results may be rephrased as follows. Fix a split maximal torus $\mathbb{T}$ and a Borel subgroup $\mathbb{B}$ containing $\mathbb{T}$ and write T and B for the corresponding groups of F-rational points. Let ${}^0T$ be the unique maximal compact subgroup of T and $\chi : {}^0T \to \mathbb{C}^\times$ be a smooth character. We construct a pair $(J, \rho) = (J_\chi, \rho_\chi)$ where J is a compact open subgroup of G and ρ is a smooth character of J (satisfying, in particular, $J \cap T = {}^0T$ and $\rho \mid J \cap T = \chi$). We show that a smooth irreducible representation $(\pi, \mathcal{V})$ of G contains ρ (on restriction to J) if and only if π is equivalent to a G-subquotient of $\mathrm{Ind}_B^G(\widetilde{\chi})$ where $\widetilde{\chi}$ is a character of T extending χ. This is precisely the statement that (J, ρ) is an $\mathfrak{s}_\chi$-type where $\mathfrak{s}_\chi$ is the principal series element of $\mathfrak{B}(G)$ canonically determined by χ. We write the resulting component of the Bernstein decomposition as $\mathfrak{R}_\chi(G)$ (i.e. $\mathfrak{R}_\chi(G) = \mathfrak{R}_{\mathfrak{s}_\chi}(G)$).

The pair (J, ρ) gives rise to an idempotent e_ρ in $\mathcal{H}(G)$, the convolution algebra of compactly supported, locally constant functions $\Phi : G \to \mathbb{C}$. If $(\pi, \mathcal{V})$ is a smooth representation of G, then $\pi(e_\rho)\mathcal{V} = e_\rho \mathcal{V}$ is the space $\mathcal{V}^\rho$ of ρ-isotypic vectors in $\mathcal{V}$ (i.e. the sum of all J-subspaces of $\mathcal{V}$ isomorphic to ρ). It is clear that $\mathcal{V}^\rho$ is a module over the subalgebra $e_\rho \mathcal{H}(G) e_\rho$ of $\mathcal{H}(G)$. The process $\mathcal{V} \mapsto \mathcal{V}^\rho$ defines a functor from $\mathfrak{R}(G)$ to $e_\rho \mathcal{H}(G) e_\rho - \mathfrak{Mod}$ (the category of left $e_\rho \mathcal{H}(G) e_\rho$-modules). From the general theory developed in Bushnell and Kutzko [13], the restriction of this functor to $\mathfrak{R}_\chi(G)$ is an equivalence of categories. Further let $\mathfrak{R}_\rho(G)$ denote the full subcategory of $\mathfrak{R}(G)$ consisting of all smooth representations of G generated by their ρ-isotypic subspaces. Then (again from [13]) the categories $\mathfrak{R}_\chi(G)$ and $\mathfrak{R}_\rho(G)$ are equal (as subcategories of $\mathfrak{R}(G)$). (In fact we rederive these results here by observing that (J, ρ) is an $\mathfrak{s}_\chi$-type in G if and only if the smooth compactly induced representation $\operatorname{ind}\rho$ is a (finitely generated, projective) generator of the category $\mathfrak{R}_\chi(G)$. We then invoke a general criterion ([3] chap. 2, thm. 1.5) giving conditions under which a category admitting a projective generator is equivalent to a module category (over the endomorphism ring of the projective generator)). When χ is trivial, J is an Iwahori subgroup of G and ρ is the trivial character. These results therefore generalise the well-known results of Borel [6] and Casselman [15] relating the unramified principal series to representations generated by their Iwahori-fixed vectors.

Since ρ is a character, the algebra $e_\rho \mathcal{H}(G) e_\rho$ is simply $\mathcal{H}(G, \rho)$ (the convolution algebra of compactly supported, ρ^{-1}-spherical functions on G). We show there exists a family of isomorphisms from $\mathcal{H}(G, \rho)$ to an algebra $\mathcal{H}_\chi$ where $\mathcal{H}_\chi$ is a certain affine Hecke algebra twisted by a certain complex group algebra (both defined in terms of χ). When χ is trivial, this reduces to the Iwahori-Matsumoto description of the Iwahori Hecke algebra. More generally when χ factors through ${}^0T \to \mathbb{T}(k_F)$ (where k_F is the residue field of F), J is again an Iwahori subgroup and a description of the Hecke algebra $\mathcal{H}(G, \rho)$ in these terms has already been given by Goldstein in [17] and (as a very special case of the work of) Morris in [27].

We also construct a split reductive group $\widetilde{H}$ (in general disconnected) such that the Hecke algebras $\mathcal{H}(G, \rho)$ and $\mathcal{H}(\widetilde{H}, 1_\mathcal{I})$ are isomorphic via a family of support-preserving isomorphisms. Here $1_\mathcal{I}$ is the trivial character of an Iwahori subgroup of H (the identity component of $\widetilde{H}$). The groups H and $\widetilde{H}$ admit natural interpretations in terms of Langlands dual groups. (I am grateful to Neil Chriss and Allen Moy for suggesting versions of this.) In particular, H is an endoscopic group of G. We discuss this in Section 8 and show that the quotient $\widetilde{H}/H$ is always abelian and is trivial when G (more properly $\mathbb{G}$) has connected centre. Combining Kazhdan-Lusztig [24] with some Mackey theory arguments, we see that the simple $\mathcal{H}(G, \rho)$-modules (and thus the irreducible objects in $\mathfrak{R}_\chi(G)$) are essentially classified. (The group H however need not have connected centre even if G has.) The dual group interpretation of $\mathcal{H}(G, \rho)$ may be used to naturally attach Langlands parameters to principal series representations via the corresponding objects for representations of H having a non-trivial Iwahori-fixed vector (which are known [24, 26] at least when H has connected centre). We will write up the details and some consequences elsewhere.

The character χ also canonically determines a component $\mathfrak{R}_\chi(T)$ of the Bernstein decomposition of T. This is the full subcategory of $\mathfrak{R}(T)$ consisting of all smooth representations of T whose restriction to ${}^0T$ is a multiple of χ. The pair $({}^0T, \chi)$ is clearly a type for $\mathfrak{R}_\chi(T)$. In particular, the functor $\mathcal{W} \mapsto \mathcal{W}^\chi : \mathfrak{R}_\chi(T) \to \mathcal{H}(T, \chi) - \mathfrak{Mod}$ is an equivalence of categories. The functors of (normalised and unnormalised) parabolic induction from $\mathfrak{R}(T)$ to $\mathfrak{R}(G)$ (with respect to B) restrict to $\mathfrak{R}_\chi(T)$ to yield corresponding functors from $\mathfrak{R}_\chi(T)$ to $\mathfrak{R}_\chi(G)$. We show in section 9 that these functors correspond via the equivalences of categories $\mathfrak{R}_\chi(T) \simeq \mathcal{H}(T, \chi) - \mathfrak{Mod}$ and $\mathfrak{R}_\chi(G) \simeq \mathcal{H}(G, \rho) - \mathfrak{Mod}$ to 'algebraic' induction functors between $\mathcal{H}(T, \chi) - \mathfrak{Mod}$ and $\mathcal{H}(G, \rho) - \mathfrak{Mod}$ induced by appropriate twists of an algebra embedding $t_B : \mathcal{H}(T, \chi) \to \mathcal{H}(G, \rho)$. Given our knowledge of the structure of the Hecke algebra $\mathcal{H}(G,\rho)$, these results are special cases of (or immediate consequences of) a general result of Bushnell and Kutzko [13].

In addition, we show that the isomorphism of Hecke algebras between $\mathcal{H}(G,\rho)$ and $\mathcal{H}(\widetilde{H},1_{\mathcal{I}})$ can be chosen to respect this extra structure. In particular, if ϖ is a fixed uniformiser in F and Y denotes the cocharacter lattice of T then the homomorphism $y \mapsto y(\varpi) : Y \to T$ splits the short exact sequence

$$1 \longrightarrow {}^0T \longrightarrow T \longrightarrow Y \longrightarrow 0.$$

Via this splitting we may view χ as a character of T (trivial on $y(\varpi)$ for $y \in Y$). The group H also has T as a maximal torus and the Borel subgroup B in G determines a Borel subgroup $B(H)$ of H (containing T). We show there is an equivalence of categories between $\mathfrak{R}_\chi(G) = \mathfrak{R}_\rho(G)$ and $\mathfrak{R}_{1_{\mathcal{I}}}(\widetilde{H})$ (induced by an appropriately chosen algebra isomorphism between $\mathcal{H}(G,\rho)$ and $\mathcal{H}(\widetilde{H},1_{\mathcal{I}})$) under which the normalised induced representation $i_B^G(\chi\nu)$ corresponds to $i_{B(H)}^{\widetilde{H}}(\nu)$. Here ν is any unramified character of T. This reduces questions concerning reducibility and composition series of general principal series representations to the unramified case plus some simple Mackey theory (taking account of the abelian component group). In the unramified case much information is available (see Reeder [30] and the references therein).

The Hecke algebras we consider carry canonical inner products which are preserved by our Hecke algebra isomorphisms. This has consequences for square-integrability and formal degrees (more generally Plancherel measure). Following a standard abuse of terminology, square-integrable here actually means square-integrable-mod-centre. We borrow an argument from [10] to show that the equivalence of categories between $\mathfrak{R}_\rho(G)$ and $\mathfrak{R}_{1_{\mathcal{I}}}(\widetilde{H})$ (induced by a norm-preserving algebra isomorphism between $\mathcal{H}(G,\rho)$ and $\mathcal{H}(\widetilde{H},1_{\mathcal{I}})$) preserves square-integrability and formal degrees (given natural choices of Haar measures on G/Z_G and $\widetilde{H}/Z_{\widetilde{H}}$). As a trivial consequence of these calculations, we note that the category $\mathfrak{R}_\chi(G) = \mathfrak{R}_\rho(G)$ contains (non-zero) square-integrable representations if and only if the endoscopic group H of G is elliptic.

The idea of studying the entire smooth dual of a p-adic group by the method of restriction to compact open subgroups was pioneered by Howe. In particular, an early paper of his [20] discusses the principal series of $GL_N(F)$ in terms of compact open data (via a generalisation of the Satake isomorphism). Bushnell and Kutzko have described the entire smooth dual of $GL_N(F)$ in terms of types [10] [11]. For general reductive groups a similar description of the level-zero situation is given in Morris [28] (cf. also Moy and Prasad [29]). Some special cases (of principal series types) for $SL_N(F)$ were also treated in Sanje-Mpacko [31].

Our presentation is arranged as follows. In section 3, we construct the compact open subgroup $J = J_\chi$ and the smooth character $\rho = \rho_\chi$. We also prove that J has an Iwahori decomposition with respect to any Borel subgroup containing T and factors as ${}^0T$ times the product of the affine root groups it contains (in any order). In section 4, we compute the G-intertwining of ρ (equivalently the set of J double cosets which support a non-zero element in $\mathcal{H}(G,\rho)$). The key step adapts an argument from Howe and Moy ([19] Lemma 4.4) to our situation. Section 5 contains a slight modification of a theorem in [13] (which we use in showing that various Hecke algebra isomorphisms preserve inner products). In section 6 we use this (along with known results in the level-zero situation) to describe the structure of $\mathcal{H}(G,\rho)$. In section 7 we recall the definition of a type and some consequences and show that (J_χ, ρ_χ) is an $\mathfrak{s}_\chi$-type. We define the groups H and $\widetilde{H}$ in section 8 and show that $\mathcal{H}(G,\rho)$ is isomorphic to the Iwahori Hecke algebra of $\widetilde{H}$. We also discuss their interpretation in terms of Langlands parameters for T and give some examples. Section 9 discusses the connection with parabolic induction. Finally, section 10 contains the calculations on square-integrability described above.

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2. Notation and some preliminaries

We continue with notation already introduced. Thus $\mathbb{G}$ is a split connected reductive algebraic group defined over F with F-split maximal torus $\mathbb{T}$ where F is a non-Archimedean local field. We write $\mathcal{O}_F$ for the valuation ring in F, $\mathcal{P}_F$ for the unique prime ideal in $\mathcal{O}_F$ and k_F for the residue field of F. Replacing $\mathbb{G}$ if necessary by an F-isomorphic group, we may (and do) assume that $\mathbb{G}$ and $\mathbb{T}$ are defined and split over $\mathbb{Z}$. Then $\mathbb{N} = \mathbb{N}_{\mathbb{G}}(\mathbb{T})$ (the normaliser of $\mathbb{T}$ in $\mathbb{G}$) is also defined over $\mathbb{Z}$. Put ${}^0T = \mathbb{T}(\mathcal{O}_F)$. It is the unique maximal compact subgroup of T. If $X = \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ is the lattice of rational characters of $\mathbb{T}$, then ${}^0T = \{t \in T \mid \mathrm{val}_F(x(t)) = 0 \ (x \in X)\}$. We write $\Phi = \Phi(\mathbb{G}, \mathbb{T})$ for the roots of $\mathbb{G}$ with respect to $\mathbb{T}$ and $\Phi^\vee = \Phi^\vee(\mathbb{G}, \mathbb{T})$ for the corresponding coroots. We fix a positive system Φ^+ in Φ and write Π for the unique simple system contained in Φ^+. We often write $\mathbb{B} = \mathbb{T}\mathbb{U}$ for the corresponding Borel subgroup (with unipotent radical $\mathbb{U}$) and $\overline{\mathbb{B}} = \mathbb{T}\overline{\mathbb{U}}$ for the opposite Borel subgroup (with unipotent radical $\overline{\mathbb{U}}$). Write G, T, N, B, U etc.. for the corresponding groups of F-rational points. We may (and do) fix root group isomorphisms $x_\alpha : \mathbb{G}_a \to \mathbb{U}_\alpha$ for $\alpha \in \Phi$ such that the following hold:

i) There exists a $\mathbb{Z}$-homomorphism $\phi_\alpha : \mathbb{SL}_2 \to \mathbb{G}$ satisfying $x_\alpha(u) = \phi_\alpha\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ and $x_{-\alpha}(u) = \phi_{-\alpha}\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$. Furthermore, $\alpha^\vee(t) = \phi_\alpha\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$.

ii) There exist integers $C_{\alpha,\beta;i,j}$ (α, $\beta \in \Phi$, $\beta \neq -\alpha$, $i\alpha + j\beta \in \Phi$) such that the commutator of $x_\alpha(u)$ and $x_\beta(v)$ is given by:

$$[x_\alpha(u), x_\beta(v)] = \prod_{i,j>0,\, i\alpha+j\beta\in\Phi} x_{i\alpha+j\beta}(C_{\alpha,\beta;i,j}u^i v^j)$$

where the roots in the product are listed in some fixed order.

iii) If $X_\alpha = dx_\alpha(1) \in \mathrm{Lie}(\mathbb{G})$ and $H_\alpha = [X_\alpha, X_{-\alpha}] \in \mathrm{Lie}(\mathbb{T})$ for $\alpha \in \Phi$, then the following equations hold:

$$\begin{aligned} [X_\alpha, X_\beta] &= C_{\alpha,\beta;1,1} X_{\alpha+\beta} && \text{if } \alpha+\beta \in \Phi \\ &= 0 && \text{if } \alpha+\beta \notin \Phi \cup \{0\}, \end{aligned} \tag{2.1}$$

$$(2.2)\qquad \mathrm{Ad}\, x_\alpha(u).X_\beta = \sum_{j\geq 0, \beta+j\alpha\in\Phi} C_{\alpha,\beta;j,1} u^j X_{\beta+j\alpha} \quad (\alpha,\, \beta\in\Phi,\, \beta\neq -\alpha),$$

$$(2.3)\qquad \mathrm{Ad}\, x_\alpha(u).X_{-\alpha} = X_{-\alpha} + uH_\alpha - u^2 X_\alpha,$$

$$(2.4)\qquad \mathrm{Ad}\, x_\alpha(u).H = H - \alpha(H)uX_\alpha \quad (H\in \mathrm{Lie}(\mathbb{T})).$$

We write $\mathfrak{g} = \mathrm{Lie}(\mathbb{G})(F)$ and $\mathfrak{t} = \mathrm{Lie}(\mathbb{T})(F)$. Finally, we write $U_{\alpha,k} = x_\alpha(\mathcal{P}_F^k)$ for $k \in \mathbb{Z}$.

3. The pair (J_χ, ρ_χ)

Let $\chi : {}^0T \to \mathbb{C}^\times$ be a smooth complex character. In this section we construct a compact open subgroup J_χ of G and a smooth irreducible representation (in fact, a smooth character) ρ_χ of J_χ. The subgroup J_χ contains ${}^0T$ and the character ρ_χ extends χ. We also establish some properties of J_χ (in particular Iwahori decompositions and root group factorizations) for use in later sections. Our proof of these properties forces some restrictions on the characteristic of the residue field k_F of F. More precisely, we assume $\mathrm{char}\, k_F$ is prime to any integers which occur as ratios of squares of root lengths for pairs of roots in the same irreducible component of Φ (i.e. $\mathrm{char}\, k_F$ is prime to any positive number of bonds connecting nodes in the Dynkin diagram of Φ). Thus $\mathrm{char}\, k_F \neq 2$ (resp. $\mathrm{char}\, k_F \neq 3$) if Φ has factors of type B_n, C_n or F_4 (resp. G_2). We begin by recalling, in a very special case, some results of Bruhat and Tits.

Suppose a function $f : \Phi \to \mathbb{Z}$ satisfies:

$$\begin{aligned} f(\alpha+\beta) &\leq f(\alpha)+f(\beta) && \text{if } \alpha,\beta,\alpha+\beta\in\Phi,\\ f(\alpha)+f(-\alpha) &\geq 1 && \text{for } \alpha\in\Phi. \end{aligned}$$

In particular, f is concave in the sense of Bruhat and Tits (see [8], 6.4.3 and 6.4.5). Let $U_f = \langle U_{\alpha,f(\alpha)} : \alpha \in \Phi\rangle$ and $J_f = \langle {}^0T, U_f\rangle$. Note $J_f = {}^0T\, U_f = U_f\, {}^0T$. Further, let $U_f^+ = U \cap U_f$, $U_f^- = U^- \cap U_f$, $U_{f,\alpha} = U_f \cap U_\alpha$ (for $\alpha \in \Phi$) and $U_f^{(\alpha)} = \langle U_{\alpha,f(\alpha)}, U_{-\alpha,f(-\alpha)}\rangle$ (for $\alpha\in\Phi$).

LEMMA 3.1. – *With notation as above:*

$$U_f^{(\alpha)} = U_{-\alpha,f(-\alpha)}\, \alpha^\vee(1+\mathcal{P}^{f(\alpha)+f(-\alpha)})\, U_{\alpha,f(\alpha)}$$

for each $\alpha\in\Phi$.

Proof. – The homomorphism $\phi_\alpha : SL_2(F) \to \langle U_\alpha, U_{-\alpha}\rangle$ satisfies $\phi_\alpha\begin{pmatrix}1 & \mathcal{P}_F^k\\ 0 & 1\end{pmatrix} = U_{\alpha,k}$ and $\phi_\alpha\begin{pmatrix}1 & 0\\ \mathcal{P}_F^k & 1\end{pmatrix} = U_{-\alpha,k}$ (for $k\in\mathbb{Z}$). By direct calculation,

$$\left\langle \begin{pmatrix}1 & \mathcal{P}_F^{f(\alpha)}\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ \mathcal{P}_F^{f(-\alpha)} & 1\end{pmatrix}\right\rangle = \begin{pmatrix}1 & 0\\ \mathcal{P}_F^{f(-\alpha)} & 1\end{pmatrix} T_{f(\alpha)+f(-\alpha)} \begin{pmatrix}1 & \mathcal{P}_F^{f(\alpha)}\\ 0 & 1\end{pmatrix}$$

where $T_{f(\alpha)+f(-\alpha)} = \left\{\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix} \mid t \in 1+\mathcal{P}_F^{f(\alpha)+f(-\alpha)}\right\}$. Applying ϕ_α, we obtain the Lemma (since $\phi_\alpha\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix} = \alpha^\vee(t)$). □

LEMMA 3.2. – *With notation as above:*

i) $U_{f,\alpha} = U_{\alpha,f(\alpha)}$ *for* $\alpha \in \Phi$.

ii) The product map $\prod_{\alpha\in\Phi^\pm} U_{f,\alpha} \to U_f^\pm$ *is bijective for any ordering of the factors in the product.*

iii) $U_f = U_f^- \, {}^0T_f \, U_f^+$ *where* ${}^0T_f = \prod_{\alpha\in\Phi} \alpha^\vee(1+\mathcal{P}_F^{f(\alpha)+f(-\alpha)})$.

iv) $J_f = U_f^- \, {}^0T \, U_f^+$.

Proof. – Since f is concave i), ii) and iii) follow from the corresponding parts of [8] prop. 6.4.9 provided we show $N_f = N \cap U_f$ coincides with ${}^0T_f$. From [8] prop. 6.4.9 iv), $N_f = \langle N_f^{(\alpha)} : \alpha \in \Phi\rangle$ where $N_f^{(\alpha)} = N \cap U_f^{(\alpha)}$. Lemma 3.1 implies $N \cap U_f^{(\alpha)} = \alpha^\vee(1+\mathcal{P}^{f(\alpha)+f(-\alpha)})$. Hence $N_f = {}^0T_f$.

Part iv) is clear. □

If $\lambda : \mathcal{O}_F^\times \to \mathbb{C}^\times$ is a smooth character, we define the conductor of λ, denoted cond(λ), to be the least integer $n \geq 1$ such that $1+\mathcal{P}_F^n \subseteq \ker(\lambda)$. (This is slightly different than the usual notion (in that the trivial character has conductor one) but more convenient for our purposes.) For each $\alpha \in \Phi$, we may view $\chi \circ \alpha^\vee$ as a smooth character of $\mathcal{O}_F^\times$. We write c_α for cond($\chi \circ \alpha^\vee$).

DEFINITION 3.3. – *Define a function* $f_\chi : \Phi \to \mathbb{Z}$ *as follows:*

$$\begin{aligned} f_\chi(\alpha) &= [c_\alpha/2] && \textit{for } \alpha \in \Phi^+, \\ &= [(c_\alpha+1)/2] && \textit{for } \alpha \in \Phi^-. \end{aligned}$$

Here $[x]$ denotes the largest integer $\leq x$.

LEMMA 3.4. – *Suppose* char $k_F \neq 2$ *(resp.* char $k_F \neq 3$*) if* Φ *has irreducible factors of type* B_n, C_n *or* F_4 *(resp.* G_2*). Then:*

i) $f_\chi(\alpha+\beta) \leq f_\chi(\alpha) + f_\chi(\beta)$ *if* $\alpha, \beta, \alpha+\beta \in \Phi$.

ii) $f_\chi(\alpha) + f_\chi(-\alpha) \geq 1$ *for* $\alpha \in \Phi$.

Proof. – It is clear that ii) holds. To see i), note that by hypothesis char k_F is prime to any integers which occur as ratios of squares of root lengths for pairs of roots in the same irreducible factor of Φ. Hence:

$$p(\alpha+\beta)^\vee = q\alpha^\vee + r\beta^\vee$$

where $p, q, r \in \mathbb{Z}$ are each prime to char k_F. Since $x \mapsto x^s : 1+\mathcal{P}_F^i \to 1+\mathcal{P}_F^i$ $(i \geq 1)$ is bijective for s prime to char k_F, it follows that $c_{\alpha+\beta} \leq \max(c_\alpha, c_\beta)$ Using this observation, a short calculation (which we omit) establishes i). □

We assume from now on that the residual characteristic of F is restricted as in Lemma 3.4. We define J_χ to be J_{f_χ}. Similarly, let $U_\chi = U_{f_\chi}$ and ${}^0T_\chi = {}^0T_{f_\chi}$. From Lemma 3.2, $J_\chi/U_\chi \simeq {}^0T/{}^0T_\chi$. By construction, ${}^0T_\chi \subseteq \ker\chi$. Hence χ defines a character of ${}^0T/{}^0T_\chi$

and so can be lifted to a character ρ_χ of J_χ. As χ is fixed, we often drop the subscripts and write the pair (J_χ, ρ_χ) as (J, ρ).

Example 3.5. – Suppose $\mathbb{G} = \mathbb{SL}_2$. Since ${}^0T \simeq \mathcal{O}_F^\times$, we may view $\chi : {}^0T \to \mathbb{C}^\times$ as a character of $\mathcal{O}_F^\times$. Let $n = \mathrm{cond}(\chi)$. Then

$$J = \begin{pmatrix} \mathcal{O}_F^\times & \mathcal{P}_F^{[n/2]} \\ \mathcal{P}_F^{[(n+1)/2]} & \mathcal{O}_F^\times \end{pmatrix} \cap SL_2(F).$$

Note that χ extends to a character of J which is trivial on $U_{\alpha,[n/2]}$ and on $U_{-\alpha,[(n+1)/2]}$ (where α is the unique positive root). By definition, ρ is this extension.

Let $\Phi^{+'}$ be a system of positive roots in Φ. Then $\Phi^{+'}$ determines a unique Borel subgroup B' containing T. We have $B' = TU'$ where $U' = \prod_{\alpha\in\Phi^{+'}} U_\alpha$. Let $\overline{B}'$ be the opposite Borel so that $\overline{B}' = T\overline{U}'$ where $\overline{U}' = \prod_{\alpha\in\Phi^{-'}} U_\alpha$ (where $\Phi^{-'} = -\Phi^{+'}$). We say J has an Iwahori decomposition with respect to the pair (T, B') or with respect to the positive system $\Phi^{+'}$ if the product map:

$$(J \cap \overline{U}') \times (J \cap T) \times (J \cap U') \to J$$

is a bijection. (It is then automatically a homeomorphism.) We record some properties of (J, ρ) in the next proposition.

PROPOSITION 3.6. – *The pair (J, ρ) has the following properties:*

i) J has Iwahori decompositions with respect to any positive system $\Phi^{+'}$ in Φ. Further, $J \cap \overline{U}' = \prod_{\alpha\in\Phi^{-'}} U_{\alpha,f_\chi(\alpha)}$, $J \cap T = {}^0T$, $J \cap U' = \prod_{\alpha\in\Phi^{+'}} U_{\alpha,f_\chi(\alpha)}$ where U' and $\overline{U}'$ denote the unipotent radicals of the pair of opposite Borel subgroups, B' and $\overline{B}'$, determined by $\Phi^{+'}$.

ii) The character $\rho : J \to \mathbb{C}^\times$ satisfies $\rho(j_-'j_0'j_+') = \chi(j_0')$ where $j_-' \in J \cap \overline{U}'$, $j_0' \in J \cap T = {}^0T$ and $j_+' \in J \cap U'$ for $\overline{U}'$ and U' as in i).

Proof. – Part i) follows from Lemma 3.2 (using the positive system $\Phi^{+'}$ in place of Φ^+). Part ii) is clear from the definition of ρ. □

4. Intertwining

In this section we calculate the G-intertwining of ρ. This serves as an essential first step in our approach to determining the algebra structure of the Hecke algebra $\mathcal{H}(G, \rho)$.

First we recall the relevant definitions. Let τ be a smooth character of a compact open subgroup K of G. An element $g \in G$ is said to intertwine τ if

$$\tau \mid K \cap {}^gK = {}^g\tau \mid K \cap {}^gK$$

where ${}^gK = gKg^{-1}$ and ${}^g\tau(x) = \tau(g^{-1}xg)$ for $x \in {}^gK$. We write $\mathcal{I}_G(\tau)$ for the set of $g \in G$ which intertwine τ. The Hecke algebra $\mathcal{H}(G, \tau)$ is the convolution algebra of compactly supported functions $\phi : G \to \mathbb{C}$ such that $\phi(k_1xk_2) = \tau(k_1)^{-1}\phi(x)\tau(k_2)^{-1}$ for $k_1, k_2 \in K, x \in G$. It is easy to see that an element $g \in G$ intertwines τ if and only if the double coset KgK supports a non-zero function in $\mathcal{H}(G, \tau)$. Such a function (if it exists) is clearly unique up to a non-zero scalar. Thus the collection of double cosets $K\backslash\mathcal{I}_G(\tau)/K$ parametrizes a vector space basis of $\mathcal{H}(G,\tau)$.

We write $\overline{W}$ for the (ordinary) Weyl group of T in G. Thus $\overline{W} \simeq N/T$. We have $N = {}^0NT$ and ${}^0N \cap T = {}^0T$ where ${}^0N = \mathbb{N}(\mathcal{O}_F)$ and ${}^0T = \mathbb{T}(\mathcal{O}_F)$. It follows that $\overline{W}$ is also isomorphic to ${}^0N/{}^0T$. We write W for the extended affine Weyl group of T in G. Thus $W = Y \rtimes \overline{W}$ where $Y = \mathrm{Hom}(\mathbb{G}_m, \mathbb{T})$ is the lattice of rational cocharacters of $\mathbb{T}$. Let $X = \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ be the lattice of rational characters of T (as above) and $\langle\, ,\rangle : X \times Y \to \mathbb{Z}$ the usual perfect pairing defined by $x \circ y(-) = -^{\langle x,y\rangle}$ for $x \in X$, $y \in Y$. We have the following short exact sequence:

$$1 \longrightarrow {}^0T \xrightarrow{\ i\ } T \xrightarrow{H_T} Y \longrightarrow 0 \tag{4.1}$$

where $i : {}^0T \to T$ is the inclusion map and the map $H_T : T \to Y$ is defined by $\langle H_T(t), x\rangle = \mathrm{val}_F(x(t))$ for $x \in X$, $t \in T$. The group ${}^0N$ acts by conjugation on T and ${}^0T$. The subgroup ${}^0T$ acts trivially. We then obtain the usual action of $\overline{W} \simeq {}^0N/{}^0T$ on T and ${}^0T$ and hence on $T/{}^0T$. Of course, $\overline{W}$ also acts on Y and it is easy to check that the isomorphism $T/{}^0T \simeq Y$ induced by H_T is $\overline{W}$-equivariant. Hence ${}^0N/{}^0T \ltimes T/{}^0T \simeq \overline{W} \ltimes Y$, i.e. $N/{}^0T \simeq \overline{W} \ltimes Y = W$.

N acts on the group of smooth complex characters $({}^0T)\widehat{\ }$ of ${}^0T$ by $n\mu(t) = \mu(n^{-1}tn)$ for $n \in N$, $t \in {}^0T$, $\mu \in ({}^0T)\widehat{\ }$. T and hence ${}^0T$ act trivially. Thus the action factors to an action of $\overline{W}$ or W on $({}^0T)\widehat{\ }$. We let $N_\chi = \{n \in N \mid n\chi = \chi\}$, $\overline{W}_\chi = \{w \in \overline{W} \mid w\chi = \chi\}$, $W_\chi = \{w \in W \mid w\chi = \chi\}$. Note $W_\chi = Y \rtimes \overline{W}_\chi$, $N_\chi/{}^0T \simeq W_\chi$, $N_\chi/T \simeq \overline{W}_\chi$.

Proposition 4.1. – *With notation as above:*

$$\mathcal{I}_G(\rho) \cap N = N_\chi$$

Proof. – Suppose $n \in N$ lies in $\mathcal{I}_G(\rho)$. The condition

$${}^n\rho \mid J \cap {}^nJ = \rho \mid J \cap {}^nJ$$

then implies $n\chi \mid {}^0T = \chi \mid {}^0T$, i.e. $n\chi = \chi$ or $n \in N_\chi$.

Suppose $n \in N_\chi$ maps to w under $N/T \simeq \overline{W}$. Let $j \in J \cap {}^nJ$. We have to show $\rho(j) = \rho(n^{-1}jn)$. Write $j = j_-j_0j_+$ where $j_- \in J \cap U^-$, $j_0 \in {}^0T$ and $j_+ \in J \cap U$. Let $w^{-1}\Phi^+ = \Phi^{+'}$, $w^{-1}\Phi^- = \Phi^{-'}$. Then by Propn. 3.6, J has an Iwahori decomposition with respect to the positive system $\Phi^{+'}$. Let U' and $\overline{U}'$ be the unipotent radicals of the corresponding pair of opposite Borels. The root space decompositions of U, $\overline{U}$, U' and $\overline{U}'$ imply $n^{-1}Un = U'$ and $n^{-1}\overline{U}n = \overline{U}'$. Hence $n^{-1}jn = n^{-1}j_-nn^{-1}j_0nn^{-1}j_+n$ expresses $n^{-1}jn \in J$ as an element of $\overline{U}'TU'$. From $J = (J \cap \overline{U}')(J \cap T)(J \cap U')$ and uniqueness of expression in $\overline{U}'TU'$, we see $n^{-1}j_-n \in J \cap \overline{U}'$ and $n^{-1}j_+n \in J \cap U'$. Since both $J \cap \overline{U}'$ and $J \cap U'$ are contained in $\ker(\rho)$, we obtain:

$$\rho(n^{-1}jn) = \chi(n^{-1}j_0n) = n\chi(j_0) = \chi(j_0) = \rho(j).$$

This proves the proposition. □

Remark 4.2. – Suppose χ is a level-zero character, i.e. χ factors through ${}^0T \to \mathbb{T}(k_F)$. Since J is then the standard Iwahori subgroup of G (with respect to Φ^+), Proposition 4.1

and the Bruhat decomposition imply $\mathcal{I}_G(\rho) = JW_\chi J$. If χ does not factor through ${}^0T \to \mathbb{T}(k_F)$, we say χ has positive level. In the remainder of this section, we show $\mathcal{I}_G(\rho) = JW_\chi J$ also holds when χ has positive level (given $\operatorname{char} F = 0$ and certain restrictions on $\operatorname{char} k_F$ – see the list preceding theorem 4.15 for a precise statement).

The map from $Y \otimes_{\mathbb{Z}} F^\times$ to T induced by $y \otimes \lambda \mapsto y(\lambda)$ for $y \in Y, \lambda \in F^\times$ is an isomorphism of (topological) groups. For $q \in \mathbb{N}$, we write T_q for the image of $Y \otimes (1+\mathcal{P}_F^q)$ under this map. We also have the isomorphism (of topological groups) from $Y \otimes_{\mathbb{Z}} F$ to $\mathfrak{t}$ induced by $y \otimes \lambda \mapsto y(\lambda)$ for $y \in Y, \lambda \in F$. We write $\mathfrak{t}_r$ for the image of $Y \otimes \mathcal{P}_F^r$ for $r \in \mathbb{Z}$. The standard congruence (compact open) subgroups K_q ($q \in \mathbb{N}$) of G and the corresponding $\mathcal{O}_F$-lattices $\mathfrak{K}_r$ ($r \in \mathbb{Z}$) of $\mathfrak{g}$ are defined as follows:

$$K_q = \langle T_q, U_{\alpha,q} : \alpha \in \Phi \rangle \qquad (q \geq 1)$$

$$\mathfrak{K}_r = \mathfrak{t}_r + \sum_{\alpha \in \Phi} \mathcal{P}_F^r X_\alpha \qquad (r \in \mathbb{Z}).$$

Fix a $\mathbb{Z}$-basis $\{y_1, \dots, y_n\}$ of Y. For $q \in \mathbb{N}$, each element of $Y \otimes \mathcal{P}_F^q$ has a unique expression of the form $y_1 \otimes \lambda_1 + \cdots + y_n \otimes \lambda_n$ where $\lambda_1, \dots, \lambda_n \in \mathcal{P}_F^q$. Define a function $\gamma' : Y \otimes \mathcal{P}_F^q \to Y \otimes (1 + \mathcal{P}_F^q)$ by:

$$\gamma'(y_1 \otimes \lambda_1 + \cdots + y_n \otimes \lambda_n) = \prod_{i=1}^{n} y_i \otimes (1 + \lambda_i).$$

This induces a function from $\mathfrak{t}_q$ to T_q which we again denote by γ'. For $q \in \mathbb{N}$ define $\gamma : \mathfrak{K}_q \to K_q$ by:

$$\gamma(a + \sum_{\alpha \in \Phi} u_\alpha X_\alpha) = \gamma'(a) \prod_{\alpha \in \Phi} x_\alpha(u_\alpha) \qquad (a \in \mathfrak{t}_q, u_\alpha \in \mathcal{P}_F^q (\alpha \in \Phi)). \tag{4.2}$$

Here the elements of Φ are listed in some fixed order. It is clear that γ is a bijection of sets. Suppose $s \in \mathbb{N}$ is such that $q \leq s \leq 2q$.Then the bijections $\gamma : \mathfrak{K}_q \to K_q$ and $\gamma : \mathfrak{K}_s \to K_s$ induce group isomorphisms:

$$\mathfrak{K}_q / \mathfrak{K}_s \simeq K_q / K_s \qquad (q \leq s \leq 2q).$$

These isomorphisms are independent of the choice of $\mathbb{Z}$-basis of Y and the listing of roots in Φ.

We now construct a non-degenerate, AdG-invariant, symmetric bilinear form B on $\mathfrak{g}$ given $\operatorname{char} F = 0$ and certain restrictions on the residual characteristic (see 4.5). The form will satisfy $\mathfrak{K}_r^\perp = \mathfrak{K}_{1-r}$ for $r \in \mathbb{Z}$ where $\mathfrak{K}_r^\perp = \{X \in \mathfrak{g} \mid B(X,Y) \in \mathcal{P}_F \, (\forall\ Y \in \mathfrak{K}_r)\}$ (see Lemma 4.3).

We assume thus that $\operatorname{char} F = 0$. Then $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}$ where $\mathfrak{g}'$ is semisimple and $\mathfrak{z} = \mathfrak{z}(\mathfrak{g})$ (i.e. $\mathfrak{z}$ is the center of $\mathfrak{g}$). The Killing form B_1 of $\mathfrak{g}$, defined by $B_1(X,Y) = \operatorname{tr}(\operatorname{ad}X.\operatorname{ad}Y)$ for $X, Y \in \mathfrak{g}$, is an AdG-invariant, symmetric bilinear form on $\mathfrak{g}$ with radical $\mathfrak{z}$.

Let $Q^\vee = \mathbb{Z}\,\Phi^\vee$ be the coroot lattice of G with respect to T. Let

$$X_0^\vee = \{y \in Y \mid \langle y, \alpha \rangle = 0 \ (\forall \alpha \in \Phi)\}.$$

Then $Q^\vee + X_0^\vee$ has finite index in Y (see Springer [32]). Let $V^\vee = Y \otimes \mathbb{Q}$, $V = X \otimes \mathbb{Q}$, $V_1^\vee = Q^\vee \otimes \mathbb{Q}$, $V_0^\vee = X_0^\vee \otimes \mathbb{Q}$. The perfect pairing $\langle\, ,\rangle : X \times Y \to \mathbb{Z}$ extends to a non-degenerate $\mathbb{Q}$-bilinear pairing $\langle\, ,\rangle : V \times V^\vee \to \mathbb{Q}$. We have $V^\vee = V_1^\vee \oplus V_0^\vee$. Let $p_1 : V^\vee \to V_1^\vee$ be projection onto the first factor. The coweight lattice $P^\vee$ is defined by

$$P^\vee = \{v \in V_1^\vee \mid \langle v, \alpha\rangle \in \mathbb{Z}\ (\forall \alpha \in \Phi)\}. \tag{4.3}$$

Then $Q^\vee \subset P^\vee$ and $P^\vee/Q^\vee$ is finite. From the definition of $X_0^\vee$, we see $Y \cap V_0^\vee = X_0^\vee$. It follows that $p_1 : V^\vee \to V_1^\vee$ induces an embedding:

$$Y/(Q^\vee + X_0^\vee) \hookrightarrow P^\vee/Q^\vee. \tag{4.4}$$

We impose the following condition on the characteristic of k_F:

$$\operatorname{char} k_F \quad \text{does not divide} \quad 2\,[P^\vee : Q^\vee] \tag{4.5}$$

Thus if Φ is irreducible the excluded residual characteristics are as follows:

- for type A_n primes dividing $2(n+1)$
- for type B_n, C_n, D_n 2
- for type E_6 2, 3
- for type E_7 2
- for type E_8, F_4, G_2 2.

If Φ is not irreducible, then the excluded primes are those attached to each of its irreducible factors.

From 4.4 and 4.5, we obtain

$$\mathfrak{t}_q = Y \otimes \mathcal{P}_F^q = (Q^\vee + X_0^\vee) \otimes \mathcal{P}_F^q = Q^\vee \otimes \mathcal{P}_F^q \oplus X_0^\vee \otimes \mathcal{P}_F^q \qquad (q \in \mathbb{Z}),$$
$$\mathfrak{t} = Y \otimes F = (Q^\vee + X_0^\vee) \otimes F = Q^\vee \otimes F \oplus X_0^\vee \otimes F.$$

Define a form $g(\, ,) : Q^\vee \times Q^\vee \to \mathbb{Z}$ as follows:

$$g(q_1, q_2) = \sum_{\beta \in \Phi} \langle \beta, q_1\rangle \langle \beta, q_2\rangle \qquad (q_1, q_2 \in Q^\vee).$$

Then $g(\, ,)$ is symmetric, $\mathbb{Z}$-bilinear, positive-definite (hence non-degenerate) and $\overline{W}$-invariant. By extension of scalars, we obtain a (non-degenerate) $\mathcal{O}_F$-bilinear form $g(\, ,) : Q^\vee \otimes \mathcal{O}_F \times Q^\vee \otimes \mathcal{O}_F \to \mathcal{O}_F$ (resp. a (non-degenerate) F-bilinear form $g(\, ,) : Q^\vee \otimes F \times Q^\vee \otimes F \to F$).

The embedding:

$$y \otimes \lambda \mapsto y(\lambda) : Q^\vee \otimes F \hookrightarrow \mathfrak{t} \qquad (y \in Q^\vee, \lambda \in F)$$

has image $\mathfrak{t}' = \mathfrak{t} \cap \mathfrak{g}'$. The element $\alpha^\vee \otimes 1$ is sent to $H_\alpha = [X_\alpha, X_{-\alpha}]$ for $\alpha \in \Phi$. Note that under this map the form $g(\, ,) : Q^\vee \otimes F \times Q^\vee \otimes F \to F$ coincides with $B_1 \mid \mathfrak{t}' \times \mathfrak{t}'$. Indeed $\operatorname{ad} H_\alpha . X_\gamma = \gamma(H_\alpha) X_\gamma = \langle \gamma, \alpha^\vee\rangle X_\gamma$ for α, $\beta \in \Phi$. Hence:

$$B_1(H_\alpha, H_\beta) = \sum_{\gamma \in \Phi} \langle \gamma, \alpha^\vee\rangle \langle \gamma, \beta^\vee\rangle \qquad (\alpha,\, \beta \in \Phi)$$

so that $B_1(H_\alpha, H_\beta) = g(\alpha^\vee, \beta^\vee)$ for α, $\beta \in \Phi$ as required.

Suppose $\Phi^\vee$ has irreducible factors $\Phi_1^\vee, \ldots, \Phi_r^\vee$. Let $g_i(\,,\,) = g(\,,\,) \mid Q_i^\vee$ where $Q_i^\vee = \mathbb{Z}\,\Phi_i^\vee$. Then $g(\,,\,) = \oplus_{i=1}^r g_i(\,,\,)$ (in the obvious notation). Let l_i be the minimum of $g_i(\alpha^\vee, \alpha^\vee)$ as $\alpha^\vee$ ranges through $\Phi_i^\vee$. We define $g'(\,,\,)$ to equal $\oplus_{i=1}^r l_i^{-1} g_i(\,,\,)$. Denote by B' the corresponding form on $\mathfrak{g}'$. Thus B' equals $g'(\,,\,)$ on $\mathfrak{t}' \times \mathfrak{t}'$ and B' coincides with $l_i^{-1} B_1$ on the subalgebra of $\mathfrak{g}'$ of type Φ_i $(i = 1, \ldots, r)$.

If $\beta \in \Phi_i$ and $\alpha^\vee \in \Phi_i^\vee$, we have $\langle \beta, \alpha^\vee \rangle = \frac{2(\alpha^\vee, \beta^\vee)}{(\beta^\vee, \beta^\vee)}$ where $(\,,\,)$ is any inner product on $V_i = Q_i^\vee \otimes \mathbb{Q}$ which is invariant under W_i (the Weyl group of $\Phi_i^\vee$). In particular, $\langle \beta, \alpha^\vee \rangle = \frac{2 g_i(\alpha^\vee, \beta^\vee)}{g_i(\beta^\vee, \beta^\vee)}$. Suppose $g_i(\gamma^\vee, \gamma^\vee) = l_i$ for $\gamma^\vee \in \Phi_i^\vee$. Then

$$\begin{aligned}\langle \beta, \alpha^\vee \rangle &= \frac{2 g_i(\gamma^\vee, \gamma^\vee)}{g_i(\beta^\vee, \beta^\vee)} l_i^{-1} g_i(\alpha^\vee, \beta^\vee) \\ &= \frac{2 g_i(\gamma^\vee, \gamma^\vee)}{g_i(\beta^\vee, \beta^\vee)} g'(\alpha^\vee, \beta^\vee).\end{aligned}$$

The quantity $\frac{g_i(\gamma^\vee, \gamma^\vee)}{g_i(\beta^\vee, \beta^\vee)}$ is a ratio of squares of root lengths in $\Phi_i^\vee$ and thus is a unit in $\mathcal{O}_F$ (see the first paragraph of section 3). In particular, the matrix $(g'(\alpha^\vee, \beta^\vee))$ $(\alpha^\vee, \beta^\vee \in \Pi^\vee)$ can be transformed into the Cartan matrix of the root system $\Phi^\vee$ by multiplying certain columns by elements of $\mathcal{O}_F^\times$. Since the determinant of the Cartan matrix of $\Phi^\vee$ equals $[P^\vee : Q^\vee]$, we have $\det(g'(\alpha^\vee, \beta^\vee)) \in \mathcal{O}_F^\times$. It follows that $g'(\,,\,)$ induces an isomorphism of $\mathcal{O}_F$-modules between $Q^\vee \otimes \mathcal{O}_F$ and $\mathrm{Hom}_{\mathcal{O}_F}(Q^\vee \otimes \mathcal{O}_F, \mathcal{O}_F)$. We use this observation in the proof of the next Lemma.

Fix a $\mathbb{Z}$-basis $\{l_1, \ldots, l_r\}$ of $X_0^\vee$. Define a non-degenerate, symmetric, $\mathbb{Z}$-bilinear form $(\,,\,)' : X_0^\vee \times X_0^\vee \to \mathbb{Z}$ by $(l_i, l_j)' = \delta_{ij}$ for $1 \le i, j \le r$. It extends to a non-degenerate, F-bilinear form $(\,,\,)' : X_0^\vee \otimes F \times X_0^\vee \otimes F \to F$.

Since $\mathfrak{t} = Q^\vee \otimes F \oplus X_0^\vee \otimes F$, $g'(\,,\,) \oplus (\,,\,)'$ defines a non-degenerate, symmetric, bilinear form on $\mathfrak{t} \times \mathfrak{t}$. This extends to the form $B = B' \oplus (\,,\,)'$ on $\mathfrak{g} \times \mathfrak{g}$ (since $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}$ and $\mathfrak{z} = X_0^\vee \otimes F$). Note that B is non-degenerate and $\mathrm{Ad}\,G$- invariant (since $\mathrm{Ad}\,G$ acts trivially on $\mathfrak{z}$ and $B \mid \mathfrak{g}' \times \mathfrak{g}'$ is a multiple of the Killing form on each of the simple factors of the semisimple algebra $\mathfrak{g}'$).

If $\mathfrak{L}$ is an $\mathcal{O}_F$-lattice in $\mathfrak{g}$ (resp. $\mathfrak{t}$), we set:

$$\mathfrak{L}^\perp = \{X \in \mathfrak{g} \text{ (resp. } \mathfrak{t}) \mid B(X, Y) \in \mathcal{P}_F \ \ (\forall Y \in \mathfrak{L})\}$$

LEMMA 4.3. – *With notation as above:*

i) $\mathfrak{t}_q^\perp = \mathfrak{t}_{1-q}$. $\quad (q \in \mathbb{Z})$

ii) $\mathfrak{K}_q^\perp = \mathfrak{K}_{1-q}$. $\quad (q \in \mathbb{Z})$

Proof. – i) It is clear that $\mathfrak{t}_{1-q} \subset \mathfrak{t}_q^\perp$. To see the opposite inclusion, let $a \in \mathfrak{t}_q^\perp$. Write $a = a_1 + z_1$ where $a_1 \in Q^\vee \otimes F$, $z_1 \in X_0^\vee \otimes F$. Then i) $B(a_1, Q^\vee \otimes \mathcal{P}_F^q) \subset \mathcal{P}_F$ and ii) $B(z_1, X_o^\vee \otimes \mathcal{P}_F^q) \subset \mathcal{P}_F$. Clearly ii) implies $z_1 \in X_0^\vee$ (since $B \mid \mathfrak{z} \times \mathfrak{z} = (\,,\,)'$). We have already observed that $g'(\,,\,)$ induces an isomorphism of $\mathcal{O}_F$-modules between $Q^\vee \otimes \mathcal{O}_F$ and $\mathrm{Hom}_{\mathcal{O}_F}(Q^\vee \otimes \mathcal{O}_F, \mathcal{O}_F)$. Hence there exist elements $y_1, \ldots, y_r$ such that $g'(y_i, \alpha_j^\vee) = \delta_{ij}$ where $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_r^\vee\}$. It follows that $F y_1 + \cdots + F y_r = Q^\vee \otimes F$ and $\mathcal{P}_F^s y_1 + \cdots + \mathcal{P}_F^s y_r = Q^\vee \otimes \mathcal{P}_F^s$ for any $s \in \mathbb{Z}$. Write $a_1 = y_1 \otimes \lambda_1 + \cdots + y_r \otimes \lambda_r$ for $\lambda_1, \ldots, \lambda_r \in F$. Then $B(a_1, \alpha_i^\vee \otimes \varpi^q) = \lambda_i \varpi^q$ implies each $\lambda_i \in \mathcal{P}_F^{1-q}$. Hence $a_1 \in Q^\vee \otimes \mathcal{P}_F^{1-q}$.

ii) Since the restriction of B to the subalgebra of $\mathfrak{g}'$ of type Φ_i $(i = 1, \dots, r)$ is a multiple of the Killing form, $B(\mathfrak{t}, \mathfrak{g}_\alpha) = 0$ for all $\alpha \in \Phi$ and $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ for $\alpha, \beta \in \Phi$ unless $\alpha + \beta = 0$. Further, $2B(X_\alpha, X_{-\alpha}) = B(H_\alpha, H_\alpha) = g'(\alpha^\vee, \alpha^\vee)$ is a unit in $\mathcal{O}_F$ (by the construction of g'). It follows easily that $\mathfrak{K}_q^\perp = \mathfrak{K}_{1-q}$. $\square$

Fix a smooth character $\psi : F \to \mathbb{C}^\times$ such that $\mathcal{P}_F \subset \ker\psi$, $\mathcal{O}_F \not\subset \ker\psi$. Let $q, s \in \mathbb{N}$ such that $q \le s \le 2q$. Then $K_q/K_s \simeq \mathfrak{K}_q/\mathfrak{K}_s$ and hence $(K_q/K_s)\widehat{} \simeq (\mathfrak{K}_q/\mathfrak{K}_s)\widehat{}$. The map $X \mapsto \psi_X : \mathfrak{K}_s^\perp/\mathfrak{K}_q^\perp \to (\mathfrak{K}_q/\mathfrak{K}_s)\widehat{}$ defined by

$$\psi_X(Y) = \psi(B(X,Y)) \qquad (\text{for } Y \in \mathfrak{K}_q)$$

is an isomorphism of groups. From Lemma 4.3, we see $\mathfrak{K}_{1-s}/\mathfrak{K}_{1-q} \cong (K_q/K_s)\widehat{}$.

Let l be the least positive integer such that $\chi \mid T_l$ is trivial. Since χ has positive level, $l \ge 2$. Then $J \supset K_{l-1} \supset K_l$. We may view $\rho \mid K_{l-1}$ as an element of $(K_{l-1}/K_l)\widehat{}$. Thus $\rho \mid K_{l-1}$ is parameterized by some element of $\mathfrak{K}_{1-l}/\mathfrak{K}_{2-l}$. It is clear that this element has the form $a + \mathfrak{K}_{2-l}$ for some $a \in \mathfrak{t}_{1-l}$ (since ρ is trivial on $U_{\alpha,l-1}$ for all $\alpha \in \Phi$). For an appropriate choice of $a \in \mathfrak{t}_{1-l}/\mathfrak{t}_{2-l}$, we will eventually show $\mathcal{I}_G(\rho) \subset J\, C_G(a)\, J$ where $C_G(a) = \{g \in G \mid \operatorname{ad} g.a = a\}$.

This is the key step in an inductive argument that determines $\mathcal{I}_G(\rho)$. The precise meaning of appropriate is contained in the following Lemma. The proof requires us to exclude some further residual characteristics. (This introduces additional restrictions only in the case of exceptional groups.) More precisely, we say a prime p is bad for a root system Ψ if $\mathbb{Z}\Psi/\mathbb{Z}\Psi_1$ has p-torsion for some closed subsystem Ψ_1 of Ψ (see Springer-Steinberg [33] 4.3). For irreducible root systems, the bad primes are as follows:

- for type A_n none
- for types B_n, C_n, D_n 2
- for types E_6, E_7, F_4, G_2 2, 3
- for type E_8 2, 3, 5.

Clearly, the bad primes for a root system are the bad primes for its irreducible factors. We assume $\operatorname{char} k_F \ne p$ if p is a bad prime for Φ.

LEMMA 4.4. – *Each coset $\bar{a} \in \mathfrak{t}_{1-l}/\mathfrak{t}_{2-l}$ has a representative a such that $\alpha(a) \equiv 0 \pmod{\mathcal{P}_F^{2-l}}$ for $\alpha \in \Phi$ implies $\alpha(a) = 0$.*

Proof. – Replacing $\bar{a}$ by $\varpi^{l-1}\bar{a}$, we may assume $l = 1$, i.e. $\bar{a} \in \mathfrak{t}_0/\mathfrak{t}_1$. Let $\Phi_{\bar{a}} = \{\alpha \in \Phi : \alpha(a_1) \equiv 0 \pmod{\mathcal{P}}$ for any (for all) $a_1 \in \bar{a}\}$. It is easy to check that $\Phi_{\bar{a}}$ is a closed subsystem of Φ (i.e. if $\alpha, \beta \in \Phi_{\bar{a}}$ then $s_\alpha(\beta) \in \Phi_{\bar{a}}$ and $\mathbb{Z}\Phi_{\bar{a}} \cap \Phi = \Phi_{\bar{a}}$).

Let $Q = \mathbb{Z}\Phi$ and $Q_{\bar{a}} = \mathbb{Z}\Phi_{\bar{a}}$. Since $\operatorname{char} k_F$ is not a bad prime for Φ, the quotient $Q \otimes \mathcal{O}_F / Q_{\bar{a}} \otimes \mathcal{O}_F$ is torsion-free. Therefore $Q_{\bar{a}} \otimes \mathcal{O}_F$ has a complement in $Q \otimes \mathcal{O}_F$. In particular, any $\mathbb{Z}$-basis $\alpha_1, \dots, \alpha_u$ of $Q_{\bar{a}}$ extends to an $\mathcal{O}_F$-basis $\alpha_1, \dots, \alpha_u, \beta_1, \dots, \beta_v$ of $Q \otimes \mathcal{O}_F$.

Let $\Pi = \{\gamma_1, \dots, \gamma_r\}$ be the set of simple roots in Φ^+ so that $r = u + v$. The determinant of the Cartan matrix of Φ is $[P : Q]$ where P is the weight lattice (defined as in 4.3). Since there is a canonical perfect pairing

$$(4.6) \qquad P/Q \times P^\vee/Q^\vee \longrightarrow \mathbb{Q}/\mathbb{Z}$$

we have $[P : Q] = [P^\vee : Q^\vee]$. Thus char k_F is prime to the determinant of the Cartan matrix of Φ so that the map

$$\begin{aligned} \mathfrak{t}'_0 &\longrightarrow \mathcal{O}_F^r \\ t &\mapsto (\gamma_1(t), \dots, \gamma_r(t)) \end{aligned}$$

is an isomorphism of $\mathcal{O}_F$-modules. It follows that the map

$$\begin{aligned} \mathfrak{t}'_0 &\longrightarrow \mathcal{O}_F^r \\ t &\mapsto (\alpha_1(t), \dots, \alpha_u(t), \beta_1(t), \dots, \beta_v(t)) \end{aligned} \tag{4.7}$$

is also an isomorphism of $\mathcal{O}_F$-modules. Under this isomorphism, $\mathfrak{t}'_1$ is identified with $\mathcal{P}_F^r$. For any $a_1 \in \overline{a}$, write $a_1 = a_0 + z$ where $a_0 \in \mathfrak{t}'_0$ and $z \in \mathfrak{z}$. Let a'_0 be the element of $\mathfrak{t}'_0$ corresponding to $(0, \dots, 0, \beta_1(a_0), \dots, \beta_v(a_0))$ under 4.7. Then $a = a'_0 + z \in \overline{a}$ satisfies the required condition. □

We assume from now on that $\rho \mid K_{l-1}$ corresponds to a coset $a + \mathfrak{K}_{2-l} \in \mathfrak{K}_{1-l}/\mathfrak{K}_{2-l}$ for a as in the Lemma. We also assume that $l = 2l'$ is even. The slight modifications needed to deal with the case in which l is odd are indicated below (see Remark 4.9).

Definition 4.5. – *Define a compact open subgroup $L = L_a$ as follows:*

$$L = \langle T_{l-1}, U_{\alpha,l-1}, U_{\beta,l'} : \alpha(a) = 0, \beta(a) \neq 0 \rangle$$

It is clear that L is a subgroup of J. Let $\mathfrak{L}$ be the $\mathcal{O}_F$-lattice in $\mathfrak{g}$ that corresponds to L. Thus:

$$\mathfrak{L} = \mathfrak{t}_{l-1} + \sum_{\alpha(a)=0} \mathcal{P}_F^{l-1} X_\alpha + \sum_{\beta(a)\neq 0} \mathcal{P}_F^{l'} X_\beta$$

Now $\rho \mid L$ factors to L/K_l and as before:

$$(L/K_l)\widehat{\ } \simeq (\mathfrak{L}/\mathfrak{K}_l)\widehat{\ } \simeq \mathfrak{K}_l^\perp/\mathfrak{L}^\perp = \mathfrak{K}_{1-l}/\mathfrak{L}^\perp.$$

Again $\rho \mid L$ is parameterized by $a + \mathfrak{L}^\perp$.

The following lemma is proved in Adler [1].

Lemma 4.6. – *An element $g \in G$ lies in $\mathcal{I}_G(\rho \mid L)$ if and only if*

$$Ad\, g\, (a + \mathfrak{L}^\perp) \cap (a + \mathfrak{L}^\perp) \neq \emptyset \tag{4.8}$$

We sometimes write $\mathcal{I}_G(a + \mathfrak{L}^\perp)$ for the set of $g \in G$ which satisfy 4.8.

Suppose $r \in \mathbb{Z}$ and $q \in \mathbb{N}$. In the proof of the next lemma, we make use of the following identity:

$$\mathrm{Ad}\,\gamma(X).Y \equiv Y + \mathrm{ad}X(Y) \pmod{\mathfrak{K}_{q+r+1}} \qquad (\text{for } X \in \mathfrak{K}_q,\ Y \in \mathfrak{K}_r). \tag{4.9}$$

A short calculation shows that if (4.9) holds for $\gamma(X_1)$ and $\gamma(X_2)$ for $X_1, X_2 \in \mathfrak{K}_q$, then it also holds for $\gamma(X_1)\gamma(X_2)$. Hence it only necessary to check (4.9) for $X \in \mathfrak{t}_q$ and $X = uX_\alpha$ for $\alpha \in \Phi$ and $u \in \mathcal{P}_F^q$. Further since it is $\mathcal{O}_F$-linear in Y, we may assume $Y \in \mathfrak{t}_r$ or $Y = vX_\alpha$ for $\alpha \in \Phi$ and $v \in \mathcal{P}_F^r$. The relation therefore follows from formulas (2.1), (2.2), (2.3) and (2.4).

LEMMA 4.7. – *Let $r \in \mathbb{Z}$ and $q \in \mathbb{N}$. Suppose $a \in \mathfrak{t}_r$ such that $\alpha(a) \neq 0$ for $\alpha \in \Phi$ implies $val_F(\alpha(a)) = r$. Suppose $Y \in a + \mathfrak{K}_{1+r}$ such that $Y \in \mathfrak{z}_\mathfrak{g}(a)$ $(mod\, \mathfrak{K}_{q+r})$ where $\mathfrak{z}_\mathfrak{g}(a) = \{Y \in \mathfrak{g} \mid [a, Y] = 0\}$. Then Y is K_q-conjugate to an element of $\mathfrak{z}_\mathfrak{g}(a)$.*

Proof. – We show Y is K_q-conjugate to an element of $\mathfrak{z}_\mathfrak{g}(a)$ (mod $\mathfrak{K}_{q+r+1}$). An obvious limiting argument then establishes the Lemma.

We have $\mathfrak{z}_\mathfrak{g}(a) = \mathfrak{t} + \sum_{\alpha(a)=0} \mathfrak{g}_\alpha$. Put $\mathfrak{g}(+) = \sum_{\substack{\alpha>0 \\ \alpha(a)\neq 0}} \mathfrak{g}_\alpha$, $\mathfrak{g}(-) = \sum_{\substack{\alpha<0 \\ \alpha(a)\neq 0}} \mathfrak{g}_\alpha$. Then $\mathfrak{g} = \mathfrak{g}(-) \oplus \mathfrak{g}_0 \oplus \mathfrak{g}(+)$ where $\mathfrak{g}_0 = \mathfrak{z}_\mathfrak{g}(a)$. For $s \in \mathbb{Z}$, put $\mathfrak{K}_s(-) = \mathfrak{g}(-) \cap \mathfrak{K}_s$ and $\mathfrak{K}_s(+) = \mathfrak{g}(+) \cap \mathfrak{K}_s$. Write $Y = Y_1 + Y'$ where $Y_1 \in \mathfrak{z}_g(a)$ and $Y' \in \mathfrak{K}_{q+r}(-) + \mathfrak{K}_{q+r}(+)$.

For each $s \in \mathbb{Z}$, ad(a) restricts to give isomorphisms of $\mathcal{O}_F$-modules:

$$\mathrm{ad}\,(a) : \mathfrak{K}_s(+) \xrightarrow{\sim} \mathfrak{K}_{s+r}(+)$$

$$\mathrm{ad}\,(a) : \mathfrak{K}_s(-) \xrightarrow{\sim} \mathfrak{K}_{s+r}(-)$$

Hence we can write $Y' = \mathrm{ad}\,(a)\, Z'$ for some $Z' \in \mathfrak{K}_q(+) + \mathfrak{K}_q(-)$. Then by (4.9):

$$\begin{aligned} \mathrm{ad}\,\gamma(Z').Y &\equiv Y + [Z', Y] && (\mathrm{mod}\ \mathfrak{K}_{q+r+1}) \\ &\equiv Y + [Z', a + W] && (\mathrm{mod}\ \mathfrak{K}_{q+r+1}) \text{ where } W \in \mathfrak{K}_{r+1}) \\ &\equiv Y - [a, Z'] && (\mathrm{mod}\ \mathfrak{K}_{q+r+1}) \\ &\equiv Y_1 + Y' - Y' && (\mathrm{mod}\ \mathfrak{K}_{q+r+1}) \\ &\equiv Y_1 && (\mathrm{mod}\ \mathfrak{K}_{q+r+1}) \end{aligned}$$

□

COROLLARY 4.8. – *For any $Y \in a + \mathfrak{L}^\perp$, there exists an $x \in L$ such that $Ad\, x.Y \in \mathfrak{z}_\mathfrak{g}(a)$.*

Proof. – Note that $\mathfrak{L}^\perp = \mathfrak{K}_{2-l} + \mathfrak{K}_{1-l'}(+) + \mathfrak{K}_{1-l'}(-)$. Therefore $Y \in a + \mathfrak{L}^\perp$ implies $Y \in a + \mathfrak{K}_{2-l}$ and $Y \in \mathfrak{z}_\mathfrak{g}(a)(\mathrm{mod}\, \mathfrak{K}_{1-l'})$. Thus we are in the situation of the previous Lemma with $r = 1 - l$ and $q = l'$. It is clear that the element $\gamma(Z')$ constructed in the proof belongs to L. The result follows. □

Remark 4.9. – Suppose now that $l = 2l' + 1$ is odd. Define L to be the subgroup generated by T_{l-1} and the following affine root groups:

$$\begin{array}{lll} U_{\alpha,l-1} & \text{if} & \alpha(a) = 0 \\ U_{\beta,l'} & \text{if} & \beta(a) \neq 0 \quad \text{and } \beta > 0 \\ U_{\beta,l'+1} & \text{if} & \beta(a) \neq 0 \quad \text{and } \beta < 0. \end{array}$$

It is clear that L is again a subgroup of J. Let $\widetilde{K}_l$ be the group generated by K_l and the affine root groups $U_{\beta,\, l-1}$ if $\beta(a) \neq 0$ and $\beta > 0$. The map γ (see equation 4.2) induces an isomorphism $L/\widetilde{K}_l \simeq \mathfrak{L}/\widetilde{\mathfrak{K}}_l$ (in the obvious notation). (In particular, $\widetilde{K}_l$ is normal in L and the quotient is abelian.) Hence

$$(L/\widetilde{K}_l)\hat{} \simeq \widetilde{\mathfrak{K}}_l^\perp / \mathfrak{L}^\perp.$$

The character $\rho \mid L$ is again parametrized by $a + \mathfrak{L}^\perp$. It is straightforward to verify that Corollary 4.8 holds.

LEMMA 4.10. – *Suppose* $\mathrm{Ad}\, g.X_1 = X_2$ *where* $X_i \in \mathfrak{K}_{1-l} \cap \mathfrak{z}_{\mathfrak{g}}(a)$ *and* $X_i \equiv a \pmod{\mathfrak{K}_{2-l}}$ *for* $i = 1, 2$. *Then* $\mathrm{Ad}\, g.a_1 = a_1$ *for some* $a_1 \in a + \mathfrak{t}_{2-l}$.

Proof. – Fix a uniformiser $\varpi \in F$. Replacing a by $\varpi^{l-1}a$ and X_i by $\varpi^{l-1}X_i$, we may assume $a \in \mathfrak{t}_0$ and $X_i \in \mathfrak{K}_0 \cap \mathfrak{z}_{\mathfrak{g}}(a)$ with $X_i \equiv a \pmod{\mathfrak{K}_1}$ $(i = 1, 2)$.

We have $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}$. Let $\{l_i\}$ be any basis of $\mathfrak{z}$. Then X_α $(\alpha \in \Phi)$, H_β $(\beta \in \Pi)$, l_i is a basis of $\mathfrak{g}$ which we fix for the rest of the proof. Consider the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}_F(\mathfrak{g})$. Then $\mathrm{ad}\, \mathfrak{t} \simeq \mathfrak{t}'$ embeds in the diagonal subalgebra $\mathfrak{d}$ of $\mathrm{End}_F(\mathfrak{g})$ with respect to our fixed basis of $\mathfrak{g}$. The trace form Tr on $\mathrm{End}_F(\mathfrak{g})$ is non-degenerate. Since $\mathrm{char}\, F = 0$ its restriction to $\mathrm{ad}\, \mathfrak{g} \simeq \mathfrak{g}'$ (the Killing form of $\mathfrak{g}'$) is again non-degenerate. Hence

$$\mathrm{End}_F(\mathfrak{g}) = \mathfrak{g}' \oplus \mathfrak{g}'^{\perp}$$

where each summand is $\mathrm{Ad}\, G$-invariant. Also $\mathrm{Tr} \mid \mathfrak{d} \times \mathfrak{d}$ is non-degenerate with non-degenerate restriction to $\mathrm{ad}\, \mathfrak{t} \simeq \mathfrak{t}'$ so that

$$\mathfrak{d} = \mathfrak{t}' \oplus \mathfrak{t}'^{\perp}.$$

Equation 2.1 and the equation

$$\mathrm{ad}\, X_\alpha.H_\beta = -\langle \alpha, \beta^\vee \rangle X_\alpha \quad (\alpha, \beta \in \Phi)$$

imply that the matrix of $\mathrm{ad}\, X_\alpha$ with respect to our fixed basis of $\mathfrak{g}$ has zeros on the diagonal. Hence $\mathrm{Tr}\,(d\, \mathrm{ad}\, X_\alpha) = 0$ for $d \in \mathfrak{d}$ and $\alpha \in \Phi$. It follows that $\mathfrak{t}'^{\perp}$ ($\perp$ in $\mathfrak{d}$) is contained in $\mathfrak{g}'^{\perp}$ ($\perp$ in $\mathrm{End}_F(\mathfrak{g})$).

Write $a = a_0 + z$ where $a_0 \in \mathfrak{t}'_0$ and $z \in \mathfrak{z}$. Note that $\mathrm{ad}\, a = \mathrm{ad}\, a_0$ viewed as an element of $\mathfrak{d}$ may have distinct diagonal entries (eigenvalues) which are equal $(\mathrm{mod}\, \mathcal{P}_F)$. Let d_0 be an element of $\mathfrak{d}$ obtained by lifting such equalities $(\mathrm{mod}\, \mathcal{P}_F)$ to equalities.

Fix an algebraic closure $\overline{F}$ of F. Write $\overline{\mathcal{O}}$ for the ring of integers of $\overline{F}$ and $\overline{\mathcal{P}}$ for the unique prime ideal in $\overline{\mathcal{O}}$. Let $\overline{\mathfrak{g}} = \overline{F} \otimes_F \mathfrak{g}$. Then $\mathrm{End}_F(\mathfrak{g})$ embeds canonically in $\mathrm{End}_{\overline{F}}(\overline{\mathfrak{g}})$. Since each $\mathrm{ad}\, X_i$ preserves the $\overline{\mathcal{O}}$-lattice $\overline{\mathfrak{K}}_0 = \overline{\mathcal{O}} \otimes_{\mathcal{O}_F} \mathfrak{K}_0$ in $\overline{\mathfrak{g}}$, their eigenvalues belong to $\overline{\mathcal{O}}$. Let $\lambda \in \overline{\mathcal{O}}$ be an eigenvalue of $\mathrm{ad}\, X_i$ and let

$$\overline{\mathfrak{g}}_\lambda(\mathrm{ad}\, X_i) = \{v \in \overline{\mathfrak{g}} : (\mathrm{ad}\, X_i - \lambda)^n\, v = 0 \quad \text{for some } n \in \mathbb{N}\}$$

be the corresponding generalised eigenspace. Since $\mathrm{ad}\, X_i = \mathrm{ad}\, a + \mathrm{ad}\, Y_i$ where $Y_i \in \mathfrak{K}_1$, we see $\lambda \equiv \mu_0 (\mathrm{mod}\, \overline{\mathcal{P}})$ for some eigenvalue μ_0 of $\mathrm{ad}\, a_0$. Hence $\lambda \equiv \mu (\mathrm{mod}\, \overline{\mathcal{P}})$ for the unique eigenvalue μ of d_0 such that $\mu \equiv \mu_0 (\mathrm{mod}\, \mathcal{P})$.

We claim $\overline{\mathfrak{g}}_\lambda(\mathrm{ad}\, X_i) \subseteq \overline{\mathfrak{g}}_\mu(d_0)$ – the μ-eigenspace of d_0. Write $v \in \overline{\mathfrak{g}}_\lambda(\mathrm{ad}\, X_i)$ as $v = v_1 + \ldots + v_l$ where $v_j \in \overline{\mathfrak{g}}_{\mu_j}(d_0)$ with $\mu_1 = \mu$ and $\mu_j \neq \mu_k$ if $j \neq k$. Since $X_i \in \mathfrak{z}_{\mathfrak{g}}(a) = \mathfrak{z}_{\mathfrak{g}}(a_0)$, each eigenspace of $\mathrm{ad}\, a = \mathrm{ad}\, a_0$ is $\mathrm{ad}\, X_i$-stable. Further since each eigenspace of d_0 is a sum of eigenspaces of $\mathrm{ad}\, a_0$, we see $\mathrm{ad}\, X_i$ also stabilises these eigenspaces. Hence $(\mathrm{ad}\, X_i - \lambda)^n\, v_j = 0$ for each j. If $v_j \neq 0$ we may assume $v_j \in \overline{\mathfrak{K}}_0 \setminus \overline{\mathfrak{K}}_1$. Then $(d_0 - \lambda)^n\, v_j \in \overline{\mathfrak{K}}_1$ and thus $\mu_j \equiv \lambda (\mathrm{mod}\, \overline{\mathcal{P}})$. Hence $v_j \neq 0$ implies $j = 1$, i.e. $\overline{\mathfrak{g}}_\lambda(\mathrm{ad}\, X_i) \subseteq \overline{\mathfrak{g}}_\mu(d_0)$.

Since $\overline{\mathfrak{g}}$ is the sum of both the generalised eigenspaces of $\mathrm{ad}\, X_i$ and the eigenspaces of d_0, we must have

$$\sum_{\lambda \equiv \mu (\mathrm{mod}\ \overline{\mathcal{P}})} \overline{\mathfrak{g}}_\lambda(\mathrm{ad}\, X_i) = \overline{\mathfrak{g}}_\mu(d_0).$$

The relation $\operatorname{Ad} g \operatorname{Ad} X_1 \operatorname{Ad} g^{-1} = \operatorname{ad} X_2$ then implies $\operatorname{Ad} g\, \overline{\mathfrak{g}}_\lambda(\operatorname{ad} X_1) = \overline{\mathfrak{g}}_\lambda(\operatorname{ad} X_2)$ for all λ. Hence $\operatorname{Ad} g$ preserves the eigenspaces of d_0 and thus $\operatorname{Ad} g\, d_0 \operatorname{Ad} g^{-1} = d_0$. Write $d_0 = \operatorname{ad} a_1' + a_1''$ where $a_1' \in \mathfrak{t}'$ and $a_1'' \in \mathfrak{t}'^{\perp} \subset \mathfrak{g}'^{\perp}$. Then

$$d_0 = \operatorname{Ad} g \operatorname{ad} a_1' \operatorname{Ad} g^{-1} + \operatorname{Ad} g\, a_1'' \operatorname{Ad} g^{-1}.$$

Since $\mathfrak{g}'$ and $\mathfrak{g}'^{\perp}$ are both $\operatorname{ad} g$-stable, we have

$$\operatorname{Ad} g \operatorname{ad} a_1' \operatorname{Ad} g^{-1} = \operatorname{ad} a_1'.$$

This gives $\operatorname{ad}(\operatorname{Ad} g\, a_1') = \operatorname{ad} a_1'$ and hence $\operatorname{Ad} g\, a_1' = a_1'$. Letting $a_1 = a_1' + z$, we have $\operatorname{Ad} g\, a_1 = a_1$.

Finally we show $a_1 \in a + \mathfrak{t}_1$ by proving $x(a_1 - a) = x(a_1' - a_0) \in \mathcal{P}_F$ for all $x \in X$. Let $p : \mathfrak{d} \to \mathfrak{t}'$ be the projection map corresponding to the decomposition $\mathfrak{d} = \mathfrak{t}' \oplus \mathfrak{t}'^{\perp}$. Suppose $d \in \mathfrak{d}$ satisfies $d(X_\alpha) = d_\alpha X_\alpha$ for $\alpha \in \Pi$. Then it is easy to check that $p(d)$ is the unique element t of $\mathfrak{t}'$ such that $\alpha(t) = d_\alpha$ for $\alpha \in \Pi$. It follows that the element $p(d_0) = a_1'$ satisfies

$$\alpha(a_1') \equiv \alpha(a_0) \pmod{\mathcal{P}_F}$$

for all $\alpha \in \Pi$ and hence also for all $\alpha \in \Phi$. Further if $x \in X_0$ where

$$X_0 = \{x \in X : \langle x, \alpha^\vee \rangle = 0\ (\forall \alpha^\vee \in \Phi^\vee)\}$$

then $x(a_1') = x(a_0) = 0$. Hence $x(a_1' - a_0) \in \mathcal{P}_F$ for any $x \in Q + X_0$. Note that $Q + X_0$ has finite index in X and the quotient embeds in P/Q where P is the weight lattice (exactly as in as in 4.4). Thus $\operatorname{char} k_F$ does not divide the index $[X : Q + X_0]$ and hence $x(a_1' - a_0) \in \mathcal{P}_F$ for all $x \in X$. □

We can now compute the G-intertwining of $\rho \mid L$.

PROPOSITION 4.11. – *With notation as above:*

$$\mathcal{I}_G(\rho \mid L) = L\, C_G(a)\, L.$$

Proof. – Let $g \in \mathcal{I}_G(\rho \mid L)$. By Lemma 4.6, there exist elements X and X' in $a + \mathfrak{L}^\perp$ such that $\operatorname{Ad} g\, X = X'$. By Corollary 4.8, we can find x and x' in L such that $X_1 = \operatorname{Ad} x\, X$ and $X_2 = \operatorname{Ad} x'\, X'$ belong to $\mathfrak{z}_{\mathfrak{g}}(a) \cap \mathfrak{K}_{1-l}$. Further $X_i \equiv a \pmod{\mathfrak{K}_{2-l}}$ for $i = 1, 2$ (since L acts as the identity via ad on $\mathfrak{K}_{1-l}/\mathfrak{K}_{2-l}$). From $\operatorname{Ad}(x' g x^{-1})\, X_1 = X_2$ and Lemma 4.10, we see $g \in L\, C_G(a_1)\, L$. Since $L\, C_G(a_1)\, L$ is trivially contained in $\mathcal{I}_G(\rho \mid L)$, we obtain

$$\mathcal{I}_G(\rho \mid L) = L\, C_G(a_1)\, L.$$

The group $C_G(a_1)$ is the group of F-rational points of the reductive subgroup $\mathbb{C}_{\mathbb{G}}(a_1)$ of $\mathbb{G}$. Since $\operatorname{char} F = 0$, $\mathbb{C}_{\mathbb{G}}(a_1)$ is connected. Hence

$$C_G(a_1) = \langle T, U_\alpha : \alpha(a_1) = 0\ (\alpha \in \Phi) \rangle.$$

Note that $\alpha(a_1) = 0$ for $\alpha \in \Phi$ implies $\alpha(a) \equiv 0 \pmod{\mathcal{P}_F}$ since $a \in a_1 + \mathfrak{t}_1$. From Lemma 4.4, we see $\alpha(a) = 0$. Hence $C_G(a_1)$ is contained in $C_G(a)$ and thus $\mathcal{I}_G(\rho \mid L)$

is contained in $L\,C_G(a)\,L$. Since the opposite containment also holds, we conclude $\mathcal{I}_G(\rho \mid L) = L\,C_G(a)\,L$. $\square$

Remark 4.12. – Our method of calculating $\mathcal{I}_G(\rho)$ is adapted from Howe and Moy [19] (using adjoint representations in place of the natural action of $GL_n(F)$ and its Lie algebra on F^n). The Proposition is valid in a wider context. See Adler [1] and Adler and Roche [2].

Remark 4.13. – We have restricted the residual characteristic p of F in several ways. More precisely, we assume

- p is prime to any integers which occur as ratios of squares of root lengths for any pair of roots in the same irreducible factor of Φ
- p is not a bad prime for Φ
- p does not divide $2[P^\vee : Q^\vee] = 2[P : Q]$.

Thus if Φ is irreducible, the excluded primes are

- for type A_n divisors of $2(n+1)$
- for types B_n, C_n, D_n 2
- for types E_6, E_7, F_4, G_2 2, 3
- for type E_8 2, 3, 5.

If Φ is not irreducible, we exclude primes attached to its irreducible factors.

Remark 4.14. – Clearly, our restrictions are most severe in the case of groups of type A_n. However (using the trace form on the Lie algebra in place of the form B) our intertwining results are easily seen to hold for GL_N without restriction on the characteristic or the residual characteristic. Our determination of the structure of $\mathcal{H}(G,\rho)$ (see section 6) is then valid for $G = GL_N$ (without restriction on F or k_F). Further, by embedding a part of the GL_N Hecke algebra in the corresponding SL_N Hecke algebra (and using some additional arguments), our description of $\mathcal{H}(G,\rho)$ may also be seen to hold also for $G = SL_N$ (without restriction on F or k_F) though we do not pursue this here.

In theorem 4.15 below we use Proposition 4.11 and an inductive argument to determine the G-intertwining of ρ. In the inductive step, we need to know that the statement of Proposition 4.11 is valid not only for G but also for Levi subgroups of G. The following argument shows that this holds given restrictions on residual characteristic only slightly stronger than those in Remark 4.13.

Suppose $\theta : G_1 \to G_2$ is a central isogeny defined over $\mathcal{O}_F$ (more precisely the restriction to G_1 of such an isogeny $\theta : \mathbb{G}_1 \to \mathbb{G}_2$). Then θ induces an isomorphism $\mathfrak{g}_1 \simeq \mathfrak{g}_2$ (since p does not divide $[P : Q]$). Write $\theta(a_1 + \mathfrak{L}_1^\perp) = a_2 + \mathfrak{L}_2^\perp$ (in the obvious notation). Then $\mathcal{I}_{G_1}(a_1 + \mathfrak{L}_1^\perp) = L_1 C_{G_1}(a_1) L_1$ if and only if $\mathcal{I}_{G_2}(a_2 + \mathfrak{L}_2^\perp) = L_2 C_{G_2}(a_2) L_2$. (The 'if' direction is immediate. To obtain the 'only if' direction, use the surjective map $G_1\mathbb{C}_1(\overline{F}) \to G_2\mathbb{C}_2(\overline{F})$ where $\mathbb{C}_i$ is the centre of $\mathbb{G}_i$).

We apply this observation to the map $Z^0 \times G_{\text{der}} \to G$ (given by multiplication) where G_{der} is the group of F-rational points of the derived group of $\mathbb{G}$ and Z^0 is the group of F-rational points of the identity component of the centre of $\mathbb{G}$. Thus the statement of Proposition 4.11 applies to G and its Levi subgroups if and only if it applies to G_{der} and its Levi subgroups. Moreover, it applies to G_{der} and its Levi subgroups if and only if it applies to each of its almost simple factors and their Levi subgroups.

If G is symplectic (resp. special orthogonal) then each Levi subgroup of G is isomorphic to a product of general linear groups and a symplectic (resp. special orthogonal) group of lower rank. Hence (using Remark 4.14) Proposition 4.11 applies to all symplectic and special orthogonal groups and all their Levi subgroups provided $p \neq 2$. Thus the same conclusion holds for groups each of whose irreducible factors has type B, C or D.

We list explicit restrictions on $p = \operatorname{char} k_F$ under which the proof of Theorem 4.15 is valid. If Φ is irreducible, we restrict p as follows

- for type A_n $\quad p > n+1$
- for types B_n, C_n, D_n $\quad p \neq 2$
- for type F_4 $\quad p \neq 2, 3$
- for types G_2, E_6 $\quad p \neq 2, 3, 5$
- for types E_7, E_8 $\quad p \neq 2, 3, 5, 7$.

If Φ is not irreducible, we exclude primes attached to each of its irreducible factors. (Of course, the restriction on groups of type A_n is far too stringent and can be significantly relaxed.)

THEOREM 4.15. – *The G-intertwining of ρ is given by:*

$$\mathcal{I}_G(\rho) = J\, W_\chi\, J$$

where $W_\chi = \{w \in W \mid w\chi = \chi\}$.

Proof. – If χ has level zero, the result follows from Prop. 4.1 (see Remark 4.2). Thus, without loss of generality, χ has positive level. Since $\mathcal{I}_G(\rho)$ is then contained in $\mathcal{I}_G(\rho \mid L)$, Prop. 4.11 implies $\mathcal{I}_G(\rho) \subset J\, C_G(a)\, J$. We first show we may assume $a \notin \mathfrak{z}$ so that $C_G(a) \neq G$.

Let T' be the group of F-rational points of the subtorus $\mathbb{T}' = \mathbb{T} \cap \mathbb{G}'$ of $\mathbb{T}$. There exists a subtorus $\mathbb{S}$ of $\mathbb{T}$ such that $\mathbb{T} = \mathbb{S} \times \mathbb{T}'$ (see e.g. Digne and Michel [16], Prop. 0.5). Further, this splitting is defined over $\mathcal{O}_F$ so that $T = S \times T'$ and $T_q = S_q \times T'_q$ for all $q \in \mathbb{N}$ (in the obvious notation). Suppose now $a \in \mathfrak{z}$. This implies $\chi \mid T'_{l-1}$ is trivial. Fix a smooth character $\chi_1 : T \to \mathbb{C}^\times$ such that $\chi_1 \mid T' = 1$ and $\chi_1 \mid S_{l-1} = \chi^{-1} \mid S_{l-1}$. Since $\chi\chi_1 \circ \alpha^\vee = \chi \circ \alpha^\vee$ for $\alpha \in \Phi$, we see $J_\chi = J_{\chi\chi_1}$. Noting that χ_1 extends to a character of G (which we again denote by χ_1), we see $\rho_{\chi\chi_1} = \rho_\chi \otimes \chi_1$. Write ρ_1 for $\rho_{\chi\chi_1}$. Then $\mathcal{I}_G(\rho_1) = \mathcal{I}_G(\rho)$. By construction $\rho_1 \mid K_{l-1}$ is trivial. Repeating this argument if necessary with $\chi\chi_1$, we may assume $a \notin \mathfrak{z}$.

As already observed (in the proof of Proposition 4.11) $C_G(a)$ is (the group of F-rational points of) a connected reductive subgroup of G containing T. It follows that $C_G(a)$ is a semistandard Levi subgroup of G, i.e. a Levi subgroup of G containing T. Indeed $C_G(a) = C_G(T_a)$ where T_a is the group of F-rational points of the subtorus $\mathbb{T}_a$ of $\mathbb{T}$ corresponding to the co-torsion-free submodule X_a of X where

$$X_a = \{x \in X \mid x(a) = 0\}.$$

In particular, $C_G(a)$ is a connected reductive subgroup of G containing T of strictly smaller semisimple rank. Further $J \cap C_G(a)$ is J for $C_G(a)$ with respect to the positive system for $\Phi(C_G(a), T)$ induced by Φ^+. Also the filtration subgroups K_q $(q \in \mathbb{N})$ (resp. filtration lattices $\mathfrak{K}_r$ ($r \in \mathbb{Z}$)) intersect with $C_G(a)$ (resp. $\mathfrak{z}_\mathfrak{g}(a)$, the Lie algebra of $C_G(a)$) to give the corresponding objects in $C_G(a)$ (resp. $\mathfrak{z}_\mathfrak{g}(a)$).

Since the theorem is clear if G has semisimple rank zero (i.e. $G = T$) we conclude (by induction on the semisimple rank) that $\mathcal{I}_G(\rho)$ is given by $JW_\chi J$. □

5. An isomorphism of Hecke algebras

In this section, we prove a slight modification of a result of Bushnell and Kutzko. We need this modification to show that certain Hecke algebra isomorphisms preserve canonical involutions and inner products.

Let $P = MN$ and $\overline{P} = M\overline{N}$ be oppposite parabolic subgroups of G having Levi factor M (and unipotent radicals N and $\overline{N}$). Let (K, τ) be a pair consisting of a compact open subgroup K of G and a smooth character τ of K. Write $K_M = K \cap M$, $K_N = K \cap N$ and $K_{\overline{N}} = K \cap \overline{N}$. We assume (K, τ) satisfies the following conditions:

i) $K = K_{\overline{N}} K_M K_N$

ii) $K_{\overline{N}}$ and K_N are contained in the kernel of τ.

Let $\tau_M = \tau \mid K_M$. We write $\mathcal{I}_G(\tau)$ for the G-intertwining of τ (as before) and assume the pair (K, τ) satisfies the further condition:

iii) $\mathcal{I}_G(\tau)$ is contained in JMJ.

From i) this is equivalent to $\mathcal{I}_G(\tau) = K\mathcal{I}_M(\tau_M)K$ (using the obvious notation). Under these hypotheses, it is a special case of a result of Bushnell and Kutzko ([13] 7.2 (ii)) that there exists an algebra isomorphism

$$t : \mathcal{H}(M, \tau_M) \xrightarrow{\simeq} \mathcal{H}(G, \tau) \tag{5.1}$$

which is support-preserving in the sense that $\operatorname{supp}(t(\phi)) = K \operatorname{supp}(\phi) K$ for $\phi \in \mathcal{H}(M, \tau_M)$.

The algebras $\mathcal{H}(M, \tau_M)$ and $\mathcal{H}(G, \tau)$ each carry a canonical involution $*$ defined by $f^*(x) = \overline{f(x^{-1})}$ ($f \in \mathcal{H}(M, \tau_M)$ or $\mathcal{H}(G, \tau)$). However the isomorphism t is not $*$-preserving. We show in this section that t may be modified (by twisting by the square-root of the modulus character of P) to obtain a $*$-preserving isomorphism t_u (which is clearly still support-preserving). More precisely, if $\delta_P(m) = |\det(\operatorname{Ad} m; \operatorname{Lie} N)|$ for $m \in M$, the map

$$f \mapsto f.\delta_P^{1/2} : \mathcal{H}(M, \tau_M) \longrightarrow \mathcal{H}(M, \tau_M)$$

defines an algebra isomorphism where $f.\delta_P^{1/2}(m) = f(m)\delta_P^{1/2}(m)$ ($m \in M$ and $f \in \mathcal{H}(M, \tau_M)$). The isomorphism $t_u : \mathcal{H}(M, \tau_M) \xrightarrow{\simeq} \mathcal{H}(G, \tau)$ is obtained by precomposing with t, i.e. $t_u(f) = t(f.\delta_P^{1/2})$ for $f \in \mathcal{H}(M, \tau_M)$.

The existence of a support-preserving, $*$-preserving isomorphism between $\mathcal{H}(M, \tau_M)$ and $\mathcal{H}(G, \tau)$ will be used repeatedly in later sections. The results of this section may also be extended (without difficulty) to the case where τ is a smooth irreducible representation of K (not necessarily one-dimensional).

We first recall briefly the construction of the isomorphism 5.1. See section 6 of Bushnell and Kutzko [13] for full details. Write $\mathcal{I}^+$ for the set of elements x in M such that

$x K_N x^{-1} \subset K_N$ and $x^{-1} K_{\overline{N}} x \subset K_{\overline{N}}$. Such elements are called positive. The collection of functions in $\mathcal{H}(M, \tau_M)$ with support contained in $K_M \mathcal{I}^+ K_M$ forms a subalgebra which we denote by $\mathcal{H}^+(M, \tau_M)$. Suppose $\phi \in \mathcal{H}(M, \tau_M$ is supported on $K_M x K_M$ $(x \in \mathcal{I}^+)$. Write Φ for the unique element of $\mathcal{H}(G, \tau)$ supported on KxK such that $\phi(x) = \Phi(x)$. The assignment $\phi \mapsto \Phi$ extends to an algebra embedding $t : \mathcal{H}^+(M, \tau_M) \to \mathcal{H}(G, \tau)$.

Now fix a strongly (P, K) positive element ζ in the center of M (see Bushnell and Kutzko [13] 6.16). In particular, ζ is positive and for each $x \in \mathcal{I}_M(\tau_M)$ there exists a positive integer n such that $\zeta^n x$ is positive. For $m \in \mathbb{Z}$, write ϕ_m for the unique element of $\mathcal{H}(M, \tau_M)$ supported on $K_M \zeta^m$ such that $\phi_m(\zeta^m) = 1$. Clearly

$$\phi_m \, \phi_n = \phi_{m+n} \qquad (m, n \in \mathbb{Z}).$$

For $m \in \mathbb{Z}$, we write Φ_m for the unique element of $\mathcal{H}(G, \tau)$ supported on $K\zeta^m K$ such that $\Phi_m(\zeta^m) = 1$. Thus $t(\phi_m) = \Phi_m$ for $m \in \mathbb{N}$. Our hypotheses on $\mathcal{I}_G(\tau)$ imply that Φ_1 (and hence each Φ_m for $m \in \mathbb{N}$) is invertible in $\mathcal{H}(G, \tau)$. Then the map $t : \mathcal{H}^+(M, \tau_M) \to \mathcal{H}(G, \tau)$ extends to a (well-defined) algebra embedding of $\mathcal{H}(M, \tau_M)$ as follows. (It is independent of the choice of strongly (P, K) positive element ζ in the center of M.) Let $\phi \in \mathcal{H}(M, \tau_M)$. There exists a positive integer m such that $\phi_m \phi \in \mathcal{H}^+(M, \tau_M)$. By definition,

$$t(\phi) = \Phi_m^{-1} \, t(\phi_m \phi).$$

Our hypotheses on $\mathcal{I}_G(\tau)$ imply that t is a support-preserving isomorphism of $\mathbb{C}$-algebras.

The isomorphism $t_u : \mathcal{H}(M, \tau_M) \xrightarrow{\sim} \mathcal{H}(G, \tau)$ is $*$-preserving if and only if it is unitary with respect to the standard inner products $(\,,\,)_M$ and $(\,,\,)_G$ on $\mathcal{H}(M, \tau_M)$ and $\mathcal{H}(G, \tau)$ (equivalently if and only if $\|\phi\|_M = \|t_u(\phi)\|_G$ for $\phi \in \mathcal{H}(M, \tau_M)$). Here $(\phi_1, \phi_2) = \int_M \phi_1(m) \overline{\phi_2(m)} \, dm$ and $\|\phi\|_M^2 = (\phi, \phi)_M$ for $\phi, \phi_1, \phi_2 \in \mathcal{H}(M, \tau_M)$ where the Haar measure dm on M gives K_M measure one (and similarly for the corresponding objects on G). To see this, first use the relation $(\phi_1, \phi_2)_M = (e_{\tau_M}, \phi_1 \phi_2^*)_M$ for ϕ_1, ϕ_2 in $\mathcal{H}(M, \tau_M)$ where e_{τ_M} is the identity element of $\mathcal{H}(M, \tau_M)$. Combining this with the corresponding relation on G shows that t_u is unitary implies t_u is $*$-preserving. The converse follows easily from the relation

$$\phi_1 \, \phi_2^*(1) = (\phi_1, \phi_2)_M$$

for $\phi_1, \phi_2 \in \mathcal{H}(M, \tau_M)$ (and similarly on G) combined with the fact that t_u is support-preserving.

PROPOSITION 5.1. – *Suppose* (K, τ) *and* $t : \mathcal{H}(M, \tau_M) \to \mathcal{H}(G, \tau)$ *are as above. Then* $t_u : \mathcal{H}(M, \tau_M) \to \mathcal{H}(G, \tau)$ *defined by*

$$t_u(\phi) = t(\phi . \delta_P^{1/2}) \qquad (\phi \in \mathcal{H}(M, \tau_M))$$

is a $*$-preserving, support-preserving isomorphism of $\mathbb{C}$*-algebras.*

Proof. – It is clear that t is a support-preserving isomorphism of $\mathbb{C}$-algebras. By the discussion preceding the statement of the proposition, it is $*$-preserving if and only if

$$\|t_u(\phi)\|_G = \|\phi\|_M \qquad (\phi \in \mathcal{H}(M, \tau_M)).$$

We first show this for $\phi \in \mathcal{H}^+(M,\tau_M)$. It is enough to consider the case where ϕ is supported on a single double coset $K_M x K_M$ ($x \in \mathcal{I}^+$). Then $t_u(\phi) = \delta_P^{1/2}(x)\Phi$ where Φ has support KxK and $\Phi(x) = \phi(x)$. Thus $\|t_u(\phi)\|_G = \|\phi\|_M$ if and only if

$$\mathrm{vol}\,(K_M x K_M) = \delta_P(x)\,\mathrm{vol}\,(KxK).$$

To see this, note

$$\begin{aligned}
\mathrm{vol}\,(KxK) &= [KxK : K] \\
&= [K : xKx^{-1} \cap K] \\
&= [K_N : xK_N x^{-1} \cap K_N][K_M : xK_M x^{-1} \cap K_M][K_{N^-} : xK_{N^-}x^{-1} \cap K_{N^-}] \\
&= [K_N : xK_N x^{-1}][K_M : xK_M x^{-1} \cap K_M] \qquad (x \in \mathcal{I}^+) \\
&= \delta_P(x)^{-1}\,\mathrm{vol}\,(K_M x K_M).
\end{aligned}$$

Now suppose $\phi \in \mathcal{H}(M,\tau_M)$ is arbitrary. There exists a positive integer m such that $\phi_m\,\phi \in \mathcal{H}^+(M,\tau_M)$. Clearly $\|\phi_m\,\phi\|_M = \|\phi\|_M$. To prove the proposition it is therefore sufficient to show

$$\|t_u(\phi_1)\,\Phi\|_G = \|\Phi\|_G \qquad (\Phi \in \mathcal{H}(G,\tau)).$$

Since $\|t_u(\phi_1)\|_G = 1$ and $\|\Phi_1\,\Phi_2\|_G \le \|\Phi_1\|_G\|\Phi_2\|_G$ (for $\Phi_1, \Phi_2 \in \mathcal{H}(G,\tau)$), this holds provided we can show

$$\|t_u(\phi_1)^{-1}\|_G = 1.$$

However

$$\begin{aligned}
t_u(\phi_1)^{-1} &= t_u(\phi_{-1}) \\
&= c\,\Phi_{-1}
\end{aligned}$$

for some constant c (since t_u is support-preserving). Further, $t_u(\phi_1\,\phi_{-1})$ equals the identity element of $\mathcal{H}(G,\tau)$ and hence $c\,\Phi_1\,\Phi_{-1}(1) = 1$. It follows that $c\,\mathrm{vol}\,(K\zeta^{-1}K) = 1$ which implies $\|t_u(\phi_{-1})\|_G = 1$ as required. □

Remark 5.2. – Suppose $\phi \in \mathcal{H}(M,\tau_M)$ is supported on $K_M x K_M$ for some $x \in M$. As above, we denote by Φ the unique element of $\mathcal{H}(G,\tau)$ supported on KxK such that $\Phi(x) = \phi(x)$. From the proof of 7.2 ii) in [13], we have

$$t(\phi) = [K_{N^-} : x^{-1}K_{N^-}x \cap K_{N^-}]\,\Phi.$$

In particular, $t(\phi) = c\,\Phi$ where c is a positive constant. Hence $t_u(\phi)$ is also a positive constant times Φ. Using $\|t_u(\phi)\|_G = \|\phi\|_M$, we now see

$$t_u(\phi) = \frac{\mathrm{vol}\,(K_M x K_M)^{1/2}}{\mathrm{vol}\,(KxK)^{1/2}}\,\Phi. \tag{5.2}$$

We use this observation in the proof of Lemma 9.3.

6. The structure of $\mathcal{H}(G,\rho)$

We can now describe the structure of the Hecke algebra $\mathcal{H}(G,\rho)$. First we recall some standard notation. Let $V = Y \otimes_{\mathbb{Z}} \mathbb{R}$. For each $\alpha \in \Phi$, $s_\alpha \in \mathrm{Aut}(Y)$ is defined by

$$s_\alpha(y) = y - \langle y, \alpha\rangle\, \alpha^\vee \qquad (y \in Y)$$

where $\langle\, ,\, \rangle$ is the canonical perfect pairing $\langle\, ,\, \rangle : X \times Y \to \mathbb{Z}$ and $\alpha^\vee \in \Phi^\vee$ is the coroot associated to $\alpha \in \Phi$. Since there is a canonical injection from $\mathrm{Aut}(Y)$ into $\mathrm{Aut}(V)$ $(= \mathrm{Aut}_{\mathbb{R}}(V))$, we may also view s_α as an element of $\mathrm{Aut}(V)$. Again we have

$$s_\alpha(v) = v - \langle v, \alpha\rangle\, \alpha^\vee \qquad (v \in V)$$

where $\langle\, ,\, \rangle$ now denotes the extension of $\langle\, ,\, \rangle : X \times Y \to \mathbb{Z}$ to a (non-degenerate, $\mathbb{R}$-bilinear) pairing between $X \otimes_{\mathbb{Z}} \mathbb{R}$ and V. Let Φ_{af} be the set of affine roots associated to Φ. Thus

$$\Phi_{\mathrm{af}} = \{\alpha + k : \alpha \in \Phi, k \in \mathbb{Z}\}$$

viewed as a set of affine functionals on V. If $a \in \Phi_{\mathrm{af}}$, we write s_a for the corresponding affine reflection. Thus if $a = \alpha + k$ $(\alpha \in \Phi, k \in \mathbb{Z})$

$$\begin{aligned} s_a(v) &= v - a(v)\, \alpha^\vee \\ &= v - \langle v, \alpha\rangle\, \alpha^\vee - k\alpha^\vee \end{aligned}$$

for $v \in V$. We have $s_a \in \mathrm{Aff}(V)$ (the set of affine-linear transformations of V). We now define some of the objects that will appear in our description of $\mathcal{H}(G,\rho)$ (cf. Goldstein [17] Definition 3.5).

DEFINITION 6.1.

$$\begin{aligned} \Phi_\chi &= \{\alpha \in \Phi : \chi \circ \alpha^\vee \mid \mathcal{O}_F^\times = 1\} \\ \Phi_{\chi,af} &= \{a \in \Phi_{\mathrm{af}} : a = \alpha + k \ \ (\alpha \in \Phi_\chi,\ k \in \mathbb{Z})\} \\ W_\chi^0 &= \langle s_a : a \in \Phi_{\chi,af}\rangle \quad \le \quad \mathit{Aff}(V) \\ \overline{W}_\chi^0 &= \langle s_\alpha : \alpha \in \Phi_\chi\rangle \quad \le \quad \mathit{Aut}(V) \end{aligned}$$

It is immediate that $\Phi_\chi^\vee = \{\alpha^\vee \in \Phi^\vee \mid \chi \circ \alpha^\vee \mid \mathcal{O}_F^\times = 1\}$ is a closed sub-root system of $\Phi^\vee$. It follows that Φ_χ is a sub-root system of Φ with Weyl group $\overline{W}_\chi^0$. Let $\Phi_\chi^+ = \Phi^+ \cap \Phi_\chi$ so that Φ_χ^+ is a positive system in Φ_χ. Define

$$\mathcal{C}_\chi = \{v \in V \mid 0 < \alpha(v) < 1 \ \ (\alpha \in \Phi_\chi^+)\}.$$

Then $\mathcal{C}_\chi$ is a chamber in the decomposition of V induced by $\Phi_{\chi,\mathrm{af}}$, i.e. $\mathcal{C}_\chi$ is a connected component of $V - \bigcup_{a\in\Phi_{\chi,\mathrm{af}}} H_a$ where $H_a = \{v \in V \mid a(v) = 0\}$. Note that $\mathcal{C}_\chi$ defines an ordering on $\Phi_{\chi,\mathrm{af}}$ via $a > 0$ if and only if $a(v) > 0$ for all $v \in \mathcal{C}_\chi$ $(a \in \Phi_{\chi,\mathrm{af}})$. Let $\Pi_{\chi,\mathrm{af}} = \{a \in \Phi_{\chi,\mathrm{af}} :$ a is a minimal positive element $\}$. The walls of the chamber $\mathcal{C}_\chi$ are then the various H_a's as a ranges through $\Pi_{\chi,\mathrm{af}}$. Let

$$S_\chi^0 = \{s_a \mid a \in \Pi_{\chi,\mathrm{af}}\}, \ \ \Omega_\chi = \{w \in W_\chi \mid w\,\mathcal{C}_\chi = \mathcal{C}_\chi\}.$$

LEMMA 6.2. – *(W_χ^0, S_χ^0) is a Coxeter system and W_χ decomposes as:*

$$W_\chi = W_\chi^0 \rtimes \Omega_\chi.$$

The group Ω_χ preserves the set of generators S_χ^0.

Proof. – It follows from Bourbaki [NB] (V, section 3) that (W_χ^0, S_χ^0) is a Coxeter system. Hence W_χ^0 acts simply transitively on the the chambers of V (with respect to W_χ^0) and thus $W_\chi = W_\chi^0 \Omega_\chi$ and $W_\chi^0 \cap \Omega_\chi = 1$. Since Ω_χ fixes $\mathcal{C}_\chi$, each element of Ω_χ permutes the walls of $\mathcal{C}_\chi$. This implies that Ω_χ preserves the set of generators S_χ^0. In particular, W_χ^0 is a normal subgroup of W_χ. □

The Hecke algebra $\mathcal{H}(W_\chi^0, S_\chi^0)$ of the Coxeter system (W_χ^0, S_χ^0) is the associative $\mathbb{C}$-algebra with $\mathbb{C}$-basis $\{e_w \mid w \in W_\chi^0\}$ and multiplication given by:

$$e_w e_{w'} = e_{ww'} \qquad \text{if } l(ww') = l(w) + l(w'), \tag{6.1}$$

$$e_s^2 = qe_1 + (q-1)e_s \qquad \text{for } s \in S_\chi^0. \tag{6.2}$$

In (6.1) l is the length function of the Coxeter system (W_χ^0, S_χ^0). It is well-known that such an algebra exists and is unique (e.g. see Humphreys [22], chap. 7).

The group Ω_χ acts by $\mathbb{C}$-algebra automorphisms on $\mathcal{H}(W_\chi^0, S_\chi^0)$ via $\omega.e_w = e_{\omega(w)}$ where $\omega(w) = \omega\, w\, \omega^{-1}$. This action clearly extends to an injective homomorphism of $\mathbb{C}$-algebras $\mathbb{C}[\Omega_\chi] \hookrightarrow \mathrm{End}_{\mathbb{C}\text{-alg}}\mathcal{H}(W_\chi^0, S_\chi^0)$. Via this homomorphism, we obtain a $\mathbb{C}$-algebra structure on the $\mathbb{C}$-vector space $\mathcal{H}(W_\chi^0, S_\chi^0) \otimes_{\mathbb{C}} \mathbb{C}[\Omega_\chi]$. Explicitly, the multiplication is determined by the formula:

$$e_w \otimes e_\omega \,.\, e_{w'} \otimes e_{\omega'} = e_w e_{\omega(w')} \otimes e_{\omega\omega'} \qquad (w,\, w' \in W_\chi^0 \text{ and } \omega,\, \omega' \in \Omega_\chi).$$

We denote this algebra by $\mathcal{H}(W_\chi^0, S_\chi^0) \tilde{\otimes} \mathbb{C}[\Omega_\chi]$. We sometimes abbreviate to $\mathcal{H}_\chi$. We frequently write the basis elements $e_w \otimes e_\omega$ as $e_{w\omega}$ where $w \in W_\chi^0$ and $\omega \in \Omega_\chi$. Thus $e_{w\omega} = e_w e_\omega$ and $e_\omega e_w e_{\omega^{-1}} = e_{\omega(w)}$ $(w \in W_\chi^0,\, \omega \in \Omega_\chi)$.

Define a function $* : \mathcal{H}_\chi \to \mathcal{H}_\chi$ by extending $\mathbb{C}$-antilinearly the assignment $e_w^* = e_{w^{-1}}$. Thus $(\alpha e_w)^* = \overline{\alpha} e_w^*$ for $w \in W_\chi$ and $\alpha \in \mathbb{C}$ where 'bar' denotes complex conjugation. The function $*$ defines an involution of $\mathcal{H}_\chi$. Define an involution $*$ on $\mathcal{H}(G, \rho)$ by $f^*(x) = \overline{f(x^{-1})}$ for $f \in \mathcal{H}(G, \rho)$ and $x \in G$. We now state the main result of this section.

THEOREM 6.3. – *i) With notation as above, there exists a family of $*$-preserving $\mathbb{C}$-algebra isomorphisms:*

$$\Psi : \mathcal{H}(W_\chi^0, S_\chi^0) \tilde{\otimes} \mathbb{C}[\Omega_\chi] \xrightarrow{\simeq} \mathcal{H}(G, \rho).$$

ii) These isomorphisms are support-preserving in the sense that the function $\Psi(e_w)$ has support $J\, w\, J$ for $w \in W_\chi$.

iii) The restriction $\Psi \mid \mathcal{H}(W_\chi^0, S_\chi^0)$ is uniquely determined.

Proof. – When χ has level zero, i) and ii) follow easily from Morris [27] (or Goldstein [17]). For completeness we include details. From [27], we obtain a set of basis elements of $\mathrm{End}_G(\mathrm{Ind}\,\rho)$ which multiply in a specified manner ([27] 7.12). It is well-known that there is an isomorphism of $\mathbb{C}$-algebras $t : \mathcal{H}(G,\rho) \xrightarrow{\simeq} \mathrm{End}_G(\mathrm{Ind}\,\rho^{-1})$ (see equation 7.2).

By Howlett and Lehrer [18] 6.11, χ extends to a character $\widetilde{\chi}$ of ${}^0N_\chi$. (This can also be proved by a more direct argument.) Note that $N_\chi = {}^0N_\chi Y$ (a semi-direct product) where Y is embedded in N_χ by $y \mapsto y(\varpi) : Y \to N_\chi$ (here ϖ is a fixed uniformiser). Setting $\widetilde{\chi}(y) = 1$ yields an extension of χ to N_χ. For $w \in W_\chi$ let $B_w = \widetilde{\chi}(n)\,\theta_n$ for any $n \in N_\chi$ projecting to w (see [27] 5.4). With this choice of B_w the cocycle λ in [27] 6.1 is trivial. It is easy to see that in our situation we have $\epsilon_w = 1$ for all $w \in W_\chi$ (see [27] 7.10). It follows that the cocycle μ in the final description of the algebra $\mathrm{End}_G(\mathrm{Ind}\,\rho)$ ([27] 7.12) is also trivial. It is clear from this description that there exists an isomorphism of $\mathbb{C}$-algebras

$$\Psi_1 : \mathcal{H}_\chi \xrightarrow{\simeq} \mathrm{End}_G(\mathrm{Ind}\,\rho).$$

(In the notation of [27] $R(\chi) = W_\chi^0$, $C(\chi) = \Omega_\chi$, $\Gamma = \Phi_{\chi,\mathrm{af}}$ and $p_a = q$ for all $a \in \Delta = \Pi_{\chi,\mathrm{af}}$). Thus there exists an isomorphism of $\mathbb{C}$-algebras

$$\Psi : \mathcal{H}_\chi \xrightarrow{\simeq} \mathcal{H}(G,\rho)$$

satisfying ii) (since $\mathcal{H}_\chi$ and $\mathcal{H}_{\chi^{-1}}$ coincide).

It remains to show that Ψ is $*$-preserving. To see this, define $*$ on $\mathrm{End}_G(\mathrm{Ind}\,\rho)$ to be the unique $\mathbb{C}$-antilinear extension of the assignment $T_w^* = T_{w^{-1}}$ for $w \in W_\chi$ (see [27] 7.9). (Thus $*$ on $\mathrm{End}_G(\mathrm{Ind}\,\rho)$ corresponds to $*$ on $\mathcal{H}_\chi$.) It suffices to show that $t \circ * = * \circ t$ where $t : \mathcal{H}(G,\rho^{-1}) \xrightarrow{\simeq} \mathrm{End}_G(\mathrm{Ind}\rho)$ is the isomorphism in equation 7.2.

For $n \in N_\chi$ let Φ_n be the unique element of $\mathcal{H}(G,\rho^{-1})$ supported on JnJ such that $\Phi_n(n) = 1$. A straightforward calculation shows that

$$t_{\Phi_n} = q^{l(w)}\,\theta_n$$

where $n \in N_\chi$ maps to $w \in W_\chi$. Hence $\widetilde{\chi}(n)\,t_{\Phi_n} = q^{l(w)}\,B_w$. For $w \in W_\chi^0$ or Ω_χ, we have

$$\begin{aligned} T_w &= q^{\frac{1}{2}(l_\chi(w)+l(w))}\,B_w \\ &= q^{\frac{1}{2}(l_\chi(w)-l(w))}\,\widetilde{\chi}(n)\,t_{\Phi_n}. \end{aligned}$$

This clearly implies $t \circ * = * \circ t$ and thus completes the proof of i) and ii) in the level-zero case.

When χ has positive level, we recall from the proof of Theorem 4.15 that either $\mathcal{I}_G(\rho)$ has $J-J$ double coset representatives contained in a proper Levi subgroup of G containing T or there exists a character χ_1 of G such that $\chi\chi_1$ (viewed as a character of ${}^0T$) has level zero. In this latter case, put $\rho_1 = \rho_{\chi\chi_1} = \rho \otimes \chi_1$. The map $f \mapsto f\chi_1 : \mathcal{H}(G,\rho) \to \mathcal{H}(G,\rho_1)$ is then a support-preserving, $*$-preserving isomorphism. Hence we may assume, without loss of generality, that $\mathcal{I}_G(\rho)$ lies in JMJ where M is a proper Levi subgroup of G (containing T).

We are thus in exactly the situation treated in section 5. By proposition 5.1, there exists a $*$-preserving, support-preserving isomorphism from $\mathcal{H}(G,\rho)$ to $\mathcal{H}(M,\rho_M)$ where $\rho_M = \rho \mid J \cap M$. Since $J \cap M$ is J for M and the semisimple rank of M is strictly smaller than that of G, we may complete our proof of the existence of a family of $*$-preserving, support-preserving isomorphisms provided we do so in the case of semisimple rank zero, i.e. when $G = T$.

We assume therefore that $G = T$. Then $J = {}^0T$ and $\rho = \chi$. Also $W_\chi^0 = 1$ and $W_\chi = \Omega_\chi = Y$. We are thus reduced to showing there exists a $*$-preserving, support-preserving isomorphism of $\mathbb{C}$-algebras

$$\Psi : \mathbb{C}[Y] \xrightarrow{\simeq} \mathcal{H}(T, \chi).$$

This is obvious.

Finally we show $\Psi \mid \mathcal{H}(W_\chi^0, S_\chi^0)$ is uniquely determined. Suppose Ψ and Ψ' are two support-preserving isomorphisms from $\mathcal{H}_\chi$ to $\mathcal{H}(G, \rho)$. Then $\Theta = \Psi^{-1} \circ \Psi' \mid \mathcal{H}(W_\chi^0, S_\chi^0)$ is a $\mathbb{C}$ -algebra automorphism of $\mathcal{H}(W_\chi^0, S_\chi^0)$ such that $\Theta(e_w) = \lambda_w e_w$ where $\lambda_w \in \mathbb{C}^\times$ for each $w \in W_\chi^0$. Now (6.1) implies $\lambda(ww') = \lambda_w \lambda_{w'}$ if $l(ww') = l(w) + l(w')$ $(w, w' \in W_\chi^0)$ and (6.2) implies $\lambda_s = 1$ for $s \in S_\chi^0$. Hence $\lambda_w = 1$ for all $w \in W_\chi^0$ as required. □

Remark 6.4. – The algebra $\mathcal{H}_\chi = \mathcal{H}(W_\chi^0, S_\chi^0) \tilde{\otimes} \mathbb{C}[\Omega_\chi]$ carries an inner product $\langle\, , \rangle$ defined by:

$$\langle e_w, e_{w'} \rangle = \delta_{w,w'}\, q^{l(w)} \qquad (w, w' \in W_\chi = W_\chi^0 \rtimes \Omega_\chi)$$

where $l(w)$ is defined to equal $l(w_1)$ if $w = w_1\omega$ $(w_1 \in W_\chi^0,\ \omega \in \Omega_\chi)$. We also use $\langle\, , \rangle$ to denote the standard inner product on $\mathcal{H}(G, \rho)$:

$$\langle \phi_1, \phi_2 \rangle = \int \phi_1(x)\overline{\phi_2(x)}\, dx \qquad (\phi_i \in \mathcal{H}(G, \rho))$$

Note $\langle \phi_1, \phi_2 \rangle = \phi_1\phi_2^*(1) = \langle \phi_1\phi_2^*, e_\rho \rangle$ where $*$ is the canonical involution on $\mathcal{H}(G, \rho)$. It is easy to check that the corresponding relation holds on $\mathcal{H}_\chi$, i.e.

$$\langle e_{w_1}, e_{w_2} \rangle = \langle e_{w_1}.e_{w_2}^*, e_1 \rangle \qquad \text{for all } w_1, w_2 \in W_\chi.$$

Thus $e_{w_1}e_{w_2}^* = \langle e_{w_1}, e_{w_2} \rangle e_1 + f$ where $f \in \operatorname{span}\{e_w : w \neq 1\}$. It follows that any support-preserving, $*$- preserving isomorphism $\Psi : \mathcal{H}_\chi \xrightarrow{\sim} \mathcal{H}(G, \rho)$ is an isometry with respect to these inner products (assuming Haar measure on G is normalised so that J has measure one).

7. Principal series types

We show in this section that (J, ρ) is a type for the component of the Bernstein decomposition determined by χ. We begin by recalling the Bernstein decomposition of $\mathfrak{R}(G)$ (the category of smooth complex representations of G). We then recall the definition of type and some consequences. Much of our presentation and notation is taken from Bushnell and Kutzko [13].

A pair (M, σ), consisting of a Levi subgroup M of G and an irreducible supercuspidal representation σ of M, is called a cuspidal pair. Two such pairs, (M_i, σ_i) $(i = 1, 2)$, are said to be equivalent if there exists a $g \in G$ such that $M_2 = {}^gM_1$ and $\sigma_2 \simeq {}^g\sigma_1$. If we denote by Irr($G$) the set of equivalence classes of smooth irreducible representations of G, then the following map is well-defined (Bernstein and Zelevinsky [5]):

$$\begin{aligned} \mathrm{Irr}\,(G) &\to \{\text{ equivalence classes of cuspidal pairs}\} \\ \pi &\mapsto \text{ class of } (M,\sigma) \text{ if } \pi \simeq \text{ a } G\text{-subquotient of } i_P^G\,\sigma \end{aligned}$$

where P is any parabolic subgroup of G with Levi factor M and i_P^G is the normalised parabolic induction functor. The image of π under this map is called the supercuspidal support of π. We say two cuspidal pairs, (M_i, σ_i) $(i = 1, 2)$, are inertially equivalent if there exists a $g \in G$ and an unramified character ν of M_2 such that:

$$M_2 = {}^gM_1 \quad \text{and} \quad {}^g\sigma_1 \simeq \sigma_2 \otimes \nu.$$

We write $[M,\sigma]$ for the inertial equivalence class of (M,σ) and $\mathfrak{B}(G)$ for the set of all inertial equivalence classes. If $(\pi, \mathcal{V})$ is an irreducible smooth representation of G, we denote by $\mathfrak{I}(\pi)$ the inertial equivalence class of its supercuspidal support.

Now fix a class $\mathfrak{s} \in \mathfrak{B}(G)$. We denote by $\mathfrak{R}_{\mathfrak{s}}(G)$ the full subcategory of $\mathfrak{R}(G)$ defined as follows:

DEFINITION 7.1. – *Let $(\pi, \mathcal{V}) \in \mathfrak{R}(G)$. Then $(\pi, \mathcal{V}) \in \mathfrak{R}_{\mathfrak{s}}(G)$ if and only if every irreducible G-subquotient π_0 of π satisfies $\mathfrak{I}(\pi_0) = \mathfrak{s}$.*

The subcategories $\mathfrak{R}_{\mathfrak{s}}(G)$ $(\mathfrak{s} \in \mathfrak{B}(G))$ split the category $\mathfrak{R}(G)$ (Bernstein and Deligne [4]). This is the Bernstein decomposition of $\mathfrak{R}(G)$. Precisely:

THEOREM 7.2. – *i) The category $\mathfrak{R}(G)$ is the direct product*

$$\mathfrak{R}(G) = \prod_{\mathfrak{s}\in\mathfrak{B}(G)} \mathfrak{R}_{\mathfrak{s}}(G)$$

of the subcategories $\mathfrak{R}_{\mathfrak{s}}(G)$ as $\mathfrak{s}$ ranges through $\mathfrak{B}(G)$.

i)′ Concretely, if $(\pi, \mathcal{V})$ is a smooth representation of G, then, for each $\mathfrak{s} \in \mathfrak{B}(G)$, $\mathcal{V}$ has a unique maximal G-subspace $\mathcal{V}_{\mathfrak{s}} \in \mathfrak{R}_{\mathfrak{s}}(G)$ and

$$\mathcal{V} = \sum_{\mathfrak{s}\in\mathfrak{B}(G)} \mathcal{V}_{\mathfrak{s}} \qquad (\textit{direct sum}).$$

If $\mathfrak{s}$, $\mathfrak{s}' \in \mathfrak{B}(G)$ and $\mathfrak{s} \neq \mathfrak{s}'$, then $Hom_G(\mathfrak{R}_{\mathfrak{s}}(G), \mathfrak{R}_{\mathfrak{s}'}(G)) = 0$ (in the obvious notation).

We can now say what we mean by an $\mathfrak{s}$-type in G.

DEFINITION 7.3. – *An $\mathfrak{s}$-type in G is a pair (K, ρ), where K is a compact open subgroup of G and ρ is an irreducible smooth representation of K, with the following property: an irreducible smooth representation π of G contains ρ if and only if $\mathfrak{I}(\pi) \in \mathfrak{s}$.*

Remark 7.4. – Clearly there is also a notion of $\mathfrak{s}$-type in G where $\mathfrak{s}$ is a subset of $\mathfrak{B}(G)$ which is not necessarily a singleton. It follows from [4](section 3) that such an $\mathfrak{s}$ is necessarily finite.

Suppose now that (ρ, W) is a smooth irreducible representation of a compact open subgroup K of G. If $(\pi, \mathcal{V})$ is a smooth representation of G, we write $\mathcal{V}^\rho$ for the ρ-isotypic subspace of $\mathcal{V}$, i.e. the sum of all K-subspaces of $\mathcal{V}$ isomorphic to ρ. We have $\mathcal{V}^\rho = e_\rho \mathcal{V}$ where e_ρ is the idempotent element of $\mathcal{H}(G)$ defined by

$$\begin{aligned} e_\rho(x) &= \mu(K)^{-1} \dim(\rho)\, \mathrm{tr}_W(\rho(x^{-1})) \qquad && (x \in K) \\ &= 0 && (x \notin K). \end{aligned}$$

We write $\mathcal{V}[\rho]$ for the G-subspace of $\mathcal{V}$ generated by $\mathcal{V}^\rho$ (i.e. $\mathcal{V}[\rho] = \mathcal{H}(G)\,\mathcal{V}^\rho$) and $\mathfrak{R}_\rho(G)$ for the full subcategory of $\mathfrak{R}(G)$ consisting of all $(\pi, \mathcal{V}) \in \mathfrak{R}(G)$ such that $\mathcal{V} = \mathcal{V}[\rho]$ (i.e. $\mathcal{V}$ is generated by its ρ-isotypic vectors).

The process $\mathcal{V} \mapsto \mathcal{V}^\rho$ defines a functor from $\mathfrak{R}(G)$ to $e_\rho\mathcal{H}(G)e_\rho - \mathfrak{Mod}$ (the category of left $e_\rho\mathcal{H}(G)e_\rho$-modules). It is well-known that this induces a bijection between smooth irreducible representations of G containing ρ and simple $e_\rho\mathcal{H}(G)e_\rho$-modules (e.g. see Bushnell and Kutzko [10] 4.2.3). However

$$\mathcal{V} \mapsto \mathcal{V}^\rho : \mathfrak{R}_\rho(G) \to e_\rho\mathcal{H}(G)e_\rho - \mathfrak{Mod}$$

is not in general an equivalence of categories. Suppose however that (K, ρ) is an $\mathfrak{s}$-type in G (for some $\mathfrak{s} \in \mathfrak{B}(G)$). In this case we show (in the next few paragraphs) that the categories $\mathfrak{R}_\mathfrak{s}(G)$ and $\mathfrak{R}_\rho(G)$ are equal (as subcategories of $\mathfrak{R}(G)$) and that the functor $\mathcal{V} \mapsto \mathcal{V}^\rho : \mathfrak{R}_\mathfrak{s}(G) \to e_\rho\mathcal{H}(G)e_\rho - \mathfrak{Mod}$ is an equivalence of categories. In particular the category $\mathfrak{R}_\rho(G)$ is closed under subquotients. Conversely, Bushnell and Kutzko prove in [13] (using an argument due to Bernstein) that if $\mathfrak{R}_\rho(G)$ is closed under subquotients, then (K, ρ) is an $\mathfrak{s}$-type in G where $\mathfrak{s}$ is a (necessarily) finite subset of $\mathfrak{B}(G)$.

Let $\mathrm{ind}\,\rho$ be the smooth representation of G compactly induced by ρ. There is a natural isomorphism of $\mathbb{C}$-vector spaces:

$$\text{(7.1)} \qquad \mathrm{Hom}_G(\mathrm{ind}\,\rho, \pi) \;\simeq\; \mathrm{Hom}_K(\rho, \pi) \qquad (\pi \in \mathfrak{R}(G)).$$

If we realise $\mathrm{ind}\,\rho$ in the usual way as the space of compactly supported functions $f : G \to W$ such that $f(kg) = \rho(k)f(g)$ $(k \in K, g \in G)$ with G acting by right translations, then the maps are given by

$$\begin{aligned} t &\mapsto t_1 : \mathrm{Hom}_G(\mathrm{ind}\,\rho, \pi) \to \mathrm{Hom}_K(\rho, \pi) \\ t &\mapsto \tilde{t} : \mathrm{Hom}_K(\rho, \pi) \to \mathrm{Hom}_G(\mathrm{ind}\,\rho, \pi) \end{aligned}$$

where

$$\begin{aligned} t_1(w) &= t(e_w) \quad (w \in W), \\ \tilde{t}(f) &= \int_G \pi(g)t(f(g^{-1}))\, dg \quad (f \in \mathrm{ind}\,\rho). \end{aligned}$$

Here $e_w \in \mathrm{ind}\,\rho$ is defined by $e_w(k) = \rho(k)w$ for $k \in K$, $e_w(g) = 0$ for $g \notin K$ (and Haar measure on G is normalised so that K has measure one).

In particular $\mathrm{Hom}_G(\mathrm{ind}\,\rho, \pi) \neq 0$ if and only if $\mathrm{Hom}_K(\rho, \pi) \neq 0$. Fix $\mathfrak{s} \in \mathfrak{B}(G)$ and suppose $(\pi, \mathcal{V}) \in \mathfrak{R}_\mathfrak{s}(G)$. Since every irreducible subquotient of $(\pi, \mathcal{V})$ is also in $\mathfrak{R}_\mathfrak{s}(G)$ and smooth representations of K are completely reducible, we see (K, ρ) is an $\mathfrak{s}$-type implies $\mathrm{Hom}_G(\mathrm{ind}\,\rho, \pi) \neq 0$. Further, (K, ρ) is an $\mathfrak{s}$-type implies $\mathrm{ind}\,\rho \in \mathfrak{R}_\mathfrak{s}(G)$. Indeed if $\mathrm{ind}\,\rho \notin \mathfrak{R}_\mathfrak{s}(G)$ there exists a G-surjection from $\mathrm{ind}\,\rho$ to a smooth representation σ where $\sigma \in \mathfrak{R}_\mathfrak{t}(G)$ for some $\mathfrak{t} \in \mathfrak{B}(G)$ with $\mathfrak{t} \neq \mathfrak{s}$. Further, σ is finitely generated (since

$\operatorname{ind}\rho$ is). Thus σ has an irreducible quotient τ. By 7.1, τ contains ρ which means (K,ρ) cannot be an $\mathfrak{s}$-type in G.

Conversely, suppose $\operatorname{ind}\rho \in \mathfrak{R}_{\mathfrak{s}}(G)$ and $\operatorname{Hom}_G(\operatorname{ind}\rho,\pi) \neq 0$ for each non-zero $\pi \in \mathfrak{R}_{\mathfrak{s}}(G)$. It is then easy to see that (K,ρ) is an $\mathfrak{s}$-type in G. Hence (K,ρ) is an $\mathfrak{s}$-type in G if and only if $\operatorname{ind}\rho$ is a generator of $\mathfrak{R}_{\mathfrak{s}}(G)$, i.e. $\operatorname{ind}\rho \in \mathfrak{R}_{\mathfrak{s}}(G)$ and $\operatorname{Hom}_G(\operatorname{ind}\rho,\pi) \neq 0$ for non-zero $\pi \in \mathfrak{R}_{\mathfrak{s}}(G)$ (equivalently, every object in $\mathfrak{R}_{\mathfrak{s}}(G)$ is a quotient of a direct sum of copies of $\operatorname{ind}\rho$).

The isomorphism (7.1) implies that $\operatorname{ind}\rho$ is a finitely generated, projective object in $\mathfrak{R}(G)$. Thus if (K,ρ) is an $\mathfrak{s}$-type in G, $\operatorname{ind}\rho$ is a finitely generated, projective generator of $\mathfrak{R}_{\mathfrak{s}}(G)$. By a well-known result (see e.g. Bass [3], chap 2 Theorem 1.5), the functor

$$\begin{aligned}\mathfrak{R}_{\mathfrak{s}}(G) &\to \operatorname{End}_G(\operatorname{ind}\rho)^{\mathrm{opp}} - \mathfrak{Mod}\\ \pi &\mapsto \operatorname{Hom}_G(\operatorname{ind}\rho,\pi)\end{aligned}$$

is an equivalence of categories (where $\operatorname{Hom}_G(\operatorname{ind}\rho,\pi)$ is given the $\operatorname{End}_G(\operatorname{ind}\rho)^{\mathrm{opp}}$-module structure defined by $t.f = f\circ t$ ($t \in \operatorname{End}_G(\operatorname{ind}\rho)$, $f \in \operatorname{Hom}_G(\operatorname{ind}\rho,\pi)$)). Thus the functor

$$\begin{aligned}\mathfrak{R}_{\mathfrak{s}}(G) &\to \operatorname{End}_G(\operatorname{ind}\rho)^{\mathrm{opp}} \otimes_{\mathbb{C}} \operatorname{End}_{\mathbb{C}}(W) - \mathfrak{Mod}\\ \pi &\mapsto \operatorname{Hom}_G(\operatorname{ind}\rho,\pi) \otimes_{\mathbb{C}} W\end{aligned}$$

is also an equivalence of categories.

We write $\mathcal{H}(G,\rho^\vee)$ (where $(\rho^\vee,W^\vee)$ is the contragredient of (ρ,W)) for the convolution algebra of compactly supported functions $\Phi : G \to \operatorname{End}_{\mathbb{C}}(W)$ such that $\Phi(k_1xk_2) = \rho(k_1)\Phi(x)\rho(k_2)$, for k_1, $k_2 \in K$, $x \in G$. Define maps between $\mathcal{H}(G,\rho^\vee)$ and $\operatorname{End}_G(\operatorname{ind}\rho)$ as follows:

$$\begin{aligned}\phi &\mapsto t_\phi : \mathcal{H}(G,\rho^\vee) \to \operatorname{End}_G(\operatorname{ind}\rho)\\ t &\mapsto \phi_t : \operatorname{End}_G(\operatorname{ind}\rho) \to \mathcal{H}(G,\rho^\vee)\end{aligned} \tag{7.2}$$

where

$$\begin{aligned}t_\phi(f)(g) &= \int_G \phi(x)(f(x^{-1}g))\,dx \quad (f \in \operatorname{Ind}\rho,\ g \in G)\\ \phi_t(g)(w) &= t(e_w)(g) \quad (g \in G,\ w \in W)\end{aligned}$$

(Here (as above) Haar measure on G is normalised so that K has measure one.) It is well-known and easy to check that these maps are inverse isomorphisms of $\mathbb{C}$-algebras. It is also easy to verify that the map $\Phi \mapsto \Phi'$ where $\Phi'(g) = \Phi(g^{-1})^\vee$ is an anti-isomorphism from $\mathcal{H}(G,\rho^\vee)$ to $\mathcal{H}(G,\rho)$ (where $\Phi(g^{-1})^\vee \in \operatorname{End}_{\mathbb{C}}(W^\vee)$ is the transpose of $\Phi(g^{-1}) \in \operatorname{End}_{\mathbb{C}}(W)$.) Thus we have an isomorphism of $\mathbb{C}$-algebras:

$$t \mapsto \phi_t' : \operatorname{End}_G(\operatorname{ind}\rho)^{\mathrm{opp}} \xrightarrow{\simeq} \mathcal{H}(G,\rho).$$

Via this isomorphism and the isomorphism $\operatorname{Hom}_G(\operatorname{Ind}\rho,\pi) \simeq \operatorname{Hom}_K(\rho,\pi)$, we can transport the left $\operatorname{End}_G(\operatorname{ind}\rho)^{\mathrm{opp}}$-module structure on $\operatorname{Hom}_G(\operatorname{ind}\rho,\pi)$ to a left $\mathcal{H}(G,\rho)$-module structure on $\operatorname{Hom}_K(\rho,\pi)$ ($\pi \in \mathfrak{R}(G)$). A short calculation shows that this is given by:

$$\Phi'.t(w) = \int_G \pi(g)t(\Phi(g^{-1})w)\,dg \quad (\Phi \in \mathcal{H}(G,\rho^\vee),\ t \in \operatorname{Hom}_K(\rho,\pi),\ w \in W).$$

It is clear that the evaluation map

$$\mathrm{ev} : \mathrm{Hom}_K(\rho, \pi) \otimes_{\mathbb{C}} W \xrightarrow{\sim} \mathcal{V}^\rho$$
$$\mathrm{ev}(t \otimes w) = t(w) \quad (t \in \mathrm{Hom}_K(\rho, \pi),\ w \in W)$$

is an isomorphism of $\mathbb{C}$-vector spaces. The space $\mathrm{Hom}_K(\rho, \pi) \otimes_{\mathbb{C}} W$ is naturally a left $\mathcal{H}(G, \rho) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(W)$-module. Recall ([10], chap. 4) that there is a canonical $\mathbb{C}$-algebra isomorphism:

$$\Psi : \mathcal{H}(G, \rho) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(W) \xrightarrow{\simeq} e_\rho \mathcal{H}(G) e_\rho$$
$$\Phi \otimes (w \otimes w^\vee) \mapsto (g \mapsto \dim \rho \langle w, \Phi(g) w^\vee \rangle) \quad (w \in W,\ w^\vee \in W^\vee).$$

Here $\langle\ ,\ \rangle$ denotes the canonical (evaluation) pairing and $w \otimes w^\vee \in \mathrm{End}_{\mathbb{C}}(W)$ is defined by $w \otimes w^\vee(w_1) = \langle w_1, w^\vee \rangle w$ for $w_1 \in W$. Via the isomorphisms Ψ and ev, $\mathcal{V}^\rho$ becomes a left $e_\rho \mathcal{H}(G) e_\rho$-module. We now show that this module structure coincides with the natural left $e_\rho \mathcal{H}(G) e_\rho$-module structure on $\mathcal{V}^\rho$ (induced by the left $\mathcal{H}(G)$-module structure on $\mathcal{V}$). This is equivalent to the commutativity of the diagram

$$\begin{array}{ccc}
\mathcal{H}(G, \rho) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(W) \times \mathrm{Hom}_K(\rho, \pi) \otimes_{\mathbb{C}} W & \longrightarrow & \mathrm{Hom}_K(\rho, \pi) \otimes_{\mathbb{C}} W \\
{\scriptstyle \Psi \times \mathrm{ev}} \downarrow & & {\scriptstyle \mathrm{ev}} \downarrow \\
e_\rho \mathcal{H}(G) e_\rho \times \mathcal{V}^\rho & \longrightarrow & \mathcal{V}^\rho
\end{array}$$

where the horizontal arrows come from the given module structures. Let

$$(\Phi \otimes (w \otimes w^\vee), t \otimes w_1) \in \mathcal{H}(G, \rho) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}} \times \mathrm{Hom}_K(\rho, \pi) \otimes_{\mathbb{C}} W.$$

A short calculation shows that the clockwise image in $\mathcal{V}^\rho$ of this element is given by

$$\langle w_1, w^\vee \rangle \int_G \pi(g)(t(\Phi(g)^\vee w))\, dg \in \mathcal{V}^\rho. \tag{7.3}$$

Another short calculation shows that the counter-clockwise image is given by

$$\dim \rho \int_G \pi(g)(t(w_1 \otimes w^\vee(\Phi(g)^\vee w)))\, dg. \tag{7.4}$$

In showing (7.3) and (7.4) are equal, we may assume the support of Φ lies in a single double coset KxK for some $x \in G$. Write $KxK = \bigcup_{i=1}^r k_i x K$ where $k_i x K \neq k_j x K$ for $i \neq j$. Then (7.3) becomes

$$\langle w_1, w^\vee \rangle \sum_{i=1}^r \int_K \pi(k_i)\pi(x)\pi(k)\big(t(\rho(k^{-1})\Phi(x)^\vee \rho(k_i^{-1}) w)\big)\, dk$$
$$= \sum_{i=1}^r \pi(k_i)\pi(x)\big(t[\langle w_1, w^\vee \rangle \Phi(x)^\vee \rho(k_i^{-1}) w]\big)$$

Equation (7.4) becomes

$$\dim \rho \sum_{i=1}^r \int_K \pi(k_i)\pi(x)\pi(k)(t(w_1)\langle w, \rho^\vee(k_i)\Phi(x)\rho^\vee(k) w^\vee \rangle\, dk$$
$$= \sum_{i=1}^r \pi(k_i)\pi(x)\bigg(t[\dim \rho \int_K \langle w, \rho^\vee(k_i)\Phi(x)\rho^\vee(k) w^\vee \rangle \rho(k) w_1\, dk]\bigg).$$

Comparing these last two equations, we are reduced to showing

$$\langle w_1, w^\vee \rangle \Phi(x)^\vee \rho(k_i^{-1}) w = \dim \rho \int_K \langle w, \rho^\vee(k_i)\Phi(x)\rho^\vee(k)w^\vee \rangle \rho(k) w_1 \, dk.$$

This is equivalent to showing

$$\int_K \langle \Phi(x)^\vee \rho(k_i^{-1}) w, \rho^\vee(k) w^\vee \rangle \, \langle \rho(k) w_1, w_1^\vee \rangle \, dk = \frac{1}{\dim \rho} \langle w_1, w^\vee \rangle \, \langle \Phi(x)^\vee \rho(k_i^{-1}) w, w_1^\vee \rangle$$

for all $w_1 \in W^\vee$ which holds by a direct application of Schur orthogonality.

Summing up, we have shown that the functor

$$\mathcal{V} \mapsto \mathcal{V}^\rho : \mathfrak{R}_\mathfrak{s}(G) \to e_\rho \mathcal{H}(G) e_\rho - \mathfrak{Mod}$$

is an equivalence of categories when (K, ρ) is an $\mathfrak{s}$-type in G.

We now show the categories $\mathfrak{R}_\mathfrak{s}(G)$ and $\mathfrak{R}_\rho(G)$ are equal (when (K, ρ) is an $\mathfrak{s}$-type). Let $(\pi, \mathcal{V}) \in \mathfrak{R}_\mathfrak{s}(G)$. We have $(\mathcal{V}/\mathcal{V}[\rho])^\rho = 0$. This implies $\mathcal{V}/\mathcal{V}[\rho] = 0$ (since every non-zero object in $\mathfrak{R}_\mathfrak{s}(G)$ contains ρ). Thus $\mathcal{V} = \mathcal{V}[\rho]$ for all $(\pi, \mathcal{V}) \in \mathfrak{R}_\mathfrak{s}(G)$. To see that $\mathcal{V} = \mathcal{V}[\rho]$ for $(\pi, \mathcal{V}) \in \mathfrak{R}(G)$ implies $(\pi, \mathcal{V}) \in \mathfrak{R}_\mathfrak{s}(G)$, we observe that $\mathcal{V} = \mathcal{V}[\rho]$ implies π is a quotient of a direct sum of copies of $\operatorname{ind} \rho$ and $\operatorname{ind} \rho \in \mathfrak{R}_\mathfrak{s}(G)$ (since by hypothesis (K, ρ) is an $\mathfrak{s}$-type in G).

We have proved parts i) and iii) of the following theorem which records some of the main properties of types (see Theorem 4.3 in [13]).

THEOREM 7.5. – *Let $\mathfrak{s} \in \mathfrak{B}(G)$ and let (K, ρ) be an $\mathfrak{s}$-type. Then:*

i) The categories $\mathfrak{R}_\rho(G)$ and $\mathfrak{R}_\mathfrak{s}(G)$ are equal as subcategories of $\mathfrak{R}(G)$. In particular, $\mathfrak{R}_\rho(G)$ is closed under subquotients.

ii) Let $(\pi, \mathcal{V})$ be a smooth representation of G. Then there exists a uniquely determined G-subspace $\mathcal{U}$ of $\mathcal{V}$ such that

$$\mathcal{V} = \mathcal{V}[\rho] \oplus \mathcal{U}.$$

iii) The functor

$$\mathcal{V} \mapsto \mathcal{V}^\rho : \mathfrak{R}_\rho(G) \to e_\rho \mathcal{H}(G) e_\rho - \mathfrak{Mod}$$

is an equivalence of categories.

Suppose now that $\chi : {}^0T \to \mathbb{C}^\times$ is a smooth character. Let $\widetilde{\chi}$ be any character of T extending χ. The inertial equivalence class $[T, \widetilde{\chi}]$ depends only on χ. We denote it by $\mathfrak{s}_\chi$. Let $\mathfrak{R}_\chi(G)$ be the resulting (principal series) component of the Bernstein decomposition, i.e. $\mathfrak{R}_\chi(G) = \mathfrak{R}_{\mathfrak{s}_\chi}(G)$. We sometimes refer to it as the component of the Bernstein decomposition corresponding to χ.

LEMMA 7.6. – *For $t \in T$, let ϕ_t be a non-zero element of $\mathcal{H}(G, \rho)$ supported on $J\,t\,J$. Then ϕ_t is invertible in $\mathcal{H}(G, \rho)$.*

Proof. – Suppose $t \in T$ corresponds to w under $N/{}^0T \simeq W$. Then $w \in W_\chi$ (in fact $w \in Y$). By Lemma 6.2, we may write $w = s_1 \dots s_r \omega$ where $s_1, \dots, s_r \in S_\chi^0$, $l(s_1 \dots s_r) = r$ and $\omega \in \Omega_\chi$. Fix a support-preserving isomorphism $\Psi : \mathcal{H}_\chi \xrightarrow{\sim} \mathcal{H}(G, \rho)$. Then $\Psi^{-1}(\phi_t)$ is a non-zero scalar times $e_{s_1} \dots e_{s_r} e_\omega$. Since each e_{s_i} is invertible and e_ω is invertible, $\Psi^{-1}(\phi_t)$ is also invertible. Hence ϕ_t is invertible. □

THEOREM 7.7. – *The pair (J_χ, ρ_χ) is an $\mathfrak{s}_\chi$-type in G.*

Proof. – We apply Theorem 8.3 of Bushnell and Kutzko [13]. It is clear that the pair $({}^0T, \chi)$ is a type for $\mathfrak{R}_\chi(T)$ (the component of the Bernstein decomposition of T determined by χ). Thus it is sufficient to verify that (J, ρ) is a G-cover of $({}^0T, \chi)$ in the language of [13] (see Definition 8.1). From Proposition 3.6, we see that (J, ρ) satisfies conditions i) and ii) in the definition of G-cover. That condition iii) is also satisfied follows immediately from Lemma 7.6. □

Remark 7.8. – Let $B = TU$ be a Borel subgroup containing T with unipotent radical U. Let $(\pi, \mathcal{V})$ be a smooth representation of G. The key point in proving Theorem 7.7 (which remains implicit in our proof since we invoke Thm. 8.3 of [13]) is to show that the Jacquet module functor $\mathcal{V} \mapsto \mathcal{V}_U$ induces an isomorphism of $\mathbb{C}$-vector spaces:

$$\mathcal{V}^\rho \xrightarrow{\simeq} \mathcal{V}_U^\chi.$$

This is a consequence of Lemma 7.6. An easy argument (using Frobenius reciprocity) then shows that (J, ρ) is an $\mathfrak{s}_\chi$-type.

COROLLARY 7.9. – *i) The categories $\mathfrak{R}_{\mathfrak{s}_\chi}(G)$ and $\mathfrak{R}_\rho(G)$ are equal as subcategories of $\mathfrak{R}(G)$. In particular, if $(\pi, \mathcal{V}) \in \mathfrak{R}_\rho(G)$ then every G-subquotient of π is also in $\mathfrak{R}_\rho(G)$.*

ii) Let $(\pi, \mathcal{V})$ be any smooth representation of G. There exists a uniquely determined G-subspace $\mathcal{U}$ of $\mathcal{V}$ such that

$$\mathcal{V} = \mathcal{V}[\rho] \oplus \mathcal{U}.$$

iii) The functor $\mathcal{V} \mapsto \mathcal{V}^\rho : \mathfrak{R}_\rho(G) \to \mathcal{H}(G, \rho) - \mathfrak{Mod}$ is an equivalence of categories.

Proof. – This is Theorem 7.5 for the type (J, ρ). □

8. A dual group interpretation of $\mathcal{H}(G, \rho)$

We begin by rephrasing Theorem 6.3 on the structure of the Hecke algebra $\mathcal{H}(G, \rho)$. We define a reductive $\mathcal{O}_F$-group $\widetilde{\mathbb{H}}$ in terms of χ (in general disconnected) with group of F-rational points $\widetilde{H}$ and show that there exists a $*$-preserving, support-preserving isomorphism from $\mathcal{H}(\widetilde{H}, 1_\mathcal{I})$ to $\mathcal{H}(G, \rho)$ where $\mathcal{I}$ is a certain Iwahori subgroup of H (the group of F-rational points of the identity component of $\widetilde{\mathbb{H}}$). The groups H and $\widetilde{H}$ have natural interpretations in terms of Langlands parameters for T. (The key idea here was suggested to me by Neil Chriss and Allen Moy.) We discuss this next and give two examples. We then show the quotient $\widetilde{H}/H$ is always abelian and is trivial when $\mathbb{G}$ has connected centre. Finally we observe that H is an endoscopic group of G.

First, we define the groups H and $\widetilde{H}$. The quadruple $\Psi_\chi = (X, \Phi_\chi, Y, \Phi_\chi^\vee)$ is a root datum. Hence there exists a connected reductive $\mathcal{O}_F$-group $\mathbb{H}$ with $\mathcal{O}_F$-split maximal torus

$\mathbb{T}_{\mathbb{H}}$ (unique up to $\mathcal{O}_F$-isomorphism) such that the root datum $\Psi(\mathbb{H},\mathbb{T}_{\mathbb{H}})$ equals Ψ_χ. We may (and will) assume that $\mathbb{T}_{\mathbb{H}} = \mathbb{T}$. As usual we denote by H and T the corresponding groups of F-rational points. Note that $\overline{W}_\chi^0 = \overline{W}(H)$ (the ordinary Weyl group of H). Let $\Phi_\chi^+ = \Phi_\chi \cap \Phi^+$. Then Φ_χ^+ is a positive system in Φ_χ. Put $C_\chi = \{w \in \overline{W}_\chi \mid w\Phi_\chi^+ = \Phi_\chi^+\}$. Write $\mathbb{B}_{\mathbb{H}}$ for the Borel subgroup of $\mathbb{H}$ corresponding to Φ_χ^+ with group of F-rational points B_H. It is clear that C_χ acts on the based root datum $\Psi_0(\mathbb{H}) = (X,\Pi_\chi,Y,\Pi_\chi^\vee)$ where Π_χ is the unique simple system contained in Φ_χ^+. Fix $\mathcal{O}_F$-isomorphisms $x_\alpha : \mathbb{G}_a \to \mathbb{U}_\alpha$ $(\alpha \in \Phi_\chi)$. Put $u_\alpha = x_\alpha(1)$ $(\alpha \in \Pi_\chi)$. The short exact sequence:

$$1 \to \operatorname{Int}\mathbb{H} \to \operatorname{Aut}\mathbb{H} \to \operatorname{Aut}\Psi_0(\mathbb{H}) \to 1.$$

is split by an isomorphism:

$$\operatorname{Aut}\Psi_0(\mathbb{H}) \quad \xrightarrow{\simeq} \quad \operatorname{Aut}(\mathbb{H},\mathbb{B}_{\mathbb{H}},\mathbb{T},(u_\alpha)_{\alpha\in\Pi_\chi}).$$

Via this splitting, C_χ embeds in $\operatorname{Aut}\mathbb{H}$. By definition, $\widetilde{\mathbb{H}} = \mathbb{H} \rtimes C_\chi$. Then $\widetilde{H} = H \rtimes C_\chi$.

As before, let $\Phi_{\chi,\mathrm{af}} = \{a \in \Phi_{\mathrm{af}} \mid \mathrm{gr}(a) \in \Phi_\chi\}$ where $\mathrm{gr}(a)$ denotes the linear part of a. Note that $\Phi_{\chi,\mathrm{af}}$ is the set of affine roots of H with respect to T. Let $W(H)$ be the extended affine Weyl group of H with respect to T. Thus $W(H) = Y \rtimes \overline{W}_\chi^0$. As above

$$\mathcal{C}_\chi = \{v \in V : 0 < \alpha(v) < 1 \ \ (\alpha \in \Phi_\chi^+)\}$$

is a chamber with respect to the decomposition of V induced by the affine functionals in $\Phi_{\chi,\mathrm{af}}$. Let

$$\Omega(H) = \{w \in W(H) \mid w\,\mathcal{C}_\chi = \mathcal{C}_\chi\}.$$

LEMMA 8.1. – *i)* $\overline{W}_\chi = \overline{W}_\chi^0 \rtimes C_\chi$.
ii) $\Omega_\chi = \Omega(H) \rtimes C_\chi$.

Proof. – i) The normality of $\overline{W}_\chi^0$ in $\overline{W}_\chi$ follows directly from the definition of $\overline{W}_\chi^0$. Let $w \in \overline{W}_\chi^0$. Then $w\,\Phi_\chi^+ \subset \Phi_\chi$ is a positive system for Φ_χ. Hence there exists a $w_1 \in \overline{W}_\chi^0$ (the Weyl group of Φ_χ) such that $w\,\Phi_\chi^+ = w_1\,\Phi_\chi^+$, i.e. $w_1^{-1}w \in C_\chi$. Hence $\overline{W}_\chi = \overline{W}_\chi^0 C_\chi$. It is clear that $\overline{W}_\chi^0 \cap C_\chi = 1$. This proves i).

ii) From $W(H) = Y \rtimes \overline{W}_\chi^0$, we see $\Omega(H) = \{(y,w) \in \Omega_\chi : w \in \overline{W}_\chi^0\}$. Since an element $v \in V$ lies in $\mathcal{C}_\chi$ if and only if $0 < \alpha(v) < 1$ for all $\alpha \in \Phi_\chi^+$ and $w\,\Phi_\chi^+ = \Phi_\chi^+$ for all $w \in C_\chi$, we see $w\,\mathcal{C}_\chi = \mathcal{C}_\chi$, i.e. C_χ is a subgroup of Ω_χ. Suppose $c \in C_\chi$ and $(y,w) \in \Omega(H)$. Then $c\,(y,w)\,c^{-1} = (c(y), cwc^{-1})$ and $cwc^{-1} \in \overline{W}_\chi^0$ (by i)). Hence C_χ normalizes $\Omega(H)$.

Suppose now that $(y,w) \in \Omega_\chi$. By i) we may write $w \in \overline{W}_\chi$ as $w = w_1w_2$ where $w_1 \in \overline{W}_\chi^0$ and $w_2 \in C_\chi$. Then $(y,w) = (y,w_1)\,w_2$ is an element of $\Omega(H)\,C_\chi$ as required. □

The decomposition $\Omega_\chi = \Omega(H) \rtimes C_\chi$ implies $\mathbb{C}[\Omega_\chi] = \mathbb{C}[\Omega(H)]\,\widetilde{\otimes}\,\mathbb{C}[C_\chi]$ where the twisted tensor product is defined by the conjugation action of C_χ on $\Omega(H)$. Hence:

$$\begin{aligned} (8.1) \qquad \mathcal{H}(W_\chi^0,S_\chi^0)\,\widetilde{\otimes}\,\mathbb{C}[\Omega_\chi] &\simeq \mathcal{H}(W_\chi^0,S_\chi^0)\,\widetilde{\otimes}\,(\mathbb{C}[\Omega(H)]\,\widetilde{\otimes}\,\mathbb{C}[C_\chi]) \\ &\simeq (\mathcal{H}(W_\chi^0,S_\chi^0)\,\widetilde{\otimes}\,\mathbb{C}[\Omega(H)])\,\widetilde{\otimes}\,\mathbb{C}[C_\chi]. \end{aligned}$$

The positive system Φ_χ^+ uniquely determines an Iwahori subgroup of H which we denote by $\mathcal{I}$. Thus

$$\mathcal{I} = \prod_{\alpha\in\Phi_\chi^-} U_{\alpha,1}\, {}^0T \prod_{\alpha\in\Phi_\chi^+} U_{\alpha,0}.$$

As in Iwahori and Matsumoto [23], there is a $*$-preserving, support-preserving isomorphism:

$$\mathcal{H}(H, 1_\mathcal{I}) \;\simeq\; \mathcal{H}(W_\chi^0, S_\chi^0)\,\widetilde{\otimes}\,\mathbb{C}[\Omega(H)]. \tag{8.2}$$

It is easy to see that there is a $*$-preserving, support-preserving isomorphism:

$$\mathcal{H}(\widetilde{H}, 1_\mathcal{I}) \;\simeq\; \mathcal{H}(H, 1_\mathcal{I})\,\widetilde{\otimes}\,\mathbb{C}[C_\chi]. \tag{8.3}$$

Combining (8.1), (8.2), (8.3) and Theorem 6.3 we obtain the following theorem.

THEOREM 8.2. – *The algebras $\mathcal{H}(\widetilde{H}, 1_\mathcal{I})$ and $\mathcal{H}(G, \rho)$ are isomorphic via a family of $*$-preserving, support-preserving (and hence inner-product preserving) isomorphisms.*

Suppose now that $\widetilde{\chi}$ is a character of T extending χ. We write $\widehat{T}$ for the complex torus dual to T in the sense of Langlands (thus $X^*(\widehat{T}) = X_*(T)$ and $X_*(\widehat{T}) = X^*(T)$). Let W_F be the (absolute) Weil group of F. The Langlands parameter of $\widetilde{\chi} : T \to \mathbb{C}^\times$ is the homomorphism $\tau_{\widetilde{\chi}} : W_F \to \widehat{T}$ defined by

$$\alpha(\tau_{\widetilde{\chi}}(\sigma)) = \widetilde{\chi}(\alpha(\tau_F(\sigma))) \tag{8.4}$$

for all $\alpha \in X^*(\widehat{T}) = X_*(T)$ and all $\sigma \in W_F$. Here $\tau_F : W_F \to F^\times$ induces the isomorphism $W_F^{\mathrm{ab}} \xrightarrow{\sim} F^\times$ of local class field theory. Since $\tau_F^{-1}(\mathcal{O}_F^\times) = I_F$ (the inertia group of F), the restriction $\tau_{\widetilde{\chi}} \mid I_F$ depends only on χ. We denote it by τ_χ.

The centraliser in $\widehat{G}$ (the Langlands dual group of G) of the image of τ_χ is the reductive subgroup of $\widehat{G}$ generated by $\widehat{T}$ and those root groups U_α for which $\alpha \mid \mathrm{im}\,\tau_\chi$ is trivial together with those Weyl group representatives n_w for which $w(t) = t$ for all $t \in \mathrm{im}\,\tau_\chi$. The identity component is then generated by $\widehat{T}$ and those U_α for which $\alpha \mid \mathrm{im}\,\tau_\chi$ is trivial (see e.g. [33] 4.1). Write $\overline{W}(\mathrm{im}\,\tau_\chi)$ for the stabiliser in $\overline{W} = W(T, G) = W(\widehat{T}, \widehat{G})$ of $\mathrm{im}\,\tau_\chi$ and $\overline{W}(\mathrm{im}\,\tau_\chi)^0$ for the normal subgroup generated by those reflections s_α for which $\alpha \mid \mathrm{im}\,\tau_\chi = 1$. Then

$$C_{\widehat{G}}(\mathrm{im}\,\tau_\chi)/C_{\widehat{G}}(\mathrm{im}\,\tau_\chi)^0 \;\simeq\; \overline{W}(\mathrm{im}\,\tau_\chi)/\overline{W}(\mathrm{im}\,\tau_\chi)^0.$$

Using 8.4, the condition $\alpha \mid \mathrm{im}\,\tau_\chi = 1$ is easily seen to be equivalent to $\chi \circ \alpha \mid \mathcal{O}_F^\times = 1$, i.e. $\alpha \in \Phi_\chi^\vee$ (viewing α as an element of $\Phi(T, G)^\vee = \Phi(\hat{T}, \hat{G})$). Again from 8.4, we see $w(\tau_\chi(\sigma)) = \tau_\chi(\sigma)$ for all $\sigma \in I_F$ if and only if $w\chi = \chi$, i.e. $w \in \overline{W}_\chi$. Thus

$$\begin{aligned}\Psi(C_{\widehat{G}}(\mathrm{im}\,\tau_\chi)^0, \widehat{T}) &= (X^*(\widehat{T}), \Phi_\chi^\vee, X_*(\widehat{T}), \Phi_\chi)\\ &= (Y, \Phi_\chi^\vee, X, \Phi_\chi)\\ &= \Psi(\mathbb{H}, \mathbb{T})^\vee.\end{aligned}$$

Hence $C_{\widehat{G}}(\mathrm{im}\,\tau_\chi)^0$ may be viewed as the Langlands dual group $\widehat{H}$ of H. The component group $\pi_0(C_{\widehat{G}}(\mathrm{im}\,\tau_\chi))$ is given by $\overline{W}_\chi/\overline{W}_\chi^0$ and so is isomorphic to $C_\chi = \pi_0(\mathbb{H})$. Fix non-trivial elements $u_\alpha \in U_\alpha$ where U_α is the α root group for $(\mathcal{G}, \mathcal{T}) = (C_{\widehat{G}}(\operatorname{im} \tau_\chi)^0, \widehat{T})$ $(\alpha \in \Pi_\chi^\vee)$. We have

$$C_\chi \hookrightarrow \operatorname{Aut}(\mathcal{G})/\operatorname{Inn}(\mathcal{G}) \simeq \operatorname{Aut}(\mathcal{G}, \mathcal{T}, (u_\alpha)_{\alpha \in \Pi_\chi^\vee}).$$

Thus we obtain an action of C_χ on $\Psi(\mathbb{H}, \mathbb{T})^\vee$. It is clear that this action coincides (under transpose inverse) with the action of C_χ on $\Psi_\chi = \Psi(\mathbb{H}, \mathbb{T})$. In rough terms $C_{\widehat{G}}(\operatorname{im} \tau_\chi)$ might be considered 'the dual group' of $\widetilde{H}$.

Example 8.3 (cf. Sanje-Mpacko [31].). – Let $G = SL(n, F)$. Then

$$^0T \quad \simeq \quad \{(\lambda_1, \ldots, \lambda_n) \in (\mathcal{O}_F^\times)^n : \lambda_1 \ldots \lambda_n = 1\}.$$

Let $\chi_1 : \mathcal{O}_F^\times \to \mathbb{C}^\times$ be a character of order n (e.g. suppose $n \mid (q-1)$ where $q = |k_F|$). Define $\chi : {}^0T \to \mathbb{C}^\times$ by

$$\chi(\lambda_1, \ldots, \lambda_n) = \chi_1(\lambda_1)\chi_1^2(\lambda_2) \ldots \chi_1^{n-1}(\lambda_{n-1}) \quad (\lambda_i \in \mathcal{O}_F^\times).$$

The root datum $\Psi_\chi = (X, \emptyset, Y, \emptyset)$ where $X \simeq \mathbb{Z}^n/\mathbb{Z}(1, \ldots, 1)$ and

$$Y \quad \simeq \quad \{(l_1, \ldots, l_n) \in \mathbb{Z}^n : \sum l_i = 0\}.$$

In particular, $\overline{W}_\chi^0 = \{1\}$. An easy calculation shows that $\overline{W}_\chi \simeq \mathbb{Z}/n\mathbb{Z}$ (with generator $(\lambda_1, \ldots, \lambda_n) \mapsto (\lambda_n, \lambda_1, \ldots, \lambda_{n-1})$). Thus $W_\chi \simeq Y \rtimes \mathbb{Z}/n\mathbb{Z}$ and the Hecke algebra $\mathcal{H}(G, \rho)$ is isomorphic to $\mathbb{C}[W_\chi]$ (the group algebra of W_χ). Finally, $H = T$ and $\widetilde{H} = H \rtimes \mathbb{Z}/n\mathbb{Z}$ (where e.g. $\overline{1} \in \mathbb{Z}/n\mathbb{Z}$ acts by $(\lambda_1, \ldots, \lambda_n) \mapsto (\lambda_n, \lambda_1, \ldots, \lambda_{n-1})$ $(\lambda_i \in F^\times)$).

Example 8.4. – Let $G = Sp_{2n}(F)$ $(n \geq 2)$. Then $^0T \simeq (\mathcal{O}_F^\times)^n$ and thus a character $\chi : {}^0T \to \mathbb{C}^\times$ is given by an n-tuple of characters of $\mathcal{O}_F^\times$. Let $\chi = (\chi_1, \chi_1, \ldots, \chi_1)$ where $\chi_1^2 = 1$, $\chi_1 \neq 1$. The root datum Ψ_χ then equals $(\mathbb{Z}^n, \{\pm e_i \pm e_j\}, \mathbb{Z}^n, \{\pm e_i \pm e_j\})$ using standard notation. Thus $\Phi_\chi = \Phi_\chi^\vee$ has type D_n for $n \geq 3$ and type $A_1 \times A_1$ for $n = 2$. Clearly, $\overline{W}_\chi = \overline{W}$ and $\overline{W}_\chi^0$ is the subgroup of index two consisting of signed permutations with an even number of sign changes. The group $H = SO_{2n}(F)$ and $\widetilde{H} = O_{2n}(F)$. In particular, $\mathcal{H}(G, \rho)$ is isomorphic to the Iwahori Hecke algebra of $O_{2n}(F)$.

We now show

- the group C_χ is always abelian,
- $\mathbb{G}$ has connected centre implies $C_\chi = \{1\}$ and hence $H = \widetilde{H}$.

Tensoring the split exact sequence

$$1 \longrightarrow 1 + \mathcal{P}_F \longrightarrow \mathcal{O}_F^\times \longrightarrow k_F^\times \longrightarrow 1.$$

with Y yields the split exact sequence

$$1 \longrightarrow Y \otimes (1 + \mathcal{P}_F) \longrightarrow Y \otimes \mathcal{O}_F^\times \longrightarrow Y \otimes k_F^\times \longrightarrow 1.$$

Hence the sequence

$$1 \longrightarrow T_1 \longrightarrow {}^0T \longrightarrow \mathbb{T}(k_F) \longrightarrow 1$$

is again split exact and thus $\chi : {}^0T \to \mathbb{C}^\times$ factors as $\chi = \chi_1\mu$ where $\chi_1 = \chi \mid T_1$ (by abuse of notation) and μ is trivial on T_1. Therefore $\tau_\chi = \tau_{\chi_1}\tau_\mu$ and $\operatorname{im}\tau_\chi = \operatorname{im}\tau_{\chi_1}\operatorname{im}\tau_\mu$. Note $\operatorname{im}\tau_\mu$ is cyclic (since $k_F^\times$ is cyclic) and $\operatorname{im}\tau_{\chi_1}$ consists of elements of p-power order (where $p = \operatorname{char} k_F$).

Suppose now that $\mathbb{G}$ has connected centre. Then $X/\mathbb{Z}\Phi$ is torsion free so that $\mathbb{Z}\Phi$ is a direct summand of X (where $X = X^*(T) = X_*(\hat{T})$). Thus the coroot lattice of $\widehat{G}$ is a direct summand of the cocharacter lattice of $\widehat{T}$. This is equivalent to $\widehat{G}$ having simply connected derived group. We use the following results of Steinberg:

- Let $\mathcal{G}$ be a connected reductive group (over an algebraically closed field) with simply connected derived group. Let t be a semisimple element of $\mathcal{G}$. Then $C_{\mathcal{G}}(t)$ (the centraliser of t in $\mathcal{G}$) is connected. (This is Steinberg's connectedness theorem (see Steinberg [34]).)
- Suppose moreover that $t^n \in Z(\mathcal{G})$ for some n divisible by no torsion prime for the root system of $\mathcal{G}$. Then the derived group of (the connected group) $C_{\mathcal{G}}(t)$ is again simply connected (see Steinberg [35]).

A prime l is a torsion prime for the root system Ψ if $\mathbb{Z}\Psi^\vee/\mathbb{Z}\Psi_1^\vee$ has l-torsion for some closed subsystem Ψ_1 of Ψ. (See Springer and Steinberg [33] pp. 178-179 for a listing of the possible torsion primes.) Our restrictions on $p = \operatorname{char} k_F$ imply that p is not a torsion prime for $\Phi^\vee$.

Applying these results, we see $\widehat{G}_1 = C_{\widehat{G}}(\operatorname{im}\tau_{\chi_1})$ is connected with simply connected derived group. The connectedness theorem then implies

$$C_{\widehat{G}_1}(\operatorname{im}\tau_\mu) = C_{\widehat{G}}(\operatorname{im}\tau_\chi)$$

is connected (since $\operatorname{im}\tau_\mu$ is cyclic). Hence $C_\chi = \{1\}$ and $H = \widetilde{H}$.

If $\mathbb{G}$ does not have connected centre then $X/\mathbb{Z}\Phi$ (the group of rational characters of the centre) has non-trivial torsion. In this case we fix a surjective homomorphism $q : L \to X/\mathbb{Z}\Phi$ where L is free abelian of finite rank. Consider the pullback diagram (cf. [25] pages 119-120):

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{p_1} & L \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{p} & X/\mathbb{Z}\Phi \end{array}$$

Thus $\widetilde{X} = \{(x,l) \in X \times L \mid p(x) = q(l)\}$ where p is the obvious surjection and p_1, p_2 are the canonical projections. Let

$$\widetilde{\Phi} = \{(\alpha, 0) \in X \times L : \alpha \in \Phi\}.$$

Note that $\widetilde{\Phi}$ is contained in $\widetilde{X}$ and that p_1 induces an isomorphism $\widetilde{X}/\mathbb{Z}\widetilde{\Phi} \xrightarrow{\sim} L$. In particular, $\widetilde{X}/\mathbb{Z}\widetilde{\Phi}$ is torsion free.

Let $\widetilde{\mathbb{T}}$ be the F-split F-torus such that $X^*(\widetilde{\mathbb{T}}) = \widetilde{X}$. Let $\widetilde{T} = \widetilde{\mathbb{T}}(F)$. Then

$$\widetilde{T} = \operatorname{Hom}(\widetilde{X}, F^\times) \supset \operatorname{Hom}(X, F^\times) = T.$$

By standard arguments (e.g. using root data) the inclusion $T \subset \widetilde{T}$ extends to an inclusion $G \subset \widetilde{G} = \widetilde{\mathbb{G}}(F)$ where $\widetilde{\mathbb{G}}$ is a connected reductive F-group with maximal torus $\widetilde{\mathbb{T}}$ and $\widetilde{\Phi} = \Phi(\widetilde{\mathbb{G}}, \widetilde{\mathbb{T}})$. By construction $\widetilde{\mathbb{G}}$ has connected centre (since $\widetilde{X}/\mathbb{Z}\widetilde{\Phi}$ is torsion free).

Let $\widetilde{\chi}$ be any extension of χ to ${}^0\widetilde{T}$. Since $\widetilde{\mathbb{G}}$ has connected centre, we have $C_{\widetilde{\chi}} = \{1\}$ or $\overline{W}_{\widetilde{\chi}} = \overline{W}^0_{\widetilde{\chi}}$. Note also that $\overline{W}^0_{\widetilde{\chi}} = \overline{W}^0_{\chi}$ (since the image of any coroot of $\widetilde{\mathbb{G}}$ with respect to $\widetilde{\mathbb{T}}$ is contained in $\mathbb{T}$). Thus for $c \in C_{\chi}$, we have $c\widetilde{\chi} = \widetilde{\chi}\mu_c$ where μ_c is trivial on ${}^0T$, i.e. μ_c belongs to the Pontryagin dual $({}^0\widetilde{T}/{}^0T)^{\wedge}$ of ${}^0\widetilde{T}/{}^0T$. Further, $\mu_c = 1$ if and only if $c \in C_{\chi} \cap \overline{W}^0_{\chi} = \{1\}$. It is easy to verify that the map

$$c \mapsto \mu_c : C_{\chi} \to ({}^0\widetilde{T}/{}^0T)^{\wedge}$$

is a homomorphism (e.g. by showing that $w(t)t^{-1} \in T$ for all $t \in \widetilde{T}$ and all $w \in \overline{W}$). Thus C_{χ} embeds in $({}^0\widetilde{T}/{}^0T)^{\wedge}$. In particular, C_{χ} is abelian.

Finally we observe that H is an endoscopic group of G. Since $\mathbb{H}$ is F-split, we only have to show that $C_{\widehat{G}}(\operatorname{im}\tau_{\chi})^0 = C_{\widehat{G}}(x)^0$ for some $x \in \widehat{T}$. We prove the following:

- Let $\mathcal{G}$ be a connected reductive group over an algebraically closed field. Suppose S_1 and S_2 are finite subgroups of a fixed maximal torus $\mathcal{T}$ of $\mathcal{G}$ where S_2 is cyclic and the orders of S_1 and S_2 are relatively prime. Finally, suppose the order of S_1 is not divisible by a bad prime for $\Phi = \Phi(\mathcal{G}, \mathcal{T})$. Then $C_{\mathcal{G}}(S)^0 = C_{\mathcal{G}}(x)^0$ for some $x \in \mathcal{T}$ where $S = S_1S_2$.

This clearly implies that H is endoscopic for G (taking $(\mathcal{G}, \mathcal{T}) = (\widehat{G}, \widehat{T})$, $S_1 = \operatorname{im}\tau_{\chi_1}$ and $S_2 = \operatorname{im}\tau_{\mu}$).

We first show $C_{\mathcal{G}}(S_1)^0 = C_{\mathcal{G}}(t)^0$ for some $t \in \mathcal{T}$. Let

$$\Phi_1 = \{\alpha \in \Phi : \alpha(s_1) = 1 \;\; \forall s_1 \in S_1\}.$$

Clearly Φ_1 is a closed subsystem of Φ. Since (by definition) $\mathbb{Z}\Phi/\mathbb{Z}\Phi_1$ can only have l-torsion for a prime l when l is a bad prime for Φ and since $s_1 \mapsto s_1^l$ is an automorphism of S_1, we see $\mathbb{Z}\Phi/\mathbb{Z}\Phi_1$ is torsion free. Hence $\mathbb{Z}\Phi_1$ is a direct summand of $\mathbb{Z}\Phi$. It follows that there exists an element $t \in \mathcal{T}$ such that $\alpha(t) = 1$ for $\alpha \in \mathbb{Z}\Phi$ if and only if $\alpha \in \mathbb{Z}\Phi_1$ (see e.g. Humphreys [21] 16.2, Lemma C). Then $C_{\mathcal{G}}(S_1)^0 = C_{\mathcal{G}}(t)^0$.

Note $C_{\mathcal{G}}(S)^0$ is the identity component of the centraliser of s in $C_{\mathcal{G}}(t)^0$ where s generates S_2. This is the group generated by $\mathcal{T}$ and those root groups $\mathcal{U}_{\alpha}$ for which $\alpha(t) = 1$ and $\alpha(s) = 1$. We show this is equivalent to $\alpha(st) = 1$ by proving $\alpha(st) = 1$ implies $\alpha(t) = 1$. It follows that $C_{\mathcal{G}}(S)^0 = C_{\mathcal{G}}(st)^0$. Thus suppose $\alpha(st) = 1$. Then $\alpha(t)^m = 1$ where m is the order of s and so $m\alpha \in \mathbb{Z}\Phi_1$. Since $\mathbb{Z}\Phi/\mathbb{Z}\Phi_1$ is torsion free, this implies $\alpha \in \mathbb{Z}\Phi_1 \cap \Phi = \Phi_1$. Hence $\alpha(t) = 1$ (and $\alpha(s) = 1$).

9. Jacquet and induction functors via Hecke algebra homomorphisms

Let $B = TU$ be a Borel subgroup containing T with unipotent radical U. Write $\operatorname{Ind}_B^G : \mathfrak{R}(T) \to \mathfrak{R}(G)$ for the unnormalised (parabolic) induction functor and $R_U : \mathfrak{R}(G) \to \mathfrak{R}(T)$ for the Jacquet restriction functor. As above, we write $\mathfrak{R}_{\chi}(T)$ for the component of the Bernstein decomposition of T determined by χ. Thus $\mathfrak{R}_{\chi}(T)$ is the full subcategory of $\mathfrak{R}(T)$ consisting of all smooth representations $(\sigma, \mathcal{W})$ of T such that each irreducible T-subquotient of σ is given by a character $\widetilde{\chi} : T \to \mathbb{C}^\times$ extending χ (equivalently, σ on restriction to ${}^0T$ is a multiple of χ).

The functor Ind_B^G restricts to $\mathfrak{R}_\chi(T)$ to yield a functor $\mathrm{Ind}_B^G : \mathfrak{R}_\chi(T) \to \mathfrak{R}_\chi(G)$. If $(\pi, \mathcal{V}) \in \mathfrak{R}_\chi(G)$, we have

$$R_U(\mathcal{V}) \in \prod_{w \in \overline{W}} \mathfrak{R}_{w\chi}(T).$$

Composing $R_U \mid \mathfrak{R}_\chi(G)$ with the projection functor $p_\chi : \prod_{w\in\overline{W}} \mathfrak{R}_{w\chi}(T) \to \mathfrak{R}_\chi(T)$, we obtain a functor $p_\chi \circ R_U : \mathfrak{R}_\chi(G) \to \mathfrak{R}_\chi(T)$. It is clear that this functor is left adjoint to $\mathrm{Ind}_B^G : \mathfrak{R}_\chi(T) \to \mathfrak{R}_\chi(G)$ (since $\mathrm{Hom}_T(\mathfrak{R}_{w\chi}(T), \mathfrak{R}_\chi(T)) = 0$ if $w\chi \neq \chi$). By abuse of notation, we often drop the p_χ and simply write $R_U : \mathfrak{R}_\chi(G) \to \mathfrak{R}_\chi(T)$.

We write $i_B^G : \mathfrak{R}(T) \to \mathfrak{R}(G)$ and $r_U : \mathfrak{R}(G) \to \mathfrak{R}(T)$ for the normalised induction and Jacquet restriction functors. Thus if $(\sigma, \mathcal{W})$ is a smooth representation of T and δ_B is the modulus character for the action of T on U by conjugation (i.e. $d(tut^{-1}) = \delta_B(t)\, du$ for $t \in T$ where du is any Haar measure on U), then $i_B^G(\sigma) = \mathrm{Ind}_B^G(\sigma \otimes \delta_B^{1/2})$. If $(\pi, \mathcal{V})$ is a smooth representation of G, $r_U(\mathcal{V}) = R_U(\mathcal{V}) \otimes \delta_B^{-1/2}$. Since δ_B is an unramified character of T, we obtain (as above) functors $i_B^G : \mathfrak{R}_\chi(T) \to \mathfrak{R}_\chi(G)$ and $r_U : \mathfrak{R}_\chi(G) \to \mathfrak{R}_\chi(T)$ (more properly $p_\chi \circ r_U$). It is clear that r_U is left adjoint to i_B^G.

From Corollary 7.9, the functor $\mathcal{V} \mapsto \mathcal{V}^\rho : \mathfrak{R}_\chi(G) \to \mathcal{H}(G,\rho) - \mathfrak{Mod}$ is an equivalence of categories. The functor $\mathcal{W} \mapsto \mathcal{W}^\chi : \mathfrak{R}_\chi(T) \to \mathcal{H}(T,\chi) - \mathfrak{Mod}$ is also an equivalence of categories ($({}^0T, \chi)$ is a type for $\mathfrak{R}_\chi(T)$). Our first aim in this section is to show that, under these equivalences, the induction and Jacquet restriction functors between $\mathfrak{R}_\chi(T)$ and $\mathfrak{R}_\chi(G)$ (unnormalised and normalised) correspond to induction and restriction functors between $\mathcal{H}(T,\chi) - \mathfrak{Mod}$ and $\mathcal{H}(G,\rho) - \mathfrak{Mod}$ induced by appropriate twists of a $\mathbb{C}$-algebra embedding $t_B : \mathcal{H}(T,\chi) \to \mathcal{H}(G,\rho)$. Given our knowledge of the structure of the Hecke algebra $\mathcal{H}(G,\rho)$, these results are special cases of (or immediate consequences of) a general result of Bushnell and Kutzko (see [13] 7.12, 8.4).

We now describe the algebra embedding $t = t_B : \mathcal{H}(T,\chi) \to \mathcal{H}(G,\rho)$. This is done in a much more general context in sections 6 and 7 of [13]. Fix a uniformiser ϖ in F. The short exact sequence (see equation 4.1)

$$1 \longrightarrow {}^0T \longrightarrow T \xrightarrow{H_T} Y \longrightarrow 0$$

is then split by the homomorphism $y \mapsto y(\varpi) : Y \to T$. For each $y \in Y$, let ϕ_y be the unique element of $\mathcal{H}(T,\chi)$ with support ${}^0Ty$ such that $\phi_y(y(\varpi)) = 1$. It is clear that the elements ϕ_y for $y \in Y$ form a vector space basis of $\mathcal{H}(T,\chi)$ (and that the $\mathbb{C}$-linear extension of the assignment $y \mapsto \phi_y$ defines an isomorphism of $\mathbb{C}$-algebras between $\mathbb{C}[Y]$ and $\mathcal{H}(T,\chi)$). For each $y \in Y$, let Φ_y be the unique element of $\mathcal{H}(G,\rho)$ with support JyJ such that $\Phi_y(y(\varpi)) = 1$. The Borel subgroup B corresponds to a system of simple roots Π_B in $\Phi(G,T)$. Define a subsemigroup Y^+ of Y as follows:

$$Y^+ = \{y \in Y \mid \langle y, \alpha\rangle \geq 0 \quad (\alpha \in \Pi_B)\}.$$

Given $y \in Y$ we may write $y = y_1 - y_2$ where $y_1, y_2 \in Y^+$. By definition

$$t(\phi_y) = \Phi_{y_1}\Phi_{y_2}^{-1}.$$

(Note Φ_{y_2} is indeed invertible in $\mathcal{H}(G,\rho)$ by Lemma 7.6.) The following lemma shows that $t_B(\phi_y)$ is well-defined (i.e. independent of the choice of $y_1, y_2 \in Y^+$ such that $y_1 - y_2 = y$) and extends to a $\mathbb{C}$-algebra embedding $t_B : \mathcal{H}(T,\chi) \to \mathcal{H}(G,\rho)$.

LEMMA 9.1. – *For $y_1, y_2 \in Y^+$, $\Phi_{y_1}\Phi_{y_2} = \Phi_{y_1+y_2}$.*

Proof. – This is straightforward (using Iwahori decompositions). □

It is easy to check that the homorphism $t : \mathcal{H}(T,\chi) \to \mathcal{H}(G,\rho)$ is independent of the choice of uniformiser ϖ (as the notation indicates).

Suppose now that $j : \mathcal{H}(T,\chi) \to \mathcal{H}(G,\rho)$ is an algebra homomorphism. Then j induces restriction and induction functors $j^* : \mathcal{H}(G,\rho) - \mathfrak{Mod} \to \mathcal{H}(T,\chi) - \mathfrak{Mod}$ and $j_* : \mathcal{H}(T,\chi) - \mathfrak{Mod} \to \mathcal{H}(G,\rho) - \mathfrak{Mod}$ as follows.

If $M \in \mathcal{H}(G,\rho) - \mathfrak{Mod}$, then $j^*(M) = M$ with $\mathcal{H}(T,\chi)$-action given by $\phi.m = j(\phi)m$ for $\phi \in \mathcal{H}(T,\chi)$ and $m \in M$. If $N \in \mathcal{H}(T,\chi) - \mathfrak{Mod}$, $j_*(N) = \mathrm{Hom}_{\mathcal{H}(T,\chi)}(\mathcal{H}(G,\rho), N)$. Here $\mathcal{H}(G,\rho)$ is viewed as an $\mathcal{H}(T,\chi)$-module via j and $\mathcal{H}(G,\rho)$ acts by right translations. Thus $\Phi.\alpha(\Phi') = \alpha(\Phi'\Phi)$ for $\alpha \in \mathrm{Hom}_{\mathcal{H}(T,\chi)}(\mathcal{H}(G,\rho), N)$ and Φ, $\Phi' \in \mathcal{H}(G,\rho)$. It is easy to verify that (j^*, j_*) is an adjoint pair, i.e. there exist natural isomorphisms (of $\mathbb{C}$-vector spaces)

$$\mathrm{Hom}_{\mathcal{H}(T,\chi)}(j^*(M), N) \simeq \mathrm{Hom}_{\mathcal{H}(G,\rho)}(M, j_*(N)) \tag{9.1}$$

for $M \in \mathcal{H}(G,\rho) - \mathfrak{Mod}$ and $N \in \mathcal{H}(T,\chi) - \mathfrak{Mod}$.

If τ is an unramified character of T, we write $t_\tau : \mathcal{H}(T,\chi) \to \mathcal{H}(G,\rho)$ for the algebra embedding given by $t_\tau(\phi) = t(\phi.\tau)$ for $\phi \in \mathcal{H}(T,\chi)$ where $\phi.\tau(t) = \phi(t)\tau(t)$ $(t \in T)$. For ease of notation, we write t_u for the algebra embedding $t_{\delta_B^{1/2}}$.

THEOREM 9.2. – *Let $B = TU$ be a Borel subgroup containing T with unipotent radical U. Let $t : \mathcal{H}(T,\chi) \to \mathcal{H}(G,\rho)$ be the associated embedding of $\mathbb{C}$-algebras (as described above). Then the following diagrams commute (up to natural equivalence).*

$$\begin{array}{ccc} \mathfrak{R}_\chi(G) & \xrightarrow{\simeq} & \mathcal{H}(G,\rho) - \mathfrak{Mod} \\ {\scriptstyle R_U}\downarrow & & \downarrow{\scriptstyle t^*_{\delta_B}} \\ \mathfrak{R}_\chi(T) & \xrightarrow{\simeq} & \mathcal{H}(T,\chi) - \mathfrak{Mod}. \end{array} \tag{9.2}$$

$$\begin{array}{ccc} \mathfrak{R}_\chi(G) & \xrightarrow{\simeq} & \mathcal{H}(G,\rho) - \mathfrak{Mod} \\ {\scriptstyle \mathrm{Ind}_B^G}\uparrow & & \uparrow{\scriptstyle (t_{\delta_B})_*} \\ \mathfrak{R}_\chi(T) & \xrightarrow{\simeq} & \mathcal{H}(T,\chi) - \mathfrak{Mod} \end{array} \tag{9.3}$$

$$\begin{array}{ccc} \mathfrak{R}_\chi(G) & \xrightarrow{\simeq} & \mathcal{H}(G,\rho) - \mathfrak{Mod} \\ {\scriptstyle r_U}\downarrow & & \downarrow{\scriptstyle t^*_u} \\ \mathfrak{R}_\chi(T) & \xrightarrow{\simeq} & \mathcal{H}(T,\chi) - \mathfrak{Mod}. \end{array} \tag{9.4}$$

$$\begin{array}{ccc} \mathfrak{R}_\chi(G) & \xrightarrow{\simeq} & \mathcal{H}(G,\rho) - \mathfrak{Mod} \\ {\scriptstyle i_B^G}\uparrow & & \uparrow{\scriptstyle (t_u)_*} \\ \mathfrak{R}_\chi(T) & \xrightarrow{\simeq} & \mathcal{H}(T,\chi) - \mathfrak{Mod} \end{array} \tag{9.5}$$

In each case the rows are given by the equivalences of categories

$$\mathcal{V} \mapsto \mathcal{V}^{\rho} : \mathfrak{R}_{\chi}(G) \xrightarrow{\simeq} \mathcal{H}(G,\rho) - \mathfrak{Mod}$$

and

$$\mathcal{W} \mapsto \mathcal{W}^{\chi} : \mathfrak{R}_{\chi}(T) \xrightarrow{\simeq} \mathcal{H}(T,\chi) - \mathfrak{Mod}.$$

Proof. – The first two diagrams are commutative by Bushnell and Kutzko [13] 8.4. In fact, if $(\pi, \mathcal{V}) \in \mathfrak{R}_{\chi}(G)$ then it follows from [13] 7.12 that the isomorphism $\mathcal{V}^{\rho} \xrightarrow{\simeq} R_U(\mathcal{V})^{\chi}$ (see Remark 7.8) satisfies

$$R_U(t_{\delta_B}(\phi).v) = \phi.R_U(v) \tag{9.6}$$

for $v \in \mathcal{V}^{\rho}, \phi \in \mathcal{H}(T,\chi)$. The hypotheses of [13] hold in our situation by Lemma 7.6. This gives 9.2 (and also 9.3 by uniqueness of adjoints).

To prove 9.4 it is enough to show

$$r_U(t_u(\phi_y).v) = \phi_y.r_U(v)$$

for $v \in \mathcal{V}^{\rho}$, $y \in Y$. We may rewrite this as

$$R_U(t(\phi_y \delta_B^{1/2}(y)).v) = \delta_B^{-1/2}(y)\phi_y.R_U(v)$$

for $v \in \mathcal{V}^{\rho}$, $y \in Y$. Equivalently

$$R_U(t(\phi_y \delta_B(y)).v) = \phi_y.R_U(v)$$

for $v \in \mathcal{V}^{\rho}$ and $y \in Y$ which we know from 9.6.

The commutativity of 9.5 follows (again by uniqueness of adjoints). □

Thus, under the equivalences of categories $\mathcal{V} \mapsto \mathcal{V}^{\rho} : \mathfrak{R}_{\chi}(G) \to \mathcal{H}(G,\rho) - \mathfrak{Mod}$ and $\mathcal{W} \mapsto \mathcal{W}^{\chi} : \mathfrak{R}_{\chi}(T) \to \mathcal{H}(T,\chi) - \mathfrak{Mod}$, the adjoint pair (R_U, Ind_B^G) corresponds to $(t^*_{\delta_B}, (t_{\delta_B})_*)$ and the pair (r_U, i_B^G) corresponds to $(t^*_{\sqrt{\delta_B}}, (t_{\sqrt{\delta_B}})_*) = (t^*_u, (t_u)_*)$.

Suppose now that the Borel subgroup B containing T corresponds to the positive system Φ^+ of Φ (as in the definition of the subgroup J). Let $B(H)$ be the Borel subgroup of H (with unipotent radical $U(H)$) corresponding to the positive system $\Phi_{\chi}^+ = \Phi_{\chi} \cap \Phi^+$. In the remainder of this section we prove a simple relation between the induction and restriction functors $i_B^G : \mathfrak{R}_{\chi}(T) \to \mathfrak{R}_{\chi}(G)$ and $r_U : \mathfrak{R}_{\chi}(G) \to \mathfrak{R}_{\chi}(T)$ and corresponding functors for $\widetilde{H}$. A precise statement is given in Theorem 9.4.

First we claim that the functor $\mathcal{V} \mapsto \mathcal{V}^{1_{\mathcal{I}}} : \mathfrak{R}_{1_{\mathcal{I}}}(\tilde{H}) \to \mathcal{H}(\tilde{H}, 1_{\mathcal{I}}) - \mathfrak{Mod}$ is an equivalence of categories. This holds provided the category $\mathfrak{R}_{1_{\mathcal{I}}}(\widetilde{H})$ is closed under subquotients (see [12] 3.9 (i)). Thus suppose $\mathcal{V} \in \mathfrak{R}_{1_{\mathcal{I}}}(\widetilde{H})$. Write $r_H^{\widetilde{H}}(\mathcal{V})$ for the smooth representation of H obtained by (ordinary) restriction. Since $\widetilde{H} = H \rtimes C_{\chi}$ and C_{χ} normalises $\mathcal{I}$, $r_H^{\widetilde{H}}(\mathcal{V}) \in \mathfrak{R}_{1_{\mathcal{I}}}(H)$. Moreover since $(\mathcal{I}, 1_{\mathcal{I}})$ is a type in H, each subquotient of $r_H^{\widetilde{H}}(\mathcal{V})$ is again in $\mathfrak{R}_{1_{\mathcal{I}}}(H)$ and thus each subquotient of $\mathcal{V}$ is (a fortiori) in $\mathfrak{R}_{1_{\mathcal{I}}}(\widetilde{H})$.

It is clear that the restriction functor $r_H^{\widetilde{H}} : \mathfrak{R}_{1_{\mathcal{I}}}(\widetilde{H}) \to \mathfrak{R}_{1_{\mathcal{I}}}(H)$ fits into the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{R}_{1_{\mathcal{I}}}(\tilde{H}) & \xrightarrow{\simeq} & \mathcal{H}(\tilde{H}, 1_{\mathcal{I}}) - \mathfrak{Mod} \\ {\scriptstyle r_H^{\tilde{H}}} \downarrow & & \downarrow {\scriptstyle i^*} \\ \mathfrak{R}_{1_{\mathcal{I}}}(H) & \xrightarrow{\simeq} & \mathcal{H}(H, 1_{\mathcal{I}}) - \mathfrak{Mod} \end{array} \tag{9.7}$$

where i^* is induced by the inclusion homomorphism $i : \mathcal{H}(H, 1_{\mathcal{I}}) \to \mathcal{H}(\widetilde{H}, 1_{\mathcal{I}})$ and the rows are given by the equivalences of categories $\mathcal{V} \mapsto \mathcal{V}^{1_{\mathcal{I}}}$.

The functor $r_H^{\widetilde{H}} : \mathfrak{R}_{1_{\mathcal{I}}}(\widetilde{H}) \to \mathfrak{R}_{1_{\mathcal{I}}}(H)$ has a right adjoint $i_H^{\widetilde{H}} : \mathfrak{R}_{1_{\mathcal{I}}}(H) \to \mathfrak{R}_{1_{\mathcal{I}}}(\widetilde{H})$ (ordinary induction from $\widetilde{H}$ to H). Also $i^* : \mathcal{H}(\widetilde{H}, 1_{\mathcal{I}}) - \mathfrak{Mod} \to \mathcal{H}(H, 1_{\mathcal{I}}) - \mathfrak{Mod}$ has a right adjoint $i_* : \mathcal{H}(H, 1_{\mathcal{I}}) - \mathfrak{Mod} \to \mathcal{H}(\widetilde{H}, 1_{\mathcal{I}}) - \mathfrak{Mod}$ (exactly as in 9.1). By uniqueness of adjoints, we obtain the commutative diagram:

$$\begin{array}{ccc} \mathfrak{R}_{1_{\mathcal{I}}}(\widetilde{H}) & \xrightarrow{\simeq} & \mathcal{H}(\widetilde{H}, 1_{\mathcal{I}}) - \mathfrak{Mod} \\ {\scriptstyle i_H^{\widetilde{H}}}\uparrow & & \uparrow{\scriptstyle i_*} \\ \mathfrak{R}_{1_{\mathcal{I}}}(H) & \xrightarrow{\simeq} & \mathcal{H}(H, 1_{\mathcal{I}}) - \mathfrak{Mod}. \end{array} \tag{9.8}$$

Since $(\mathcal{I}, 1_{\mathcal{I}})$ is a type for $\mathfrak{R}_1(H)$, we have $\mathfrak{R}_1(H) = \mathfrak{R}_{1_{\mathcal{I}}}(H)$. Applying Theorem 9.2 iii) and iv) (in this context) and writing $t_{u,H}$ for $t_{\delta_{B_H}^{1/2}}$, we obtain further commutative diagrams:

$$\begin{array}{ccc} \mathfrak{R}_{1_{\mathcal{I}}}(H) & \xrightarrow{\simeq} & \mathcal{H}(H, 1_{\mathcal{I}}) - \mathfrak{Mod} \\ {\scriptstyle r_{U(H)}}\downarrow & & \downarrow{\scriptstyle t_{u,H}^*} \\ \mathfrak{R}_{1_{^0T}}(T) & \xrightarrow{\simeq} & \mathcal{H}(T, 1_{^0T}) - \mathfrak{Mod} \end{array} \tag{9.9}$$

and

$$\begin{array}{ccc} \mathfrak{R}_{1_{\mathcal{I}}}(H) & \xrightarrow{\simeq} & \mathcal{H}(H, 1_{\mathcal{I}}) - \mathfrak{Mod} \\ {\scriptstyle i_{B(H)}^H}\uparrow & & \uparrow{\scriptstyle (t_{u,H})_*} \\ \mathfrak{R}_{1_{^0T}}(T) & \xrightarrow{\simeq} & \mathcal{H}(T, 1_{^0T}) - \mathfrak{Mod}. \end{array} \tag{9.10}$$

Combining 9.7 and 9.9 and also 9.8 and 9.10 (and using the obvious notation) yields the commutativity of

$$\begin{array}{ccc} \mathfrak{R}_{1_{\mathcal{I}}}(\widetilde{H}) & \xrightarrow{\simeq} & \mathcal{H}(\widetilde{H}, 1_{\mathcal{I}}) - \mathfrak{Mod} \\ {\scriptstyle r_{U(H)}}\downarrow & & \downarrow{\scriptstyle t_{u,H}^*} \\ \mathfrak{R}_{1_{^0T}}(T) & \xrightarrow{\simeq} & \mathcal{H}(T, 1_{^0T}) - \mathfrak{Mod} \end{array} \tag{9.11}$$

and

$$\begin{array}{ccc} \mathfrak{R}_{1_{\mathcal{I}}}(\widetilde{H}) & \xrightarrow{\simeq} & \mathcal{H}(\widetilde{H}, 1_{\mathcal{I}}) - \mathfrak{Mod} \\ {\scriptstyle i_{B(H)}^{\widetilde{H}}}\uparrow & & \uparrow{\scriptstyle (t_{u,H})_*} \\ \mathfrak{R}_{1_{^0T}}(T) & \xrightarrow{\simeq} & \mathcal{H}(T, 1_{^0T}) - \mathfrak{Mod}. \end{array} \tag{9.12}$$

As above, let ϕ_y for $y \in Y$ denote the unique element of $\mathcal{H}(T, \chi)$ supported on ${}^0Ty$ such that $\phi_y(y(\varpi)) = 1$. Write χ_y for the characteristic function of ${}^0Ty$. It is clear that there is a unique isomorphism of $\mathbb{C}$-algebras

$$\psi = \psi_\varpi : \mathcal{H}(T, \chi) \longrightarrow \mathcal{H}(T, 1_{^0T})$$

satisfying $\psi(\phi_y) = \chi_y$. The key step in our argument is contained in the following lemma.

LEMMA 9.3. – *Let $\psi\,(=\psi_\varpi):\mathcal{H}(T,\chi)\to\mathcal{H}(T,1_{{}^0T})$ be the $\mathbb{C}$-algebra isomorphism described above. Then there exists a support- preserving, $*$-preserving $\mathbb{C}$-algebra isomorphism $\Psi\,(=\Psi_\varpi):\mathcal{H}(G,\rho)\to\mathcal{H}(\widetilde{H},1_{\mathcal{I}})$ such that the following diagram commutes:*

$$\begin{array}{ccc}\mathcal{H}(G,\rho) & \xrightarrow{\Psi} & \mathcal{H}(\widetilde{H},1_{\mathcal{I}})\\ {\scriptstyle t_u}\uparrow & & \uparrow{\scriptstyle t_{u,H}}\\ \mathcal{H}(T,\chi) & \xrightarrow{\psi} & \mathcal{H}(T,1_{{}^0T}).\end{array}$$

Proof. – Clearly it is sufficient to show there exists an algebra isomorphism as in the statement of the lemma such that

$$\Psi(t_u(\phi_y))=t_{u,H}(\psi(\phi_y))\qquad(y\in Y^+).$$

If $J^+=J\cap U$, then $\delta_B(y)=[J^+:yJ^+y^{-1}]^{-1}$. Using the Iwahori decomposition of J with respect to B, we have $\delta_B(y)=\mathrm{vol}\,(JyJ)^{-1}$. Similarly $\delta_{B(H)}(y)=\mathrm{vol}\,(\mathcal{I}y\mathcal{I})^{-1}$. Thus we need to show

$$\Psi(\mathrm{vol}\,(JyJ)^{-1/2}\,\Phi_y)=\mathrm{vol}\,(\mathcal{I}y\mathcal{I})^{-1/2}\,\chi_{\mathcal{I}y\mathcal{I}}\qquad(y\in Y^+)\tag{9.13}$$

where $\chi_{\mathcal{I}y\mathcal{I}}$ is the characteristic function of $\mathcal{I}y\mathcal{I}$.

Let M be minimal among Levi subgroups of G containing T such that $\mathcal{I}_G(\rho)\subset JMJ$. From Proposition 5.1 and Remark 5.2, there exists a $*$- preserving, support-preserving Hecke algebra isomorphism

$$\Psi^M:\mathcal{H}(G,\rho)\xrightarrow{\simeq}\mathcal{H}(M,\rho_M)$$

such that

$$\Psi^M(\mathrm{vol}\,(JyJ)^{-1/2}\,\Phi_y)=\mathrm{vol}\,(J_MyJ_M)^{-1/2}\,\Phi_y^M\qquad(y\in Y^+)$$

where $\rho_M=\rho\mid J_M$ ($J_M=J\cap M$) and Φ_y^M is the unique function in $\mathcal{H}(M,\rho_M)$ with support J_MyJ_M such that $\Phi_y^M(y(\varpi))=1$. Moreover from the proof of theorem 4.15, there exists a character χ_1 of M such that $\chi\chi_1$ viewed as a character of ${}^0T$ has level zero. We may also assume that the character χ_1 satisfies $\chi_1(y(\varpi))=1$. Using the isomorphism $f\mapsto f\chi_1:\mathcal{H}(M,\rho_M)\to\mathcal{H}(M,\rho_M\chi_1)$, we are thus reduced to the proving 9.13 in the level-zero case.

We assume therefore that χ is a level zero character of ${}^0T$. We use Morris [27] to show there exists a ($*$-preserving, support-preserving) $\mathbb{C}$-algebra isomorphism satisfying 9.13.

As in the proof of Theorem 6.3, we fix an extension $\widetilde{\chi}$ of χ to N_χ satisfying $\widetilde{\chi}(y(\varpi))=1$ for all $y\in Y$. We then have a basis B_x for $x\in W_\chi=W_{\chi^{-1}}$ of $\mathrm{End}_G(\mathrm{Ind}\,\rho^{-1})$ given by

$$B_x=\widetilde{\chi}(n)^{-1}\,\theta_n$$

for any $n\in N_\chi$ projecting to $x\in W_\chi$ (see [27] 5.4, 5.5). Write $y=w\omega$ where $w\in W_\chi^0$ and $\omega\in\Omega_\chi$. In the notation of [27], $C(\chi)=C(\chi^{-1})=\Omega_\chi$. Further the cocycle λ (in [27] 6.2) is trivial (by our choice of B_x for $x\in W_\chi$). Applying [27] Propn. 7.6 a),

$$\begin{aligned}B_wB_\omega&=\left(\frac{\mathrm{Ind}\,w\omega}{\mathrm{Ind}\,w\;\mathrm{Ind}\,\omega}\right)^{1/2}B_{w\omega}\\&=q^{\frac12(l(y)-l(w)-l(\omega))}\,B_y.\end{aligned}\tag{9.14}$$

Since $\epsilon_x = 1$ for all $x \in W_\chi$ and $p_w = q^{l_\chi(w)}$, we have

$$T_w = q^{\frac{1}{2}(l(w)+l_\chi(w))} B_w \qquad (9.15)$$
$$T_\omega = q^{\frac{1}{2}l(\omega)} B_\omega$$

(see [27] 7.9, 7.10). A straightforward calculation shows that the isomorphism

$$t : \mathcal{H}(G,\rho) \xrightarrow{\simeq} \mathrm{End}_G(\mathrm{Ind}\,\rho^{-1})$$

of equation 7.2 satisfies $t_{\Phi_n} = q^{l(x)}\,\theta_n$ for any $n \in N_\chi$ projecting to $x \in W_\chi$. Here Φ_n is the unique element of $\mathcal{H}(G,\rho)$ supported on JnJ satisfying $\Phi_n(n) = 1$. Thus

$$\begin{aligned} q^{-\frac{1}{2}l(y)}\Phi_y \overset{t}{\mapsto} q^{-\frac{1}{2}l(y)}\, t_{\Phi_y} \\ &= q^{\frac{1}{2}l(y)}\,\theta_{y(\varpi)} \\ &= q^{\frac{1}{2}l(y)}\,B_y \quad (\text{since } \widetilde{\chi}(y(\varpi)) = 1) \\ &= q^{\frac{1}{2}l(w)}B_w\, q^{\frac{1}{2}l(\omega)}B_\omega \quad (\text{by } 9.14) \\ &= q^{-\frac{1}{2}l_\chi(w)}\,T_w\,T_\omega \quad (\text{by } 9.15). \end{aligned}$$

Applying the composition of the isomorphisms between $\mathrm{End}_G(\mathrm{Ind}\,\rho^{-1})$ and $\mathcal{H}_\chi$ and between $\mathcal{H}_\chi$ and $\mathcal{H}(\widetilde{H}, 1_\mathcal{I})$, this last element is sent to

$$q^{-\frac{1}{2}l_\chi(w)}\chi_{\mathcal{I}w\mathcal{I}}\,\chi_{\mathcal{I}\omega\mathcal{I}} = q^{-\frac{1}{2}l_\chi(y)}\chi_{\mathcal{I}y\mathcal{I}}.$$

This proves 9.13 in the level-zero case and thus completes the proof of the lemma. □

We now fix a $\mathbb{C}$-algebra isomorphism $\Psi : \mathcal{H}(G,\rho) \xrightarrow{\simeq} \mathcal{H}(\widetilde{H}, 1_\mathcal{I})$ as in the Lemma. We immediately obtain the following commutative diagrams on module categories (where the top (resp. bottom) rows are induced by Ψ (resp. ψ)):

$$\begin{array}{ccc} \mathcal{H}(G,\rho) - \mathfrak{Mod} & \xrightarrow{\simeq} & \mathcal{H}(\widetilde{H}, 1_\mathcal{I}) - \mathfrak{Mod} \\ {\scriptstyle t_u^*}\downarrow & & \downarrow{\scriptstyle t_{u,H}^*} \\ \mathcal{H}(T,\chi) - \mathfrak{Mod} & \xrightarrow{\simeq} & \mathcal{H}(T, 1_{\circ T}) - \mathfrak{Mod} \end{array} \qquad (9.16)$$

and

$$\begin{array}{ccc} \mathcal{H}(G,\rho) - \mathfrak{Mod} & \xrightarrow{\simeq} & \mathcal{H}(\widetilde{H}, 1_\mathcal{I}) - \mathfrak{Mod} \\ {\scriptstyle (t_u)_*}\uparrow & & \uparrow{\scriptstyle (t_{u,H})_*} \\ \mathcal{H}(T,\chi) - \mathfrak{Mod} & \xrightarrow{\simeq} & \mathcal{H}(T, 1_{\circ T}) - \mathfrak{Mod}. \end{array} \qquad (9.17)$$

The isomorphism Ψ induces an equivalence of categories between $\mathfrak{R}_\chi(G)$ and $\mathfrak{R}_{1_\mathcal{I}}(\widetilde{H})$ (by composing the inverse of the equivalence $\mathcal{W} \mapsto \mathcal{W}^{1_\mathcal{I}} : \mathfrak{R}_{1_\mathcal{I}}(\widetilde{H}) \xrightarrow{\simeq} \mathcal{H}(\widetilde{H}, 1_\mathcal{I}) - \mathfrak{Mod}$ with the composition of $\Psi_* : \mathcal{H}(G,\rho) - \mathfrak{Mod} \xrightarrow{\simeq} \mathcal{H}(\widetilde{H}, 1_\mathcal{I}) - \mathfrak{Mod}$ and $\mathcal{V} \mapsto \mathcal{V}^\rho : \mathfrak{R}_\chi(G) \xrightarrow{\simeq} \mathcal{H}(G,\rho) - \mathfrak{Mod}$). Similarly ψ induces an equivalence of categories between $\mathfrak{R}_\chi(T)$ and $\mathfrak{R}_{1_{\circ T}}(T)$. These are compatible with induction and restriction as described in the following theorem.

THEOREM 9.4. – *The equivalences of categories* $\mathfrak{R}_\chi(G) \xrightarrow{\simeq} \mathfrak{R}_{1_\mathcal{I}}(\widetilde{H})$ *(induced by* Ψ*) and* $\mathfrak{R}_\chi(T) \xrightarrow{\simeq} \mathfrak{R}_{1_{0_T}}(T)$ *(induced by* ψ*) fit into the following commutative diagrams:*

$$\begin{array}{ccc} \mathfrak{R}_\chi(G) & \xrightarrow{\simeq} & \mathfrak{R}_{1_\mathcal{I}}(\widetilde{H}) \\ {\scriptstyle r_U}\downarrow & & \downarrow{\scriptstyle r_{U(H)}} \\ \mathfrak{R}_\chi(T) & \xrightarrow{\simeq} & \mathfrak{R}_{1_{0_T}}(T). \end{array} \tag{9.18}$$

$$\begin{array}{ccc} \mathfrak{R}_\chi(G) & \xrightarrow{\simeq} & \mathfrak{R}_{1_\mathcal{I}}(\widetilde{H}) \\ {\scriptstyle i_B^G}\uparrow & & \uparrow{\scriptstyle i_{B(H)}^G} \\ \mathfrak{R}_\chi(T) & \xrightarrow{\simeq} & \mathfrak{R}_{1_{0_T}}(T). \end{array} \tag{9.19}$$

In particular, if ν *is an unramified character of* T*, then the representation* $i_B^G(\chi.\nu)$ *corresponds to* $i_{B(H)}^{\widetilde{H}}(\nu)$.

Proof. – Combining 9.4, 9.16 and 9.11, we immediately obtain (the commutativity of) 9.18. Similarly, (the commutativity of) 9.19 follows immediately from (the commutativity of) 9.5, 9.17 and 9.12. □

10. Square-integrability and formal degrees

Throughout this section we fix a $*$-preserving, support-preserving (and hence inner-product preserving) isomorphism of $\mathbb{C}$-algebras $\Psi : \mathcal{H}(\widetilde{H}, 1_\mathcal{I}) \xrightarrow{\simeq} \mathcal{H}(G, \rho)$. This induces an equivalence of categories between $\mathcal{H}(\widetilde{H}, 1_\mathcal{I}) - \mathfrak{Mod}$ and $\mathcal{H}(G, \rho) - \mathfrak{Mod}$. Via the equivalences $\mathcal{W} \mapsto \mathcal{W}^{1_\mathcal{I}} : \mathfrak{R}_{1_\mathcal{I}}(\widetilde{H}) \xrightarrow{\simeq} \mathcal{H}(\widetilde{H}, 1_\mathcal{I}) - \mathfrak{Mod}$ and $\mathcal{V} \mapsto \mathcal{V}^\rho : \mathfrak{R}_\chi(G) \xrightarrow{\simeq} \mathcal{H}(G, \rho) - \mathfrak{Mod}$, we also obtain an equivalence of categories between $\mathfrak{R}_{1_\mathcal{I}}(\widetilde{H})$ and $\mathfrak{R}_\chi(G) = \mathfrak{R}_\rho(G)$. Hence there is a bijection between smooth irreducible representations of $\widetilde{H}$ containing $1_\mathcal{I}$ and smooth irreducible representations of G containing ρ. In this section we show that this correspondence preserves square-integrability provided Z_H/Z_G is compact. (Here Z_H denotes the centre of H and Z_G the centre of G.) This condition is equivalent to the endoscopic group H of G being elliptic. When Z_H/Z_G is non-compact, it is a trivial consequence of our calculations that there exist no (non-zero) square-integrable representations of G containing ρ. (Following a standard abuse of terminology we use the terms square-integrable and square-integrability in place of the more accurate (but cumbersome) square-integrable-mod-centre and square-integrability-mod-centre.) We also observe that (when square-integrability is preserved) the formal degrees of corresponding square-integrable representations are equal (given certain natural choices of Haar measures). It is well-known that Hecke algebra isomorphisms may be used to make observations of this sort. This first appeared in the p-adic setting in Howe and Moy [19]. In section 7.7 of [10] Bushnell and Kutzko obtain analagous results for a simple type in $GL_N(F)$ (in place of ρ). Their treatment however is general and we have merely transposed it to our context (often almost word for word).

Remark 10.1. – It would not be difficult to recast the arguments of this section without explicit mention of the groups H or $\widetilde{H}$. Instead one would work with the 'abstract' Hecke algebra $\mathcal{H}_\chi$ or appropriate quotients (if $\mathbb{G}$ is not semisimple) and directly define square-integrability and formal degrees for modules over such algebras via their canonical inner products (see 6.4).

Let $(\pi, \mathcal{V})$ be a smooth irreducible representation of G with unitary central character $\omega_\pi = \omega$ (thus $|\omega(z)| = 1$ for $z \in Z = Z_G$). We say $(\pi, \mathcal{V})$ is square-integrable (by abuse of terminology) if

$$\int_{G/Z} |\langle \pi(g)v, \widetilde{v}\rangle|^2 \, d\dot{g} < \infty \qquad (v \in \mathcal{V},\ \widetilde{v} \in \widetilde{\mathcal{V}}).$$

Clearly we may replace Z by any cocompact subgroup Z_1 of Z.

Whenever Z_1 is such a subgroup we fix Haar measures on Z_1 and G/Z_1 as follows. Let dz be the Haar measure on Z_1 such that ${}^0Z_1 = Z_1 \cap {}^0T$ has measure one. Let dg be the Haar measure on G giving J measure one. We then choose the induced Haar measure $d\dot{g}$ on G/Z_1 (i.e. $dz\, d\dot{g} = dg$).

Let $\mathcal{H}(G, \omega)$ be the convolution algebra (with respect to $d\dot{g}$) of smooth functions $\phi : G \to \mathbb{C}$ which are compactly supported modulo Z_1 and satisfy $\phi(zg) = \omega(z)^{-1}\phi(g)$ ($z \in Z_1, g \in G$). Further, let $\mathcal{H}^2(G, \omega)$ be the space of (left and right) smooth functions $\phi : G \to \mathbb{C}$ which satisfy $\phi(zg) = \omega(z)^{-1}\phi(g)$ and are also square-integrable modulo Z_1. We view $\mathcal{H}^2(G, \omega)$ as a smooth G-representation by left translations (or equivalently as a left $\mathcal{H}(G, \omega)$-module). We write $*$ for the usual involution on $\mathcal{H}(G, \omega)$ (i.e. $\phi^*(g) = \overline{\phi(g^{-1})}$ for $g \in G, \phi \in \mathcal{H}(G, \omega)$). This is related to the canonical inner product $(\ ,\ )$ on $\mathcal{H}(G, \omega)$ by the standard formula

$$\begin{aligned}(\phi, \psi) &= \int_{G/Z_1} \phi(g)\overline{\psi(g)} d\dot{g} \\ &= \phi\psi^*(1) \qquad (\phi, \psi \in \mathcal{H}(G, \omega)).\end{aligned}$$

It is clear that $*$ and $(\ ,\ )$ extend to $\mathcal{H}^2(G, \omega)$.

Suppose now that $(\pi, \mathcal{V})$ contains ρ. Then $\omega \mid Z_1 \cap J = \rho \mid Z_1 \cap J$ and thus $\rho\omega$ is a well-defined representation of $Z_1 J$. The subalgebra $\mathcal{H}(G, \rho\omega)$ of $\mathcal{H}(G, \omega)$ is defined in the obvious way. We have $\mathcal{H}(G, \rho\omega) = e_{\rho\omega}\mathcal{H}(G, \omega)e_{\rho\omega}$ where

$$\begin{aligned}e_{\rho\omega}(x) &= \dot{\mu}(JZ_1/Z_1)\rho\omega(x)^{-1} (x \in JZ_1) \\ &= 0 \qquad\qquad (x \notin JZ_1).\end{aligned}$$

The volume factor $\dot{\mu}(JZ_1/Z_1)$ is in fact one (given our choice of Haar measure on G/Z_1). The completion of $\mathcal{H}(G, \rho\omega)$ with respect to the (restriction of the) inner product $(\ ,\ )$ is $\mathcal{H}^2(G, \rho\omega) = e_{\rho\omega}\mathcal{H}^2(G, \omega)e_{\rho\omega}$.

From [10] 7.7.4 (or 7.7.5) we have the following Proposition.

PROPOSITION 10.2. – *Let $(\pi, \mathcal{V})$ be a smooth irreducible representation of G containing ρ with central character ω. Then $(\pi, \mathcal{V})$ is square-integrable if and only if $\mathcal{V}^\rho = \mathcal{V}^{\rho\omega}$ is isomorphic to a left $\mathcal{H}(G, \rho\omega)$-submodule of $\mathcal{H}^2(G, \rho\omega)$.*

For a fixed uniformiser ϖ in F, the exact sequence (see 4.1)

$$1 \longrightarrow {}^0T \longrightarrow T \xrightarrow{H_T} Y \longrightarrow 0$$

is split by the homomorphism $y \mapsto y(\varpi) : Y \to T$. The resulting isomorphism $T \simeq {}^0T \times Y$ restricts to Z to give $Z \simeq {}^0Z \times L$ where ${}^0Z = Z \cap {}^0T$ and

$$L = \{y \in Y \mid \langle y, \alpha\rangle = 0 \ (\alpha \in \Phi)\}.$$

We also write L for its image in Z under the embedding $l \mapsto l(\varpi) : L \to Z$. Thus we may view L as a cocompact subgroup of Z. As above, $\rho\omega$ defines a representation of $JL = JZ$. There is a canonical surjective algebra isomorphism

$$P_\omega : \mathcal{H}(G,\rho) \longrightarrow \mathcal{H}(G,\rho\omega)$$

given by

$$\begin{aligned} P_\omega(\phi)(g) &= \int_Z \omega(z)\phi(zg)dz \\ &= \sum_{l\in L} \omega(l)\phi(lg) \end{aligned}$$

for $\phi \in \mathcal{H}(G,\rho)$, $x \in G$. For $l \in L$, define $\phi_l \in \mathcal{H}(G,\rho)$ by $\operatorname{supp}\phi_l = Jl$ and $\phi_l(l) = \omega(l)^{-1}$.

LEMMA 10.3. – *The kernel of P_ω is the two-sided ideal of $\mathcal{H}(G,\rho)$ generated by the elements $\phi_l - e_\rho$ $(l \in L)$.*

Proof. – Fix a $\mathbb{Z}$-basis $l_1, \ldots, l_r$ of L. Write $\phi_1 = \phi_{l_1}, \ldots, \phi_r = \phi_{l_r}$. A simple inductive argument using the relation

$$\phi_{l+l'} - e_\rho = (\phi_l - e_\rho)(\phi_{l'} - e_\rho) + (\phi_l - e_\rho) + (\phi_{l'} - e_\rho) \qquad (l, l' \in L)$$

shows that the two-sided ideal of $\mathcal{H}(G,\rho)$ generated by $\phi_1 - e_\rho, \ldots, \phi_r - e_\rho$ equals the two-sided ideal generated by $\phi_l - e_\rho$ $(l \in L)$. It suffices therefore to show that $\operatorname{Ker} P_\omega$ is generated by $\phi_1 - e_\rho, \ldots, \phi_r - e_\rho$.

A straightforward calculation shows $\phi_i - e_\rho \in \operatorname{Ker} P_\omega$ $(1 \le i \le r)$. Suppose $P_\omega(\phi) = 0$ for $\phi \in \mathcal{H}(G,\rho)$. We may assume $\operatorname{supp}\phi \subset ZJxJ$ for some $x \in G$. Since ϕ has compact support, there exist integers $m_i \le n_i$ $(1 \le i \le r)$ such that

$$\operatorname{supp}\phi \subset \bigcup_{\substack{m_1 \le i_1 \le n_1 \\ m_r \le i_r \le n_r}} l_1^{i_1} \ldots l_r^{i_r} JxJ.$$

Note we may assume $\sum_{i=1}^r (n_i - m_i) > 0$. Indeed if $n_i = m_i$ $(1 \le i \le r)$, then $\operatorname{supp}\phi$ is contained in a single J-double coset and so $P_\omega(\phi) = 0$ implies $\phi = 0$. Suppose $m_1 \le i_1 < n_1$ and $\phi(l_1^{m_1} \ldots l_r^{m_r} x) = \alpha \ne 0$. There exists a (unique) function $\psi \in \mathcal{H}(G,\rho)$ such that $\psi(l_1^{m_1} \ldots l_r^{m_r} x) = \alpha$ and $\operatorname{supp}\psi = l_1^{m_1} \ldots l_r^{m_r} JxJ$. Consider $\phi + \psi(\phi_1 - e_\rho)$. The support of this function is contained in

$$\bigcup_{\substack{m_1 \le i_1 \le n_1 \\ m_r \le i_r \le n_r}} l_1^{i_1} \ldots l_r^{i_r} JxJ.$$

Further,

$$\begin{aligned}\psi\phi_1(l_1^{m_1}\dots l_r^{m_r}x) &= \int_G \psi(l_1^{m_1}\dots l_r^{m_r}xy^{-1})\phi_1(y)dy\\ &= \int_J \psi(l_1^{m_1}\dots l_r^{m_r}l_1^{-1}j^{-1})\phi_1(jl_1)dj\\ &= \psi(l_1^{m_1-1}l_2^{m_2}\dots l_r^{m_r}x)\omega(l_1)^{-1}\\ &= 0.\end{aligned}$$

Therefore $(\phi+\psi(\phi_1-e_\rho))(l_1^{m_1}\dots l_r^{m_r}x)=0$ and hence

$$\operatorname{supp}(\phi+\psi(\phi_1-e_\rho))\subset \bigcup_{\substack{m_1+1\le i_1\le n_1\\ m_2\le i_2\le n_2\\ m_r\le C_r\le n_r}} l_1^{i_1}\dots l_r^{i_r}JxJ.$$

Continuing in this way we eventually obtain a function in $\operatorname{Ker}P_\omega$ supported on the single double coset JxJ. Since such a function must be the zero function, this proves the lemma. □

We also have the embedding $l\mapsto l(\varpi):L\to Z_{\widetilde{H}}$ (where $Z_{\widetilde{H}}$ denotes the centre of $\widetilde{H}$). Note that L (viewed as a subgroup of $Z_{\widetilde{H}}$ via this embedding) is discrete but not necessarily cocompact. We assume the convolution algebra $\mathcal{H}(\widetilde{H},1_{\mathcal{I}})$ is defined with respect to the Haar measure on $\widetilde{H}$ giving $\mathcal{I}$ measure one. Then

$$\Psi^{-1}(\phi_l)=\alpha_l\chi_l\quad(l\in L)$$

where χ_l is the characteristic function of $\mathcal{I}l$ and $|\alpha_l|=1$. We define a homomorphism $\omega':L\to\mathbb{C}^\times$ by

$$\omega'(l)=\alpha_l^{-1}\qquad(l\in L).$$

Then $1_{\mathcal{I}}\omega'$ is a representation of $\mathcal{I}L$. Using the obvious notation, we form the convolution algebra $\mathcal{H}(\widetilde{H},1_{\mathcal{I}}\omega')$ (with respect to the Haar measure $d\dot{h}$ on $\widetilde{H}/L$ such that $dl\,d\dot{h}=dh$ where dh is our fixed Haar measure on $\tilde{H}$ (giving $\mathcal{I}$ measure one) and dl is the counting measure on L). Again we have a surjective algebra homomorphism

$$P_{\omega'}:\mathcal{H}(\widetilde{H},1_{\mathcal{I}})\longrightarrow\mathcal{H}(\widetilde{H},1_{\mathcal{I}}\omega')$$

given by

$$P_{\omega'}(\phi)(x)=\sum_{l\in L}\omega'(l)\phi(lx)\qquad(\phi\in\mathcal{H}(\widetilde{H},1_{\mathcal{I}}),x\in\widetilde{H}).$$

Arguing exactly as in the proof of Lemma 10.3, we obtain the following lemma.

LEMMA 10.4. – *The kernel of $P_{\omega'}$ is the two-sided ideal of $\mathcal{H}(\widetilde{H},1_{\mathcal{I}})$ generated by the elements $\alpha_l\chi_l-e$ $(l\in L)$ where e denotes the identity element of $\mathcal{H}(\widetilde{H},1_{\mathcal{I}})$.*

COROLLARY 10.5. – *There exists a unique ($*$-preserving, support-preserving) isomorphism of $\mathbb{C}$-algebras*

$$\Psi_\omega:\mathcal{H}(\tilde{H},1_{\mathcal{I}}\omega')\xrightarrow{\simeq}\mathcal{H}(G,\rho\omega)$$

such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{H}(\widetilde{H}, 1_{\mathcal{I}}) & \xrightarrow{\Psi} & \mathcal{H}(G, \rho) \\ {\scriptstyle P_{\omega'}}\downarrow & & \downarrow{\scriptstyle P_{\omega}} \\ \mathcal{H}(\widetilde{H}, 1_{\mathcal{I}}\omega') & \xrightarrow{\Psi_\omega} & \mathcal{H}(G, \rho\omega). \end{array}$$

Proof. – This is clear from Lemma 10.3 and Lemma 10.4. □

We now verify that Ψ_ω is unitary with respect to the standard inner products on $\mathcal{H}(\widetilde{H}, 1_{\mathcal{I}}\omega')$ and $\mathcal{H}(G, \rho\omega)$.

LEMMA 10.6. – *The canonical inner products* (,) *on* $\mathcal{H}(\widetilde{H}, 1_{\mathcal{I}}\omega')$ *and* $\mathcal{H}(G, \rho\omega)$ *satisfy*

$$(\Psi_\omega\phi, \Psi_\omega\theta) = (\phi, \theta) \qquad (\phi, \theta \in \mathcal{H}(\widetilde{H}, 1_{\mathcal{I}}\omega')).$$

Proof. – Fix $\phi_1, \theta_1 \in \mathcal{H}(\widetilde{H}, 1_{\mathcal{I}})$. Then

$$(\Psi_\omega P_{\omega'}\phi_1, \Psi_\omega P_{\omega'}\theta_1) = \Psi_\omega P_{\omega'}(\phi_1\theta_1^*)(1).$$

Write $P_{\omega'}(\phi_1\theta_1^*) = \alpha e + \eta$ where $\alpha \in \mathbb{C}$, e is the unit element of $\mathcal{H}(\widetilde{H}, 1_{\mathcal{I}}\omega')$ and $\operatorname{supp}\eta \cap \mathcal{I}L = \emptyset$. Then

$$\Psi_\omega P_{\omega'}(\phi_1\theta_1^*)(1) = \alpha e_{\rho\omega} + \Psi_\omega(\eta)$$

and $\operatorname{supp}\Psi_\omega(\eta) \cap JL = \emptyset$. Therefore

$$\Psi_\omega P_{\omega'}(\phi_1\theta_1^*)(1) = \alpha\dot{\mu}(JL/L)^{-1} \tag{10.1}$$

where $\dot{\mu}(JL/L)$ denotes the measure of JL/L with respect to our fixed Haar measure on G/L.

We also have

$$\begin{aligned} (P_{\omega'}\phi_1, P_{\omega'}\theta_1) &= P_{\omega'}(\phi_1\theta_1^*)(1) \\ &= \alpha e(1) \\ &= \alpha\dot{\mu}(\mathcal{I}L/L)^{-1} \end{aligned} \tag{10.2}$$

where $\dot{\mu}(\mathcal{I}L/L)$ now denotes the measure of $\mathcal{I}L/L$ with respect to our fixed Haar measure on $\widetilde{H}/L$.

Comparing 10.1 and 10.2 gives

$$(P_{\omega'}\phi_1, P_{\omega'}\theta_1) = \frac{\dot{\mu}(JL/L)}{\dot{\mu}(\mathcal{I}L/L)}(\Psi_\omega P_{\omega'}\phi_1, \Psi_\omega P_{\omega'}\theta_1).$$

Since $P_{\omega'}$ is surjective, we have

$$(\phi, \theta) = c(\Psi_\omega\phi, \Psi_\omega\theta) \qquad (\phi, \theta \in \mathcal{H}(\widetilde{H}, 1_{\mathcal{I}}\omega'))$$

where $c = \frac{\dot\mu(JL/L)}{\dot\mu(\mathcal{I}L/L)}$. However, by our choices of Haar measures, we have $\dot\mu(JL/L) = \dot\mu(\mathcal{I}L/L) = 1$. Indeed, if $f \in L^1(G)$

$$\int_G f(g)dg = \int_{G/L} \sum_{l\in L} f(lg)\, d\dot g.$$

Taking f to be the characteristic function of J gives $\dot\mu(JL/L) = 1$. Similarly $\dot\mu(\mathcal{I}L/L) = 1$ and hence $c = 1$ as required. □

Since Ψ_ω preserves the canonical inner products on $\mathcal{H}(\widetilde H, 1_{\mathcal{I}}\omega')$ and $\mathcal{H}(G, \rho\omega)$, it extends to an isomorphism

$$\Psi_\omega : \mathcal{H}^2(\widetilde H, 1_{\mathcal{I}}\omega') \xrightarrow{\simeq} \mathcal{H}^2(G, \rho\omega)$$

on completions and therefore induces a bijection between simple $\mathcal{H}(G, \rho\omega)$-submodules of $\mathcal{H}^2(G, \rho\omega)$ and simple $\mathcal{H}(\widetilde H, 1_{\mathcal{I}}\omega')$-submodules of $\mathcal{H}^2(\widetilde H, 1_{\mathcal{I}}\omega')$.

It is convenient to temporarily use the following terminology. We say a smooth irreducible representation $(\sigma, \mathcal{W})$ of $\widetilde H$ is L-square-integrable if its central character ω_σ is unitary on L (i.e. $|\omega_\sigma(l)| = 1$ $(l \in L)$) and

$$\int_{\tilde H/L} |\langle \sigma(h)w, \tilde w\rangle|^2\, \dot h < \infty \qquad (w \in \mathcal{W}, \tilde w \in \tilde{\mathcal{W}}).$$

(If L is not cocompact in $Z(\widetilde H)$, this notion is vacuous in that only the zero representation is L-square integrable. Otherwise, L-square-integrability is equivalent to square-integrability.) From the analogue of Proposition 10.2, the process $\mathcal{W} \mapsto \mathcal{W}^{1_{\mathcal{I}}\omega'}$ establishes a bijection between smooth irreducible L-square-integrable representations of $\widetilde H$ containing $1_{\mathcal{I}}$ and simple $\mathcal{H}(\widetilde H, 1_{\mathcal{I}}\omega')$-submodules of $\mathcal{H}^2(\widetilde H, 1_{\mathcal{I}}\omega')$. Using Proposition 10.2 again, we obtain a bijection between (smooth, irreducible) square-integrable representations of G containing ρ and (smooth irreducible) L-square-integrable representations of $\widetilde H$ containing $1_{\mathcal{I}}$.

Exactly as on G, we may write $Z_H = {}^0Z_H L_H$ where ${}^0Z_H = Z_H \cap {}^0T$ and

$$L_H = \{y \in Y \mid \langle y, \alpha\rangle = 0\ (\alpha \in \Phi_\chi)\}.$$

Let $L_{\tilde H} = L_H^{C_\chi} = \{l \in L_H \mid cl = l\ (c \in C_\chi)\}$. Note that $L_{\widetilde H}$ embeds in $Z_{\widetilde H}$ (recall $\widetilde H = H \rtimes C_\chi$) and $Z_{\widetilde H}/L_{\widetilde H}$ is compact. Note also that we have the chain of inclusions

$$Z_H \supset Z_{\widetilde H} \supset Z_G$$

(viewing each group as a subgroup of T).

It is clear that if $L_{\widetilde H}/L$ is infinite (equivalently if $Z_{\widetilde H}/Z_G$ is noncompact) there exist no (non-zero) L-square-integrable representations of $\widetilde H$. Therefore, in this case, there exist no (non-zero) square-integrable representations of G containing ρ.

Suppose however that $L_{\widetilde H}/L$ is finite. Then $Z_{\widetilde H}/L$ is compact and hence L-square-integrability on $\widetilde H$ is equivalent to square-integrability. Further, it is easy to see (e.g. by using Clifford theory) that there exist (non-zero) square-integrable representations of $\widetilde H$

if and only if $L_H/L_{\widetilde{H}}$ is finite (equivalently if and only if $Z_H/Z_{\widetilde{H}}$ is compact). From the exact sequence

$$0 \longrightarrow L_{\widetilde{H}}/L \longrightarrow L_H/L \longrightarrow L_H/L_{\widetilde{H}} \longrightarrow 0$$

we see $L_{\widetilde{H}}/L$ and $L_H/L_{\widetilde{H}}$ are both finite if and only if L_H/L is finite (equivalently if and only if Z_H/Z_G is compact).

Putting these observations together, we obtain the following theorem. Before we state it we note (for later use) that $L_H/L_{\widetilde{H}}$ is finite if and only if $L_H = L_{\widetilde{H}}$. Indeed, $L_{\widetilde{H}} = L_H^{C_\chi}$ and C_χ is a group of automorphisms of L_H. Thus it is sufficient to observe that if L is a $\mathbb{Z}$-lattice and $\alpha \in \mathrm{Aut}_{\mathbb{Z}}(L)$ then $[L : L^\alpha] < \infty$ if and only if $L = L^\alpha$.

THEOREM 10.7. – *i) If Z_H/Z_G is non-compact, there exist no (non-zero) square-integrable representations of G containing ρ.*

ii) If Z_H/Z_G is compact, there exist square-integrable representations of G containing ρ. Further, a smooth irreducible representation of G containing ρ is square-integrable if and only if it corresponds to a square-integrable representation of $\widetilde{H}$ under the equivalence of categories between $\mathfrak{R}_\rho(G)$ and $\mathfrak{R}_{1_\mathcal{I}}(\widetilde{H})$.

Remark 10.8. – For $L_1 \subset Y$, let $L_1^\perp = \{x \in X : \langle x, l_1\rangle = 0,\ \forall l_1 \in L_1\}$. Then Z_H/Z_G is compact if and only if L_H/L is finite which holds if and only if $L^\perp/L_H^\perp$ is finite, i.e. if and only if $\mathbb{Z}\Phi/\mathbb{Z}\Phi_\chi$ is finite. This in turn is equivalent to the statement that $\widehat{H}$ is not contained in a proper Levi subgroup of $\widehat{G}$ (see Digne and Michel [16] 14.11), i.e. that the endoscopic group H of G is elliptic.

Remark 10.9. – If $\mathbb{Z}\Phi/\mathbb{Z}\Phi_\chi$ is finite, our restrictions on residual characteristic (in particular, the fact that $p = \mathrm{char}\, k_F$ is not a bad prime for Φ) imply that χ is essentially tamely ramified, i.e. some twist of χ by a character of G is trivial on T_1. Thus if χ is not essentially tamely ramified, the category $\mathfrak{R}_\chi(G) = \mathfrak{R}_\rho(G)$ contains no (non-zero) square-integrable representations. The following example of Tadic [36] (Section 10) shows that the restriction on residual characterisic (implicit in theorem 10.7) is not always necessary. (I am grateful to the referee for pointing this out.) Let $\nu(x) = |x|_F$ for $x \in F^\times$ where F is now any non-archimedean field of characteristic not equal to two. Suppose $\psi_1, \ldots, \psi_n$ are n non-trivial quadratic characters of $F^\times$ linearly independent over $\mathbb{Z}/2\mathbb{Z}$. Consider the character of the standard maximal torus of $Sp(4n, F)$ corresponding to the $2n$-tuple of characters $(\nu\psi_1, \psi_1, \ldots, \nu\psi_n, \psi_n)$. The resulting induced representation of $Sp(4n, F)$ (obtained by normalised induction from the standard Borel) has exactly 2^n square-integrable subquotients. When $n = 2$, one may recover Tadic's computation of the subquotients of this induced representation by straightforward Hecke algebra computations - the relevant Hecke algebras are the Iwahori Hecke algebras of $Sp(4, F) \times Sp(4, F)$ and $Sp(4, F) \times O(4, F)$. When $n > 2$, the methods of this paper do not apply (since the existence of n quadratic characters ψ_i as above forces $p = 2$).

We assume for the remainder of this section that the quotient Z_H/Z_G is compact. Let $(\pi, \mathcal{V})$ be a (smooth, irreducible) square-integrable representation of G containing ρ and $(\sigma, \mathcal{W})$ be the corresponding (smooth, irreducible) square-integrable representation of $\widetilde{H}$ containing $1_\mathcal{I}$.

The formal degree $d(\pi)$ of π with respect to the Haar measure $d\dot{g}$ on G/Z is defined by

$$\int_{G/Z} (\pi(g)v_1, w_1)\overline{(\pi(g)v_2, w_2)}d\dot{g} = d(\pi)^{-1}(v_1, v_2)\overline{(w_1, w_2)} \tag{10.3}$$

for $v_i, w_i \in \mathcal{V}$ where $(\, , \,)$ is a G-invariant inner product on $\mathcal{V}$. It is clear (given our choices of Haar measures on G/Z and G/L) that we may also integrate over G/L. Since there is an embedding of $\mathcal{V}^{\rho\omega}$ into $\mathcal{H}^2(G, \rho\omega)$, we can regard $\mathcal{V}$ as a subspace of $\mathcal{H}^2(G, \omega)$. Rewriting 10.3 using functions $\phi_i, \theta_i \in \mathcal{H}^2(G, \omega) \cap \mathcal{V}$ and noting $(\pi(g)\phi, \theta) = \phi\theta^*(g^{-1})$, we obtain

$$(\phi_1\theta_1^*, \phi_2\theta_2^*) = d(\pi)^{-1}(\phi_1, \phi_2)\overline{(\theta_1, \theta_2)} \tag{10.4}$$

where $(\, , \,)$ now denotes the canonical inner product on $\mathcal{H}^2(G, \omega)$.

Let dz be the Haar measure on $Z_{\widetilde{H}}$ giving ${}^0Z_{\widetilde{H}} = Z_{\widetilde{H}} \cap {}^0T$ measure one. We then obtain quotient measures $d\dot{z} = \frac{dz}{dl}$ on $Z_{\widetilde{H}}/L$ and $d\dot{h} = \frac{dh}{dz}$ on $\widetilde{H}/Z_{\tilde{H}}$. (Here dh is the Haar measure on $\widetilde{H}$ giving $\mathcal{I}$ measure one and dl is the counting measure on L.) There is a positive constant c such that

$$\int_{\widetilde{H}/L} f(\dot{h})\frac{dh}{dl} = c\int_{\widetilde{H}/Z_{\widetilde{H}}}\int_{Z_{\widetilde{H}}/L} f(\dot{z}\dot{h})\frac{dz}{dl}\frac{dh}{dz} \qquad (f \in L^1(\widetilde{H}/L)). \tag{10.5}$$

Taking f to be the characteristic function of $\mathcal{I}L/L$, an easy calculation yields

$$1 = c\,\mu_{Z_{\widetilde{H}}/L}({}^0Z_{\widetilde{H}}L_H/L)$$

(using the obvious notation). Here we have used $Z_{\widetilde{H}} = {}^0Z_{\widetilde{H}}L_H$ (which holds since $L_{\widetilde{H}} = L_H$). By our choice of Haar measure on $Z_{\widetilde{H}}/L$, $\mu_{Z_{\widetilde{H}}/L}({}^0Z_{\widetilde{H}}L/L) = 1$ and hence $\mu_{Z_{\widetilde{H}}/L}({}^0Z_{\widetilde{H}}L_H/L) = [L_H : L]$ so that $c = [L_H : L]^{-1}$.

Returning to 10.4, we assume $\phi_i, \theta_i \in \mathcal{H}^2(G, \rho\omega) \cap \mathcal{V}$. Write $\Psi_\omega^{-1}(\phi_i) = \xi_i$, $\Psi_\omega^{-1}(\theta_i) = \zeta_i$. Since Ψ_ω preserves inner products, we have

$$(\xi_1\zeta_1^*, \xi_2\zeta_2^*) = d(\pi)^{-1}(\xi_1, \xi_2)\overline{(\zeta_1, \zeta_2)} \tag{10.6}$$

where $(\, , \,)$ is now the canonical inner product on $\mathcal{H}^2(\widetilde{H}, 1_{\mathcal{I}}\omega')$. We also have $\xi_i, \theta_i \in \mathcal{H}^2(\widetilde{H}, 1_{\mathcal{I}}\omega_\sigma)$ where ω_σ is the central character of $(\sigma, \mathcal{W})$ $(\omega_\sigma|_L = \omega')$. Furthermore, if $f_1, f_2 \in \mathcal{H}^2(\widetilde{H}, 1_{\mathcal{I}}\omega_\sigma)$ then 10.5 gives

$$\int_{\widetilde{H}/L} f_1(\dot{h})\overline{f_2(\dot{h})}\frac{dh}{dl} = [L_H : L]^{-1}\mu_{Z_{\widetilde{H}}/L}(Z_{\widetilde{H}}/L)\int_{\widetilde{H}/Z_{\widetilde{H}}} f_1(\dot{h})\overline{f_2(\dot{h})}\frac{dh}{dz}$$
$$= \int_{\widetilde{H}/Z_{\widetilde{H}}} f_1(\dot{h})\overline{f_2(\dot{h})}\frac{dh}{dz}$$

using $\mu_{Z_{\widetilde{H}}/L}(Z_{\widetilde{H}}/L) = \mu_{Z_{\widetilde{H}}/L}({}^0Z_{\widetilde{H}}L_H/L) = [L_H : L]$. Therefore 10.6 also holds when $(\, , \,)$ is the canonical inner product on $\mathcal{H}^2(\widetilde{H}, 1_{\mathcal{I}}\omega_\sigma)$. We also have that the formal degree $d(\sigma)$ of $(\sigma, \mathcal{W})$ (with respect to our fixed Haar measure on $\widetilde{H}/Z_{\widetilde{H}}$) satisfies

$$(\xi_1\zeta_1^*, \xi_2\zeta_2^*) = d(\sigma)^{-1}(\xi_1, \xi_2)\overline{(\zeta_1, \zeta_2)}$$

where $(\, , \,)$ is still the canonical inner product on $\mathcal{H}^2(\widetilde{H}, 1_{\mathcal{I}}\omega_\sigma)$. Hence $d(\pi) = d(\sigma)$. Thus the equivalence of categories between $\mathfrak{R}_\rho(G)$ and $\mathfrak{R}_{1_{\mathcal{I}}}(\widetilde{H})$ preserves formal degrees (with respect to our choices of Haar measures on G/Z and $\widetilde{H}/Z_{\widetilde{H}}$).

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