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The Hodge de Rham theory of relative Malcev completion

THE HODGE DE RHAM THEORY OF
RELATIVE MALCEV COMPLETION (*)

By Richard M. HAIN

ABSTRACT. – Suppose that X is a smooth manifold and $\rho : \pi_1(X, x) \to S$ is a representation of the fundamental group of X into a real reductive group with Zariski dense image. To such data one can associate the Malcev completion $\mathcal{G}$ of $\pi_1(X, x)$ relative to ρ. In this paper we generalize Chen's iterated integrals and show that the H^0 of a suitable complex of these iterated integrals is the coordinate ring of $\mathcal{G}$. This is used to show that if X is a complex algebraic manifold and ρ is the monodromy representation of a variation of Hodge structure over X, then the coordinate ring of $\mathcal{G}$ has a canonical mixed Hodge structure. © Elsevier, Paris

RÉSUMÉ. – Soit X une variété différentiable et soit $\rho : \pi_1(X, x) \to S$ une représentation du groupe fondamental de X dans un groupe réductif réel. Quand l'image de ρ est Zariski dense, on a la complétion de Malcev $\mathcal{G}$ de $\pi_1(X, x)$ relative à ρ. Nous donnons une généralisation des intégrales itérées de Chen et nous montrons que le H^0 d'un complexe convenable de ces intégrales itérées est l'anneau des coordonnées de $\mathcal{G}$. Quand X est une variété algébrique complexe et ρ est la monodromie d'une variation de Hodge sur X, nous montrons que l'anneau des coordonnées de $\mathcal{G}$ est muni d'une structure de Hodge mixte canonique. © Elsevier, Paris

1. Introduction

Suppose that π is an abstract group, that S is a reductive algebraic group defined over a field F of characteristic zero, and that $\rho : \pi \to S(F)$ is a homomorphism with Zariski dense image. The completion of π relative to ρ is a proalgebraic group $\mathcal{G}$ which is an extension

$$1 \to \mathcal{U} \to \mathcal{G} \xrightarrow{p} S \to 1$$

where $\mathcal{U}$ is prounipotent, and a homomorphism $\tilde{\rho} : \pi \to \mathcal{G}(F)$ which lifts ρ:

$$
\begin{array}{ccc}
\pi & \xrightarrow{\rho} & S \\
\tilde{\rho}\downarrow & & \| \\
\mathcal{G} & \xrightarrow{p} & S
\end{array}
$$

It is characterized by the following universal mapping property. If ϕ is a homomorphism of π to a (pro)algebraic group G over F which is an extension

$$1 \to U \to G \to S \to 1$$

of S by a unipotent group U, and if the composite

$$\pi \to G \to S$$

is ρ, then there is a unique homomorphism $\mathcal{G} \to G$ of F-proalgebraic groups which commutes with the projections to S and through which ϕ factors.

When S is the trivial group, $\mathcal{G}$ is simply the classical Malcev (or unipotent) completion of π. In this case, with $F = \mathbb{R}$ or $\mathbb{C}$, and π the fundamental group of a smooth manifold, there is a de Rham theorem for $\mathcal{O}(\mathcal{G})$ which was proved by K.-T. Chen [4]. In these notes we generalize Chen's de Rham Theorem from the unipotent case to the general case. Our approach is based on the notes [4] of Deligne where an approach to computing the Lie algebra of the prounipotent radical of $\mathcal{G}$ via Sullivan's minimal models is sketched. Before explaining our result in general, we recall Chen's de Rham Theorem in the unipotent case.

If M is a smooth manifold and $w_1, \ldots, w_r$ are smooth 1-forms on M, then Chen defined

$$\int_\gamma w_1 \ldots w_r = \int \cdots \int_{0 \le t_1 \le \cdots \le t_r \le 1} f_1(t_1) \ldots f_r(t_r) \, dt_1 \ldots dt_r$$

where $\gamma : [0,1] \to M$ is a piecewise smooth path and $\gamma^* w_j = f_j(t) \, dt$. These are viewed as functions on the path space of M. An iterated integral is a linear combination of such functions and the constant function. Fix a base point $x \in M$. Set $\pi = \pi_1(M, x)$. Denote the iterated integrals on the space of loops in M based at x by $\mathcal{I}_x$. Denote by $H^0(\mathcal{I}_x)$ those elements of $\mathcal{I}_x$ whose value on a loop depends only on its homotopy class. Then Chen's π_1 de Rham Theorem asserts that integration induces a Hopf algebra isomorphism

$$\mathcal{O}(\mathcal{U}) \cong H^0(\mathcal{I}_x)$$

where $\mathcal{U}$ denotes the real points of the unipotent completion of π and $\mathcal{O}(\mathcal{U})$ its coordinate ring. Another important ingredient of Chen's theorem is that it gives an algebraic description of $\mathcal{I}_x$ and $H^0(\mathcal{I}_x)$ in terms of the (reduced) bar construction on the de Rham complex of M and the augmentation induced by the base point.

In this paper we generalize the definition of iterated integrals and prove a more general de Rham theorem in which the Hopf algebra $\mathcal{O}(\mathcal{G})$ of functions on the completion of $\pi_1(M, x)$ relative to a homomorphism $\rho : \pi_1(M, x) \to S$ is isomorphic to a Hopf algebra of "locally constant iterated integrals," defined algebraically in terms of a suitable (2-sided) bar construction on a complex $E^\bullet_{\text{fin}}(M, \mathcal{O}(P))$. The complex $E^\bullet_{\text{fin}}(M, \mathcal{O}(P))$ plays a central role in all our constructions and was introduced by Deligne in his notes [4], the main result of which is that the pronilpotent Lie algebra associated to its 1-minimal model is the Lie algebra of the prounipotent radical $\mathcal{U}$ of $\mathcal{G}$.

In Section 12 we define the completion of the fundamental groupoid of a manifold M with respect to the representation ρ. This is a category (in fact, a groupoid) whose objects are the points of X and where the Hom sets are proalgebraic varieties; the automorphism of the object $x \in M$ is the completion of $\pi_1(M, x)$ relative to ρ. There is a canonical functor of the fundamental groupoid of M to this category. We give a de Rham description of the coordinate ring of each Hom variety in terms of a suitable 2-sided bar

construction on $E^\bullet_{\mathrm{fin}}(M, \mathcal{O}(P))$ and of the functor from the fundamental groupoid to its relative completion using iterated integrals.

One of the main applications of Chen's π_1 de Rham Theorem is to give a direct functorial construction of Morgan's mixed Hodge structure [15] on the unipotent completion of the fundamental group of a pointed complex algebraic variety as is explained in [7]. In this paper we prove that if X is a smooth complex algebraic variety (or the complement of a normal crossings divisor in a compact Kähler manifold) and $\mathbb{V} \to X$ is a variation of Hodge structure with polarization $\langle \ , \ \rangle$ whose monodromy representation

$$\rho : \pi_1(X, x) \to S := \mathrm{Aut}(V_x, \langle \ , \ \rangle)$$

has Zariski dense image ([1]), then the coordinate ring $\mathcal{O}(\mathcal{G})$ of the completion of $\pi_1(X, x)$ relative to ρ has a natural mixed Hodge structure. More generally, we show that the coordinate rings of the Hom sets of the relative completion of the fundamental groupoid of X with respect to ρ have canonical mixed Hodge structures which are compatible with the groupoid structure.

Our principal application of the Hodge theorem for relative completion appears in [11] where we use it to prove that the unipotent completion of each Torelli group (genus $\neq 2$) has a canonical mixed Hodge structure given the choice of a smooth projective curve of genus g. Another application suggested by Ludmil Kartzarkov, and proved in Section 13, is a generalization of the theorem of Deligne-Griffiths-Morgan-Sullivan (DGMS) on fundamental groups of compact Kähler manifolds: If X is a compact Kähler manifold and $\mathbb{V} \to X$ is a polarized variation of Hodge structure with Zariski dense monodromy, then the prounipotent radical of the completion of $\pi_1(X, x)$ relative to the monodromy representation has a presentation with only quadratic relations. The theorem of DGMS is recovered by taking $\mathbb{V}$ to be the trivial variation $\mathbb{Q}_X$.

In Section 14 we show that if X is a smooth variety and $\mathbb{V}$ is a variation of Hodge structure over X with Zariski dense monodromy representation ρ, then there is a canonical integrable 1-form

$$\omega \in E^1(X') \mathbin{\widehat{\otimes}} \mathrm{Gr}^W_\bullet \mathfrak{u}$$

where X' is the Galois covering of X with Galois group $\mathrm{im}\rho$, and $\mathfrak{u}$ the Lie algebra of the prounipotent radical $\mathcal{U}$ of the completion $\mathcal{G}$ of $\pi_1(X, x)$ with respect to ρ. This form is $\mathrm{im}\rho$ invariant under the natural actions of $\mathrm{im}\rho$ on X' and $\mathfrak{u}$. It can be integrated to the canonical representation

$$\tilde\rho : \pi_1(X, x) \to S \ltimes \mathcal{U} \cong \mathcal{G}.$$

In the particular case where X is the complement of the discriminant locus in $\mathbb{C}^n$, where $\pi_1(X, x)$ is the braid group B_n and S the symmetric group, this connection is the standard

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([1]) The assumption that the monodromy have Zariski dense monodromy can probably be removed. What one needs to know is that the Zariski closure of the image of ρ is reductive and that its coordinate ring has a natural real Hodge structure – *see* Remark 13.13. This should follow from the work of Simpson and Corlette as each of them has pointed out.

one

$$\omega = \sum_{i<j} d\log(x_i - x_j)\, X_{ij}$$

on X', the complement in $\mathbb{C}^n$ of the hyperplanes $x_i = x_j$. Kohno [13] used the Σ_n invariant form ω and finite dimensional representations of $\mathrm{Gr}_\bullet^W \mathfrak{u}$ to construct Jones's representations of B_n. Our construction is used in [11] to construct an analogous "universal projectively flat connection" for the mapping class groups in genus ≥ 3.

2. Conventions

Here, to avoid confusion later on, we make explicit our basic conventions and review some basic constructions that depend, to some extent, on these conventions.

Throughout these notes, X will be a connected smooth manifold. By a path in X from $x \in X$ to $y \in X$, we shall mean a piecewise smooth map $\alpha : [0,1] \to X$ with $\alpha(0) = x$ and $\alpha(1) = y$. The set of all paths in X will be denoted by PX. There is a natural projection $PX \to X \times X$; it takes α to its endpoints $(\alpha(0), \alpha(1))$. The fiber of this map over (x,y) will be denoted by $P_{x,y}X$, and the inverse image of $\{x\} \times X$ will be denoted by $P_{x,-}$. The sets PX, $P_{x,y}X$, $P_{x,-}X$, each endowed with the compact-open topology, are topological spaces.

We shall multiply paths in their natural order, as distinct from the functional order. That is, if α and β are two paths in X with $\alpha(1) = \beta(0)$, then the path $\alpha\beta$ is defined and is the path obtained by first traversing α, and then β.

Suppose that $(\widetilde{X}, \tilde{x}_o) \to (X, x_o)$ is a pointed universal covering of X. With our path multiplication convention, $\pi_1(X, x_o)$ acts on the *left* of $\widetilde{X}$. One way to see this clearly is to note that there is a natural bijection

$$\coprod_{y \in X} \pi_0(P_{x_o,y}X) \to \widetilde{X}.$$

This bijection is constructed by taking the homotopy class of the path α in X that starts at x_o to the endpoint $\tilde{\alpha}(1)$ of the unique lift $\tilde{\alpha}$ of α to $\widetilde{X}$ that starts at $\tilde{x}_o$. With respect to this identification, the action of $\pi_1(X, x_o)$ is by left multiplication.

Another consequence of our path multiplication convention is that $\pi_1(X, x_o)$ naturally acts on the *right* of the fiber over x_o of a flat bundle over X, as can be seen from an elementary computation. Conversely, if

$$\rho : F \times \pi_1(X, x_o) \to F$$

is a right action of $\pi_1(X, x_o)$ on F, then one can define $F \times_\rho \widetilde{X}$ to be the quotient space $F \times \widetilde{X} / \sim$, where the equivalence relation is defined by

$$(f, gx) \sim (fg, x)$$

for all $g \in \pi_1(X, x_o)$. This bundle has a natural flat structure – namely the one induced by the trivial flat structure on the bundle $F \times \widetilde{X} \to \widetilde{X}$. The composite

$$F \cong F \times \{\tilde{x}_o\} \hookrightarrow F \times \widetilde{X} \to F \times_\rho \widetilde{X}$$

gives a natural identification of the fiber over x_o with F. With respect to this identification, the monodromy representation of the flat bundle $F \times_\rho \widetilde{X} \to X$ is ρ.

Of course, left actions can be converted into right actions by using inverses. Presented with a natural left action of $\pi_1(X, x_o)$ on a space, we will convert it, in this manner, into a right action in order to form the associated flat bundle.

The flat bundle over X corresponding to the right $\pi_1(X, x_o)$-module V will be denoted by $\mathbb{V}$. For a flat vector bundle $\mathbb{V}$ over X, we shall denote the complex of smooth forms with coefficients in the corresponding C^∞ vector bundle by $E^\bullet(X, \mathbb{V})$. This is a complex whose cohomology is naturally isomorphic to $H^\bullet(X, \mathbb{V})$. In particular, the C^∞ de Rham complex of X will be denoted by $E^\bullet(X)$.

By definition, mixed Hodge structures (MHSs) are usually finite dimensional. When studying MHSs on completions of fundamental groups, one encounters two kinds of infinite dimensional MHSs

$$((V_\mathbb{R}, W_\bullet), (V_\mathbb{C}, W_\bullet, F^\bullet)).$$

In both cases, the weight graded quotients are finite dimensional. In one, the weight filtration is bounded below (*i.e.* $W_l V = 0$, for some l) so that each $W_m V$ is finite dimensional. In this case we require that each $W_m V$ with the induced filtrations be a finite dimensional MHS in the usual sense. The other case is dual. Here the weight filtration is bounded above (*i.e.*, $V = W_l V$ for some l). In this case, each $V/W_m V$ is finite dimensional. We require that V be complete in the topology defined by the weight filtration (*i.e.*, V is the inverse limit of the $V/W_m V$), that each part of the Hodge filtration be closed in V, and that each $V/W_m V$ with the induced filtrations be a finite dimensional MHS in the usual sense. Such mixed Hodge structures form an abelian category, as is easily verified.

Finally, if $V^\bullet$ is a graded module and r is a integer, $V[r]^\bullet$ denotes the graded module with

$$V[r]^n = V^{r+n}.$$

3. The coordinate ring of a reductive linear algebraic group

Suppose that S is a reductive linear algebraic group over a field F of characteristic zero. The right and left actions of S on itself induce commuting left and right actions of S on its coordinate ring $\mathcal{O}(S)$.

If V is a right S module, its dual $V^* := \mathrm{Hom}_F(V, F)$ is a left S module via the action

$$(s \cdot \phi)(v) := \phi(v \cdot s),$$

where $s \in S$, $\phi \in \mathrm{Hom}_F(V, F)$ and $v \in V$.

The following result generalizes to reductive groups a well known fact about the group ring of a finite group.

PROPOSITION 3.1. – *If $(V_\alpha)_\alpha$ is a set of representatives of the isomorphism classes of irreducible right S-modules, then, as an (S, S) bimodule, $\mathcal{O}(S)$ is canonically isomorphic to*

$$\bigoplus_\alpha V_\alpha^* \boxtimes V_\alpha.$$

Proof. – This follows from the following facts:

1. If V is an S module, then the set of matrix entries of V is the dual $(\mathrm{End} V)^*$ of $\mathrm{End} V$. It has commuting right and left S actions. The right action is induced by left multiplication of S on itself by left translation, and the left action by the right action of S on itself.

2. As a vector space, $(\mathrm{End}_F V)^*$ is naturally isomorphic to $V^* \otimes V$. The isomorphism takes $\phi \otimes v \in V^* \otimes V$ to the matrix entry

$$\{f : V \to V\} \mapsto \left\{ F \xrightarrow{v} V \xrightarrow{f} V \xrightarrow{\phi} F \right\}.$$

   It is easily checked that this isomorphism gives an isomorphism $(\mathrm{End} V)^* \cong V^* \boxtimes V$ of (S, S)-bimodules.

3. By standard arguments (*cf.* [1]), the fact that S is reductive implies that the subspace of $\mathcal{O}(S)$ spanned by the matrix entries of all irreducible linear representations is a subalgebra of $\mathcal{O}(S)$. That is, the image of the linear map

$$\Phi : \sum_\alpha V_\alpha^* \boxtimes V_\alpha \to \mathcal{O}(S)$$

   is a subalgebra of $\mathcal{O}(S)$. Since Φ is $S \times S$ equivariant, and since the $V_\alpha^* \boxtimes V_\alpha$ are pairwise non-isomorphic irreducible representations of $S \times S$, Φ is injective.

4. Since S is linear, it has a faithful linear representation V_0, say and $\mathcal{O}(S)$ is generated by the matrix entries of V_0. It follows that Φ is surjective, and therefore an algebra isomorphism.   □

Recall that if G is an affine algebraic group over F, then the Lie algebra $\mathfrak{g}$ of G can be recovered from $\mathcal{O}(G)$ as follows: Denote the maximal ideal in $\mathcal{O}(G)$ of functions that

vanish at the identity by $\mathfrak{m}$. Then, as a vector space, $\mathfrak{g}$ is isomorphic to the dual $\mathfrak{m}/\mathfrak{m}^2$ of the Zariski tangent space of G at the identity. The bracket is induced by the comultiplication

$$\Delta : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$$

as we shall now explain. Evaluation at the identity and inclusion of scalars give linear maps $\mathcal{O}(G) \to F$ and $F \to \mathcal{O}(G)$. There is therefore a canonical isomorphism

$$\mathcal{O}(G) \cong F \oplus \mathfrak{m}.$$

Using this decomposition, we see that the diagonal induces a diagonal map

$$\overline{\Delta} : \mathfrak{m} \to \mathfrak{m} \otimes \mathfrak{m}.$$

Denote the involution $f \otimes g \mapsto g \otimes f$ of $\mathfrak{m} \otimes \mathfrak{m}$ by τ. The map

$$\overline{\Delta} - \tau \circ \overline{\Delta} : \mathfrak{m} \to \mathfrak{m} \otimes \mathfrak{m}$$

induces the map

$$\Delta^c : \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2 \otimes \mathfrak{m}/\mathfrak{m}^2$$

dual to the bracket $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$.

4. A basic construction

From this point on S will be a linear algebraic group defined over $\mathbb{R}$. We will abuse notation and also denote its group of real points by S. We will assume now that we have a representation

$$\rho : \pi_1(X, x_o) \to S.$$

We do not assume that ρ has Zariski dense as it is not necessary for the preliminary constructions in this and the next few sections. We will, however, assume that ρ has Zariski dense image in Section 8 and subsequent sections. We will fix a set of representatives $(V_\alpha)_\alpha$ of the isomorphism classes of rational representations of S.

Composing ρ with the action of S on itself by *right* multiplication, we obtain a right action of $\pi_1(X, x_o)$ on S. Denote the corresponding flat bundle by

$$p : P \to X.$$

This is a left principal S bundle whose fiber $p^{-1}(x_o)$ over x_o comes with an identification with S ; the S action and the marking of $p^{-1}(x_o)$ are induced by the obvious left action of S on $S \times \widetilde{X}$ and by the composite

$$S \cong S \times \{x_o\} \hookrightarrow S \times \widetilde{X} \to P.$$

The point $\tilde{x}_o$ of $p^{-1}(x_o)$ corresponding to $1 \in S$ will be used as a basepoint of P.

Each rational representation of S gives rise to a representation of $\pi_1(X, x_o)$, and therefore to a local system over X. We shall call such a local system a *rational local system*.

The action of $\pi_1(X, x_o)$ on S by right multiplication induces a left action of S on $\mathcal{O}(S)$, the coordinate ring of S. Convert this to a right action using inverses:

$$(f\gamma)(s) = f(s\gamma^{-1}),$$

where $f \in \mathcal{O}(S)$, $\gamma \in \pi_1(X, x_o)$, and $s \in S$. Denote the associated flat bundle by

$$\mathcal{O}(P) \to X.$$

This is naturally a *right* flat principal S bundle over X. It follows from (3.1) that it is the direct sum of its rational sub-local systems:

$$\text{(1)} \qquad \mathcal{O}(P) = \bigoplus_\alpha \mathsf{V}_\alpha^* \otimes V_\alpha.$$

In particular, it is the direct limit of its rational sub-local systems. Define

$$E_{\mathrm{fin}}^\bullet(X, \mathcal{O}(P)) = \varinjlim E^\bullet(X, \mathsf{M}),$$

where M ranges over the rational sub-local systems of $\mathcal{O}(P)$. Denote the cohomology

$$\varinjlim H^\bullet(X, \mathsf{M})$$

of this complex by $H^\bullet(X, \mathcal{O}(P))$. The right action of S on $\mathcal{O}(P)$ induces a right action of S on

$$H^\bullet(X, \mathcal{O}(P)).$$

From (1), it follows that there is a natural isomorphism

$$E_{\mathrm{fin}}^\bullet(X, \mathcal{O}(P)) \cong \bigoplus_\alpha E^\bullet(X, \mathsf{V}_\alpha^*) \otimes V_\alpha$$

of right S modules. The following result is an immediate consequence.

PROPOSITION 4.1. – *For each irreducible representation V of S, there is a natural isomorphism*

$$\left[H^k(X, \mathcal{O}(P)) \otimes V \right]^S \cong H^k(X, \mathsf{V}). \qquad \square$$

The bundle $P \to X$ is foliated by its locally flat sections. Denote this foliation by $\mathcal{F}$. We view it as a sub-bundle of TP, the tangent bundle of P. Denote by $E^k(P, \mathcal{F})$ the vector space consisting of C^∞ sections of the dual of the bundle

$$\Lambda^k \mathcal{F} \to P.$$

One can differentiate sections along the leaves to obtain an exterior derivative map

$$d : E^k(P, \mathcal{F}) \to E^{k+1}(P, \mathcal{F}).$$

With this differential, $E^\bullet(P, \mathcal{F})$ is a differential graded algebra. Moreover, the left action of S on P induces a natural right action of S on it, and the natural restriction map

$$(2) \qquad\qquad\qquad E^\bullet(P) \to E^\bullet(P, \mathcal{F})$$

is an S-equivariant homomorphism of differential graded algebras.

The base point $\tilde{x}_o \in P$ induces augmentations

$$E_{\text{fin}}^\bullet(X, \mathcal{O}(P)) \to \mathbb{R} \text{ and } E^\bullet(P, \mathcal{F}) \to \mathbb{R}.$$

PROPOSITION 4.2. – *There is a natural, augmentation preserving d.g. algebra homomorphism*

$$E_{\text{fin}}^\bullet(X, \mathcal{O}(P)) \to E^\bullet(P, \mathcal{F})$$

which is injective and S-equivariant with respect to the natural right S actions.

Proof. – The bundle $(P, \tilde{x}_o) \to (X, x_o)$ is the quotient of $(S \times \tilde{X}, (1, x_o)) \to (\tilde{X}, x_o)$ by the action on $S \times \tilde{X}$ of $\pi_1(X, x_o)$ given by

$$\gamma : (s, x) \mapsto (s\rho(\gamma)^{-1}, \gamma x).$$

Via the pullback, we can identify $E_{\text{fin}}^\bullet(X, \mathcal{O}(P))$ with the $\pi_1(X, x_o)$ invariant elements of $E^\bullet(\tilde{X}) \otimes \mathcal{O}(S)$ and the augmentation with evaluation at $(1, \tilde{x}_o)$. Since the foliation $\mathcal{F}$ is the quotient by $\pi_1(X, x_o)$ of the foliation $\tilde{\mathcal{F}}$ on $S \times \tilde{X}$ with leaves $\{s\} \times \tilde{X}$, we can identify $E^\bullet(P, \mathcal{F})$ with the $\pi_1(X, x_o)$ invariant elements of $E^\bullet(S \times \tilde{X}, \tilde{\mathcal{F}})$. The d.g. algebra homomorphism

$$E^\bullet(\tilde{X}) \otimes \mathcal{O}(S) \to E^\bullet(\tilde{P}, \tilde{\mathcal{F}})$$

defined by $w \otimes \phi \mapsto \phi w$ is $\pi_1(X, x_o)$ equivariant, and therefore induces a d.g. algebra homomorphism

$$E_{\text{fin}}^\bullet(X, \mathcal{O}(P)) \to E^\bullet(P, \mathcal{F}).$$

Since both augmentations correspond to evaluation at $(1, \tilde{x}_o)$, this homomorphism is augmentation preserving. $\qquad\qquad\square$

5. Iterated integrals and monodromy of flat bundles

Consider the category $\mathcal{B}(X, S)$ whose objects are flat vector bundles $\mathbb{V}$ over X that admit a finite filtration

$$\mathbb{V} = \mathbb{V}^0 \supset \mathbb{V}^1 \supset \mathbb{V}^2 \supset \cdots$$

by sub-local systems with the properties:

1. the intersection of the $\mathbb{V}^i$ is trivial;

2. each graded quotient $\mathbb{V}^i/\mathbb{V}^{i+1}$ is the local system associated with a rational representation of S.

Denote the fiber over the base point x_o by V_o. It has a filtration corresponding to the filtration $\mathbb{V}^\bullet$ of $\mathbb{V}$:

$$V_o = V_o^0 \supset V_o^1 \supset V_o^2 \supset \cdots$$

The second condition above implies that there are rational representations $\tau_i : S \to \mathrm{AutGr}^i V_o$ such that the representation of $\pi_1(X, x_o)$ on $\mathrm{Gr}^i V_o$ is the composite

$$\pi_1(X, x_o) \xrightarrow{\rho} S \xrightarrow{\tau_i} \mathrm{AutGr}^i V_o.$$

Let $\tau : S \to \prod \mathrm{AutGr}^i V_o$ be the product of the representations τ_i. Let

$$G = \{\phi \in \mathrm{Aut}V_o : \phi \text{ preserves } V_o^\bullet \text{ and } \mathrm{Gr}^\bullet \phi \in \mathrm{im}\tau\}.$$

This is a linear algebraic group which is an extension of $\mathrm{im}\tau$ by the unipotent group

$$U = \{\phi \in \mathrm{Aut}V_o : \phi \text{ preserves } V_o^\bullet \text{ and acts trivially on } \mathrm{Gr}^\bullet V_o\}$$

whose Lie algebra we shall denote by $\mathfrak{u}$. We shall denote the monodromy representation at x_o of $\mathbb{V}$ by

$$\tilde{\rho} : \pi_1(X, x_o) \to G.$$

Denote the C^∞ vector bundles associated to the flat bundles $\mathbb{V}$ and $\mathbb{V}^i$ by $\mathcal{V}$ and $\mathcal{V}^i$, respectively. We would like to trivialize $\mathcal{V}$. In order to do this, we pull it back to P along the projection $p : P \to X$.

PROPOSITION 5.1. – *There is a trivialization*

$$p^*\mathcal{V} \xrightarrow{\cong} P \times V_o$$

and a splitting of the natural map $G \to \mathrm{im}\tau$ which satisfy

1. *the corresponding connection form* [2] *$\tilde{\omega}$ satisfies*

$$\tilde{\omega} \in E^1(P) \otimes \mathfrak{u};$$

2. *the isomorphism $V_o \to V_o$, induced by the trivialization of $p^*\mathcal{V}$ between the fiber over the points $\tilde{x}_o$ and $s \cdot \tilde{x}_o$ of $p^{-1}(x_o)$, is $\tau(s)^{-1}$.*

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[2] Our convention is that the connection form associated to the trivialized bundle $V \times X \to X$ with connection ∇ is the 1-form ω on X with values in $\mathrm{End}V$ which is characterized by the property that for all sections $f : X \to V$

$$\nabla f = df - f\omega \in E^1(X) \otimes V.$$

Note that the second condition implies that the isomorphism $V_o \to V_o$, induced by the trivialization of $p^*\mathcal{V}$ between the fiber over the points $a \cdot \tilde{x}_o$ and $sa \cdot \tilde{x}_o$ of $p^{-1}(x_o)$, is $\tau(s)^{-1}$.

The first step in the proof is the following elementary result. It can be proved by induction on the length of the filtration. It gives the splitting of $G \to \mathrm{im}\tau$.

LEMMA 5.2. – *There is an isomorphism*

$$\mathcal{V} \cong \bigoplus_{i \geq 0} \mathrm{Gr}^i \mathcal{V}$$

of C^∞ vector bundles that splits the filtration $\mathcal{V}^\bullet$. That is,

1. *the sub-bundle $\mathcal{V}^i$ corresponds to $\oplus_{j \geq i} \mathrm{Gr}^j \mathcal{V}$;*
2. *the isomorphism*

$$\mathrm{Gr}^i \mathcal{V} \to \mathcal{V}^i / \mathcal{V}^{i+1}$$

induced by the trivialization is the identity. □

Proof of (5.1). – Pulling back the splitting given by (5.2) of the filtration $\mathcal{V}^i$ to P, we obtain a splitting

$$p^*\mathcal{V} \cong \bigoplus_i p^* \mathrm{Gr}^i \mathcal{V}$$

of $p^*\mathcal{V}$. So it suffices to trivialize each $p^* \mathrm{Gr}^i \mathcal{V}$.

To do this, we first do it on a single leaf $\mathcal{L}$ of P. The restriction of the monodromy representation τ to $\mathcal{L}$ is clearly trivial. Consequently, the restriction of $p^*\mathcal{V}$ to $\mathcal{L}$ is trivial as a flat bundle. Observe that if this leaf contains $\tilde{x}_o$, then this trivialization satisfies condition (2) in the statement of (5.1).

Next, change the trivialization of $p^* \mathrm{Gr}^i \mathcal{V}$ on $p^{-1}(x_o)$ so that it satisfies condition (2) in the statement of (5.1). Extend this to a trivialization of $p^{-1} \mathrm{Gr}^i \mathcal{V}$ on all of P by parallel transport along the leaves of P. This gives a well defined local trivialization which is a global trivialization by the argument in the previous paragraph.

We thus obtain a trivialization of $p^*\mathcal{V}$ which is compatible with the filtration $\mathcal{V}^\bullet$ and which is flat on each $\mathrm{Gr}^i \mathcal{V}$. It follows that the connection form $\tilde{\omega}$ associated to this trivialization satisfies $\tilde{\omega} \in E^1(P) \otimes \mathfrak{u}$. □

If S is not finite, this connection is not flat as it is not flat in the vertical direction. We can make it flat by restricting it to the leaves of the foliation $\mathcal{F}$ of P. Denote the image of $\tilde{\omega}$ under the restriction homomorphism

$$E^1(P) \otimes \mathfrak{u} \to E^1(P, \mathcal{F}) \otimes \mathfrak{u}$$

by ω. It defines the connection in the leaf direction. This connection is clearly flat, and it follows that ω is integrable.

The following assertion is a consequence of (4.2) and the properties (1) and (2) in the statement of Proposition 5.1. Note that we view S as acting on the left of $\mathfrak{u}$ via the adjoint action – that is, via the composite $S \to \mathrm{im}\tau \hookrightarrow G \to \mathrm{Aut}\,\mathfrak{u}$.

PROPOSITION 5.3. – *The connection form ω is integrable and lies in the subspace $E^1_{\text{fin}}(X, \mathcal{O}(P)) \otimes \mathfrak{u}$ of $E^1(P, \mathcal{F}) \otimes \mathfrak{u}$. Moreover, if $s \in S$, then $s^*\omega = Ad(s)\omega$.* $\qquad\square$

Remark 5.4. – There is a converse to this result. Suppose that $\mathfrak{u}$ is a nilpotent Lie algebra in the category of rational representations of S. Then we can form the semi-direct product $G = S \ltimes U$, where U is the corresponding unipotent group. If V is a G module, and if

$$\omega \in E^1_{\text{fin}}(X, \mathcal{O}(P)) \otimes \mathfrak{u}$$

satisfies the conditions

1. $d\omega + \omega \wedge \omega = 0$;
2. $s^*\omega = Ad(s)\omega$;

then we can construct an object of $\mathcal{B}(X, S)$ with fiber V over x_o whose pullback to P has connection form ω with respect to an appropriate trivialization.

We are now ready to express the monodromy representation of V in terms of iterated integrals of ω. Recall that K.-T. Chen [3] defined, for 1-forms w_i on a manifold M taking values in an associative algebra A,

$$\int_\gamma w_1 w_2 \ldots w_r$$

to be the element

$$\int \cdots \int_{0 \leq t_1 \leq \ldots \leq t_r \leq 1} f_1(t_1) f_2(t_2) \ldots f_r(t_r) \, dt_1 dt_2 \ldots dt_r$$

of A, where $\gamma * w_j = f_j(t)dt$. This is regarded as an A-valued function $PM \to A$ on the path space of M. An A-valued iterated integral is a function $PM \to A$ which is a linear combination of functions of this form together with a constant function.

Suppose that $V \times M \to M$ is a trivial bundle with a connection given by the connection form

$$\omega \in E^1(M) \otimes \text{End}(V).$$

In this case we can define the parallel transport map

$$T : PM \to \text{Aut}(V)$$

where PM denotes the space of piecewise smooth paths in M. A path goes to the linear transformation of V obtained by parallel transporting the identity along it. Chen [3] obtained the following expression for T in terms of ω.

PROPOSITION 5.5. – *With notation as above, we have*

$$T(\gamma) = 1 + \int_\gamma \omega + \int_\gamma \omega\omega + \int_\gamma \omega\omega\omega + \cdots \qquad\square$$

Note that since $\mathfrak{u}$ is nilpotent, this is a finite sum. Armed with this formula, we can express the monodromy of $V \to X$ in terms of $\omega \in E^1_{\text{fin}}(X, \mathcal{O}(P))$. Suppose that $\gamma \in P_{x_o, x_o} X$. Denote the unique lift of γ to P which is tangent to $\mathcal{F}$ and begins at $\tilde{x}_o \in p^{-1}(x_o)$ by $\tilde{\gamma}$.

PROPOSITION 5.6. – *The monodromy of* $\mathbb{V} \to X$ *takes* $\gamma \in P_{x_o,x_o}X$ *to*

$$\tilde{\rho}(\gamma) = \left(1 + \int_{\tilde{\gamma}} \omega + \int_{\tilde{\gamma}} \omega\omega + \int_{\tilde{\gamma}} \omega\omega\omega + \cdots \right)\tau(\rho(\gamma)) \in G.$$

The proof is a straightforward consequence of Chen's formula (5.5) and condition (2) of (5.1).

This formula motivates the following generalization of Chen's iterated integrals.

DEFINITION 5.7. – For $\phi \in \mathcal{O}(S)$ and $w_1, \ldots, w_r$ elements of $E^1_{\text{fin}}(X, \mathcal{O}(P))$, we define

$$\int (w_1 \ldots w_r | \phi) : P_{x_o,x_o}X \to \mathbb{R}$$

by

$$\int_\gamma (w_1 \ldots w_r | \phi) = \phi(\rho(\gamma)) \int_{\tilde{\gamma}} w_1 \ldots w_r.$$

We will call linear combinations of such functions *iterated integrals with coefficients in* $\mathcal{O}(S)$. They will be regarded as functions $P_{x_o,x_o}X \to \mathbb{R}$. We will denote the set of them by $I(X, \mathcal{O}(S))_{x_o}$. Such an iterated integral will be said to be *locally constant* if it is constant on each connected component of $P_{x_o,x_o}X$. We shall denote the set of locally constant iterated integrals on $P_{x_o,x_o}X$ by $H^0(I(X, \mathcal{O}(S))_{x_o})$. Evidently, each such locally constant iterated iterated integral defines a function

$$\pi_1(X, x_o) \to \mathbb{R}.$$

By taking matrix entries in (5.6), we obtain the following result.

COROLLARY 5.8. – *Each matrix entry of the monodromy representation*

$$\tilde{\rho} : \pi_1(X, x_o) \to G$$

of an object of $\mathcal{B}(X, S)$ *can be expressed as a locally constant iterated integral on* X *with coefficients in* $\mathcal{O}(S)$. $\qquad\square$

The following results imply that $H^0(I(X, \mathcal{O}(S))_{x_o})$ is a Hopf algebra with coproduct dual to the multiplication of paths, and antipode dual to the involution of $P_{x_o,x_o}X$ that takes each path to its inverse.

PROPOSITION 5.9. – *Suppose that* γ *and* μ *are in* $P_{x_o,x_o}X$, *that* $\phi, \psi \in \mathcal{O}(S)$ *and that* $w_1, w_2, \ldots \in E^1_{\text{fin}}(X, \mathcal{O}(P))$. *Then we have:*

$$\int_\gamma (w_1 \ldots w_p | \phi) \int_\gamma (w_{p+1} \ldots w_{p+q} | \phi) = \sum_{\sigma \in Sh(p,q)} \int_\gamma \left(w_{\sigma(1)} \ldots w_{\sigma(p+q)} | \phi\psi\right)$$

where $Sh(p, q)$ *denotes the set of shuffles of type* (p, q);

$$\int_{\gamma^{-1}} (w_1 \ldots w_r | \phi) = (-1)^r \int_\gamma \left(\rho(\gamma^{-1})^* w_r \ldots \rho(\gamma^{-1})^* w_1 | i_S^* \phi\right)$$

where $i_S^* : \mathcal{O}(S) \to \mathcal{O}(S)$ is the antipode of $\mathcal{O}(S)$;

$$\int_{\gamma\mu} (w_1 \ldots w_r | \phi) = \sum_{i=0}^{r} \sum_j \int_\gamma (w_1 \ldots w_i | \phi_j') \int_\mu (\rho(\gamma)^* w_{i+1} \ldots \rho(\gamma)^* w_r | \phi_j'')$$

where $\Delta_S : \mathcal{O}(S) \to \mathcal{O}(S) \otimes \mathcal{O}(S)$ is the coproduct of $\mathcal{O}(S)$, and

$$\Delta_S \phi = \sum_j \phi_j' \otimes \phi_j''.$$

Proof. – This proof is a straightforward using the definition (5.7) and basic properties of classical iterated integrals due to Chen [3]. □

COROLLARY 5.10. – *The set of iterated integrals $I(X, \mathcal{O}(S))_{x_o}$ is a commutative Hopf algebra.*

Remark 5.11. – Suppose that ρ has Zariski dense image. Let $\pi_1(X, x_o) \to \mathcal{G}$ be the completion of $\pi_1(X, x_o)$ relative to $\rho : \pi_1(X, x_o) \to S$. Since the coordinate ring $\mathcal{O}(\mathcal{G})$ of $\mathcal{G}$ is the ring of matrix entries of representations of G, it follows from (5.8) that there is a Hopf algebra inclusion

$$\mathcal{O}(\mathcal{G}) \hookrightarrow H^0(I(X, \mathcal{O}(S))_{x_o}).$$

This should be an isomorphism. To prove this assertion, it would suffice to show that

$$H^0(I(X, \mathcal{O}(P))) \otimes_{\mathcal{O}(S)} \mathbb{R}$$

is the direct limit of coordinate rings of a directed system of unipotent groups, each with an S action. This is surely true, but we seek a more algebraic de Rham theorem for $\mathcal{O}(\mathcal{G})$ which is more convenient for Hodge theory.

6. Higher iterated integrals

As a preliminary step to defining the algebraic analogue of $I(X, \mathcal{O}(S))_{x_o}$, we generalize the definition of iterated integrals with values in $\mathcal{O}(S)$ to higher dimensional forms.

Denote by $E^n(P_{x_o,x_o}X)$ the differential forms of degree n on the loop space $P_{x_o,x_o}X$. One can surely use any "reasonable" definition of differential forms on $P_{x_o,x_o}X$, but we will use Chen's definition from [3] where, to specify a differential form on $P_{x_o,x_o}X$, it is enough to specify its pullback along each "smooth map" $\alpha : U \to P_{x_o,x_o}X$ from an open subset U of some finite dimensional euclidean space. By a smooth map, we mean a map $\alpha : U \to P_{x_o,x_o}X$ whose "suspension"

$$\widehat{\alpha} : [0,1] \times U \to X; \quad (t, u) \mapsto \alpha(u)(t)$$

is continuous and smooth on each $[t_{j-1}, t_j] \times U$ for some partition

$$0 = t_0 \leq t_1 \leq \ldots \leq t_m = 1$$

of $[0, 1]$.

DEFINITION 6.1. – Suppose that $\phi \in \mathcal{O}(S)$, and that $w_j \in E_{\text{fin}}^{n_j}(X, \mathcal{O}(P))$ with each $n_j > 0$. Set $n = -r + \sum_j n_j$. Define

$$\int (w_1 \ldots w_r | \phi) \in E^n(P_{x_o, x_o} X)$$

by specifying that for each smooth map $\alpha : U \to P_{x,x} X$,

$$\alpha^* \int (w_1 \ldots w_r | \phi)$$

is the element

$$\int \cdots \int_{0 \leq t_1 \leq \cdots \leq t_r \leq 1} \widehat{w}_1(t_1) \wedge \ldots \wedge \widehat{w}_r(t_r) \, dt_1 dt_2 \ldots dt_r \, \phi(\rho(\alpha(u)))$$

of $E^n(U)$, where

$$\widehat{w}_j : (\partial/\partial t) \dashv \widetilde{\alpha}^* w_j$$

and $\widetilde{\alpha} : [0,1] \times U \to P$ is the smooth map with the property that for each $x \in U$, the map $t \mapsto \widetilde{\alpha}(t,x)$ is the unique lift of $t \mapsto \widehat{\alpha}(t,x)$ that begins at $\widetilde{x}_o$ and is tangent to $\mathcal{F}$.

These iterated integrals form a subspace $I^\bullet(X, \mathcal{O}(S))_{x_o}$ of $E^\bullet(P_{x_o, x_o} X)$. Chen's arguments [3] can be adapted easily to show that this is, in fact, a sub d.g. Hopf algebra of $E^\bullet(P_{x_o, x_o} X)$. In particular, we have:

PROPOSITION 6.2. – *The space of locally constant iterated integrals on X with coefficients in $\mathcal{O}(S)$ is $H^0(I^\bullet(X, \mathcal{O}(S))_{x_o})$.* $\square$

7. The reduced bar construction

In this section we review Chen's definition of the reduced bar construction which he described in [2].

Suppose that $A^\bullet$ is a commutative differential graded algebra (hereafter denoted d.g.a.) and that $M^\bullet$ and $N^\bullet$ are complexes which are modules over $A^\bullet$. That is, the structure maps

$$A^\bullet \otimes M^\bullet \to M^\bullet \qquad \text{and} \qquad A^\bullet \otimes N^\bullet \to N^\bullet$$

are chain maps. We shall suppose that $A^\bullet$, $M^\bullet$ and $N^\bullet$ are all positively graded. Denote the subcomplex of $A^\bullet$ consisting of elements of positive degree by A^+.

The *(reduced) bar construction* $B(M^\bullet, A^\bullet, N^\bullet)$ is defined as follows. We first describe the underlying graded vector space. It is a quotient of the graded vector space

$$T(M^\bullet, A^\bullet, N^\bullet) := \bigoplus_s M^\bullet \otimes \left(A^+[1]^{\otimes r}\right) \otimes N^\bullet.$$

We will use the customary notation $m[a_1|\ldots|a_r]n$ for

$$m \otimes a_1 \otimes \ldots \otimes a_r \otimes n \in T(M^\bullet, A^\bullet, N^\bullet).$$

To obtain the vector space underlying the bar construction, we mod out by the relations

$$m[dg|a_1|\ldots|a_r]n = m[ga_1|\ldots|a_r]n - m \cdot g[a_1|\ldots|a_r]n;$$

$$m[a_1|\ldots|a_i|dg|a_{i+1}|\ldots|a_r]n = m[a_1|\ldots|a_i|g\,a_{i+1}|\ldots|a_r]n$$
$$- m[a_1|\ldots|a_i\,g|a_{i+1}|\ldots|a_r]n \qquad 1 \le i < s;$$

$$m[a_1|\ldots|a_r|dg]n = m[a_1|\ldots|a_r]g \cdot n - m[a_1|\ldots|a_r\,g]n;$$
$$m[dg]n = 1 \otimes g \cdot n - m \cdot g \otimes 1$$

Here each $a_i \in A^+$, $g \in A^0$, $m \in M^\bullet$, $n \in N^\bullet$, and r is a positive integer.

Before defining the differential, it is convenient to define an endomorphism J of each graded vector space by $J : v \mapsto (-1)^{\deg v}v$. The differential is defined as

$$d = d_M \otimes 1_T \otimes 1_N + J \otimes d_B \otimes 1 + J_M \otimes J_T \otimes d_N + d_C.$$

Here T denotes the tensor algebra on $A^+[1]$, d_B is defined by

$$d_B[a_1|\ldots|a_r] = \sum_{1 \le i \le r} (-1)^i[Ja_1|\ldots|Ja_{i-1}|da_i|a_{i+1}|\ldots|a_r]$$
$$+ \sum_{1 \le i < r} (-1)^{i+1}[Ja_1|\ldots|Ja_{i-1}|Ja_i \wedge a_{i+1}|a_{i+2}|\ldots|a_r]$$

and d_C is defined by

$$d_C m[a_1|\ldots|a_r]n = (-1)^s Jm[Ja_1|\ldots|Ja_{r-1}]a_r \cdot n - Jm \cdot a_1[a_2|\ldots|a_r]n.$$

One can check that these differentials are well defined.

If both $M^\bullet$ and $N^\bullet$ are themselves d.g.a.s over $A^\bullet$, then $B(M^\bullet, A^\bullet, N^\bullet)$ is also a differential graded algebra. The product is defined by

$$(3) \qquad m'[a_1|\ldots|a_p]n' \otimes m''[a_{p+1}|\ldots|a_{p+q}]n'' \mapsto$$
$$\sum_{\sigma \in \Sigma(p,q)} \pm m' \wedge m''[a_{\sigma(1)}|a_{\sigma(2)}|\ldots|a_{\sigma(p+q)}]n' \wedge n''.$$

Here $\Sigma(p,q)$ denotes the set of shuffles of type (p,q). The sign in front of each term on the right hand side is determined by the usual sign conventions that apply when moving a symbol of degree k past one of degree l – one considers each a_j to be of degree $-1 + \deg a_j$.

The reduced bar construction $B(M^\bullet, A^\bullet, N^\bullet)$ has a standard filtration

$$M^\bullet \otimes N^\bullet = B_0(M^\bullet, A^\bullet, N^\bullet) \subseteq B_1(M^\bullet, A^\bullet, N^\bullet) \subseteq B_2(M^\bullet, A^\bullet, N^\bullet) \subseteq \cdots$$

which is often called the *bar filtration*. The subspace

$$B_s(M^\bullet, A^\bullet, N^\bullet)$$

is defined to be the span of those $m[a_1|\dots|a_r]n$ with $r \leq s$. When $A^\bullet$ has connected homology (*i.e.*, $H^0(A^\bullet) = \mathbb{R}$), the corresponding spectral sequence, which is called the *Eilenberg-Moore spectral sequence*, has E_1 term

$$E_1^{-s,t} = \left[M^\bullet \otimes H^+(A^\bullet)^{\otimes s} \otimes N^\bullet\right]^t.$$

A proof of this can be found in [2]. It is at this point that the density of ρ begins to plays a role in our constructions.

PROPOSITION 7.1. – *If the image of $\rho : \pi_1(X, x_o) \to S$ is Zariski dense image, then $E_{\mathrm{fin}}^\bullet(X, \mathcal{O}(P))$ is connected.*

Proof. – The density of $\mathrm{im}\rho$ implies that $H^0(X, \mathbb{V}_\alpha)$ vanishes for all non-trivial irreducible representations V_α of S. The result follows from (4.1). □

The following basic property of the reduced bar construction is a special case of a result proved in [2]. It is easily proved using the Eilenberg-Moore spectral sequence. Suppose that $\psi : A_1^\bullet \to A_2^\bullet$ is a d.g.a. homomorphism, and that $M^\bullet$ is a right $A_2^\bullet$ module and $N^\bullet$ a right $A_2^\bullet$ module. Then $M^\bullet$ and $N^\bullet$ can be regarded as $A_1^\bullet$ modules via ψ. We therefore have a chain map

(4) $$B(M^\bullet, A_1^\bullet, N^\bullet) \to B(M^\bullet, A_2^\bullet, N^\bullet).$$

PROPOSITION 7.2. – *If ψ is a quasi-isomorphism, then so is (4).* □

8. The construction of $\mathcal{G}^{\mathrm{DR}}$

From this point on, we will assume that the representation $\rho : \pi_1(X, x_o) \to S$ has Zariski dense image. In this section we construct a proalgebraic group $\mathcal{G}^{\mathrm{DR}}$ which is an extension

$$1 \to \mathcal{U}^{\mathrm{DR}} \to \mathcal{G}^{\mathrm{DR}} \xrightarrow{p} S \to 1,$$

where $\mathcal{U}^{\mathrm{DR}}$ is prounipotent, and a homomorphism

$$\tilde{\rho} : \pi_1(X, x_o) \to \mathcal{G}^{\mathrm{DR}}$$

whose composition with $p : \mathcal{G}^{\mathrm{DR}} \to S$ is ρ. We do this by constructing the coordinate ring of $\mathcal{G}^{\mathrm{DR}}$ using the bar construction. In the two subsequent sections, we will show that $\tilde{\rho} : \pi_1(X, x_o) \to \mathcal{G}^{\mathrm{DR}}$ is the Malcev completion of $\pi_1(X, x_o)$ relative to ρ.

The fixed choice of a base point $\tilde{x}_o \in p^{-1}(x_o)$ determines augmentations

$$\epsilon_{\tilde{x}_o} : E_{\mathrm{fin}}^\bullet(X, \mathcal{O}(P)) \to \mathbb{R}$$

and

$$\delta_{x_o} : E_{\mathrm{fin}}^\bullet(X, \mathcal{O}(P)) \to \mathcal{O}(S)$$

as we shall now explain. Since these augmentations are compatible with restriction, it suffices to give them in a neighbourhood of x_o. Over a contractible neighbourhood U of x_o, the local system P is trivial and may therefore by identified with the trivial flat bundle $S \times U \to S$ in such a way that $\tilde{x}_o$ corresponds to $(1, x_o) \in S \times U$. The restriction of an element of $E_{\mathrm{fin}}^k(X, \mathcal{O}(P))$ to U is then of the form

$$\sum_i \phi_i \otimes w_i$$

where $\phi_i \in \mathcal{O}(S)$, and $w_i \in E^k(U)$. Denote the augmentation $E^\bullet(U) \to \mathbb{R}$ induced by x_o by μ_{x_o}. Then the augmentations δ_{x_o} and $\epsilon_{\tilde{x}_o}$ are defined by

$$\delta_{x_o} : \sum_i \phi_i \otimes w_i \mapsto \sum_i \mu_{x_o}(w_i)\, \phi_i$$

and

$$\epsilon_{\tilde{x}_o} : \sum_i \phi_i \otimes w_i \mapsto \sum_i \mu_{x_o}(w_i)\, \phi_i(1).$$

One can regard $\mathbb{R}$ and $\mathcal{O}(S)$ as algebras over $E_{\mathrm{fin}}^\bullet(X, \mathcal{O}(P))$ where the actions of $E_{\mathrm{fin}}^\bullet(X, \mathcal{O}(P))$ on these is defined using these two augmentations. We can therefore form the bar construction

$$B(\mathbb{R}, E_{\mathrm{fin}}^\bullet(X, \mathcal{O}(P)), \mathcal{O}(S))$$

which we shall denote by $B(E_{\mathrm{fin}}^\bullet(X, \mathcal{O}(P))_{\tilde{x}_o,(x_o)})$. It is a commutative d.g.a. when endowed with the product (3). It is, in fact, a d.g. Hopf algebra, with coproduct defined as follows:

$$\Delta : [w_1| \ldots |w_r]\phi \mapsto \sum_{i,j} \sum_{l>i} \sum_{k_l} [w_1| \ldots |w_i](\psi_{k_{i+1}}^{(i+1)} \ldots \psi_{k_r}^{(r)} \phi_j') \otimes [w_{k_{i+1}}^{(i+1)}| \ldots |w_{k_r}^{(r)}]\phi_j''$$

where

$$\Delta_S(\phi) = \sum_j \phi_j' \otimes \phi_j''$$

is the diagonal of $\mathcal{O}(S)$, and the map

$$E_{\mathrm{fin}}^\bullet(X, \mathcal{O}(P)) \to \mathcal{O}(S) \otimes E_{\mathrm{fin}}^\bullet(X, \mathcal{O}(P)),$$

which is induced by the left action of S on P, takes w_l to

$$\sum_{k_l} \psi_{k_l}^{(l)} \otimes w_{k_l}^{(l)}.$$

The following proposition is a direct consequence of the definition (6.1) and the basic properties of iterated integrals which may be found in [3].

PROPOSITION 8.1. – *The map*

$$B(E_{\text{fin}}^\bullet(X, \mathcal{O}(P))_{\tilde{x}_o,(x_o)}) \to I^\bullet(X, \mathcal{O}(S))_{x_o}$$

defined by

$$[w_1|w_2|\dots|w_r]\phi \mapsto \int (w_1 w_2 \dots w_r|\phi)$$

is a well defined d.g. Hopf algebra homomorphism. □

Remark 8.2. – At this point, it may be helpful to note that if $G = S \ltimes U$ is an affine algebraic group, then $\mathcal{O}(G) \cong \mathcal{O}(U) \otimes \mathcal{O}(S)$ as an algebra, but the coproduct is twisted by the map $\mathcal{O}(U) \to \mathcal{O}(U) \otimes \mathcal{O}(S)$ dual to the action of S on U.

PROPOSITION 8.3. – *If* $\pi_1(X, x_o)$ *is finitely generated, then*

$$H^0(B(E_{\text{fin}}^\bullet(X, \mathcal{O}(P))_{\tilde{x}_o,(x_o)}))$$

is the coordinate ring of a linear proalgebraic group which is an extension of S by a prounipotent group.

In the proof, we shall need the following technical result, the proof of which is a modification of Sullivan's proof of the existence of minimal models (*cf.* [17].) It is needed to show that

$$\operatorname{Spec} H^0(B(E_{\text{fin}}^\bullet(X, \mathcal{O}(P))_{\tilde{x}_o,(x_o)})) \cong \operatorname{Spec} H^0(B(E_{\text{fin}}^\bullet(X, \mathcal{O}(P))_{\tilde{x}_o,x_o})) \times S.$$

PROPOSITION 8.4. – *There is a d.g. subalgebra $A^\bullet$ of $E_{\text{fin}}^\bullet(X, \mathcal{O}(P))$ with $A^0 = \mathbb{R}$, which is also an S-submodule, with the properties that the inclusion is a quasi-isomorphism.*

Proof. – We shall write $\mathcal{O}$ for $\mathcal{O}(P)$. First recall that $E_{\text{fin}}^\bullet(X, \mathcal{O})$ is the direct sum of its isotypical pieces

$$E_{\text{fin}}^\bullet(X, \mathcal{O}) = \bigoplus_\alpha E^\bullet(X, \mathsf{V}_\alpha^*) \otimes V_\alpha$$

and that each of the summands $E^\bullet(X, \mathsf{V}_\alpha^*) \otimes V_\alpha$ is a subcomplex.

The algebra $A^\bullet$ is constructed as an ascending union of d.g. subalgebras $A^\bullet(n)$, each of which is an S-module, and where the inclusion $A^\bullet(n) \hookrightarrow E_{\text{fin}}^\bullet(X, \mathcal{O})$ is S-equivariant, an isomorphism on homology in dimensions $\leq n$, and injective on homology in degree $n + 1$. We begin by defining $A^\bullet(0) = \mathbb{R}$. That this satisfies these conditions follows from (7.1). The most difficult of these subalgebras to construct is $A^\bullet(1)$. It is the ascending union of d.g. subalgebras $A^\bullet(1, m)$, each of which is also an S-submodule of $E_{\text{fin}}^\bullet(X, \mathcal{O})$. We leave the construction of the $A^\bullet(n)$ when $n > 1$ to the reader as the ideas are easily worked out from the construction of Sullivan's minimal model [17] and the construction of $A^\bullet(1)$ given below.

To construct $A^\bullet(1, 1)$, choose a subspace $Z_\alpha(1)$ of the closed elements of $E^1(X, \mathsf{V}_\alpha^*)$ such that inclusion $Z_\alpha(1) \to E^1(X, \mathsf{V}_\alpha^*)$ induces an isomorphism on homology. Define

$A^\bullet(1,1)$ to be the d.g. subalgebra generated by $\oplus_\alpha Z_\alpha(1) \otimes V_\alpha$. Then the inclusion $A^\bullet(1,1) \to E^\bullet_{\mathrm{fin}}(X, \mathcal{O})$ induces in isomorphism on H^k when $k \leq 1$.

Now suppose that we have constructed d.g. subalgebras $A^\bullet(1,k)$ of $E^\bullet_{\mathrm{fin}}(X, \mathcal{O})$ when $1 \leq k < m$, each of which is an S-submodule of with the property that $A^\bullet(1, k-1) \subseteq A^\bullet(1,k)$ and

$$(5) \quad \ker\{H^2(A^\bullet(1, k-1)) \to H^2(A^\bullet(1,k))\} = \ker\{H^2(A^\bullet(1, k-1)) \to H^2(X, \mathcal{O})\}$$

when $1 < k < m$. To construct $A^\bullet(1, m)$, note that

$$\ker\{H^2(A^\bullet(1, m-1)) \to H^2(X, \mathcal{O})\}$$

is an S module. Since $A^\bullet(1, m-1)$ is the direct sum of its isotypical pieces, we can find subspaces W_α of the closed elements of $E^2(X, \mathbb{V}^*_\alpha)$ such that $W_\alpha \otimes V_\alpha \subseteq A^2(1, m-1)$, and such that

$$\bigoplus_\alpha W_\alpha \otimes V_\alpha \hookrightarrow A^2(1, m-1)$$

induces an isomophism

$$\bigoplus_\alpha W_\alpha \otimes V_\alpha \to \ker\{H^2(A^\bullet(1, m-1)) \to H^2(X, \mathcal{O})\}$$

We can therefore find subspaces $Z(m)_\alpha$ of $E^1(X, \mathbb{V}^*_\alpha)$ such that $d : Z(m)_\alpha \to W_\alpha$ is an isomorphism. Now define $A^\bullet(1, m)$ to be the subalgebra of $E^\bullet_{\mathrm{fin}}(X, \mathcal{O})$ generated by $A^\bullet(1, m-1)$ and the $Z_\alpha(m) \otimes V_\alpha$. This completes the inductive construction of the $A^\bullet(1, m)$. Now define $A^\bullet(1)$ to be the union of the $A^\bullet(1, m)$. It follows from equation (5) that the inclusion of $A^\bullet(1)$ into $E^\bullet_{\mathrm{fin}}(X, \mathcal{O})$ induces an isomorphism on H^1 and an injection on H^2. $\square$

Remark 8.5. – A similar argument can be used to show that there is a (Sullivan) minimal model $\phi : \mathcal{M}^\bullet \to E^\bullet_{\mathrm{fin}}(X, \mathcal{O}(P))$ where $\mathcal{M}^\bullet$ is a direct limit of finite dimensional S-modules and where ϕ is S equivariant.

Proof of (8.3). – Choose a d.g. subalgebra $A^\bullet$ of $E^\bullet_{\mathrm{fin}}(X, \mathcal{O}(P))$ as given by (8.4). It follows from (7.2) that the natural map

$$H^0(B(\mathbb{R}, A^\bullet, \mathcal{O}(S))) \to H^0(B(E^\bullet_{\mathrm{fin}}(X, \mathcal{O}(P))_{\tilde{x}_o, (x_o)}))$$

is an isomorphism. Since $A^0 = \mathbb{R}$, we have that

$$H^0(B(\mathbb{R}, A^\bullet, \mathcal{O}(S))) = H^0(B(\mathbb{R}, A^\bullet, \mathbb{R})) \otimes \mathcal{O}(S).$$

It is not difficult to check that $\mathcal{O}(S)$ is a sub Hopf algebra, and that this is a tensor product of algebras, but where the coproduct is twisted by the action of S on $H^0(B(\mathbb{R}, A^\bullet, \mathbb{R}))$.

So, if we can show that $H^0(B(\mathbb{R}, A^\bullet, \mathbb{R}))$ is the limit of the coordinate rings of an inverse system of unipotent groups, each with an S action, then we will have shown that

$$H^0(B(E_{\mathrm{fin}}^\bullet(X, \mathcal{O}(P))_{\tilde{x}_o, (x_o)}))$$

is the coordinate ring of

$$S \ltimes \mathrm{Spec} H^0(B(\mathbb{R}, A^\bullet, \mathbb{R}))$$

and therefore proved the proposition. From [6], we know that there is a canonical splitting (in particular, it is S equivariant) of the projection

$$H^0(B(\mathbb{R}, A^\bullet, \mathbb{R})) \to Q H^0(B(\mathbb{R}, A^\bullet, \mathbb{R})) =: Q$$

onto the indecomposable elements Q. This splitting induces an S-equivariant algebra isomorphism

$$\mathbb{R}[Q] \to H^0(B(\mathbb{R}, A^\bullet, \mathbb{R})).$$

The bar filtration induces a filtration

$$Q_1 \subseteq Q_2 \subseteq Q_3 \subseteq \cdots \subseteq Q$$

of the indecomposables such that $Q = \cup Q_r$, where $Q^1 = H^1(X, \mathcal{O}(P))$ and where Q_r/Q_{r-1} is a submdodule of $Q_1^{\otimes r}$. Each Q_r is a Lie coalgebra, and the cobracket Δ^c satisfies

$$\Delta^c : Q_r \to \sum_{i+j=r} Q_i \otimes Q_j$$

and is injective when $r > 1$. Since $\pi_1(X, x_o)$ is finitely generated, each of the cohomology groups $H^1(X, \mathbb{V})$ is finite dimensional for each rational local system $\mathbb{V}$ over X. It follows from (4.1) that each isotypical component of $H^1(X, \mathcal{O}(P))$ is finite dimensional. Since Q_r/Q_{r-1} is a submodule of

$$Q_1^{\otimes r} \cong H^1(X, \mathcal{O}(P))^{\otimes r},$$

it follows that each S-isotypical component of each Q_r is finite dimensional. One can now prove by induction on r using the nilpotence, that as an S-module, each Q_r is the direct limit of duals of nilpotent Lie algebras, each of which has an S action. This completes the proof. □

DEFINITION 8.6. – Define proalgebraic groups $\mathcal{G}^{\mathrm{DR}}$ and $\mathcal{U}^{\mathrm{DR}}$ by

$$\mathcal{G}^{\mathrm{DR}} = \mathrm{Spec} H^0(B(E_{\mathrm{fin}}^\bullet(X, \mathcal{O}(P))_{\tilde{x}_o, (x_o)}))$$

and

$$\mathcal{U}^{\mathrm{DR}} = \mathrm{Spec} H^0(B(\mathbb{R}, E_{\mathrm{fin}}^\bullet(X, \mathcal{O}(P)), \mathbb{R})).$$

Evidently, we have an extension

$$1 \to \mathcal{U}^{\mathrm{DR}} \to \mathcal{G}^{\mathrm{DR}} \to S \to 1$$

of proalgebraic groups, where $\mathcal{U}^{\mathrm{DR}}$ is prounipotent.

When we want to emphasize the dependence of $\mathcal{G}^{\mathrm{DR}}$ and $\mathcal{U}^{\mathrm{DR}}$ on (X, x), we will write them as $\mathcal{G}^{\mathrm{DR}}(X, x)$ and $\mathcal{U}^{\mathrm{DR}}(X, x)$, respectively.

PROPOSITION 8.7. – *There is a natural homomorphism* $\tilde{\rho} : \pi_1(X, x_o) \to \mathcal{G}^{\mathrm{DR}}$ *whose composition with* $\mathcal{G}^{\mathrm{DR}} \to S$ *is* ρ.

Proof. – Define a map from $P_{x_o, x_o} X$ to the linear functionals on

$$B(E_{\mathrm{fin}}^{\bullet}(X, \mathcal{O}(P))_{\tilde{x}_o, (x_o)})$$

by

$$\gamma : [w_1 | \dots | w_r] \phi \mapsto \int_\gamma (w_1 \dots w_r | \phi).$$

This induces a function

$$\Phi : \pi_1(X, x_o) \to \mathrm{Hom}_{\mathbb{R}}(\mathcal{O}(\mathcal{G}^{\mathrm{DR}}), \mathbb{R}).$$

Define $\tilde{\rho}$ by taking the class of γ in $\pi_1(X, x_o)$ to the maximal ideal of

$$H^0(B(E_{\mathrm{fin}}^{\bullet}(X, \mathcal{O}(P))_{\tilde{x}_o, (x_o)}))$$

consisting of those elements on which γ vanishes. (Note that γ acts via integration.) That this is a group homomorphism follows from (5.9). $\qquad\square$

Remark 8.8. – By standard rational homotopy theory,

$$H^1(\mathcal{U}^{\mathrm{DR}}) \cong H^1(X, \mathcal{O}(P)) \qquad \text{and} \qquad H^2(\mathcal{U}^{\mathrm{DR}}) \subseteq H^2(X, \mathcal{O}(P)).$$

It follows that if $H^2(X, \mathbb{V}_\alpha)$ vanishes for all α, then $\mathcal{U}^{\mathrm{DR}}$ is freely generated by

$$\prod_\alpha H_1(X, \mathbb{V}_\alpha) \otimes V_\alpha^*$$

as a prounipotent group. This holds, for example, when Γ is a noncompact curve, such as a modular curve. (Similar statements hold when X is replaced by Γ, so that $\mathcal{U}$ is always free when Γ is.)

9. Construction of homomorphisms from $\mathcal{G}^{\mathrm{DR}}$

Suppose that G is a linear algebraic group which can be expressed as an extension

$$1 \to U \to G \to S \to 1$$

where U is unipotent. Choose an isomorphism of G with $S \ltimes U$. Denote the Lie algebra of U by $\mathfrak{u}$.

PROPOSITION 9.1. – *Each one form* $\omega \in E_{\mathrm{fin}}^1(X, \mathcal{O}(P)) \otimes \mathfrak{u}$ *that satisfies*

1. $d\omega + \omega \wedge \omega = 0$;
2. *for all* $s \in S$, $s^*\omega = \mathrm{Ad}(s)\omega$;

determines a homomorphism $\mathcal{G}^{\mathrm{DR}} \to G$ *that commutes with projection to* S.

Proof. – First note that since the exponential map $\mathfrak{u} \to U$ is a polynomial isomorphism, $\mathcal{O}(U)$ is isomorphic to the polynomials $\mathbb{R}[\mathfrak{u}]$ on the vector space $\mathfrak{u}$. Further, there is a natural isomorphism

$$
(6) \qquad \mathcal{O}(U) \cong \mathbb{R}[\mathfrak{u}] \to \varinjlim \operatorname{Hom}(U\mathfrak{u}/I^n, \mathbb{R})
$$

which is defined by noting that $U\mathfrak{u}$ is, by the PBW Theorem, the symmetric coalgebra $S^c\mathfrak{u}$ on $\mathfrak{u}$. The isomorphism (6) is an isomorphism of Hopf algebras.

Set

$$
T = 1 + [\omega] + [\omega|\omega] + [\omega|\omega|\omega] + \cdots
$$

which we view as an element of

$$
B(E_{\mathrm{fin}}^\bullet(X, \mathcal{O}(P))_{\tilde{x}_o,(x_o)}) \mathbin{\widehat{\otimes}} \widehat{U}\mathfrak{u}
$$

of degree zero. Here $\widehat{U}\mathfrak{u}$ denotes the completion

$$
\varprojlim U\mathfrak{u}/I^n
$$

of $U\mathfrak{u}$ with respect to the powers of its augmentation ideal, and $\widehat{\otimes}$ denotes the completed tensor product

$$
\varprojlim B(E_{\mathrm{fin}}^\bullet(X, \mathcal{O}(P))_{\tilde{x}_o,(x_o)}) \otimes \widehat{U}\mathfrak{u}/I^n.
$$

The coordinate ring of G is isomorphic to $\mathcal{O}(U) \otimes \mathcal{O}(S)$. Define a linear map

$$
\Theta : \mathcal{O}(G) \to B(E_{\mathrm{fin}}^\bullet(X, \mathcal{O}(P))_{\tilde{x}_o,(x_o)})^0
$$

by

$$
f \otimes \phi \mapsto \langle T, f \rangle \cdot \phi.
$$

It is not difficult to check that Θ is a well defined Hopf algebra homomorphism. This uses the fact that $s^*\omega = Ad(s)\omega$. That ω satisfies the integrability condition

$$
d\omega + \omega \wedge \omega = 0
$$

implies that $\operatorname{im}\Theta$ is contained in $H^0(B(E_{\mathrm{fin}}^\bullet(X, \mathcal{O}(P))_{\tilde{x}_o,(x_o)}))$. It follows that Θ induces a Hopf algebra homomorphism

$$
\mathcal{O}(G) \to H^0(B(E_{\mathrm{fin}}^\bullet(X, \mathcal{O}(P))_{\tilde{x}_o,(x_o)}))
$$

and therefore a group homomorphism

$$
\theta : \mathcal{G}^{\mathrm{DR}} \to G
$$

which commutes with the projections to S.

Finally, it follows from (5.4), (5.5) and (8.7) that the composite

$$\pi_1(X, x_o) \to \mathcal{G}^{\mathrm{DR}} \to G$$

is the homomorphism induced by ω.   □

COROLLARY 9.2. – *If* $\mathbb{V}$ *is a local system in* $\mathcal{B}(X, S)$, *then the monodromy representation*

$$\tau : \pi_1(X, x_o) \to \mathrm{Aut} V_o$$

factors through $\tilde{\rho} : \pi_1(X, x_o) \to \mathcal{G}^{\mathrm{DR}}$.   □

COROLLARY 9.3. – *If* $\tau : \pi_1(X, x_o) \to G$ *is a homomorphism into a linear algebraic group which is an extension of* S *by a unipotent group, and whose composite with the projection to* S *is* ρ, *then there is a homomorphism* $\mathcal{G}^{\mathrm{DR}} \to G$ *whose composite with*

$$\tilde{\rho} : \pi_1(X, x_o) \to \mathcal{G}^{\mathrm{DR}}$$

is τ.

Proof. – Denote the kernel of $G \to S$ by U. One can construct a faithful, finite dimensional representation V of G which has a filtration

$$V = V^0 \supset V^1 \supset V^2 \supset \cdots$$

by G-submodules whose intersection is zero and where each V^j/V^{j+1} is a trivial U-module. The corresponding local system over X lies in $\mathcal{B}(X, S)$. The result now follows from (9.2).   □

10. Isomorphism with the relative completion

Denote $\pi_1(X, x_o)$ by π. In Section 8 we constructed a homomorphism $\pi \to \mathcal{G}^{\mathrm{DR}}$. In this section, we prove:

THEOREM 10.1. – *If* π *is finitely generated, then the homomorphism* $\pi \to \mathcal{G}^{\mathrm{DR}}$ *is the completion of* π *relative to* ρ.

To prove the theorem, we first fix a completion $\pi \to \mathcal{G}$ of π relative to ρ. The universal mapping property of the relative completion gives a homomorphism $\mathcal{G} \to \mathcal{G}^{\mathrm{DR}}$ of proalgebraic groups that commutes with the canonical projections to S. It follows from (9.3), that there is a natural homomorphism

(7) $\mathcal{G}^{\mathrm{DR}} \to \mathcal{G}$

that also commutes with the projections to S. It follows from the universal mapping property of the relative completion that the composite

$$\mathcal{G} \to \mathcal{G}^{\mathrm{DR}} \to \mathcal{G}$$

is the identity.

Denote the prounipotent radical of $\mathcal{G}$ by $\mathcal{U}$. Since π is finitely generated, each of the groups $H^1(\pi, V)$ is finite dimensional for each rational representation V of S.

In view of the following proposition and the assumption that π is finitely generated, all we need do to show that the natural homomorphism $\mathcal{G} \to \mathcal{G}^{\mathrm{DR}}$ is an isomorphism is to show that either of the induced maps

$$\mathrm{Hom}_S(H_1(\mathcal{U}), V) \to \mathrm{Hom}_S(H_1(\mathcal{U}^{\mathrm{DR}}), V) \to \mathrm{Hom}_S(H_1(\mathcal{U}), V)$$

is an isomorphism for all S-modules V.

PROPOSITION 10.2. – *Suppose that G_1 and G_2 are extensions of the reductive group S by unipotent groups U_1, U_2, respectively:*

$$1 \to U_j \to G_j \to S \to 1.$$

Suppose that $\theta : G_1 \to G_2$ is a split surjective homomorphism of algebraic groups that commutes with the projections to S. If either of the induced maps

$$\mathrm{Hom}_S(H_1(U_1), V) \to \mathrm{Hom}_S(H_1(U_2), V) \to \mathrm{Hom}_S(H_1(U_1), V)$$

is an isomorphism for all S modules V, then both are, and θ is an isomorphism.

Proof. – The proof reduces to the basic fact that a split surjective homomorphism between nilpotent Lie algebras is an isomorphism if and only if it induces an isomorphism on H_1. The details are left to the reader. $\square$

Our first task in the proof of Theorem 10.1 is to compute $\mathrm{Hom}_S(H_1(\mathcal{U}), V)$.

PROPOSITION 10.3. – *For all S-modules V, there is a canonical isomorphism*

$$H^1(\pi, V) \cong \mathrm{Hom}_S(H_1(\mathcal{U}), V).$$

Proof. – We introduce an auxiliary group for the proof. Let

$$\mathrm{Hom}_\rho(\pi, S \ltimes V)$$

be the set of group homomorphisms $\pi \to S \ltimes V$ whose composite with the projection $S \ltimes V \to S$ is ρ. Then there is a natural bijection between $\mathrm{Hom}_\rho(\pi, S \ltimes V)$ and the set of splittings $\pi \to \pi \ltimes V$ of the projection $\pi \ltimes V \to \pi$: the splitting σ corresponds to $\tilde{\rho} : \pi \to S \ltimes V$ if and only if the diagram

$$
\begin{array}{ccc}
\pi & \xrightarrow{\ \sigma\ } & \pi \ltimes V \\
\| & & \downarrow{\scriptstyle \rho \ltimes id} \\
\pi & \xrightarrow{\ \tilde{\rho}\ } & S \ltimes V
\end{array}
$$

commutes.

The kernel V acts on both $\mathrm{Hom}_\rho(\pi, S \ltimes V)$ and the set of splittings, in both cases by inner automorphisms. The action commutes with the bijection. Since $H^1(\pi, V)$ is naturally

isomorphic to the set of splittings of $\pi \ltimes V \to \pi$ modulo conjugation by V [14, p. 106], the bijection induces a natural isomorphism

$$H^1(\pi, V) \cong \mathrm{Hom}_\rho(\pi, S \ltimes V)/\sim .$$

On the other hand, by the universal mapping property of the relative completion, each element of $\mathrm{Hom}_\rho(\pi, S \ltimes V)$ induces a homomorphism $\mathcal{G} \to S \ltimes V$ which commutes with the projections to S. Such a homomorphism induces a homomorphism $\mathcal{U} \to V$, and therefore an S-equivariant homomorphism $H_1(\mathcal{U}) \to V$. Since V is central, this induces a homomorphism

$$\mathrm{Hom}_\rho(\pi, S \ltimes V) \to \mathrm{Hom}_S(H_1(\mathcal{U}), V).$$

To complete the proof, we show that this is an isomorphism. Denote the commutator subgroup of $\mathcal{U}$ by $\mathcal{U}'$. Then the quotient $\mathcal{G}/\mathcal{U}'$ is an extension of S by $H_1(\mathcal{U})$; the latter being a possibly infinite product of representations of S in which each isotypical factor is finite dimensional. Using the fact that every extension of S by a rational representation in the category of algebraic groups splits and that any two such splittings are conjugate by an element of the kernel, we see that the extension

$$(8) \qquad\qquad 0 \to H_1(\mathcal{U}) \to \mathcal{G}/\mathcal{U}' \to S \to 1$$

is split and that any two splittings are conjugate by an element of $H_1(\mathcal{U})$. Choose a splitting of this sequence. This gives an isomorphism

$$\mathcal{G}/\mathcal{U}' \cong S \ltimes H_1(\mathcal{U}).$$

An S-equivariant homomorphism $H_1(\mathcal{U}) \to V$ induces a homomorphism

$$\mathcal{G}/\mathcal{U}' \cong S \ltimes H_1(\mathcal{U}) \to S \ltimes V$$

of proalgebraic groups. Composing this with the homomorphism

$$\pi \to \mathcal{G} \to \mathcal{G}/\mathcal{U}',$$

we obtain an element of $\mathrm{Hom}_\rho(\pi, S \ltimes V)/\sim$. Since all splittings of (8) differ by an inner automorphism by an element of $H_1(\mathcal{U})$, we have constructed a well defined map

$$\mathrm{Hom}_S(H_1(\mathcal{U}), V) \to \mathrm{Hom}_\rho(\pi, S \ltimes V).$$

This is easily seen to be the inverse of the map constructed above. This completes the proof. $\qquad\qquad\square$

The following result completes the proof of Theorem 10.1.

PROPOSITION 10.4. – *The map*

$$H_1(\mathcal{U}) \to H_1(\mathcal{U}^{\mathrm{DR}})$$

induced by (7) *is an isomorphism. Consequently, by* (8.8),

$$H_1(\mathcal{U}) \cong \prod_\alpha H_1(\Gamma, \mathbf{V}_\alpha) \otimes V_\alpha^*.$$

Proof. – It suffices to show that for all rational representations V of S, the map

$$\left[H^1(\mathcal{U}^{\mathrm{DR}}) \otimes V\right]^S \to \left[H^1(\mathcal{U}) \otimes V\right]^S$$

is an isomorphism. Both groups are isomorphic to $H^1(X, \mathbf{V})$. We just have to show that it corresponds to the identity.

Choose a de Rham representative $w \in E^1(X, \mathbf{V})$ of a class in $H^1(X, \mathbf{V})$. Let $\delta \in V^* \otimes V$ be the element corresponding to the identity $V \to V$. Set

$$\omega := w \otimes \delta \in E^1(X, \mathbf{V}) \otimes V^* \otimes V.$$

Regard V as an abelian Lie algebra. Then

$$\omega \in E_{\mathrm{fin}}^1(X, \mathcal{O}(P)) \otimes V.$$

It is closed, and therefore satisfies the integrability condition $d\omega + \omega \wedge \omega = 0$. Since the identity $V \to V$ is S equivariant,

$$s^* \omega = Ad(s)\omega$$

for all $s \in S$.

Set $W = V \oplus \mathbb{R}$. Filter this by

$$W = W^0 \supset W^1 \supset W^2 = 0$$

where $W^1 = V$. Then $V \subset \mathrm{End}W$. It follows from (5.4) that ω defines a connection on $P \times V$ which is flat along the leaves of the foliation $\mathcal{F}$ and descends to a flat bundle over X. The monodromy representation of this bundle is a homomorphism

$$\tau : \pi_1(X, x_o) \to S \ltimes V \subset \mathrm{Aut}W.$$

It follows from the monodromy formula (5.5) that τ takes the class of the loop γ to

$$\left(\rho(\gamma), \int_{\tilde{\gamma}} w\right) \in S \ltimes V.$$

The result follows. $\qquad\square$

11. Naturality

Suppose that π_X and π_Y are two groups, and that $\rho_X : \pi_X \to S_X$ and $\rho_Y : \pi_Y \to S_Y$ are homomorphisms into the F-points of reductive algebraic groups, each with Zariski dense image. We have the two corresponding relative completions

$$\tilde{\rho}_X : \pi_X \to \mathcal{G}_X \qquad \text{and} \qquad \tilde{\rho}_Y : \pi_Y \to \mathcal{G}_Y.$$

Fix an algebraic group homomorphism $\Psi : S_X \to S_Y$.

PROPOSITION 11.1. – *If $\psi : \pi_X \to \pi_Y$ is a homomorphism such that the diagram*

$$
\begin{array}{ccc}
\pi_X & \xrightarrow{\;\rho_X\;} & S_X \\
\psi \downarrow & & \downarrow \Psi \\
\pi_Y & \xrightarrow{\;\rho_Y\;} & S_Y
\end{array}
$$

commutes, then there is a canonical homomorphism $\widehat{\Psi} : \mathcal{G}_X \to \mathcal{G}_Y$ such that the diagram

$$
\begin{array}{ccc}
\pi_X & \xrightarrow{\;\tilde{\rho}_X\;} & \mathcal{G}_X \\
\psi \downarrow & & \downarrow \widehat{\Psi} \\
\pi_Y & \xrightarrow{\;\tilde{\rho}_Y\;} & \mathcal{G}_Y
\end{array}
$$

commutes.

Proof. – Let $\Psi^* \mathcal{G}_Y$ be the pullback of G_Y along Ψ:

$$
\begin{array}{ccc}
\Psi^* \mathcal{G}_Y & \longrightarrow & S_X \\
\downarrow & & \downarrow \Psi \\
\mathcal{G}_Y & \longrightarrow & S_Y.
\end{array}
$$

This group is an extension of S_X by the prounipotent radical of $\mathcal{G}_Y$. The homomorphisms $\pi_X \to S_X$ and $\pi_X \to \pi_Y \to \mathcal{G}_Y$ induce a homomorphism $\pi_X \to \Psi^* \mathcal{G}_Y$. By the universal mapping property of $\tilde{\rho}_X : \pi_X \to \mathcal{G}_X$, there is a homomorphism $\mathcal{G}_X \to \Psi^* \mathcal{G}_Y$ which extends the homomorphism $\pi_X \to \Psi^* \mathcal{G}_Y$. The sought after homomorphism $\widehat{\Psi}$ is the composite $\mathcal{G}_X \to \Psi^* \mathcal{G}_Y \to \mathcal{G}_Y$. $\qquad\qquad\square$

Next we explain how to realize $\widehat{\Psi}$ using the bar construction. Suppose that (X, x) and (Y, y) are two pointed manifolds. Denote $\pi_1(X, x)$ and $\pi_1(Y, y)$ by π_X and π_Y, respectively. Suppose that $f : (X, x) \to (Y, y)$ is a smooth map which induces the homomorphism $\psi : \pi_X \to \pi_Y$ on fundamental groups. Denote the principal bundles associated to ρ_X and ρ_Y by $P_X \to X$ and $P_Y \to Y$. We have the d.g.a.s

$$E_{\mathrm{fin}}^\bullet(X, \mathcal{O}(P_X)) \qquad \text{and} \qquad E_{\mathrm{fin}}^\bullet(Y, \mathcal{O}(P_Y)).$$

Since the diagram in Proposition 11.1 commutes, f and ψ induce a d.g.a. homomorphism

$$(f, \phi)^* : E_{\mathrm{fin}}^\bullet(Y, \mathcal{O}(P_Y)) \to E_{\mathrm{fin}}^\bullet(X, \mathcal{O}(P_X)).$$

This homomorphism respects the augmentations induced by $x \in X$ and $y \in Y$, and therefore induces a d.g. Hopf algebra homomorphism

$$B(E_{\text{fin}}^{\bullet}(Y, \mathcal{O}(P_Y))_{\tilde{y},(y)}) \to B(E_{\text{fin}}^{\bullet}(X, \mathcal{O}(P_X))_{\tilde{x},(x)}).$$

This induces a homomorphism

$$(9) \qquad \qquad \mathcal{G}^{\text{DR}}(X, x) \to \mathcal{G}^{\text{DR}}(Y, y)$$

after taking H^0 and then Spec.

PROPOSITION 11.2. – *Under the canonical identifications of $\mathcal{G}^{\text{DR}}(X, x)$ with $\mathcal{G}(X, x)$ and $\mathcal{G}^{\text{DR}}(Y, y)$ with $\mathcal{G}(Y, y)$, the homomorphism $\widehat{\Psi} : \mathcal{G}(X, x) \to \mathcal{G}(Y, y)$ corresponds to the homomorphism* (9).

Proof. – If γ is a loop in X based at x, $w_1, \ldots, w_r \in E_{\text{fin}}^{\bullet}(Y, \mathcal{O}(P_Y))$, and $\phi \in \mathcal{O}(S_Y)$, then

$$\int_{f \circ \gamma} (w_1 \ldots w_r | \phi) = \int_{\gamma} ((f, \phi)^* w_1 \ldots (f, \phi)^* w_r | \Psi^* \phi).$$

It follows that (9) is the homomorphism $\widehat{\Psi}$ induced by f and ψ. □

12. Relative completion of the fundamental groupoid

In this section we explain how the fundamental groupoid of X can be completed with respect to $\rho : \pi_1(X, x) \to S$ and we give a de Rham construction of it. In the unipotent case, the de Rham theorem is implicit in Chen's work [3], and is described explicitly in [12].

Recall that the fundamental groupoid $\pi(X)$ of a topological space X is the category whose objects are the points of X and whose morphisms from $a \in X$ to $b \in X$ are homotopy classes $\pi(X; a, b)$ of paths $[0, 1] \to X$ from a to b. We can think of $\pi(X)$ as a torsor over $X \times X$; the fiber over (a, b) being $\pi(X; a, b)$. Observe that there is a canonical isomorphism between the fiber over (a, a) and $\pi_1(X, a)$. The torsor is the one over $X \times X$ corresponding to the representation

$$(10) \qquad \pi_1(X \times X, (a, a)) \cong \pi_1(X, a) \times \pi_1(X, a) \to \text{Aut} \pi_1(X, a)$$

where

$$(\gamma, \mu) \mapsto \{g \mapsto \gamma^{-1} g \mu\}.$$

As in previous sections, X will be a connected smooth manifold and x_o a distinguished base point. Suppose, as before, that $\rho : \pi_1(X, x_o) \to S$ is a Zariski dense homomorphism to a reductive real algebraic group. Denote the completion of $\pi_1(X, x_o)$ relative to ρ by $\pi_1(X, x_o) \to \mathcal{G}$. The representation (10) extends to a representation

$$\pi_1(X \times X, (x_o, x_o)) \cong \pi_1(X, x_o) \times \pi_1(X, x_o) \to \text{Aut} \mathcal{G}$$

Denote the corresponding torsor over $X \times X$ by $\boldsymbol{G}$. This is easily seen to be a torsor of real proalgebraic varieties. Denote the fiber of $\boldsymbol{G}$ over (a,b) by $\mathcal{G}_{a,b}$. There is a canonical map

$$\pi(X; a, b) \to \mathcal{G}_{a,b}$$

which induces a map of torsors. It follows from standard arguments that, for all a, b and c in X, there is a morphism of proalgebraic varieties

(11) $$\mathcal{G}_{a,b} \times \mathcal{G}_{b,c} \to \mathcal{G}_{a,c}$$

which is compatible with the multiplication map

$$\pi(X; a, b) \times \pi(X; b, c) \to \pi(X; a, c).$$

An efficient way to summarize the properties of $\boldsymbol{G}$ and the multiplication maps is to say that they form a category (in fact, a groupoid) whose objects are the elements of X and where $\mathrm{Hom}(a,b)$ is $\mathcal{G}_{a,b}$ with composition defined by (11). In addition, the natural map $\pi(X; a, b) \rightsquigarrow \mathcal{G}_{a,b}$ from the fundamental groupoid of X to this category is a functor. We shall call this functor the *relative completion of the fundamental groupoid of X with respect to ρ* ([3]). Our goal is to give a description of it in terms of differential forms.

We also have the torsor $\boldsymbol{P}$ over $X \times X$ associated to the representation

$$\pi_1(X \times X, x_o) \cong \pi_1(X, x_o) \times \pi_1(X, x_o) \to \mathrm{Aut}\, S$$

where

$$(\gamma, \mu) \to \left\{ g \mapsto \rho(\gamma)^{-1} g \rho(\mu) \right\}.$$

given by ρ. Denote the fiber of $\boldsymbol{P}$ over (a,b) by $\mathcal{P}_{a,b}$. As above, we have a category whose objects are the points of X and where $\mathrm{Hom}(a,b)$ is $\mathcal{P}_{a,b}$. There is also a functor from the fundamental groupoid of X to this category which is the identity on objects. Denote the restriction of this torsor to $\{a\} \times X$ by $\boldsymbol{P}_{a,\_}$. We shall view this as a torsor over (X, a). The image id_a of the identity in $\pi_1(X, a)$ in $\mathcal{P}_{a,a}$ gives a canonical lift of the base point a of X to $\boldsymbol{P}_{a,\_}$. Observe that $\boldsymbol{P}_{x_o,\_}$ is the principal S bundle P used in the construction of $\mathcal{G}$.

For each $a \in X$, we can from the corresponding local system $\mathcal{O}(\mathcal{P}_{a,\_})$ whose fiber over $b \in X$ is the coordinate ring of $\mathcal{P}_{a,b}$. We can form the complex

$$E_{\mathrm{fin}}^\bullet(X, \mathcal{O}(\mathcal{P}_{a,\_})) = \varinjlim E^\bullet(X, \mathbb{M})$$

where $\mathbb{M}$ ranges over all finite dimensional sub-local systems of $\mathcal{O}(\mathcal{P}_{a,\_})$. This has augmentations

$$\epsilon_a : E_{\mathrm{fin}}^\bullet(X, \mathcal{O}(\mathcal{P}_{a,\_})) \to \mathbb{R}$$

and

$$\delta_{a,b} : E_{\mathrm{fin}}^\bullet(X, \mathcal{O}(\mathcal{P}_{a,\_})) \to \mathcal{O}(\mathcal{P}_{a,b})$$

([3]) We shall see that the torsor $\mathcal{G}$ is independent of the choice of base point x_o, so it may have been better to call $\mathcal{G}$ the completion of the fundamental groupoid relative to the principal bundle P.

given by evaluation at id_a and on the fiber over b, respectively. We view $\mathbb{R}$ as a right $E^\bullet_{\mathrm{fin}}(X, \mathcal{O}(\mathcal{P}_{a,\_}))$ module via ϵ_a and $\mathcal{O}(\mathcal{P}_{a,b})$ as a left $E^\bullet_{\mathrm{fin}}(X, \mathcal{O}(\mathcal{P}_{a,\_}))$ module via $\delta_{a,b}$. We can therefore form the two sided bar construction

$$B(E^\bullet_{\mathrm{fin}}(X, \mathcal{O}(\mathcal{P}_{a,\_}))_{\mathrm{id}_a,(b)}) := B(\mathbb{R}, E^\bullet_{\mathrm{fin}}(X, \mathcal{O}(\mathcal{P}_{a,\_})), \mathcal{O}(\mathcal{P}_{a,b}))$$

Define

$$(12) \qquad \Delta : B(E^\bullet_{\mathrm{fin}}(X, \mathcal{O}(\mathcal{P}_{a,\_}))_{\mathrm{id}_a,(c)})$$
$$\to B(E^\bullet_{\mathrm{fin}}(X, \mathcal{O}(\mathcal{P}_{a,\_}))_{\mathrm{id}_a,(b)}) \otimes B(E^\bullet_{\mathrm{fin}}(X, \mathcal{O}(\mathcal{P}_{b,\_}))_{\mathrm{id}_b,(c)})$$

by

$$\Delta : [w_1|\ldots|w_r]\phi \mapsto \sum_{i,j} \sum_{l>i} \sum_{k_l} [w_1|\ldots|w_i](\psi^{(i+1)}_{k_{i+1}} \ldots \psi^{(r)}_{k_r} \phi'_j) \otimes [w^{(i+1)}_{k_{i+1}}|\ldots|w^{(r)}_{k_r}]\phi''_j$$

where

$$\Delta_P(\phi) = \sum_j \phi'_j \otimes \phi''_j$$

is the map $\mathcal{O}(\mathcal{P}_{a,c}) \to \mathcal{O}(\mathcal{P}_{a,b}) \otimes \mathcal{O}(\mathcal{P}_{b,c})$ dual to the multiplication map $\mathcal{P}_{a,b} \times \mathcal{P}_{b,c} \to \mathcal{P}_{a,c}$; and the map

$$E^\bullet_{\mathrm{fin}}(X, \mathcal{O}(\mathcal{P}_{a,\_})) \to \mathcal{O}(\mathcal{P}_{a,b}) \otimes E^1_{\mathrm{fin}}(X, \mathcal{O}(\mathcal{P}_{b,\_}))$$

induced by multiplication $\mathcal{P}_{a,b} \times \boldsymbol{\mathcal{P}}_{b,\_} \to \boldsymbol{\mathcal{P}}_{a,\_}$ takes w_l to

$$\sum_{k_l} \psi^{(l)}_{k_l} \otimes w^{(l)}_{k_l}.$$

Definition 5.7 generalizes:

DEFINITION 12.1. – For γ a path in X from a to b, $\phi \in \mathcal{O}(\mathcal{P}_{a,b})$ and $w_1, \ldots, w_r$ elements of $E^1_{\mathrm{fin}}(X, \mathcal{O}(\mathcal{P}_{a,\_}))$, we define

$$\int_\gamma (w_1 \ldots w_r | \phi) = \phi(\tilde{\gamma}(1)) \int_{\tilde{\gamma}} w_1 \ldots w_r$$

where $\tilde{\gamma}$ is the unique lift of γ to a horizontal section of $\boldsymbol{\mathcal{P}}_{a,\_}$ which begins at $\mathrm{id}_a \in \mathcal{P}_{a,a}$.

There is an analogous extension of the definition of higher iterated integrals (6.1) to this situation. As in that case, one has a d.g. algebra homomorphism

$$B(E^\bullet_{\mathrm{fin}}(X, \mathcal{O}(\mathcal{P}_{a,\_}))_{\mathrm{id}_a,(b)}) \to E^\bullet(P_{a,b}X)$$

to the de Rham complex of $P_{a,b}X$, the space of paths in X from a to b. It is defined by

$$[w_1|\ldots|w_r]\phi \mapsto \int (w_1 \ldots w_r|\phi).$$

By taking a homotopy class $\gamma \in \pi(X; a, b)$ to the ideal of functions that vanish on it, we obtain a function

$$\pi(X; a, b) \to \mathrm{Spec} H^0(B(E^\bullet_{\mathrm{fin}}(X, \mathcal{O}(\mathcal{P}_{a,\_}))_{\mathrm{id}_a,(b)})).$$

THEOREM 12.2. – *This function gives a natural algebra isomorphism*

$$\mathcal{O}(\mathcal{G}_{a,b}) \cong H^0(B(E^\bullet_{\mathrm{fin}}(X, \mathcal{O}(\mathcal{P}_{a,\_}))_{a,(b)})).$$

Moreover, the map

$$\mathcal{O}(\mathcal{G}_{a,c}) \to \mathcal{O}(\mathcal{G}_{a,b}) \otimes \mathcal{O}(\mathcal{G}_{b,c})$$

induced by (11) *corresponds to* (12) *under this isomorphism.*

Sketch of Proof. – Define $\mathcal{G}^{\mathrm{DR}}_{a,b}$ by

$$\mathcal{G}^{\mathrm{DR}}_{a,b} = \mathrm{Spec} H^0(B(E^\bullet_{\mathrm{fin}}(X, \mathcal{O}(\mathcal{P}_{a,\_}))_{a,(b)})).$$

The coproduct above induces morphisms

$$\mathcal{G}^{\mathrm{DR}}_{a,b} \times \mathcal{G}^{\mathrm{DR}}_{b,c} \to \mathcal{G}^{\mathrm{DR}}_{a,c}.$$

We therefore have a groupoid whose objects are the points of X and where $\mathrm{Hom}(a, b)$ is $\mathcal{G}^{\mathrm{DR}}_{a,b}$ and a function

$$\pi(X; a, b) \to \mathcal{G}^{\mathrm{DR}}_{a,b}.$$

This map is easily seen to be compatible with path multiplication (use the generalization of the last property of (5.9)), and is therefore a functor of groupoids. Since X is connected, it suffices to prove that $\mathcal{G}^{\mathrm{DR}}_{a,b}$ is isomorphic to $\mathcal{G}_{a,b}$ for just one pair of points a, b of X. But these are isomorphic in the case $a = b = x_o$ by Theorem 10.1. $\square$

13. Hodge theory

Now suppose that X is a smooth complex algebraic variety (or the complement of a normal crossings divisor in a compact Kähler manifold) and that $\mathbb{V}$ is a variation of Hodge structure over X. Denote the semisimple group associated to the fiber V_o over the base point $x_o \in X$ by S. This is the "orthogonal" group

$$S = \mathrm{Aut}(V_o, \langle \ , \ \rangle)$$

associated to the polarization $\langle \ , \ \rangle$. It is semi-simple. Suppose that the image of the monodromy representation

$$\rho : \pi_1(X, x_o) \to S$$

is Zariski dense. Denote the completion of $\pi_1(X, x_o)$ relative to ρ by

$$\tilde{\rho} : \pi_1(X, x_o) \to \mathcal{G}(X, x_o).$$

THEOREM 13.1. – *Under these assumptions, the coordinate ring $\mathcal{O}(\mathcal{G}(X, x_o))$ of the completion of $\pi_1(X, x_o)$ with respect to ρ has a canonical real mixed Hodge structure with weights ≥ 0 for which the product, coproduct, antipode and the natural inclusion*

$$\mathcal{O}(S) \hookrightarrow \mathcal{O}(\mathcal{G}(X, x_o))$$

are all morphisms of mixed Hodge structure. Moreover the canonical homomorphism $\mathcal{G}(X, x_o) \to S$ induces an isomorphism $\mathrm{Gr}_0^W \mathcal{O}(\mathcal{G}(X, x_o)) \cong \mathcal{O}(S)$.

Denote the Lie algebra of S by $\mathfrak{s}$. This has a canonical Hodge structure of weight 0. The following result is an important corollary of the proof of Theorem 13.1. It follows immediately from the theorem and the standard description of the Lie algebra of an affine algebraic group given at the end of Section 3.

COROLLARY 13.2. – *Under the assumptions of the theorem, the Lie algebra $\mathfrak{g}(X, x_o)$ of $\mathcal{G}(X, x_o)$ has a canonical MHS with weights ≤ 0, and the homomorphism $\mathfrak{g}(X, x_o) \to \mathfrak{s}$ is a morphism of MHS which induces an isomorphism*

$$\mathrm{Gr}_0^W \mathfrak{g} \cong \mathfrak{s}.$$

In particular, there is a canonical MHS with weights < 0 on $\mathfrak{u}(X, x_o)$, the Lie algebra of the prounipotent radical of $\mathcal{G}(X, x_o)$. □

The principal assertion of Theorem 13.1 is a special case of the following result when $a = b = c = x_o$.

THEOREM 13.3. – *With X, $\mathbb{V}$ and S as above, if $a, b \in X$, then the coordinate ring $\mathcal{O}(\mathcal{G}_{a,b})$ of the completion of $\pi(X; a, b)$ relative to ρ has a canonical mixed Hodge structure with weight ≥ 0 and whose multiplication is a morphism of MHS. If a, b and c are three points of X, then the map*

$$\mathcal{O}(\mathcal{G}_{a,c}) \to \mathcal{O}(\mathcal{G}_{a,b}) \otimes \mathcal{O}(\mathcal{G}_{b,c})$$

induced by path multiplication is a morphism of MHS. Moreover, the mixed Hodge structure on $\mathcal{O}(\mathcal{G}_{a,b})$ depends only on the variation $\mathbb{V}$ and not on the choice of the base point x_o.

Because of the last assertion, it may be more appropriate to say that $\mathcal{G}_{a,b}$ is the completion of $\pi(X; a, b)$ with respect to the variation $\mathbb{V}$.

The reader is assumed to be familiar with the basic methods for constructing mixed Hodge structures on the cohomology of bar constructions as described in [8, §3]. In the previous section we showed how to express $\mathcal{O}(\mathcal{G}_{a,b})$ as the 0th cohomology group of a suitable reduced bar construction. So in order to show that it has a canonical MHS we need only find a suitable augmented, multiplicative mixed Hodge complex $\mathbf{A}^\bullet$ which is quasi-isomorphic to $E_{\mathrm{fin}}^\bullet(X, \mathcal{O}(\mathcal{P}_{a,\_}))$. To do this, we shall use the work of M. Saito on Hodge modules.

First, some notation: Assume that $X = \overline{X} - D$, where $\overline{X}$ is a compact Kähler manifold and D is a normal crossings divisor. Denote the inclusion $X \hookrightarrow \overline{X}$ by j. Denote Deligne's canonical extension of $\mathbb{V} \otimes \mathcal{O}_X$ to $\overline{X}$ by $\overline{\mathcal{V}}$. Saito proves that there is a Hodge module over $\overline{X}$ canonically associated to $\mathbb{V}$, whose complex part is a bifiltered D-module $(M, W_\bullet, F^\bullet)$, and whose real part is $Rj_* \mathbb{V}_\mathbb{R}$ endowed with a suitable weight filtration. There is a canonical inclusion

$$\Omega^\bullet_{\overline{X}}(\overline{X} \log D) \otimes_{\mathcal{O}} \overline{\mathcal{V}} \hookrightarrow M \otimes_{\mathcal{O}} \Omega^\bullet_{\overline{X}}.$$

Saito defines Hodge and weight filtrations on $\Omega^\bullet_{\overline{X}}(\overline{X} \log D) \otimes_{\mathcal{O}} \overline{\mathcal{V}}$ by restricting those of M. The Hodge filtration is simply the tensor product of those of $\Omega^\bullet_{\overline{X}}(\overline{X} \log D)$ and $\overline{\mathcal{V}}$. The weight filtration is more difficult to describe.

THEOREM 13.4. – (Saito [16, (3.3)]). *The pair*

$$(13) \qquad M^\bullet(X, \mathbb{V}) := ((Rj_* \mathbb{V}_\mathbb{R}, W_\bullet), (\Omega^\bullet_{\overline{X}}(\overline{X} \log D) \otimes_{\mathcal{O}} \overline{\mathcal{V}}, F^\bullet, W_\bullet))$$

is a cohomological mixed Hodge complex whose cohomology is canonically isomorphic to $H^\bullet(X, \mathbb{V})$. $\qquad\square$

We can therefore obtain a mixed Hodge complex which computes $H^\bullet(X, \mathbb{V})$ by taking the standard fine resolution of these sheaves by C^∞ forms. (So the complex part of this will be the C^∞ log complex $E^\bullet(\overline{X} \log D, \overline{\mathcal{V}})$ with suitable Hodge and weight filtrations.)

To apply Saito's machinery to the current situation, we will need to know that $\mathcal{O}(\mathcal{P}_{x_o, \_})$ is a direct limit of variations over X.

LEMMA 13.5. – *The local system associated with an irreducible representation of $S(\mathbb{R})$ underlies a variation of Hodge structure over X. These structures are compatible with the decomposition of tensor products. Moreover, these variations are unique up to Tate twist.*

Proof. – The connected component of the identity of $S(\mathbb{R})$ is a real form of $Sp_n(\mathbb{C})$ when $\mathbb{V}$ has odd weight, and $SO_n(\mathbb{C})$ when $\mathbb{V}$ has even weight. In both cases each irreducible representations of the complex group can be constructed by applying a suitable Schur functor the the fundamental representation and then taking the intersection of the kernels of all contractions with the polarization. (This is Weyl's construction of the irreducible representations; it is explained, for example, in [5, §17.3, §19.2].) Since these operations preserve variations of Hodge structure, it follows that a local system corresponding to an irreducible representation of S underlies a variation of Hodge structure. Since the monodromy representation of $\mathbb{V}$ is Zariski dense, the structure of a polarized variation of Hodge structure on this local system is unique up to Tate twist. (*Cf.* the proof of [10, (9.1)].) $\qquad\square$

This, combined with (3.1) yields:

COROLLARY 13.6. – *With our assumptions, $\mathcal{O}(\mathcal{P}_{x_o, \_})$ is a direct limit of variations of Hodge structure over X of weight 0, and the multiplication map is a morphism.* $\qquad\square$

COROLLARY 13.7. – *For each $b \in X$, there is a canonical Hodge structure on $\mathcal{O}(P_{x_o, b})$.* $\qquad\square$

Combining (13.5) with (10.3), we obtain:

COROLLARY 13.8. – *The local system over X whose fiber over $x \in X$ is $H^1(\mathcal{U}(X, x))$ is an admissible variation of mixed Hodge structure whose weights are positive.*

Using Saito's machine [16], we see that there is a mixed Hodge complex (MHC) $\mathbf{A}^\bullet$ which is quasi-isomorphic to $E^\bullet_{\text{fin}}(X, \mathcal{O}(\mathcal{P}_{a,\_}))$. The complex part of this MHC is simply the complex of C^∞ forms on $\overline{X}$ with logarithmic singularities along D and which have coefficients in the canonical extension $\overline{\mathcal{O}}$ of $\mathcal{O}(\mathcal{P}_{a,\_})$ to $\overline{X}$. The Hodge filtration is the obvious one induced by the Hodge filtration of $\overline{\mathcal{O}}$ and the Hodge filtration of forms on $\overline{X}$. The weight filtration is not so easily described, and we refer to Saito's paper for that.

We need to know that the multiplication is compatible with the Hodge and weight filtrations. This follows from the next result.

PROPOSITION 13.9. – *If $\mathbf{V}_1 \otimes \mathbf{V}_2 \to \mathbf{W}$ is a pairing of variations of Hodge structure over X, then the multiplication map*

$$M^\bullet(X, \mathbf{V}_1) \otimes M^\bullet(X, \mathbf{V}_2) \to M^\bullet(X, \mathbf{W})$$

is a morphism of cohomological mixed Hodge complexes.

Proof. – This follows immediately from the naturality of Saito's construction, its compatibility with exterior tensor products, and the fact that variations of Hodge structure are closed under exterior products – use restriction to the diagonal. $\square$

There are two augmentations

$$\mathbb{R} \leftarrow \mathbf{A}^\bullet \to \mathcal{O}(\mathcal{P}_{a,b})$$

corresponding to the inclusions $\mathcal{P}_{a,b} \hookrightarrow \mathcal{P}_{a,\_}$ and $\text{id}_a \in \mathcal{P}_{a,a}$, and these are compatible with all filtrations. It follows from [8, (3.2.1)], (7.2) and (12.2) that

$$B(\mathbb{R}, \mathbf{A}^\bullet, \mathcal{O}(\mathcal{P}_{a,b}))$$

is a MHC whose H^0 is isomorphic to $\mathcal{O}(\mathcal{G}_{a,b})$. Moreover, the multiplication is compatible with the Hodge and weight filtrations. Consequently,

$$\mathcal{O}(\mathcal{G}_{a,b}) \cong H^0(B(\mathbb{R}, \mathbf{A}^\bullet, \mathcal{O}(\mathcal{P}_{a,b})))$$

has a canonical MHS and its multiplication is a morphism of MHS. Since the MHS on $P_{a,b}$ depends only on $\mathbf{V}$ and not on x_o, the same is true of the MHS on $\mathcal{O}(\mathcal{G}_{a,b})$.

If a, b, and c are three points of X, then it follows directly from the definitions that the map

$$B(\mathbb{R}, \mathbf{A}^\bullet, \mathcal{O}(\mathcal{P}_{a,c})) \to B(\mathbb{R}, \mathbf{A}^\bullet, \mathcal{O}(\mathcal{P}_{a,b})) \otimes B(\mathbb{R}, \mathbf{A}^\bullet, \mathcal{O}(\mathcal{P}_{b,c}))$$

corresponding to path multiplication is a morphism of MHCs. It follows that the induced map

$$\mathcal{O}(\mathcal{G}_{a,c}) \to \mathcal{O}(\mathcal{G}_{a,b}) \otimes \mathcal{O}(\mathcal{G}_{b,c})$$

is a morphism of MHS. This completes the proof of Theorem 13.3; Theorem 13.1 follows by taking $a = b = x_o$ except for the assertion that $\mathcal{O}(S) \hookrightarrow \mathcal{O}(\mathcal{G})$ is a morphism of MHS. This follows as this is induced by the natural inclusion

$$\mathcal{O}(S) \hookrightarrow B(\mathbb{R}, \mathbf{A}^\bullet, \mathcal{O}(S)),$$

that takes ϕ to $[\ ]\phi$. It is a morphism of MHCs. This completes the proofs of Theorems 13.1 and 13.3. $\square$

We now turn our attention to the variation of the Hodge filtration. Suppose that X and S are as above. Consider the real local system over $X \times X$ whose fiber over (a, b) is $\mathcal{O}(\mathcal{G}_{a,b})$. Denote it by $\mathcal{O}$. Next we establish that this underlies a "pre-variation of MHS." Denote by $\mathcal{F}^p\mathcal{O}$ the subset of the associated complex local system with fiber $F^p\mathcal{O}(\mathcal{G}_{a,b}(\mathbb{C}))$ over (a, b). Denote by $W_m\mathcal{O}$ the subset of $\mathcal{O}$ with fiber $W_m\mathcal{O}(\mathcal{G}_{a,b})$ over b.

THEOREM 13.10. – *The subset $W_m\mathcal{O}$ is a flat sub-bundle of $\mathcal{O}$, and $F^p\mathcal{O}$ is a holomorphic sub-bundle of $\mathcal{O}_\mathbb{C}$ whose corresponding sheaf of sections $\mathcal{F}^p$ satisfies Griffiths transversality:*

$$\nabla : \mathcal{F}^p \to \mathcal{F}^{p-1} \otimes \Omega^1_X.$$

Sketch of Proof. – We will prove the result for the restriction $\mathcal{O}_a$ of $\mathcal{O}$ to $\{a\} \times X$. The result for the restriction of $\mathcal{O}$ to $X \times \{b\}$ is proved similarly. The general result then follows as the tangent spaces of $\{a\} \times X$ and $X \times \{b\}$ span the tangent space of $X \times X$ at (a, b).

First we need a formula for the connection on $\mathcal{O}_a$ at the point b in X. Fix a path γ in X from a to b. Suppose that $\mu : [-\epsilon, \epsilon] \to X$ is a smooth path with $\mu(0) = b$. For $s \in [-\epsilon, \epsilon]$ let $\gamma_s : [0, 1] \to X$ be the piecewise smooth path obtained by following γ and then μ from $t = 0$ to $t = s$. Suppose that $w_1, \ldots, w_r$ are in $E^\bullet_{\text{fin}}(X, \mathcal{O}(\mathcal{P}_{a,b}))$. Suppose that U is a contractible neighbourhood of b in X. With respect to a flat trivialization of the restriction of $\mathcal{P}_{a,\_}$ to U, we have

$$w_r|_U = \sum_j w_r^j \otimes \psi_j$$

where $w_r^j \in E^1(U)$ and $\psi_j \in \mathcal{O}(\mathcal{P}_{a,b})$.

It follows from the analogue of (5.9) in this situation that

$$\left.\frac{d}{ds}\right|_{s=0} \int_{\gamma_s} (w_1 \ldots w_r | \phi) = \sum_j \int_\gamma (w_1 \ldots w_{r-1} | \phi\psi_j) \langle w_r^j, \dot\mu(0) \rangle.$$

The restriction of the connection on $\mathcal{O}_a$ to the stalk at b is therefore induced by the map

$$[w_1| \ldots |w_r]\phi \mapsto \sum_j [w_1| \ldots |w_{r-1}]\phi\psi_j \otimes w_r^j$$

on the bar construction. The flatness of the weight filtration follows immediately from the definition of the weight filtration on the bar construction. Further, if $(z_1, \ldots, z_n)$ is a

holomorphic coordinate in X centered at b, then it follows immediately from the definition of the Hodge and weight filtrations on $\mathcal{O}(\mathcal{G}_{a,b}(\mathbb{C}))$ and the formula for the connection that

$$\nabla_{\partial/\partial \bar{z}_k} : \mathcal{F}^p \to \mathcal{F}^p$$

for each k, so that the Hodge filtration varies holomorphically at b. Similarly, on the stalk of F^p at b we have

$$\nabla_{\partial/\partial z_k} : \mathcal{F}^p \to \mathcal{F}^{p-1}$$

as each w_r^j contributes at most 1 to the Hodge filtration of $\mathcal{O}(\mathcal{G}_{a,b})$. $\square$

When X is compact, we have:

COROLLARY 13.11. – *If X is a compact Kähler manifold, then $\boldsymbol{O}$ is an admissible variation of MHS over $X \times X$.* $\square$

In order to prove the corresponding result in the non-compact case, it is necessary to study the asymptotic behaviour of $\boldsymbol{O}$. I plan to consider this in a future paper.

We now consider naturality. Suppose that X and $\mathbb{V}$ are as above, and that Y is a smooth variety and that $\mathbb{W}$ is an variations of Hodge structure over Y. We will now denote S by S_X:

$$S_X = \mathrm{Aut}(V_o, \langle \ , \ \rangle).$$

Set

$$S_Y = \mathrm{Aut}(W_o, \langle \ , \ \rangle)$$

where W_o denotes the fiber of $\mathbb{W}$ over y_o. Suppose that the monodromy representation

$$\rho_Y : \pi_1(Y, y_o) \to S_Y$$

has Zariski dense image. Denote the completion of $\pi_1(X, x_o)$ with respect to ρ_X : $\pi_1(X, x_o) \to S_X$ by $\pi_1(X, x_o) \to \mathcal{G}_X$, and the completion of $\pi_1(Y, y_o)$ with respect to ρ_Y by $\pi_1(Y, y_o) \to \mathcal{G}_Y$.

Suppose that $f : (Y, y_o) \to (X, x_o)$ is a morphism of varieties, and that we have fixed an inclusion

$$\mathrm{End}\,\mathbb{V} \hookrightarrow \mathrm{End}\,f^*\mathbb{W}$$

of variations of Hodge structure. This fixes a group homomorphism

$$\Psi : S_X \hookrightarrow S_Y$$

that is compatible with the Hodge theory. By (11.1), there is a homomorphism $\widehat{\Psi}$: $\mathcal{G}_X \to \mathcal{G}_Y$ such that the diagram

$$
\begin{array}{ccccc}
\pi_1(X, x_o) & \longrightarrow & \mathcal{G}_X & \longrightarrow & S_X \\
f_* \downarrow & & \downarrow \widehat{\Psi} & & \downarrow \Psi \\
\pi_1(Y, y_o) & \longrightarrow & \mathcal{G}_Y & \longrightarrow & S_Y
\end{array}
$$

commutes.

THEOREM 13.12. – *Under these hypotheses, the induced map*

$$\widehat{\Psi}^* : \mathcal{O}(\mathcal{G}_Y) \to \mathcal{O}(\mathcal{G}_X)$$

is a morphism of MHS.

Proof. – First, choose smooth compactifications $\overline{X}$ of X and $\overline{Y}$ of Y such that $X = \overline{X} - D$ and $Y = \overline{Y} - E$, where D and E are normal crossings divisors. We may choose these such that f extends to a morphism $\overline{X} \to \overline{Y}$, which we shall also denote by f.

Denote by P_X the flat left S_X principal bundle over X associated the the representation of $\pi_1(X, x_o)$ on S_X via ρ_X. Denote the analogous principal S_Y principal bundle over Y by P_Y. Associated to these we have the local systems $\mathcal{O}(P_X)$ over X and $\mathcal{O}(P_Y)$ over Y.

The construction above gives multiplicative mixed Hodge complexes

$$\mathbf{A}^\bullet(X, \mathcal{O}(P_X)), \quad \mathbf{A}^\bullet(X, f^*\mathcal{O}(P_Y)), \quad \text{and} \quad \mathbf{A}^\bullet(Y, \mathcal{O}(P_Y))$$

which compute the canonical mixed Hodge structures on

$$H^\bullet(X, \mathcal{O}(P_X)), \quad H^\bullet(X, f^*\mathcal{O}(P_Y)), \quad \text{and} \quad H^\bullet(Y, \mathcal{O}(P_Y))$$

respectively. The map f induces a morphism

$$\mathbf{A}^\bullet(Y, \mathcal{O}(P_Y)) \to \mathbf{A}^\bullet(X, f^*\mathcal{O}(P_Y))$$

of MHCs, while the inclusion $S_X \hookrightarrow S_Y$ induces a morphism

$$\mathbf{A}^\bullet(X, f^*\mathcal{O}(P_Y)) \to \mathbf{A}^\bullet(X, \mathcal{O}(P_X))$$

of MHCs. The composition of these corresponds to the induced map

$$E^\bullet_{\mathrm{fin}}(Y, \mathcal{O}(P_Y)) \to E^\bullet_{\mathrm{fin}}(X, \mathcal{O}(P_X))$$

induced by f under the canonical quasi-isomorphisms. It follows that the induced map

$$B(\mathbb{R}, \mathbf{A}^\bullet(Y, \mathcal{O}(P_Y)), \mathcal{O}(P_Y)) \to B(\mathbb{R}, \mathbf{A}^\bullet(X, \mathcal{O}(P_X)), \mathcal{O}(P_X))$$

is a morphism of MHCs and that the induced the map

$$f^* : \mathcal{O}(\mathcal{G}_Y) \to \mathcal{O}(\mathcal{G}_X)$$

on H^0 is the ring homomorphism induced by f. The result follows. $\square$

Remark 13.13. – Suppose that $\mathbb{V}$ is a variation of Hodge structure over the complement X of a normal crossings divisor in a compact Kähler manifold. We will say that the pair $(X, \mathbb{V})$ is *neat* if the Zariski closure S of the image of the monodromy map

$$\rho : \pi_1(X, x_o) \to \mathrm{Aut}(V_o, \langle \ , \ \rangle)$$

is semi-simple, and that the canonical MHS on the coordinate ring of

$$\mathrm{Aut}(V_o, \langle \ , \ \rangle)$$

induces one on S. For example, every variation where S is finite is neat. I believe that every (X, V) is neat, but have not yet found a proof.

The results (13.1), (13.2), (13.3), (13.8), (13.10), (13.11), (13.13) and their proofs are valid with the assumption that $\mathrm{im}\rho$ be Zariski dense in $\mathrm{Aut}(V_o, \langle \ , \ \rangle)$ replaced by the assumption that the pairs (X, V) and (Y, W) be neat.

The following is an application suggested by Ludmil Kartzarkov.

THEOREM 13.14. – *Suppose that X is a compact Kähler manifold and that V is a polarized variation of Hodge structure over X whose monodromy representation ρ has Zariski dense image. Then the prounipotent radical of the completion of $\pi_1(X, x_o)$ relative to ρ has a quadratic presentation.*

Proof. – It is well known that if X is compact Kähler and V is a polarized variation of Hodge structure over X of weight m, then $H^k(X, \mathsf{V})$ has a pure Hodge structure of weight $k + m$. In particular, as the variation $\mathcal{O}(P)$ over X has weight zero, $H^k(X, \mathcal{O}(P))$ is pure of weight weight k for all k.

Denote the Lie algebra of the prounipotent radical of the relative completion of $\pi_1(X, x_o)$ by $\mathfrak{u}$. It follows from (8.6) and (10.1) that $\mathfrak{u}$ is the Lie algebra canonically associated to the d.g.a. $E_{\mathrm{fin}}^\bullet(X, \mathcal{O}(P))$ by rational homotopy theory (either via Sullivan's theory of minimal models, or via Chen's theory as the dual of the indecomposables of the bar construction on $E_{\mathrm{fin}}^\bullet(X, \mathcal{O}(P))$.) There are canonical maps

$$H^1(\mathfrak{u}) \cong H^1(X, \mathcal{O}(P)) \qquad \text{and} \qquad H^2(\mathfrak{u}) \hookrightarrow H^2(X, \mathcal{O}(P)).$$

These are morphisms of MHS [11, (7.2)]. It follows that $H^1(\mathfrak{u})$ is pure of weight 1 and $H^2(\mathfrak{u})$ is pure of weight 2. The result now follows from [11, (5.2), (5.7)]. $\qquad \square$

Remark 13.15. – It is not necessarily true that $\mathfrak{u}$ is a quotient of the unipotent completion of $\ker \rho$. A criterion for surjectivity is given in [9, (4.6)]. For this reason it may not be easy to apply this result in general situations without artificially restrictive hypotheses.

14. A canonical connection

For the time being, let X, x_o, V, ρ, etc. be as in the previous section. However, all groups and Lie algebras in this section will be complex, and $\mathcal{G}$, $\mathcal{U}$, $\mathfrak{u}$, etc. denote the *complex* points of the relative completion of $\pi_1(X, x_o)$, its prounipotent radical, its Lie algebra, etc. Denote the image of ρ by Γ, and the Galois covering of X with Galois group Γ by X'. In this section, we show how the Hodge theory of $\mathcal{G}$ gives a canonical (given the choice of x_o), Γ invariant integrable 1-form

$$\omega \in E^1(X') \,\widehat{\otimes}\, \mathrm{Gr}_\bullet^W \mathfrak{u}$$

on X' which can be integrated to the canonical representation

$$\tilde{\rho} : \pi_1(X, x_o) \to S \ltimes \mathcal{U} \cong \mathcal{G}.$$

Here $\widehat{\otimes}$ denotes the completed tensor product, which is defined below.

At the end of the section, we shall explain what this means when X is the complement of the discriminant locus in $\mathbb{C}^n$ and S is the symmetric group Σ_n. In this case, X' is the complement of the hyperplanes $x_i = x_j$ in $\mathbb{C}^n$, $\pi_1(X, x_o)$ is the classical braid group B_n, and the form is

$$\omega = \sum_{i<j} d\log(x_i - x_j).$$

First, we shall define the completed tensor product $\widehat{\otimes}$. Suppose that $\mathfrak{u}$ is a topological Lie algebra and that

$$\mathfrak{u} = \mathfrak{u}^1 \supseteq \mathfrak{u}^2 \supseteq \mathfrak{u}^3 \supseteq \cdots$$

is a base of neighbourhoods of 0. Suppose that E is a vector space. Define

$$E \widehat{\otimes} \mathfrak{u} = \varprojlim E \otimes \mathfrak{u}/\mathfrak{u}^m.$$

We can regard a graded Lie algebra $\mathfrak{u} = \oplus_{m<0}\mathfrak{u}_m$ as a topological Lie algebra where the basic neighbourhoods of 0 are

$$\bigoplus_{l \leq m} \mathfrak{u}_l, \quad m < 0.$$

The definition of completed tensor product therefore extends to the case where $\mathfrak{u}$ is graded. Finally, if $\mathfrak{u}$ is a Lie algebra in the category of mixed Hodge structures where $\mathfrak{u} = W_{-1}\mathfrak{u}$ which is complete with respect to the weight filtration, and if E is a complex vector space, then there is a canonical isomorphism

$$E \widehat{\otimes} \mathfrak{u}_\mathbb{C} \cong E \widehat{\otimes} \mathrm{Gr}^W_\bullet \mathfrak{u}_\mathbb{C}$$

as $\mathfrak{u}_\mathbb{C}$ is canonically isomorphic to $\prod \mathrm{Gr}^W_m \mathfrak{u}_\mathbb{C}$. (*Cf.* [11, (5.2)].)

We view a principal bundle with structure group a proalgebraic group to be the inverse limit of the principal bundles whose structure groups are the finite dimensional quotients of the proalgebraic group. A connection on a principal bundle with proalgebraic structure group is the inverse limit of compatible connections on the corresponding bundles with finite dimensional structure group.

14.1. The unipotent case

We begin with the unipotent case, $S = 1$. Suppose that X is a smooth manifold with distinguished base point x_o. Denote the complex form of the unipotent completion of $\pi_1(X, x_o)$ by $\mathcal{G}$ and the complex points of the completion of $\pi(X; x_o, x)$ by $\mathcal{G}_{x_o, x}$. The family

$$(\mathcal{G}_{x_o, x})_{x \in X}$$

forms a flat principal left $\mathcal{G}$ bundle over X that we shall denote by $\mathcal{G}_{x_o, \_}$. Since the structure group is contractible, (more precisely, an inverse limit of contractible groups), this bundle has a section. Pulling back the canonical connection form, we obtain an integrable connection form

$$\omega \in E^\bullet(X) \mathbin{\widehat{\otimes}} \mathfrak{g}$$

where $\mathfrak{g}$ denotes the Lie algebra of $\mathcal{G}$. The monodromy representation of this form is the monodromy of $\mathcal{G}_{x_o, \_}$, which is the canonical homomorphism

$$\pi_1(X, x_o) \to \mathcal{G}.$$

When X is an algebraic manifold, there is a canonical choice of section and therefore a canonical connection form. To see this, note that for each $a \in X$, the weights on $\mathcal{O}(\mathcal{G}_{x_o, a})$ are ≥ 0 and that

$$\mathrm{Gr}_0^W \mathcal{O}(\mathcal{G}_{x_o, a}) \cong \mathbb{C}.$$

Since there is a canonical ring isomorphism

$$\mathcal{O}(\mathcal{G}_{x_o, a}) \cong \bigoplus_{l \geq 0} \mathrm{Gr}_l^W \mathcal{O}(\mathcal{G}_{x_o, a})$$

there is a canonical augmentation

$$\mathcal{O}(\mathcal{G}_{x_o, a}) \to \mathbb{C}$$

whose kernel is

$$\bigoplus_{l > 0} \mathrm{Gr}_l^W \mathcal{O}(\mathcal{G}_{x_o, a}).$$

This determines a canonical point in $\mathcal{G}_{x_o, a}$. Since the family $\{\mathcal{O}(\mathcal{G}_{x_o, a})\}_{a \in X}$ is a variation of MHS over X (*see* [12]), these distinguished points vary smoothly as a varies. They therefore determine a smooth section of $\mathcal{G}_{x_o, \_}$. We therefore have a canonical integrable 1-form

$$\omega \in E^1(X) \mathbin{\widehat{\otimes}} \mathfrak{g} \cong E^1(X) \mathbin{\widehat{\otimes}} \mathrm{Gr}_\bullet^W \mathfrak{g}.$$

14.2. The general case

The first step in doing this in general is to explain the necessary constructions in the C^∞ case. So suppose for the time being that X is a smooth manifold; ρ, S, $\mathcal{G}$ and $P \to X$ are as before. We also have the torsor

$$\mathcal{G}_{x_o,\_} \to X.$$

It is a flat principal left $\mathcal{G}$ bundle over X. There is a map

$$\begin{array}{ccc}
\mathcal{G}_{x_o,\_} & \xrightarrow{\ \pi\ } & P \\
\downarrow & & \downarrow \\
X & = & X
\end{array}$$

of flat bundles. It is compatible with the canonical homomorphism $\mathcal{G} \to S$. Denote by X' the leaf of P containing the distinguished lift $\tilde{x}_o$ to P of x_o. The projection $P \to X$ induces a covering map $X' \to X$. It is the Galois covering corresponding to $\ker \rho$. Define $\mathcal{U}_{x_o,\_}$ to be the subset $\pi^{-1}X'$ of $\mathcal{G}_{x_o,\_}$. There is a natural projection $\mathcal{U}_{x_o,\_} \to X'$ induced by π. Note that the fiber of this over $\tilde{x}_o$ is $\mathcal{U}$, the prounipotent radical of $\mathcal{G}$. Denote the fiber of $\mathcal{U}_{x_o,\_}$ over $a \in X'$ by $\mathcal{U}_{x_o,a}$.

Each point a of P determines an augmentation

$$\epsilon_a : E_{\mathrm{fin}}^\bullet(X, \mathcal{O}(P)) \to \mathbb{C}.$$

Given two points a and b of P, we may form the two sided bar construction

(14) $$B(\mathbb{C}, E_{\mathrm{fin}}^\bullet(X, \mathcal{O}(P)), \mathbb{C})$$

where the left hand $\mathbb{C}$ is viewed as a module over $E_{\mathrm{fin}}^\bullet(X, \mathcal{O}(P))$ via ϵ_a, and the right hand $\mathbb{C}$ via ϵ_b. We shall denote the d.g.a. (14) by $B(E_{\mathrm{fin}}^\bullet(X, \mathcal{O}(P))_{a,b})$.

PROPOSITION 14.1. – *Each $\mathcal{U}_{x_o,a}$ is a proalgebraic variety with coordinate ring*

$$\mathcal{O}(\mathcal{U}_{x_o,a}) \cong H^0(B(E_{\mathrm{fin}}^\bullet(X, \mathcal{O}(P))_{\tilde{x}_o,a})).$$

Moreover, $\mathcal{U}_{x_o,\_} \to X'$ is a principal $\mathcal{U}$ bundle with respect to the natural $\mathcal{U}$ action on $\mathcal{G}_{x_o,\_}$.

Choose a splitting $S \to \mathcal{G}$ of the natural homomorphism $\mathcal{G} \to S$. This induces an isomorphism $\mathcal{G} \cong S \ltimes \mathcal{U}$. The splitting enables us to lift the action of S to $\mathcal{G}_{x_o,\_}$ in such a way that the projection $\mathcal{G}_{x_o,\_} \to P$ is S equivariant. Since Γ is a subgroup of S, and since it preserves $X' \subset P$, it follows that there is a natural left action of Γ on $\mathcal{U}_{x_o,\_}$ and that, with respect to this action, the projection $\mathcal{U}_{x_o,\_}$ is Γ equivariant.

Denote the pullback of the extension

$$1 \to \mathcal{U} \to \mathcal{G} \to S \to 1$$

along $\Gamma \hookrightarrow S$ by $\mathcal{G}_\Gamma$. This is an extension

$$1 \to \mathcal{U} \to \mathcal{G}_\Gamma \to \Gamma \to 1.$$

The splitting $S \to \mathcal{G}$ induces a splitting $\Gamma \to \mathcal{G}_\Gamma$, and therefore a semi-direct product decomposition $\mathcal{G}_\Gamma \cong \Gamma \ltimes \mathcal{U}$. The image of the canonical homomorphism $\pi_1(X, x_o) \to \mathcal{G}$ lies in $\mathcal{G}_\Gamma$. The composite $\mathcal{U}_{x_o,\_} \to X' \to X$ is a flat principal left $\mathcal{G}_\Gamma$ bundle over X. The associated monodromy representation is the canonical homomorphism $\pi_1(X, x_o) \to \mathcal{G}_\Gamma$. The monodromy therefore induces the canonical homomorphism

$$\pi_1(X, x_o) \to \mathcal{G} \cong S \ltimes \mathcal{U}.$$

Next, we explain that the pullback of this bundle to X' is trivial, and therefore given by an integrable 1-form.

PROPOSITION 14.2. – *There is a Γ equivariant section of $\mathcal{U}_{x_o,\_} \to X'$.*

Proof. – The action of Γ on X' is free. It follows that the action of Γ on $\mathcal{U}_{x_o,\_}$ is also free. Consequently, the square

$$\begin{array}{ccc} \mathcal{U}_{x_o,\_} & \longrightarrow & \Gamma \backslash \mathcal{U}_{x_o,\_} \\ \downarrow & & \downarrow \\ X' & \underset{p}{\longrightarrow} & X \end{array}$$

is a pullback square. Since the fibers of $\Gamma \backslash \mathcal{U}_{x_o,\_} \to X$ are connected, it has a C^∞ section. This section pulls back to a Γ invariant section of $\mathcal{U}_{x_o,\_} \to X'$.

Let Γ act on $\mathcal{U}$ on the left via the adjoint action:

$$Ad(\gamma) : u \mapsto \gamma u \gamma^{-1}.$$

Then Γ acts on $X' \times \mathcal{U}$ on the left via the diagonal action. It follows from the previous result that the flat principal bundle $\mathcal{U}_{x_o,\_} \to X'$ has a Γ invariant trivialization. We therefore have a connection form

$$\omega \in E^1(X') \,\widehat{\otimes}\, \mathfrak{u}.$$

PROPOSITION 14.3. – *This connection form satisfies $\gamma^* \omega = Ad(\gamma)\omega$ for all $\gamma \in \Gamma$.*

Proof. – Since the bundle is trivial, its (locally defined) sections can be identified with (locally defined) functions $X' \to \mathcal{U}$. Since Γ preserves the connection, we see that for each $\gamma \in \Gamma$ the local section u is flat if and only if the local section $(\gamma^{-1})^* Ad(\gamma)(u)$ is flat. That is, $Ad(\gamma)(u)$ is flat if and only if $\gamma^* u$ is flat. The result now follows from a standard and straightforward computation. □

Now suppose that X is an algebraic manifold. We have to show that this construction can be made canonical. Note that, given the choice of the base point x_o, the only choices made in the construction of ω were the choice of a splitting of the homomorphism $\mathcal{G} \to S$, and the choice of a Γ invariant section of $\mathcal{U}_{x_o,\_} \to X'$. We will now explain how Hodge theory provides canonical choices of both.

It follows from (13.2) that $\mathrm{Gr}_0^W \mathfrak{g} \cong \mathfrak{s}$. Consequently, there is a canonical splitting of the homomorphism $\mathfrak{g} \to \mathfrak{s}$. This induces a canonical splitting of the homomorphism $\mathcal{G} \to S$, and therefore a canonical action of Γ on $\mathcal{U}_{x_o,\_}$ and a canonical identification $\mathcal{G} \cong S \ltimes \mathcal{U}$.

It remains to explain why there is a canonical choice of a Γ equivariant section of $\mathcal{U}_{x_o,\_}$. This is an elaboration of the argument in the unipotent case.

For each $b \in X$, Hodge theory provides canonical ring isomorphisms

$$\mathcal{O}(G_{x_o,b}) \cong \bigoplus_{m \geq 0} \mathrm{Gr}_m^W \mathcal{O}(\mathcal{G}_{x_o,b})$$

and

$$\mathcal{O}(\mathcal{P}_{x_o,b}) \cong \mathrm{Gr}_0^W \mathcal{O}(\mathcal{G}_{x_o,b}).$$

Moreover, it follows from (13.10) that these identifications depend smoothly on b. Consequently, there is a smooth section σ of the canonical projection $\mathcal{G}_{x_o,\_} \to \mathcal{P}_{x_o,\_}$. Restricting to X', we obtain a canonical smooth section of the projection $\mathcal{U}_{x_o,\_} \to X'$.

PROPOSITION 14.4. – *This section is Γ equivariant.*

Proof. – For each $x \in X$ we have the action $\mathcal{G} \times \mathcal{G}_{x_o,x} \to \mathcal{G}_{x_o,x}$. By (12.2) the corresponding map of coordinate rings is a morphism of MHS. By the choice of splitting of $\mathcal{G} \to S$, the action of S given by the splitting preserves the canonical isomorphism

$$\mathcal{O}(\mathcal{G}_{x_o,x}) \cong \bigoplus_{l \geq 0} \mathrm{Gr}_l^W \mathcal{O}(\mathcal{G}_{x_o,x}).$$

It follows that the section of $\mathcal{G}_{x_o,\_} \to \mathcal{P}_{x_o,\_}$ defined above is equivariant with respect to the left S actions. It follows that the restriction of this section to X' is Γ equivariant. $\square$

EXAMPLE 14.5. – In this example, we take X to be the complement in $\mathbb{C}^n$ of the universal discriminant locus. (View $\mathbb{C}^n$ as the space of monic polynomials of degree n.) Pick a base point x_o. The fundamental group of this space is the classical braid group. Denote the symmetric group on n letters by Σ_n. There is a natural homomorphism $\rho : B_n \to \Sigma_n$. Denote the corresponding covering of X by $\pi : X' \to X$. Its fundamental group is the pure braid group P_n. As is well known, X' is the complement of the hyperplanes $x_i = x_j$ in $\mathbb{C}^n$ where $i \neq j$. The projection takes $(x_1, \ldots, x_n)$ to the monic polynomial $\prod(T - x_j)$. The natural left action of Σ_n on X' is given by

$$\sigma : (x_1, \ldots, x_n) \mapsto (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}).$$

The local system $\pi_* \mathbb{Q}_{X'}$ is a variation of Hodge structure over X of weight 0, rank n, and type $(0,0)$. The closure of the image of the monodromy is Σ_n, a semi-simple group. So we can apply Theorem 13.13 to deduce the existence of a MHS on the relative completion, and the existence of a universal connection. In this case, the canonical connection is well known by the work [13] of Kohno.

Denote the free Lie algebra over $\mathbb{C}$ generated by the Y_j by $\mathbb{L}(Y_1, \ldots, Y_m)$. Denote the unipotent completion of P_n by $\mathbb{P}_n$ and its Lie algebra by $\mathfrak{p}_n$. The associated graded of $\mathfrak{p}_n$ of is the graded Lie algebra

$$\mathbb{L}(X_{ij} : ij \text{ is a two element subset of } \{1, \ldots, n\})/R$$

where R is the ideal generated by the quadratic relations

$$[X_{ij}, X_{kl}] \text{ when } i, j, k \text{ and } l \text{ are distinct};$$
$$[X_{ij}, X_{ik} + X_{jk}] \text{ when } i, j \text{ and } k \text{ are distinct}.$$

The natural (left) action of the symmetric group on it is defined by

$$Ad(\sigma) : X_{ij} \mapsto X_{\sigma(ij)}.$$

The canonical invariant form

$$\omega \in E^1(X') \otimes \mathrm{Gr}_\bullet^W \mathfrak{p}_n$$

is

$$\omega = \sum_{ij} d\log(x_i - x_j) X_{ij}.$$

It is invariant because

$$\sigma^* \omega = \sum_{ij} d\log(x_{\sigma^{-1}(i)} - x_{\sigma^{-1}(j)}) X_{ij} = \sum_{ij} d\log(x_i - x_j) X_{\sigma(ij)} = Ad(\sigma)\omega.$$

We therefore obtain a homomorphism

$$B_n \to \Sigma_n \ltimes \mathcal{P}_n$$

where $\mathcal{P}_n$ denotes the complex form of the Malcev completion of P_n. This is the completion of B_n relative to $\rho : B_n \to \Sigma_n$.

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