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THE HILBERT SCHEMES OF DEGREE THREE CURVES

BY SCOTT NOLLET

ABSTRACT. – In this paper we show that the Hilbert scheme $H(3, g)$ of locally Cohen-Macaulay curves in $\mathbb{P}^3$ of degree three and genus g is connected. This is achieved by giving a classification of these curves, determining the irreducible components of $H(3, g)$, and giving certain specializations to show connectedness. As a byproduct, we find that there are curves which lie in the closure of each irreducible component.

RÉSUMÉ. – Le but de cet article est de montrer que $H(3, g)$ (le schéma de Hilbert des courbes localement de Cohen-Macaulay de l'espace projectif $\mathbb{P}^3$ de degré 3 et de genre arithmétique g) est connexe pour toutes valeurs de $g \leq 1$. Nous arrivons à cette conclusion en donnant une classification des courbes in $H(3, g)$, déterminant les composantes irréductibles (il existe $\lfloor \frac{4-g}{3} \rfloor$ composantes, de dimensions différentes), et finalement, exhibant certaines courbes qui sont dans la clôture de chacune des composantes.

0. Introduction

In his thesis [6], Hartshorne showed that the (full) Hilbert scheme for projective subschemes with a fixed Hilbert polynomial is connected. Often one studies certain subsets of the Hilbert scheme which parametrize subschemes satisfying a certain property. For example, one can consider the Hilbert scheme of smooth curves in $\mathbb{P}^3$. The smooth curves of degree 9 and genus 10 afford an example for which this Hilbert scheme is *not* connected (*see* [5], IV, ex. 6.4.3). It is not known for which properties the corresponding Hilbert scheme is connected.

In the present paper, we are interested in the Hilbert scheme $H(d, g)$ of locally Cohen-Macaulay curves in $\mathbb{P}^3$ of degree d and arithmetic genus g. By work of several authors [7, 14, 15], it is known that $H(d, g)$ is nonempty precisely when $d > 0, g = \frac{1}{2}(d-1)(d-2)$ or $d > 1, g \leq \frac{1}{2}(d-2)(d-3)$. In a recent paper [13], Martin-Deschamps and Perrin prove that $H(d, g)$ is reduced only when $d = 2$ or $g = \frac{1}{2}(d-1)(d-2)$ or $g = \frac{1}{2}(d-2)(d-3)$ or $(d, g) = (3, -1)$. For all other (d, g) pairs for which $H(d, g)$ is nonempty, there is a nonreduced irreducible component corresponding to curves which are extremal in the sense that their Rao modules have the largest possible dimension.

It can be gleaned from several sources [13, 3, 4] that $H(d, g)$ is irreducible precisely in the cases $d = 2$ or $g > \frac{1}{2}(d-3)(d-4) + 1$ or $(d, g) \in \{(4, 1), (3, 1), (3, 0), (3, -1)\}$. In particular, $H(d, g)$ is connected in these cases. In the present paper we show that $H(3, g)$ is connected for all g. This is the first interesting case in the sense that these Hilbert schemes have several irreducible components. Curiously, there are certain extremal curves which lie in the closure of each irreducible component.

The paper is organized as follows. In the first section, we review several results of Banica and Forster [2] on multiplicity structures on smooth curves in a smooth threefold and classify space curves of degree two as an example. We also briefly review the extremal curves studied by Martin-Deschamps and Perrin. In the second section, we classify the multiplicity three structures on a line. This is used in the third section, where we classify all locally Cohen-Macaulay curves of degree 3 in $\mathbb{P}^3$. In particular, the irreducible components of the Hilbert scheme are determined. We also produce some flat families of triple lines, which show that the Hilbert scheme is connected.

In this paper, we work over an algebraically closed field k of arbitrary characteristic. $S = k[x,y,z,w]$ denotes the homogeneous coordinate ring of $\mathbb{P}^3_k$. If $V \subset S$ is a closed subvariety, then S_V denotes the the homogeneous coordinate ring S/I_V of V. We often use the abbreviation CM to mean locally Cohen-Macaulay. $H(d,g)$ denotes the Hilbert scheme of locally Cohen-Macaulay curves in $\mathbb{P}^3$ of degree d and arithmetic genus g.

I would like to thank Robin Hartshorne for useful discussions, and for various editorial suggestions. The fact that theorem 1.7 holds when char $k = p > 0$ was brought to my attention by Enrico Schlesinger. I appreciate the very careful reading of the referee, who caught several small errors.

1. Preliminaries

In this section we review the results of Banica and Forster [2] on multiplicity structures on smooth curves in smooth threefolds. As an example, we give the classification of double lines in $\mathbb{P}^3$, which will be used in section two when we classify the triple lines in $\mathbb{P}^3$. We also recall a few notions from linkage theory and summarize the results of Martin-Deschamps and Perrin on extremal curves.

DEFINITION 1.1. – If Y is a scheme, then a *locally Cohen-Macaulay multiplicity structure Z on Y* is a locally Cohen-Macaulay scheme Z which contains Y and has the same support as Y. For short, we simply say that Z is a *multiplicity structure* on Y.

In [2], Banica and Forster consider a smooth curve Y inside a smooth threefold X. Starting with a multiplicity structure $Y \subset Z \subset X$, they define a filtration on Z as follows. Let $Y^{(i)}$ denote the subscheme of X defined by ${I_Y}^i$. Let Z_i denote the subscheme of X obtained from $Z \cap Y^{(i)}$ by removing the embedded points. This gives the (unique) largest Cohen-Macaulay subscheme contained in $Z \cap Y^{(i)}$. If k is the smallest integer such that $Z \subset Y^{(k)}$, we obtain the *Cohen-Macaulay filtration* for $Y \subset Z$

$$Y = Z_1 \subset Z_2 \subset \ldots \subset Z_k = Z.$$

Letting $\mathcal{I}_i = \mathcal{I}_{Z_i}$, there are sheaves $L_j = \mathcal{I}_j/\mathcal{I}_{j+1}$ associated to this filtration. For any $i,j \geq 1$, it turns out that $\mathcal{I}_i\mathcal{I}_j \subset \mathcal{I}_{i+j}$, and hence the L_j are $\mathcal{O}_Y$-modules. In fact, the L_j are shown to be locally free $\mathcal{O}_Y$-modules. Further, there are induced multiplication maps $L_i \otimes L_j \to L_{i+j}$, which are generically surjective (because $\mathcal{I}_j = \mathcal{I}_Z + {\mathcal{I}_Y}^i$ on an open set). In particular, we get generically surjective maps $L_1^{\otimes j} \to L_j$.

From the above, we see that if Z is a multiplicity structure on Y, then there is a filtration $\{Z_j\}$ and exact sequences

$$0 \to \mathcal{I}_{Z_{j+1}} \to \mathcal{I}_{Z_j} \to L_j \to 0$$

where the L_j are vector bundles on Y. If Y is connected, the multiplicity of Z is defined by $\mu(Z) = \dim_K(\mathcal{O}_{Z,\eta})$, where η is the generic point of Y and $K = \mathcal{O}_{Y,\eta}$ is the function field of Y. The sequences above show that $\mu(Z) = 1 + \sum_{j=1}^{k-1} \operatorname{rank} L_j$.

REMARK 1.2. – The above constructions can be carried out in Z (instead of X), and would yield the same filtration as above. Thus the Cohen-Macaulay filtration is well-defined for abstract (non-embedded) multiplicity structures.

Now we use the fact that X is a smooth threefold. In this case, the conormal sheaf $\mathcal{I}_Y/\mathcal{I}_Y^2$ is a rank two bundle on Y. Since the surjection $\mathcal{I}_Y \to L_1$ factors through the conormal bundle, we see that $L = L_1$ has rank zero, one, or two. If the rank is zero, then $L = 0$ and the generically surjective maps show that all the $L_j = 0$ for all j, hence $Z = Y$. If the rank is two, then the surjection $\mathcal{I}_Y/\mathcal{I}_Y^2 \to L$ becomes an isomorphism, hence $Y^{(2)} \subset Z$ and Z has generic embedding dimension three.

We are mainly interested in the case $\operatorname{rank}(L) = 1$, in which case we say the extension $Y \subset Z$ is *quasi-primitive*. Here the generically surjective maps $L^{\otimes j} \to L_j$ show that there are effective divisors D_j on Y such that $L_j = L^j(D_j)$ for $j < k$ ($L_j = 0$ for $j \geq k$). The multiplication maps $L_i \otimes L_j \to L_{i+j}$ show that $D_i + D_j \leq D_{i+j}$ for all $i, j \geq 1$ with $i + j < k$ (define $D_1 = 0$). We say that $(L, D_2, \ldots D_{k-1})$ is the *type* of the extension Z.

Now we specialize to the case of multiplicity structures on a line in $\mathbb{P}^3$, as this is crucial to the classification of curves of degree three. We make some elementary observations before describing the double lines.

LEMMA 1.3. – *Let $Z \subset \mathbb{P}^3$ be a quasiprimitive multiplicity structure on a line Y of type $(L, D_2, \ldots, D_{d-1})$ and degree d. Then*

(*a*) *$L \cong \mathcal{O}_Y(a)$ for some $a \geq -1$.*

(*b*) *If Z has embedding dimension two, then $D_i = 0$ for $1 \leq i \leq d-1$.*

(*c*) *Z is planar if and only if $a = -1$ and $D_i = 0$ for each $1 \leq i \leq d-1$.*

Proof. – The Picard group of Y is $\mathbf{Z}$, generated by $\mathcal{O}_Y(1)$, hence $L \cong \mathcal{O}_Y(a)$ for some $a \in \mathbf{Z}$. The surjection $\mathcal{O}_Y(-1)^2 \cong \mathcal{I}_Y/\mathcal{I}_Y^2 \to L$ shows that $a \geq -1$, proving part (*a*). The exact sequences

$$0 \to \mathcal{I}_{Z_{j+1}} \to \mathcal{I}_{Z_j} \to L^j(D_j) \to 0 \tag{1}$$

show that $p_a(Z) = -a\frac{d(d-1)}{2} - d + 1 - \sum \deg D_j$. This takes on its maximum value of $\frac{1}{2}(d-1)(d-2)$ when $a = -1$ and $D_j = 0$ for all j. On the other hand, [7], theorem 3.1 states that Z is planar if and only if $p_a(Z) = \frac{1}{2}(d-1)(d-2)$, proving part (*c*).

For part (*b*), assume that Z has embedding dimension two. Fixing a point $P \in Z$, the Zariski tangent space of Z at P is two dimensional. It follows that there is an open affine neighborhood U of P and a smooth surface $S \subset U$ which contains $Z \cap U$. Working on S, the line Y is defined by a single equation t at P and hence Z_j is defined by t^j for each $1 \leq j \leq d$. The maps $L^j \to L_j$ are induced from diagrams

$$\begin{array}{ccc} \mathcal{I}_Y^j & \to & \mathcal{I}_{Z_j} \\ \downarrow & & \downarrow \\ L^j & \to & L_j. \end{array}$$

Restricting this diagram to S shows that the top map is surjective. Since the vertical maps are surjective, the bottom map is surjective on U. Since Z is covered by such open sets U, the bottom map is surjective (everywhere) and $D_j = 0$ for $2 \leq j < k$.

The simplest multiple curves are the double lines in $\mathbb{P}^3$.

PROPOSITION 1.4. – *Let Y be the line $\{x = y = 0\}$ in $\mathbb{P}^3$ and let $a \geq -1$ be an integer. Let f and g be two homogeneous polynomials of degree $a+1$ which have no common zeros along Y. Then f and g define a surjection $u : \mathcal{I}_Y \to \mathcal{O}_Y(a)$ by $x \mapsto f, y \mapsto g$. The kernel of u gives the ideal sheaf of a multiplicity two structure Z on Y. Further, we have*

(*a*) $p_a(Z) = -1 - a$

(*b*) $H^1_*(\mathcal{I}_Z) \cong (S/(x, y, f, g))(a)$.

(*c*) $I_Z = (x^2, xy, y^2, xg - yf)$.

(*d*) *If f', g' define another two structure Z', then $Z = Z'$ if and only if there exists $c \in k^*$ such that $f' = cf \operatorname{mod} I_Y$ and $g' = cg \operatorname{mod} I_Y$.*

(*e*) *Each multiplicity two structure Z on Y arises by the construction above.*

Proof. – This can be found in work of Migliore [9] and also by work of Martin-Deschamps and Perrin ([11], IV, example 6.9) in the context of linkage theory. From the above theory of multiplicity structures, we see that giving a double structure Z on Y is equivalent to finding a surjection $u : \mathcal{I}_Y \to \mathcal{L}$, where $\mathcal{L}$ is a line bundle on Y. Since such a surjection must factor through $\mathcal{I}_Y/\mathcal{I}_Y^2 \cong \mathcal{O}_Y(-1)^2$, we see that $\mathcal{L} \cong \mathcal{O}_Y(a)$ with $a \geq -1$ and that the map is given by two polynomials f, g of degree $a+1$.

REMARK 1.5. – If Z is a double line from proposition 1.4 above, the exact sequence

$$0 \to \mathcal{O}_Y(a) \to \mathcal{O}_Z \to \mathcal{O}_Y \to 0$$

shows that Z has local embedding dimension two at each point. In fact, Z is contained in a smooth (global) surface of degree $a+2$. To see this, one can choose the polynomials f, g in the variables z, w. Since f and g have no common zeros along Y, the surface with equation $xg - yf$ is smooth along Y. When $a = -1$, this surface is a (smooth) plane which contains Y. When $a \geq 0$, there exist surfaces of degree $a+2$ which contain Z and are smooth away from Z (take a union of planes). Intersecting these open conditions in $\mathbb{P}H^0(\mathcal{I}_Z(a+2))$, we find that there are surfaces of degree $a+2$ containing Z which are smooth. The general surface of higher degree containing Z will have a finite number of singularities along Y.

COROLLARY 1.6. – *Description of $H(2, g)$:*

(*a*) *If $g > 0$, then $H(2, g)$ is empty.*

(b) If $g = 0$, then $H(2, g)$ is irreducible of dimension 8. All curves in $H(2, g)$ are planar, and the general member is a smooth conic. The reduced reducible curves (two lines meeting at a point) form an irreducible family of dimension 7, and the multiplicity two structures on a line form an irreducible family of dimension 5.

(c) If $g = -1$, then $H(2, g)$ is irreducible of dimension 8. The general curve is a union of two skew lines. The multiplicity two structures on a line form an irreducible family of dimension 7.

(d) If $g < -1$, then $H(2, g)$ is irreducible of dimension 5-2g. All curves are multiplicity two structures on a line with $a = -1 - g$.

Proof. – (*a*) is known, since $\frac{1}{2}(d-1)(d-2)$ is an upper bound on the genus of locally Cohen-Macaulay curves (*see* [7] or [14]). The descriptions of the families of reduced curves is standard. To describe the moduli for the double lines of genus $g \leq 0$, we use Proposition 1.4. The choice of the line Y is given by a 4-dimensional (irreducible) Grassman variety. Given the line Y, the multiplicity structure Z is uniquely determined by the open set of $(f, g) \in H^0(\mathcal{O}_Y(a+1))^2/k^*$ where f and g have no common multiple. This is an irreducible choice of dimension $2a + 3 = 1 - 2g$. Adding the choice of the line Y gives an irreducible family of dimension $5 - 2g$.

In their excellent book [11], Martin-Deschamps and Perrin build a strong foundation for linkage theory of locally Cohen-Macaulay curves in $\mathbb{P}^3$. Perhaps the most important result there is the structure theorem for even linkage classes (*see* [11], IV, Theorem 5.1). It states that if $\mathcal{L}$ is an even linkage class of curves which are not arithmetically Cohen-Macaulay, then there exists a curve $C_0 \in \mathcal{L}$ such that any other curve $C \in \mathcal{L}$ is obtained from C_0 by a sequence of ascending elementary double links (*see* [11], III, Definition 2.1) followed by a deformation with constant cohomology through curves in $\mathcal{L}$. In particular, C_0 achieves the smallest degree and genus among curves in $\mathcal{L}$. C_0 is called a *minimal curve* for $\mathcal{L}$.

A practical aspect of [11] (*see* chapter IV) is an algorithm for finding a minimal curve associated to a finite length graded S-module. If M is such a module, there exists a minimal graded free resolution

$$0 \to L_4 \xrightarrow{\sigma_4} L_3 \xrightarrow{\sigma_3} L_2 \xrightarrow{\sigma_2} L_1 \xrightarrow{\sigma_1} L_0 \to M \to 0$$

which sheafifies to an exact sequence of direct sums of line bundles. Let $\mathcal{N}_0 = \ker\tilde{\sigma}_2$ and $\mathcal{E}_0 = \ker\tilde{\sigma}_3$. The algorithm of Martin-Deschamps and Perrin gives a way to split $\mathcal{L}_2$ into $\mathcal{P} \oplus \mathcal{Q}$, where $\mathcal{P}$ and $\mathcal{Q}$ are also direct sums of lines bundles. If C_0 is a minimal curve, then there exists an integer h_0 and exact sequences

$$\begin{aligned} 0 &\to \mathcal{P} \to \mathcal{N}_0 \to \mathcal{I}_{C_0}(h_0) \to 0 \\ 0 &\to \mathcal{E}_0 \to \mathcal{Q} \to \mathcal{I}_{C_0}(h_0) \to 0. \end{aligned}$$

One consequence of this is that if

$$0 \to L_4 \xrightarrow{\theta} L_3 \to Q \to I_{C_0} \to 0$$

is a minimal graded S-resolution for the total ideal of the minimal curve C_0, then $L_3^\vee \xrightarrow{\theta^\vee} L_4^\vee$ begins a minimal resolution for M^*.

In a later paper [12], Martin-Deschamps and Perrin tackled the problem of bounding the dimensions of the Rao module of a curve in terms of the degree and genus. For a curve C, define the *Rao function* ρ_C by $\rho_C(n) = h^1(\mathcal{I}_C(n))$. If C is not an arithmetically Cohen-Macaulay curve, then r_a (resp. r_o) denotes the smallest (resp. largest) value n for which $\rho_C(n) \neq 0$. The bound can be stated as follows:

PROPOSITION 1.7. – *Let $C \subset \mathbb{P}^3$ be a curve of degree d and genus g which is not arithmetically Cohen-Macaulay. Set*

$$l = d-2, \qquad a = \frac{1}{2}(d-2)(d-3) - g.$$

Then $a \geq 1, l \geq 0$, and the Rao function is bounded by

(1) $r_a \geq -a+1$.

(2) $\rho_C(n) \leq a$ *for* $0 \leq n \leq l$.

(3) $r_o \leq a+l-1$.

Proof. – Suppose that C is not arithmetically Cohen-Macaulay, hence not planar. If $d=2$, then the bound follows easily from Proposition 1.4, so we may assume $d \geq 3$. If char $k=0$, then the result is given in [12] Theorem 2.5 and Corollary 2.6, so we may assume char $k = p > 0$. In this case, the proof given in [12] applies to C if the general plane section $C \cap H$ is not contained in a line. On the other hand, if the general plane section $C \cap H$ is contained in a line, then Hartshorne's restriction theorem ([7], Theorem 2.1) shows that C is supported on a line and has embedding dimension two. Thus it suffices to prove the proposition for multiple lines of embedding dimension two.

Let Y be the supporting line of C, and let $(L, D_2, \ldots, D_{d-1})$ be the type of the extension $Y \subset C$. By Lemma 1.3 (*b*), $D_j = 0$ for $1 \leq j \leq d-1$. Writing $L = \mathcal{O}_Y(a)$ with $a \geq -1$ as in 1.3 (*a*), we have that $a \geq 0$ by 1.3 (*c*). Now we prove the result by induction on *d*. Since the case $d=2$ follows from Proposition 1.4, we procede to the induction step. Letting $\{C_j\}$ be the Cohen-Macaulay filtration of C, we may assume that the proposition holds for C_{d-1}. By [12], Proposition 2.3, it suffices to prove that $h^1(\mathcal{I}_C(n)) \leq \frac{1}{2}(d-2)(d-3) - g$ for $0 \leq n \leq d-2$. From the exact sequence

$$0 \to \mathcal{I}_C \to \mathcal{I}_{C_{d-1}} \to \mathcal{O}_Y((d-1)a) \to 0$$

we find that $h^1(\mathcal{I}_C(n)) \leq h^1(\mathcal{I}_{C_{d-1}}(n)) + (d-1)a + n + 1$ for $n \geq 0$. The exact sequence also shows that $g(C_{d-1}) = g(C) + (d-1)a + 1$. Applying the induction hypothesis to C_{d-1}, we find that

$$h^1(\mathcal{I}_C(n)) \leq \frac{1}{2}(d-3)(d-4) - g(C) + n$$

for $n \geq 0$. For $n \leq d-3$, this proves the bound claimed. For $n = d-2$, [12], Proposition 2.3 shows that $h^1(\mathcal{I}_{C_{d-1}}(d-2))$ is strictly smaller than the bound used above, and we conclude similarly.

The question of sharpness for Proposition 1.7 has a nice answer: equalities in (1), (2) and (3) can be realized by a single curve. Martin-Deschamps and Perrin call such a curve *extremal*. These curves are characterized in Theorem 1.8 below. Notationally, a finite length graded S-module M is said to be a *Koszul module parametrized by* $a \geq 1$ *and* $l \geq 0$ if M is isomorphic to a complete intersection module $S/(f_1, f_2, f_3, f_4)$ with $\deg f_1 = \deg f_2 = 1$, $\deg f_3 = a$ and $\deg f_4 = a + l$.

THEOREM 1.8. – *Characterization of extremal curves:*

(*a*) *Fix* $a \geq 1, l \geq 0$, *and let* M *be a Koszul module parametrized by* a *and* l. *Then any minimal curve for the even linkage class* $\mathcal{L}(M)$ *is an extremal curve of degree* $d = l + 2$ *and genus* $g = -a + \frac{1}{2}(d-2)(d-3)$.

(*b*) *Conversely, let* $C \subset \mathbb{P}^3$ *be an extremal curve of degree* $d \geq 2$ *and genus* $g < \frac{1}{2}(d-2)(d-3)$. *If* $l = d - 2$ *and* $a = \frac{1}{2}(d-2)(d-3) - g$, *then* C *is a minimal curve for a Koszul module parametrized by* a *and* l.

Proof. – Part (*a*) can be calculated from the minimal curve algorithm above (*see* [13], Proposition 0.5 for details). Part (*b*) is proven when char $k = 0$ in [13], Theorem 1.1, however their proof applies to any curve C whose general plane section $C \cap H$ is not contained in a line. If char $k = p > 0$, C is not planar, and $C \cap H$ is contained in a line for general H, then C is a multiplicity structure on a line Y of embedding dimension two as in the proof of Proposition 1.7 above. In particular, if C is of type $(L, D_2, \ldots, D_{d-1})$, then $D_j = 0$ for each j and $L \cong \mathcal{O}_Y(a)$ with $a \geq 0$. In this case the exact sequences (1) show that $r_a = -a(d-1) > \frac{1}{2}(d-2)(d-3) - g(C) = \frac{1}{2}(d-2)(d-3) - d + 1 - \frac{1}{2}d(d-1)a$, hence C is not extremal, and the proof is complete.

THEOREM 1.9. – *Let* $d \geq 3$ *and* $g < \frac{1}{2}(d-2)(d-3)$ *and assume that char* $k = 0$. *Then the family of extremal curves forms an irreducible component of the Hilbert scheme* $H(d, g)$ *of dimension* $\frac{3}{2}d(d-3) + 9 - 2g$. *This component is nonreduced except when* $(d, g) = (3, -1)$.

This is proved in [13], Theorem 4.2 and Theorem 4.3. I can find no use of the hypothesis char $k = 0$, so probably this statement holds in finite characteristic as well. We will see later that the extremal curves always give an irreducible component of dimension $9 - 2g$ when $d = 3$ (*see* Propositions 3.1, 3.4 and 3.5).

2. Triple Lines in $\mathbb{P}^3$

In this section we classify the multiplicity three structures on a fixed line $Y \subset \mathbb{P}^3$. If W is a quasiprimitive multiplicity three structure of type (L, D_2), then we have two exact sequences

$$0 \to \mathcal{I}_Z \to \mathcal{I}_Y \to \mathcal{O}_Y(a) \to 0 \tag{2}$$

$$0 \to \mathcal{I}_W \to \mathcal{I}_Z \to \mathcal{O}_Y(2a + b) \to 0 \tag{3}$$

where Z is one of the multiplicity two structures on Y described in Proposition 1.4, $L = \mathcal{O}_Y(a), a \geq -1, \deg D_2 = b \geq 0$. We loosely say that W is of *type* (a, b).

In classifying the triple lines of type (a, b), we will handle the case $a = -1$ separately. This is because the corresponding double line Z is a complete intersection when $a = -1$, while this is not the case for $a > 0$.

PROPOSITION 2.1. – *Let $Y \subset \mathbb{P}^3$ be the line $\{x = y = 0\}$ and let Z be the multiplicity two structure $\{x = y^2 = 0\}$ on Y. Let p, q be two homogeneous polynomials of degrees $b-1, b$ which have no common zeros along Y. Then p and q define a surjection $u : \mathcal{I}_Z \to \mathcal{O}_Y(b-2)$ by $x \mapsto p, y^2 \mapsto q$. The kernel of u is the ideal sheaf of a multiplicity three structure W on Y. Further, we have*

(*a*) $p_a(W) = 1 - b$

(*b*) $H^1_*(\mathcal{I}_W) \cong (S/(x, y, p, q))(b-2)$

(*c*) $I_W = (x^2, xy, y^3, xq - y^2p)$

(*d*) *If p', q' define another three structure W', then $W = W'$ if and only if there exists $c \in k^*$ such that $p' = cp \operatorname{mod} I_Y$ and $q' = cq \operatorname{mod} I_Y$.*

(*e*) *W is quasiprimitive with second CM filtrant Z, unless $b = 1$ and $q = 0$, in which case $W = Y^{(2)}$.*

Proof. – Since Z is a global complete intersection with total ideal (x, y^2), $I_Z \otimes S_Y = I_Z/I_Z I_Y \cong S_Y(-1) \oplus S_Y(-2)$ is freely generated by the images of x and y^2. The map $x \mapsto \overline{p}, y^2 \mapsto \overline{q}$ defines a graded homomorphism $\phi : I_Z \to I_Z/I_Z I_Y \to S_Y(b-2)$. Since $(\overline{p}, \overline{q})$ form a regular sequence in S_Y, the kernel of the map $I_Z/I_Z I_Y \to S_Y(b-2)$ is given by the Koszul relation $x\overline{q} - y^2\overline{p}$, hence $\ker\phi = (I_Z I_Y, xq - y^2p) = (x^2, xy, y^3, xq - y^2p)$. The cokernel $\operatorname{coker}\phi = S_Y(b-2)/(\overline{p}, \overline{q}) \cong (S/(x, y, p, q))(b-2)$ has finite length, hence ϕ sheafifies to a surjection $u : \mathcal{I}_Z \to \mathcal{O}_Y(b-2)$.

Letting W be the subscheme defined by $\mathcal{I}_W = \ker u$, we have an exact sequence

$$0 \to \mathcal{I}_W \to \mathcal{I}_Z \to \mathcal{O}_Y(b-2) \to 0.$$

Since $H^0_*(u) = \phi$ and $H^1_*(\mathcal{I}_Z) = 0$, we immediately deduce properties (*b*) and (*c*). The snake lemma provides a second exact sequence

$$0 \to \mathcal{O}_Y(b-2) \to \mathcal{O}_W \to \mathcal{O}_Z \to 0,$$

which shows that W is supported on Y and that $\operatorname{depth}\mathcal{O}_W \geq 1$, hence W is a CM multiplicity three structure on Y with $p_a(W) = 1 - b$. If the polynomials p', q' also define W by the construction above, then (q, p) and (q', p') generate the same principal S_Y-submodule of $S_Y(-1) \oplus S_Y(-2)$, hence property (*d*) holds.

If $q = 0$, then p must be a unit, as otherwise p and q will have common zeros along Y. In this case, we see that $b - 1 = \deg p = 0$ and that $I_W = I_Y^2$, whence $W = Y^{(2)}$. If $q \neq 0$, we use part (*c*) to see that $I_W + I_Y^2 = (x^2, xy, y^2, xq) = (I_Y^2, xq)$. At the points on Y where $q \neq 0$, this ideal is simply (y^2, x), so the cokernel of the inclusion $(I_Y^2, xq) \subset (y^2, x)$ has finite support. Since the latter ideal defines the multiplicity two structure Z on Y, we see that Z is the second CM filtrant for W, proving part (*e*).

COROLLARY 2.2. – *Triple lines of type $a = -1, b \geq 0$: Let W be a quasiprimitive multiplicity three structure on a line Y of type $(-1, b)$ or the second infinitesimal* *neighborhood* $Y^{(2)}$. *Then, after a suitable change of coordinates,* W *is constructed by Proposition* 2.1. *The family of such multiplicity three structures is irreducible of dimension* $5+2b$.

Proof. – $W=Y^{(2)}$ is given by the construction in taking $b=1, q=0, p=1$. If W is quasiprimitive, then we have the exact sequence 3, and the construction above gives all surjections $u:\mathcal{I}_Z\to\mathcal{O}_Y(b-2)$. To parametrize this family, we first choose the double line Z (an irreducible choice of dimension 5 by Corollary 1.6), and then we must choose $(p,q)\in H^0(\mathcal{O}_Y(b-1))\times H^0(\mathcal{O}_Y(b))/k^*$, which is an irreducible choice of dimension $2b$. Thus the family is irreducible of dimension $5+2b$.

PROPOSITION 2.3. – *Let* $Z\subset\mathbb{P}^3$ *be the double line with total ideal* $I_Z=(I_Y^2, xg-yf)$, *where* Y *is the line* $\{x=y=0\}$ *and* f,g *are homogeneous polynomials of degree* $a+1$ *having no common zeros along* Y, *as in Proposition* 1.4. *Let* p *and* q *be homogeneous polynomials of degrees* $b, 3a+b+2$ *having no common zeros along* Y. *Then* p *and* q *define a surjection* $u:\mathcal{I}_Z\to\mathcal{O}_Y(2a+b)$ *by* $x^2\mapsto pf^2, xy\mapsto pfg, y^2\mapsto pg^2$ *and* $xg-yf\mapsto q$. *The kernel of* u *is the ideal sheaf of a quasiprimitive multiplicity three structure on* Y *with second Cohen-Macaulay filtrant* Z. *Further, we have*

(*a*) $p_a(W)=-2-3a-b$.

(*b*) $I_W=(I_Y^3, x(xg-yf), y(xg-yf), p(xg-yf)-rx^2-sxy-ty^2)$, *where* r,s,t *are chosen so that* $q=rf^2+sfg+tg^2 \operatorname{mod} I_Y$.

(*c*) *If* p',q' *define the multiplicity three structure* W', *then* $W=W'$ *if and only if there exists* $d\in k^*$ *such that* $p'=dp\operatorname{mod}I_Y$ *and* $q'=dq\operatorname{mod}I_Y$.

Proof. – The ideal $I_Z=(x^2,xy,y^2,xg-yf)$ has S-presentation

$$S(-3)^2\oplus S(-a-3)^2\xrightarrow{\varphi}S(-2)^3\oplus S(-a-2)\to I_Z\to 0$$

given by the matrix

$$\varphi=\begin{pmatrix} y & 0 & -g & 0\\ -x & y & f & -g\\ 0 & -x & 0 & f\\ 0 & 0 & x & y\end{pmatrix}.$$

Tensoring with S_Y, we see that $I_Z/I_ZI_Y=\operatorname{coker}\varphi\otimes S_Y$ is isomorphic to $S_Y(-a-2)\oplus(f^2,fg,g^2)(2a)$, where $\overline{x}^2,\overline{xy},\overline{y}^2$ are identified with f^2,fg,g^2. Making this identification, we have an inclusion $I_Z/I_ZI_Y\subset S_Y(-a-2)\oplus S_Y(2a)$ whose cokernel has finite length. It follows that the sheafification of I_Z/I_ZI_Y is isomorphic to $\mathcal{O}_Y(-a-2)\oplus\mathcal{O}_Y(2a)$, freely generated by $xg-yf$ and an element e such that $ef^2=\overline{x}^2, efg=\overline{xy}$ and $eg^2=\overline{y}^2$.

The polynomials p and q give a graded homomorphism

$$\phi: I_Z\to I_Z/I_ZI_Y\subset S_Y(-a-2)\oplus S_Y(2a)\xrightarrow{(\overline{q},\overline{p})}S_Y(2a+b).$$

The kernel of the map $(\overline{q},\overline{p})$ is given by the Koszul relation $qe-p(xg-yf)$. Since f and g are relatively prime of degree $a+1$, the map $S_Y(-2a-2)^3\to S_Y$ given by (f^2, fg, g^2) is surjective in degrees $\geq 3(a+1)-1$, and hence there exist (r,s,t) such that $q = rf^2 + sfg + tg^2 \mathrm{mod} I_Y$ (because $\deg q \geq 3a+2$). We can now write $(rx^2 + sxy + ty^2 - p(xg-yf)) = I_Z/I_Z I_Y \cap \ker(\overline{q}, \overline{p})$, and hence

$$\ker\phi = (x^3, x^2y, xy^2, y^3, x(xg-yf), y(xg-yf), rx^2 + sxy + ty^2 - p(xg-yf)).$$

The cokernel of ϕ is of finite length, so ϕ sheafifies to a surjection $u : \mathcal{I}_Z \to \mathcal{O}_Y(2a+b)$.

Letting W be the subscheme whose ideal sheaf is the kernel of u, we get an exact sequence

$$0 \to \mathcal{O}_Y(2a+b) \to \mathcal{O}_W \to \mathcal{O}_Z \to 0$$

which shows that $\mathrm{Supp}W = Y$ and $\mathrm{depth}\mathcal{O}_W \geq 1$, hence W is a multiplicity three structure on Y. Since $p_a(Z) = -1-a$, the exact sequence shows that $p_a(W) = -2-3a-b$. The exact sequence

$$0 \to \mathcal{I}_W \to \mathcal{I}_Z \to \mathcal{O}_Y(2a+b) \to 0$$

shows that $I_W = \ker\phi$. If p' and q' define W' by the above construction and $W' = W$, then $eq' - (xg-yf)p'$ generates the same S_Y-submodule of $I_Z/I_Z I_Y \subset S_Y(-a-2) \oplus S_Y(2a)$ as $eq - (xg-yf)p$. Since e and $xg - yf$ are free generators, it follows that there exists $d \in k^*$ such that $p' = dp \mathrm{mod} I_Y$ and $q' = dq \mathrm{mod} I_Y$. This proves (*a*),(*b*) and (*c*).

From part (*b*), we find that $I_W + I_Y^2 = (I_Y^2, h(xg-yf))$. The cokernel of the inclusion $(I_Y^2, h(xg-yf)) \subset I_Z$ is supported on the zeros of h along Y. Since Z has no embedded points, the second CM filtrant of W is Z and the extension $Y \subset W$ is quasi-primitive.

REMARK 2.4. – Note that if $u : \mathcal{I}_Z \to \mathcal{O}_Y(2a+b)$ is the surjection above, then $H^1_*(u)$ is the zero map. Indeed, $H^1_*(\mathcal{I}_Z)$ is generated in degree $-a$ by Proposition 1.4 while $H^1(\mathcal{O}_Y(a+b)) = 0$. In particular, we have an exact sequence

$$0 \to \mathrm{coker}\phi \to M_W \to M_Z \to 0$$

which shows that the Rao module M_W is 2-generated. Since all curves of degree ≤ 2 have Rao modules which are zero or principal, W is a minimal curve. Further, the exact sequences 2 and 3 now give that

$$h^2(\mathcal{I}_W(l)) = h^1(\mathcal{O}_Y(l)) + h^1(\mathcal{O}_Y(a+l)) + h^1(\mathcal{O}_Y(2a+b+l))$$

for all $l \in \mathbf{Z}$.

From the total ideal I_W given in part (*b*), one can compute a minimal graded S-resolution for I_W, which has the form

$$\begin{array}{c} S(-a-b-4) \oplus S(-a-5)^2 \\ \downarrow^{\theta} \\ S(-a-b-3)^2 \oplus S(-a-4)^4 \oplus S(-4)^3 \\ \downarrow \\ S(-a-b-2) \oplus S(-a-3)^2 \oplus S(-3)^4 \end{array}$$

This resolution determines $h^0(\mathcal{I}_W(l))$ for all $l \in \mathbf{Z}$. Combining with the dimensions $h^2(\mathcal{I}_W(l))$ found above, all the $h^i(\mathcal{I}_W(l))$ can be computed.

The machinery for minimal curves of Martin-Deschamps and Perrin shows that $\theta^\vee$ begins a minimal resolution for M_W^*. Completing this resolution and dualizing the last map gives a presentation for M_W. Carrying this out (we suppress the calculation here), one finds that $H^1_*(\mathcal{I}_W) \cong \operatorname{coker}\psi$, where

$$\psi : S(2a+b-1)^2 \oplus S(a-1)^2 \oplus S(-1)^2 \to S(2a+b) \oplus S(a)$$

is the map given by the matrix

$$\begin{pmatrix} x & y & fp & gp & -gt & -fr-gs \\ 0 & 0 & x & y & f & g \end{pmatrix}$$

Here r, s and t are chosen as in part (*b*) of the proposition.

REMARK 2.5. – In the case when $b = 0$, $\overline{p}$ must be a unit. It follows that the generators $x(xg - yf)$ and $y(xg - yf)$ are not needed for the total ideal I_W (*see* also [1], p. 24). In this case it is clear that W is the unique triple line supported on Y and contained in the surface defined by $rx^2 + sxy + ty^2 - p(xg - yf)$.

COROLLARY 2.6. – *Triple lines of type* $a, b \geq 0$*: Let* W *be a quasi-primitive multiplicity three structure on a line* $Y \subset \mathbb{P}^3$ *of type* (a, b) *with* $a, b \geq 0$*. Then* W *arises from the construction of Proposition* 2.3 *after a suitable change of coordinates. The family of these triple lines is irreducible of dimension* $10 + 5a + 2b$.

Proof. – Since W is of type (a, b) with $a \geq 0$, there is an exact sequence

$$0 \to \mathcal{I}_W \to \mathcal{I}_Z \xrightarrow{u} \mathcal{O}_Y(2a+b) \to 0$$

where Z is a double line of type $a \geq 0$. By Proposition 1.4, we may change coordinates so that $I_Z = (x^2, xy, y^2, xf - yg)$ where f, g are homogeneous polynomials of degree $a + 1$ with no common zeros along Y. As in the proof of 2.3 above, $\mathcal{I}_Z \otimes \mathcal{O}_Y \cong \mathcal{O}_Y(2a) \oplus \mathcal{O}_Y(-a-2)$ is freely generated by e and the image of $xg - yf$, where $ef^2 = x^2, efg = xy$ and $eg^2 = y^2$. From this it is clear that such a map u is given by homogeneous polynomials p, q of degrees $b, 3a + b + 2$ which have no common zero along Y, and hence W arises by the construction of Proposition 2.3.

3. The Hilbert Scheme

In this section we describe the Hilbert scheme $H(3, g)$ of locally Cohen-Macaulay curves of degree 3 and arithmetic genus $g \leq 1$. In particular, we classify all CM curves of degree 3 and describe the irreducible components of $H(3, g)$. We also show that certain extremal curves lie in the closure of each irreducible component, hence that $H(3, g)$ is connected. We begin with the curves of genus $-1 \leq g \leq 1$, which have been described elsewhere.

PROPOSITION 3.1. – *For* $-1 \leq g \leq 1$*, the Hilbert scheme* $H(3, g)$ *is smooth and irreducible of dimension* 12.

Proof. – $H(3,1)$ parametrizes plane cubic curves, and is a $\mathbb{P}^9$ bundle over $(\mathbb{P}^3)^*$, hence is irreducible of dimension 12. That $H(3,0)$ and $H(3,-1)$ are smooth and irreducible of dimension 12 is part of [13], Theorem 4.1. $H(3,0)$ consists of arithmetically Cohen-Macaulay curves and $H(3,-1)$ consists of extremal curves.

For $g \leq -2$, the Hilbert scheme is not irreducible, and more work is required to show connectedness. Our first task is to describe how the unions of double lines and reduced lines fit in with the irreducible families of triple lines.

PROPOSITION 3.2. – *Fix $g \leq -2$. Then*

(*a*) *The family of curves $W = Z \cup_{2P} L$ formed by taking the union of a double line Z with $p_a(Z) = g - 1$ and a line L which meets Z in a double point form an irreducible family of dimension $9 - 2g$.*

(*b*) *The family of curves $W = Z \cup_P L$ formed by taking the union of a double line Z with $p_a(Z) = g$ and a line L which meets Z in a reduced point form an irreducible family of dimension $8 - 2g$.*

(*c*) *The family of curves W which are triple lines of type $(-1, 1-g)$ form an irreducible family of dimension $7 - 2g$.*

Each curve above is an extremal curve, hence is a minimal curve for a Koszul module parametrized by $a = -g$ and $l = 1$. The families (b) and (c) lie in the closure of the family (a).

Proof. – Let $W = Z \cup_{2P} L$ be a curve from family (*a*) above. After a change of coordinates we may write $I_L = (x, z)$ and $I_Z = (x^2, xy, y^2, xg - yf)$. We have an exact sequence

$$0 \to \mathcal{I}_W \to \mathcal{I}_Z \oplus \mathcal{I}_L \xrightarrow{\pi} \mathcal{I}_{2P} \to 0$$

where $2P = Z \cap L$ denotes the double point. Noting that $I_L + I_Z = (x, z, y^2, yf)$ and that $I_{2P} = (x, z, y^2)$ ($2P$ is a complete intersection), we see that $H_0^*(\pi)$ is surjective. Since $H_*^1(\mathcal{I}_{2P})$ vanishes in positive degrees, we conclude that $r_o(W) = r_o(Z) = -g$ and that $\rho_W(1) = -g$. The complete intersection $Z(x^2, y^2 z)$ links W to $W' = Z' \cup_{2Q} L$, which is also from family (*a*). Applying the argument above and using the isomorphism $M_{W'}^* \cong M_W(1)$ shows that $r_a(W) = 1 + g$ and $\rho_W(0) = -g$. Thus W is extremal.

To parametrize this family of curves, one first chooses the double line Z (an irreducible choice of dimension $7 - 2g$ by Corollary 1.6, since $p_a(Z) = g - 1$), then a point $P \in Z$ (1 parameter), and finally a line L through P lying in the tangent plane to Z at P (1 parameter). This shows that this family is irreducible of dimension $9 - 2g$.

The argument for $W = Z \cup_P L$ is similar. We choose suitable coordinates and write $I_L = (x, z), I_Z = (x^2, xy, y^2, xg - yf)$. We have an exact sequence

$$0 \to \mathcal{I}_W \to \mathcal{I}_Z \oplus \mathcal{I}_L \xrightarrow{\pi} \mathcal{I}_P \to 0$$

where $P = Z \cap L$. Writing $I_P = (x, y, z)$ and $I_L + I_Z = (x, z, y^2, yf)$ we see that $\dim \operatorname{coker} H_*^0(\pi(l)) = 1$ for $1 \leq l \leq -g = \deg yf - 1$. It follows again that $r_o(W) = r_o(Z) - 1 = -g$ and $\rho_W(1) = \rho_Z(1) - 1 = -g$. The complete intersection $Z(x^2, y^2 z)$ links W to a curve W' from family (*b*), so we find that W is extremal.

To parametrize these curves, we first choose a double line Z (an irreducible choice of dimension $5-2g$, since $p_a(Z)=g$), and then choose a general line L which meets Z (3 parameters). This shows that (b) is an irreducible family of dimension $8-2g$.

If W is a triple line of type $(-1,1-g)$, then from Corollary 2.2 we have $M_W=(S/(x,y,p,q))(-1-g)$ where $\deg p=1-g$ and $\deg q=2-g$. It follows that W is extremal. The family of such triple lines W is irreducible of dimension $7-2g$ by Corollary 2.2.

Let W be a curve in family (b) or (c). Since W is extremal, W is a minimal curve for a Koszul module M parametrized by $a=-g$ and $l=1$ by Theorem 1.8 (b). By [13], Proposition 0.6, we may change variables to write $M_W=S/(x,y,F,G)$ and $I_W=(x^2,xy,hy^2,xG+hyF)$ for some $h\in k[y,z,w]$ (here we have $\deg F=a, \deg G=a+1, \deg h=1$). There exists $h_0\in k[y,z,w]$ such that Y does not divide h_0 and h_0 does not divide G. With this choice, the ideal

$$I_{W_0}=(x^2,xy,h_0y^2,xG+h_0yF)=(x,h_0)\cap(x^2,xy,y^2,xG+h_0yF)$$

gives a curve W_0 from family (a) (note that the sum of the ideals intersected is (x,h_0,y^2), which defines a double point of the intersection). In $\mathbb{P}(k[y,z,w]_1)$ we can find a straight line $\mathbb{P}^1$ which goes through h and h_0. An open subset $U\subset\mathbb{P}^1$ parametrizes a family of curves in $H_{\gamma,M}$ whose special member is W and whose general member is a curve from family (a) (because the conditions that Y not divide h_0 and h_0 not divide G are open in $\mathbb{P}(k[y,z,w]_1)$.).

PROPOSITION 3.3. – *Fix $g\leq -2$. Then*

(a) *The family of curves $W=Z\cup L$ formed by taking the union of a double line Z with $p_a(Z)=g+1$ and a disjoint line L form an irreducible family of dimension $7-2g$.*

(b) *The family of curves W which are triple lines of type $(0,-2-g)$ form an irreducible family of dimension $6-2g$.*

The curves above are all minimal, and each curve in family (b) is obtained from curves in family (a) by a deformation which preserves cohomology.

Proof. – Let $W=Z\cup L$ be a curve from family (a) above. We begin by computing the total ideal and cohomology for W. In suitable coordinates, we may write $I_L=(z,w), I_Z=(x^2,xy,y^2,xg-yf)$ with $g,f\in k[z,w]$. In particular, $xg-yf\in I_L$ and hence $J=((x,y)^2(z,w),xg-yf)\subset I_L\cap I_Z$. One can compute that the minimal graded S-resolution of J is of the form

$$S(g-2)\oplus S(-5)^2\to S(g-1)^2\oplus S(-4)^7\to S(g)\oplus S(-3)^6$$

and hence J is the total ideal for W. Comparing with the resolution of Remark 2.4 (with $a=0, b=-2-g$) shows that W has the same Hilbert function as a curve in family (b). Moreover, the exact sequence

$$0\to\mathcal{I}_W\to\mathcal{I}_Z\oplus\mathcal{I}_L\to\mathcal{O}\to 0$$

shows that $h^2(\mathcal{I}_W(l)) = h^2(\mathcal{I}_L(l)) + h^2(\mathcal{I}_Z(l)) = h^1(\mathcal{O}_L(l)) + h^1(\mathcal{O}_Y(l)) + h^1(\mathcal{O}_Y(-g-2+l))$. This agrees with the second cohomology dimensions found in Remark 2.4 for triple lines of type $(0, -2-g)$, hence the dimensions of the cohomology groups for families (*a*) and (*b*) are the same. This same exact sequence also shows that M_W is 2-generated.

The family (*a*) is parametrized by first choosing Z (an irreducible choice of dimension $3-2g$ by Corollary 1.6) and then choosing a general line L (4 parameters), hence the family (*a*) is irreducible of dimension $7-2g$. Family (*b*) is irreducible of dimension $6-2g$ by Corollary 2.6. These curves are minimal because they are of degree three and their Rao modules are 2-generated.

Let W be a triple line from family (*b*). By Corollary 2.6, we can change coordinates and write $I_W = ((x,y)^3, xq, yq, hq - ax^2 - bxy - cy^2)$, where $q = xg - yf$ is a quadric surface containing the underlying second CM filtrant Z. By Remark 1.5, q may be chosen to be the equation of a smooth quadric Q. We may choose z and w so that $q = xz - yw$.

On the smooth quadric Q the family of lines $L_t = Z(x + wt, y + zt)$ give a flat family over $\mathbb{A}^1$ with $L_0 = Y$. $D_t = L_t \cup Y$ forms a flat family such that $D_0 = Z$ is the double line Z on Q supported on Y, the second Cohen-Macaulay filtrant of W. Writing this family as $D \subset \mathbb{P}^3 \times \mathbb{A}^1 \xrightarrow{\pi} \mathbb{A}^1$, we see by Grauert's theorem, $\pi_*(\mathcal{I}_D(-g))$ is locally free on $\mathbb{A}^1$, hence globally free. In particular, if $s_1 \in \mathcal{I}_{D_1}(-g)$ is the equation of a smooth surface containing $D_1 = L_1 \cup Y$, we can find a section s_t extending s_1 such that $s_0 = hq - ax^2 - bxy - cy^2$.

Now consider the family $C_t = S_t \cap (Y^{(2)} \cup L_t)$. Let $U \subset \mathbb{A}^1$ be the open set where C_t is locally Cohen-Macaulay. For $t \neq 0$, C_t is the disjoint union of a double line on Y and the line L_t. The ideal of C_t is given by $I_t = ((x,y)^2(x + wt, y + zt), s_t)$. Note that $x^2(y + zt) - xy(x + wt) = xqt \in I_t$ and similarly $yqt \in I_t$. Flattening over U, we must add xq and yq to I_t. In particular, the limit ideal I_0 contains $((x,y)^3, xq, yq, s_0)$, and hence gives W.

PROPOSITION 3.4. – *The Hilbert scheme* $H(3,-2)$ *consists of the following two irreducible components:*

(*a*) *The irreducible family* H_{-1} *of dimension* 13 *from Proposition* 3.2.

(*b*) *The closure* H_0 *of the irreducible family of sets of three disjoint lines. This closure is* 12*-dimensional and contains the curves from Proposition* 3.3.

Proof. – Clearly both of these families are irreducible of the dimensions claimed. It suffices that these two families give all of $H(3,-2)$ and are irreducible components. Let $W \in H(3,-2)$. Since $g(W) < 0$, W is not integral, hence is reducible or nonreduced. If W is reduced, then W is the union of 3 disjoint lines (any union of a conic and line has $g \geq -1$), hence lies in family (*b*).

If W is not reduced, then $\deg \operatorname{Supp} W < 3$. If $\deg \operatorname{Supp} W = 2$, then W is a union of a double line Z and a reduced line L (the support of W cannot be irreducible, since a multiple conic has degree ≥ 4). If Z misses L, then $p_a(Z) = -1$ and hence Z is a limit of pairs of skew lines by Corollary 1.6. It follows that Z is in the closure of family (*b*). If Z meets L in a scheme of length 1 or 2, then Z is in family (*a*) by Proposition 3.2. If $\deg \operatorname{Supp} W = 1$, then W is a triple structure on a line Y of arithmetic genus -2, which must be quasiprimitive (otherwise $W = Y^{(2)}$, when $g(W) = 0$). The only possible types are $(-1,3)$ or $(0,0)$, which are limits of unions $Z \cup L$ described above by Propositions 3.2 and 3.3, hence these lie in families (*a*) and (*b*) respectively. Family (*a*) cannot lie in the closure of family (*b*) by reason of dimension. Family (*b*) cannot lie in the closure of family (*a*) by reason of semicontinuity; the curves in family (*a*) are extremal, while the curves in family (*b*) are not (If C is in family (*b*), one checks that $h^1(\mathcal{I}_C(-1)) = 0$).

PROPOSITION 3.5. – *Let $g \leq -3$. Then the Hilbert scheme $H(3,g)$ consists of the following irreducible components:*

(*a*) *The irreducible family of dimension $9-2g$ from Proposition* 3.2, *which we now denote H_{-1}.*

(*b*) *The closure of the irreducible family of dimension $7-2g$ from Proposition* 3.3, *which we now denote H_0.*

(*c*) *For each $0 < a \leq (-2-g)/3$, the closure of the irreducible family H_a of dimension $6-2g-a$ consisting of triple lines of type $(a,-2-3a-g)$.*

Proof. – Let $C \in H(3,g)$. Then C is not integral because $g \leq -3$. If C were reduced, it would be a union of 3 lines (these have genus ≥ -2, hence are ruled out) or the union of a conic and a line (which has genus ≥ -1, hence is ruled out). Thus C is not reduced and $\dim \operatorname{Supp} C < 3$. If C has support of degree 2, the support cannot be irreducible, since a multiplicity structure on a conic has degree at least 4. Hence the support of C consists of two lines, and all possible configurations are covered in families (*a*) and (*b*) above. If C has support of degree 1, then C is a triple line and Corollaries 2.2 and 2.6 show that C is among the families listed above.

Now we show that the H_i are irreducible components. Let $-1 \leq i < j \leq (-2-g)/3$. H_i is not contained in the closure of H_j because $\dim H_i > \dim H_j$. On the other hand, semicontinuity shows that H_j is not contained in the closure of H_i. Indeed, from Corollaries 2.2 and 2.6, we see that the Rao module for a triple line of type $(a,\, b)$ has a generator of minimal degree $-2a-b$, and hence a minimal degree generator for the Rao module of a curve in H_a occurs in degree $g+2+a$. This shows that for $C \in H_i$ we have $h^0(\mathcal{O}_C(g+2+i)) \neq 0$ while for $C \in H_j$ we have $h^0(\mathcal{O}_C(g+2+i)) = 0$. Hence there can be no specialization from a family of curves in H_i to a curve in H_j.

PROPOSITION 3.6. – *For each $a \geq 0$ and $b \geq 0$, there exists a flat family $W \subset \mathbb{P}^3 \times \mathbb{A}^1$ whose general member W_t is a triple line of type (a,b) for $t \neq 0$ and whose special member W_0 is a triple line of type $(-1, 3a+b+3)$.*

Proof. – Consider the family defined by the ideal I_t with generators

$$x^3, x^2y, xy^2, y^3, x(xz^{a+1} - tyw^{a+1}), y(xz^{a+1} - tyw^{a+1}),$$

$$z^b t^2 (xz^{a+1} - tyw^{a+1}) - x^2 w^{a+b}.$$

We flatten this family over t by adding to the ideal those elements p such that $pt \in I_t$. Let A, B, C denote the last three generators given for the ideal. Then we must add

$$D = (w^{a+b}A + z^{a+1}C)/t = -xyw^{2a+b+1} + z^{a+b+1}t(xz^{a+1} - tyw^{a+1})$$

to the ideal. We must also add

$$E = (w^{2a+b+1}B + z^{a+1}D)/t = -y^2w^{3a+b+2} + z^{2a+b+2}(xz^{a+1} - tyw^{a+1})$$

to the ideal. Setting $t = 0$, we find that the limit ideal I_0 contains the generators

$$x^3, x^2y, xy^2, y^3, xyz^{a+1}, x^2w^{a+b}, xyw^{2a+b+1}, xz^{3a+b+3} - y^2w^{3a+b+2}.$$

It follows that the saturation of I_0 contains the ideal

$$(x^2, xy, y^3, xz^{3a+b+3} - y^2w^{3a+b+2}),$$

but this is the total ideal of a triple structure of type $(-1, 3a+b+3)$ by Corollary 2.2. On the other hand, Corollary 2.6 shows that the ideal I_t for $t \neq 0$ is the total ideal of a triple line of type (a, b). This gives the flat family W.

REMARK 3.7. – The commutative algebra in the proof above was inspired by a geometric example of Robin Hartshorne. He gave an example of a deformation of three disjoint lines to a triple line of type $(-1, 3)$ by deforming the unique quadric containing the three lines to a double plane while at the same time bringing the lines together.

THEOREM 3.8. – *The Hilbert scheme $H(3, g)$ is connected if it is nonempty.*

Proof. – By Proposition 3.1, it suffices to consider the case $g \leq -2$. In this case $H(3, g)$ has irreducible components $\{H_a\}_{a \geq -1}$ by Propositions 3.4 and 3.5. Let H_a be one of these components with $a \geq 0$. Choosing $b = -2 - 3a - g$, Proposition 3.6 gives a family of triple lines whose general member lies in H_a and whose special member lies in H_{-1}.

REMARK 3.9. – The Proof of Proposition 3.6 shows that a triple line W with total ideal $(x^2, xy, y^3, xz^{1-g} - y^2w^{-g})$ lies in the closure of each irreducible component of $H(3, g)$.

EXAMPLE 3.10. – Hartshorne has shown that the Hilbert scheme $H(4, 0)$ is also connected. Here we give an independent proof using the methods of this paper. $H(4, 0)$ has two irreducible components ([13], §4): $H_1 =$ the extremal curves (these have Rao module of Koszul type parametrized by $a = 1$ and $l = 2$) and $H_2 =$ the curves with Rao module k in degree 1. We will give a specialization from quadruple lines in H_2 to quadruple lines in H_1.

Let Y be the line $\{x = y = 0\}$ and W be the planar triple line with total ideal $I_W = (x, y^3)$. As in Proposition 2.2, a pair (h, k) of homogeneous polynomials of degrees 1 and 3 with no common zeros along Y determines a map $I_W \xrightarrow{\phi} S_Y$ by $x \mapsto h, y \mapsto k$ which sheafifies to a surjection u. $\ker u = \mathcal{I}_T$ defines a multiplicity four line Z_1 such that $p_a(Z_1) = 0$, $I_{Z_1} = (x^2, xy, y^4, xk - y^2h)$, and $H^1_*(\mathcal{I}_{Z_1}) \cong S/(x, y, h, k)$. It follows that $Z_1 \in H_1$.

Letting V be a quasiprimitive multiplicity three structure of type $(-1, 1)$ on Y, Proposition 2.2 shows that we may write $I_V = (x^2, xy, xq - y^2)$, with $q \notin I_Y$ (p is unit in this case). As in the proof of Proposition 2.6, a pair (f, g) of forms with no common zeros along Y determines a map $I_V \to S_Y(-1)$ by $x^2 \mapsto 0, xy \mapsto f, xq - y^2 \mapsto g$ which sheafifies to a surjection w. $\ker w = \mathcal{I}_{Z_2}$ defines a multiplicity four line Z_2 such that $p_a(Z_2) = 0$,

$I_{Z_2} = (x^2, xy^2, xyq - y^3, gxy - f(xq - y^2))$ and $H^1_*(\mathcal{I}_{Z_2}) \cong S/(x, y, f, g))(-1)$, hence $Z_2 \in H_2$.

Now consider the ideal

$$I_t = (x^2, xy^2, ty^3 - xyz, xyw - tz(y^2 t - xz))$$

in the ring $k[t][x, y, z, w]$. For $t \neq 0$, this gives the total ideal of a curve in H_2 (see Z_2 above). Flattening over t, we add to this ideal the polynomials p such that $pt \in I_t$. Letting A, B, C denote the last three generators listed, we add

$$D = (wB + zC)/t = y^3 w + xz^3 - ty^2 z$$
$$E = (zA + yB)/t = y^4$$

to I_t. Setting $t = 0$, it follows that

$$(x^2, xy^2, xyz, xyw, y^3 w + xz^3, y^4) \subset I_0$$

and hence $(x^2, xy, y^4, y^3 w + xz^3) \subset \overline{I}_0$. This ideal gives a multiplicity four line in H_1 (see Z_1 above). This shows that $H(4, 0)$ is connected.

REMARK 3.11. – The results in this paper raise several questions:

(1) Does each irreducible component of $H(d, g)$ contain quasiprimitive multiple lines?

(2) Can each quasiprimitive multiplicity structure on a line be deformed to an extremal multiplicity structure on the same line?

(3) Is $H(d, g)$ connected for all (d, g)?

The answers are yes when $d = 2$ and $d = 3$. Positive answers to (1) and (2) would give a positive answer to (3).

REMARK 3.12. – One consequence of our classification is that every curve of degree three can be deformed with constant cohomology to a quasiprimitive multiplicity structure on a line. A recent result from the PhD thesis of Rich Liebling ([8], Corollary 4.1.5) shows that if C is a curve whose Rao module is annihilated by the total ideal of a line, then C deforms with constant cohomology and Rao module to a quasiprimitive multiplicity structure on a line. This generalizes an observation of Juan Migliore on Buchsbaum curves ([10], remark 3.3.1 (*a*)). The following example shows that there are not such deformations in general.

EXAMPLE 3.13. – Let C be a disjoint union of a line L and a triple line W of type $(-1, 2)$. The genus of W is -1, so $g(C) = -2$. Using the standard exact sequence

$$0 \to \mathcal{I}_C \to \mathcal{I}_W \oplus \mathcal{I}_L \to \mathcal{O}_{\mathbb{P}^3} \to 0$$

one can compute that the Rao module has type $\{2, 3, 1\}$ starting in degree 0. On the other hand, it is not difficult to see that there are just four possible Rao functions for the multiple lines in $H(4, -2)$. There are quasiprimitive multiple lines of type $(-1, 0, 5)$ (resp. $(-1, 1, 4), (-1, 2, 3)$) which whose Rao module has type $\{1, 2, 3, 3, 3, 2, 1\}$ (resp. $\{1, 2, 3, 2, 1\}, \{2, 3, 2\}$) starting in degree -2 (resp. $-1, 0$). There are also two Rao functions arising from multiplicity four lines which are not quasiprimitive. If Y is the underlying line of support, the ideals for these multiple lines occur as the kernels of maps $\mathcal{I}_{Y^{(2)}} \to \mathcal{O}_Y(1)$. These yield Rao modules of types $\{1, 2, 3, 2, 1\}, \{1, 2, 3, 1\}$ starting in degree -1. Thus C cannot specialize with constant cohomology and Rao module to a multiple line.

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