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ALGEBRAIC COVERS: FIELD OF MODULI VERSUS FIELD OF DEFINITION (*)

BY PIERRE DÈBES AND JEAN-CLAUDE DOUAI

ABSTRACT. – The field of moduli of a finite cover $f : X \to B$ *a priori* defined over the separable closure K_s of a field K, with B defined over K, need not be a field of definition. This paper provides a cohomological measure of the obstruction. The case of G-covers, *i.e.*, Galois covers given together with their automorphisms, was fairly well-known. But no such cohomological measure was available for mere covers. In that situation, the problem is shown to be controlled not by one, as for G-covers, but by several characteristic classes in $H^2(K_m, Z(G))$, where K_m is the field of moduli and $Z(G)$ is the center of the group of the cover. Furthermore our approach reveals a more hidden obstruction coming on top of the main one, called the first obstruction and which does not exist for G-covers. In contrast with previous works, our approach is not based on Weil's descent criterion but rather on some elementary techniques in Galois cohomology. Furthermore the base space B can be an algebraic variety of any dimension and the ground field K a field of any characteristic. Our main result yields concrete criteria for the field of moduli to be a field of definition. Our main result also leads to some local-global type results. For example we prove this local-to-global principle: a G-cover $f : X \to B$ is defined over $\mathbb{Q}$ if and only if it is defined over $\mathbb{Q}_p$ for all primes p.

RÉSUMÉ. – Le corps des modules K d'un revêtement $f : X \to B$, défini *a priori* sur la clôture séparable de K, avec B défini sur K, n'est pas forcément un corps de définition. Cet article donne une mesure cohomologique de l'obstruction. Le cas des G-revêtements, *i.e.*, des revêtements galoisiens donnés avec leurs automorphismes, était assez bien connu. Mais pour les revêtements seuls, on n'avait pas de tels résultats. On montre dans ce cas que le problème est contrôlé par plusieurs classes caractéristiques dans $H^2(K, Z(G))$ (à valeurs dans le centre $Z(G)$ du groupe du revêtement), et non pas par une seule comme pour les G-revêtements. On déduit de nouveaux critères très concrets pour que le corps des modules soit un corps de définition. Une autre application est le principe local-global suivant, conjecturé par E. Dew : un G-revêtement est défini sur $\mathbb{Q}$ si et seulement si il est défini sur chaque $\mathbb{Q}_p$.

1. Introduction

Let B be an algebraic variety defined over a field K and $f : X \to B$ be a finite cover *a priori* defined over the separable closure K_s of K. Assume that this cover is *isomorphic* to each of its conjugates under $G(K_s/K)$. The field K is said to be the *field of moduli* of the cover. Does it follow that the given cover can be defined over K? The answer is "No" in general. The field of moduli is not necessarily a field of definition: an example was recently given by Couveignes and Granboulan [CouGr]. Still, in many circumstances,

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the field of moduli *is* a field of definition, in which case it is the smallest field of definition containing K. Studying the obstruction for the field of moduli to be a field of definition is the main topic of this paper.

The case of covers of the projective line $\mathbb{P}^1$ in characteristic 0 has been much studied due to the connection with the regular form of the inverse Galois problem –does each finite group occur as the Galois group of a regular Galois extension of $\mathbb{Q}(T)$? The general question classically covers two situations: the first one is concerned with covers –we use the phrase "mere covers" in the sequel– whereas the second one considers G-covers, *i.e.*, Galois covers given with their automorphisms. The notion of field of moduli goes back to Weil and has been investigated then in particular by Belyi, Fried, Harbater and the first author. The main known results are the following ones. For mere covers of $\mathbb{P}^1$, the field of moduli is a field of definition if the cover f has no automorphisms (Weil [We], Fried [Fr]), or, if the cover is Galois (Coombes-Harbater [CoHa]). For a G-cover of automorphism group G, the field of moduli is a field of definition if the center $Z(G)$ is a direct summand of G (*e.g.* if $Z(G) = \{1\}$ or G is abelian). Furthermore, for G-covers, the obstruction to the field of moduli being a field of definition can be measured by a specific characteristic class in the second cohomological group $H^2(K, Z(G))$ of K with values in the center $Z(G)$ and with trivial action (Belyi [Be], Dèbes [Db1], [Db2]).

No such cohomological characterization of the obstruction was known for mere covers. Filling up this gap was one motivation of this paper. We present here a general approach that shows that the problem is indeed entirely of a cohomological nature. A simplified form of our Main Theorem is this.

MAIN THEOREM. – *Let $f : X \rightarrow B$ be a mere cover defined over K_s with K as field of moduli. Let G denote the automorphism group of the Galois closure of the cover f. Then there exists an action L of $G(K_s/K)$ on the center $Z(G)$ of G and a family $(\Omega_\delta)_{\delta\in\Delta}$ of characteristic classes*

$$\Omega_\delta \in H^2(K, Z(G), L)$$

indexed by a certain set Δ and with the property that the field of moduli K is a field of definition if and only if at least one out of the Ω_δs is trivial in $H^2(K, Z(G), L)$.

In contrast with the G-cover case, the problem is controlled not by one but by several characteristic classes in $H^2(K, Z(G), L)$. In addition, the action L need not be the trivial action, as it is for G-covers. Basically, the difference between mere covers and G-covers is this. By definition of "G-cover", all K-models of a G-cover are regular and Galois over K: the extension of constants is trivial. Unlikewise, a mere cover may have several models over K with different non trivial extensions of constants *in the Galois closure* over K. In fact, the various characteristic classes $\Omega_\delta \in H^2(K, Z(G), L)$ correspond to the "possible" extensions of constants in the Galois closure over K of a K-model of f (Main Theorem (II)). In a next paper [DbDo2], we will show that the problem can be even more highly structured by using the theory of *gerbes* of Giraud.

On the other hand, in the mere cover case, the index set Δ of the Main Theorem may be empty, that is, there may be *a priori* no possible extension of constants (in Galois closure) for a K-model. In that case of course, the cover cannot be defined over K. This is an additional obstruction, which does not exist for G-covers. It will be shown to correspond to the solvability of a certain embedding problem, a condition that is denoted by (λ/Lift) and plays a central role in the rest of the paper. We will give some practical criteria for (λ/Lift) to hold, *i.e.*, for Δ to be nonempty (Prop. 3.1). We will also give an *iff* criterion in terms of the vanishing of certain cohomological data (Thm. 4.7).

Contrary to most previous works, we do not need such assumptions as the existence of unramified K-rational points on the base space B, which can be a quite restrictive condition, even for $B = \mathbb{P}^1$ (*e.g.* K is a finite field and $\mathbb{P}^1(K)$ consists only of branch points of the cover). Classically, such conditions imply that the exact sequence of arithmetic fundamental groups

$$(1) \qquad 1 \to \Pi_{K_s}(B^*) \to \Pi_K(B^*) \to G(K_s/K) \to 1$$

is split. Here, given a field F over which B and the ramification locus are defined, B^* denotes the space B with the ramification locus removed and $\Pi_F(B^*)$ the F-arithmetic fundamental group of B^*. That splitting condition is denoted by (Seq/Split) in the sequel. We do not assume in the Main Theorem that condition (Seq/Split) holds. Thus the base space B can be a curve with no K-rational points. The base space B can actually be an algebraic variety of any dimension and the ground field K a field of any characteristic.

This has this other application. The covers were so far assumed to be defined *a priori* over a separably closed field K_s. The question was that of the descent from K_s to the field of moduli K. There is a more general form of the problem for which the covers are assumed to be *a priori* defined over an arbitrary Galois extension F of K. A more general notion of field of moduli relative to the extension F/K can be defined and the question is that of the descent from F to this relative field of moduli. The Main Theorem will be established in this more general context. The exact sequence of concern then is obtained by replacing K_s by F in (1). It is not split in general even in the case B has K-rational points. Our approach allows to handle this more general form of the problem. To our knowledge it was only investigated by E. Dew in some special cases [Dew]. We will refer to the initial form of the problem as the *absolute* one and to the more general one as the *relative* one.

Another innovation is that we handle simultaneously both mere covers and G-covers. The specific objects we will be dealing with are the following ones. Given three groups Π, G, N such that G is normal in N, they are the surjective homomorphisms $\phi : \Pi \to G$ regarded modulo the equivalence that identifies two such homomorphisms that are conjugate by an element of N. Both mere covers and G-covers of a base space B correspond to the special case that Π is the algebraic fundamental group of B with the ramification locus removed, G is the automorphism group of the Galois closure of the cover. The difference between mere covers and G-covers is this: for G-covers, $N = G$ whereas for mere covers, the group N should be taken to be the normalizer $Nor_{S_d}G$ of G in the representation $G \hookrightarrow S_d$ given by the action of G on an unramified fiber of the cover.

Our Main Theorem leads to quite concrete criteria for the field of moduli to be a field of definition. For example, we obtain the following one for mere covers, which, to our knowledge, is new: under condition (λ/Lift), a mere cover is defined over its field of moduli if $Z(G) = \{1\}$ (Cor. 3.2). Our criteria contain all classical results as special cases. The conclusion of Coombes-Harbater theorem –a mere cover that is Galois is defined over its field of moduli– is shown to hold under the only condition (Seq/Split). An example in [DbEm] shows that the same conclusion does not hold if condition (Seq/Split) is removed. Under (Seq/Split), we also obtain, for G-covers and mere covers, some upper bounds for the degree $[K_d : K]$ of some field of definition over the field of moduli K.

As first shown in [Db2], there is a connection between the problem "field of moduli vs field of definition" and some local-global type properties of the field of definition of covers. Our Main Theorem allows to prove this *local-to-global* principle for G-covers.

THEOREM (Thm. 3.8). – *Let $f : X \to B$ be a* G*-cover defined over $\overline{\mathbb{Q}}$. Then $f : X \to B$ is defined over $\mathbb{Q}$ if and only if it is defined over $\mathbb{Q}_p$ for all primes p (including $p = \infty$).*

In particular, finding a regular Galois extension of $\mathbb{Q}(T)$ of given group G –the regular inverse Galois problem– is tantamount to finding a G-cover of $\mathbb{P}^1$ of group G that has a model over each $\mathbb{Q}_p$. Thm. 3.8 was conjectured by E. Dew and proved in [Db2] for G-covers of $\mathbb{P}^1$. Here we extend this result to G-covers of a variety of arbitrary dimension. The main difficulty was to handle the case where condition (Seq/Split) does not necessarily holds. The Main Theorem is the main tool.

Questions of interest remain open. More generally, Thm. 3.8 holds with an arbitrary number field K replacing $\mathbb{Q}$ except possibly in a special case coming from Grunwald's theorem (*see* §3.5). It is unknown whether the local-to-global principle holds in this special case; no counter-example has yet been found. It is also unknown whether the local-to-global principle holds for mere covers in place of G-covers. We devote a forthcoming paper to these questions [DbDo1]. We will establish the local-to-global principle for mere covers under additional assumptions on the group G and the embedding $G \subset S_d$. We think however that the local-to-global principle is very unlikely to hold in general for mere covers. We suggest why in §3.5.

The paper is organized as follows. In §2, we give the basic definitions and recall the dictionary between covers and representations of (arithmetic) fundamental groups. This is very much classical for covers of curves in characteristic 0. In §3 we state the main results and give the applications. The Main Theorem divides into three parts. Part I is concerned with the *first obstruction*, *i.e.*, condition (λ/Lift). When this condition holds, there is a second obstruction, the *main obstruction*. It is described in Part II. Finally, Part III reformulates the whole result under the additional assumption (Seq/Split). Concrete criteria for the field of moduli to be a field of definition are derived in §3.4. §3 ends with the proof of the local-to-global principle for G-covers over $\overline{\mathbb{Q}}$. §4 is devoted to the proof of the Main Theorem: the problem is entirely rephrased in algebraic terms; we can then handle it with cohomological techniques. The same techniques allow to investigate the basic condition (λ/Lift): an *iff* cohomological criterion is given in §4.3.

We end this introduction by indicating how this paper applies to more general situations. Firstly, because they are the most classical situations, we are in this paper mostly concerned with G-covers and mere covers. In a final note we explain that our paper actually applies to any kind of covers $f : X \to B$ given with some "extra structure". Secondly, the notion of field of moduli can be defined for other kinds of structure than covers, *e.g.* an algebraic variety X. [DbEm] shows that, under suitable conditions, the obstruction that the field of moduli is a field of definition is the same as for the cover $X \to X/Aut(X)$. Consequently results of the current paper yield results about fields of moduli of curves in particular. Finally we consider here covers *a priori* defined over an algebraic extension of the base field K. Various notions of field of moduli can also be defined for objects *a priori* defined over a transcendental extension of K. A subsequent paper will show how to unify these various notions and will explain that the essential problem is the one considered here, *i.e.*, the algebraic descent from K_s to the field of moduli.

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2. Preliminaries on mere covers and G-covers

NOTATION 2.1. – Given a Galois extension E/k, its Galois group is denoted by $G(E/k)$. We let elements τ of Galois groups act to the right ($x \rightarrow x^\tau$). Given a field k, we denote by k_s a separable closure of k and by $G(k)$ the absolute Galois group $G(k_s/k)$ of k. As usual in Galois cohomology, we write $H^n(k,-,-)$ for $H^n(G(k),-,-)$. In a group G, conjugation by an element $g \in G$ is the homomorphism $x \rightarrow x^g = gxg^{-1}$ ($x \in G$). As for Galois actions, our notation has the group act to the right.

2.1. Mere covers and G-covers over a field K

Let K be a field and B be a regular projective geometrically irreducible K-variety. By *mere cover of B over K*, we mean a finite and generically unramified morphism $f : X \rightarrow B$ defined over K with X a normal and geometrically irreducible K-variety. The term "mere" is meant to distinguish mere covers from G-covers defined below.

The associated field extension $K(X)/K(B)$ is a finite separable field extension that is regular over K (*i.e.*, $K(X) \cap K_s = K$ (in a common separable closure of $K(B)$)). The degree of a cover is $d = [K(X) : K(B)] = [K_s(X) : K_s(B)]$. The cover is said to be *Galois* over K if the field extension $K(X)/K(B)$ is Galois.

Mere covers $f : X \rightarrow B$ over K and finite separable regular field extensions $K(X)/K(B)$ actually correspond to one another through the function field functor.

[Indeed, let $E/K(B)$ be a finite separable extension regular over K. For each affine open subset $U = Spec(R)$ of B, let $\widetilde{R}$ be the integral closure of R in E. The associated morphisms $Spec(\widetilde{R}) \rightarrow Spec(R)$ can be patched together to give a finite and generically unramified morphism $f : X \rightarrow B$ over K with X a normal and irreducible variety.

Furthermore, if D is the (reduced) ramification divisor (*see* definition just below) of the extension $EK_s/K_s(B)$, then the morphism is étale above $B^* = B - D$. This follows from the Purity of Branch Locus ([Mi], [SGA1; Exp. 10]).

Finally, if in addition D is a divisor with normal crossings, then the morphism is also flat. For certain authors, flatness is part of the definition of covers. We will not need it. So we have not included it so not to restrict the generality of our results.]

A cover $f : X \rightarrow B$ over a separably closed field K_s has two basic geometric invariants, which only depend on the isomorphism class of the cover. First the *group* G of the cover, *i.e.*, the automorphism group of the Galois closure $\hat{f} : \widehat{X} \rightarrow B$ of f, or, equivalently, the Galois group $G(K_s(\widehat{X})/K_s(B))$. Second, the *ramification divisor* D of the cover, which is defined as follows. Since B is normal, the local ring at each hypersurface of B is a discrete valuation ring. Say that an hypersurface of B is ramified if the associated discrete valuation ramifies in the extension $E/K(B)$. Then define the ramification divisor as the formal sum of all ramified hypersurfaces. By invariants of a cover over a non separably closed field K, we always mean the invariants of the cover over K_s obtained by extension of scalars. The ramification divisor D of a cover over K is invariant under the action of $G(K)$.

By G-*cover of* B *of group* G *over* K, we mean a Galois cover $f : X \to B$ over K given together with an isomorphism $h : G \to G(K(X)/K(B))$. The capital letter "G" in "G-cover" indicates that the Galois group is part of the data; the "G" (italicized) is here the name of the group. G-covers of B of group G over K correspond to regular Galois extensions $K(X)/K(B)$ given with an isomorphism of the Galois group $G(K(X)/K(B))$ with G.

2.2. Isomorphisms of mere covers and G-covers

An *isomorphism between two mere covers* $f : X \to B$ and $f' : X' \to B$ over a field K is an algebraic morphism $\chi : X \to X'$ that induces a $K(B)$-isomorphism between the function field extensions $K(X)/K(B)$ and $K(X')/K(B)$. In a common algebraic closure of $K(B)$, this condition amounts to this: the extensions $K(X)/K(B)$ and $K(X')/K(B)$ should be conjugate by χ. Equivalently, an isomorphism $\chi : X \to X'$ of mere covers is an algebraic isomorphism $X \to X'$, defined over K, such that $\chi \circ f' = f$.

Isomorphisms of mere covers of B over K and $K(B)$-isomorphisms of extensions of $K(B)$ correspond to one another: just extend the argument of §2.1 for "objects" to "morphisms". More precisely, the function field functor is an equivalence of categories.

An *isomorphism between two G-covers* $f : X \to B$ and $f' : X' \to B$ of group G over K is a map $\chi : X \to X'$ with the following properties:

– χ is an isomorphism of mere covers over K.

– χ commutes with the given actions of G.

In a common algebraic closure of $K(B)$, the extensions $K(X)/K(B)$ and $K(X')/K(B)$ are necessarily equal and χ induces an element of $G(K(X)/K(B))$. Isomorphisms of G-covers of B over K correspond to $K(B)$-automorphisms of regular Galois extensions of $K(B)$.

If F/K is a field extension and f is a mere cover (resp. G-cover) over K, the mere cover (resp. G-cover) over F obtained from f by extension of scalars is denoted by $f \otimes_K F$ (that $f \otimes_K F$ is indeed a mere cover (resp. G-cover) over F follows from our definition, in particular from the regularity condition). Assume the base space B is defined over K. A mere cover (resp. a G-cover) $f : X \to B$ over F is said to be *defined over* K if there exists a mere cover (resp. a G-cover) $f_K : X \to B$ over K such that $f_K \otimes_K F$ is isomorphic to f over F.

2.3. K-arithmetic fundamental group

Assume that the base variety B is defined over K and fix a $G(K)$-invariant divisor D of B with only simple components. Set $B^* = B - D$. The *K-arithmetic fundamental group* of B^* is denoted by $\Pi_K(B^*)$ or simply by Π_K when the context is clear. In the notation of [SGA1], $\Pi_K(B^*)$ is $\pi_1(B^*, \xi)$, where ξ is the geometric generic point of

B corresponding to $Spec(K(B)_s) \to Spec(K(B)) \to B^*$. The group $\Pi_K(B^*)$ can be defined in the following way.

Fix a separable closure $(K(B))_s$ of $K(B)$. Take then for K_s the separable closure of K inside $(K(B))_s$. Let $\Omega_D \subset (K(B))_s$ be the maximal algebraic separable extension of $K_s(B)$ unramified above B^*. Then $\Pi_K(B^*)$ is the Galois group of the extension $\Omega_D/K(B)$. The K_s-fundamental group $\Pi_{K_s}(B^*)$ is also called the *geometric fundamental group*. From Galois theory, for each Galois extension F/K, we have the following exact sequence of fundamental groups

$$1 \to \Pi_F(B^*) \to \Pi_K(B^*) \to G(F/K) \to 1$$

REMARK 2.2. – Assume K is of characteristic 0. From Grauert-Remmert's theorem and the GAGA theorems, covers of B that are unramified outside D correspond to analytic unramified finite covers of $B \setminus D$, which, in turn correspond to topological covers of $B \setminus D$ (for the complex topology). This implies that $\Pi_{K_s}(B^*)$ is the profinite completion of the topological fundamental group of B.

2.4. Dictionary "covers/homomorphisms"

2.4.1. *Mere covers.* – Degree d mere covers of B over a field K with ramification divisor in D correspond to transitive representations

$$\Psi : \Pi_K(B^*) \to S_d$$

such that the restriction to $\Pi_{K_s}(B^*)$ is transitive. Here S_d denotes the symmetric group in d letters.

[**Correspondences**. A degree d mere cover $f : X \to B$ over K with ramification divisor in D corresponds, *via* the functor "function fields", to a finite regular separable subextension $E/K(B)$ of $\Omega_D/K(B)$. Its Galois closure $\widehat{E}/K(B)$ corresponds to a quotient of $\Pi_K = \Pi_K(B^*)$, or, equivalently, to a surjective homomorphism $\Phi : \Pi_K(B^*) \to G$ where $G = G(\widehat{E}/K(B))$. *Via* Galois theory, the extension $E/K(B)$ corresponds to a specific subgroup H of G. Label the left cosets of G modulo H by the integers $1, \ldots, d$ in such a way that H corresponds to 1. The action of G by left multiplication on the left cosets of G modulo H provides a representation $i : G \hookrightarrow S_d$. The composed homomorphism $\Psi = i \circ \Phi : \Pi_K \to S_d$ is the desired representation. The representation Ψ is only defined up to conjugation by an element of S_d. But the image group $G = \Psi(\Pi_K)$ is well-defined.

Conversely, given a representation $\Psi : \Pi_K(B^*) \to S_d$ as above, denote the stabilizer of 1 by $\Pi_K(1)$. Consider the fixed field $E = \Omega_D^{\Pi_K(1)}$ of $\Pi_K(1)$ in Ω_D. Then the extension $E/K(B)$ is the function field extension associated to a degree d mere cover $\tilde{f} : \tilde{X} \to B$ over K.]

Two covers over K are isomorphic if and only if the corresponding representations Ψ and Ψ' are conjugate by an element φ in the normalizer $\mathrm{Nor}_{S_d}G$ in S_d of the image group $G = \Psi(\Pi_{K_s}) = \Psi'(\Pi_{K_s})$, *i.e.*,

$$\Psi'(x) = \varphi\Psi(x)\varphi^{-1} \qquad \text{for all} \quad x \in \Pi_K(B^*)$$

2.4.2. G-*covers.* – G-covers of B of group G over K correspond to surjective homomorphisms

$$\Phi : \Pi_K(B^*) \to G$$

such that $\Phi(\Pi_{K_s}(B^*)) = G$.

[**Correspondences**. A G-cover $f : X \to B$ over K of group G and with ramification divisor in D corresponds, *via* the functor "function fields" to a specific finite regular Galois subextension of $\Omega_D/K(B)$, *i.e.*, *via* Galois theory, to a specific normal subgroup H of $\Pi_K(B^*)$. The group Π_K/H is canonically identified with the Galois group $G(K(X)/K(B))$. Composing the natural surjection $\Pi_K \to \Pi_K/H$ with the given isomorphism $G(K(X)/K(B) \to G$ provides a surjective homomorphism $\Phi : \Pi_K(B^*) \to G$ as above.

Conversely, given such an homomorphism, consider the fixed field $E = \Omega_D^{Ker(\Phi)}$ of $Ker(\Phi)$ in Ω_D. Then the extension $E/K(B)$ is the function field extension associated to a Galois cover $f : X \to B$ over K of group $\Pi_K/Ker(\Phi)$. The isomorphism $\Pi_K/Ker(\Phi) \to G$ endows f with a structure of G-cover over K.]

Two G-covers over K are isomorphic if and only if the corresponding homomorphisms Φ and Φ' are conjugate by an element of $\varphi \in G$, *i.e.*,

$$\Phi'(x) = \varphi\Phi(x)\varphi^{-1} \qquad \text{for all} \quad x \in \Pi_K(B^*)$$

2.4.3. *Mere cover induced by a* G-*cover.* – If a G-cover corresponds to an homomorphism $\Phi : \Pi_K(B^*) \to G$ as above, then the associated mere cover corresponds to the homomorphism $\Psi : \Pi_K(B^*) \to S_d$ obtained by composing Φ with the regular representation $G \hookrightarrow S_d$ of G (where $d = |G|$).

2.4.4. G-*cover attached to a mere cover.* – Conversely let $\Psi : \Pi_K(B^*) \to S_d$ be the representation of Π_K associated to a degree d mere cover $f : X \to B$. Set $G = \Psi(\Pi_K)$. The induced map $\Psi : \Pi_K(B^*) \to G$ corresponds to the Galois closure $\widehat{K(X)}/K(B)$ over K of the function field extension $K(X)/K(B)$. It does not necessarily correspond to a G-cover over K. It does if and only if $\Psi(\Pi_K) = \Psi(\Pi_{K_s}) = G$, which amounts to saying that the extension $\widehat{K(X)}/K(B)$ is a regular extension.

2.5. (G-)covers

We frequently use the word "(G-)cover" for the phrase "mere cover (resp. G-cover)" in statements holding for both mere covers and G-covers. We will consider descent problems for the field of definition of (G-)covers from a Galois extension F of a field K down to the field K. In the mere cover situation, we will always assume that the Galois closure over F of the mere cover is, as G-cover, defined over F. This insures that the group of the cover is the same over F as over F_s. This is of course not restrictive in the absolute situation, *i.e.*, when F is separably closed.

Both mere covers and G-covers can be handled simultaneously. Namely mere covers correspond to transitive representations $\phi : \Pi_F(B^*) \to S_d$ whereas G-covers $f : X \to B$ correspond to surjective homomorphisms $\phi : \Pi_F(B^*) \to G$. In both cases let G denote the group of the cover. Then set

$$N = \begin{cases} G & \text{in the G-cover case} \\ Nor_{S_d}G & \text{in the mere cover case} \end{cases}$$

$$C = Cen_N G = \begin{cases} Z(G) & \text{in the G-cover case} \\ Cen_{S_d}G & \text{in the mere cover case} \end{cases}$$

where $Z(G)$ is the center of G and $Nor_{S_d}G$ and $Cen_{S_d}G$ are respectively the normalizer and the centralizer of G in S_d. Thus, both mere covers and G-covers over F correspond to homomorphisms (or representations) $\phi : \Pi_F(B^*) \twoheadrightarrow G \subset N$.

Suppose given a Galois extension F/K and a (G-)cover $f : X \to B$ such that the base B is defined over K and the ramification divisor D is $G(K)$-invariant. Then the mere cover f (resp. G-cover f) can be defined over K if and only if the representation $\phi : \Pi_F(B^*) \to G \subset N$ can be extended to a representation $\Pi_K(B^*) \to N$.

Finally we always regard N as a subgroup of S_d where d is the degree of f: in the mere cover case, an embedding $N \hookrightarrow S_d$ is given by definition; in the G-cover case, embed $N = G$ in S_d by the regular representation of G.

2.6. Galois action

The Galois group $G(F/K)$ has a natural action on F-varieties; in particular, $G(F/K)$ acts on (G-)covers of B over F. Let $f : X \to B$ be a (G-)cover and $\tau \in G(F/K)$. The corresponding conjugate (G-)cover will be denoted by $f^\tau : X^\tau \to B^\tau$. The group of the cover f^τ is the same as the group of f; if D is the ramification divisor of f, then D^τ is the ramification divisor of f^τ. Assume now that B is defined over K and that D is $G(K)$-invariant. Let $\phi : \Pi_F(B^*) \to G$ be the homomorphism corresponding to f. Pick an element $\hat{\tau} \in \Pi_K(B^*)$ above $\tau \in G(F/K)$ and consider the homomorphism $\phi^{\hat{\tau}} : \Pi_F(B^*) \to G$ defined by

$$\phi^{\hat{\tau}}(x) = \phi(x^{\hat{\tau}^{-1}}) \qquad \text{for all} \quad x \in \Pi_F(B^*)$$

where $x^{\hat{\tau}} = \hat{\tau}x(\hat{\tau})^{-1}$. Then the homomorphism $\phi^{\hat{\tau}}$ corresponds to a (G-)cover that is isomorphic over F to the (G-)cover f^τ. Note that because our notation has the groups $G(F/K)$ and $\Pi_K(B^*)$ act to the right, we have $f^{uv} = (f^v)^u$ and $\phi^{\hat{u}\hat{v}} = \left(\phi^{\hat{v}}\right)^{\hat{u}}$, $(u, v \in G(F/K))$.

2.7. Field of moduli

Fix a Galois extension F/K. Let $f : X \to B$ be a mere cover (resp., G-cover) *a priori* defined over F. Consider the subgroup $M(f)$ (resp. $M_G(f)$) of $G(F/K)$ consisting of all the elements $\tau \in G(F/K)$ such that the covers (resp., the G-covers) f and f^τ are isomorphic over F. Then the *field of moduli* of the cover f (resp., the G-cover f) *relative to the extension* F/K is defined to be the fixed field

$$F^{M(f)} \text{ (resp. } F^{M_G(f)})$$

of $M(f)$ (resp. $M_G(f)$) in F. The field of moduli relative to the extension K_s/K is called the *absolute field of moduli* (relative to K). The field of moduli of a (G-)cover is easily seen to be contained in each field of definition containing K (in particular, it is a finite extension of K). So it is the smallest field of definition containing K provided that *it is* a field of definition. The ramification divisor D of f is automatically invariant under $M(f)$ (resp. $M_G(f)$).

Let K_m be the field of moduli of f relative to the extension F/K. Then the field of moduli of f relative to the extension F/K_m is K_m (this essentially follows from the fact that $M(f)$ (resp. $M_G(f)$) is a closed subgroup of $G(F/K)$ for the Krull topology (*e.g.*

[DbEm]). That observation generally allows to reduce to the situation where the base field K is the field of moduli of the given (G-)cover f.

Assume $\phi : \Pi_F \to G \subset N$ is the homomorphism corresponding to the (G-)cover $f : X \to B$ over F. Then K is the field of moduli of the (G-)cover f relative to the extension F/K if and only if the ramification divisor D is $G(K)$-invariant and the following condition –called the *field of moduli condition*–, holds.

(FMod) For each $\mathrm{U} \in \Pi_K(B^*)$, there exists $\varphi_{\mathrm{U}} \in N$ such that

$$\phi(x^{\mathrm{U}}) = \phi(x)^{\varphi_{\mathrm{U}}} \qquad \text{(for all } x \in \Pi_F(B^*))$$

For each $\mathrm{U} \in \Pi_K(B^*)$, φ_{U} is well-defined modulo $Cen_N G = C$. Denote the map $\Pi_K(B^*) \to N/C$ that sends each $\mathrm{U} \in \Pi_K(B^*)$ to the coset of φ_{U} modulo C by $\overline{\varphi}$. It follows immediately from the definition that $\overline{\varphi} : \Pi_K(B^*) \to N/C$ is a group homomorphism. Also if ϕ is changed to ϕ^α with $\alpha \in N$, then $\overline{\varphi}$ should be replaced by $\overline{\varphi}^{\overline{\alpha}}$, where $\overline{\alpha}$ is the coset of α modulo C.

The map $\overline{\varphi} : \Pi_K(B^*) \to N/C$ is a key data of the problem. If K is the field of moduli, *i.e.*, if condition (FMod) holds, then the map $\overline{\varphi}$ is uniquely attached to the original representation $\phi : \Pi_F \to G \subset N$. In the sequel, the homomorphism $\overline{\varphi} : \Pi_K(B^*) \to N/C$ is called *the representation of* Π_K *modulo* C *given by the field of moduli condition.*

2.8. Extension of constants in the Galois closure

Let F/K be a Galois extension and $f : X \to B$ be a (G-)cover defined over F. Let $\phi_F : \Pi_F(B^*) \to G \subset N$ be the associated homomorphism. Assume that the (G-)cover f can be defined over K, *i.e.*, that the (G-)cover f has a model $f_K : X_K \to B$ over K. Let $\phi_K : \Pi_K(B^*) \to N$ be the associated extension of ϕ_F to $\Pi_K(B^*)$. Consider the function field extensions $F(X)/F(B)$ and $K(X_K)/K(B)$ respectively associated to f and f_K. Denote the Galois closure of the extension $F(X)/F(B)$ (resp. $K(X_K)/K(B)$) by $\widehat{F(X)}/F(B)$ (resp. $\widehat{K(X_K)}/K(B)$). The field $\widehat{F(X)}$ (resp. $\widehat{K(X_K)}$) is the fixed field in Ω_D (defined in §2.3) of the kernel of

$$\phi_F : \Pi_F(B^*) \to G$$

$$\text{(resp. } \phi_K : \Pi_K(B^*) \to N)$$

Consider then the field $\widehat{K} = \widehat{K(X_K)} \cap F$ (inside Ω_D). The extension $\widehat{K}/K$ is called the *extension of constants in the Galois closure* of the model f_K of f. The phrase "in the Galois closure" should be stressed: the function field extension $K(X_K)/K(B)$ itself is a regular extension, *i.e.*, the field $K(X_K) \cap F$ of constants in $K(X_K)$ is equal to K.

Denote by Λ the unique homomorphism $G(F/K) \to N/G$ that makes the following diagram commute. Existence of Λ follows from $\phi_K(\Pi_F) = \phi_F(\Pi_F) \subset G$ and uniqueness from the surjectivity of $\Pi_K \to G(F/K)$.

$$\begin{array}{ccc} \Pi_K(B^*) & \longrightarrow & G(F/K) \\ {\scriptstyle \phi_K}\downarrow & & \downarrow{\scriptstyle \Lambda} \\ N & \longrightarrow & N/G \end{array}$$

PROPOSITION 2.3. – *The homomorphism $\Lambda : G(F/K) \to N/G$ induced by ϕ_K on $G(F/K)$ corresponds to the extension of constants $\widehat{K}/K$ in the Galois closure of the model f_K of f_F. In other words, we have*

$$G(F/\widehat{K}) = Ker(\Lambda)$$

Consequently, $G(\widehat{K}/K)$ identifies with a subgroup of N/G.

The homomorphism $\Lambda : G(F/K) \to N/G$ is called the *constant extension map (in Galois closure)* of the K-model f_K of f. For G-covers, $N/G = \{1\}$, the map Λ is trivial and $\widehat{K} = K$. By definition, G-covers over K are required to be Galois over K with the same Galois group as over K_s. Thus they do not have any extension of constants in their Galois closure.

Proof. – The restriction $G(\widehat{F(X)}/F(B)) \to G(\widehat{K(X_K)}/\widehat{K}(B))$ is an isomorphism. Consequently so is the restriction $r : G(\widehat{F(X)}/\widehat{K(X_K)}) \to G(F/\widehat{K})$. Now the first group $G(\widehat{F(X)}/\widehat{K(X_K)})$ is equal to the quotient group $Ker(\phi_K)/Ker(\phi_F)$. The second group $G(F/\widehat{K})$, *via* the isomorphism r, corresponds then to $Ker(\Lambda)$. □

2.9. Arithmetic action of $G(F/K)$ on a fiber

In this paragraph we fix a Galois extension F/K and we assume that the exact sequence of fundamental groups

$$1 \to \Pi_F(B^*) \to \Pi_K(B^*) \to G(F/K) \to 1$$

splits. We will call this condition (Seq/Split) for short. We let $\mathrm{s} : G(F/K) \to \Pi_K(B^*)$ denote a section to the map $\Pi_K(B^*) \twoheadrightarrow G(F/K)$.

For $F = K_s$ (*i.e.*, for the absolute form of the problem), then condition (Seq/Split) classically holds if the base space B has K-rational points off the branch point set D. Indeed each such K-rational point provides a section $\mathrm{s} : G(K) \to \Pi_K(B^*)$. On the other hand, condition (Seq/Split) does not always hold: an example in which it does not is given in [DbEm].

Let $f : X \to B$ be a (G-)cover defined over the small field K. Let $\phi_K : \Pi_K(B^*) \to N$ be the associated homomorphism. Under condition (Seq/Split), the homomorphism ϕ_K is determined by its restriction ϕ_F to $\Pi_F(B^*)$ and by the homomorphism $\phi_K \circ \mathrm{s} : G(F/K) \to N$. That is, the model f_K of f over K is determined by the (G-)cover $f \otimes_K F$ and the homomorphism $\phi_K \circ \mathrm{s}$.

This homomorphism $\phi_K \circ \mathrm{s} : G(F/K) \to N$ can be interpreted as the action of $G(F/K)$ on an unramified fiber of the cover. Consider first the special case for which $F = K_s$, B is a curve and $\mathrm{s} = \mathrm{s}_{t_o}$ is the section given by an unramified K-rational point t_o on B. Recall that N comes equipped with an embedding $G \hookrightarrow S_d$. Then Prop. 2.1 of [Db2] shows that, for each $\tau \in G(K)$, the element $\phi_K \circ \mathrm{s}_{t_o}(\tau)$ is conjugate in S_d to the action of τ on the fiber $f_K^{-1}(t_o)$.

Return to the general case. Each element of $\Pi_K(B^*)$ induces a permutation of the different embeddings of the function field $K(X_K)$ in a separable closure $(K(B))_s$ of

$K(B)$. This set of embeddings $K(X_K) \hookrightarrow (K(B))_s$ can be viewed as the geometric generic fiber of the cover. By analogy with the case $\mathrm{s} = \mathrm{s}_{t_o}$, for each $\tau \in G(F/K)$, the element $\phi_K \circ \mathrm{s}(\tau)$ will be called the *arithmetic action of τ on the generic fiber associated with the section s.* Such actions will be called *actions of $G(F/K)$ on an unramified fiber.* The most important case is the case $\mathrm{s} = \mathrm{s}_{t_o}$: the generic fiber associated with s_{t_o} is the special fiber above t_o. Another important special case is for $K = \mathbb{F}_p$: the absolute Galois group $G(K)$ is the profinite group $\widehat{\mathbb{Z}}$ generated by the Frobenius, the corresponding action of $G(K)$ on the generic fiber is given by the action of (a lifting of) the Frobenius.

Actions of $G(F/K)$ on unramified fibers are more precise invariants of the K-models of the cover than the extension of constants (in Galois closure). Indeed, if $\mathrm{s} : G(F/K) \to \Pi_K(B^*)$ is a section, the constant extension map $\Lambda : G(F/K) \to N/G$ is equal to $\phi_K \circ \mathrm{s} : G(F/K) \to N$ composed with $N \twoheadrightarrow N/G$.

3. Main results

Let F/K be a Galois extension. The *absolute* situation (as considered in the introduction) corresponds to the special case $F = K_s$. Here we consider more generally a *relative* situation: F/K is an arbitrary Galois extension. In a rough way, descending the field of definition of a (G-)cover from F to K consists in enriching the given model of f over F with some extra arithmetical data relative to K. There may not exist such arithmetical data which are compatible with the model over F. But we will see that the fact that K is the field of moduli insures that such arithmetical data exist at least *modulo the group C* (defined in §2.5). So the whole problem can be regarded as a lifting problem: one wishes to lift some arithmetical data given in a quotient group N/C up to the group N. More precisely, this arithmetical data consists of the *representation of Π_K modulo C* given by the field of moduli condition (defined in §2.7). In practice, this data can be reached in two ways:

– **through the extension of constants in the Galois closure** (§2.8): §3.1/3.2 use this to classify the various K-models of a cover. This viewpoint divides the problem into two steps. The first one is to find all the possible extensions of constants in Galois closure and leads to a first obstruction: there must be at least one. This obstruction, which does not exist in the case of G-covers, is called the *first obstruction* (§3.1). When this first obstruction vanishes, *i.e.*, when there is a possible extension of constants, there remains the *main obstruction*, namely, the obstruction for the existence of a K-model with *this* extension of constants. The Main Theorem (Parts I and II) gives a cohomological description of these obstructions.

– **through the arithmetic action of $G(F/K)$ on an unramified fibe**ı (§2.9): such actions are more precise arithmetic invariants than the extension of constants. But they are defined under the extra condition (Seq/Split). This condition, however, is a quite natural condition. Part III of the Main Theorem corresponds to the special case for which condition (Seq/Split) holds.

The Main Theorem is proved in §4. The last two paragraphs of §3 are devoted to applications of the Main Theorem. In §3.4 we give some practical criteria for the field of moduli to be a field of definition. In §3.5 we prove the local-to-global principle.

We fix a Galois extension F/K and a (G-)cover $f : X \to B$ defined over F and with K as field of moduli. More precisely, we fix a representation $\phi_F : \Pi_F(B^*) \twoheadrightarrow G \subset N$ associated with the (G-)cover f (*cf.* §2.5). We will explain how our results depend on the chosen model. The base B is assumed to be defined over K. Notation is that of §2.

3.1. The first obstruction

The (G-)cover $f : X \to B$ may have several models over K (and may have none). Attached to each of these models is a constant extension $\widehat{K}/K$. This data is actually a significant arithmetic invariant of the K-model. It is in fact one of the few data that can possibly distinguish the different models over K. For example, a Galois cover may have a model over K that is Galois over K (*i.e.*, may be defined over K as G-cover) and another model that is not; both models are isomorphic over F; the extension of constants (in Galois closure) is trivial in the former case and not in the latter. It seems natural that this data arises when trying to find all the K-models of a (G-)cover f. Thus a first problem is to determine all the possible constant extensions $\widehat{K}/K$, or, equivalently, all the possible constant extension maps (in Galois closure) $\Lambda : G(F/K) \to N/G$ (§2.8). On the other hand there may be none, but in that case of course, the cover cannot be defined over K. We will see that the fact that K is the field of moduli insures that the map Λ exists and is uniquely determined *modulo the group* C. More specifically, we have this first obstruction.

MAIN THEOREM (I) (First obstruction). – *Assume that K is the field of moduli of the* (G-)*cover f relative to the extension F/K. Let $\overline{\varphi} : \Pi_K(B^*) \to N/C$ be the representation of Π_K modulo C given by the field of moduli condition* (§2.7).

(*a*) *There exists a unique homomorphism $\lambda : G(F/K) \to N/CG$ that makes the following diagram commute*

$$\begin{array}{ccc} \Pi_K(B^*) & \longrightarrow & G(F/K) \\ {\scriptstyle\overline{\varphi}}\big\downarrow & & \big\downarrow{\scriptstyle\lambda} \\ N/C & \longrightarrow & N/CG \end{array}$$

(*b*) *The constant extension map (in Galois closure) $\Lambda : G(F/K) \to N/G$ of each K-model of the* (G-)*cover is a lift of λ.*

(*c*) *In particular, the following condition*

(λ/Lift) *There exists at least one lifting $\Lambda : G(F/K) \to N/G$* of *$\lambda : G(F/K) \to N/CG$. is a necessary condition for the field of moduli K to be a field of definition of the* (G-)*cover.*

The map $\lambda : \Pi_K(B^*) \to N/C$ is uniquely determined by the representation $\phi_F : \Pi_F(B^*) \twoheadrightarrow G \subset N$ associated with the (G-)cover $f : X \to B$. If ϕ_F is changed to ϕ_F^α with $\alpha \in N$, then λ should be replaced by $\lambda^{\overline{\alpha}}$, where $\overline{\alpha}$ is the coset of α modulo CG. The homomorphism $\lambda : G(F/K) \to N/CG$ is called *the constant extension map (in Galois closure) modulo C* given by the field of moduli condition.

The case of G-covers. In this case we have $N/CG = N/G = \{1\}$. Consequently the constant extension map λ modulo C is the trivial one and the trivial map $1 : \Gamma \to N/C$ is the only possible constant extension map lifting λ. Thus condition (λ/Lift) holds trivially.

The case of mere covers is different. The map λ may have no liftings Λ, (*i.e.*, condition (λ/Lift) may not hold), and may have several ones. In §4.3 we give an *iff* cohomological criterion for (λ/Lift) to hold. However we prefer to keep it as a basic condition because it is necessary, natural and is likely to hold in practice. Here are some practical criteria.

PROPOSITION 3.1. – *Condition (λ/Lift) holds in each of the following situations* (the third one anticipates §4.3).

(*a*) *The Galois group $G(F/K)$ is a projective profinite group.*

(*b*) *The group CG/G has a complement in the group N/G, (in other words, the natural map $N/G \twoheadrightarrow N/CG$ splits). This holds in particular for mere covers that are Galois of group G such that $Inn(G)$ has a complement in $Aut(G)$.*

(*c*) *The group $C/Z(G)$ is a centerless group and the "band" of the problem is representable. The latter holds for example if $Inn(C/Z(G))$ has a complement in $Aut(C/Z(G))$ (e.g. $C = Z(G)$).*

Proof. – (*a*) is immediate. (c) is proved in §4.3 (where "band" is defined). In (b), only the part relative to the application to Galois covers needs more details. For Galois covers, the group G acts freely and transitively on each unramified fiber of the cover. Thus the embedding $G \hookrightarrow S_d$ can be taken to be the left regular representation $\gamma : G \hookrightarrow S_d$ of G (*i.e.*, $\gamma(g)(x) = g.x$ $(g, x \in G)$). The following facts (*) and (**), which we will use in a couple of occasions, are more or less classical. The desired result follows immediately from (**). Identify G with $\gamma(G)$.

(*) The group $N = \mathrm{Nor}_{S_d}(G)$ is the semi-direct product $C \times^s Aut(G)$ of $C = \mathrm{Cen}_{S_d} G$ and $Aut(G)$.

(**) $N/G \simeq Aut(G)$ and $N/CG \simeq Aut(G)/Inn(G)$.

[Proof of (*). The group $C = \mathrm{Cen}_{S_d} G$ is the image of the right regular representation $\delta : G \hookrightarrow S_d$ (given by $\delta(g)(x) = x.g$ $(g, x \in G)$). For each $\sigma \in S_d$, there exists a unique $c_\sigma \in C$ such that $c_\sigma \sigma(1) = 1$. It is easily checked that $\sigma \in N = \mathrm{Nor}_{S_d}(G)$ if and only if $c_\sigma \sigma \in Aut(G)$. The rest of the proof of (*) is straightforward. □

Proof of (**). For each $\nu = \delta(g).\chi \in N = C \times^s Aut(G)$ (with $g \in G$, $\chi \in Aut(G)$), we have $\gamma(g)^{-1}.(\delta(g).\chi) \in Aut(G)$: indeed, $\gamma(g)^{-1} \cdot \delta(g)$ is the conjugation by g^{-1}. This shows that $N/G = GAut(G)/G \simeq Aut(G)$. The rest of the proof readily follows. □]

3.2. The main obstruction

Assume that condition (λ/Lift) holds, *i.e.*, that there is a possible extension of constants for a K-model of the (G-)cover f. The main question remains: does there actually exist a K-model with *this* extension of constants? Part II of the Main Theorem is concerned this second part of the problem, which we call the *main obstruction*. More notation is needed.

The actions L, L_Λ, L_Λ^* and the operator δ^1. Since elements of CG commute with those of $Z(G)$, the action of N by conjugation on $Z(G)$ factors through the map $N \twoheadrightarrow N/CG$ to yield an action of N/CG on $Z(G)$. Similarly, the action of N by conjugation on C factors through the map $N \twoheadrightarrow N/G$ to yield an action of N/G on C. We call these actions "actions by conjugation (via N)" of N/CG on $Z(G)$ and of N/G on C.

The **action L of $G(F/K)$ on $Z(G)$** is the action obtained by composing the map $\lambda : G(F/K) \to N/CG$ with the action by conjugation of N/CG on $Z(G)$. Given a lifting $\Lambda : G(F/K) \to N/G$ of λ, the **action L_Λ of $G(F/K)$ on C** is the action obtained by composing $\Lambda : G(F/K) \to N/G$ with the action by conjugation of N/G on C. It is readily checked that the action L_Λ, restricted to $Z(G)$, coincides with the action L.

Finally, we denote by L_Λ^* the action obtained by composing $\Lambda : G(F/K) \to N/G$ with the action by conjugation of N/G on CG/G.

Fix a lifting $\Lambda : G(F/K) \to N/G$ of λ. Consider the exact sequence

$$1 \to Z(G) \to C \to CG/G \to 1$$

Its kernel $Z(G)$ is abelian and central (*i.e.*, $Z(G) \subset Z(C)$). Thus the **coboundary operator**

$$H^1(G(F/K), CG/G, L_\Lambda^*) \to H^2(G(F/K), Z(G), L)$$

is well-defined. It will be denoted by δ^1.

The *main obstruction* to the field of moduli being a field of definition can be described as follows. Conclusion (*e*) is the key part. The 2-cocycle Ω_Λ involved is defined in (*b*).

MAIN THEOREM (II) (Main obstruction). – *Assume that K is the field of moduli of the* (G-)*cover f relative to the extension F/K and that condition (λ/Lift) holds. Fix a lifting $\Lambda : G(F/K) \to N/G$ of λ.*

(*a*) *Let $s : G(F/K) \to \Pi_K(B^*)$ be an arbitrary set-theoretic section to the map $\Pi_K(B^*) \to G(F/K)$ and $\overline{\varphi} : \Pi_K(B^*) \to N/C$ be the representation of Π_K modulo C given by the field of moduli condition (§2.7). For each $u \in G(F/K)$, there exists an element $\phi_u \in N$, unique modulo $Z(G)$, such that*

$$\begin{cases} \phi_u = \Lambda(u) \text{ modulo } G \\ \phi_u = \overline{\varphi} \circ \mathrm{s}(u) \text{ modulo } C \end{cases}$$

(*b*) *Consider the 2-cochain $(\Omega_{u,v})_{u,v\in\Gamma}$ defined by:*

$$\Omega_{u,v} = \left(\phi_u\ \phi_v\ \phi_{uv}^{-1}\right)\left(\Phi_F(\mathrm{s}(u)\mathrm{s}(v)\mathrm{s}(uv)^{-1}\right)^{-1} \qquad (u, v \in \Gamma)$$

The 2-*cochain $(\Omega_{u,v})_{u,v\in\Gamma}$ induces a* 2-*cocycle $\Omega_\Lambda \in H^2(G(F/K), Z(G), L)$, which is independent of the choice of $\phi_u \in N$ modulo $Z(G)$ ($u \in G(F/K)$) in (a) above and of the set-theoretic section* s.

(*c*) *The set of all liftings $\Lambda' : G(F/K) \to N/G$ of the constant extension map $\lambda : G(F/K) \to N/CG$ modulo C exactly consists of those maps Λ' of the form $\Lambda' = \tilde{\theta} \cdot \Lambda$ where $\tilde{\theta}$ is any 1-cochain in $Z^1(G(F/K), CG/G, L_\Lambda^*)$* (1).

(*d*) *If $\tilde{\theta}$ is any* 1-*cochain in $Z^1(G(F/K), CG/G, L_\Lambda^*))$ and θ is the induced 1-cocycle in $H^1(G(F/K), CG/G, L_\Lambda^*)$, then the following conditions are equivalent:*

(i) $\Omega_\Lambda^{-1} = \delta^1(\theta)$

(ii) *There exists a K-model of the* (G-)*cover f with constant extension map (in Galois closure) equal to the map $\tilde{\theta} \cdot \Lambda : G(F/K) \to N/G$* (2).

(1) In particular, from the Main Theorem (I), the constant extension map of each K-model of f is of the form $\tilde{\theta} \cdot \Lambda$.

(2) In particular, condition (ii) actually depends only on θ in $H^1(G(F/K), CG/G, L_\Lambda^*)$ and not on the particular cochain representative $\tilde{\theta}$ in $Z^1(G(F/K), CG/G, L_\Lambda^*)$.

(*e*) *In particular the following conditions are equivalent:*

(i) $\Omega_\Lambda^{-1} \in \delta^1(H^1(G(F/K), CG/G, L_\Lambda^*))$

(ii) *The field of moduli* K *is a field of definition of the* (G-)*cover* f (3).

In other words, the field of moduli K is a field of definition if and only if at least one out of the 2-cocycles $\delta^1(\theta) \cdot \Omega_\Lambda$, where θ ranges over $H^1(G(F/K), CG/G, L_\Lambda^*)$, is trivial in $H^2(G(L/K), Z(G), L)$. We said in the introduction that the whole problem of whether the field of moduli K is a field of definition was controlled by several characteristic classes in $H^2(G(L/K), Z(G), L)$. These several characteristic classes are the 2-cocycles $\delta^1(\theta) \cdot \Omega_\Lambda$ ($\theta \in H^1(G(F/K), CG/G, L_\Lambda^*)$). Under condition ($\lambda$/Lift), the index set Δ of our simplified form of the Main Theorem given in the Introduction can be taken to be $\Delta = \delta^1(H^1(G(F/K), CG/G, L_\Lambda^*)$.

In the case of G-covers, we have $CG/G = \{1\}$. Thus there is only one characteristic class in $H^2(G(L/K), Z(G), L)$. Furthermore, in that case, the action L is the trivial action. The case of mere covers is different. There may be several possible constant extension maps Λ and to each of them corresponds a chance to descend the field of definition; the actual test is the vanishing of a well-defined 2-cocycle in $H^2(G(F/K), Z(G), L)$ attached to Λ.

3.3. Special case for which (Seq/Split) holds

In this paragraph we assume that condition (Seq/Split) holds (§2.9) and reformulate the Main Theorem in that situation. We let $\mathrm{s} : G(F/K) \to \Pi_K(B^*)$ denote a group-theoretic section to the map $\Pi_K(B^*) \twoheadrightarrow G(F/K)$. Under condition (Seq/Split), there is a more precise invariant of the K-models of the cover than the extension of constants (in Galois closure), namely, the action of $G(F/K)$ on the generic fiber associated with the section s (§2.9). In a rough way, descending the field of definition from F to K consists then in finding an action $G(F/K) \to N \subset S_d$ that is compatible with the given F-model of the (G-)cover f, *i.e.*, the given representation $\phi_F : \Pi_F(B^*) \twoheadrightarrow G \subset N$. "Compatible" means that the action should respect the semi-direct product structure $\Pi_K \simeq \Pi_F \times^s G(F/K)$ given by the section s. More precisely we have the following result.

MAIN THEOREM (III) (under (Seq/Split)). – *Assume that* K *is the field of moduli of the* (G-)*cover* f *relative to the extension* F/K *and that condition (Seq/Split) holds. Fix a section* $\mathrm{s} : G(F/K) \to \Pi_K(B^*)$. *Let* $\overline{\varphi} : \Pi_K(B^*) \to N/C$ *be the representation of* Π_K modulo C *given by the field of moduli condition* (§2.7).

(*a*) *For each lifting* $\Lambda : G(F/K) \to N/G$ *of* $\lambda : G(F/K) \to N/CG$, *the 2-cocycle* $\Omega_\Lambda \in H^2(G(F/K), Z(G), L)$ *of Part II is trivial if and only if there exists an homomorphism* $\varphi : G(F/K) \to N$ *that lifts the homomorphism* $\overline{\varphi} \circ s$ *and that induces* Λ *modulo* G.

(*b*) *In particular, the field of moduli K is a field of definition of the (G-)cover if and only if the homomorphism* $\overline{\varphi} \circ \mathrm{s} : G(F/K) \to N/C$ *has at least one lifting* $\varphi : G(F/K) \to N$ (4).

(3) In particular, condition (i) does not depend on the fixed lift Λ of λ.

(4) In particular, the latter condition does not depend on the choice of the section s.

(c) *More precisely, to each lifting* $\varphi : G(F/K) \to N$ *of the map* $\overline{\varphi} \circ s$ *corresponds a model over* K *of the* (G-)*cover* f, *which has the property that the action* $\varphi : G(F/K) \to N \subset S_d$ *is the arithmetic action of* $G(F/K)$ *on the generic fiber associated with the section* s. *Namely, this* K*-model is the one associated with the extension of* $\phi_F : \Pi_F(B^*) \twoheadrightarrow G \subset N$ *to* $\Pi_K(B^*) = \Pi_F(B^*) \times^s G(F/K)$ *that is equal to* φ *on* $G(F/K)$.

3.4. Concrete criteria

In this paragraph, we present some practical criteria for the field of moduli of a (G-)cover to be a field of definition. We recover all classical criteria as special cases. The main improvements are that our results are concerned with both G-covers and mere covers, of a base space B of arbitrary dimension, over a ground field of arbitrary characteristic and for F/K an arbitrary Galois extension. Furthermore, results of §3.4.1 do not assume that condition (Seq/Split) holds.

3.4.1. *Consequences of Main Theorem (Part II).* – It follows immediately from the Main Theorem (II) that, under condition (λ/Lift), the field of moduli K is a field of definition if the cohomological group $H^2(G(F/K), Z(G), L)$ is trivial. Recall that condition (λ/Lift) is automatically satisfied in the situation of G-covers.

COROLLARY 3.2. – *Under condition (λ/Lift), the field of moduli K is a field of definition if the center $Z(G)$ of G is trivial.*

This was well-known for G-covers of $\mathbb{P}^1$ but seems to be new for mere covers for which previous results involved so far the *centralizer* $C = \mathrm{Cen}_{S_d} G$. Combined with Prop. 3.1, Cor. 3.2 yields this criterion: a mere Galois cover with centerless group G such that $Inn(G)$ has a complement in $Aut(G)$ is automatically defined over its field of moduli.

COROLLARY 3.3. – *The field of moduli K is a field of definition if $G(F/K)$ is a projective profinite group.*

Indeed, if $G(F/K)$ is projective, then condition (λ/Lift) holds (Prop. 3.1 (a)) and the group $H^2(G(F/K), Z(G), L)$ is trivial.

Take $F = K_s$, that is, consider the absolute form of the problem. Cor. 3.3 requires then that $G(K)$ be projective. This holds if K is of cohomological dimension ≤ 1. Finite fields, $\overline{k}(T)$, $\overline{k}((T))$ (formal power series) with k any field, $\mathbb{Q}_p^{ur}$ (maximal unramified algebraic extension of $\mathbb{Q}_p$), $\mathbb{Q}^{ab}$, PAC fields are some classical examples of fields of cohomological dimension ≤ 1. Over these fields, the absolute field of moduli of a (G-)cover is a field of definition. Another consequence of Cor. 3.3 is that the field of moduli of a (G-)cover defined over $\overline{\mathbb{Q}}$ is the intersection of its fields of definition: indeed, from Artin-Scheier's theorem, each number field is the intersection of fields of cohomological dimension ≤ 1 (*see* Prop. 2.7 of [CoHa] which proves this for G-covers).

3.4.2. *Consequences of Main Theorem (Part III).* – In this paragraph we assume that condition (Seq/Split) holds. The three corollaries below use conclusion (b) of the Main Theorem (III): the field of moduli K is a field of definition of the (G-)cover if and only if the homomorphism $\overline{\varphi} \circ \mathrm{s} : G(F/K) \to N/C$ has at least one lifting $\varphi : G(F/K) \to N$. The first one is an immediate consequence.

COROLLARY 3.4. – *Under condition (Seq/Split), the field of moduli K is a field of definition if $C = \mathrm{Cen}_N G$ has a complement in N. This holds in particular in each of the following situations:*

– *G-covers for which $Z(G)$ is a direct summand of G (e.g. G abelian).*

– *Mere covers that are Galois* (use statement (*) in proof of Prop. 3.1).

The last situation is a generalization of Coombes-Harbater's theorem, which was established for covers of $\mathbb{P}^1$.

COROLLARY 3.5 (generalizes [Db2]). – *Under condition (Seq/Split), there exists a field of definition K_d with degree $[K_d : K]$ over the field of moduli bounded as follows.*

$$(for\ \text{G-}covers): \qquad [K_d : K] \leq \frac{|G|}{|Z(G)|}$$

$$(for\ mere\ covers): \qquad [K_d : K] \leq \frac{|\mathrm{Nor}_{S_d} G|}{|\mathrm{Cen}_{S_d} G|}$$

Proof. – Let K_d be the fixed field in F of $Ker(\overline{\varphi} \circ \mathrm{s})$. Apply conclusion (*b*) of the Main Theorem (III) with K taken to be K_d. The restriction of $(\overline{\varphi} \circ \mathrm{s})$ to $(G(F/K_d))$ is the trivial map and so can be lifted to an homomorphism $G(F/K_d) \to N$. Conclude that K_d is a field of definition of the (G-)cover. The desired estimates follow from

$$[K_d : K] = |G(F/K)/Ker(\overline{\varphi}_\mathrm{S})| = |(\overline{\varphi}_\mathrm{S})(G(F/K))| \leq |N/C|$$

and the definitions of N and C. $\square$

COROLLARY 3.6 (generalizes [Dew]). – *Under condition (Seq/Split), the field of moduli K is a field of definition if $G(F/K)$ is a finite cyclic group of order divisible by the exponent $exp(N)$ of the group N.*

Proof. – Let ζ be a generator of $G(F/K)$ and $n = [F : K]$. It is easy to lift the homomorphism $\overline{\varphi} \circ \mathrm{s} : G(F/K) \to N/C$ to some homomorphism $G(F/K) \to N$. Indeed it suffices to lift $\overline{\varphi} \circ \mathrm{s}(\zeta)$ up to an element $g \in N$ such that $g^n = 1$. Now because of the assumption on $exp(N)$, all elements $g \in N$ satisfy $g^n = 1$. $\square$

REMARK 3.7. – The main idea in Cor.3.6 about the use of cyclic extensions is due to E. Dew. He also notes that the hypotheses only have to be satisfied for an extension F' of F. Namely we have this more general result.

(1) *Let $f_F : X \to B$ be a* (G-)*cover over F with K as field of moduli. Assume that there exists a field F' such that the extension F'/K is cyclic of order a multiple of $exp(N)$ and (Seq/Split) holds with F' replacing F. Then K is a field of definition of the* (G-)*cover $f_F \otimes_F F'$ (but not necessarily of the original* (G-)*cover f_F).*

This may be used to give an alternate proof of the fact that over a field K with pro-cyclic absolute Galois group $G(K)$, the absolute field of moduli is a field of definition.

3.5. The local-to-global principle

The Main Theorem has the following application which we call the *local-to-global principle* for G-covers over $\overline{\mathbb{Q}}$. The base variety B is a regular projective irreducible variety defined over $\mathbb{Q}$.

THEOREM 3.8. – *A* G-*cover* $f : X \to B$ *over* $\overline{\mathbb{Q}}$ *is defined over* $\mathbb{Q}$ *if and only if it is defined over each completion* $\mathbb{Q}_p$ *of* $\mathbb{Q}$ *(including* $p = \infty$*). More generally, the same conclusion holds with* $\mathbb{Q}$ *replaced by any number field* K *such that the following special case does not hold.*

SPECIAL CASE. – *The special case comes from the special case of Grunwald's theorem* [ArTa]. *For each integer* $r > 0$, ζ_r *is a primitive* 2^r*th root of* 1 *and* $\eta_r = \zeta_r + \zeta_r^{-1}$. *Then denote by* s *the smallest integer such that* $\eta_s \in K$ *and* $\eta_{s+1} \notin K$. *The special case is defined by these three simultaneous conditions:*

1. $-1,\ 2+\eta_s,\ -(2+\eta_s)$ *are non-squares in* K.
2. *For each prime* $\mathfrak{p}$ *of* K *dividing* 2, *at least one out of the elements* $-1,\ 2+\eta_s$, $-(2+\eta_s)$ *is a square in* $K_{\mathfrak{p}}$.
3. *The abelian group* $Z(G)$ *contains an element of order a multiple of* 2^t *with* $t > s$.

If $K = \mathbb{Q}$, then $s = 2$ and $\eta_s = 0$. Since -1, 2 and -2 are non-squares in $\mathbb{Q}_2$, condition 2 cannot be satisfied. Therefore the special case does not occur if $K = \mathbb{Q}$. Similarly the special case does not occur if K contains $\sqrt{-1}$ or if K contains $\sqrt{-2}$ (5) or if $Z(G)$ is of odd order. Examples for which the special case holds are actually quite rare (*see* [ArTa]).

The local-to-global principle for G-covers between curves over an arbitrary number field was conjectured by E. Dew [Dew]. The special case $B = \mathbb{P}^1$ was first proved in [Db2] (except in the special case of Grunwald's theorem). The proof took advantage of the cohomological nature of the obstruction to the field of moduli being a field of definition. Thm. 3.8 extends this result to G-covers of more general base spaces B. In particular, B does not need to have a K-rational point.

It is unknown whether the local-to-global principle holds in the special case of Grunwald's theorem; no counter-example has yet been found. It is also unknown whether the local-to-global principle holds for mere covers in place of G-covers. We will devote a forthcoming paper to this question [DbDo1] (*see* also Remark 3.9). We will establish the local-to-global principle for mere covers under additional assumptions on the group G and the embedding $G \subset S_d$. We think however that the local-to-global principle is very unlikely to hold in general for mere covers. There is indeed for mere covers an extra obstruction, which is as before related to the fact that, unlike G-covers, mere covers may have several models with essentially distinct extensions of constants in Galois closure.

More precisely, if a given mere cover $f : X \to B$ over $\overline{\mathbb{Q}}$ has a model $f_K : X_K \to B$ over some number field K, then the local covers $f_K \otimes_K K_v$ obtained by extension of scalars from K to each completion K_v of K have this property: the extensions of constants

(5) Case "$\sqrt{-2} \in K$": In that case, either $\sqrt{-1} \in K$ and then condition 1 is not satisfied, or $\sqrt{-1} \notin K$, but then $\sqrt{2} \notin K$, which yields $s = 2$, $\eta_s = 0$ and $-(2+\eta_s) = -2$ is a square in K, *i.e.*, condition 1 not satisfied either.

in Galois closure $\widehat{K_v}/K_v$ all come by extension of scalars from a same extension $\widehat{K}/K$. Now, if instead it is only assumed that the mere cover f has a local model f_v over each K_v, one can hardly expect that the same be true for the associated constant extensions $\widehat{K_v}/K_v$. On the contrary one can imagine that they come from the constant extensions $\widehat{K_i}/K_i$ of *several* models over several fields K_i but that no K-model exists.

Proof of Thm. 3.8. – Let $f: X \to B$ be a G-cover defined over each completion K_v of a number field K. Since the (absolute) field of moduli is contained in each field of definition, it can be embedded in each completion K_v, where v runs over the set M_K of all places of K. It is a classical consequence of Cebotarev's theorem that this forces the field of moduli to be K. Consequently the ramification divisor is $G(K)$-invariant. From the Main Theorem then, the obstruction to K being a field of definition of f is measured by a certain 2-cocycle $\Omega \in H^2(K, Z(G))$ (with trivial action). By hypothesis, this 2-cocycle vanishes in $H^2(K_v, Z(G))$ for each place $v \in M_K$. Therefore the element Ω lies in the kernel of the map

$$H^2(K, Z(G)) \to \prod_{v \in M_K} H^2(K_v, Z(G)) \tag{2}$$

The rest of the proof consists in showing that this map is injective except possibly in the special case. By writing $Z(G)$ as a product of cyclic groups, one may reduce to the case $Z(G) = \mathbb{Z}/n\mathbb{Z}$. Then from the Tate-Poitou theorem [Se; II-§6.3], the kernel of the map (3) is in duality with the kernel of the map

$$H^1(K, \mu_n) \to \prod_{v \in M_K} H^1(K_v, \mu_n)$$

where $\mu_n = Hom(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m)$ is the group of nth roots of 1 in $\overline{K}$. Classically we have $H^1(K, \mu_n) \simeq K^\times/(K^\times)^n$. The result then follows from Grunwald's theorem [ArTa; Ch. 10]: for a global field, the natural map

$$K^\times/(K^\times)^n \to \prod_{v \in M_K} K_v^\times/(K_v^\times)^n$$

is injective except possibly in the special case described above (which corresponds to the special case of Grunwald's theorem in [ArTa] p. 96 with the extra condition $S = \emptyset$). $\square$

REMARK 3.9. – The proof extends to the case of mere covers for which the obstruction "field of moduli vs field of definition" can be measured by a *single* 2-cocycle $\Omega \in H^2(K, Z(G))$ with *trivial* action. In general the question is controlled by a family $(\Omega_\Lambda)_{\Lambda \in \Delta}$ of elements of $H^2(K, Z(G), L)$. The parametrizing set Δ, either is empty, or can be taken to be $\delta^1(H^1(K, CG/G, L_\Lambda^*))$. Therefore the local-global principle holds for mere covers for which

(1) Condition (λ/Lift) holds (which insures $\Delta \neq \emptyset$).

(2) $Z(G)$ is a direct summand of $C = \mathrm{Cen}_{S_d}(G)$ (which insures that the set $\delta^1(H^1(K, CG/G, L_\Lambda^*))$ is trivial).

(3) Elements of $Z(G)$ commute with those of $N = \mathrm{Nor}_{S_d} G$ (*i.e.*, $Z(G) \subset Z(N)$) (which insures that the action L is the trivial one).

We will elaborate on this in [DbDo1].

4. Proofs of the main results

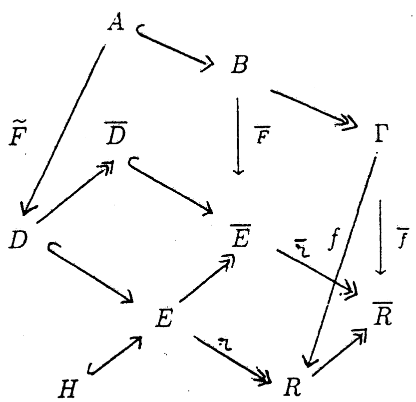
This section contains the proof of the Main Theorem. We start with a pure group theoretical problem in §4.1. This paragraph is the technical core of the paper. We then prove the Main Theorem in §4.2: appropriately formulated, it appears as a special case of §4.1. A basic assumption of the Main Theorem is condition (λ/Lift). In §4.3 we prove, under a minimal condition denoted (Band/Rep), an *iff* cohomological criterion for (λ/Lift) to hold.

4.1. A pure group theoretical problem

The problem below is about the possibility of extending a given group homomorphism with some extra constraints. We will give a cohomological solution (Thm. 4.3): more precisely we will produce a characteristic class Ω in a certain cohomological group with the property that the vanishing of it is equivalent to the possibility of extending the given homomorphism.

4.1.1. *Basic problem.*

DATA. – *A commutative diagram*



– *where all arrows are group homomorphisms,*

– *where* $\hookrightarrow$ *means that the group homomorphism in question is injective (we may then regard these maps as inclusions),*

– *where* $\twoheadrightarrow$ *means that the group homomorphism in question is surjective,*

– *where all sequences in which arrows are lined up are exact.*

HYPOTHESIS. – *We assume that* $H \subset \mathrm{Cen}_E D$.

QUESTION. – *Does there exist a group homomorphism* $F : B \to E$ *such that the enlarged diagram commutes, that is, such that:*

(i) F *extends* $\widetilde{F} : A \to D$, *i.e., the restriction of* F *to* A *equals* $\widetilde{F}$, *and*

(ii) F *induces* $\overline{F} : B \to \overline{E}$ *modulo* H *or, equivalently,* F *is a lifting of* $\overline{F} : B \to \overline{E}$, *and*

(iii) F *induces* $f : \Gamma \to R$ *over* Γ, *i.e.,* F *composed with the map* $E \twoheadrightarrow R$ *coincides with the map* $B \twoheadrightarrow \Gamma$ *composed with the map* f?

4.1.2. *Notation*

(a) **Cosets modulo** H. Given any element $n \in E$, the corresponding element in $\overline{E}$ via the map $E \twoheadrightarrow \overline{E}$ will be denoted by $\overline{n}$. For simplicity, we use the same notation, *i.e.*, we also use "bars", for images of elements of D via the map $D \twoheadrightarrow \overline{D}$ and for images of elements of R via the map $R \twoheadrightarrow \overline{R}$. Finally, given any map g with values in D or E or R, we denote by $\overline{g}$ the map defined by $\overline{g}(x) = \overline{g(x)}$.

(b) **The abelian group** $Z = D \cap H$. Under the assumption $H \subset \mathrm{Cen}_E D$, the group $D \cap H$ is an abelian group, which we denote by Z. The group $D \cap H$ is actually equal to the intersection $Z(D) \cap Z(H)$ of the respective centers of D and H.

(c) **Actions by conjugation**

– **Action of** $\overline{R}$ **on** Z: The kernel of the map $E \twoheadrightarrow \overline{R}$ obtained by composing the maps $E \twoheadrightarrow \overline{E}$ and $\overline{E} \twoheadrightarrow \overline{R}$ is the group DH. Since from (b) above, elements of DH commute with those of Z, the action of E by conjugation on Z factors through the map $E \twoheadrightarrow \overline{R}$ to yield an action of $\overline{R}$ on Z.

– **Action of** R **on** H: it follows from $H \subset \mathrm{Cen}_E D$ that the action of E by conjugation on H factors through the map $E \twoheadrightarrow R$ to yield an action of R on H.

– **Action of** $\overline{E}$ **on** D: it follows from $H \subset \mathrm{Cen}_E D$ that the action of E by conjugation on D factors through the map $E \twoheadrightarrow \overline{E}$ to yield an action of $\overline{E}$ on D.

For simplicity, we call these actions *actions by conjugation via* E.

(d) **Actions of** Γ

– The **action** L **of** Γ **on** Z is defined to be the action obtained by composing $f : \Gamma \to \overline{R}$ with the action by conjugation of $\overline{R}$ on Z (*via* E). This action plays a quite important role. It can be defined equivalently as follows: for each $d \in Z$ and each $c \in \Gamma$,

$$d^{L(c)} = d^e$$

where $e \in E$ is any preimage of $\overline{f}(c)$ via the map $E \twoheadrightarrow \overline{R}$.

– The **action** L_f **of** Γ **on** H is the action obtained by composing $f : \Gamma \to R$ with the action by conjugation of R on H (*via* E). It is readily checked that the action L_f, restricted to Z, coincides with the action L.

– The **action** L_f^* **of** Γ **on** HD/D is the action obtained by composing $f : \Gamma \to R$ with the action by conjugation of R on the kernel of $R \twoheadrightarrow \overline{R}$, which identifies to HD/D. It is readily checked that the action L_f of Γ on H induces an action of Γ on HD/D and that that action is the action L_f^* just defined.

4.1.3. *Conditions (FMod) and (rest/mod)*

The following conditions are important:

(FMod) *For each* $a \in A$ *and for each* $b \in B$ *we have*

$$\widetilde{F}(a^b) = \widetilde{F}(a)^{\overline{F}(b)}$$

(rest/mod) *The map* $\overline{\widetilde{F}} : A \to \overline{D}$ *induced by* $\widetilde{F}$ *modulo* H *coincides with the restriction* $\widetilde{\overline{F}} : A \to \overline{D}$ *of* $\overline{F}$ *to* A.

Condition (FMod) is the "Field of moduli" condition. This condition exactly corresponds to condition (FMod) of §2.7 in the context of (G-)covers. Condition (rest/mod) is concerned with the corner $A \to \overline{D}$ in diagram 4.1.1. Maps $\overline{\widetilde{F}}$ and $\widetilde{\overline{F}}$ are two natural maps $A \to \overline{D}$. Condition (rest/mod) insures that both maps are equal. The name (rest/mod) is meant to suggest that "restriction" commutes with "modding out by H".

PROPOSITION 4.1. – *Assume that Basic Problem* 4.1.1 *has an affirmative answer,* i.e., *that there is an homomorphism F extending $\widetilde{F}$ and inducing $\overline{F}$ and f. Then both conditions (FMod) and (rest/mod) hold true.*

Proof. – Assume that F exists. Let $a \in A$ and $b \in B$. We have:

$$\widetilde{F}(a^b) = F(a^b) = F(a)^{F(b)} = \widetilde{F}(a)^{F(b)} = \widetilde{F}(a)^{\overline{F}(b)}$$

thus proving condition (FMod). Next we have $F(a) = \widetilde{F}(a)$. So $\overline{F}(a) = \overline{\widetilde{F}}(a)$. This proves that $\overline{\widetilde{F}}$ coincides with the restriction of $\overline{F}$ to A, *i.e.*, that condition (rest/mod) holds. □

In addition to condition (FMod) and (rest/mod), what else is needed for Basic Problem 4.1.1 to have an affirmative answer? The answer is Thm. 4.3 below.

4.1.4. *Solution to Basic Problem 4.1.1*

We respectively denote the maps $E \twoheadrightarrow R$ and $\overline{E} \twoheadrightarrow \overline{R}$ by r and $\overline{r}$.

LEMMA 4.2. – *Let $s : \Gamma \to B$ be an arbitrary set-theoretic section to the map $B \twoheadrightarrow \Gamma$. For each $u \in \Gamma$, there exists an element $\phi_u \in E$, unique modulo Z, such that*

$$\begin{cases} r(\phi_u) = f(u) \\ \overline{\phi_u} = \overline{F} \circ \mathrm{s}(u) \end{cases} \tag{1}$$

Proof. – Pick $\phi_u^o \in E$ such that $r(\phi_u^o) = f(u)$. Then we have

$$\begin{aligned} \overline{r}(\overline{\phi_u^o}) &= \overline{r(\phi_u^o)} \\ &= \overline{f(u)} \\ &= \overline{f}(u) \\ &= \overline{r} \circ \overline{F}(\mathrm{s}(u)) \end{aligned}$$

Consequently $\overline{\phi_u^o} = \overline{F}(\mathrm{s}(u)).\ \overline{d_u}$ for some $d_u \in D$. Set

$$\phi_u = \phi_u^o.(d_u)^{-1}$$

Then ϕ_u satisfies (1). Uniqueness of ϕ_u modulo $Z = D \cap H$ is clear. □

For each choice of a set-theoretic section s and a family $(\phi_u)_{u\in\Gamma}$ as in Lemma 4.2, we now define elements $\Phi_{u,v} \in E$, $\gamma_{u,v} \in A$ and $\Omega_{u,v} \in E$ $(u, v \in \Gamma)$, in the following way.

$$\begin{cases} \Phi_{u,v} = \phi_u\ \phi_v\ \phi_{uv}^{-1} \\ \gamma_{u,v} = \mathrm{s}(u)\ \mathrm{s}(v)\ \mathrm{s}(uv)^{-1} \\ \Omega_{u,v} = \Phi_{u,v}\ (\widetilde{F}(\gamma_{u,v}))^{-1} \end{cases} \tag{2}$$

THEOREM 4.3. – *Assume that both conditions (FMod) and (rest/mod) hold.*

(*a*) *The* 2*-cochain $(\Omega_{u,v})_{u,v\in\Gamma}$ defines a* 2*-cocycle of* Γ *with values in the abelian group $Z = D \cap H$ and for the action L on Z.*

(*b*) *Modulo the coboundaries, the* 2*-cocycle* $(\Omega_{u,v})_{u,v\in\Gamma}$ *is independent*

– of the set-theoretic section $s:\Gamma\rightarrow B$*, and*

– of the family $(\phi_u)_{u\in\Gamma}$ *satisfying (1) in Lemma* 4.2.

(*c*) *Basic Problem 4.1.1 has an affirmative answer*, i.e., *there is an homomorphism* F *extending* $\widetilde{F}$ *and inducing* $\overline{F}$ *and* f*, if and only the 2-cocycle* Ω *is trivial in* $H^2(\Gamma, Z, L)$.

Proof of Thm. 4.3 (*a*).

Step 1: Ω *has value in* $Z = D\cap H$: Let $u,v\in\Gamma$. Classically $\gamma_{u,v}$ is in A. Consequently $\widetilde{F}(\gamma_{u,v})$ is defined and lies in D. *A priori* $\Phi_{u,v}$ is in E but using (1) we obtain

$$\begin{aligned} r(\Phi_{u,v}) &= r(\phi_u)r(\phi_v)r(\phi_{uv})^{-1}\\ &= f(u)f(v)f(uv)^{-1}\\ &= 1\end{aligned}$$

which proves that $\Phi_{u,v}$ is in D. Thus $\Omega_{u,v}\in D$. It remains to prove that $\Omega_{u,v} = \Phi_{u,v}.(\widetilde{F}(\gamma_{u,v}))^{-1}$ is in H. This follows from

$$\begin{aligned} \overline{\Phi_{u,v}} &= \overline{\phi_u}\ \overline{\phi_v}\ \overline{\phi_{uv}}^{-1}\\ &= \overline{F}\circ \mathrm{s}(u)\ \overline{F}\circ \mathrm{s}(v)\ (\overline{F}\circ \mathrm{s}(uv))^{-1} && \text{(from (1))}\\ &= \overline{F}(\gamma_{u,v})\\ &= \overline{\widetilde{F}}(\gamma_{u,v}) && \text{(from (rest/mod))}\\ &= \overline{\widetilde{F}(\gamma_{u,v})}\end{aligned}$$

Step 2: *Cocycle condition.* The following formulas are straightforward:

$$(3)\qquad \begin{cases}\Phi_{g,h}^{-1}\ (\Phi_{h,k})^{\phi_g}\ \Phi_{g,hk}\ \Phi_{gh,k}^{-1} = 1\\ \gamma_{g,h}^{-1}\ (\gamma_{h,k})^{\mathrm{S}(g)}\ \gamma_{g,hk}\ \gamma_{gh,k}^{-1} = 1\end{cases}$$

Apply $\widetilde{F}$ to the second one and use that

$$\begin{aligned}\widetilde{F}((\gamma_{h,k})^{\mathrm{S}(g)}) &= (\widetilde{F}(\gamma_{h,k}))^{\overline{F}(\mathrm{S}(g))} && \text{(from (FMod))}\\ &= (\widetilde{F}(\gamma_{h,k}))^{\overline{\phi}_g} && \text{(from Lemma 4.2)}\end{aligned}$$

to obtain this formula

$$(4)\qquad \widetilde{F}(\gamma_{g,h})^{-1}\ \widetilde{F}(\gamma_{h,k})^{\phi_g}\ \widetilde{F}(\gamma_{g,hk})\ \widetilde{F}(\gamma_{gh,k})^{-1} = 1$$

Combine it with the first formula of (3) to obtain that $\Omega_{u,v} = \Phi_{u,v}.(\widetilde{F}(\gamma_{u,v}))^{-1}$ satisfies

$$(5)\qquad \Omega_{g,h}^{-1}\ (\Omega_{h,k})^{\phi_g}\ \Omega_{g,hk}\ \Omega_{gh,k}^{-1} = 1$$

[Hint: prove the equivalent formula $(\Omega_{h,k})^{\phi_g}\Omega_{g,hk} = \Omega_{g,h}\Omega_{gh,k}$. Note that the $\Omega_{u,v}$s commute with elements of both D and H.]

Note eventually that, since $\Omega_{h,k} \in Z$, we have

$$\Omega_{h,k}^{\phi_g} = \Omega_{h,k}^{\overline{r(\phi_g)}} = \Omega_{h,k}^{\overline{f}(g)} = \Omega_{h,k}^{L(g)}$$

Conclude that condition (5) is indeed the cocycle condition in $H^2(\Gamma, Z, L)$. $\square$

Proof of Thm. 4.3 (b). – Let $\mathrm{s}' : \Gamma \to B$ be another set-theoretic section of $B \twoheadrightarrow \Gamma$. From Lemma 4.2, for each $u \in \Gamma$, we may choose an element $\phi'_u \in N$, uniquely determined modulo Z, such that

$$\begin{cases} r(\phi'_u) = f(u) \\ \overline{\phi'_u} = \overline{F} \circ \mathrm{s}'(u) \end{cases}$$

As above define a 2-cocycle $\Omega' = (\Omega'_{u,v})_{u,v\in\Gamma}$ by

$$\begin{cases} \Phi'_{u,v} = \phi'_u \ \phi'_v \ (\phi'_{uv})^{-1} \\ \gamma'_{u,v} = \mathrm{s}'(u) \ \mathrm{s}'(v) \ (\mathrm{U}s'(uv))^{-1} \\ \Omega'_{u,v} = \Phi'_{u,v} \ (\widetilde{F}(\gamma'_{u,v}))^{-1} \end{cases}$$

We need to show that the 2-cocycles $(B_{u,v})_{u,v\in\Gamma}$ and $(B'_{u,v})_{u,v\in\Gamma}$ differ by a coboundary.

Let $\delta : \Gamma \to B$ be the set-theoretic map defined by $\delta(u) = \mathrm{s}(u)^{-1}\mathrm{s}'(u)$. We have $\delta(\Gamma) \subset A$. Let $u \in \Gamma$. Using the uniqueness modulo Z in Lemma 4.2, one obtains

$$\phi'_u = \phi_u \ \widetilde{F}(\delta(u)) \ \zeta_u \qquad \text{for some } \zeta_u \in Z$$

Then set

$$c_u = \widetilde{F}(\delta(u)) \ \zeta_u = \zeta_u \ \widetilde{F}(\delta(u)) \tag{6}$$

The following formulas are straightforward:

$$\begin{cases} \Omega'_{u,v} = \Omega_{u,v} \ \widetilde{F}(\gamma_{u,v})(c_u^{\phi_v^{-1}} \ c_v \ c_{uv}^{-1})^{\phi_{uv}}(\widetilde{F}(\gamma'_{u,v}))^{-1} \\ \gamma'_{u,v} = \gamma_{u,v}(\delta(u)^{\mathrm{S}(v)^{-1}} \delta(v) \ \delta(uv)^{-1})^{\mathrm{S}(uv)} \end{cases} \tag{7}$$

Apply $\widetilde{F}$ to the second one to obtain

$$\widetilde{F}(\gamma'_{u,v}) = \widetilde{F}(\gamma_{u,v})(\widetilde{F}(\delta(u))^{\phi_v^{-1}} \widetilde{F}(\delta(v)) \ \widetilde{F}(\delta(uv))^{-1})^{\phi_{uv}}$$

Substituting back in the first one yields

$$\Omega'_{u,v} = \Omega_{u,v}[(c_u^{\phi_v^{-1}} \ c_v \ c_{uv}^{-1})^{\phi_{uv}}[(\widetilde{F}(\delta(u))^{\phi_v^{-1}} \widetilde{F}(\delta(v)) \ \widetilde{F}(\delta(uv))^{-1})^{\phi_{uv}}]^{-1}]^{\widetilde{F}(\gamma_{u,v})}$$

Now (6) gives

$$c_u^{\phi_v^{-1}} \ c_v \ c_{uv}^{-1} = (\zeta_u^{\phi_v^{-1}} \ \zeta_v \ \zeta_{uv}^{-1})(\widetilde{F}(\delta(u))^{\phi_v^{-1}} \widetilde{F}(\delta(v)) \ \widetilde{F}(\delta(uv))^{-1})$$

Whence

$$\Omega'_{u,v} = \Omega_{u,v}[(\zeta_u^{\phi_v^{-1}} \ \zeta_v \ \zeta_{uv}^{-1})^{\phi_{uv}}]^{\widetilde{F}(\gamma_{u,v})}$$

Finally set $\xi_u = \zeta_u^{\phi_u} = \zeta_u^{L(u)}$ for each $u \in \Gamma$ to obtain

$$\Omega'_{u,v} = \Omega_{u,v} \ [\xi_u \xi_v^{L(u)} \xi_{uv}^{-1}]^{\ \widetilde{F}(\gamma_{u,v})}$$

The 2-cochain $([\xi_u \xi_v^{L(u)} \xi_{uv}^{-1}]\ {}^{\widetilde{F}(\gamma_{u,v})})_{u,v\in\Gamma}$ is as desired a coboundary of Γ with values in the abelian group Z and for the action L on Z. $\square$

Proof of Thm. 4.3 (c).

Step 1: *Necessary form of F.* Denote the map $B \twoheadrightarrow \Gamma$ by ρ and fix a set-theoretic section $\mathrm{s} : \Gamma \to B$ of ρ. Then each element $\mathrm{u} \in B$ can be written in a unique way

$$\mathrm{u} = a\ \mathrm{s}(u) \text{ where } a \in A \qquad \text{and} \qquad u = \rho(\mathrm{u})$$

If $F : B \to E$ is any homomorphism extending $\widetilde{F} : A \to D$ and inducing $\overline{F} : B \to \overline{E}$ and $f : \Gamma \to R$, then we have, on one hand $F(a) = \widetilde{F}(a)$, and, on the other hand,

$$\begin{cases} r(F(\mathrm{s}(u)) = f(u) \\ \overline{F(\mathrm{s}(u))} = \overline{F} \circ \mathrm{s}(u) \end{cases}$$

Lemma 4.2 then provides

$$F(\mathrm{s}(u)) = \phi_u c_u \qquad \text{for some } \ c_u \in Z$$

Conclude that $F(\mathrm{u})$ is necessarily of the form

$$F(\mathrm{u}) = \widetilde{F}(a)\ \phi_u c_u \qquad \text{for some } \ c_u \in Z \tag{8}$$

Step 2: *Iff condition for F to be a group homomorphism.* Consider a set-theoretic map $F : B \to E$ defined as in (8). By construction, F extends $\widetilde{F}$ and induces $\overline{F}$ and f. It remains to find out whether F is a group homomorphism. Let

$$\begin{cases} \mathrm{u} = x\ \mathrm{s}(u) \\ \mathrm{v} = y\ \mathrm{s}(v) \end{cases} \quad \text{with} \quad \begin{matrix} x \in A \text{ and } u = \rho(\mathrm{u}) \\ y \in A \text{ and } v = \rho(\mathrm{v}) \end{matrix}$$

be two arbitrary elements of B. We have

$$\begin{aligned} \mathrm{uv} &= x\ \mathrm{s}(u)\ y\ \mathrm{s}(v) \\ &= x\ y^{\mathrm{S}(u)}\ \mathrm{s}(u)\mathrm{s}(v)\mathrm{s}(uv)^{-1}\ \mathrm{s}(uv) \\ &= x\ y^{\mathrm{S}(u)}\ \gamma_{u,v}\ \mathrm{s}(uv) \end{aligned}$$

whence

$$\begin{aligned} F(\mathrm{uv}) &= \widetilde{F}\Big(x\ y^{\mathrm{S}(u)}\ \gamma_{u,v}\Big)\ \phi_{uv} c_{uv} \\ &= \widetilde{F}(x)\ \widetilde{F}(y)^{\phi_u}\ \widetilde{F}(\gamma_{u,v})\ \phi_{uv} c_{uv} \end{aligned}$$

On the other hand we have

$$\begin{aligned} F(\mathrm{u})F(\mathrm{v}) &= \widetilde{F}(x)\ \phi_u c_u\ \widetilde{F}(y)\ \phi_v c_v \\ &= \widetilde{F}(x)\ \widetilde{F}(y)^{\phi_u c_u}\ \phi_u c_u\ \phi_v c_v \\ &= \widetilde{F}(x)\ \widetilde{F}(y)^{\phi_u}\ \phi_u c_u\ \phi_v c_v \end{aligned}$$

Therefore F is a group homomorphism if and only if for all $u, v \in \Gamma$ we have

$$\widetilde{F}(\gamma_{u,v})\ \phi_{uv}c_{uv} = \phi_u c_u\ \phi_v c_v$$

Multiplying both terms to the right by the inverse of the left hand side term leads to the equivalent formula

$$(c_u^{\phi_v^{-1}}\ c_v\ c_{uv}^{-1})^{\phi_u\phi_v}\ \Omega_{u,v} = 1$$

Finally set $d_u = (c_u^{-1})^{\phi_u} = (c_u^{-1})^{L(u)}$ ($u \in \Gamma$) to conclude that F is a group homomorphism if and only if for all $u, v \in \Gamma$ we have

$$\Omega_{u,v} = d_u\ d_v^{L(u)}\ d_{uv}^{-1} \tag{9}$$

Step 3: *Conclusion.* Assume Basic Problem 4.1.1 has an affirmative answer, *i.e.*, that there is an homomorphism F extending $\widetilde{F}$ and inducing $\overline{F}$ and f. Then $F(\mathrm{u})$ ($\mathrm{u} \in B$) is of the form (8) (Step 1) and (9) holds (Step 2). So the 2-cocycle Ω is trivial in $H^2(\Gamma, Z, L)$. Conversely, if Ω is trivial in $H^2(\Gamma, Z, L)$, then (9) holds for some family $(d_u)_{u\in\Gamma}$ of elements of Z. Set $c_u = (d_{\underline{u}}^{-1})^{\phi_u^{-1}}$ ($u \in \Gamma$) and define F as in (8). From Step 2, F is an homomorphism extending $\widetilde{F}$ and inducing $\overline{F}$ and f. □

4.1.5. *Dependence in* f

In the diagram of Basic Problem 4.1.1, the map $f : \Gamma \to R$ is a lifting of $\overline{f} : \Gamma \to \overline{R}$. In this section we study how the 2-cocycle $\Omega = \Omega_f$ is changed when f is changed to another lifting f' of $\overline{f}$.

PROPOSITION 4.4. – (*a*) *The set of all liftings of* $\overline{f}$ *exactly consists of those maps* $f' : \Gamma \to R$ *of the form* $f' = \theta \cdot f$ *where* θ *is any* 1*-cocycle in* $Z^1(\Gamma, HD/D, L_f^*)$.

(*b*) *Let* $\theta_1, \theta_2 \in Z^1(\Gamma, HD/D, L_f^*)$. *Let* $f_i' = \theta_i \cdot f$ *be the corresponding liftings of* $\overline{f}$, $i = 1, 2$. *If the* 1*-cocycles* θ_1, θ_2 *are cohomologous, then the* 2*-cocycles* $\Omega_{f_1'}, \Omega_{f_2'} \in H^2(\Gamma, Z, L)$ *are equal.*

Proof. – (*a*) follows straightforwardly from the definition of $Z^1(\Gamma, HD/D, L_f^*)$, which is the set of all maps $\theta : \Gamma \to HD/D$ satisfying the cocycle condition

$$\theta(uv) = \theta(u)\ \theta(v)^{f(u)}$$

(*b*) Assume that θ_1 and θ_2 are cohomologous, *i.e.*, there exists $h \in HD/D$ such that

$$\theta_2(u) = h^{-1}\theta_1(u)h^{f(u)} \qquad \text{for all } u \in \Gamma$$

For each $u \in \Gamma$, denote by $\phi_{i,u}'$ the unique element modulo Z satisfying the conclusion of Lemma 4.2 with f_i' replacing f. Lift each element $\theta_i(u) \in HD/D$ to an element $\widetilde{\theta_i(u)} \in H$ and lift the element $h \in HD/D$ to an element $\tilde{h} \in H$. Then it is readily checked that one can take, for each $u \in \Gamma$

$$\begin{cases} \phi_{1,u}' = \widetilde{\theta_1(u)}\ \phi_u \\ \phi_{2,u}' = (\phi_{1,u}')^{(\tilde{h})^{-1}} \end{cases}$$

It follows that

$$\begin{aligned}
\left(\Omega_{f_2'}\right)_{u,v} &= \phi'_{2,u}\ \phi'_{2,v}\ (\phi'_{2,uv})^{-1}\ (\widetilde{F}(\gamma_{u,v}))^{-1} \\
&= (\phi'_{1,u})^{(\tilde{h})^{-1}}\ (\phi'_{1,v})^{(\tilde{h})^{-1}}\ ((\phi'_{1,uv})^{(\tilde{h})^{-1}})^{-1}\ (\widetilde{F}(\gamma_{u,v}))^{-1} \\
&= \left(\Omega_{f_1'}\right)_{u,v}^{(\tilde{h})^{-1}} \\
&= \left(\Omega_{f_1'}\right)_{u,v} \qquad (\text{since } \tilde{h} \in H \subset Cen_E D) \qquad \square
\end{aligned}$$

From Prop. 4.4, the set of characteristic classes $\Omega_{f'}$, when f' ranges over the lifts of $\overline{f}$, can be parametrized by $H^1(\Gamma, HD/D, L_f^*)$. Prop. 4.5 below is more precise. The kernel $Z = D \cap H$ of the exact sequence

$$1 \to Z \to H \to HD/D \to 1$$

is abelian and central in H. Consequently, the correspondence that maps each 1-cocycle $\theta \in Z^1(\Gamma, HD/D, L_f^*)$ to the 2-cochain $\tilde{\theta}(u)\tilde{\theta}(v)^{L_f(u)}\tilde{\theta}(uv)^{-1}$, where, for each $h \in \Gamma$, $\tilde{\theta}(h) \in H$ is a lift of $\theta(h) \in HD/D$, induces a map –the coboundary operator

$$\delta^1 : H^1(\Gamma, HD/D, L_f^*) \to H^2(\Gamma, Z, L)$$

For each $\theta \in Z^1(\Gamma, HD/D, L_f)$, $\delta^1(\theta)$ is the obstruction to the possibility of lifting θ up to a 1-cocycle $\tilde{\theta} \in Z^1(\Gamma, H, L_f)$.

PROPOSITION 4.5. – *Let $\theta \in Z^1(\Gamma, HD/D, L_f)$ and $f' = \theta \cdot f$ be the corresponding lifting of $\overline{f}$. Then we have*

$$\Omega_{f'} = \delta^1(\theta)\ \Omega_f$$

Proof. – For each $u \in \Gamma$, denote by ϕ'_u the unique element modulo Z satisfying the conclusion of Lemma 4.2 with f' replacing f. It is readily checked that one can take $\phi'_u = \tilde{\theta}(u)\ \phi_u$. It follows that

$$\begin{aligned}
(\Omega_{f'})_{u,v} &= \phi'_u\ \phi'_v\ (\phi'_{uv})^{-1}\ (\widetilde{F}(\gamma_{u,v}))^{-1} \\
&= \tilde{\theta}(u)\phi_u\ \tilde{\theta}(v)\phi_v\ (\tilde{\theta}(uv)\phi_{uv})^{-1}\ (\widetilde{F}(\gamma_{u,v}))^{-1} \\
&= \tilde{\theta}(u)\phi_u\ \tilde{\theta}(v)\phi_v\ ((\phi_u\phi_v)^{-1}\ (\Omega_f)_{u,v}\ \widetilde{F}(\gamma_{u,v}))\ \tilde{\theta}(uv)^{-1}\ (\widetilde{F}(\gamma_{u,v}))^{-1} \\
&= \tilde{\theta}(u)\ \tilde{\theta}(v)^{\phi_u}\ (\Omega_f)_{u,v}\ \tilde{\theta}(uv)^{-1} \\
&= \tilde{\theta}(u)\ \tilde{\theta}(v)^{L_f(u)}\ \tilde{\theta}(uv)^{-1}\ (\Omega_f)_{u,v} \\
&= (\delta^1(\theta)\ \Omega_f)_{u,v} \qquad \square
\end{aligned}$$

4.2. Proof of the Main Theorem

Notation is that of §2 and §3. We fix a Galois extension F/K and a (G-)cover $f : X \to B$ defined over F. The base space B is assumed to be defined over K. Let $\phi_F : \Pi_F(B^*) \to G \subset N$ be the homomorphism associated to the (G-)cover f. Assume that K is the field of moduli of the (G-)cover f. Then the ramification divisor D is $G(K)$-invariant and condition (FMod) holds, that is,

(FMod) *For each* $\mathrm{U} \in \Pi_K(B^*)$, *there exists* $\varphi_{\mathrm{U}} \in N$ *such that*

$$\phi_F(x^{\mathrm{U}}) = \phi_F(x)^{\varphi_{\mathrm{U}}} \qquad \text{(for all } x \in \Pi_F(B^*))$$

As in §2.7, denote by $\overline{\varphi} : \Pi_K(B^*) \to N/C$ the representation of Π_K modulo C given by the field of moduli condition. Recall the definition of $\overline{\varphi}$. Given any element $n \in N$, the corresponding coset modulo C will be denoted by $\overline{n}$. Then for each $\mathrm{U} \in \Pi_K(B^*)$, $\overline{\varphi}(\mathrm{U})$ is defined to be the coset $\overline{\varphi_{\mathrm{U}}}$ of any element $\varphi_{\mathrm{U}} \in N$ satisfying the formula above in the (FMod) condition. The right-hand side term then rewrites

$$\phi_F(x)^{\varphi_{\mathrm{U}}} = \phi_F(x)^{\overline{\varphi}(\mathrm{U})}$$

PROPOSITION 4.6 (Condition (rest/mod)). – *For each* $u \in \Pi_F(B^*)$, *we have*

$$\overline{\phi_F(u)} = \overline{\varphi}(u)$$

That is, the restriction of the map $\overline{\varphi}$ *to* $\Pi_F(B^*)$ *coincides with the map* $\overline{\phi_F} : \Pi_F(B^*) \to N/C$ *induced by* $\phi_F : \Pi_F(B^*) \to N$ *modulo* C.

Proof. – For $u \in \Pi_F(B^*)$, we have $\phi_F(x^u) = \phi_F(x)^{\phi_F(u)}$ for all $x \in \Pi_F(B^*)$. That is, one can take $\varphi_u = \phi_F(u)$ in condition (FMod). □

4.2.1. *Proof of Main Theorem (I).* – From Prop. 4.6, $\overline{\varphi}(\Pi_F(B^*)) \subset G/C \cap G \simeq CG/C$. Thus there exists an homomorphism $\lambda : G(F/K) \to N/CG$ such that the following diagram commutes

$$\begin{tikzcd} \Pi_K(B^*) \arrow[r] \arrow[d, "\overline{\varphi}"'] & G(F/K) \arrow[d, "\lambda"] \\ N/C \arrow[r] & N/CG \end{tikzcd}$$

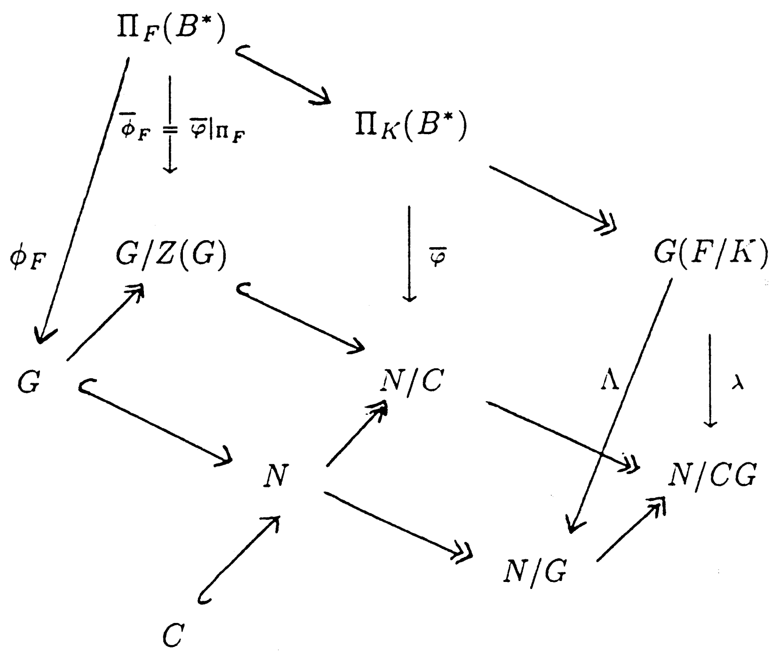
Furthermore, since the map $\Pi_K \to G(F/K)$ is onto, such an homomorphism λ is unique. This proves (a).

Let f_K be a K-model of the (G-)cover f. Let $\phi_K : \Pi_K \to N$ be the associated extension of $\phi_F : \Pi_F \to N$. As in Prop. 4.6 above, it is shown that, for each $\mathrm{U} \in \Pi_K(B^*)$,

$$\overline{\phi_K(\mathrm{U})} = \overline{\varphi}(\mathrm{U})$$

That is, $\phi_K : \Pi_K \to N$ is a lifting of $\overline{\varphi} : \Pi_K(B^*) \to N/C$ ([6]). Consider the constant extension map (in Galois closure) $\Lambda : G(F/K) \to N/G$ of the K-model of the (G-)cover f. From §2.8, it is the map induced by ϕ_K on $G(F/K)$ modulo G. Therefore the map $\Lambda : G(F/K) \to N/G$ should necessarily be a lifting of $\lambda : G(F/K) \to N/CG$. This proves ($b$). Finally ($c$) follows immediately from (b). □

4.2.2. *Proof of Main Theorem (II).* – We assume here that condition (λ/Lift) holds. We fix a lifting $\Lambda : G(F/K) \to N/G$ of $\lambda : G(F/K) \to N/CG$. The situation corresponds to the following diagram,



which is a diagram as in Basic Problem 4.1.1. The hypothesis of Basic Problem 4.1.1, here "$C \subset \mathrm{Cen}_N G$" follows from the definition of $C = \mathrm{Cen}_N G$. Both conditions (FMod) and (rest/mod) have been checked above. The existence of a solution $\phi_K : \Pi_K(B^*) \to N$ to Basic Problem 4.1.1 in that situation exactly corresponds to the possibility of descending the field of definition of the (G-)cover f from F to its field of moduli K with the extra property that the constant extension in Galois closure of the K-model be given by the map Λ. Part II of the Main Theorem corresponds to the conclusions of §4.1 in the specific context of (G-)covers. More precisely, conclusion (a) corresponds to Lemma 4.2, conclusion (b) to Thm. 4.3 (a) and (b), conclusion (c) to Prop. 4.4 (a), conclusion (d) to Thm. 4.3 (c) combined with Prop. 4.4 (b) and Prop. 4.5.

([6]) The converse is not true: an arbitrary lifting $\phi_K : \Pi_K \to N$ of $\overline{\varphi} : \Pi_K(B^*) \to N/C$ need not extend $\phi_F : \Pi_F \to G$.

Finally conclusion (*e*) is a straightforward consequence of conclusions (*c*) and (*d*) (along with the Main Theorem (I)). Namely, assume that the field of moduli K is a field of definition of the (G-)cover f. Then from the Main Theorem (I), the constant extension map Λ' of a K-model of f is a lifting of λ. Therefore, from the Main Theorem (II)(*c*), Λ' is of the form $\Lambda' = \tilde{\theta} \cdot \Lambda$ for some $\tilde{\theta} \in Z^1(G(F/K), CG/G, L_\Lambda^*)$. Denote by θ the element of $H^1(G(F/K), CG/G, L_\Lambda^*)$ induced by $\tilde{\theta}$. From the Main Theorem (II) (*d*), we have $\Omega_\Lambda^{-1} = \delta^1(\theta) \in \delta^1(H^1(G(F/K), CG/G, L_\Lambda^*))$. The converse follows even more immediately from (i) $\Rightarrow$ (ii) in conclusion (*d*). $\square$

4.2.3. *Proof of Main Theorem (III).* – We assume condition (Seq/Split) holds. We fix a group-theoretic section $\mathrm{s} : \Pi_K(B^*) \to G(F/K)$.

Proof of (a). – Fix a lifting Λ of λ. The set-theoretic section $\Gamma \to B$ of §4.1 can be taken to be here the group-theoretic section s. The 2-cocycle $\Omega = \Omega_\Lambda$ gets simpler since, with notation of §4.1, we have then $\gamma_{u,v} = 1$ and so $\Omega_{u,v} = \phi_u\, \phi_v\, \phi_{uv}^{-1}$. It follows immediately from that and the definition of ϕ_u that the vanishing of Ω corresponds to the existence of a lifting $\varphi : G(F/K) \to N$ of $\overline{\varphi} \circ \mathrm{s}$ that induces Λ modulo G.

Proof of (b). – If the field of moduli K is a field of definition, then the map λ has at least one lifting Λ and the associated 2-cocycle Ω_Λ is trivial. From (a) above, the homomorphism $\overline{\varphi} \circ \mathrm{s}$ has some lifting φ. Conversely, assume that $\overline{\varphi} \circ \mathrm{s}$ has some lifting $\varphi : G(F/K) \to N$. Then this homomorphism φ induces an homomorphism $\Lambda : G(F/K) \to N/G$ modulo G. This map Λ is a lift of λ. From (a) above, the associated 2-cocycle Ω_Λ is trivial. Therefore, from the Main Theorem (II), the field of moduli K is a field of definition.

Proof of (c). – (*c*) follows immediately from the definition of the arithmetic action of $G(F/K)$ on the generic fiber associated with the section s (§2.9). $\square$

4.3. Condition (λ/Lift)

The following condition.

(λ/Lift) *There exists at least one lifting $\Lambda : G(F/K) \to N/G$ of the constant extension map (in Galois closure) $\lambda : G(F/K) \to N/CG$ given by the field of moduli condition.*

is a basic assumption of the Main Theorem (II). Recall also that it is anyhow a necessary condition for the field of moduli to be a field of definition. In this paragraph, we give, under a minimal assumption denoted (Band/Rep), an *iff* cohomological criterion for condition (λ/Lift) to hold.

Condition (λ/Lift) corresponds to the weak solvability of the following embedding problem (by "weak" we mean that the solution Λ need not be surjective).

$$(\mathrm{EP}_o) \qquad \begin{array}{ccccc} & & & & G(F/K) \\ & & & \swarrow^{\Lambda} & \big\downarrow \lambda \\ CG/G & \hookrightarrow & N/G & \xrightarrow{\overline{p}} & N/CG \end{array}$$

This problem is trivial for G-covers since for G-covers, we have $N = G$ and so $N/CG = N/G = \{1\}$. The question is also quite classical in the case that the kernel CG/G is abelian and contained in the center of N/G. Namely the question is equivalent to the vanishing of a characteristic class in $H^2(\Gamma, CG/G)$ (with trivial action).

The question is a little more delicate when the kernel CG/G is an arbitrary group. First, there is no natural action of N/CG on the kernel CG/G, but only an *outer action*, *i.e.*, an homomorphism $\overline{\kappa} : N/CG \to Out(CG/G)$ from N/CG to the outer automorphism group

$$Out(CG/G) = Aut(CG/G)/Inn(CG/G)$$

of CG/G. Here $Aut(CG/G)$ (resp. $Inn(CG/G)$) denotes the automorphism group (resp. inner automorphism group) of CG/G. In a general way, given a normal subgroup H of a group E, denote the subgroup of $Aut(H)$ consisting of all the automorphisms of H obtained by conjugation by an element of E by $Autc_E(H)$. Similarly denote the quotient group $Autc_E(H)/Inn(H)$ by $Outc_E(H)$.

A necessary condition for the embedding problem (EP$_o$) to have a solution is that the following one does.

(EP$_1$)

$$\begin{tikzcd} & & G(F/K) \arrow[dl, dashed] \arrow[d, "\overline{\kappa}\circ\lambda"] \\ Inn(CG/G) \arrow[r, hook] & Autc_{N/G}CG/G \arrow[r, two heads] & Outc_{N/G}CG/G \end{tikzcd}$$

Giraud [Gi] calls the homomorphism $\overline{\kappa} \circ \lambda : G(F/K) \to Outc(CG/G))$ the *band* of the original embedding problem (EP$_o$) and says that the *band* is *representable* when

(Band/Rep) *The homomorphism* $\overline{\kappa} \circ \lambda : G(F/K) \to Outc_{N/G}(CG/G))$ *can be lifted up to a real action* $G(F/K) \to Autc_{N/G}(CG/G)$, (i.e., *the embedding problem (EP$_1$) has a solution*).

Condition (Band/Rep) is a necessary condition for (Λ/Lift). Conversely we have

THEOREM 4.7. – *Assume that condition (Band/Rep) holds. Fix a solution* $\ell : G(F/K) \to Autc_{N/G}CG/G$ *to the embedding problem* (EP$_1$). *Then there exists*

– *an action* $\tilde{\ell}$ *of* $G(F/K)$ *on* CG/G *with restriction* χ *on* $Z(CG/G)$ *independent of the choice of* ℓ,

– *a* 2-*cocycle* $\omega_\ell \in H^2(G(F/K), Z(CG/G), \chi)$,

which are explicitly described in the proof below and which have the property that condition (λ/Lift) is equivalent to the condition

$$\omega_\ell^{-1} \in \delta^1(H^1(G(F/K), \mathrm{Inn}(CG/G), \tilde{\ell}^*))$$

where $\tilde{\ell}^*$ *is the action of* $G(F/K)$ *on* $Inn(CG/G)$ *naturally induced by* $\tilde{\ell}$ *and* δ^1 *is the coboundary operator*

$$H^1(G(F/K), \mathrm{Inn}(CG/G), \tilde{\ell}^*) \to H^2(G(F/K), Z(CG/G), \chi)$$

associated with the exact sequence

$$1 \to Z(CG/G) \to CG/G \to Inn(CG/G) \to 1$$

For example, condition (λ/Lift) holds if condition (Band/Rep) holds and CG/G is a centerless group. Of course Thm. 4.7 leaves us with condition (Band/Rep). This condition can be regarded as the very first condition for the field of moduli to be a field of definition. It holds for example if $Inn(CG/G)$ has a complement in $Aut(CG/G)$.

REMARK 4.8. – The procedure that was applied above to the embedding problem (EP_o) can actually be in turn applied to the embedding problem (EP_1), which gives rise to a new embedding problem (EP_2), etc. This inductive procedure stops after a finite number of stages. Indeed, at each stage, the kernel H of the embedding problem is replaced by $Inn(H)$ which is of order $|Inn(H)| \leq |H|$. Thus the sequence of the orders of the kernels of the successive embedding problems is a decreasing sequence of positive integers. Therefore, at some stage, we will have $|Inn(H)| = |H|$, *i.e.*, H is a centerless group.

Proof of Thm. 4.7. – Under condition (Band/Rep), the embedding problem (EP_1) can be viewed as a Basic Problem 4.1.1. Denote by κ the natural map $N/G \to Autc_{N/G}CG/G$. We have the following diagram

$$\begin{tikzcd}
 & 1 \arrow[rr, hook] \arrow[d, "1"] \arrow[ddl, "1"'] & & G(F/K) \arrow[drr, two heads] \arrow[dd, "\lambda"] & & \\
 & Ker(\overline{\kappa}) \arrow[drr, hook] & & & & G(F/K) \arrow[dd, "\overline{\kappa}\circ\lambda"] \arrow[dddl, "\ell"'] \\
Ker(\kappa) \arrow[ur, two heads] \arrow[drr, hook] & & & N/CG \arrow[ddr, two heads, "\overline{\kappa}"] & & \\
 & & N/G \arrow[ur, two heads] \arrow[drr, two heads, "\kappa"] & & & Outc_{N/G}CG/G \\
 & CG/G \arrow[ur, hook] & & & Autc_{N/G}CG/G \arrow[ur, two heads] &
\end{tikzcd}$$

The action $\ell : G(F/K) \to Aut(CG/G)$ being fixed, the problem is to find an homomorphism $\Lambda : G(F/K) \to N/G$ that lifts λ, that induces ℓ (and that extends the trivial map). The hypothesis of Basic Problem 4.1.1, here $CG/G \subset \mathrm{Cen}_{N/G}Ker(\kappa)$ follows from the definition of κ. Both conditions (FMod) and (rest/mod) hold here trivially since the upper corner map is the trivial map $1 \to Ker(\overline{\kappa})$.

Thus we can use the general results of §4.1. Here Z is the abelian group $CG/G \cap Ker(\kappa) = Z(CG/G)$. The action $\tilde{\ell}$ of $G(F/K)$ on CG/G is the action L_f

of Γ on HD/D of §4.1. The restricted action χ of $G(F/K)$ on $Z(CG/G)$ is the action L of Γ on Z. The 2-cocycle $\omega_\ell \in H^2(G(F/K), Z(CG/G), \chi)$ is the 2-cocycle $\Omega \in H^2(\Gamma, Z, L)$ of §4.1. The action $\tilde{\ell}^*$ of $G(F/K)$ on $Inn(CG/G)$ is the action L_f^* of Γ on HD/D of §4.1. The main conclusion of Thm. 4.3 is the existence of a 2-cocycle $\omega = \omega_\ell \in H^2(G(F/K), Z(CG/G), \chi)$ which vanishes if and only λ has a lift Λ that induces ℓ.

It remains to make ℓ vary among the solutions of the embedding problem (EP_1). From Prop. 4.5, a solution ℓ being fixed, the existence of a solution ℓ' for which the 2-cocycle $\omega_{\ell'}$ vanishes in $H^2(G(F/K), Z(CG/G), \chi)$ is equivalent to the condition:

$$\omega_\ell^{-1} \in \delta^1(H^1(G(F/K), Inn(CG/G), \tilde{\ell}^*)) \qquad \square$$

Final Note

The goal of this final note is to explain that our paper applies not only to G-covers and mere covers but to any kind of covers $f : X \to B$ given with some "extra structure". By extra structure, we mean for example: an action of some given finite group on X trivial on B; or, an unramified point on X; or, if X is an abelian variety, some level structure on X, etc.

Thanks to the generality of §4, our paper applies to any objects that can be viewed as surjective homomorphisms $\phi : \Pi_{K_s}(B^*) \to G$ regarded modulo conjugation by elements of a group N normalizing G. The main results of §4 are Thm. 4.3, Prop. 4.4 and 4.5. Our main examples are G-covers ($N = G$) and mere covers (G is then given as a subgroup of S_d and N is the normalizer of G in S_d). The Main Theorem corresponds to a translation of these results in those situations. But the main results of §4 can be immediately used for covers given with some other kind of extra structure. The only problem is to identify the group N.

Consider for example the situation of covers $f : X \to B$ defined over K_s given with the action of a group Γ on X trivial on B. The action of Γ can be specified in two ways:

(1) elements of Γ are given as acting on points of X: the action of Γ corresponds to an homomorphism $\Gamma \hookrightarrow Aut(X/B)$ of Γ in the automorphism group $Aut(X/B)$ of the cover.

(2) elements of Γ are given as acting on functions of X: the action of Γ corresponds to an homomorphism $\Gamma \hookrightarrow Aut(K_s(X)/K_s(B))$ of Γ in the automorphism group $Aut(K_s(X)/K_s(B))$ of the function field extension $K_s(X)/K_s(B)$.

Let $f : X \to B$ be a degree d cover over K_s. Denote the Galois group of the Galois closure of the function field extension $K_s(X)/K_s(B)$ by G and the action of G on conjugates of a primitive element of $K_s(X)/K_s(B)$ by $G \subset S_d$. Then we have:

$$\begin{cases} Aut(K_s(X)/K_s(B)) \simeq \dfrac{Nor_G G(1)}{G(1)} \\ Aut(X/B) \simeq Cen_{S_d}(G) \end{cases}$$

and the anti-isomorphism $Aut(K_s(X)/K_s(B)) \simeq Aut(X/B)$ corresponds to the classical anti-isomorphism

$$* : \frac{Nor_G G(1)}{G(1)} \to Cen_{S_d} G$$

Thus giving an action of a group Γ on X trivial on B is equivalent to giving a subgroup Γ_{fct} of $Nor_G G(1)/G(1)$, or, to giving a subgroup Γ_{pt} of $Cen_{S_d}(G)$.

With this notation, isomorphism classes of covers $f : X \to B$ with action of Γ correspond to surjective homomorphisms $\phi : \Pi_{K_s}(B^*) \to G$ regarded modulo conjugation by elements of the following group N

$$N = Nor_{S_d} G \cap Cen_{S_d}(\Gamma_{pt}) = Nor_{S_d} G \cap Cen_{S_d}((\Gamma_{fct})^*) \tag{1}$$

Namely two "covers f and f' with action of Γ" and with corresponding representations ϕ and ϕ' are isomorphic if

(a) they are isomorphic as mere covers, *i.e.*, if there exists an element $\omega \in Nor_{S_d} G$ such that $\phi' = \omega\phi\omega^{-1}$, and

(b) the isomorphism is compatible with the actions of Γ, *i.e.*, if the element ω can be picked in such a way that it commutes with the elements of Γ_{pt}.

For G-covers of group G, we have $G = \Gamma_{fct}$ and $G \subset S_d$ is the regular representation of G. So $Nor_G G(1)/G(1) = G$ and $(\Gamma_{fct})^* = Cen_{S_d} G$. Whence $N = Cen_{S_d}(Cen_{S_d} G) = G$ as expected. For mere covers, $\Gamma_{pt} = 1$, so the formula yields $N = Nor_{S_d}(G)$ as expected.

Applying the results of §4 for E equal to the group N above and H equal to $C = Cen_N G$, one obtains that, for a cover $f : X \to B$ with the action of a group Γ, the obstruction to the field of moduli being a field of definition "lies in" the group $H^2(K, Z, L)$ with values in the abelian group $Z = C \cap G$ (for a certain action of $G(K)$ on Z).

Formula (1) generalizes to any given "extra structure": take for N the subgroup of $Nor_{S_d}(G)$ consisting of those elements $\omega \in Nor_{S_d} G$ which "respect the extra structure". For example, for covers $f : X \to B$ given with a base point on X, the group N is

$$N = Nor_{S_d} G \cap S_d(1)$$

where $S_d(1)$ is the stabilizer of 1 in S_d. But since G acts transitively on $\{1, \dots, d\}$, the group $C = Cen_N G$ is trivial. This leads to this classical conclusion: the field of moduli a cover $f : X \to B$ with an unramified marked point on X is a field of definition.

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