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Hyper-Lie Poisson structures

HYPER-LIE POISSON STRUCTURES (*)

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ABSTRACT. – The main purpose of the paper is to study hyperkähler structures from the viewpoint of symplectic geometry. We introduce a notion of hypersymplectic structures which encompasses that of hyperkähler structures. Motivated by the work of Kronheimer on (co)adjoint orbits of semi-simple Lie algebras [10], [11], we define hyper-Lie Poisson structures associated with a compact semi-simple Lie algebra and give criterion which implies their existence. We study an explicit example of a hyper-Lie Poisson structure, in which the moduli spaces of solutions to Nahm's equations assocaited to Lie algebra $\mathfrak{su}(2)$ are realized as hypersymplectic leaves and are related to the (co)adjoint orbits of $\mathfrak{sl}(2, \mathbb{C})$.

RÉSUMÉ. – L'objet essentiel de cet article est l'étude des structures hyper-kähleriennes du point de vue de la géométrie symplectique. Nous introduisons une notion de structure hypersymplectique qui englobe celle de structure hyper-kählerienne. Motivés par le travail de Kronheimer sur les orbites coadjointes des algèbres de Lie semisimples [Kronheimer:LMS] [Kronheimer:JDG1990], nous définissons des structures « hyper-Lie Poisson associées » à une algèbre de Lie semisimple compacte et donnons un critère qui implique leur existence. Nous étudions un exemple explicite d'une telle structure « hyper-Lie Poisson » dans lequel les espaces de modules des solutions des équations de Nahm associées à l'algèbre de Lie $\mathfrak{su}(2)$ sont réalisées comme feuilles hypersymplectiques et sont reliées aux orbites coadjointes de $\mathfrak{sl}(2, \mathbb{C})$.

1. Introduction

Due to its rich structure and close connection with gauge theory, hyperkähler manifolds have attracted increasing interest [1], [2], [7]. Roughly speaking, a hyperkähler manifold is a Riemannian manifold with three compatible complex structures I, J and K. The compatibility means that I, J, K satisfy the quaternion identities $I^2 = J^2 = K^2 = IJK = -1$, and the metric is kählerian with respect to I, J and K. While it is easy to find examples of Kähler manifolds, hyperkähler manifolds are in general more difficult to construct. The two main often used routes are twistor theory [13] and hyperkähler reduction [7], a generalization of Marsden-Weinstein reduction [12] in the hyper-context.

To any Kähler manifold there associates a symplectic structure, namely its Kähler form. For a hyperkähler manifold, there are three symplectic structures (or equivalently, Poisson structures) compatible with one another in a certain sense (described in Section 2). However, there is an essential difference between Kähler and hyperkähler manifolds. For

Kähler manifolds, there might be more than one metric compatible with the same symplectic structure. In contrast, hyperkähler structures are more rigid. Namely, the three symplectic structures completely determine the hyperkähler metric, whence the corresponding complex structures. This observation suggests another possible way of constructing a hyperkähler manifold, namely, by constructing three compatible symplectic structures on the manifold.

By focusing on symplectic structures rather than the metrics, we arrive at a new way to define the hyperkähler condition. This leads to our definition of hypersymplectic manifolds. More precisely, a hypersymplectic manifold is a manifold which admits three symplectic structures satisfying the same compatibility condition as usual hyperkähler manifolds. Hypersymplectic manifolds and their basic properties will be studied in detail in Section 2.

A powerful method of constructing symplectic manifolds in symplectic geometry is by means of Poisson manifolds. Every Poisson manifold admits a natural foliation, called the symplectic foliation, whose leaves are all symplectic [15]. For example, symplectic structures on coadjoint orbits are induced from the Lie-Poisson structure on the Lie algebra dual $\mathfrak{g}^*$ [15]. It is reasonable to expect that there exist some examples of "hyper-Poisson manifolds". By a hyper-Poisson manifold, we mean a manifold which admits three Poisson structures satisfying certain compatibility condition such that each leaf is hypersymplectic. Instead of developing general theory of hyper-Poisson structures, the present paper will be focused on finding some interesting examples. In particular, we will consider hyper-Lie Poisson structures, an analogue of Lie-Poisson structures in the hyper-context, which are presumably the simplest and most interesting examples.

There are at least two reasons that hyper-Lie Poisson structures are important. The first one comes from our attempt to understand general hyperkähler reduction. According to the author's knowledge, a successful reduction theory only exists at 0 so far. While Lie-Poisson structures (or their symplectic leaves) play an essential role in general symplectic reduction, carrying out general hyperkähler reduction will challenge our knowledge of hyper-Lie Poisson structures. In this aspect, an open question is

Open Problem: Suppose that X is a hyperkähler manifold, on which the Lie group G acts preserving the hyperkähler structure. Assume that the action is "good" enough so that the quotient X/G is a manifold. Then the three Poisson structures on X corresponding to the three Kähler forms will descend to three Poisson structures on the quotient X/G. What are the relation between these reduced Poisson structures, hyper-Poisson structures and general hyperkähler reduction?

Understanding Kronheimer's recent work on adjoint orbit hyperkähler structures provides another strong motivation. Using gauge theory and infinite dimensional hyperkähler reduction, Kronheimer proved that certain adjoint orbits of complex semi-simple Lie algebras admit hyperkähler metrics [10], [11]. Later on, his result was generalized by Biquard and Kovalev to arbitrary adjoint orbits [3], [9], and has been used to understand the Kostant-Sekiguchi correspondence [14]. However, the hyperkähler metrics and symplectic structures obtained in this way are quite mysterious and elusive. Recall that an intrinsic reason for each coadjoint orbit to admit a symplectic structure is that the dual of any Lie algebra is a Poisson manifold, and each coadjoint orbit happens to be a symplectic leaf of this Lie-Poisson structure. Inspired by this fact as well as the results of Kronheimer and

others, it is quite reasonable to expect that there exists a hyper-Lie Poisson structure such that the orbits studied by Kronheimer et al. occur as its hyper-symplectic leaves. Then, this will provide us a natural source and symplectic explanation for those hyperkähler structures on adjoint orbits. To explore the connection between the work of Kronheimer and symplectic geometry was indeed the initial motivation for us to consider hyper-Lie Poisson structures.

In this paper, as an example, we will consider in detail a hyper-Lie Poisson structure associated with $\mathfrak{g} = \mathfrak{su}(2)$. In the meantime, we will take some tentative steps toward hyper-Lie Poisson structures associated with general compact semi-simple Lie algebras. For this purpose, we will keep the discussion general from Section 2 through Section 3 while the last two sections will be devoted to the special case $\mathfrak{g} = \mathfrak{su}(2)$.

To explain our approach, we need to rephrase the definition of hyperkähler manifolds in a way slightly different from the literature. Note that there is in fact no preferred choice of complex structures on a hyperkähler manifold. The bundle maps I', J' and K' given by $(I', J', K') = (I, J, K)O$, for any orthogonal matrix $O \in SO(3)$, will satisfy exactly the same quaternion relations. Therefore the map: $O \in SO(3) \longrightarrow I'$ assigns a complex structure to every orthogonal matrix in $SO(3)$. In particular, under such an assignment, I, J and K are the complex structures corresponding to the identity matrix, the matrix of the cyclic permutation: $\{e_1, e_2, e_3\} \longrightarrow \{e_2, e_3, e_1\}$ and the matrix of the cyclic permutation: $\{e_1, e_2, e_3\} \longrightarrow \{e_3, e_1, e_2\}$, respectively. Since a matrix in $SO(3)$ can be naturally identified with a standard orthonormal basis in $\mathfrak{su}(2)$ (such are also called frames in this paper), intrinsically we can think of a hyperkähler manifold as a manifold with a family of complex structures (or equivalently symplectic structures), parameterized by frames. This point of view is different from the conventional one, in which complex structures (or symplectic structures) on a hyperkähler manifold are considered to be parameterized by the unit sphere S^2. This is the crux in our approach.

Now the question arises in which space a hyper-Lie Poisson structure should live. To answer this question, we first recall that a Lie-Poisson space $\mathfrak{g}^*$ emerges as the target space of a momentum mapping of a hamiltonian G-space. A momentum mapping of a hyperkähler G-space X (*i.e.* a hyperkähler manifold X admitting a G-action which preserves the hyperkähler structure) is usually considered, when the three complex structures I, J, K are chosen, as a map from X to $\mathfrak{g}^* \times \mathfrak{g}^* \times \mathfrak{g}^*$ [7]. However, when I, J, K are replaced by any other three complex structures I', J', K' related by an orthogonal matrix in $SO(3)$ as described earlier, the corresponding momentum mapping changes accordingly. Intrinsically, the momentum mapping of a hyperkähler G-manifold should therefore be considered as a map $X \longrightarrow L(\mathfrak{g}, \mathfrak{su}(2))$, where $L(\mathfrak{g}, \mathfrak{su}(2))$ is the space of all linear maps from $\mathfrak{g}$ to $\mathfrak{su}(2)$. It is therefore reasonable to expect that $L(\mathfrak{g}, \mathfrak{su}(2))$, as the target space of the momentum mapping of a hyperkähler G-manifold, should carry a hyper-Lie Poisson structure. Another possibly useful way to think of $L(\mathfrak{g}, \mathfrak{su}(2))$ as a natural generalization of $\mathfrak{g}^*$ is to note that this space is obtained from $\mathfrak{g}^* = L(\mathfrak{g}, \mathbb{R})$ by replacing $\mathbb{R}$ by $\mathfrak{su}(2)$.

An elegant way of obtaining a family of Poisson structures on the space $L(\mathfrak{g}, \mathfrak{su}(2))$ goes as follows. The space $L(\mathfrak{g}, \mathfrak{su}(2))$ can be identified with $\mathfrak{g}^* \times \mathfrak{g}^* \times \mathfrak{g}^*$, once an orthonormal basis of $\mathfrak{su}(2)$, *i.e.*, a frame, is fixed. If we can define a Poisson structure π on the space $\mathfrak{g}^* \times \mathfrak{g}^* \times \mathfrak{g}^*$, by pulling back π to $L(\mathfrak{g}, \mathfrak{su}(2))$ under such an identification, we then

obtain a Poisson structure on the space $L(\mathfrak{g}, \mathfrak{su}(2))$. This construction in fact enables us to obtain a family of Poisson structures simply by varying the frames. Throughout the paper, we shall identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via the Killing form, hence $\mathfrak{g}^* \times \mathfrak{g}^* \times \mathfrak{g}^*$ with $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ for simplicity. We note that any bivector field on $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ is determined by its corresponding brackets of all linear functions l_ξ^i, where $l_\xi^1(a, b, c) = \langle \xi, a \rangle$ etc. Therefore, to define the Poisson tensor π, it suffices to find its corresponding brackets of linear functions. There are certain natural conditions that the Poisson tensor π has to satisfy. One of them requires that both projections $pr_{12}, pr_{13} : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}^\mathbb{C}$ be Poisson maps. Here pr_{12}, pr_{13} are the maps defined by $pr_{12}(a, b, c) = a + ib$ and $pr_{13}(a, b, c) = a + ic$ respectively, and meanwhile $\mathfrak{g}^\mathbb{C}$ is identified with its dual and equipped with the Lie Poisson structure as a real Lie algebra. By using this condition for π, one can easily write down its brackets of all linear functions except for the bracket of l_ξ^2 and l_η^3, which should correspond to a $S^2(\mathfrak{g})$-valued function A on $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$. Therefore, the entire problem reduces to that of finding a suitable function A.

Section 3 is devoted to a detailed discussion of this problem as well as an investigation of when π defines a Poisson tensor. Moreover, we will derive the criterion on A that the induced family of Poisson structures on $L(\mathfrak{g}, \mathfrak{su}(2))$ is compatible and have the desired properties as outlined above.

Section 4 is devoted to the study of the case $\mathfrak{g} = \mathfrak{su}(2)$, where a satisfactory function A is explicitly constructed on an open submanifold of $L(\mathfrak{g}, \mathfrak{su}(2))$. The induced hypersymplectic foliation is also explicitly described, and a complete set of casimirs is obtained. The corresponding pseudo-metric on each leaf is also computed, and in fact it is shown that all hypersymplectic leaves are hyperkähler.

Section 5 is a continuation of Section 4, where the hyper-Poisson structure is extended to a certain critical set C. It is shown that hyper-symplectic leaves of this extended hyper-Lie Poisson structure are diffeomorphic to (co)adjoint orbits of $\mathfrak{sl}(2, \mathbb{C})$. In this way, we obtain a symplectic proof for the existence of hyperkähler structures on (co)adjoint orbits of $\mathfrak{sl}(2, \mathbb{C})$. Although $\mathfrak{sl}(2, \mathbb{C})$ is the simplest semi-simple Lie algebra, the existence of hyperkähler structures on its (co)adjoint orbits is somewhat already nontrivial (*see* [6]).

2. Hypersymplectic structures

The purpose of the present section is to introduce a notion called hypersymplectic structures, which includes hyperkähler manifolds as a special case and is much more natural from the viewpoint of symplectic geometry. Our definition of hyperkähler structures here is slightly different from the one in the literature, where complex structures and metrics have received much more attention. Our interests in this paper mainly lie in symplectic forms and their Poisson tensors.

By $\Omega_+^2(S)$, we denote the space of nondegenerate 2-forms on a manifold S, and $\Gamma_+(\wedge^2 TS)$ the space of non-degenerate bivector fields on S. By κ, we denote the map $\Omega_+^2(S) \longrightarrow \Gamma_+(\wedge^2 TS)$, which is the inversion when elements in $\Omega_+^2(S)$ and $\Gamma_+(\wedge^2 TS)$ are considered as bundle maps. For any $\omega \in \Omega_+^2(S)$, we write $\omega^{-1} = \kappa(\omega) \in \Gamma_+(\wedge^2 TS)$. A $\mathfrak{su}(2)$-valued 2-form Ω on S is said to be nondegenerate if the form $\omega_\xi = \langle \xi, \Omega \rangle$ is

nondegenerate for any non-trivial $\xi \in \mathfrak{su}(2)$, where the pairing is with respect to the Killing form on $\mathfrak{su}(2)$.

DEFINITION 2.1. – *A hypersymplectic structure on a manifold S is a closed non-degenerate $\mathfrak{su}(2)$-valued two-form Ω such that $\xi \longrightarrow \omega_\xi^{-1}$ is linear when $\|\xi\| = 1$. That is, for any unit vectors $\xi, \xi_1, \xi_2 \in \mathfrak{su}(2)$ satisfying $\xi = k_1\xi_1 + k_2\xi_2$,*

$$(1) \qquad\qquad \omega_\xi^{-1} = k_1\omega_{\xi_1}^{-1} + k_2\omega_{\xi_2}^{-1}.$$

Suppose that $\Omega \in \Omega^2(S) \otimes \mathfrak{su}(2)$ is a hypersymplectic structure on S. By choosing an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathfrak{su}(2)$, also called *a frame* in the sequel, Ω can be written as $\Omega = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3$, where ω_1, ω_2 and ω_3 are symplectic forms on S. By π_1, π_2 and π_3 we denote their corresponding bivector fields on S.

As usual, for each i, ω_i^b denotes the bundle map $TS \longrightarrow T^*S$ given by $\langle \omega_i^b(v), u \rangle = \omega_i(v, u)$, $\forall u, v \in TS$, X_f^i the vector field on S defined by $\omega_i^b(df)$ for any $f \in C^\infty(S)$, and $\{f, g\}_i = X_f^i g$ for any $f, g \in C^\infty(S)$.

PROPOSITION 2.2. – *Suppose that $\Omega = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3 \in \Omega^2(S) \otimes \mathfrak{su}(2)$ is a hypersymplectic structure on S. Then,*

(1).

$$(2) \qquad\qquad [\omega_i^b \circ (\omega_j^b)^{-1}]^2 = -1, \qquad \forall i \neq j,$$

where 1 denotes the identity map $TS \longrightarrow TS$.

(2).

$$[\pi_i, \pi_j] = 0, \qquad for\ any\ \ i, j,$$

where the bracket is the Schouten bracket on multivector fields [8].

Proof. – For any k_1, k_2, k_3 such that $k_1^2 + k_2^2 + k_3^2 = 1$, $k_1 e_1 + k_2 e_2 + k_3 e_3$ is a unit vector in $\mathfrak{su}(2)$. Thus, it follows from definition that

$$(k_1\omega_1^b + k_2\omega_2^b + k_3\omega_3^b)^{-1} = k_1(\omega_1^b)^{-1} + k_2(\omega_2^b)^{-1} + k_3(\omega_3^b)^{-1}.$$

That is,

$$\begin{aligned}
1 &= (k_1\omega_1^b + k_2\omega_2^b + k_3\omega_3^b)[k_1(\omega_1^b)^{-1} + k_2(\omega_2^b)^{-1} + k_3(\omega_3^b)^{-1}] \\
&= (k_1^2 + k_2^2 + k_3^2) + k_1 k_2[\omega_1^b(\omega_2^b)^{-1} + \omega_2^b(\omega_1^b)^{-1}] \\
&\quad + k_2 k_3[\omega_2^b(\omega_3^b)^{-1} + \omega_3^b(\omega_2^b)^{-1}] + k_1 k_3[\omega_1^b(\omega_3^b)^{-1} + \omega_3^b(\omega_1^b)^{-1}].
\end{aligned}$$

Equations (2) thus follow immediately.

Also, from the argument above, we see that $k_1\pi_1 + k_2\pi_2 + k_3\pi_3$ is still a Poisson tensor for any (k_1, k_2, k_3) in the 2-sphere, hence for arbitrary k_1, k_2, k_3 as well. It thus follows that $[\pi_1, \pi_2] = [\pi_2, \pi_3] = [\pi_1, \pi_3] = 0$. $\qquad \square$

An immediate consequence is the following:

COROLLARY 2.3. – *Let S be a hypersymplectic manifold with hypersymplectic form $\Omega \in \Omega^2(S) \otimes \mathfrak{su}(2)$. Then for any $\xi, \eta \in \mathfrak{g}$,*

$$[\pi_\xi, \pi_\eta] = 0,$$

where $\pi_\xi = \omega_\xi^{-1}$ and $\pi_\eta = \omega_\eta^{-1}$.

Sometimes the following equivalent version is more often used.

PROPOSITION 2.4. – *Equations (2) are equivalent to*

$$(3) \qquad \omega_i(X_f^j, X_g^j) = \{f, g\}_i, \qquad i \neq j, \quad for\ any\ f, g \in C^\infty(S).$$

Proof. – It is quite obvious that

$$[\omega_i^b \circ (\omega_j^b)^{-1}]^2 = -1 \iff (\omega_j^b)^{-1} \omega_i^b (\omega_j^b)^{-1} = -(\omega_i^b)^{-1}$$
$$\iff \langle (\omega_j^b)^{-1} \omega_i^b (\omega_j^b)^{-1} df, dg \rangle = -\langle (\omega_i^b)^{-1} df, dg \rangle,$$

which is equivalent to

$$\omega_i(X_f^j, X_g^j) = \{f, g\}_i. \qquad \square$$

In fact, Equations (2), or equivalently Equations (3), are also sufficient to construct a hypersymplectic structure on S.

PROPOSITION 2.5. – *Suppose that S is a manifold with three symplectic structures ω_1, ω_2 and ω_3 such that Equations (2) (or Equations (3)) hold. Then for any orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathfrak{su}(2)$, the $\mathfrak{su}(2)$-valued 2-form $\Omega = \omega_1 e_1 + \omega_2 e_2 + \omega_2 e_2$ defines a hypersymplectic structure on S.*

The proof is quite straightforward, and is left for the reader.

To each hypersymplectic manifold, we associate a natural pseudo-metric as we will see below.

Let $g : TS \longrightarrow T^*S$ be the bundle map given by

$$(4) \qquad\qquad g = \omega_3^b (\omega_1^b)^{-1} \omega_2^b,$$

and I, J, K the bundle maps form TS to itself defined by

$$(5) \qquad I = g^{-1} \omega_1^b, \qquad J = g^{-1} \omega_2^b \quad and \quad K = g^{-1} \omega_3^b.$$

THEOREM 2.6. – (i) *g is a pseudo-metric on S;*
(ii) *g can also be written as $g = \omega_1^b (\omega_2^b)^{-1} \omega_3^b$, or $g = \omega_2^b (\omega_3^b)^{-1} \omega_1^b$;*
(iii) *I, J, K satisfy the quaternion relation:*

$$I^2 = J^2 = K^2 = IJK = -1.$$

Proof. – (1) It follows from Equations (2), by taking the dual, that

$$(6) \qquad\qquad [(\omega_j^b)^{-1} \omega_i^b]^2 = -1, \qquad \forall i \neq j.$$

Hence, by using Equations (2) and (6) repeatedly, we have $g^* = -\omega_2^b(\omega_1^b)^{-1}\omega_3^b = [\omega_1^b(\omega_2^b)^{-1}]\omega_3^b = -\omega_1^b[(\omega_3^b)^{-1}\omega_2^b] = [\omega_3^b(\omega_1^b)^{-1}]\omega_2^b = g$. That is, g is symmetric. Furthermore, it is evident that g is nondegenerate since $\omega_1, \omega_2, \omega_3$ are all nondegenerate.

(2) It follows from Part (1) that $g = g^* = -\omega_2^b(\omega_1^b)^{-1}\omega_3^b = \omega_1^b(\omega_2^b)^{-1}\omega_3^b$. The other equation can be obtained in a similar way.

(3) Using Part (2), we have $I = g^{-1}\omega_1^b = [\omega_1^b(\omega_2^b)^{-1}\omega_3^b]^{-1}\omega_1^b = (\omega_3^b)^{-1}\omega_2^b$. Similarly, $J = (\omega_1^b)^{-1}\omega_3^b$ and $K = (\omega_2^b)^{-1}\omega_1^b$. Therefore, $I^2 = J^2 = K^2 = -1$. Furthermore,

$$\begin{aligned}
IJ &= (\omega_3^b)^{-1}\omega_2^b(\omega_1^b)^{-1}\omega_3^b \\
&= [\omega_1^b(\omega_2^b)^{-1}\omega_3^b]^{-1}\omega_3^b \\
&= g^{-1}\omega_3^b \\
&= K.
\end{aligned}$$

This concludes the proof. □

It looks as if our definition of g depends on a particular choice of the frame. However, the following theorem indicates that g is in fact independent of frames up to a sign.

THEOREM 2.7. – *If two frames are of the same orientation, then their corresponding pseudo-metrics coincide.*

Proof. – Let $\mathcal{F} = \{e_1, e_2, e_3\}$ and $\mathcal{T} = \{e_1', e_2', e_3'\}$ be any two frames. Suppose that $(e_1', e_2', e_3') = (e_1, e_2, e_3)O$ for some orthogonal matrix $O \in SO(3)$. Let I, J, K be the induced almost complex structures on S corresponding to the frame $\mathcal{F}$. We define I', J', K' by the equation:

$$(I', J', K') = (I, J, K)O.$$

It follows from the quaternion relation of I, J, K that I', J', K' also satisfy the same relation. Since $\omega_1 e_1 + \omega_2 e_2 + \omega_2 e_2 = \omega_1' e_1' + \omega_2' e_2' + \omega_2' e_2'$, it follows that $(\omega_1', \omega_2', \omega_3') = (\omega_1, \omega_2, \omega_3)O$. Hence, we have

(7) $\qquad I' = g^{-1}(\omega_1')^b, \qquad J' = g^{-1}(\omega_2')^b \qquad$ and $\qquad K' = g^{-1}(\omega_3')^b.$

By using the quaternion relation of I', J', K', we can easily deduce that $g = (\omega_3')^b((\omega_1')^b)^{-1}(\omega_2')^b$, which is g' by definition. □

Because of this result, we call g the pseudo-metric associated to the hypersymplectic structure despite of an ambiguity of signs. In particular, if g is positive (or negative) definite, the hypersymplectic structure becomes hyperkähler. We refer the reader to [1], [2], [6], [7] for the background on the subject of hyperkähler structures.

The following result is well-known for hyperkähler structures, and is however still valid in our general context. Readers can find a proof in, for example, [2]. For completeness, we outline a proof here.

THEOREM 2.8. – *If S is a hypersymplectic manifold, then the almost complex structures I, J, K corresponding to any frame are integrable.*

Proof. – For any vector fields $X, Y \in \mathcal{X}(S)$,

$$\omega_2(X, Y) = g(JX, Y) = g(KIX, Y) = \omega_3(IX, Y).$$

Hence, we have the relation:

$$(8) \qquad\qquad\qquad\qquad X \lrcorner \omega_2 = IX \lrcorner \omega_3.$$

It follows that a complex vector field X is of type $(1, 0)$ with respect to I iff

$$(9) \qquad\qquad\qquad\qquad X \lrcorner \omega_2 = iX \lrcorner \omega_3.$$

Suppose that X, Y are complex vector fields of type $(1, 0)$. In order to show that I is integrable it suffices to show that their bracket $[X, Y]$ is of type $(1, 0)$ according to the Newlander-Nirenberg theorem. However,

$$[X, Y] \lrcorner \omega_2 = L_X(Y \lrcorner \omega_2) - Y \lrcorner (L_X \omega_2).$$

Now,

$$\begin{aligned}
L_X \omega_2 &= (d\iota_X + \iota_X d)\omega_2 \\
&= d(i\iota_X \omega_3) \\
&= iL_X \omega_3,
\end{aligned}$$

and from Equation (9),

$$Y \lrcorner \omega_2 = i(\iota_Y \omega_3).$$

Thus,

$$[X, Y] \lrcorner \omega_2 = i[L_X(\iota_Y \omega_3) - \iota_Y(L_X \omega_3)] = i([X, Y] \lrcorner \omega_3).$$

Thus, I is integrable. Similarly, J and K are also integrable. $\square$

Remark. – It is not difficult to check that (π_2, I), (π_3, I) and all other similar pairs are Poisson-Nijenhuis structures in the sense of Kosmann-Scharzbach and Magri [8]. Poisson-Nijenhuis structures are introduced by Kosmann-Scharzbach and Magri in the study of integrable systems. Therefore, it would be very interesting to explore the relation between hypersymplectic manifolds and integrable systems.

3. Hyper-Lie Poisson structures

This section is devoted to the introduction of hyper-Lie Poisson structures. The main idea is to define on a suitable space M a family of Poisson structures parameterized by frames, which will coherently depend on the parameterization in a proper sense. We shall analyses the condition under which the induced symplectic foliations are independent of frames so that each leaf becomes hypersymplectic.

To begin with, let $\mathfrak{g}$ be a semisimple Lie algebra with Killing form $\langle \cdot, \cdot \rangle$, and A a $S^2(\mathfrak{g})$-valued function on $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$, where $S^2(\mathfrak{g})$ denotes the space of second order

symmetric tensors on $\mathfrak{g}$. We note that any element in $S^2(\mathfrak{g})$ can be naturally considered as a symmetric bilinear form on $\mathfrak{g}$. So contracting with $\xi, \eta \in \mathfrak{g}$, there corresponds to a function on $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$: $A_{\xi,\eta} = \xi \lrcorner A \lrcorner \eta$. For any $\xi \in \mathfrak{g}$, we denote by $l_\xi^1, l_\xi^2, l_\xi^3$ the linear functions on $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ defined by $l_\xi^1(a, b, c) = \langle \xi, a \rangle$, etc. Our first step is to define a Poisson structure on $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$. In order to do so (or even just to define a bivector field on $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$), it suffices to define its corresponding brackets among all linear functions l_ξ^i, $i = 1, 2, 3$, since they span the function space $C^\infty(\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g})$.

DEFINITION 3.1. – *Let A be a $S^2(\mathfrak{g})$-valued function on $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$. The following bracket defines a bivector field π on $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$.*

$$\{l_\xi^1, l_\eta^1\} = l_{[\xi,\eta]}^1$$
$$\{l_\xi^1, l_\eta^2\} = \{l_\xi^2, l_\eta^1\} = l_{[\xi,\eta]}^2$$
$$\{l_\xi^2, l_\eta^2\} = -l_{[\xi,\eta]}^1$$
$$\{l_\xi^1, l_\eta^3\} = \{l_\xi^3, l_\eta^1\} = l_{[\xi,\eta]}^3$$
$$\{l_\xi^3, l_\eta^3\} = -l_{[\xi,\eta]}^1$$
$$\{l_\xi^2, l_\eta^3\} = -\{l_\xi^3, l_\eta^2\} = A_{\xi,\eta}.$$

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Then G acts on $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ diagonally, with adjoint action on each factor.

PROPOSITION 3.2. – *The following statements are equivalent:*

(i) *The bivector field π is G-invariant;*

(ii) *the map $A : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \longrightarrow S^2(\mathfrak{g})$ is G-equivariant, where G acts on $S^2(\mathfrak{g})$ by the adjoint action;*

(iii) *for any $\xi, \eta, \zeta \in \mathfrak{g}$,*

$$\hat{\xi} A_{\eta,\zeta} = A_{[\xi,\eta],\zeta} + A_{\eta,[\xi,\zeta]}.$$

Proof. – That (1) and (2) are equivalent is quite evident.

(2) $\Longleftrightarrow$ (3). Suppose that A is G-equivariant. That is, $A(gx) = Ad_g A(x)$. It follows, by taking derivative, that for any $\xi \in \mathfrak{g}$, $\hat{\xi} A = ad_\xi A$. Hence for any $\eta, \zeta \in \mathfrak{g}$,

$$\hat{\xi} A_{\eta,\zeta} = \eta \lrcorner \hat{\xi} A \lrcorner \zeta$$
$$= \eta \lrcorner ad_\xi A \lrcorner \zeta$$
$$= A_{[\xi,\eta],\zeta} + A_{\eta,[\xi,\zeta]}.$$

The converse is also true by using the same argument backwards. ☐

The following theorem gives a necessary and sufficient condition for the bivector field π to be a Poisson tensor. As usual, for any $f \in C^\infty(\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g})$ we write X_f for the vector field $\pi^\#(df)$.

THEOREM 3.3. – *π is a Poisson tensor iff A is equivariant and at any point $(a, b, c) \in \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$,*

(10) $$\xi \lrcorner X_{l_\eta^2} A - \eta \lrcorner X_{l_\xi^2} A = [c, [\xi, \eta]],$$

(11) $$\xi \lrcorner X_{l_\eta^3} A - \eta \lrcorner X_{l_\xi^3} A = [b, [\eta, \xi]],$$

for any $\xi, \eta \in \mathfrak{g}$.

Proof. – By pr_{12}, we denote the projection $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}^{\mathbb{C}}$ given by $pr_{12}(a, b, c) = a + ib$. Similarly, pr_{13} denotes the projection from $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ to $\mathfrak{g}^{\mathbb{C}}$ given by $pr_{13}(a, b, c) = a + ic$. It is simple to see that $Tpr_{12}\pi = Tpr_{13}\pi = \pi^{\mathbb{C}}$, where $\pi^{\mathbb{C}}$ is the Lie-Poisson tensor on $\mathfrak{g}^{\mathbb{C}}$, which is identified with its dual as a real Lie algebra. Hence the Jacobi identity

$$\{\{f_1, f_2\}, f_3\} + c.p. = 0$$

holds if f_i, $i = 1, 2, 3$, are linear functions of the form l_ξ^1, l_ξ^2, or of the form l_ξ^1, l_ξ^3, where $c.p.$ stands for the cyclic permutation. It remains to check the following three cases: (1) $f_1 = l_\xi^1$, $f_2 = l_\eta^2$ and $f_3 = l_\zeta^3$; (2) $f_1 = l_\xi^2$, $f_2 = l_\eta^2$ and $f_3 = l_\zeta^3$; and (3) $f_1 = l_\xi^3$, $f_2 = l_\eta^3$ and $f_3 = l_\zeta^2$.

It is simple to see that the Jacobi identity in Case (1) is equivalent to that A is G-equivariant according to Proposition 3.2. As for Case (2),

$$\begin{aligned}
\{\{l_\xi^2, l_\eta^2\}, l_\zeta^3\} + c.p. &= -\{l_{[\xi,\eta]}^1, l_\zeta^3\} + \{A_{\eta,\zeta}, l_\xi^2\} + \{-A_{\xi,\zeta}, l_\eta^2\} \\
&= -l_{[[\xi,\eta],\zeta]}^3 - X_{l_\xi^2} A_{\eta,\zeta} + X_{l_\eta^2} A_{\xi,\zeta} \\
&= -\langle [c, [\xi, \eta]], \zeta \rangle - \langle \eta \lrcorner X_{l_\xi^2} A, \zeta \rangle + \langle \xi \lrcorner X_{l_\eta^2} A, \zeta \rangle.
\end{aligned}$$

Thus, the Jacobi identity follows iff Equation (10) holds. Similarly, Equation (11) is equivalent to the Jacobi identity for Case (3).   □

In the proof above, we have in fact shown the following:

PROPOSITION 3.4. – *Assume that both Equation* (10) *and Equation* (11) *hold for any* $\xi, \eta \in \mathfrak{g}$. *Then, both* $pr_{12} : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}^{\mathbb{C}}$ *and* $pr_{13} : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}^{\mathbb{C}}$ *are Poisson maps, where* $\mathfrak{g}^{\mathbb{C}}$, *identified with its dual, is equipped with the Lie Poisson structure as a real Lie algebra.*

Consider the space $M = L(\mathfrak{g}, \mathfrak{su}(2))$, which consists of all linear maps from $\mathfrak{g}$ to $\mathfrak{su}(2)$. Here $\mathfrak{su}(2)$ is only considered as a vector space without using any Lie algebra structure. M admits a natural G-action induced from the adjoint action on $\mathfrak{g}$. Whenever a frame $\mathcal{F}$, *i.e.*, an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathfrak{su}(2)$, is chosen, M is identified with $\mathfrak{g}^* \times \mathfrak{g}^* \times \mathfrak{g}^*$, which can also be identified with $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ using the Killing form on $\mathfrak{g}$. We shall denote such an identification $M \longrightarrow \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ by $\Psi_{\mathcal{F}}$. From now on, we will always identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$. Using the map $\Psi_{\mathcal{F}}$, the Poisson structure π on $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ is pulled back to a Poisson structure $\pi_{\mathcal{F}}$ on M. When the choice of frames $\mathcal{F}$ varies, we thus obtain a family of Poisson structures on M parameterized by frames. This is the very structure we are interested in. Corresponding to any frame $\mathcal{F} = \{e_1, e_2, e_3\}$, there exist two frames $\mathcal{F}_2$ and $\mathcal{F}_3$ obtained by the cyclic permutations: $\{e_2, e_3, e_1\}$, and $\{e_3, e_1, e_2\}$, respectively. We also often use $\mathcal{F}_1$ to denote $\mathcal{F}$. The Poisson structures corresponding to $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ are denoted by $\pi_{\mathcal{F}_1}, \pi_{\mathcal{F}_2}$ and $\pi_{\mathcal{F}_3}$, respectively.

By choosing a frame, any vector-valued function F on $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ can be pulled back to a function $F_{\mathcal{F}}$ on M via the map $\Psi_{\mathcal{F}}$. If furthermore, F is invariant under the action of $O(3)$, where $O(3)$ acts on $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ by $(a, b, c) \longrightarrow (a, b, c)O$ for any (a, b, c) and $O \in O(3)$, $F_{\mathcal{F}}$ is independent of the frame $\mathcal{F}$, and therefore can be considered as a well-defined function on M. However, in most cases, the function F is only invariant under the $SO(3)$-action. In this case, the pull back $F_{\mathcal{F}}$ depends on the orientation of $\mathcal{F}$. Whenever the orientation of frames is fixed, we shall still get a well-defined function on M.

In the sequel, we shall always assume that the $S^2(\mathfrak{g})$-valued function A on $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ is invariant under the $SO(3)$-action, and therefore can be considered as a function on M when an orientation of frames is fixed.

THEOREM 3.5. – $\Pi = \pi_{\mathcal{F}_1} e_1 + \pi_{\mathcal{F}_2} e_2 + \pi_{\mathcal{F}_3} e_3$ *does not depend on the choice of frames of the same orientation, and therefore is a well-defined section of the vector bundle* $\wedge^2 TM \otimes \mathfrak{su}(2)$.

Proof. – Assume that $\mathcal{T} = \{e_1', e_2', e_3'\}$ is another frame and $(e_1', e_2', e_3') = (e_1, e_2, e_3)O$ for some $O \in SO(3)$. It suffices to show that

$$(12) \qquad (\pi_{\mathcal{T}_1}, \pi_{\mathcal{T}_2}, \pi_{\mathcal{T}_3}) = (\pi_{\mathcal{F}_1}, \pi_{\mathcal{F}_2}, \pi_{\mathcal{F}_3})O,$$

or equivalently,

$$(13) \qquad (\{f, g\}_{\mathcal{T}_1}, \{f, g\}_{\mathcal{T}_2}, \{f, g\}_{\mathcal{T}_3}) = (\{f, g\}_{\mathcal{F}_1}, \{f, g\}_{\mathcal{F}_2}, \{f, g\}_{\mathcal{F}_3})O$$

for any $f, g \in C^\infty(M)$.

M can be identified with $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ under both $\Psi_{\mathcal{F}}$ and $\Psi_{\mathcal{T}}$. If (a, b, c) and $(a', b', c') \in \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ are, respectively, the coordinates of any point in M under these two identifications, they should be related by $(a', b', c') = (a, b, c)O$. We assume that $O = \{(a_{ij})\}$. As an example, we shall show below that

$$(14) \qquad \{f, g\}_{\mathcal{T}_1} = a_{11}\{f, g\}_{\mathcal{F}_1} + a_{21}\{f, g\}_{\mathcal{F}_2} + a_{31}\{f, g\}_{\mathcal{F}_3},$$

for $f = \langle \xi, a \rangle$ and $g = \langle \eta, a \rangle$. All the other cases can be proved similarly.

In this case, it is simple to see that

$$\begin{aligned}
\{f, g\}_{\mathcal{T}_1} &= (a_{11}, a_{12}, a_{13}) \begin{pmatrix} \langle[\xi,\eta],a'\rangle & \langle[\xi,\eta],b'\rangle & \langle[\xi,\eta],c'\rangle \\ \langle[\xi,\eta],b'\rangle & -\langle[\xi,\eta],a'\rangle & A_{\xi,\eta} \\ \langle[\xi,\eta],c'\rangle & -A_{\xi,\eta} & -\langle[\xi,\eta],a'\rangle \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \end{pmatrix} \\
&= (a_{11}, a_{12}, a_{13}) \begin{pmatrix} \langle[\xi,\eta],a'\rangle & \langle[\xi,\eta],b'\rangle & \langle[\xi,\eta],c'\rangle \\ \langle[\xi,\eta],b'\rangle & -\langle[\xi,\eta],a'\rangle & 0 \\ \langle[\xi,\eta],c'\rangle & 0 & -\langle[\xi,\eta],a'\rangle \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \end{pmatrix} \\
&= \langle[\xi,\eta], (a_{11}^2 - a_{12}^2 - a_{13}^2)a' + 2a_{11}a_{12}b' + 2a_{12}a_{13}c'\rangle \\
&= \langle[\xi,\eta], (2a_{11}^2 - 1)a' + 2a_{11}a_{12}b' + 2a_{12}a_{13}c'\rangle \\
&= \langle[\xi,\eta], 2a_{11}a - a'\rangle.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
a_{11}\{f, g\}_{\mathcal{F}_1} &+ a_{21}\{f, g\}_{\mathcal{F}_2} + a_{31}\{f, g\}_{\mathcal{F}_3} \\
&= \langle[\xi,\eta], a_{11}a - a_{21}b - a_{31}c\rangle \\
&= \langle[\xi,\eta], 2a_{11}a - a'\rangle.
\end{aligned}$$

This completes the proof.   □

An immediate consequence is the following:

THEOREM 3.6. – *Assume that Equations* (10) *and* (11) *hold. Then the Poisson tensors* $\pi_{\mathcal{F}}$, $\pi_{\mathcal{T}}$ *corresponding to any two frames commute.*

Proof. – For any $k_1, k_2, k_3 \in \mathbb{R}$ such that $k_1^2 + k_2^2 + k_3^2 = 1$, it follows from Theorem 3.5 that $k_1 \pi_{\mathcal{F}_1} + k_2 \pi_{\mathcal{F}_2} + k_3 \pi_{\mathcal{F}_3}$ is still a Poisson tensor. Hence, $\pi_{\mathcal{F}_1}$, $\pi_{\mathcal{F}_2}$ and $\pi_{\mathcal{F}_3}$ all commute. Again according to Theorem 3.5, $\pi_{\mathcal{T}}$ is a linear combination of $\pi_{\mathcal{F}_1}$, $\pi_{\mathcal{F}_2}$ and $\pi_{\mathcal{F}_3}$, and therefore commutes with $\pi_{\mathcal{F}}(= \pi_{\mathcal{F}_1})$. □

Remark. – Under the assumption of Theorem 3.3, the Poisson structure $\pi_{\mathcal{F}}$ on M is G-invariant, and $pr_1 \circ \Psi_{\mathcal{F}} : M \longrightarrow \mathfrak{g}$, the composition of $\Psi_{\mathcal{F}}$ with $pr_1 : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$, the projection onto its first factor, is a G-equivariant momentum mapping. Similarly, $\pi_{\mathcal{F}_1}$ and $\pi_{\mathcal{F}_2}$ are G-invariant with equivariant momentum mappings $pr_2 \circ \Psi_{\mathcal{F}}$ and $pr_3 \circ \Psi_{\mathcal{F}}$, respectively.

The rest of the section is devoted to the investigation of the symplectic foliation of $\pi_{\mathcal{F}}$. For simplicity, whenever a frame $\mathcal{F}$ is fixed, we shall omit the subscript $\mathcal{F}$ when denoting the Poisson structure under the circumstance without confusion. *I.e.*, we will use π_i to denote $\pi_{\mathcal{F}_i}$. By X_f^i, for $i = 1, 2, 3$, we denote the vector field $\pi_i^{\#}(df)$ for $f \in C^{\infty}(M)$.

THEOREM 3.7. – *The symplectic foliation of* $\pi_{\mathcal{F}}$ *is independent of the choice of frames* $\mathcal{F}$, *and the induced family of symplectic structures on each leaf is hyper-symplectic iff for any* $(a, b, c) \in \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ *and* $\xi \in \mathfrak{g}$, *the following system of equations for* (u, v, w) *has a solution:*

$$(15) \quad \begin{cases} A^{\#}u + [v, c] - [w, b] = [\xi, a] \\ -[u, c] + A^{\#}v + [w, a] = [\xi, b] \\ [u, b] - [v, a] + A^{\#}w = [\xi, c] \\ -[u, a] - [v, b] - [w, c] = A^{\#}\xi, \end{cases}$$

where for any $u \in \mathfrak{g}$, $A^{\#}u$ is the $\mathfrak{g}$-valued function on M obtained by contracting with u, and M is identified with $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ under $\Psi_{\mathcal{F}}$.

Remark. – The first three equations can be written in terms of a single equation as

$$X_{l_u^3}^2 + X_{l_v^1}^3 + X_{l_w^2}^1 = -\hat{\xi}.$$

If such an A exists, we shall call the corresponding family of Poisson structures a *hyper-Lie Poisson structure* and their symplectic foliation *hypersymplectic foliation.*

We need several lemmas before we can prove this theorem.

The next lemma indicates that whether System (15) is solvable is independent of the choice of frames. Therefore, the statement in Theorem 3.7 is well justified.

LEMMA 3.8. – *For any fixed* $\xi \in \mathfrak{g}$, (u, v, w) *is a solution of System* (15) *for* $(a, b, c) \in \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ *iff* $(u', v', w') = (u, v, w)O$ *is a solution of the same system for* (a', b', c'), *where* O *is any matrix in* $SO(3)$ *and* $(a', b', c') = (a, b, c)O$.

This can be proved by a straightforward verification, and is left to the reader. The proof of the following two lemmas is also quite straightforward from definition.

LEMMA 3.9. – *For the Poisson structure π on $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$, the hamiltonian vector fields of linear functions are given by*

$$X_{l_\xi^1} = (-[\xi, a], -[\xi, b], -[\xi, c]),$$

$$X_{l_\xi^2} = (-[\xi, b], [\xi, a], A^\# \xi),$$

$$X_{l_\xi^3} = (-[\xi, c], -A^\# \xi, [\xi, a]),$$

for any $\xi \in \mathfrak{g}$, where (a, b, c) is any point in $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ and the tangent space at this point is naturally identified with $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$.

LEMMA 3.10. – *For any $\xi \in \mathfrak{g}$ and $i, j = 1, 2, 3$,*

$$X_{l_\xi^i}^j = -X_{l_\xi^j}^i, \qquad i \neq j;$$

$$X_{l_\xi^i}^i = -\hat{\xi}.$$

Proof of Theorem 3.7. – Assume that System (15) has a solution. We divide our proof into several steps.

(1) $\pi_{\mathcal{F}_1}^\# T^* M = \pi_{\mathcal{F}_2}^\# T^* M = \pi_{\mathcal{F}_3}^\# T^* M$.

From Lemma (3.10), it follows that $X_{l_\xi^1}^2 = -X_{l_\xi^2}^1$ and $X_{l_\xi^2}^2 = -\hat{\xi} = X_{l_\xi^1}^1$. Hence, both $X_{l_\xi^1}^2$ and $X_{l_\xi^2}^2$ are in $\pi_{\mathcal{F}_1}^\# T^* M$. Also, it is easy to see that $X_{l_\xi^3}^2 = (A^\# \xi, -[\xi, c], [\xi, b])$. On the other hand, for any $(u, v, w) \in \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \cong T_{(a,b,c)}^* (\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g})$, we have

$$\pi_{\mathcal{F}_1}^\# (u, v, w) = (-[u, a] - [v, b] - [w, c], -[u, b] + [v, a] - A^\# w, -[u, c] + A^\# v + [w, a])$$

by Lemma 3.9. It is equal to $X_{l_\xi^3}^2$ if (u, v, w) is a solution of System (15). Hence, we have $X_{l_\xi^3}^2 \in \pi_{\mathcal{F}_1}^\# T^* M$. This shows that $\pi_{\mathcal{F}_2}^\# T^* M \subseteq \pi_{\mathcal{F}_1}^\# T^* M$. Similarly $\pi_{\mathcal{F}_1}^\# T^* M \subseteq \pi_{\mathcal{F}_2}^\# T^* M$, so do the other relations as well.

(2) The symplectic foliation of $\pi_{\mathcal{F}}$ does not depend on the choice of frames.

Let $\mathcal{T}$ be another frame, and $\pi_{\mathcal{T}}$ its corresponding Poisson structure. According to Theorem 3.5, $\pi_{\mathcal{T}}$ can be expressed as a linear combination of $\pi_{\mathcal{F}_1}$, $\pi_{\mathcal{F}_2}$ and $\pi_{\mathcal{F}_3}$. Therefore, it follows that $\pi_{\mathcal{T}}^\# T^* M \subseteq \pi_{\mathcal{F}}^\# T^* M$ from Step (1). According to Lemma 3.8 and Step (1), the symplectic foliations of $\pi_{\mathcal{T}_i}$, $i = 1, 2, 3$, also coincide. Hence, exchanging $\mathcal{F}$ and $\mathcal{T}$, we obtain the other inclusion: $\pi_{\mathcal{F}}^\# T^* M \subseteq \pi_{\mathcal{T}}^\# T^* M$.

(3) Let ω_1, ω_2 and ω_3 be the symplectic structures on a hypersymplectic leaf corresponding to $\pi_{\mathcal{F}_1}$, $\pi_{\mathcal{F}_2}$ and $\pi_{\mathcal{F}_3}$. Then $\omega_1, \omega_2, \omega_3$ satisfy Equations (3).

Below we only prove the following equation: $\forall \ f, g \in C^\infty(M)$,

$$(16) \qquad \omega_1(X_f^2, X_g^2) = \{f, g\}_1.$$

The other equations can also be proved similarly.

In fact, it suffices to show this equation when f and g are linear functions. If either f or g is l^1_ξ (for example $f = l^1_\xi$), we have

$$\begin{aligned}
\text{LHS} &= \omega_1(X^2_{l^1_\xi}, X^2_g) \quad \text{(by Lemma 3.10)} \\
&= -\omega_1(X^1_{l^2_\xi}, X^2_g) \\
&= -\langle dl^2_\xi, X^2_g \rangle \\
&= -\{g, l^2_\xi\}_2 \\
&= X^2_{l^2_\xi}(g) \quad \text{(by Lemma 3.10)} \\
&= -\hat{\xi}(g) \\
&= \{l^1_\xi, g\}_1.
\end{aligned}$$

Similarly, Equation (16) holds if either f or g is l^2_ξ.

It remains to check Equation (16) for $f = l^3_\xi$ and $g = l^3_\eta$. In this case, we know that $X^2_{l^3_\xi} \lrcorner \omega_1 = (u, v, w)$ according to the proof in Step (1), where (u, v, w) is a solution of System (15) and is considered as a cotangent vector at $(a, b, c) \in \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$. We also know that $X^2_{l^3_\eta} = (A^\# \eta, -[\eta, c], [\eta, b])$. Therefore,

$$\begin{aligned}
\omega_1(X^2_{l^3_\xi}, X^2_{l^3_\eta}) &= (X^2_{l^3_\xi} \lrcorner \omega_1)(X^2_{l^3_\eta}) \\
&= \langle u, A^\# \eta \rangle + \langle v, -[\eta, c] \rangle + \langle w, [\eta, b] \rangle \\
&= \langle A^\# u + [v, c] - [w, b], \eta \rangle \quad \text{(by Equation (15))} \\
&= \langle [\xi, a], \eta \rangle \\
&= \langle [\eta, \xi], a \rangle.
\end{aligned}$$

On the other hand, $\{l^3_\xi, l^3_\eta\}_1 = l^1_{[\eta, \xi]} = \langle [\eta, \xi], a \rangle$. Hence, $\omega_1(X^2_{l^3_\xi}, X^2_{l^3_\eta}) = \{l^3_\xi, l^3_\eta\}_1$. This completes the proof of Equation (16).

Finally, it is quite transparent from the proof above that the assumption in the statement of Theorem 3.7 should also be necessary. $\qquad\square$

To end this section, we give the following result which reveals the connection between the $S^2(\mathfrak{g})$-valued function A and the induced pseudo-metric on the hypersymplectic leaves.

THEOREM 3.11. – *Under the assumptions as in Theorem* 3.3 *and Theorem* 3.7, *the pseudo-metric* g *on each hyper-symplectic leaf is* G-*invariant, and for any* $\xi \in \mathfrak{g}$, $g(\hat{\xi}, \hat{\xi}) = -A_{\xi, \xi}$.

Proof. – Since the Poisson structures $\pi_{\mathcal{F}_1}$, $\pi_{\mathcal{F}_2}$ and $\pi_{\mathcal{F}_3}$ are all G-invariant, so are their induced symplectic structures on each hyper-symplectic leaf. Hence, the pseudo-metric g is G-invariant. According to Equation (4),

$$\begin{aligned}
g(\hat{\xi}, \hat{\xi}) &= -\pi_1(\omega^{\flat}_2 \hat{\xi}, \omega^{\flat}_3 \hat{\xi}) \\
&= -\pi_1(-dl^2_\xi, -dl^3_\xi) \\
&= -\{l^2_\xi, l^3_\xi\}_1 \\
&= -A_{\xi, \xi}. \qquad\square
\end{aligned}$$

We have seen that the vector-valued function A plays a fundamental role in defining a hyper-Lie Poisson structure. The theorem above leads to some nondegenerate criterion that A should satisfy, *i.e.*, $A_{\xi,\xi} = 0$ iff $\hat{\xi} = 0$. The work of Kronheimer [10], [11] very much supports the existence of A for compact semi-simple Lie algebras. A satisfactory solution to this problem should provide us a symplectic approach, and therefore an intrinsic explanation, on the existence of hyperkähler structures on adjoint orbits. The work on this project is still in progress. In the rest of the paper, instead we will consider the case that $\mathfrak{g} = \mathfrak{su}(2)$. This case can be handled relatively more easily because of its special character as a three-dimensional Lie algebra. However, we shall see that certain nontrivial results, some of which are already quite striking, can be deduced even in such a simple case.

4. The case of $\mathfrak{g} = \mathfrak{su}(2)$

From now on, we will work on the special case that $\mathfrak{g} = \mathfrak{su}(2)$. In this case, a function A can be explicitly constructed on an open submanifold of M, and the corresponding hyper-Lie Poisson structures are studied under the general set-up in the previous section.

By Φ, we denote the function on M defined by:

$$\Phi(a, b, c) = \langle a, [b, c] \rangle, \qquad \forall (a, b, c) \in \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}.$$

Here again M is identified with $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ under some chosen frame. This equation defines the well known Lie algebra 3-cocycle corresponding to the Killing form. However, here we consider it as a function on $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ instead of $\wedge^3 \mathfrak{g}$. Clearly, Φ is independent of the choice of frames, provided that they have the same orientation. Hence, Φ can still be considered as a well-defined function on M. Let M_o be the open submanifold of M consisting of all points where $\Phi \neq 0$. In other words, M_o consists of triples (a, b, c) which are linearly independent. Let $A : M_o \subset \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \longrightarrow S^2(\mathfrak{g})$ be the map given by

$$(17) \qquad A(a, b, c) = \frac{1}{\Phi}([a, b] \otimes [a, b] + [b, c] \otimes [b, c] + [c, a] \otimes [c, a]), \quad \forall (a, b, c) \in M_o.$$

It is not difficult to check that the rhs of Equation (17) is invariant under the natural action of $SO(3)$, so A can indeed be considered as a well-defined map from M_o to $S^2(\mathfrak{g})$.

THEOREM 4.1. – *A is G-equivariant and satisfies the condition in Theorem* 3.3

Proof. – That A is G-equivariant can be verified directly.

We note that A is uniquely characterized by the following relations:

$$(18) \qquad a \lrcorner A = [b, c], \qquad b \lrcorner A = [c, a], \qquad \text{and} \qquad c \lrcorner A = [a, b]$$

for any $(a, b, c) \in M_o$. Applying the vector field $X_{l_\eta^2}$ on both sides of the equation $a \lrcorner A = [b, c]$, we obtain

$$X_{l_\eta^2} a \lrcorner A + a \lrcorner X_{l_\eta^2} A = [X_{l_\eta^2} b, c] + [b, X_{l_\eta^2} c],$$

where both sides are considered as a $\mathfrak{g}$-valued function on $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$. Using Lemma 3.9, we have

$$-[\eta, b] \lrcorner A + a \lrcorner X_{l_\eta^2} A = [[\eta, a], c] + [b, A^\# \eta].$$

Hence,

$$a \lrcorner X_{l_\eta^2} A = [[\eta, a], c] + [b, A^\# \eta] + A^\#[\eta, b].$$

By contracting with $\xi \in \mathfrak{g}$, it follows that

$$
\begin{aligned}
\langle a, \xi \lrcorner X_{l_\eta^2} A \rangle &= \langle \xi, [[\eta, a], c] \rangle + \langle \xi, [b, A^\# \eta] \rangle + \langle \xi, A^\#[\eta, b] \rangle \\
&= \langle \xi, [[\eta, a], c] \rangle + \langle [\xi, b], A^\# \eta \rangle + \langle \xi, A^\#[\eta, b] \rangle \\
&= \langle \xi, [[\eta, a], c] \rangle + A_{\eta, [\xi, b]} + A_{\xi, [\eta, b]}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\langle a, (\xi \lrcorner X_{l_\eta^2} A - \eta \lrcorner X_{l_\xi^2} A) \rangle &= \langle \xi, [[\eta, a], c] \rangle - \langle \eta, [[\xi, a], c] \rangle \\
&= \langle a, [[\xi, \eta], c] \rangle.
\end{aligned}
$$

Using the other two identities in Equation (18), similarly we deduce that

$$
\begin{aligned}
\langle b, (\xi \lrcorner X_{l_\eta^2} A - \eta \lrcorner X_{l_\xi^2} A) \rangle &= \langle b, [[\xi, \eta], c] \rangle, \\
\langle c, (\xi \lrcorner X_{l_\eta^2} A - \eta \lrcorner X_{l_\xi^2} A) \rangle &= 0.
\end{aligned}
$$

Since the Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$ is three dimensional and $\Phi(a, b, c) \neq 0$, $\{a, b, c\}$ constitutes a basis of $\mathfrak{su}(2)$ at any point in M_o. Equation (10) thus follows immediately, similarly for Equation (11). $\square$

In fact, A also satisfies the assumption as in Theorem 3.7.

THEOREM 4.2. – *For the function A defined by Equation* (17), *System* (15) *always has a solution for any $\xi \in \mathfrak{g}$ and $(a, b, c) \in M_o$. So M_o has a hyper-Lie Poisson structure. In particular, its symplectic leaves are hyper-symplectic.*

Proof. – Fix any point $(a, b, c) \in M_o$. Since $\mathfrak{su}(2)$ is three-dimensional and System (15) is linear with respect to u, v, w, it suffices to prove this statement for any three linearly independent $\xi \in \mathfrak{g} = \mathfrak{su}(2)$. Therefore, it is sufficient to prove this for $\xi = a, b$, and c. For this purpose, one can check directly that $u = v = 0, w = -b$ is a solution for $\xi = a$; $u = -c, v = w = 0$ is a solution for $\xi = b$; and $u = 0, v = -a, w = 0$ is a solution for $\xi = c$. $\square$

In fact, in this special case, the corresponding hypersymplectic foliation can be described quite explicitly. By X, we denote the gradient vector field of Φ, where M is equipped with the standard metric induced from the Killing form on $\mathfrak{g}$. As a frame is chosen and M is identified with $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$, the vector field X at any point (a, b, c) can be written as

$$(19) \qquad\qquad X = ([b, c], [c, a], [a, b]).$$

Since both the standard metric on M and the function Φ are G-invariant, the gradient vector field X is also G-invariant. Therefore, it follows that

$$[X, \hat{\xi}] = 0, \qquad \forall \xi \in \mathfrak{g}.$$

THEOREM 4.3. – *The symplectic foliation of $\pi_{\mathcal{F}}$ on M_o coincides with the orbits of the Lie algebra action of the direct product Lie algebra $\mathbb{R} \times \mathfrak{su}(2)$, with X and $\hat{\xi}, \xi \in \mathfrak{g}$ being its generators.*

Proof. – The symplectic distribution at any point (a, b, c) is spanned by $\pi_{\mathcal{F}}^{\#}(T^*M)$. By identifying $\mathfrak{g}^*$ with $\mathfrak{g}$, $T^*_{(a,b,c)}M$ is identified with $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ as a vector space. To compute the symplectic distribution, it is sufficient to compute the image of a basis of $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ under the map $\pi_{\mathcal{F}}^{\#}$. Since $\mathfrak{g}$ is three dimensional, $\{a, b, c\}$ can be considered as a basis of $\mathfrak{g}$. Hence, it suffices to do the computation for its corresponding basis in $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$. Using Lemma 3.9, we have $\pi_{\mathcal{F}}^{\#}(\xi, 0, 0) = -\hat{\xi} = -([\xi, a], [\xi, b], [\xi, c])$. It is also easy to see that $\pi_{\mathcal{F}}^{\#}(0, a, 0) = \hat{b}$, $\pi_{\mathcal{F}}^{\#}(0, b, 0) = -\hat{a}$ and $\pi_{\mathcal{F}}^{\#}(0, c, 0) = X$; $\pi_{\mathcal{F}}^{\#}(0, 0, a) = \hat{c}$, $\pi_{\mathcal{F}}^{\#}(0, 0, b) = -X$ and $\pi_{\mathcal{F}}^{\#}(0, 0, c) = -\hat{a}$. This concludes the proof of the theorem. □

PROPOSITION 4.4. – *The Lie algebra action defined as in Theorem 4.3 is locally free on M_o, so its orbits are all 4-dimensional.*

Proof. – If not, there is $\xi \in \mathfrak{g}$ and $k \in \mathbb{R}$ not all zero, such that $\hat{\xi} + kX = 0$. If $k \neq 0$, it follows that $[b, c] = \frac{1}{k}[\xi, a]$. Hence, $\Phi = \langle a, [b, c] \rangle = \frac{1}{k}\langle a, [\xi, a] \rangle = 0$. This contradicts to the definition of M_o. If $k = 0$, we have $\hat{\xi} = -([\xi, a], [\xi, b], [\xi, c]) = 0$. It thus follows that $\xi = 0$ since $\{a, b, c\}$ is a basis of $\mathfrak{g}$. □

The coming result gives us a complete set of casimir functions for the hypersymplectic foliation.

PROPOSITION 4.5. – *The following functions $\langle a, b \rangle, \langle b, c \rangle, \langle c, a \rangle, \langle a, a \rangle - \langle b, b \rangle$ and $\langle b, b \rangle - \langle c, c \rangle$ form a complete set of casimirs for the Poisson structure $\pi_{\mathcal{F}}$ on M_o.*

Proof. – It is simple to see that these functions are all G-invariant. To show that they are casimirs, it suffices to show that they are killed by the vector field X, which can be checked directly. It is also easy to see that these functions are all independent, so it follows from dimension counting that this set of casimirs is complete. □

To end this section, we look at the induced metric on each hyper-symplectic leaf. The metric on the infinitesimal generators $\hat{\xi}$ of the G-action is already given by Theorem 3.11. In order to describe the metric, we only need to know its evaluation on the vector field X, which is the content of the following:

PROPOSITION 4.6.

$$(20) \qquad\qquad g(X, X) = -\Phi.$$

Proof. – We already know that $X^2_{l^3_\xi} = (A^{\#}\xi, -[\xi, c], [\xi, b])$. Thus, $\omega_2^b X = (0, 0, a)$. According to Lemma 3.9 and Lemma 3.10, we have $\omega_3^b X = (b, 0, 0)$. Here, in both equations, the right hand sides are considered as elements in the cotangent space of M, being identified with $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$. Therefore, $g(X, X) = -\pi_1(\omega_2^b X, \omega_3^b X) = -\pi_1((0, 0, a), (b, 0, 0)) = -\langle \pi_1^{\#}(0, 0, a), (b, 0, 0) \rangle = -\langle [c, a], b \rangle = -\Phi.$ □

The following consequence follows immediately from this result combining with Theorem 3.11.

THEOREM 4.7. – *When $\mathfrak{g} = \mathfrak{su}(2)$ and A is defined by Equation (17), each hypersymplectic leaf of M_o is a 4-dimensional hyperkähler manifold.*

5. Moduli spaces of solutions to Nahm's equations and (co)adjoint orbits

This is a continuation of the last section. When M is identified with $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ by choosing a frame $\mathcal{F}$, the vector field X is written as $X = ([b, c], [c, a], [a, b])$. Its flow is thus given by the following system of equations:

$$(21) \qquad \begin{cases} \dot{a} = [b, c] \\ \dot{b} = [c, a] \\ \dot{c} = [a, b]. \end{cases}$$

Such a system is called Nahm's equations, which was studied by Kronheimer [10] modelled on the study of general Nahm's equations made by Donaldson [5]. For any $x \in M$, we denote by $\varphi_t(x)$ the flow generated by X through the point x. We denote by S the set of all points $x \in M$ such that the flow $\varphi_t(x)$ converges as $t \longrightarrow -\infty$. S has a natural foliation according to the limit points. Note that all the limit points are critical points of Φ. We denote by C the critical set of Φ. The G-action on M leaves C invariant. For any orbit $\mathcal{O}$, we let $S_{\mathcal{O}}$ be the submanifold of S consisting of all points in S whose trajectory under the gradient vector field X converges to a point in $\mathcal{O}$, as $t \longrightarrow -\infty$, so that in particular S_0 be such a submanifold corresponding to the zero orbit. It is clear that $S = \bigcup_{\mathcal{O}} S_{\mathcal{O}}$, where the sum is over all the G-orbits in C. Kronheimer proved, using gauge theory, for a general semi-simple Lie algebra that certain $S_{\mathcal{O}}$ are hyperkähler manifolds and are diffeomorphic to adjoint orbits of $\mathfrak{g}^{\mathbb{C}}$ [10], [11]. Below we will prove this result for the special case of $\mathfrak{su}(2)$, as a consequence of the hyper-Lie Poisson structure on S. Our approach is quite elementary, and the family of symplectic structures on each leaf $S_{\mathcal{O}}$ is rather transparent.

To start with, let us introduce a function F on M by

$$F(a, b, c) = \langle a, a \rangle + \langle b, b \rangle + \langle c, c \rangle,$$

where M is identified with $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ by choosing a frame. It is simple to see that F is indeed a well-defined function on M.

LEMMA 5.1.

$$L_X \Phi = \|X\|^2,$$

and

$$L_X F = 6\Phi.$$

Proof. – The first identity follows from a general property of a gradient flow. As for the second one, we have

$$\begin{aligned} L_X F &= L_X(\langle a, a \rangle + \langle b, b \rangle + \langle c, c \rangle) \\ &= 2\langle a, [b, c] \rangle + 2\langle b, [c, a] \rangle + 2\langle c, [a, b] \rangle \\ &= 6\Phi. \end{aligned} \qquad \square$$

The following result is crucial for characterizing the elements in S.

PROPOSITION 5.2. – (i) *If $\varphi_t(x)$ converges as $t \longrightarrow -\infty$, x is either a critical point of Φ or $\Phi(x) > 0$. In the latter case, we in fact have $\Phi(\varphi_t(x)) > 0$ for all t whenever $\varphi_t(x)$ is defined.*

(ii) *If $\varphi_t(x)$ does not converge as $t \longrightarrow -\infty$ or is not defined for all $t \leq 0$ (i.e., X is incomplete in the $-\infty$ direction), Φ cannot be always nonnegative along the flow.*

Proof. – By Lemma 5.1, $\frac{d}{dt}\Phi(\varphi_t(x)) = L_X\Phi = \|X\|^2 \geq 0$. Hence, $\Phi(\varphi_t(x))$ is an increasing function with respect to t. If $\varphi_t(x)$ converges as $t \longrightarrow -\infty$, the limit point must be a critical point. However, the critical points of Φ are defined by the system of equations:

$$(22) \qquad\qquad [b, c] = [c, a] = [a, b] = 0.$$

So Φ vanishes at any critical point. This yields that $\Phi(x) \geq 0$. If $\Phi(x) = 0$, it follows that $\Phi(\varphi_t(x)) = 0$, for all $t \leq 0$. By taking derivative, we have $\|X\| = 0$. Thus, x is a critical point. In the case that $\Phi(x) > 0$, it is not difficult to see that $\Phi(\varphi_t(x))$ has to stay positive for all t whenever $\varphi_t(x)$ is defined, otherwise x will be a critical point according to the same argument above.

If $\varphi_t(x)$ is not defined for all $t \leq 0$, it must be unbounded as t approaches to a finite number. If $\varphi_t(x)$ is defined for all $t \leq 0$ but does not converge as $t \longrightarrow -\infty$, it must be unbounded as t is sufficiently negative since Φ is a real analytic function. In both cases, $F(\varphi_t(x)) \longrightarrow \infty$, as $t \longrightarrow -\lambda$ (λ is either a positive number or ∞). Assume that Φ is always nonnegative along the flow. It follows from Lemma 5.1 that $\frac{d}{dt}F(\varphi_t(x)) = 6\Phi \geq 0$. So $F(\varphi_t(x)) \leq F(x)$ when $t \leq 0$, which is a contradiction. This concludes the proof. $\square$

By M_+, we denote the submanifold of M consisting of all points where Φ is positive. The theorem above yields that $S - C$ is contained in M_+. Moreover, the vector field X is complete in $S - C$. It is clear that $S - C$ is invariant under the G-action, hence invariant under the action of the product Lie algebra $\mathbb{R} \times \mathfrak{su}(2)$ as defined in Theorem 4.3. In other words, $S - C$ is a hyper-Poisson submanifold of M_o. To extend this hyper-Poisson structure to entire S, it suffices to extend $\pi_{\mathcal{F}}$ to the critical set C. For this, one only needs to extend the vector valued function A to the critical set C. Since C is the limit set, a natural way to extend A is to take its limit along the flow X. This is in fact how we derive the formula below.

When M is identified with $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ under a chosen frame, a point $x_0 = (a_0, b_0, c_0) \in \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ is a critical point iff a_0, b_0, c_0 are parallel. Hence, for any critical point, we can always choose a frame so that the critical point is of the form $(a_0, 0, 0)$ for some $a_0 \in \mathfrak{g}$ under the identification: $M \cong \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ using this frame. Such a frame is called a standard frame. Clearly, the element a_0 is unique up to a sign. We then define the function A on C under a standard frame by:

$$(23) \qquad A : C \longrightarrow S^2(\mathfrak{g}), \quad (a_0, 0, 0) \longrightarrow \|a_0\| \sum I_i \otimes I_i - \frac{1}{\|a_0\|}(a_0 \otimes a_0),$$

where $I_i, i = 1, 2, 3$, is an orthogonal basis for $\mathfrak{g} \cong \mathfrak{su}(2)$. We also let $A = 0$ at $x = 0$. A is clearly well-defined on C.

To show that such an extension is smooth, we need to give an alternate description of S, which is much easier to deal with.

For any given nontrivial critical point x_0, let $\mathcal{F}_{x_0}$ be a standard frame such that $x_0 = (a_0, 0, 0) \in \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ when M is identified with $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ under $\mathcal{F}_{x_0}$. We denote,

by Σ_{x_0}, the subset of M consisting of all the points $x = (a, b, c) \in \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \, (\cong M$ under $\mathcal{F}_{x_0})$, satisfying the condition:

(24)
$$\begin{cases} \langle a, b \rangle = \langle b, c \rangle = \langle a, c \rangle = 0, \\ \langle b, b \rangle = \langle c, c \rangle, \\ \langle a, a \rangle - \langle b, b \rangle = \langle a_0, a_0 \rangle, \\ \Phi \geq 0. \end{cases}$$

It is easy to check that this definition is well-justified, *i.e.*, does not depend on the choice of the standard frame $\mathcal{F}_{x_0}$. Also, we define Σ_0 as the subspace of $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ consisting of all points (a, b, c) such that

(25) $\langle a, b \rangle = \langle b, c \rangle = \langle a, c \rangle = 0$, $\langle a, a \rangle = \langle b, b \rangle = \langle c, c \rangle$,   and $\Phi \geq 0$.

It is clear that these relations are preserved under the transformation $(a, b, c) \longrightarrow (a', b', c') = (a, b, c)O$ for any $O \in SO(3)$. Therefore, Σ_0 can also be considered as a subset of M.

LEMMA 5.3. – *For any nontrivial critical point x_0, $\Sigma_{x_0} = S_{G \cdot x_0}$.*

Proof. – Σ_{x_0} is obviously a closed submanifold of M. It is clear that if $x \in \Sigma_{x_0}$, then $\varphi_t(x)$ will stay in Σ_{x_0} for all the t whenever the flow is defined, since X is tangent to Σ_{x_0}. Since the intersection of Σ_{x_0} with the hypersurface $\Phi = 0$ is contained in the critical set C, we conclude that $\Phi(\varphi_t(x)) > 0$ if x is not a critical point in Σ_{x_0}. Thus, according to Proposition 5.2, $\varphi_t(x)$ exists for all $t \leq 0$ and converges as $t \longrightarrow -\infty$. Let us assume that y is the limit point of $\varphi_t(x)$. Then y is a critical point and $y \in \Sigma_{x_0}$. Assume that $y = (u, v, w)$ under the standard frame $\mathcal{F}_{x_0}$. It is not difficult to see by using Equation (24) that $v = w = 0$ and $\langle u, u \rangle = \langle a_0, a_0 \rangle$. The latter implies that $y \in G \cdot x_0$. Hence $\Sigma_{x_0} \subseteq S_{G \cdot x_0}$.

Conversely, assume that x is any point in $S_{G \cdot x_0}$ and $\varphi_t(x) \longrightarrow y \in G \cdot x_0$ as $t \longrightarrow -\infty$. Fix a standard frame $\mathcal{F}_{x_0}$ so that under it $x_0 = (a_0, 0, 0)$. Then under this frame $y = (u, 0, 0)$ with $u \in G \cdot a_0$. Hence, $\langle u, u \rangle = \langle a_0, a_0 \rangle$. Suppose that $x = (a, b, c)$ under the frame $\mathcal{F}_{x_0}$. From Proposition 4.5, it thus follows that $\langle a, b \rangle = \langle u, 0 \rangle = 0$, $\langle b, c \rangle = \langle 0, 0 \rangle = 0$, $\langle a, c \rangle = \langle u, 0 \rangle = 0$, $\langle b, b \rangle - \langle c, c \rangle = 0$ and $\langle a, a \rangle - \langle b, b \rangle = \langle u, u \rangle = \langle a_0, a_0 \rangle$. That is, x is in Σ_{x_0}. This completes the proof. $\square$

From this lemma, it follows that for any $x_0, y_0 \in C$, Σ_{x_0} and Σ_{y_0} are either disjoint, or equal. In the latter case, x_0 and y_0 must lie in the same G-orbits of C. For this reason, we shall use $\Sigma_{\mathcal{O}}$ to denote the space Σ_{x_0} for any $x_0 \in \mathcal{O}$. The lemma above shows that $S_{\mathcal{O}} = \Sigma_{\mathcal{O}}$. In fact, this is also valid when $\mathcal{O}$ is the trivial orbit.

LEMMA 5.4.

$$S_0 = \Sigma_0.$$

Proof. – That $S_0 \subseteq \Sigma_0$ follows immediately from the fact that the functions: $\langle a, b \rangle$, $\langle a, c \rangle$, $\langle b, c \rangle$, $\langle a, a \rangle - \langle b, b \rangle$, and $\langle b, b \rangle - \langle c, c \rangle$ are all preserved by the vector field X.

As for the other direction, let us assume that (a, b, c) is any nontrivial point in Σ_0. By the definition of Σ_0, we can write $a = \lambda e_1, b = \lambda e_2$ and $c = \lambda e_3$, where λ is a positive

number and $\{e_1, e_2, e_3\}$ satisfies the standard $\mathfrak{su}(2)$-relation: $[e_1, e_2] = e_3$, etc. Therefore, it is not difficult to see that $a_t = -\frac{\lambda e_1}{\lambda t - 1}$, $b_t = -\frac{\lambda e_2}{\lambda t - 1}$, $c_t = -\frac{\lambda e_3}{\lambda t - 1}$ is the flow through the point (a, b, c). Obviously, it goes to zero as $t \longrightarrow -\infty$. That is, $(a, b, c) \in S_0$. $\quad\square$

Combining the two lemmas above, we have

PROPOSITION 5.5. – *For any G-orbit $\mathcal{O}$ in C, we have*

$$S_{\mathcal{O}} = \Sigma_{\mathcal{O}}.$$

Now we are ready to prove the smoothness of the extension.

THEOREM 5.6. – *$\pi_{\mathcal{F}}$ is smooth when restricted to each $S_{\mathcal{O}}$ for any nontrivial G-orbit $\mathcal{O}$.*

Proof. – It suffices to show that the extension $A_{\xi,\xi}$ as defined by Equation (23) is smooth for any $\xi \in \mathfrak{g}$ in a neighborhood of $\mathcal{O}$ in $S_{\mathcal{O}}$. Let r be the norm of the elements in $\mathcal{O}$. Then by Proposition 5.5, under a standard frame, points (a, b, c) in $S_{\mathcal{O}}$ are characterized by the equations:

$$\langle a, b \rangle = \langle b, c \rangle = \langle a, c \rangle = 0, \qquad \langle b, b \rangle = \langle c, c \rangle, \qquad \langle a, a \rangle - \langle b, b \rangle = r^2.$$

Therefore, when a point (a, b, c) is in $S_{\mathcal{O}}$ but not in $\mathcal{O}$, we can always write $a = \sqrt{\lambda^2 + r^2} e_1, b = \lambda e_2, c = \lambda e_3$, where $\lambda = \sqrt{\langle b, b \rangle}$ and $\{e_1, e_2, e_3\}$ is an orthonormal basis of $\mathfrak{su}(2)$ satisfying the standard relation. Write $\xi = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3$. Thus a routine calculation yields that:

$$A_{\xi,\xi}(a, b, c) = \sqrt{\lambda^2 + r^2}(\xi_2^2 + \xi_3^2) + \frac{\lambda^2}{\sqrt{\lambda^2 + r^2}}\xi_1^2.$$

Substituting $\xi_2^2 + \xi_3^2 = \langle \xi, \xi \rangle - \xi_1^2$, we have

$$A_{\xi,\xi}(a, b, c) = \sqrt{\lambda^2 + r^2}\langle \xi, \xi \rangle - \frac{r^2}{\sqrt{\lambda^2 + r^2}}\xi_1^2.$$

It is trivial to see that the extension of A as given by Equation (23) coincides with the equation above when $\lambda = 0$. Since both $\lambda^2 = \langle b, b \rangle$ and $\xi_1 = \frac{\langle \xi, a \rangle}{\langle a, a \rangle}$ are smooth functions on $S_{\mathcal{O}}$, $A_{\xi,\xi}$ is clearly smooth on $S_{\mathcal{O}}$ as well. $\quad\square$

For any nontrivial orbit $\mathcal{O} \subset C$, it is obvious that $S_{\mathcal{O}}$ is invariant under the G-action, as well as that of the additive group $\mathbb{R}$ generated by X. Therefore, for any point $x \in S_{\mathcal{O}} - \mathcal{O}$, its hypersymplectic leaf $\mathcal{L}_x$, defined as in the previous section, is contained in $S_{\mathcal{O}}$. Since $\mathcal{L}_x$ is a 4-dimensional manifold according to Proposition 4.4, it can be considered as an open neighborhood of x in $S_{\mathcal{O}}$. Clearly, $S_{\mathcal{O}}$ is a union of these leaves together with their boundary $\mathcal{O}$. As observed early, there is a standard frame $\mathcal{F}$ such that when M is identified with $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ under this frame, any point in $\mathcal{O}$ is written as $x = (a, 0, 0)$ with $a \in \mathfrak{g}$. In this way, $\mathcal{O}$ is naturally identified with a (co)adjoint orbit of $\mathfrak{g}$. Although there is an ambiguity for the choice of the frame $\mathcal{F}$, such an adjoint orbit is uniquely determined, and the identification is unique up to a sign. In the following, we will fix any such a frame $\mathcal{F}$, and denote the Poisson structures $\pi_{\mathcal{F}_1}, \pi_{\mathcal{F}_2}$ and $\pi_{\mathcal{F}_3}$ simply by π_1, π_2 and π_3, respectively, and the induced symplectic structures on any leaf by ω_1, ω_2 and ω_3 for simplicity.

THEOREM 5.7. – *For any nontrivial G-orbit $\mathcal{O} \subset C$, the extended hyper-Poisson structure on S induces a hyperkähler structure on $S_{\mathcal{O}}$. Furthermore, if we choose a frame as in the*

observation above, then $\mathcal{O}$ is a symplectic submanifold with respect to ω_1. In fact, it is a Kähler submanifold with respect to I and its Kähler metric coincides with the one on $\mathcal{O}$ when it is naturally identified with a (co)adjoint orbit. In the meantime, $\mathcal{O}$ is a lagrangian submanifold with respect to ω_2 and ω_3.

Proof. – It remains to consider points in $\mathcal{O}$. For any $x \in \mathcal{O}$, one can directly verify that $\pi_i^\# T_x^* M = T_x S_{\mathcal{O}}$, for $i = 1, 2, 3$, by using a local coordinate chart under a chosen standard frame $\mathcal{F}$. Hence, the bivector fields $\pi_i, i = 1, 2, 3$, are all tangent to $S_{\mathcal{O}}$ and nondegenerate along $\mathcal{O}$. According to Theorem 4.2, the corresponding symplectic structures ω_i are all compatible along $\mathcal{O}$ by continuity, hence compatible in entire $S_{\mathcal{O}}$. Since the induced metric is negative definite on $S_{\mathcal{O}} - C$, it is also negative definite along $\mathcal{O}$. Hence, the extended hyper-Poisson structure induces a hyperkähler structure on $S_{\mathcal{O}}$. The rest of the conclusion can be verified directly, again by using local coordinates.   □

In fact, $S_{\mathcal{O}}$ is closely related to adjoint orbits of $\mathfrak{g}^{\mathbb{C}}$. To see this, let $\mathcal{F}$ be any frame, and $pr_{12} \circ \Psi_{\mathcal{F}} : M \longrightarrow \mathfrak{g}^{\mathbb{C}}$ the composition of the identification $\Psi_{\mathcal{F}} : M \longrightarrow \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ with the projection pr_{12} as defined in the proof of Theorem 3.3. According to Proposition 3.4, $pr_{12} \circ \Psi_{\mathcal{F}}$ is a Poisson map with respect to $\pi_{\mathcal{F}}$. Hence, the image of $S_{\mathcal{O}}$ is a Poisson submanifold of $\mathfrak{g}^{\mathbb{C}}$. The following theorem indicates that in generic case the image is in fact a single adjoint orbit. A general result was proved by Kronheimer [10] using gauge theory. However, our proof in this special case here is quite elementary and only uses some well-known facts in symplectic geometry. By $\mathcal{O}_{12}$ we denote the adjoint orbit in $\mathfrak{g}^{\mathbb{C}}$ containing the image $(pr_{12} \circ \Psi_{\mathcal{F}})(\mathcal{O})$.

THEOREM 5.8. – *If $\mathcal{O}_{12}$ is a regular orbit in $\mathfrak{g}^{\mathbb{C}}$, then $(pr_{12} \circ \Psi_{\mathcal{F}})(S_{\mathcal{O}}) = \mathcal{O}_{12}$. In fact, $pr_{12} \circ \Psi_{\mathcal{F}} : S_{\mathcal{O}} \longrightarrow \mathcal{O}_{12}$ is a symplectic diffeomorphism, where $S_{\mathcal{O}}$ is equipped with the symplectic structure induced from the Poisson structure $\pi_{\mathcal{F}}$ and $\mathcal{O}_{12}$ is equipped with the (co)adjoint orbit symplectic structure.*

We need some lemmas first.

LEMMA 5.9. – *Under the map $pr_{12} \circ \Psi_{\mathcal{F}} : S_{\mathcal{O}} \longrightarrow \mathfrak{g}^{\mathbb{C}}$, the inverse image of any bounded region is bounded.*

Proof. – According to Proposition 5.5 and by the definition of $\Sigma_{\mathcal{O}}$, there is a standard frame $\mathcal{T}$, under which any point $(a', b', c') \in \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ in $S_{\mathcal{O}}$ is characterized by

$$\langle a', b' \rangle = \langle b', c' \rangle = \langle a', c' \rangle = 0, \qquad \langle b', b' \rangle = \langle c', c' \rangle, \langle a', a' \rangle - \langle b', b' \rangle = r^2,$$

where r denotes the norm of elements in $\mathcal{O}$. Suppose that $O = (a_{ij}) \in SO(3)$ is the transformation matrix between the given frame $\mathcal{F}$ and this standard frame $\mathcal{T}$. That is, $(a, b, c) = (a', b', c')O$. It is simple to see that $\langle a, a \rangle = a_{11}^2 r^2 + \langle b', b' \rangle$, $\langle b, b \rangle = a_{12}^2 r^2 + \langle b', b' \rangle$, and $\langle c, c \rangle = a_{13}^2 r^2 + \langle b', b' \rangle$. Therefore, if $\langle a, a \rangle$ and $\langle b, b \rangle$ are bounded, $\langle b', b' \rangle$ has to be bounded. This implies that $\langle c, c \rangle$ is bounded as well.   □

Two immediate consequences are the following:

COROLLARY 5.10. – *The map $pr_{12} \circ \Psi_{\mathcal{F}} : S_{\mathcal{O}} \longrightarrow \mathfrak{g}^{\mathbb{C}}$ is a proper map.*

COROLLARY 5.11. – *The image of $S_{\mathcal{O}}$ under $pr_{12} \circ \Psi_{\mathcal{F}}$ is closed.*

Proof. – Using the map $\Psi_{\mathcal{F}}$, we may identify M with $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$. Suppose that under this identification (a_n, b_n, c_n) is a sequence in $S_{\mathcal{O}}$ such that $a_n \longrightarrow a_0$ and $b_n \longrightarrow b_0$. We need to show that $(a_0, b_0) \in (pr_{12} \circ \Psi_{\mathcal{F}})(S_{\mathcal{O}})$. It follows from Lemma 5.9 that c_n is bounded. Therefore there exists a convergent subsequence c_{n_k}, whose limit is denoted by c_0. Then, (a_0, b_0, c_0) is in $S_{\mathcal{O}}$ since $S_{\mathcal{O}}$ is closed. This concludes the proof. $\square$

Proof of Theorem 5.8. – Again, let us identify M with $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ by $\Psi_{\mathcal{F}}$. It is easy to see that at any point $(a, b, c) \in \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$, $T(pr_{12})X$ is the tangent vector, at $a + ib$, generated by the adjoint action ad_{ia}. Hence, the projection of entire flow: $(pr_{12} \circ \Psi_{\mathcal{F}})(\varphi_t(x))$ lies in a single orbit $\mathcal{C}$ for all t. Since $\varphi_t(x)$ converges to $\mathcal{O}$ as t goes to $-\infty$, then $(pr_{12} \circ \Psi_{\mathcal{F}})(\mathcal{O}) \subseteq \bar{\mathcal{C}}$. Therefore, $\mathcal{O}_{12} \subseteq \bar{\mathcal{C}}$. Since $\mathcal{O}_{12}$ is a regular orbit, it thus follows that $\mathcal{O}_{12} = \mathcal{C}$. This means that the image $(pr_{12} \circ \Psi_{\mathcal{F}})(S_{\mathcal{O}})$ is contained in $\mathcal{O}_{12}$ (this part of the argument is due to Kronheimer [10]).

On the other hand, according to Corollary 5.11, $(pr_{12} \circ \Psi_{\mathcal{F}})(S_{\mathcal{O}})$ is a closed Poisson submanifold in $\mathcal{O}_{12}$. Hence, it must be the entire orbit $\mathcal{O}_{12}$. That is, $(pr_{12} \circ \Psi_{\mathcal{F}}) : S_{\mathcal{O}} \longrightarrow \mathcal{O}_{12}$ is onto. This map is automatically a submersion since it is a Poisson map. By dimension counting, it must be a local diffeomorphism. However, by Corollary 5.10, it is a proper map. Therefore, it must be a covering. Since $\mathcal{O}_{12}$ is simply connected, $(pr_{12} \circ \Psi_{\mathcal{F}}) : S_{\mathcal{O}} \longrightarrow \mathcal{O}_{12}$ is thus a diffeomorphism. $\square$

Finally, we will show that $S_0 - \{0\}$ is diffeomorphic to the nilpotent orbit of $\mathfrak{sl}(2, \mathbb{C})$.

THEOREM 5.12. – $S_0 - \{0\}$ *is a hypersymplectic leaf of* M_o, *and therefore is a hyperkähler manifold. For any frame* $\mathcal{F}$, $pr_{12} \circ \Psi_{\mathcal{F}}$ *is a symplectic diffeomorphism between* $S_0 - \{0\}$ *and the nilpotent orbit of* $\mathfrak{sl}(2, \mathbb{C})$, *where* $S_0 - \{0\}$ *is equipped with the symplectic structure corresponding to* $\pi_{\mathcal{F}}$ *and the nilpotent orbit is equipped with the standard coadjoint symplectic structure.*

Proof. – It is clear that $S_0 - \{0\}$ is a union of hypersymplectic leaves since it is invariant under both the G-action and the flow of X. Each hypersymplectic leaf is 4-dimensional, and therefore must be open in $S_0 - \{0\}$. Since $S_0 - \{0\}$ itself is connected, it must be a single hypersymplectic leaf. Thus, it is hyperkähler according to Theorem 4.7. By Proposition 5.5, under the identification $\Psi_{\mathcal{F}} : M \longrightarrow \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$, a point $(a, b, c) \in \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ is in $S_0 - \{0\}$ iff $a = \lambda e_1, b = \lambda e_2$ and $c = \lambda e_3$ for some standard orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathfrak{su}(2)$, and $\lambda > 0$. Hence, its image under pr_{12}: $a + ib$, is clearly in the nilpotent orbit of $\mathfrak{sl}(2, \mathbb{C})$. A similar argument as in Corollary 5.11 shows that $(pr_{12} \circ \Psi_{\mathcal{F}})(S_0 - \{0\})$ is in fact closed. Hence it has to be the whole nilpotent orbit since $pr_{12} \circ \Psi_{\mathcal{F}}$ is a Poisson map. Finally, it is quite obvious that $pr_{12} \circ \Psi_{\mathcal{F}}$ is injective on $S_0 - \{0\}$. In fact, we always have $c = [a, b]/\sqrt{\langle a, a \rangle}$. This concludes our proof. $\square$

Remark. – As we have seen, (co)adjoint orbits of $\mathfrak{sl}(2, \mathbb{C})$ are related to the points in M_0 which have bounded trajectories (in the $-\infty$ direction) under the gradient vector field X and are contained in M_+. However, according to Theorem 4.7, there are other hypersymplectic leaves of M_0 which are contained in $M_- = M_0 - M_+$. It would be interesting to explore further the geometric structures for those leaves, and in particular the connection with the hyperkähler metrics on the cotangent bundles of hermitian symmetric spaces of noncompact type studied recently by Biquard and Gauduchon [4].

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