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QUANTIZATION OF THE MARSDEN-WEINSTEIN REDUCTION FOR EXTENDED DYNKIN QUIVERS

By Martin P. HOLLAND

ABSTRACT. – Let Γ be a finite subgroup of $\mathrm{SL}_2(k)$, for k an algebraically closed field of characteristic zero. W. Crawley-Boevey and the author [8] have introduced some noncommutative quantizations $\mathcal{O}^\lambda$ of the coordinate ring of the associated Kleinian singularity k^2/Γ indexed by those λ in $Z(k\Gamma)$ which have trace one on the regular representation. Let Q be the quiver obtained by orienting the extended Dynkin graph associated to Γ by the McKay correspondence. Let $\mathrm{Rep}(Q,\delta)$ denote the space of representations of Q with dimension vector equal to the minimal imaginary root δ of the corresponding affine root system. The group $\mathrm{GL}(\delta)=\prod_i \mathrm{GL}(\delta_i)$ acts naturally on $\mathrm{Rep}(Q,\delta)$. It is shown that each $\mathcal{O}^\lambda$ may be realised as a certain quotient of the algebra of $\mathrm{GL}(\delta)$-invariant differential operators on $\mathrm{Rep}(Q,\delta)$. © Elsevier, Paris

RÉSUMÉ. – Soit Γ un sous-groupe fini de $\mathrm{SL}_2(k)$, pour k un corps algébriquement clos de caractéristique nulle. W. Crawley-Boevey et l'auteur ont introduit [8] des quantifications non commutatives $\mathcal{O}^\lambda$ d'algèbre des fonctions régulières sur la singularité kleinienne k^2/Γ, indexées par les éléments $\lambda \in Z(k\Gamma)$ de trace égale à 1 sur la représentation régulière. On oriente, de façon arbitraire, le graphe de Dynkin étendu associé à Γ par la correspondance de McKay pour obtenir un carquois Q. Soit $\mathrm{Rep}(Q,\delta)$ l'espace de représentations de Q avec vecteur de dimension égale à δ la racine imaginaire minimale du système de racines affine associé. Il existe une action naturelle du groupe $\mathrm{GL}(\delta)=\prod_i \mathrm{GL}(\delta_i)$ sur $\mathrm{Rep}(Q,\delta)$. Nous montrons que chaque $\mathcal{O}^\lambda$ se réalise comme un certain quotient de l'algèbre des opérateurs différentiels $\mathrm{GL}(\delta)$-équivariante sur $\mathrm{Rep}(Q,\delta)$. © Elsevier, Paris

1. Introduction

Let k be an algebraically closed field of characteristic zero. If Γ is a finite non-trivial subgroup of $\mathrm{SL}_2(k)$ there are some non-commutative deformations $\mathcal{O}^\lambda$ of the coordinate ring of the associated Kleinian singularity k^2/Γ. These deformations are indexed by those λ in $Z(k\Gamma)$ which have trace one on the regular representation. They were introduced and studied by W. Crawley-Boevey and the author in [8]. In this paper we give a completely new description of these deformations using invariant differential operators on representation spaces of quivers. This new description should prove useful in investigating the deeper properties of $\mathcal{O}^\lambda$. In particular, we exhibit $\mathcal{O}^\lambda$ as the global sections of a sheaf of algebras on the projective line and prove a Beilinson-Bernstein type theorem. Thus $\mathcal{O}^\lambda$ is analogous to a minimal primitive factor of the enveloping algebra of a semisimple Lie algebra. In the earliest work of W. Crawley-Boevey and the author on deformations of Kleinian singularities we constructed representations of $\mathcal{O}^\lambda$ by differential operators in one

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variable. These representations were found case-by-case and relied on extensive computer calculations. We decided not to publish these calculations in [8] hoping for a more natural explanation. The main theorem below can be used to give such a representation of $\mathcal{O}^\lambda$. W. Crawley-Boevey exhibits a different approach to ours in [7]. Let $N_0, \dots, N_n$ be the irreducible representations of Γ, with N_0 trivial, and let V be the natural 2-dimensional representation of Γ. The McKay graph of Γ is the graph with vertex set $I = \{0, 1, \dots, n\}$, and with the number of edges between i and j being the multiplicity of N_i in $V \otimes N_j$. (Since V is self dual this is the same as the multiplicity of N_j in $V \otimes N_i$.) McKay observed that this graph is an extended Dynkin diagram [23]. Let Q be the quiver obtained from the McKay graph by choosing any orientation of the edges, and let $\delta \in \mathbb{N}^I$ be the vector with $\delta_i = \dim N_i$. This is the minimal imaginary root for the affine root system associated to Q. The simple roots for this root system are the usual basis ϵ_i of k^I and the real roots are the images of these simple roots under sequences of suitable reflections. Henceforth we always identify $Z(k\Gamma) = k^I$, with $\lambda \in Z(k\Gamma)$ corresponding to the vector whose i-th component λ_i is the trace of λ on N_i. Thus, for example, the dot product $\lambda \cdot \delta = \sum_i \lambda_i \delta_i$ is the same as the trace of $\lambda \in Z(k\Gamma)$ on the regular representation $k\Gamma$. Let $\mathrm{Rep}(Q, \delta)$ denote the vector space of representations of Q with dimension vector δ. The group $\mathrm{GL}(\delta) = \prod_{i \in I} \mathrm{GL}_{\delta_i}(k)$ acts naturally on $\mathrm{Rep}(Q, \delta)$ by conjugation, with kernel k^*. Write $\mathfrak{pgl}(\delta)$ for the Lie algebra of $\mathrm{GL}(\delta)/k^*$. We identify k^I and the characters of the Lie algebra of $\mathrm{GL}(\delta)$ via $\lambda \mapsto ((A_0, \dots, A_n) \mapsto \sum \lambda_i \mathrm{tr} A_i$. The Marsden-Weinstein reduction is the affine scheme $\mu^{-1}(0) /\!\!/ \mathrm{GL}(\delta)$, where $\mu : T^\star \mathrm{Rep}(Q, \delta) \to \mathfrak{pgl}(\delta)^*$ is the moment map. Let $\mathcal{D}(\mathrm{Rep}(Q, \delta))$ denote the algebra of differential operators on $\mathrm{Rep}(Q, \delta)$. Differentiating the action of $\mathrm{GL}(\delta)$ on $\mathrm{Rep}(Q, \delta)$ we obtain a Lie algebra map $\iota : \mathfrak{pgl}(\delta) \to \mathrm{Der}_k k[\mathrm{Rep}(Q, \delta)]$. Thus, for a character χ of $\mathfrak{pgl}(\delta)$ we can form

$$\mathfrak{A}^\chi = \frac{\mathcal{D}(\mathrm{Rep}(Q, \delta))^{\mathrm{GL}(\delta)}}{(\mathcal{D}(\mathrm{Rep}(Q, \delta))(\iota - \chi)(\mathfrak{pgl}(\delta)))^{\mathrm{GL}(\delta)}}.$$

The moment map is flat [8] and it follows from this that $\mathfrak{A}^\chi$ is a quantization of the Marsden-Weinstein reduction in the sense that, with its natural filtered structure, $\mathrm{gr} \mathfrak{A}^\chi \cong k[\mu^{-1}(0) /\!\!/ \mathrm{GL}(\delta)]$. If a is an arrow of Q we write $h(a)$ for its head and $t(a)$ for its tail. The defect $\partial \in \mathbb{Z}^I$ is defined by $\partial_i = -\delta_i + \sum_{t(a)=i} \delta_{h(a)}$. Our main result (see Corollary 4.7) on deformations of Kleinian singularities is:

THEOREM 1. – *Let $\lambda \in k^I$ satisfy $\lambda \cdot \delta = 1$. There is a k-algebra isomorphism $\mathcal{O}^\lambda \cong \mathfrak{A}^{\lambda - \partial - \epsilon_0}$.*

We are also able to construct a sheaf of algebras analogous to a sheaf of twisted differential operators on the flag variety. To put this in context we need to recall some facts about moduli spaces of quivers. We orient Q so that there are no oriented cycles. After King [14] one can form a coarse moduli space $\mathfrak{N}$ for families of ∂-semistable representations of Q with dimension vector δ. Here, $M \in \mathrm{Rep}(Q, \delta)$ is ∂-semistable if whenever N is a submodule of M with dimension vector β we have $\partial \cdot \beta \leq 0$. The space $\mathfrak{N}$ is obtained as a categorical quotient $\pi : \mathrm{Rep}(Q, \delta)^{ss}_\partial \to \mathfrak{N}$ of the semistable points by $\mathrm{GL}(\delta)$. It goes back to Ringel [29] that $\mathfrak{N} \cong \mathbb{P}^1$. It was proved by Kronheimer [16] for $k = \mathbb{C}$, and by Crawley-Boevey and the author [8], in general, that the Marsden-Weinstein reduction is the Kleinian singularity associated to the extended Dynkin diagram by the McKay correspondence. One can construct a certain partial resolution of this singularity by considering the ∂-semistable points of $\mu^{-1}(0)$ (thought of as a closed subvariety of $\mathrm{Rep}(\overline{Q},\delta) = T^\star\mathrm{Rep}(Q,\delta)$). There is a categorical quotient of $\mu^{-1}(0)^{ss}_\partial$ by $\mathrm{GL}(\delta)$, denoted by $\mathfrak{M}$. The natural morphism $f : \mathfrak{M} \to \mu^{-1}(0)/\!\!/\mathrm{GL}(\delta)$ is proper and birational. Using results of Crawley-Boevey [7] one shows that $\mathfrak{M}$ has rational singularities, and that there is a natural affine map $p : \mathfrak{M} \to \mathfrak{N}$. This means we can recover $\mathfrak{M} = \mathbf{Spec}\, p_*\mathcal{O}_{\mathfrak{M}}$. We construct a sheaf of algebras $\mathcal{A}^\chi$ on $\mathfrak{N}$, for each character χ of $\mathfrak{pgl}(\delta)$, by

$$\mathcal{A}^\chi = \frac{(\pi_*\mathcal{D}_{\mathrm{Rep}(Q,\delta)^{ss}_\partial})^{\mathrm{GL}(\delta)}}{((\pi_*\mathcal{D}_{\mathrm{Rep}(Q,\delta)^{ss}_\partial})(\iota-\chi)(\mathfrak{gl}(\delta)))^{\mathrm{GL}(\delta)}}.$$

The sheaf $\mathcal{A}^\chi$ is a quantization of $\mathfrak{M}$, in the sense that $\mathrm{gr}\mathcal{A}^\chi \cong p_*\mathcal{O}_{\mathfrak{M}}$. We complete the analogy with minimal primitive factors of enveloping algebras of semisimple Lie algebras with the theorem.

THEOREM 2. – *Let χ be a character of $\mathfrak{pgl}(\delta)$. Then*

$$\Gamma(\mathfrak{N},\mathcal{A}^\chi) \cong \mathfrak{A}^\chi \quad \textit{and} \quad \mathrm{H}^i(\mathfrak{N},\mathcal{A}^\chi) = 0, \textit{ for } i > 0.$$

If further, $\chi\cdot\alpha + \langle\partial,\alpha\rangle + \alpha_0 \neq 0$, for all real roots α, then $\Gamma(\mathfrak{N},\_\_)$ induces an equivalence of categories between the category of left $\mathcal{A}^\chi$-modules which are quasi-coherent as $\mathcal{O}_{\mathfrak{N}}$-modules and the category of left $\mathfrak{A}^\chi$-modules.

This result is proved in Theorems 5.9 and 5.12 below. Theorem 1 is obtained from a more general result. Let Q be an arbitrary quiver with vertex set I. For λ lying in a hyperplane of k^I defined below we obtain a k-algebra homomorphism from the deformed preprojective algebra Π^λ, of [8], to a suitable matrix-valued version of $\mathfrak{A}^\chi$.

When the parameter $\lambda = 0$, the algebra Π^0 is the preprojective algebra as in [9] and [2]. Let $\alpha \in \mathbb{N}^I$ and let $\mathrm{Rep}(Q,\alpha)$ be the space of representations of Q with dimension vector α. The group $\mathrm{GL}(\alpha) = \prod_{i\in I}\mathrm{GL}_{\alpha_i}(k)$ acts naturally on $\mathrm{Rep}(Q,\alpha)$ with kernel k^* and we write $\mathfrak{gl}(\alpha)$ for the Lie algebra of $\mathrm{GL}(\alpha)$. The preprojective algebra arises naturally because $\mathrm{Rep}(\Pi^0,\alpha) = \mu^{-1}(0)$, where $\mu : T^\star\mathrm{Rep}(Q,\alpha) \to \mathfrak{gl}(\alpha)^*$ is the moment map. The nilpotent cone inside $\mu^{-1}(0)$ has an important rôle for the quantum group associated to Q [20]. The Marsden-Weinstein reduction $\mu^{-1}(0)/\!\!/\mathrm{GL}(\alpha)$ has been studied in [16], [27], [6]. Set $k^\alpha = \oplus_{i\in I}k^{\alpha_i}$, which is naturally a $\mathrm{GL}(\alpha)$-module. Thus, $E_\alpha = k^\alpha \otimes k[\mathrm{Rep}(Q,\alpha)]$ is an equivariant $k[\mathrm{Rep}(Q,\alpha)]$-module. It follows that we can form $\mathcal{D}(E_\alpha)$, the algebra of k-linear differential operators on E_α. Let $\tau : \mathfrak{gl}(\alpha) \to \mathcal{D}(E_\alpha)$ denote the natural map. If χ is a character of $\mathfrak{gl}(\alpha)$ we define

$$\mathfrak{T}^\chi = \frac{\mathcal{D}(E_\alpha)^{\mathrm{GL}(\alpha)}}{(\mathcal{D}(E_\alpha)(\tau-\chi)(\mathfrak{gl}(\alpha)))^{\mathrm{GL}(\alpha)}}.$$

Before we can explain our homomorphism we need to recall some of the root system combinatorics associated to Q. The Ringel form on $\mathbb{Z}^I$ is defined by

$$\langle\alpha,\beta\rangle = \sum_{i\in I}\alpha_i\beta_i - \sum_{a\in Q}\alpha_{t(a)}\beta_{h(a)}.$$

The Tits form is the quadratic form on $\mathbb{Z}^I$ with $q(\alpha) = \langle\alpha,\alpha\rangle$. The defect $\partial_\alpha \in \mathbb{Z}^I$ is defined by $\partial_\alpha\cdot\beta = -\langle\beta,\alpha\rangle$. We identify k^I and the characters of $\mathfrak{gl}(\alpha)$ under the trace pairing.

THEOREM 3. – *Let $\lambda \in k^I$ satisfy $\lambda \cdot \alpha = 1 - q(\alpha)$. There is a k-algebra homomorphism $\Pi^\lambda \to \mathfrak{T}^{\lambda - \partial_\alpha}$.*

This is proved in Theorem 3.14 below. In the special case when Q is extended Dynkin. The homomorphism of the theorem is an isomorphism and, after suitable identifications have been made, induces the isomorphism of Theorem 1. I am grateful to Alex Glencross for showing me how to simplify the original proof of Theorem 3. I thank Bill Crawley-Boevey for helpful conversations. I am also grateful to the referee for a number of suggestions which improved the exposition.

The diagrams in this paper have been typeset using Paul Taylor's commutative diagram package.

2. Invariant differential operators

2.1. We first recall some generalities about differential operators. Fix k an algebraically closed field of characteristic zero. Let X be a smooth affine variety with coordinate ring $T = k[X]$. Let $\mathrm{Der}_k T$ denote the T-module of k-linear derivations of T. The algebra of k-linear *differential operators* on T (or X), written $\mathcal{D}(T)$ (or $\mathcal{D}(X)$), is the subalgebra of $\mathrm{End}_k T$ generated by $\mathrm{End}_T T$ and $\mathrm{Der}_k T$. We usually identify $T = \mathrm{End}_T T$. We shall also need the concept of differential operators on a free module. If V is a finite-dimensional vector space, write $E_V = V \otimes_k T$. Then $\mathrm{End}_k V \otimes_k \mathcal{D}(T)$ embeds naturally in $\mathrm{End}_k(E_V)$. This subalgebra is denoted by $\mathcal{D}(E_V)$. Of course, $\mathcal{D}(E_k) = \mathcal{D}(T)$. There is a filtration on $\mathcal{D}(E_V)$, called the *order* filtration, defined by:

$$\mathcal{D}^0(E_V) = \mathrm{End}_k V \otimes_k T, \qquad \mathcal{D}^1(E_V) = \mathrm{End}_k V \otimes (T + \mathrm{Der}_k T),$$

and $\mathcal{D}^n(E_V) = \mathcal{D}^{n-1}(E_V)\mathcal{D}^1(E_V)$, for $n > 1$. Now,

$$\mathcal{D}^n(E_V)/\mathcal{D}^{n-1}(E_V) \cong \mathrm{End}_k V \otimes_k S^n(\mathrm{Der}_k T),$$

where $S^n(\mathrm{Der}_k T)$ is the n-th symmetric power of $\mathrm{Der}_k T$. In particular,

$$\mathrm{gr}\mathcal{D}(E_V) \cong \mathrm{End}_k V \otimes_k S(\mathrm{Der}_k T) = \mathrm{End}_k V \otimes_k k[T^\star X].$$

Write $\pi : T^\star X \to X$, for the canonical map and set $k[T^\star X]_n = S^n(\mathrm{Der}_k T)$. If $\theta \in \mathcal{D}^n(E_V) \setminus \mathcal{D}^{n-1}(E_V)$ its *principal symbol* is $\theta + \mathcal{D}^{n-1}(E_V)$ in $\mathrm{End}_k V \otimes_k k[T^\star X]_n$. If X is a vector space, there is an alternative filtration $\mathcal{B}^n$ on $\mathcal{D}(E_V)$, called the *Bernstein* filtration. Identify X^* with the space spanned by the coordinate functions on X, via $k[X] = S(X^*)$, and X with the space of derivations defined by $x(\theta) = \theta(x)$, for $x \in X, \theta \in X^*$. Then $\mathcal{B}^0 = \mathrm{End}_k V$, $\mathcal{B}^1 = \mathrm{End}_k V \otimes_k (X^* \oplus X)$ and $\mathcal{B}^n = \mathcal{B}^1\mathcal{B}^{n-1}$, for $n > 1$. Again $\mathrm{gr}^{\mathcal{B}}\mathcal{D}(X) \cong k[T^\star X]$. We always use the order filtration, unless otherwise specified, although for the most part this doesn't make a lot of difference, except in the final section of the paper.

2.2. Next we suppose that a connected reductive algebraic group G acts on X. Then G acts on T and so one can differentiate this action to get a Lie algebra map $\iota : \mathfrak{g} \to \mathrm{End}_k T$. In fact, the elements of $\mathfrak{g}$ are acting as derivations of T, so one can extend ι to an algebra homomorphism $\iota : U(\mathfrak{g}) \to \mathcal{D}(T)$. Filtering $\mathcal{D}(T)$ and $U(\mathfrak{g})$ as usual, this homomorphism is filtered and the associated graded homomorphism $S(\mathfrak{g}) \to k[T^\star X]$ arises from the moment map

$$\mu : T^\star X \to \mathfrak{g}^* \qquad \theta \mapsto (x \mapsto \theta(\iota(x)_{\pi(\theta)})).$$

The action of G on $\mathrm{End}_k T$, ${}^g\theta = g\theta g^{-1}$, for $g \in G, \theta \in \mathrm{End}_k T$, restricts to an action on $\mathcal{D}(T)$. Differentiating, we get an action of $\mathfrak{g}$ on $\mathcal{D}(T)$ by

$$x \cdot \theta := [\iota(x), \theta],$$

for $x \in \mathfrak{g}, \theta \in \mathcal{D}(T)$. For any G-module the fixed points under G are annihilated by $\mathfrak{g}$. It follows that the derivations $\iota(\mathfrak{g})$ commute with elements of $\mathcal{D}(T)^G$. Thus, $(\mathcal{D}(T)\iota(\mathfrak{g}))^G$ is an ideal of $\mathcal{D}(T)^G$. More generally, if χ is a character of $\mathfrak{g}$ then $(\mathcal{D}(T)(\iota - \chi)(\mathfrak{g}))^G$ is an ideal of $\mathcal{D}(T)^G$ and we write

$$\mathfrak{A}^\chi(X, G) = \frac{\mathcal{D}(T)^G}{(\mathcal{D}(T)(\iota - \chi)(\mathfrak{g}))^G}.$$

Of course $\mathfrak{A}^\chi(X, G) = 0$ if χ doesn't vanish on $\mathrm{Ker}\iota$. Since $\mathrm{gr}\mathfrak{A}^\chi$ is evidently a homomorphic image of $k[\mu^{-1}(0)/\!\!/G]$, one sees that $\mathfrak{A}^\chi$ is Noetherian. Note that here, $\mu^{-1}(0)$ means the *scheme-theoretic* fibre. Knop [15] has shown that $\mathcal{D}(T)^G$ is free over its centre, a polynomial algebra. One expects that the properties of $\mathcal{D}(T)^G$ should be similar to those of the enveloping algebra of a semisimple Lie algebra and that the $\mathfrak{A}^\chi$ should be like minimal primitive factors. Musson and Van den Bergh have worked out the case when G is a torus [26]. Schwarz [30], [31] has found criteria for the natural map $\epsilon : \mathfrak{A}^0 \to \mathcal{D}(X/\!\!/G)$, to be an isomorphism (this holds for what Schwarz terms "very good" actions [30, 3.24]), generalising earlier work of Levasseur-Stafford, Levasseur, and Musson [19], [18], [25]. This can easily be generalised to equivariant modules. For example, suppose that we are given a representation V of G. Write $j : \mathfrak{g} \to \mathrm{End}_k V$, for the corresponding Lie algebra map. Now $E_V := V \otimes_k T$ is a G-equivariant T-module. One checks that G acts on $\mathcal{D}(E_V)$ by k-algebra automorphisms via its identification with $\mathrm{End}_k V \otimes_k \mathcal{D}(T)$. Thus there is an induced Lie algebra map $\tau : \mathfrak{g} \to \mathcal{D}(E_V)$. Note that

$$\tau(X) = j(X) \otimes 1 + 1 \otimes \iota(X), \quad \text{for } X \in \mathfrak{g}. \tag{1}$$

In particular, $\mathrm{Ker}\tau = \mathrm{Ker}j \cap \mathrm{Ker}\iota$. We define

$$\mathfrak{T}^\chi(X, G, V) = \frac{\mathcal{D}(E_V)^G}{(\mathcal{D}(E_V)(\tau - \chi)(\mathfrak{g}))^G},$$

whenever χ is a character of $\mathfrak{g}$. We usually omit X, G, V from the notation. Again, if χ doesn't vanish on $\mathrm{Ker}\tau$ we have $\mathfrak{T}^\chi = 0$. In the case of the trivial module we have $\mathfrak{T}^\chi(X, G, k) = \mathfrak{A}^\chi(X, G)$. Note that $\mathfrak{T}^\chi$ is Noetherian since its associated graded ring is a homomorphic image of $(\mathrm{End}_k V \otimes_k k[\mu^{-1}(0)])^G$ which is a finite module over $k[\mu^{-1}(0)/\!\!/G]$. Observe that if ρ is a character of G then $\mathcal{D}(E_V) \cong \mathcal{D}(E_{V \otimes_k k_\rho})$ as k-algebras and G-modules. Thus,

$$\mathfrak{T}^\chi(X, G, V) \cong \mathfrak{T}^{\chi+\rho}(X, G, V \otimes_k k_\rho). \tag{2}$$

Now suppose that X is a representation of the reductive group G.

2.3. Lemma. – *The moment map* $\mu : T^\star X \to \mathfrak{g}^*$ *is flat if and only if* $\dim \mu^{-1}(0) = 2\dim X - \dim G$.

Proof. – If μ is flat then the zero fibre has pure dimension $\dim T^\star X - \dim \mathfrak{g}^*$, by [11, Corollary III.9.6]. On the other hand, if $\dim \mu^{-1}(0) = 2\dim X - \dim G$ then μ is flat at 0 by the local criterion for flatness [1, Proposition V.3.5]. Since the set of points at which μ is flat is open [10, 11.1.1] and k^*-invariant, we are done. □

Let $\chi : \mathfrak{g} \to k$ be a character of $\mathfrak{g}$. Note that $\mathrm{gr}\mathcal{D}(E_V) \cong \mathrm{End}_k V \otimes_k k[T^\star X]$ contains a diagonal copy of $k[T^\star X]$ (i.e. $1 \otimes_k k[T^\star X]$) and, by (1), taking symbols maps $\mathfrak{g}$ into this diagonal copy. Thus, there is a natural surjective $k[T^\star X]$-module map

$$\mathrm{gr}\mathcal{D}(E_V)/(\mathrm{gr}\mathcal{D}(E_V))\mathfrak{g} \to \mathrm{gr}(\mathcal{D}(E_V)/\mathcal{D}(E_V)(\tau - \chi)(\mathfrak{g})).$$

We need a slight generalisation of a result of Schwarz [30, Proposition 8.11].

2.4. Proposition. – *Suppose that the moment map* $\mu : T^\star X \to \mathfrak{g}^*$ *is flat. Then*

$$\mathrm{gr}\left(\frac{\mathcal{D}(E_V)}{\mathcal{D}(E_V)(\tau - \chi)(\mathfrak{g})}\right) \cong \frac{\mathrm{gr}\mathcal{D}(E_V)}{(\mathrm{gr}\mathcal{D}(E_V))\mathfrak{g}}$$

as $k[T^\star X]$*-modules. In particular,*

$$\mathrm{gr}\mathfrak{T}^\chi \cong (\mathrm{End}_k V \otimes_k k[\mu^{-1}(0)])^G$$

as k-algebras.

Proof. – The hypothesis ensures that ι (hence τ) is injective so that $\mathfrak{g}$ can be identified with a subspace of $k[T^\star X]$ via the symbol map. Let $M = \mathcal{D}(E_V)/\mathcal{D}(E_V)(\tau - \chi)(\mathfrak{g})$ and let k_χ denote the one-dimensional $U(\mathfrak{g})$-module defined by the character χ. Then $M \cong \mathcal{D}(E_V) \otimes_{U(\mathfrak{g})} k_\chi$. Consider the chain complex $B_\bullet$ obtained from the classical Chevalley-Eilenberg resolution [33, Theorem 7.7.2] of the trivial module by tensoring with k_χ. Thus, $B_p = U(\mathfrak{g}) \otimes_k \wedge^p \mathfrak{g}$ and the differential is defined by

$$\begin{aligned} d(u \otimes x_1 \wedge \ldots \wedge x_n) = &\sum_{i=1}^{p} (-1)^{i+1} u(x_i - \chi(x_i)) \otimes x_1 \wedge \ldots \wedge \widehat{x_i} \wedge \ldots \wedge x_p \\ &+ \sum_{1 \le i < j \le p} (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \wedge \ldots \wedge \widehat{x_i} \wedge \ldots \wedge \widehat{x_j} \wedge \ldots \wedge x_p. \end{aligned}$$

$B_\bullet$ is exact in positive degrees and $h_0(B_\bullet) = k_\chi$. Let $C_\bullet = \mathcal{D}(E_V) \otimes_{U(\mathfrak{g})} B_\bullet = \mathcal{D}(E_V) \otimes_k \wedge^\bullet \mathfrak{g}$. Then $h_0(C_\bullet) = M$. We filter this complex by $F_m C_t = \mathcal{D}^{m-t}(E_V) \otimes_k \wedge^t \mathfrak{g}$. As usual this filtration induces one on $h(C_\bullet)$ and in particular on $M = h_0(C_\bullet)$. It is easy to see that this filtration on M is the natural one coming from the order filtration on $\mathcal{D}(E_V)$. There is a spectral sequence with $E^0_{pq} = F_p C_{p+q}/F_{p-1} C_{p+q}$. Since the filtration on $C_\bullet$ is exhaustive and bounded below it converges to $h(C_\bullet)$. Further observe that

$$E^0_{pq} = \mathcal{D}^{-q}(E_V) \otimes_k \wedge^{p+q}\mathfrak{g}/\mathcal{D}^{-q-1}(E_V) \otimes_k \wedge^{p+q}\mathfrak{g} \cong \mathrm{End}_k V \otimes_k k[T^\star X]_{-q} \otimes_k \wedge^{p+q}\mathfrak{g}.$$

In particular, E^0_{pq} is zero outside the region bounded by $p \ge 0$, $q \le 0$, $p + q \ge 0$. Let $x_1, \ldots, x_r$ be a basis of $\mathfrak{g}$. We can identify these via their symbols with elements of

$k[T^\star X]$ which are homogeneous of degree one. Since μ is flat, these elements form a regular sequence. Now the differential on E^0 is given by

$$d(u \otimes x_1 \wedge \ldots \wedge x_{p+q}) = \sum_{i=1}^{p+q} ux_i \otimes x_1 \wedge \ldots \wedge \widehat{x_i} \wedge \ldots \wedge x_{p+q}.$$

In the p-th column the only non-trivial homology it generates is in the $(p,-p)$ position, viz $\mathrm{End}_k V \otimes_k (k[T^\star X]_p / k[T^\star X]_{p-1}\mathfrak{g})$. To see this, note that the usual Koszul complex $K_\bullet(x_1,\ldots,x_r)$ is exact except in degree zero, and splits into homogeneous components because $x_1,\ldots,x_r$ are a regular sequence of homogeneous elements of degree one. Finally, $E^0_{p\bullet}$ is $\mathrm{End}_k V$ tensored with one of these components. Thus, the spectral sequence collapses at E^1 and we see that

$$(\mathrm{gr}M)_p \cong \mathrm{End}_k V \otimes_k \frac{k[T^\star X]_p}{k[T^\star X]_{p-1}\mathfrak{g}} \cong \left(\frac{\mathrm{gr}\mathcal{D}(E_V)}{(\mathrm{gr}\mathcal{D}(E_V))\mathfrak{g}}\right)_p,$$

as needed. □

2.5. Remark. – Note that $\mu^{-1}(0)$ means the *scheme*-theoretic fibre. Also

$$(\mathrm{End}_k V \otimes_k k[\mu^{-1}(0)])^G \cong \{G\text{-equivariant morphisms } f : \mu^{-1}(0) \to \mathrm{End}_k V\}$$

as k-algebras.

2.6. Remark. – Note that with the hypothesis of the lemma $\mathcal{D}(T)$ is flat (on either side) as a $U(\mathfrak{g})$-module. See [4, Proposition 2.3.12].

3. Representation of the deformed preprojective algebra by differential operators

In this section we give a representation of the deformed preprojective algebra by differential operators. We begin with the notation that we will use concerning quivers. Let Q be any quiver (oriented graph) on the vertex set $I = \{0,\ldots,n\}$. We write $a \in Q$ for a is an edge of Q. Then we write $h(a) \in I$ for the head of a and $t(a)$ for its tail. Let $\alpha,\beta \in \mathbb{Z}^I$. The *Ringel form* is the bilinear form on $\mathbb{Z}^I$ with

$$\langle \alpha,\beta\rangle = \sum_{i\in I} \alpha_i\beta_i - \sum_{a\in Q} \alpha_{t(a)}\beta_{h(a)}.$$

The *symmetric bilinear form* is defined by $(\alpha,\beta) = \langle\alpha,\beta\rangle + \langle\beta,\alpha\rangle$. The *Tits form* is the quadratic form given by $q(\alpha) = \langle\alpha,\alpha\rangle = \frac{1}{2}(\alpha,\alpha)$. The *Dynkin quivers* of types A_n ($n \geq 1$), D_n ($n \geq 4$), E_6, E_7, E_8 are the connected quivers for which the quadratic form is positive definite. The *extended Dynkin quivers* of types $\widetilde{A_n}$ ($n \geq 0$), $\widetilde{D_n}$ ($n \geq 4$), $\widetilde{E_6}$, $\widetilde{E_7}$, $\widetilde{E_8}$ are the connected quivers for which the quadratic form is positive semi-definite (where $\widetilde{A_0}$ is the quiver with one vertex and one loop, and a quiver of type $\widetilde{A_1}$ has two vertices joined by two arrows, in either direction). In this case we denote by δ the minimal vector in $\mathbb{N}^I \setminus \{0\}$ in the radical of $(,)$. Any vertex i with $\delta_i = 1$ is called an *extending vertex*, and deleting i one obtains the corresponding Dynkin quiver. If Q is extended Dynkin then we always number the vertices $I = \{0,1,\ldots,n\}$ with 0 an extending vertex. Let $\epsilon_i : i \in I$

be the usual basis of $\mathbb{Z}^I$. We say a vertex i is *loopfree* if there are no arrows $a : i \to i$. If i is a loopfree vertex the *simple reflection* $s_i \in \mathrm{GL}_n(\mathbb{Z})$ is defined by $s_i(\alpha) = \alpha - (\alpha, \epsilon_i)$, for $\alpha \in \mathbb{Z}^I$. The *Weyl group* is the subgroup of $\mathrm{GL}_n(\mathbb{Z})$ generated by $s_i : i \in I$ and is loopfree. The *simple roots* are ϵ_i, for i loopfree. The *real roots* are the images of the simple roots under the Weyl group. For more details on root systems in this context, see [13]. Let $\overline{Q}$ denote the *double* of Q. This is the quiver on the same vertex set as Q but with an additional arrow $a^* : h(a) \to t(a)$ for each edge $a \in Q$. The *opposite* quiver to Q is denoted Q^*. It has the same vertex set as Q but has arrows $a^* : a \in Q$. The path algebra of $\overline{Q}$ is denoted $k\overline{Q}$. We write e_i for the trivial path in $k\overline{Q}$ corresponding to the vertex i. If $\lambda \in k^I$ we can think of it as the element $\sum_{i\in I} \lambda_i e_i$ of $k\overline{Q}$. Recall from [8] the definition of the *deformed preprojective algebra* with parameter λ. Let $c = \sum_{a\in Q}[a, a^*]$ and write $(c - \lambda)$ for the two-sided ideal of $k\overline{Q}$ it generates. Then define

$$\Pi^\lambda = \frac{k\overline{Q}}{(c-\lambda)}.$$

Again following [8] we denote $\mathcal{O}^\lambda = e_0\Pi^\lambda e_0$. The isomorphism class of Π^λ (and of $\mathcal{O}^\lambda$) is unchanged if we replace λ by a non-zero scalar multiple. For technical reasons we also need to consider a global version of the deformed preprojective algebra. If S is a commutative ring and $\lambda \in S^I$ we define $\Pi^{S,\lambda} = S\overline{Q}/(c - \lambda)$. Likewise $\mathcal{O}^{S,\lambda} = e_0\Pi^{S,\lambda}e_0$. Note that $k\overline{Q}$ is a graded algebra if for $a \in Q$ we put a in degree zero and a^* in degree one. This induces a filtered structure on Π^λ and hence on $\mathcal{O}^\lambda$. (Equally well, except in the last section, we could use the grading and induced filtrations starting with a and a^* in degree one. In that case instead of using the operator filtration on the rings of differential operators we consider, one works with the Bernstein filtration.) Let $\alpha \in \mathbb{N}^I$ and write $\mathrm{Rep}(Q, \alpha)$ for the space of representations of Q with dimension vector α. This is the vector space

$$\mathrm{Rep}(Q,\alpha) = \prod_{a\in Q} \mathrm{Hom}_k(k^{\alpha_{t(a)}}, k^{\alpha_{h(a)}}).$$

Note that

$$T^\star\mathrm{Rep}(Q,\alpha) = \mathrm{Rep}(\overline{Q},\alpha) = \mathrm{Rep}(Q,\alpha) \oplus \mathrm{Rep}(Q^*,\alpha).$$

Thus,

$$k[T^\star\mathrm{Rep}(Q,\alpha)] = k[t^a_{p,q} : a \in \overline{Q}, 1 \le p \le h(a), 1 \le q \le t(a)].$$

Let

$$T = k[\mathrm{Rep}(Q,\alpha)] = k[t^a_{p,q} : a \in Q, 1 \le p \le h(a), 1 \le q \le t(a)].$$

Observe that the principal symbol of the differential operator $\partial/\partial t^a_{p,q}$ is $t^{a^*}_{q,p}$. Let $\mathrm{GL}(\alpha) = \prod_{i\in I} \mathrm{GL}_{\alpha_i}(k)$. It has Lie algebra $\mathfrak{gl}(\alpha) = \prod_{i\in I} M_{\alpha_i}(k)$. Let e^i_{pq} denote the pq-th matrix unit in the i-th summand, for $i \in I$ and $1 \le p, q \le \alpha_i$. $\mathrm{GL}(\alpha)$ acts naturally on $\mathrm{Rep}(Q,\alpha)$ by conjugation with kernel k^*. Denote by $\mathfrak{pgl}(\alpha)$ the quotient of $\mathfrak{gl}(\alpha)$ which is the Lie algebra of $\mathrm{GL}(\alpha)/k^*$. Let us differentiate the action of $\mathrm{GL}(\alpha)$ on $\mathrm{Rep}(Q,\alpha)$.

3.1. LEMMA. – *Differentiating the action of* $\mathrm{GL}(\alpha)$ *on* $\mathrm{Rep}(Q,\alpha)$ *gives rise to the Lie algebra map* $\iota : \mathfrak{gl}(\alpha) \to \mathcal{D}(T)$ *defined by*

$$e^i_{pq} \mapsto \sum_{a\in Q:t(a)=i} \sum_{j=1}^{\alpha_{h(a)}} t^a_{jp} \frac{\partial}{\partial t^a_{jq}} - \sum_{a\in Q:h(a)=i} \sum_{j=1}^{\alpha_{t(a)}} t^a_{qj} \frac{\partial}{\partial t^a_{pj}}.$$

Proof. – Note that if $\mathrm{GL}_n(k) \times k^n \to k^n$ is given by $(g, v) \mapsto gv$ then differentiating its action on the coordinate ring $k[x_1, \ldots, x_n]$ one gets:

$$e_{ij} \mapsto -x_j \partial/\partial x_i.$$

Likewise if $\mathrm{GL}_n(k) \times k^n \to k^n$ is given by $(g, v) \mapsto vg^{-1}$ then one has:

$$e_{ij} \mapsto x_i \partial/\partial x_j.$$

To see where the formula in the statement comes from, think of, for $a \in Q$ with $h(a) = i$, the action of $\mathrm{GL}_{\alpha_i}(k)$ on the component

$$\mathrm{Hom}(k^{\alpha_{t(a)}}, k^{\alpha_i}) = M_{\alpha_i \times \alpha_{t(a)}}(k)$$

of $\mathrm{Rep}(Q, \alpha)$. This is just $\alpha_{t(a)}$ copies of the standard left action of $\mathrm{GL}_{\alpha_i}(k)$ on k^{α_i}. The second term in the display follows from this. The first term is obtained similarly. □

3.2. NOTATION. – If $\lambda \in k^I$ then we identify it with the character of $\mathfrak{gl}(\alpha)$ defined by $(A_0, \ldots, A_n) \mapsto \sum \lambda_i \mathrm{tr} A_i$. If $\lambda \cdot \alpha = 0$ we regard λ as a character of $\mathfrak{pgl}(\alpha)$.

3.3. DEFINITION. – Let $\chi \in k^I$ satisfy $\chi \cdot \alpha = 0$. Then we define

$$\mathfrak{A}^\chi := \mathfrak{A}^\chi(\mathrm{Rep}(Q, \alpha), \mathrm{GL}(\alpha)).$$

3.4. REMARK. – The reason we choose $\chi \cdot \alpha = 0$ is to avoid $\mathfrak{A}^\chi$ collapsing to zero. Note that $\mathrm{Ker}\iota = k(I_{\alpha_i})_{i \in I}$, where I_{α_i} denotes the $\alpha_i \times \alpha_i$ identity matrix. (This is because ι factors through the natural map $\mathfrak{g} \to \mathrm{End}_k \mathrm{Rep}(Q, \alpha)^*$ and $\mathrm{GL}(\alpha)$ acts on $\mathrm{Rep}(Q, \alpha)$ with kernel k^*.) So our assumption that $\chi \cdot \alpha = 0$ is to ensure that χ vanishes on $\mathrm{Ker}\iota$.

Recall that if Q has no oriented cycles then $\mathrm{Rep}(Q, \alpha) /\!\!/ \mathrm{GL}(\alpha)$ is a point [17, Theorem 1]. Thus, the natural map $\epsilon : \mathfrak{A}^0 \to \mathcal{D}(\mathrm{Rep}(Q, \alpha) /\!\!/ \mathrm{GL}(\alpha)) = k$ of (2.2) is trivially surjective, but this is not very interesting. For the most part, the examples considered in this paper are quite different from the ones which Schwarz is interested in [30], [31].

It is perhaps worth making explicit the action of $\mathrm{GL}(\alpha)$ on $\mathcal{D}(T)$. If $g = (g_0, \ldots, g_n)$ is in $\mathrm{GL}(\alpha)$ and $a \in Q$ then

$$(3) \qquad g \cdot t^a = g_{h(a)}^{-1} t^a g_{t(a)} \qquad \text{and} \qquad g \cdot \partial^a = g_{t(a)}^{-1} \partial^a g_{h(a)}.$$

Here t^a is the $\alpha_{h(a)} \times \alpha_{t(a)}$ matrix with p, q entry t^a_{pq} and ∂^a is the $\alpha_{t(a)} \times \alpha_{h(a)}$ matrix with p, q entry $\partial/\partial t^a_{qp}$.

3.5. NOTATION. – We denote $k^\alpha = \oplus_i k^{\alpha_i}$.

3.6. DEFINITION. – Let $\chi \in k^I$ satisfy $\chi \cdot \alpha = 1$. Then we define

$$\mathfrak{T}^\chi := \mathfrak{T}^\chi(\mathrm{Rep}(Q, \alpha), \mathrm{GL}(\alpha), k^\alpha).$$

3.7. REMARK. – The reason we assume $\chi \cdot \alpha = 1$ is as follows. Choose $\theta \in \mathbb{Z}^I$ (so that θ arises from a character of $\mathrm{GL}(\alpha)$) with $\theta \cdot \alpha = -1$, and put $V = k^\alpha \otimes_k k_\theta$. Now $\mathrm{Ker} j_V = \mathrm{Ker}\iota$ (and hence both are equal to $\mathrm{Ker}\tau_V$). For, if $x = (x_i)_{i \in I}$ is in the kernel of j_V then each x_i must be a scalar matrix and the value of this scalar is the same for all i, namely $-\sum_{j\in I}\theta_j \mathrm{tr}(x_j)$. On the other hand, if each $x_i = sI_{\alpha_i}$, for a scalar s, then the fact that $\theta\cdot\alpha = -1$ tells us that $x \in \mathrm{Ker} j_V$. Finally, using (2), $\mathfrak{T}^\chi \cong \mathfrak{T}^{\chi+\theta}(\mathrm{Rep}(Q,\alpha), \mathrm{GL}(\alpha), k^\alpha \otimes_k k_\theta)$. To avoid $\mathfrak{T}^\chi$ collapsing to zero, $\chi+\theta$ must vanish on $\mathrm{Ker}\tau_V$. In other words, $\chi\cdot\alpha = 1$.

3.8. We recall a result of Crawley-Boevey on the preprojective algebra Π^0 and show how it can be used to compute the GK dimensions of $\mathfrak{A}^\chi$ and $\mathfrak{T}^\chi$ for some cases. Partially order $\mathbb{Z}^I$, by $\alpha \le \beta$ if $\alpha_i \le \beta_i$, for all $i \in I$. Following Crawley-Boevey [6] let Σ_0 consist of those $\alpha \in \mathbb{N}^I$ such that $\alpha \ne 0$ and $(\beta, \alpha-\beta) \le -2$ whenever $0 \lneqq \beta \lneqq \alpha$ with $\beta, \alpha-\beta \in \mathbb{N}^I$. For $\alpha \in \Sigma_0$ [6] proves that $\mu : T^\star\mathrm{Rep}(Q,\alpha) \to \mathfrak{pgl}(\alpha)^*$ is flat with integral fibres and that $\dim \mu^{-1}(0)/\!\!/\mathrm{GL}(\alpha) = 2 - 2q(\alpha)$.

3.9. EXAMPLE. – Let $\alpha \in \Sigma_0$ and let $\chi \in k^I$.

(a) If $\chi\cdot\alpha = 0$ then $\mathfrak{A}^\chi$ is a domain and has GK dimension $2-2q(\alpha)$.

(b) If $\chi\cdot\alpha = 1$ then $\mathfrak{T}^\chi$ has GK dimension $2-2q(\alpha)$.

Proof. – Crawley-Boevey's result says the moment map is flat with integral fibres. By Proposition 2.4 this tells us that, for $\chi\cdot\alpha = 0$, we have

$$\mathrm{gr}\mathfrak{A}^\chi \cong k[\mu^{-1}(0)/\!\!/\mathrm{GL}(\alpha)]. \tag{4}$$

This shows that $\mathfrak{A}^\chi$ is a domain. Since the filtration of $\mathfrak{A}^\chi$ is not finite-dimensional there is a potential problem in deducing that $\mathfrak{A}^\chi$ has the same GK dimension as its associated graded ring, namely $2-2q(\alpha)$. However, [22, Corollary 1.4] applies to remove this difficulty, proving (a). Now let $\chi \in k^I$ satisfy $\chi\cdot\alpha = 1$. Again, we apply Crawley-Boevey's result and Proposition 2.4 to get

$$\mathrm{gr}\mathfrak{T}^\chi \cong (\mathrm{End}_k k^\alpha \otimes_k k[\mu^{-1}(0)])^{\mathrm{GL}(\alpha)}. \tag{5}$$

(Since the moment map for $\mathfrak{pgl}(\alpha)$, not $\mathfrak{gl}(\alpha)$, is flat, this application of the Proposition occurs after twisting by θ as in Remark 3.7.) Another application of [22] completes the proof of (b). Alternatively, one can avoid the use of [22] by proving isomorphisms similar to (4) and (5) using the Bernstein filtration, rather than the operator filtration. The result then follows by [21, Proposition 8.6.5]. □

3.10. REMARK. – With the hypotheses of Example 3.9(b), W. Crawley-Boevey and the author have proved that $\mathfrak{T}^\chi$ is prime.

Obviously $\mathfrak{T}^\chi$ and $\mathfrak{A}^\chi$ depend on the orientation of Q, but not seriously.

3.11. LEMMA. – *Let $a \in Q$ and let Q' be the quiver obtained from Q by reversing the arrow a. Then there are k-algebra isomorphisms*

$$\mathfrak{A}^\chi \to \mathfrak{A}^{\chi+\alpha_{t(a)}\epsilon_{h(a)}-\alpha_{h(a)}\epsilon_{t(a)}} \quad \textit{and} \quad \mathfrak{T}^\chi \to \mathfrak{T}^{\chi+\alpha_{t(a)}\epsilon_{h(a)}-\alpha_{h(a)}\epsilon_{t(a)}}.$$

These isomorphisms are filtered, for the Bernstein filtration.

Proof. – There is an isomorphism

$$\mathcal{D}(\mathrm{Rep}(Q,\alpha)) \to \mathcal{D}(\mathrm{Rep}(Q',\alpha))$$

that is the identity on t^b_{ij} and $\partial/\partial t^b_{ji}$, for all $1 \leq i \leq \alpha_{h(b)}$, $1 \leq j \leq \alpha_{t(b)}$, $b \neq a$, and maps t^a_{pq} to $-\partial/\partial t^{a^*}_{qp}$ and $\partial/\partial t^a_{qp}$ to $t^{a^*}_{pq}$, for $1 \leq p \leq \alpha_{h(a)}$, $1 \leq q \leq \alpha_{t(a)}$. It is easy to check that this isomorphism induces the claimed ones. □

3.12. NOTATION. – We write $E_\alpha = k^\alpha \otimes_k T$.

3.13. PROPOSITION. – *There is a k-algebra homomorphism $\Phi : k\overline{Q} \to \mathcal{D}(E_\alpha)^{\mathrm{GL}(\alpha)}$. If $\alpha_0 = 1$ this restricts to a k-algebra homomorphism $e_0 k\overline{Q} e_0 \to \mathcal{D}(T)^{\mathrm{GL}(\alpha)}$.*

Proof. – Define a representation of $\overline{Q}$ by placing T^{α_i} at vertex i. Now, if a is an arrow of Q then we associate to it the map $T^{\alpha_{t(a)}} \to T^{\alpha_{h(a)}}$ given by the matrix t^a. On the other hand, if a^* is the opposite arrow then we associate to it the map $T^{\alpha_{h(a)}} \to T^{\alpha_{t(a)}}$ given by the matrix ∂^a. This certainly gives a k-algebra homomorphism $\theta : k\overline{Q} \to \mathrm{End}_k E_\alpha$. It is clear that that the image is contained in the subalgebra $\mathcal{D}(E_\alpha)$. The formula (3) and the definition of the action of $\mathrm{GL}(\alpha)$ on $\mathcal{D}(E_\alpha)$ show that θ has image in $\mathcal{D}(E_\alpha)^{\mathrm{GL}(\alpha)}$ and so induces the homomorphism Φ of the statement. Finally, $\Phi(e_0)\mathcal{D}(E_\alpha)\Phi(e_0) = \mathcal{D}(T)$, if $\alpha_0 = 1$. For, identifying $\mathcal{D}(E_\alpha) = M_{\sum \alpha_i}(\mathcal{D}(T))$, we see that $\Phi(e_0)$ is the obvious matrix unit. □

Recall from §3 the definition of the preprojective algebra Π^0. In [8, Lemma 8.1 and the subsequent paragraph] it was shown that there is a naturally defined graded k-algebra homomorphism

$$\psi : \Pi^0 \to M_{\sum \alpha_i}(k[\mu^{-1}(0)])^{\mathrm{GL}(\alpha)}.$$

Let us briefly recall the construction. First one defines a map

$$\Psi : k\overline{Q} \to M_{\sum \alpha_i}(k[T^\star \mathrm{Rep}(Q, \alpha)]).$$

This map sends an arrow a of $\overline{Q}$ to t^a. Now $\mathrm{GL}(\alpha)$ acts naturally on

$$M_{\sum \alpha_i}(k[T^\star \mathrm{Rep}(Q, \alpha)]) \cong \mathrm{End}_k k^\alpha \otimes_k k[T^\star \mathrm{Rep}(Q, \alpha)].$$

The image of Ψ clearly lies in the invariant matrices. Of course, the map Φ of the lemma quantizes Ψ. For, Φ is a filtered homomorphism and its associated graded map is Ψ. The map ψ is defined by observing that Ψ descends to the quotient $\Pi^0 \to M_{\sum \alpha_i}(k[\mu^{-1}(0)])$. Restricting to the invariants one obtains ψ. In this paper an attempt is made to quantize ψ. The codomain of Φ is $\mathcal{D}(E_\alpha)$, a simple ring, and so one cannot make "a descent to the quotient", as for Ψ, by just directly factoring out the relation $\sum[a, a^*] - \lambda$. Instead we have to work in $\mathcal{D}(E_\alpha)^{\mathrm{GL}(\alpha)}$. Let us explain precisely what we mean by a quantization of ψ. Let $\lambda \in k^I$ and let $\chi \in k^I$ satisfy $\chi \cdot \alpha = 1$ (χ will depend on λ). There are natural graded, surjective k-algebra homomorphisms $\Pi^0 \to \mathrm{gr}\Pi^\lambda$ and $M_{\sum \alpha_i}(k[\mu^{-1}(0)])^{\mathrm{GL}(\alpha)} \to \mathrm{gr}\mathfrak{T}^\chi$. We are looking for a filtered k-algebra homomorphism $\phi^\lambda : \Pi^\lambda \to \mathfrak{T}^\chi$ such that the diagram

$$\begin{array}{ccc} \mathrm{gr}\,\Pi^\lambda & \xrightarrow{\mathrm{gr}\,\phi^\lambda} & \mathrm{gr}\,\mathfrak{T}^\chi \\ \uparrow & & \uparrow \\ \Pi^0 & \xrightarrow{\psi} & M_{\sum \alpha_i}(k[\mu^{-1}(0)])^{\mathrm{GL}(\alpha)} \end{array} \tag{6}$$

commutes. We need one more piece of notation. Define $\partial_\alpha \in \mathbb{Z}^I$ by

$$\partial_\alpha \cdot \beta = -\langle \beta, \alpha \rangle.$$

Since $\partial_\alpha \cdot \alpha = -q(\alpha)$ we have a bijection

$$\{\lambda \in k^I : \lambda \cdot \alpha = 1 - q(\alpha)\} \xrightarrow[\cong]{\lambda \mapsto \lambda - \partial_\alpha} \{\chi \in (\mathfrak{gl}(\alpha)^*)^{\mathrm{GL}(\alpha)} : \chi \cdot \alpha = 1\}.$$

3.14. THEOREM. – *Let $\lambda \in k^I$ satisfy $\lambda \cdot \alpha = 1 - q(\alpha)$. Then there is a filtered k-algebra homomorphism*

$$\phi^\lambda : \Pi^\lambda \to \mathfrak{T}^{\lambda - \partial_\alpha}.$$

Further, with $\chi = \lambda - \partial_\alpha$, diagram (6) commutes. If $\alpha_0 = 1$ then ϕ^λ restricts to give a k-algebra homomorphism

$$\mathcal{O}^\lambda \to \mathfrak{A}^{\lambda - \epsilon_0 - \partial_\alpha}$$

In order to prove the main theorem we need a lemma. First some notation.

3.15. NOTATION. – The direct sum decomposition

$$\mathrm{End}_k E_\alpha = \bigoplus_{v,w \in I} \mathrm{Hom}_k(T^{\alpha_v}, T^{\alpha_w})$$

restricts to another one

$$\mathcal{D}(E_\alpha) = \bigoplus_{v,w \in I} \mathcal{D}(T^{\alpha_v}, T^{\alpha_w}).$$

The component $\mathcal{D}(T^{\alpha_v}, T^{\alpha_w})$ of $\mathcal{D}(E_\alpha)$ identifies with $M_{\alpha_w \times \alpha_v}(\mathcal{D}(T))$. In particular, if θ is an element of this component we speak of its ij entry θ_{ij}. Recall from Proposition 3.13 that there is a homomorphism $\Phi : k\overline{Q} \to \mathcal{D}(E_\alpha)$. We write e_i for $\Phi(e_i)$. Finally, if $v \in I$ and $1 \le i, j \le \alpha_v$ then the element e^v_{ij} of $\mathfrak{gl}(\alpha) = \prod_{w \in I} M_{\alpha_w}(k)$ can be identified with an element of $\mathcal{D}(T^{\alpha_v}, T^{\alpha_v})$. On the other hand, $\iota(e^v_{ij})$ is an element of $\mathcal{D}(T)$ and this can be identified with an element of $\mathcal{D}(E_\alpha)$ via the diagonal inclusion. With these conventions, and using (1), we have $\tau(e^v_{ij}) = \iota(e^v_{ij}) + e^v_{ij}$.

The proof of the theorem boils down to the following lemma. My original proof was somewhat over-complicated. I am grateful to Alex Glencross for pointing out to me this more concise argument.

3.16. LEMMA. – *For $i \in I$, we have*

$$\Phi\left(\sum_{h(a)=i} aa^* - \sum_{t(a)=i} a^*a + \lambda_i e_i\right) = \sum_{1 \le p,q \le \alpha_i} e^i_{pq}(\chi - \tau)(e^i_{qp}),$$

where $\chi = \lambda - \partial_\alpha$.

Proof. – Using the notation in (3) and the definition of Φ in Proposition 3.13 we have

$$\Phi\left(\sum_{h(a)=i} aa^* - \sum_{t(a)=i} a^*a + \lambda_i e_i\right) = \left(\sum_{h(a)=i} t^a \partial^a - \sum_{t(a)=i} \partial^a t^a\right) + \lambda_i e_i.$$

Note that this is an element of $\mathcal{D}(T^{\alpha_i}, T^{\alpha_i}) = M_{\alpha_i}(T)$. Consider, for $1 \le p, q \le \alpha_i$, the pq-th entry of this matrix:

$$\left(\sum_{h(a)=i} \sum_{j=1}^{\alpha_{t(a)}} t^a_{pj} \frac{\partial}{\partial t^a_{qj}} - \sum_{t(a)=i} \sum_{j=1}^{\alpha_{h(a)}} \frac{\partial}{\partial t^a_{jp}} t^a_{jq} \right) + \lambda_i \delta_{pq}$$
$$= \left(\sum_{h(a)=i} \sum_{j=1}^{\alpha_{t(a)}} t^a_{pj} \frac{\partial}{\partial t^a_{qj}} - \sum_{t(a)=i} \sum_{j=1}^{\alpha_{h(a)}} \left(t^a_{jq} \frac{\partial}{\partial t^a_{jp}} + \partial_{pq} \right) \right) + \lambda_i \delta_{pq}$$
$$= -\iota(e^i_{qp}) + (\lambda_i - \sum_{t(a)=i} \alpha_{h(a)}) \partial_{pq} = -\iota(e^i_{qp}) + (\chi_i - \alpha_i) \delta_{pq}.$$

The last equality uses Lemma 3.1 and the fact that $\chi = \lambda - \partial_\alpha$. It follows that

$$\Phi\left(\sum_{h(a)=i} aa^* - \sum_{t(a)=i} a^*a + \lambda_i e_i \right) = -\left(\sum_{1 \le p,q \le \alpha_i} e^i_{pq} \iota(e^i_{qp}) \right) + (\chi_i - \alpha_i) e_i. \tag{7}$$

On the other hand,

$$\sum_{1 \le p,q \le \alpha_i} e^i_{pq} (\chi - \tau)(e^i_{qp}) = - \sum_{1 \le p,q \le \alpha_i} e^i_{pq} (\iota(e^i_{qp}) + e^i_{qp} - \chi_i \delta_{pq})$$
$$= -\left(\sum_{1 \le p,q \le \alpha_i} e^i_{pq} \iota(e^i_{qp}) + e^i_{pp} \right) + \chi_i e_i$$
$$= -\left(\sum_{1 \le p,q \le \alpha_i} e^i_{pq} \iota(e^i_{qp}) \right) + (\chi_i - \alpha_i) e_i.$$

Now equation (7) completes the proof of the lemma. □

Proof of Theorem 3.14. – Recall that $\Pi^\lambda \cong \Pi^{-\lambda}$. In order to prove the first part of the theorem we must show that

$$\Phi\left(\sum [a, a^*] + \lambda\right) \in \left(\mathcal{D}(E_\alpha)(\tau - \chi)(\mathfrak{gl}(\alpha))\right)^{\mathrm{GL}(\alpha)}.$$

Since the image of Φ consists of invariants and

$$\sum_{a \in Q} [a, a^*] = \sum_{i \in I} \left(\sum_{h(a)=i} aa^* - \sum_{t(a)=i} a^*a \right)$$

the lemma proves this. To prove the second part, note that for for $v \in I$ and $1 \le i, j \le \alpha_v$ one has

$$\tau(e^v_{ij}) e_0 = (\iota(e^v_{ij}) + e^v_{ij}) e_0 = \begin{cases} \iota(e^v_{ij}) e_0 & v \neq 0 \\ \iota(e^v_{ij}) e_0 + e_0 & v = 0. \end{cases}$$

□

3.17. Example. – Let Q be the quiver

$$\begin{tikzcd} 0 \bullet \arrow[r, shift left=4, "a_0"] \arrow[r, shift left=2, "a_1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=3, "a_n"'] & \bullet 1 \end{tikzcd}$$

and let $\alpha = (1,1)$. Let $\lambda \in k^2$ be such that $\lambda_0 + \lambda_1 = n$. Then

$$\phi^\lambda : \mathcal{O}^\lambda \to \mathfrak{A}^{\lambda - \epsilon_0 - \partial_\alpha}$$

is surjective. Further, the image is isomorphic to the global sections of a sheaf of twisted differential operators on $\mathbb{P}^n$.

Proof. – Note that $q(\alpha) = 1 - n$. Thus the condition $\lambda_0 + \lambda_1 = n$ just says that $\lambda \cdot \alpha = 1 - q(\alpha)$. Now, $\mathcal{D}(T) = k[t_0, \dots, t_n, \partial_0, \dots, \partial_n]$ and $\mathrm{GL}(\alpha)/k^* \cong k^*$. The homomorphism

$$\psi^\lambda : \mathcal{O}^\lambda \to \frac{k[t_0, \dots, t_n, \partial_0, \dots, \partial_n]^{k^*}}{(k[t_0, \dots, t_n, \partial_0, \dots, \partial_n](\sum_{i=0}^n t_i \partial_i - \lambda_0 + n + 1))^{k^*}}$$

is given by $a_j^* a_i \mapsto \partial_j t_i$. A simple calculation shows that the associated graded homomorphism of this map is surjective. Finally, it is well known and easy to show that the image is isomorphic to the global sections of a sheaf of twisted differential operators on $\mathbb{P}^n$ (cf [5, Example 3.10(a)], [32, Theorem 6.1.2]). □

4. Extended Dynkin case

Consider now the extended Dynkin case. That is, let Q be a quiver of type $\widetilde{A_n}$, $n \geq 0$, $\widetilde{D_n}$, $n \geq 4$, or $\widetilde{E_n}$, $n = 6, 7, 8$ and suppose that 0 is an extending vertex. Let δ denote the minimal imaginary root. i.e. $\delta \in \mathbb{N}^I \setminus \{0\}$ is minimal with $q(\delta) = 0$. Note that $\delta_0 = 1$. For $\chi \in k^I$ let $\mathfrak{A}^\chi = \mathfrak{A}^\chi(\mathrm{Rep}(Q, \delta), \mathrm{GL}(\delta))$ and $\mathfrak{T}^\chi = \mathfrak{T}^\chi(\mathrm{Rep}(Q, \delta), \mathrm{GL}(\delta), k^\delta))$. We are going to show that the k-algebra homomorphisms of Theorem 3.14 are isomorphisms, when $\alpha = \delta$. After [8, Lemma 8.3, Corollary 3.6] and Proposition 2.4 the vertical maps in diagram (6) are isomorphisms and so it will be enough to show that the graded map ψ is an isomorphism. To do this it is necessary to introduce a flat family of algebras with special fibre $M_{\sum \delta_i}(k[\mu^{-1}(0)])^{\mathrm{GL}(\delta)}$. Let $\lambda \in k^I$ satisfy $\lambda \cdot \delta = 0$. As usual we regard λ as a character of $\mathfrak{pgl}(\delta)$. Define

$$T^\lambda = M_{\sum \delta_i}(k[\mu^{-1}(\lambda)])^{\mathrm{GL}(\delta)} \qquad \text{and} \qquad A^\lambda = k[\mu^{-1}(\lambda) /\!\!/ \mathrm{GL}(\delta)].$$

If we grade $k[T^\star \mathrm{Rep}(Q, \delta)]$ by putting t_{pq}^a in degree zero and $t_{pq}^{a^*}$ in degree one, for $a \in Q$, we get induced filtrations of A^λ and T^λ. Observe that

$$\mathrm{gr} T^\lambda \cong (\mathrm{gr} M_{\sum \delta_i}(k[\mu^{-1}(\lambda)]))^{\mathrm{GL}(\delta)} \cong M_{\sum \delta_i}(\mathrm{gr} k[\mu^{-1}(\lambda)])^{\mathrm{GL}(\delta)}.$$

But $\mathrm{gr}k[\mu^{-1}(\lambda)] \cong k[\mu^{-1}(0)]$. This is because μ is flat [8, Lemma 8.3]. Thus, $\mathrm{gr}T^{\lambda} \cong T^0$. By [8, Lemma 8.1], the map $\Psi : k\overline{Q} \to M_{\sum \delta_i}(k[T^{\star}\mathrm{Rep}(Q,\delta)])$ induces a filtered k-algebra homomorphism $\psi^{\lambda} : \Pi^{\lambda} \to T^{\lambda}$ which makes the diagram

$$
\begin{tikzcd}
\Pi^0 \arrow[r, "\psi^0"] \arrow[d, "\cong"] & T^0 \arrow[d, "\cong"] \\
\mathrm{gr}\,\Pi^{\lambda} \arrow[r, "\mathrm{gr}\,\psi^{\lambda}"] & \mathrm{gr}\,T^{\lambda}
\end{tikzcd}
\tag{8}
$$

commute. (That $\Pi^0 \to \mathrm{gr}\Pi^{\lambda}$ is an isomorphism is shown in [8, Corollary 3.6].) Note that ψ^{λ} restricts to a map $\psi_0^{\lambda} : \mathcal{O}^{\lambda} \to A^{\lambda}$ since $\mathcal{O}^{\lambda} = e_0\Pi^{\lambda}e_0$ and $A^{\lambda} = e_0T^{\lambda}e_0$. In [8, Theorem 8.10] it is shown that ψ^0 is injective. It follows from the diagram above that ψ^{λ} is injective. In [8, Corollary 8.11] it is shown that ψ_0^{λ} is an isomorphism.

4.1. LEMMA. – *Suppose that $\lambda \cdot \alpha \neq 0$, for all real roots α. Then ψ^{λ} is an isomorphism.*

Proof. – The hypothesis on λ ensures that $\Pi^{\lambda}e_0\Pi^{\lambda} = \Pi^{\lambda}$ [8]. Clearly then $T^{\lambda}e_0T^{\lambda} = T^{\lambda}$ and so e_0T^{λ} is a progenerator as an T^{λ}-module with endomorphism ring A^{λ}. As a consequence of the proof of [17, Theorem 1] (see (ii) at the end of §3) we have that e_0T^{λ} is generated as a module over A^{λ} by the image of $e_0\Pi^{\lambda}$. It follows that ψ^{λ} restricted to a map $e_0\Pi^{\lambda} \to e_0T^{\lambda}$ is a bijection. Clearly then we obtain an isomorphism $\Pi^{\lambda} = \mathrm{End}_{\mathcal{O}^{\lambda}} e_0\Pi^{\lambda} \to \mathrm{End}_{A^{\lambda}} e_0T^{\lambda} = T^{\lambda}$, as needed. □

Let S be a commutative ring and let $\lambda \in S^I$ satisy $\lambda \cdot \delta = 0$. Consider the fibre product

$$
\begin{tikzcd}
Y_{S,\lambda} \arrow[r] \arrow[d] & \mathrm{Spec}\, S \arrow[d, "\hat{\lambda}"] \\
T^{\star}\,\mathrm{Rep}(Q,\delta) \arrow[r, "\mu"] & \mathfrak{pgl}(\delta)
\end{tikzcd}
$$

where $\hat{\lambda} : \mathrm{Spec}S \to \mathfrak{pgl}(\delta)$ is the map corresponding to λ. Observe that $\mathrm{GL}(\delta)$ acts naturally on $Y_{S,\lambda}$, in such a way that all maps in the fibre product are equivariant (where the action on $\mathrm{Spec}S$ is trivial). We define $T^{S,\lambda} = M_{\sum \delta_i}(k[Y_{S,\lambda}])^{\mathrm{GL}(\delta)}$. There is a natural ring homomorphism $\psi^{S,\lambda} : \Pi^{S,\lambda} \to T^{S,\lambda}$ which induces one $\psi_0^{S,\lambda} : \mathcal{O}^{S,\lambda} \to k[Y_{S,\lambda}]^{GL(\delta)}$. Observe that if $f : S \to U$ is a ring homomorphism then

$$\Pi^{U,f(\lambda)} \cong \Pi^{S,\lambda} \otimes_S U, \quad T^{U,f(\lambda)} \cong T^{S,\lambda} \otimes_S U, \quad \text{and} \quad \psi^{U,f(\lambda)} = \psi^{S,\lambda} \otimes 1.$$

4.2. LEMMA. – *$T^{S,\lambda}$ is flat over S.*

Proof. – $Y_{S,\lambda} \to \mathrm{Spec}S$ is flat by [8, Lemma 8.6]. Thus $M_{\sum \delta_i}(k[Y_{S,\lambda}])$ is flat over S. Then $T^{S,\lambda}$ is flat over S, for, by the Reynolds operator [24], it is a direct summand of $M_{\sum \delta_i}(k[Y_{S,\lambda}])$. □

4.3. LEMMA. – *If S is a domain with quotient field F, and $\lambda\cdot\alpha \neq 0$ for all real roots α, then*

$$\psi^{S,\lambda} \otimes 1 : \Pi^{R,\lambda} \otimes F \to T^{S,\lambda} \otimes F$$

is an isomorphism.

Proof. – It suffices to show that the map is an isomorphism on tensoring with an algebraic closure $\overline{F}$ of F. But now, Lemma 4.1 applies with the base field k replaced by $\overline{F}$. □

4.4. LEMMA. – *Suppose S is a Dedekind domain, $\mathfrak{m}$ is a maximal ideal and C is a torsion S-module, not necessarily finitely generated. If $\mathrm{Tor}_1^S(C, S/\mathfrak{m}) = 0$ then $C \otimes S/\mathfrak{m} = 0$.*

Proof. – [8, Lemma 8.9] □

4.5. THEOREM. – *The map $\psi^0 : \Pi^0 \to T^0$ is an isomorphism.*

Proof. – Recall that ψ^0 is injective. Now fix some $\nu \in k^I$ such that $\nu \cdot \alpha \neq 0$ for all real roots α. Let $S = k[t]$ and let $\lambda = \nu t \in S^I$. Since $\Pi^{S,\lambda}$ is flat over S and $\psi^{S,\lambda}$ gives an isomorphism on tensoring with the quotient field of S, one has an exact sequence

$$0 \to \Pi^{S,\lambda} \to T^{S,\lambda} \to C \to 0.$$

and C is a torsion S-module. Tensoring this sequence with the module k on which t acts as zero, the first map becomes ψ^0, so remains injective. Thus, $\mathrm{Tor}_1^S(C, k) = 0$, since $T^{S,\lambda}$ is flat over S. By the previous lemma, $C \otimes k = 0$, so ψ^0 is an isomorphism. □

4.6. COROLLARY. – *For any λ with $\lambda \cdot \delta = 0$ the map $\psi^\lambda : \Pi^\lambda \to T^\lambda$ is an isomorphism.*

Proof. – By Equation (8), the map $\mathrm{gr}\psi^\lambda : \mathrm{gr}\Pi^\lambda \to \mathrm{gr}T^\lambda$ is surjective and we had already established that it was injective. □

4.7. COROLLARY. – *For any λ with $\lambda \cdot \delta = 1$ the k-algebra homomorphisms $\phi^\lambda : \Pi^\lambda \to \mathfrak{T}^{\lambda-\partial_\delta}$ and $\mathcal{O}^\lambda \to \mathfrak{A}^{\lambda-\partial_\delta-\epsilon_0}$ are isomorphisms.*

Proof. – As we saw above, the natural map $\Pi^0 \to \mathrm{gr}\Pi^\lambda$ is an isomorphism, by [8, Corollary 3.6]. The moment map $\mathrm{Rep}(Q,\delta) \to \mathfrak{pgl}(\delta)^*$ is flat, by [8, Lemma 8.3]. Thus, by Proposition 2.4 the natural map $T^0 \to \mathrm{gr}\mathfrak{T}^\chi$ is an isomorphism. The result follows from the commuting diagram (6) and Theorem 4.5. □

5. A sheaf of algebras

Retain the assumption of the first paragraph of the previous section that Q is an extended Dynkin quiver, δ is the minimal imaginary root etc. We also assume that Q has no oriented cycles. We have seen earlier that $\mathfrak{A}^\chi$, for χ a character of $\mathfrak{pgl}(\delta)$, quantizes the Marsden-Weinstein reduction $\mu^{-1}(0)/\!\!/GL(\delta)$. Note that, by [16], [8], $\mu^{-1}(0)/\!\!/GL(\delta)$ is the Kleinian singularity associated to the extended Dynkin diagram by the McKay correspondence [23]. In this section we construct a sheaf of algebras on $\mathbb{P}^1$ that quantizes a certain partial resolution of $\mu^{-1}(0)/\!\!/GL(\delta)$, defined in terms of moduli spaces of representations of Π^0 of dimension vector δ. This sheaf of algebras is analogous to a sheaf of twisted differential operators on the flag variety of a semisimple algebraic group. If θ is a character of $GL(\delta)/k^*$, we identify it with an element of $\mathbb{Z}^I : \theta \cdot \delta = 0$. We denote by $k[\mathrm{Rep}(Q,\delta)]_\theta$, the set of *semi-invariants* of weight θ i.e. those functions on which $GL(\delta)$ acts by the character θ. We write $\partial = \partial_\delta$, the *defect*. We say that $y \in \mathrm{Rep}(Q,\delta)$ is *∂-semistable* if there exists a semi-invariant $f \in k[\mathrm{Rep}(Q,\delta)]_{n\partial}$, with $n \geq 1$, such that $f(y) \neq 0$. We say that y is *∂-stable* if, in addition, the stabiliser of y in $GL(\delta)$ is k^* and the action of $GL(\delta)$ on the subset $\{x \in \mathrm{Rep}(Q,\delta) : f(x) \neq 0\}$ is closed. Write $\mathrm{Rep}(Q,\delta)^{ss}_\partial$ for the set of ∂-semistable points in $\mathrm{Rep}(Q,\delta)$. King [14] has shown that $M \in \mathrm{Rep}(Q,\delta)$ is ∂-semistable (resp. ∂-stable) if and only if $\partial \cdot \underline{\dim}\, N \leq 0$ (resp. < 0) for all proper submodules (i.e. $N \neq 0$, $N \neq M$) of M (Note that King's definition of the weight of a semi-invariant differs from ours by a sign). Further, [14], [24, Theorem 1.10] there is a categorical quotient $\pi : \mathrm{Rep}(Q,\delta)^{ss}_\partial \to \mathfrak{R}$. One description of $\mathfrak{R}$ is as $\mathrm{Proj}A$, where

$$A = \bigoplus_{n \geq 0} k[\mathrm{Rep}(Q,\delta)]_{n\delta}.$$

By construction, there is a projective morphism $\mathfrak{R} \to \mathrm{Spec}A_0 = pt$; the latter equality holds as Q has no oriented cycles. The points of $\mathrm{Rep}(Q,\delta)^{ss}_\partial$ have another description. Let τ denote the Auslander-Reiten translate. If M is an indecomposable finite-dimensional kQ-module then M is either *preprojective*, *regular* or *preinjective* according to whether $\tau^n(M) = 0$, for $n \gg 0$, $\tau^n(M) \neq 0$, for all $n \in \mathbb{Z}$, or $\tau^{-n}M = 0$, for $n \gg 0$. These three possibilities occur when $\partial \cdot \underline{\dim}\, M$ is strictly less than, equal to, or strictly greater than zero. An arbitrary finite-dimensional module is called regular if and only if all its indecomposable direct summands are regular. It is not difficult to prove the following.

5.1. LEMMA. – *The regular modules in* $\mathrm{Rep}(Q,\delta)$ *are exactly the ∂-semistable points.*

As was first pointed out by Ringel [29]:

5.2. LEMMA. – $\mathfrak{R} \cong \mathbb{P}^1$.

Now consider $\mu^{-1}(0) = \mathrm{Rep}(\Pi^0,\delta) \subset \mathrm{Rep}(\overline{Q},\delta)$. Just as for $\mathrm{Rep}(Q,\delta)$, one can consider ∂-semistable and ∂-stable points of $\mu^{-1}(0)$. Again one can form a categorical quotient $\mu^{-1}(0)^{ss}_\partial \to \mathfrak{M}$. Further $\mathfrak{M}$ is constructed as $\mathrm{Proj}B$, where $B = \oplus_{n\geq 0} k[\mu^{-1}(0)]_{n\partial}$. This time, $\mathrm{Spec}B_0 = \mu^{-1}(0)/\!\!/GL(\delta)$ and so we get a projective morphism $f : \mathfrak{M} \to \mu^{-1}(0)/\!\!/GL(\delta)$. For this, see [28]. Since there is a simple Π^0-module of dimension δ [6], it is easy to see that f is birational. Note that identifying $\mathrm{Rep}(\overline{Q},\delta) = T^\star\mathrm{Rep}(Q,\delta)$ we can regard $T^\star\mathrm{Rep}(Q,\delta)^{ss}_\partial$, the cotangent bundle of $\mathrm{Rep}(Q,\delta)^{ss}_\partial$, as an open subset of $\mathrm{Rep}(\overline{Q},\delta)$. It is perhaps worth remarking that $T^\star\mathrm{Rep}(Q,\delta)^{ss}_\partial$ is not, in general, equal to $\mathrm{Rep}(\overline{Q},\delta)^{ss}_\partial$. However we do have the following result of Crawley-Boevey [7, Lemma 12.1].

5.3. LEMMA. – $T^\star\mathrm{Rep}(Q,\delta)^{ss}_\partial \cap \mu^{-1}(0) = \mu^{-1}(0)^{ss}_\partial$.

By virtue of the lemma there is a natural affine map $\mu^{-1}(0)^{ss}_\partial \to \mathrm{Rep}(Q,\delta)^{ss}_\partial$ obtained by composing the inclusion $\mu^{-1}(0)^{ss}_\partial \to T^\star\mathrm{Rep}(Q,\delta)^{ss}_\partial$ and the projection $T^\star\mathrm{Rep}(Q,\delta)^{ss}_\partial \to \mathrm{Rep}(Q,\delta)^{ss}_\partial$. Taking categorical quotients this induces an affine map $p : \mathfrak{M} \to \mathfrak{R}$. We need another result of Crawley-Boevey [7, Theorem 12.3].

5.4. THEOREM. – $\mathfrak{M}$ *is normal.*

5.5. REMARK. – Crawley-Boevey actually shows that $\mu^{-1}(0)^{ss}_\partial$ is normal. Of course, the theorem follows from that. Interestingly though, Ringel has observed that $\mu^{-1}(0)$ need not be normal.

As f is birational and projective, it gives a *partial* desingularisation of

$$\mu^{-1}(0)/\!\!/GL(\delta);$$

in general, $\mathfrak{M}$ is not smooth. Since $\mu^{-1}(0)/\!\!/GL(\delta)$ has rational singularities we can compute the cohomology of the structure sheaf of $\mathfrak{M}$ (Actually, $\mathfrak{M}$ itself has rational singularities but we will not need this fact).

5.6. LEMMA. – *The cohomology of the structure sheaf of $\mathfrak{M}$ is given by*

$$k[\mathfrak{M}] = k[\mu^{-1}(0)/\!\!/GL(\delta)] \qquad \textit{and} \qquad \mathrm{H}^i(\mathfrak{M}, \mathcal{O}_{\mathfrak{M}}) = 0, \quad \textit{for } i > 0.$$

Proof. – Let $g : \tilde{\mathfrak{M}} \to \mathfrak{M}$ be a resolution of singularities and consider the Leray spectral sequence for the composite $h := fg$. Note that h is a resolution of $\mu^{-1}(0)/\!\!/GL(\delta)$. We get that

$$R^p f_* R^q g_* \mathcal{O}_{\tilde{\mathfrak{M}}} \Longrightarrow R^{p+q} h_* \mathcal{O}_{\tilde{\mathfrak{M}}}.$$

Of course, as $\mu^{-1}(0)/\!\!/GL(\delta)$ has rational singularities we obtain that the right-hand-side is zero except for $p = q = 0$ when it is $k[\mu^{-1}(0)/\!\!/GL(\delta)]$. Since the $1, 0$ entry of the E_2-page is clearly unchanged by any subsequent differentials we see that

$$R^1 f_*(g_* \mathcal{O}_{\tilde{\mathfrak{M}}}) = 0.$$

As $\mathfrak{M}$ is normal and g is birational we have, by Zariski's Main Theorem, that

$$g_* \mathcal{O}_{\tilde{\mathfrak{M}}} = \mathcal{O}_{\mathfrak{M}}$$

and so we deduce that $\mathrm{H}^1(\mathfrak{M}, \mathcal{O}_{\mathfrak{M}}) = 0$. Finally, $\mathrm{H}^i(\mathfrak{M}, \mathcal{O}_{\mathfrak{M}}) = 0$, for $i \geq 2$ by the Leray spectral sequence for the affine map p. □

Our aim is to give a noncommutative quantization of $\mathfrak{M}$. Let $\mathcal{D}_{\mathrm{Rep}(Q,\delta)^{ss}_{\partial}}$ denote the sheaf of k-linear differential operators on $\mathrm{Rep}(Q,\delta)^{ss}_{\partial}$. For χ a character of $\mathfrak{pgl}(\delta)$, define a sheaf of k-algebras on $\mathfrak{N}$ by

$$\mathcal{A}^{\chi} = \frac{(\pi_* \mathcal{D}_{\mathrm{Rep}(Q,\delta)^{ss}_{\partial}})^{GL(\delta)}}{((\pi_* \mathcal{D}_{\mathrm{Rep}(Q,\delta)^{ss}_{\partial}})(\iota - \chi)(\mathfrak{gl}(\delta)))^{GL(\delta)}}.$$

It is quasi-coherent as a sheaf of left $\mathcal{O}_{\mathfrak{N}}$-modules.

5.7. PROPOSITION.

$$\mathrm{gr}\mathcal{A}^{\chi} \cong p_* \mathcal{O}_{\mathfrak{M}}.$$

Proof. – Let $T = k[\mathrm{Rep}(Q,\delta)]$ and $\mathfrak{g} = \mathfrak{pgl}(\delta)$. Since $\mu : \mathrm{Rep}(Q,\delta) \to \mathfrak{g}^*$ is flat, Proposition 2.4 tells us that the natural map

$$\mathrm{gr}\mathcal{D}(T)/\mathrm{gr}\mathcal{D}(T)\mathfrak{g} \to \mathrm{gr}(\mathcal{D}(T)/\mathcal{D}(T)(\iota - \chi)(\mathfrak{g}))$$

is an isomorphism. This isomorphism clearly passes to the localisation where one replaces T by T_s, for some semi-invariant s of weight ∂^n. Taking $GL(\delta)$-invariants we obtain an isomorphism

$$p_*\mathcal{O}_{\mathfrak{M}}|_{D_+(s)} \to \mathrm{gr}\mathcal{A}^\chi|_{D_+(s)}$$

where $D_+(s)$ is the open set $\mathrm{Spec}T_{(s)}$ of $\mathfrak{R}$. Since these isomorphisms are evidently compatible we have the isomorphism of the statement. □

Note that the proposition says that $\mathcal{A}^\chi$ is a quantization of the partial desingularisation $\mathfrak{M}$. For, since p is affine we can recover $\mathfrak{M} = \mathbf{Spec}\, p_*\mathcal{O}_{\mathfrak{M}} \cong \mathbf{Spec}\,\mathrm{gr}\mathcal{A}^\chi$. It would be interesting to quantize a minimal resolution of singularities of

$$\mu^{-1}(0)/\!\!/GL(\delta).$$

Next we investigate some of the properties of $\mathcal{A}^\chi$. In its relation to $\mathfrak{A}^\chi$ it behaves in a similar way as does the sheaf of twisted differential operators on the flag variety to the corresponding minimal primitive factor of the enveloping algebra of a semisimple Lie algebra. We need a general result.

5.8. LEMMA. – *Let $\mathcal{A}$ be a sheaf of k-algebras with a quasi-coherent filtration over a variety X. Suppose that* $\mathrm{H}^1(X, \mathrm{gr}\mathcal{A}) = 0$. *Then* $\mathrm{H}^1(X, \mathcal{A}) = 0$ *and*

$$\mathrm{gr}\Gamma(X, \mathcal{A}) \cong \Gamma(X, \mathrm{gr}\mathcal{A}).$$

Proof. – Applying the global sections functor to the short exact sequence

$$0 \to \mathcal{A}_{i-1} \to \mathcal{A}_i \to \mathcal{A}_i/\mathcal{A}_{i-1} \to 0$$

one obtains the exactness of

$$\mathrm{H}^1(X, \mathcal{A}_{i-1}) \to \mathrm{H}^1(X, \mathcal{A}_i) \to \mathrm{H}^1(X, (\mathrm{gr}\mathcal{A})_i).$$

By induction, we see that $\mathrm{H}^1(X, \mathcal{A}_i) = 0$, for all $i \geq 0$. Thus,

$$0 \to \Gamma(X, \mathcal{A}_{i-1}) \to \Gamma(X, \mathcal{A}_i) \to \Gamma(X, (\mathrm{gr}\mathcal{A})_i) \to 0$$

is exact. □

5.9. THEOREM. – *The cohomology of $\mathcal{A}^\chi$ is given by*

$$\Gamma(\mathfrak{R}, \mathcal{A}^\chi) \cong \mathfrak{A}^\chi \qquad \textit{and} \qquad \mathrm{H}^i(\mathfrak{R}, \mathcal{A}^\chi) = 0,$$

for $i > 0$.

Proof. – There is a natural filtered homomorphism

$$\mathfrak{A}^\chi \to \Gamma(\mathfrak{R}, \mathcal{A}^\chi).$$

To prove the first claim it is enough to show that the associated graded map is an isomorphism. Thus, we must show that

$$\mathrm{gr}\Gamma(\mathfrak{R}, \mathcal{A}^\chi) \cong k[\mu^{-1}(0)/\!\!/GL(\delta)]. \tag{9}$$

By Proposition 5.7 we know that $\mathrm{gr}\mathcal{A}^\chi \cong p_*\mathcal{O}_{\mathfrak{M}}$. Since p is affine it follows that $\mathrm{H}^i(\mathfrak{R}, \mathrm{gr}\mathcal{A}^\chi) \cong \mathrm{H}^i(\mathfrak{M}, \mathcal{O}_{\mathfrak{M}})$. By Lemma 5.6, we see that this cohomology group is $k[\mu^{-1}(0)/\!\!/GL(\delta)]$, for $i = 0$, and vanishes, for $i > 0$. Now, by Lemma 5.8 we obtain (9), as needed. Further we see that $\mathrm{H}^i(\mathfrak{R}, \mathcal{A}^\chi) = 0$, for $i = 1$. Since $\mathfrak{R} \cong \mathbb{P}^1$, the vanishing for $i > 1$ is clear. □

5.10. Lemma.

If U is an open affine subset of $\mathfrak{R}$ then the restriction map $\Gamma(\mathfrak{R}, \mathcal{A}^\chi) \to \Gamma(U, \mathcal{A}^\chi)$ is injective and induces an isomorphism on division rings of fractions.

Proof. – Firstly observe that, $\Gamma(\mathfrak{R}, \mathcal{A}^\chi)$ and $\Gamma(U, \mathcal{A}^\chi)$ are both Noetherian domains and so have division rings of fractions, by equation (9) and Proposition 5.7. The first claim follows once one has that

$$\mathrm{gr}\Gamma(\mathfrak{R}, \mathcal{A}^\chi) \to \mathrm{gr}\Gamma(U, \mathcal{A}^\chi)$$

is injective. But

$$\mathrm{gr}\Gamma(\mathfrak{R}, \mathcal{A}^\chi) = \Gamma(\mathfrak{R}, \mathrm{gr}\mathcal{A}^\chi),$$

and likewise on U, so this is clear. Finally, as $\Gamma(\mathfrak{R}, \mathrm{gr}\mathcal{A}^\chi)$ and $\Gamma(U, \mathrm{gr}\mathcal{A}^\chi)$ have the same fractions we easily deduce that the same is true for $\Gamma(\mathfrak{R}, \mathcal{A}^\chi)$ and $\Gamma(U, \mathcal{A}^\chi)$. □

5.11. Corollary. – *$\mathfrak{A}^\chi$ has a faithful representation by differential operators on an open subset of $\mathfrak{R}$. In particular, the quotient division algebra of $\mathfrak{A}^\chi$ is the first Weyl division algebra $D_1(k)$.*

Proof. – We can choose U as in the lemma so that $GL(\delta)/k^*$ is acting freely on $\pi^{-1}(U)$. An argument along the same lines as [30, Corollary 4.5] shows that $\Gamma(U, \mathcal{A}^\chi) \cong \mathcal{D}(U)$, hence the result. □

We can now show that there is an analogue of the Beilinson-Bernstein theorem [3], at least when $\mathfrak{A}^\chi$ is hereditary. Our approach uses the main theorem of Hodges and Smith [12]. However, first we need to point out that one of their hypotheses is not needed. Recall their notation. Thus, X is an irreducible variety over a field K, $\mathcal{R}$ is a sheaf of Noetherian K-algebras over X. It is supposed that $R = \Gamma(X, \mathcal{R})$ has a classical ring of quotients Q. Further it is assumed that (i) The structure sheaf $\mathcal{O}$ of X is a subsheaf of $\mathcal{R}$ and that $\mathcal{R}$ is a quasi-coherent sheaf of left $\mathcal{O}$-modules. (ii) If U is an open affine subset of X then $\mathcal{R}(U)$ is a subalgebra of Q containing R and is generated as a right or left R-module by $\Gamma(U, \mathcal{O})$. (iii) There is a finite open affine cover (U_α) for X such that the diagonal embedding $R \to \oplus\mathcal{R}(U_\alpha)$ obtained from the restriction maps makes $\oplus\mathcal{R}(U_\alpha)$ a faithfully flat right R-module. The reader will check that Hodges and Smith use the assumption in (ii) that $\mathcal{R}(U)$ is generated as a right or left R-module by $\Gamma(U, \mathcal{O})$ in only one place in the proof of their main theorem, viz to obtain the conclusion of [12, Lemma 2.6]. However, this use is unecessary. For, if U and V are open affine subsets of X then

$$\begin{aligned}\mathcal{R}(U \cap V) = \mathcal{O}(U \cap V)\mathcal{R}(U \cap V) = &\mathcal{O}(U \cap V)\mathcal{R}(V) = \\ &\mathcal{O}(U)\mathcal{O}(V)\mathcal{R}(V) = \mathcal{O}(U)\mathcal{R}(V) = \mathcal{R}(U)\mathcal{R}(V).\end{aligned}$$

Here, the first equality is clear, the second uses that $\mathcal{R}$ is quasi-coherent, the third uses that X is a variety, and the remaining equalities are clear. This, together with the proof of [12, Lemma 2.5], yields the conclusion of [12, Lemma 2.6].

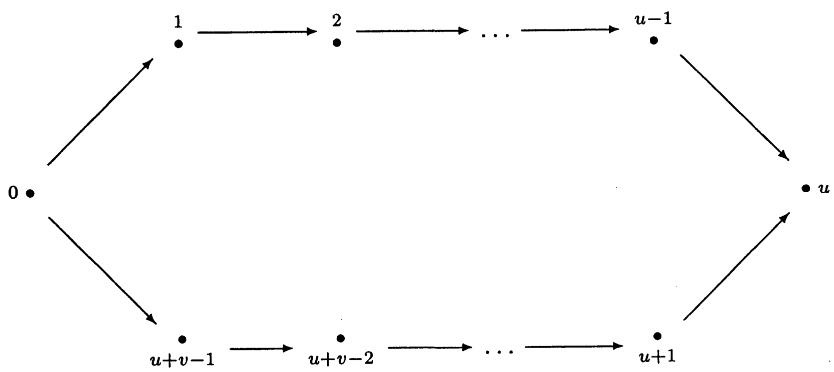
5.12. THEOREM. – *If $\chi \cdot \alpha + \langle \partial, \alpha \rangle + \alpha_0 \neq 0$, for all real roots α, then $\Gamma(\mathfrak{R}, \_\_)$ gives an equivalence of categories between the category of left $\mathcal{A}^\chi$-modules which are quasi-coherent over $\mathcal{O}_{\mathfrak{R}}$ and the category of left $\mathfrak{A}^\chi$-modules.*

Proof. – For such a χ, $\mathfrak{A}^\chi$ is hereditary [8, Theorem 4]. Now, if U is any open affine subset of $\mathfrak{R}$, then $\mathcal{A}^\chi(U)$ has the same quotient division ring as $\mathfrak{A}^\chi$ and so is a torsionfree $\mathfrak{A}^\chi$-module. It follows that $\mathcal{A}^\chi(U)$ is a flat right $\mathfrak{A}^\chi$-module. Consider the complex

$$0 \to \mathfrak{A}^\chi \to \Gamma(U_0, \mathcal{A}^\chi) \oplus \Gamma(U_1, \mathcal{A}^\chi) \to \Gamma(U_0 \cap U_1, \mathcal{A}^\chi) \to 0.$$

Since $\mathrm{H}^1(\mathfrak{R}, \mathcal{A}^\chi) = 0$, this complex is exact. Let M be a left $\mathfrak{A}^\chi$-module. Apply $\_\_ \otimes_{\mathfrak{A}^\chi} M$ to the above exact sequence. The flatness mentioned above ensures that the resulting sequence is exact. Thus $\Gamma(U_0, \mathcal{A}^\chi) \oplus \Gamma(U_1, \mathcal{A}^\chi)$ is a faithfully flat right $\mathfrak{A}^\chi$-module. Finally, we apply the main result of [12]. $\square$

5.13. REMARK. – A Beilinson-Bernstein theorem should hold more generally than this. For example, for the quiver $Q = \widetilde{A_{u,v}}$



one can obtain the conclusion of Theorem 5.12 provided that for all integers i, j with $0 \leq i \leq u-1$, $u \leq j \leq u+v-1$ one has $\sum_{t=i+1}^{j} \chi_t \notin \mathbb{N} \setminus \{0\}$.

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