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POLYNOMIAL INVARIANTS FOR FIBERED 3-MANIFOLDS AND TEICHMÜLLER GEODESICS FOR FOLIATIONS

BY CURTIS T. MCMULLEN [1]

ABSTRACT. – Let $F \subset H^1(M^3, \mathbb{R})$ be a fibered face of the Thurston norm ball for a hyperbolic 3-manifold M.

Any $\phi \in \mathbb{R}_+ \cdot F$ determines a measured foliation $\mathcal{F}$ of M. Generalizing the case of Teichmüller geodesics and fibrations, we show $\mathcal{F}$ carries a canonical *Riemann surface* structure on its leaves, and a transverse *Teichmüller flow* with pseudo-Anosov expansion factor $K(\phi) > 1$.

We introduce a polynomial invariant $\Theta_F \in \mathbb{Z}[H_1(M,\mathbb{Z})/\text{torsion}]$ whose roots determine $K(\phi)$. The Newton polygon of Θ_F allows one to compute fibered faces in practice, as we illustrate for closed braids in S^3. Using fibrations we also obtain a simple proof that the shortest geodesic on moduli space $\mathcal{M}_g$ has length $O(1/g)$. © 2000 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – Soit M une variété hyperbolique de dimension 3, et $F \subset H^1(M^3, \mathbb{R})$ une face fibrée de la boule unité dans la norme de Thurston.

Chaque $\phi \in \mathbb{R}_+ \cdot F$ détermine un feuilletage mesuré $\mathcal{F}$ de M. Généralisant le cas des géodésiques de Teichmüller et des fibrations, nous démontrons que $\mathcal{F}$ porte une structure complexe canonique sur les feuilles, et admet un *flot transverse de Teichmüller*, avec facteur d'expansion pseudo-Anosov $K(\phi) > 1$.

Nous introduisons un invariant polynomial $\Theta_F \in \mathbb{Z}[H_1(M,\mathbb{Z})/\text{torsion}]$, dont les racines déterminent $K(\phi)$. Le polygone de Newton de Θ_F permet le calcul pratique des faces fibrées, comme nous l'illustrons pour les tresses fermées dans S^3. Nous obtenons aussi, en utilisant les fibrations, une preuve simple du fait que la géodésique la plus courte sur l'espace de modules $\mathcal{M}_g$ est de longueur $O(1/g)$. © 2000 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

Every fibration of a 3-manifold M over the circle determines a closed loop in the moduli space of Riemann surfaces. In this paper we introduce a polynomial invariant for M that packages the Teichmüller lengths of these loops, and we extend the theory of Teichmüller geodesics from fibrations to measured foliations.

Riemann surfaces and fibered 3-manifolds. Let M be a compact oriented 3-manifold, possibly with boundary. Suppose M fibers over the circle $S^1 = \mathbb{R}/\mathbb{Z}$, with fiber S and pseudo-

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Anosov monodromy $\psi : S \to S$:

$$\begin{array}{ccc} \psi \circlearrowright S & \longrightarrow & M \\ & & \Big\downarrow \pi \\ & & S^1. \end{array}$$

Then there is:

- a natural complex structure J_s along the fibers $S_s = \pi^{-1}(s)$, and
- a flow $f : M \times \mathbb{R} \to M$, circulating the fibers at unit speed,

such that the conformal distortion of f is minimized.

Indeed, the mapping-class ψ determines a loop in the moduli space of complex structures on S, represented by a unique Teichmüller geodesic

$$\gamma : S^1 \to \mathcal{M}_{g,n}.$$

The complex structure on the fibers is given by $(S_s, J_s) = \gamma(s)$. The time t map of the flow f is determined by the condition that on each fiber, $f_t : (S_s, J_s) \to (S_{s+t}, J_{s+t})$ is a Teichmüller mapping. Outside a finite subset of S_s, f_t is locally an affine stretch of the form

$$f_t(x + iy) = K^t x + i K^{-t} y, \tag{1.1}$$

where $K > 1$ is the *expansion factor* of the monodromy ψ. The Teichmüller length of the loop γ in moduli space is $\log K$.

This well-known interplay between topology and complex analysis was developed by Teichmüller, Thurston and Bers (see [4]). The fibration π, the resulting geometric structure on M and the expansion factor K are all determined (up to isotopy) by the cohomology class $\phi = [S] \in H^1(M, \mathbb{R})$.

Fibered faces. In this paper we extend the theory of Teichmüller geodesics from fibrations to measured foliations.

The Thurston norm $\|\phi\|_T$ on $H^1(M, \mathbb{R})$ leads to a coherent picture of all the cohomology classes represented by fibrations and measured foliations of M. To describe this picture, we begin by defining the Thurston norm, which is a generalization of the genus of a knot; it measures the minimal complexity of an embedded surface in a given cohomology class. For an integral cohomology class ϕ, the norm is given by:

$$\|\phi\|_T = \inf\bigl\{|\chi(S_0)| : (S, \partial S) \subset (M, \partial M) \text{ is dual to } \phi\bigr\},$$

where $S_0 \subset S$ excludes any S^2 or D^2 components of S. The Thurston norm is extended to real classes by homogeneity and continuity. The unit ball of the Thurston norm is a polyhedron with rational vertices.

An embedded, oriented surface $S \subset M$ is a *fiber* if it is the preimage of a point under a fibration $M \to S^1$. Any fiber minimizes $|\chi(S)|$ in its cohomology class. Moreover, $[S]$ belongs to the cone $\mathbb{R}_+ \cdot F$ over an open *fibered face* F of the unit ball in the Thurston norm. Every integral class in $\mathbb{R}_+ \cdot F$ is realized by a fibration $M^3 \to S^1$; more generally, every real cohomology class $\phi \in \mathbb{R}_+ \cdot F$ is represented by a *measured foliation* $\mathcal{F}$ of M. Such a foliation is determined by a closed, nowhere-vanishing 1-form ω on M, with $T\mathcal{F} = \operatorname{Ker} \omega$ and with measure

$$\mu(T) = \left| \int_T \omega \right|$$

for any connected transversal T to $\mathcal{F}$. For an integral class, the leaves of $\mathcal{F}$ are closed and come from a fibration $\pi: M \to S^1$ with $\omega = \pi^*(dt)$.

Generalizing the case of fibrations, we will show (Section 9):

THEOREM 1.1. – *For any measured foliation $\mathcal{F}$ of M, there is a complex structure J on the leaves of $\mathcal{F}$, a unit speed flow*

$$f:(M,\mathcal{F}) \times \mathbb{R} \to (M,\mathcal{F}),$$

and a $K > 1$, such that f_t maps leaves to leaves by Teichmüller mappings with expansion factor $K^{|t|}$.

The foliation $\mathcal{F}$, the complex structure J along its leaves, the transverse flow f and the stretch factor K are all determined up to isotopy by the cohomology class $[\mathcal{F}] \in H^1(M,\mathbb{R})$.

Here f has *unit speed* if it is generated by a vector field v with $\omega(v) = 1$, where ω is the defining 1-form of $\mathcal{F}$. The complex structure J makes each leaf $\mathcal{F}_\alpha$ of $\mathcal{F}$ into a Riemann surface, and

$$f_t: \mathcal{F}_\alpha \to \mathcal{F}_\beta$$

is a *Teichmüller mapping* with *expansion factor* K if

$$\mu(f_t) = \frac{\bar{\partial} f_t}{\partial f_t} = \left(\frac{K^2-1}{K^2+1}\right)\frac{\bar{q}}{|q|}$$

for some holomorphic quadratic differential $q(z)\,dz^2$ on $\mathcal{F}_\alpha$. Away from the zeros of q, such a mapping has the form of an affine stretch as in (1.1).

Quantum geodesics. Theorem 1.1 provides, for a general measured foliation $\mathcal{F}$ with typical leaf S, a 'quantum geodesic'

$$\gamma: \mathbb{R}/H_1(M,\mathbb{Z}) \to \mathrm{Teich}(S)/H_1(M,\mathbb{Z}).$$

Here $H_1(M,\mathbb{Z})$ acts on $\mathbb{R}$ by translation by the periods Π of ω, and on Teich(S) by monodromy around loops in M. Generically Π is a dense subgroup of $\mathbb{R}$, in which case $\mathbb{R}/\Pi$ and Teich(S)$/H_1(M,\mathbb{Z})$ are 'quantum spaces' in the sense of Connes [12]. The map γ plays the role of a closed Teichmüller geodesic for the virtual mapping class determined by $\mathcal{F}$.

The Teichmüller polynomial. Next we introduce a polynomial invariant Θ_F for a fibered face $F \subset H^1(M,\mathbb{R})$. This polynomial determines the Teichmüller expansion factors $K(\phi)$ for all $\phi = [\mathcal{F}] \in \mathbb{R}_+ \cdot F$.

Like the Alexander polynomial, Θ_F naturally resides in the group ring $\mathbb{Z}[G]$, where $G = H_1(M,\mathbb{Z})/\text{torsion}$. Observe that $\mathbb{Z}[G]$ can be thought of as a ring of complex-valued functions on the character variety $\widehat{G} = \mathrm{Hom}(G,\mathbb{C}^*)$, with

$$\left(\sum a_g \cdot g\right)(\rho) = \sum a_g \rho(g).$$

To define Θ_F, we first show F determines a 2-dimensional lamination $\mathcal{L} \subset M$, transverse to every fiber $[S] \in \mathbb{R}_+ \cdot F$ and with $S \cap \mathcal{L}$ equal to the expanding lamination for the monodromy $\psi: S \to S$. Next we define, for every character $\rho \in \widehat{G}$, a group of twisted cycles $Z_2(\mathcal{L},\mathbb{C}_\rho)$. Here a *cycle* μ is simply an additive, holonomy-invariant function $\mu(T)$ on compact, open transversals T to $\mathcal{L}$, with values in the complex line bundle specified by ρ.

The *Teichmüller polynomial* $\Theta_F \in \mathbb{Z}[G]$ defines the largest hypersurface $V \subset \widehat{G}$ such that

$$\dim Z_2(\mathcal{L}, \mathbb{C}_\rho) > 0 \quad \text{for all } \rho \in V. \tag{1.2}$$

More precisely, we associate to $\mathcal{L}$ a module $T(\widetilde{\mathcal{L}})$ over $\mathbb{Z}[G]$, and (Θ_F) is the smallest principal ideal containing all the minor determinants in a presentation matrix for $T(\widetilde{\mathcal{L}})$. Thus Θ_F is well-defined up to multiplication by a unit $\pm g \in \mathbb{Z}[G]$.

Information packaged in Θ_F. Let $\Theta_F = \sum a_g \cdot g$ be the Teichmüller polynomial of a fibered face F of the Thurston norm ball in $H^1(M, \mathbb{R})$. In Sections 3–6 we will show:

(1) *The Teichmüller polynomial is symmetric; that is, $\Theta_F = \sum a_g \cdot g^{-1}$ up to a unit in $\mathbb{Z}[G]$.*
(2) *For any fiber $[S] = \phi \in \mathbb{R}_+ \cdot F$, the expansion factor $k = K(\phi)$ of its monodromy ψ is the largest root of the polynomial equation*

$$\Theta_F(k^\phi) = \sum a_g k^{\phi(g)} = 0. \tag{1.3}$$

(3) *Eq.* (1.3) *also determines the expansion factor for any measured foliation $[\mathcal{F}] = \phi \in \mathbb{R}_+ \cdot F$.*
(4) *The function $1/\log K(\phi)$ is real-analytic and strictly concave on $\mathbb{R}_+ \cdot F$.*
(5) *The cone $\mathbb{R}_+ \cdot F$ is dual to a vertex of the Newton polygon*

$$N(\Theta_F) = \big(\text{the convex hull of } \{g\colon\ a_g \neq 0\}\big) \subset H_1(M, \mathbb{R}).$$

To see the relation of Θ_F to expansion factors, note that a fibration $M \to S^1$ with fiber S determines a measured lamination $(\lambda, \mu_0) \in \mathcal{ML}(S)$, such that the transverse measure μ_0 on λ is expanded by a factor $K > 1$ under monodromy. Thus the suspension of μ_0 gives a cycle $\mu \in Z_2(\mathcal{L}, \mathbb{C}_\rho)$ with character

$$\rho(\gamma) = K^{[S]\cdot[\gamma]}$$

for loops $\gamma \subset M$. Therefore $\Theta_F(\rho) = 0$ (as in (1.2) above), and thus K can be recovered from the zeros of Θ_F.

The relation between F and the Newton polygon of Θ_F ((1) above) comes from the fact that $K(\phi) \to \infty$ as $\phi \to \partial F$.

A formula for $\Theta_F(t, u)$. One can also approach the Teichmüller polynomial from a 2-dimensional perspective. Let $\psi\colon S \to S$ be a pseudo-Anosov mapping, and let $(t_1, \ldots, t_b)$ be a multiplicative basis for

$$H = \operatorname{Hom}\big(H^1(S, \mathbb{Z})^\psi, \mathbb{Z}\big) \cong \mathbb{Z}^b,$$

where $H^1(S, \mathbb{Z})^\psi$ is the ψ-invariant cohomology of S. (When ψ acts trivially on cohomology, we can identify H with $H_1(S, \mathbb{Z})$.) By evaluating cohomology classes on loops, we obtain a natural map $\pi_1(S) \to H$. Choose a lift

$$\widetilde{\psi}\colon \widetilde{S} \to \widetilde{S}$$

of ψ to the H-covering space of S.

Let $M = S \times [0,1]/\langle (x,1) \sim (\psi(x), 0)\rangle$ be the mapping torus of ψ, let

$$G = H_1(M, \mathbb{Z})/\text{torsion} \cong H \oplus \mathbb{Z},$$

and let $F \subset H^1(M,\mathbb{R})$ be the fibered face with $[S] \in \mathbb{R}_+ \cdot F$. Then we can regard Θ_F as a Laurent polynomial

$$\Theta_F(t,u) \in \mathbb{Z}[G] = \mathbb{Z}[H] \oplus \mathbb{Z}[u] = \mathbb{Z}\big[t_1^{\pm 1},\ldots,t_b^{\pm 1},u^{\pm 1}\big],$$

where u corresponds to $[\widetilde{\psi}]$.

To give a concrete expression for Θ_F, let E and V denote the edges and vertices of an invariant train track $\tau \subset S$ carrying the expanding lamination of ψ. Then $\widetilde{\psi}$ acts by matrices $P_E(t)$ and $P_V(t)$ on the free $\mathbb{Z}[H]$-modules generated by the lifts of E and V to $\widetilde{S}$. In terms of this action we show (Section 3):

(6) *The Teichmüller polynomial is given by*

$$\Theta_F(t,u) = \frac{\det(uI - P_E(t))}{\det(uI - P_V(t))}.$$

Using this formula, many of the properties of Θ_F follow from the theory of Perron–Frobenius matrices over a ring of Laurent polynomials, developed in Appendix A.

Fixed-points on $\mathbb{P}\mathcal{ML}_s(S)$. Let $\mathcal{ML}_s(S)$ denote the space of measured laminations $\Lambda = (\lambda,\mu)$ on S *twisted* by $s \in H^1(S,\mathbb{R})$, meaning μ transforms by $e^{s(\gamma)}$ under $\gamma \in \pi_1(S)$.

The mapping-class ψ acts on $\mathcal{ML}_s(S)$ for all $s \in H^1(S,\mathbb{R})^\psi$, once we have chosen the lift $\widetilde{\psi}$. As in the untwisted case, ψ has a unique pair of fixed-points $[\Lambda_\pm]$ in $\mathbb{P}\mathcal{ML}_s(S)$, whose supports $\lambda_\pm$ are independent of s. In Section 8 we show:

(7) *The eigenvector $\Lambda_+ \in \mathcal{ML}_s(S)$ satisfies*

$$\psi \cdot \Lambda_+ = k(s)\Lambda_+,$$

where $u = k(s) > 0$ is the largest root of the polynomial $\Theta_F(e^s,u) = 0$. The function $\log k(s)$ is convex on $H^1(S,\mathbb{R})^\psi$.

Short geodesics on moduli space. It is known that the shortest geodesic loop on moduli space $\mathcal{M}_g$ has Teichmüller length $L(\mathcal{M}_g) \asymp 1/g$ (see [40]). In Section 10 we show mapping-classes with invariant cohomology provide a natural source of such short geodesics.

More precisely, let $\psi : S \to S$ be a pseudo-Anosov mapping on a closed surface of genus $g \geqslant 2$, leaving invariant a primitive cohomology class

$$\xi_0 : \pi_1(S) \to \mathbb{Z}.$$

Let $\widetilde{S} \to S$ be the corresponding $\mathbb{Z}$-covering space, with deck group generated by $h : \widetilde{S} \to \widetilde{S}$, and fix a lift $\widetilde{\psi}$ of ψ to $\widetilde{S}$. Then for all $n \gg 0$, the surface $R_n = \widetilde{S}/\langle h^n \widetilde{\psi}\rangle$ has genus $g_n \asymp n$, and $h : \widetilde{S} \to \widetilde{S}$ descends to a pseudo-Anosov mapping-class $\psi_n : R_n \to R_n$.

This renormalization construction gives mappings ψ_n with expansion factors satisfying

$$K(\psi_n) = K(\phi)^{1/n} + O\big(1/n^2\big),$$

and hence produces closed Teichmüller geodesics of length

$$L(\psi_n) = \frac{L(\psi)}{n} + O\big(n^{-2}\big) \asymp \frac{1}{g_n}.$$

This estimate is obtained by realizing the surfaces R_n as fibers in the mapping torus of ψ; see Section 10.

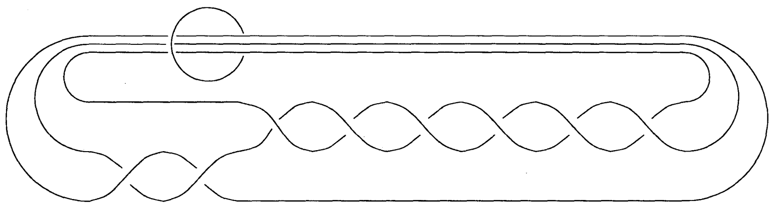

Fig. 1. The 4 component fibered link $L(\beta)$, for the pure braid $\beta = \sigma_1^2\sigma_2^{-6}$.

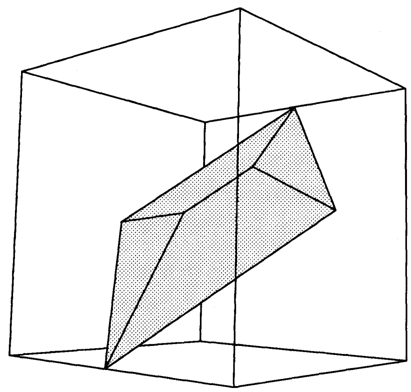

Fig. 2. The fibered face of Thurston norm ball for $M = S^3 - L(\beta)$.

Closed braids. The Teichmüller polynomial leads to a practical algorithm for computing a fibered face $F \subset H^1(M,\mathbb{R})$ from the dynamics on a particular fiber $[S] \in \mathbb{R}_+ \cdot F$.

Closed braids in S^3 provide a natural source of fibered 3-manifolds to which this algorithm can be applied, as we demonstrate in Section 11. For example, Fig. 1 shows a 4-component link $L(\beta)$ obtained by closing the braid $\beta = \sigma_1^2\sigma_2^{-6}$ after passing it through the unknot α. The disk spanned by α meets β in 3 points, providing a fiber $S \subset M = S^3 - L(\beta)$ isomorphic to a 4-times punctured sphere.

The corresponding fibered face is a 3-dimensional polyhedron

$$F \subset H^1(M,\mathbb{R}) \cong \mathbb{R}^4;$$

its projection to $H^1(S,\mathbb{R}) \cong \mathbb{R}^3$ is shown in Fig. 2. Details of this example and others are presented in Section 11.

Comparison with the Alexander polynomial. In [33] we defined a norm $\|\cdot\|_A$ on $H^1(M,\mathbb{R})$ using the Alexander polynomial of M, and established the inequality

$$\|\phi\|_A \leqslant \|\phi\|_T$$

between the Alexander and Thurston norms (when $b_1(M) > 1$). This inequality suggested that the Thurston norm should be refined to polynomial invariant, and Θ_F provides such an invariant for the fibered faces of the Thurston norm ball.

The Alexander polynomial Δ_M and the Teichmüller polynomial Θ_F are compared in Table 1. Both polynomials are attached to modules over $\mathbb{Z}[G]$, namely $A(M)$ and $T(\widetilde{\mathcal{L}})$. These modules give rise to groups of (co)cycles with twisted coefficients, and Δ and Θ_F describe the locus of characters $\rho \in \widehat{G}$ where $\dim Z^1(M,\mathbb{C}_\rho) > 1$ and $\dim Z_2(\mathcal{L},\mathbb{C}_\rho) > 0$ respectively.

Table 1

Alexander	Teichmüller
3-manifold M	Fibered face F for M
Alexander module $A(M)$	Teichmüller module $T(\widetilde{\mathcal{L}})$
$\mathrm{Hom}(A(M), B) = Z^1(M, B)$	$\mathrm{Hom}(T(\widetilde{\mathcal{L}}), B) = Z_2(\widetilde{\mathcal{L}}, B)$
Alexander polynomial Δ_M	Teichmüller polynomial Θ_F
Alexander norm on $H^1(M,\mathbb{Z})$	Thurston norm on $H^1(M,\mathbb{Z})$
$\|\phi\|_A = b_1(\mathrm{Ker}\phi) + p(M)$	$\|\phi\|_T = \inf\{\|\chi(S)\|: [S] = \phi\}$
$\|\phi\|_A = \|\phi\|_T$ for the cohomology class of a fibration $M \to S^1$	
Extended Torelli group of S acts on $H^1(S)$ with twisted coefficients	Extended Torelli group acts on $\mathcal{ML}(S)$ with twisted coefficients

The polynomials Δ and Θ_F are related to the Alexander and Thurston norms on $H^1(M,\mathbb{R})$, and these norms agree on the cohomology classes of fibrations. Moreover, if the lamination $\mathcal{L}$ for the fibered face F has transversally oriented leaves, then Δ_M divides Θ_F and F is also a face of the Alexander norm ball (Section 7).

From a 2-dimensional perspective, the polynomials attached to a fibered manifold M can be described in terms of a mapping-class $\psi \in \mathrm{Mod}(S)$. The description is most uniform for ψ in the *Torelli group* $\mathrm{Tor}(S)$, the subgroup of $\mathrm{Mod}(S)$ that acts trivially on $H = H_1(S,\mathbb{Z})$. By providing ψ with a lift $\widetilde{\psi}$ to the H-covering space of S, we obtain the *extended Torelli group* $\widetilde{\mathrm{Tor}}(S)$, a central extension satisfying:

$$0 \to H_1(S,\mathbb{Z}) \to \widetilde{\mathrm{Tor}}(S) \to \mathrm{Tor}(S) \to 0.$$

The lifted mappings $\widetilde{\psi} \in \widetilde{\mathrm{Tor}}(S)$ preserve twisted coefficients for any $s \in H^1(S,\mathbb{R})$, so we obtain a *linear* representation of $\widetilde{\mathrm{Tor}}(S)$ on $H^1(S,\mathbb{C}_s)$ and a *piecewise-linear* action on $\mathcal{ML}_s(S)$. For example, when S is a sphere with $n+1$ boundary components, the pure braid group P_n is a subgroup of $\widetilde{\mathrm{Tor}}(S)$, and its action on $H^1(S,\mathbb{C}_s)$ is the *Gassner representation* of P_n [6].

Characteristic polynomials for these actions then give the Alexander and Teichmüller invariants Δ_M and Θ_F.

Other foliations. Gabai has shown that every norm-minimizing surface $S \subset M$ is the leaf of a taut foliation $\mathcal{F}$ (see [21]), and the construction of pseudo-Anosov flows transverse to taut foliations is a topic of current research. It would be interesting to obtain polynomial invariants for these more general foliations, and in particular for the non-fibered faces of the Thurston norm ball.

Notes and references. Contributions related to this paper have been made by many authors.

For a pseudo-Anosov mapping with transversally orientable foliations, Fried investigated a twisted Lefschetz zeta-function $\zeta(t,u)$ similar to $\Theta_F(t,u)$. For example, the homology directions of these special pseudo-Anosov mappings can be recovered from the support of $\zeta(t,u)$, just as $\mathbb{R}_+ \cdot F$ can be recovered from Θ_F; and the concavity of $1/\log(K(\phi))$ holds in a general setting. See [18,20].

Laminations, foliations and branched surfaces with affine invariant measures have been studied in [25,13,31,8,38] and elsewhere. The Thurston norm can also be studied using taut foliations [22], branched surfaces [37,34] and Seiberg–Witten theory [27]. Another version of Theorem 1.1 is presented by Thurston in [45, Theorem 5.8].

Background on pseudo-Anosov mappings, laminations and train tracks can be found, for example, in [16], [42, §8.9], [44,4,24,5] and the references therein. Additional notes and references are collected at the end of each section.

2. The module of a lamination

Laminations. Let λ be a Hausdorff topological space. We say λ is an n-dimensional *lamination* if there exists a collection of compact, totally disconnected spaces K_α such that λ is covered by open sets U_α homeomorphic to $K_\alpha \times \mathbb{R}^n$.

The *leaves* of λ are its connected components.

A compact, totally disconnected set $T \subset \lambda$ is a *transversal* for λ if there is an open neighborhood U of T and a homeomorphism

$$(U,T) \cong \big(T \times \mathbb{R}^n, T \times \{0\}\big). \tag{2.1}$$

Any compact open subset of a transversal is again a transversal.

Modules and cycles. We define the *module of a lamination*, $T(\lambda)$, to be the $\mathbb{Z}$-module generated by all transversals $[T]$, modulo the relations:

(i) $[T] = [T'] + [T'']$ if T is the disjoint union of T' and T''; and

(ii) $[T] = [T']$ if there is a neighborhood U of $T \cup T'$ such (2.1) holds for both T and T'.

Equivalently, (ii) identifies transversals that are equivalent under holonomy (sliding along the leaves of the lamination).

For any $\mathbb{Z}$-module B, we define the space of *n-cycles* on an n-dimensional lamination λ with values in B by:

$$Z_n(\lambda, B) = \operatorname{Hom}\big(T(\lambda), B\big).$$

For example, cycles $\mu \in Z_n(\lambda, \mathbb{R})$ correspond to finitely-additive transverse signed measures; the measure of a transversal $\mu(T)$ is holonomy invariant by relation (ii), and it satisfies

$$\mu(T \sqcup T') = \mu(T) + \mu(T')$$

by relation (i).

Action of homeomorphisms. Let $\psi : \lambda_1 \to \lambda_2$ be a homeomorphism between laminations. Then ψ determines an isomorphism

$$\psi^* : T(\lambda_2) \to T(\lambda_1),$$

defined by pulling back transversals:

$$\psi^*([T]) = \big[\psi^{-1}(T)\big].$$

Applying $\operatorname{Hom}(\cdot, B)$, we obtain a pushforward map on cycles,

$$\psi_* : Z_n(\lambda_1, B) \to Z_n(\lambda_2, B),$$

satisfying $(\psi_*(\mu))(T) = \mu(\psi^{-1}(T))$ and thus generalizing the pushforward of measures.

The mapping-torus. Now let $\psi:\lambda\to\lambda$ be a homeomorphism of an n-dimensional lamination to itself. The *mapping torus* $\mathcal{L}$ of ψ is the $(n+1)$-dimensional lamination defined by

$$\mathcal{L}=\lambda\times[0,1]/\langle(x,1)\sim(\psi(x),0)\rangle.$$

The lamination $\mathcal{L}$ fibers over S^1 with fiber λ and monodromy ψ. Since cycles on $\mathcal{L}$ correspond to ψ-invariant cycles on λ, we have:

PROPOSITION 2.1. – *The module of the mapping torus of $\psi:\lambda\to\lambda$ is given by*

$$T(\mathcal{L})=\operatorname{Coker}(\psi^*-I)=T(\lambda)/(\psi^*-I)\big(T(\lambda)\big).$$

Example: $(\mathbb{Z}_p,x+1)$. – Let $\lambda=\mathbb{Z}_p$ be the p-adic integers, considered as a 0-dimensional lamination, and let $\psi:\lambda\to\lambda$ be the map $\psi(x)=x+1$. Then the mapping torus $\mathcal{L}$ of ψ is a 1-dimensional solenoid, satisfying

$$T(\mathcal{L})\cong\mathbb{Z}[1/p],$$

where $\mathbb{Z}[1/p]\subset\mathbb{Q}$ is the subring generated by $1/p$. Indeed, the transversals $T_n=p^n\mathbb{Z}_p$ and their translates generate $T(\lambda)$, so their images $[T_n]$ generate $T(\mathcal{L})$. Since T_n is the union of p translates of T_{n+1}, we have $[T_n]=p[T_{n+1}]$, and therefore $T(\mathcal{L})\cong\mathbb{Z}[1/p]$ by the map sending $[T_n]$ to p^{-n}.

Observe that

$$Z_1(\mathcal{L},\mathbb{R})=\operatorname{Hom}\big(\mathbb{Z}[1/p],\mathbb{R}\big)=\mathbb{R},$$

showing there is a unique finitely-additive probability measure on $\mathbb{Z}_p$ invariant under $x\mapsto x+1$.

Twisted cycles. Next we describe cycles with twisted coefficients.

Let $\widetilde{\lambda}\to\lambda$ be a Galois covering space with abelian deck group G. Then G acts on $T(\widetilde{\lambda})$, making the latter into a module over the *group ring* $\mathbb{Z}[G]$. Any G-module B determines a bundle of twisted local coefficients over λ, and we define

$$Z_n(\lambda,B)=\operatorname{Hom}_G\big(T(\widetilde{\lambda}),B\big).$$

For example, any homomorphism

$$\rho:G\to\mathbb{R}_+$$

makes $\mathbb{R}$ into a module $\mathbb{R}_\rho$ over $\mathbb{Z}[G]$. The cycles $\mu\in Z_n(\lambda,\mathbb{R}_\rho)$ can then be interpreted as either:

(i) cycles on $\widetilde{\lambda}$ satisfying $g_*\mu=\rho(g)\mu(T)$ for all $g\in G$; or

(ii) cycles on λ with values (locally) in the real line bundle over λ determined by $\rho\in H^1(\lambda,\mathbb{R}_+)$.

Geodesic laminations on surfaces. Now let S be a compact orientable surface with $\chi(S)<0$. Fix a complete hyperbolic metric of finite volume on $\operatorname{int}(S)$.

A *geodesic lamination* $\lambda\subset S$ is a compact lamination whose leaves are hyperbolic geodesics.

A *train track* $\tau\subset S$ is a finite 1-complex such that

(i) every $x\in\tau$ lies in the interior of a smooth arc embedded in τ,

(ii) any two such arcs are tangent at x, and

(iii) for each component U of $S-\tau$, the double of U along the smooth part of ∂U has negative Euler characteristic.

A geodesic lamination λ is *carried* by a train track τ if there is a continuous *collapsing map* $f:\lambda\to\tau$ such that for each leaf $\lambda_0\subset\lambda$,

(i) $f|\lambda_0$ is an immersion, and
(ii) λ_0 is the geodesic representative of the path or loop $f:\lambda_0 \to S$.
Collapsing maps between train tracks are defined similarly. Every geodesic lamination is carried by some train track [24, 1.6.5].

The *vertices* (or switches) of a train track, $V \subset \tau$, are the points where 3 or more smooth arcs come together. The *edges* E of τ are the components of $\tau - V$; some 'edges' may be closed loops.

A train track is *trivalent* if only 3 edges come together at each vertex. A trivalent train track has *minimal complexity* for λ if it has the minimal number of edges among all trivalent τ carrying λ.

The module of a train track. Let $T(\tau)$ denote the $\mathbb{Z}$-module generated by the edges E of τ, modulo the relations

$$[e_1] + \cdots + [e_r] = [e'_1] + \cdots + [e'_s]$$

for each vertex $v \in V$ with incoming edges (e_i) and outgoing edges (e'_j). (The distinction between incoming and outgoing edges depends on the choice of a direction along τ at v.) Since there is one relation for each vertex, we obtain a presentation for $T(\tau)$ of the form:

$$\mathbb{Z}^V \xrightarrow{D} \mathbb{Z}^E \to T(\tau) \to 0. \tag{2.2}$$

As for a geodesic lamination, we define the 1-cycles on τ with values in B by

$$Z_1(\tau, B) = \mathrm{Hom}\big(T(\tau), B\big).$$

THEOREM 2.2. – *Let $\lambda \subset S$ be a geodesic lamination, and let τ be a train track carrying λ with minimal complexity. Then there is a natural isomorphism*

$$T(\lambda) \cong T(\tau).$$

COROLLARY 2.3. – *For any geodesic lamination λ, the module $T(\lambda)$ is finitely-generated.*

COROLLARY 2.4. – *If λ is connected and carried by a train track τ of minimal complexity, then we have*

$$T(\lambda) \cong \mathbb{Z}^{|\chi(\tau)|} \oplus \begin{cases} \mathbb{Z} & \text{if } \tau \text{ is orientable,} \\ \mathbb{Z}/2 & \text{otherwise.} \end{cases}$$

(Here $\chi(\tau)$ is the Euler characteristic of τ.)

Proof. – Use the fact that the transpose $D^* : \mathbb{Z}^E \to \mathbb{Z}^V$ of the presentation matrix (2.2) for $T(\tau)$ behaves like a boundary map, and $\sum n_i v_i$ is in the image of D^* iff $\sum n_i = 0$ (in the orientable case) or $\sum n_i = 0 \pmod 2$ (in the non-orientable case). □

Proof of Theorem 2.2. – Let $\tau_0 = \tau$. The collapsing map $f_0 : \lambda \to \tau_0$ determines a map of modules

$$f_0^* : T(\tau_0) \to T(\lambda)$$

sending each edge $e \in E$ to the transversal defined by

$$T = f_0^*(e) = f_0^{-1}(x)$$

for any $x \in e$. We will show f_0^* is an isomorphism.

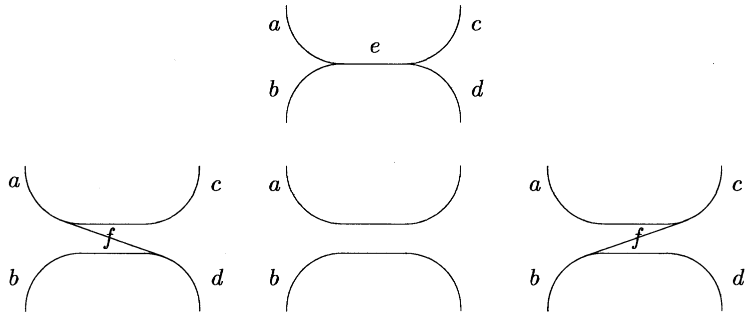

Fig. 3. Three possible splittings.

We begin by using λ to guide a sequence of splittings of τ_0 into finer and finer train tracks τ_n, converging to λ itself, in the sense that there are collapsing maps $f_n : \lambda \to \tau_n$ converging to the inclusion $\lambda \subset S$. We will also have collapsing maps $g_n : \tau_{n+1} \to \tau_n$ such that $f_n = g_n \circ f_{n+1}$. Each τ_n will be of minimal complexity.

The train track τ_{n+1} is constructed from τ_n as follows. First, observe that each edge of τ_n carries at least one leaf of λ (since τ_n has minimal complexity). Thus each cusp of a component U of $S - \tau$ (where tangent edges a, b in τ come together) corresponds to pair of adjacent leaves λ_a, λ_b of λ. Choose a particular cusp, and split τ_n between a and b so that the train track continues to follow λ_a and λ_b. When we split past a vertex, we obtain a new trivalent train track τ_{n+1}. There are 3 possible results of splitting, recorded in Fig. 3.

In the middle case, the leaves λ_1 and λ_2 diverge, and we obtain a train track τ_{n+1} carrying λ but with fewer edges than τ_n; this is impossible, since τ_n has minimal complexity.

In the right and left cases, we obtain a train track τ_{n+1} of the same complexity as τ_n, with a natural collapsing map $g_{n+1} : \tau_{n+1} \to \tau_n$. Since the removed and added edges e and f are both in the span of $\langle a, b, c, d \rangle$, the module map

$$g_n^* : T(\tau_n) \to T(\tau_{n+1}) \tag{2.3}$$

is an isomorphism.

By repeatedly splitting every cusp of $S - \tau$, we obtain train tracks with longer and longer edges, following the leaves of λ more and more closely; thus the collapsing maps can be chosen such that $f_n : \lambda \to \tau_n$ converges to the identity. Compare [42, Proposition 8.9.2], [24, §2].

To prove $T(\lambda) \cong T(\tau_0)$, we will define a map

$$\phi : T(\lambda) \to T_\infty = \varinjlim T(\tau_n)$$

(where the direct limit is taken with respect to the collapsing maps g_n^*). Given any transversal T to λ, there is a neighborhood U of T in λ homeomorphic to $T \times \mathbb{R}$. Then for all $n \gg 0$, we have

$$\sup_{x \in \lambda} d\big(f_n(x), x\big) < d(T, \partial U),$$

and thus all the leaves of λ carried by $\tau \cap U$ are accounted for by T. Therefore T is equivalent to a finite sum of edges in $T(\tau_n)$:

$$f_n^*\big([e_1] + \cdots + [e_i]\big) = [T],$$

and we define $\phi(T) = [e_1] + \cdots + [e_i]$.

It is now straightforward to verify that ϕ is a map of modules, inverting the map $T_\infty \to T(\lambda)$ obtained as the inverse limit of the collapsings $f_n^* : T(\tau_n) \to T(\lambda)$. But the maps g_n^* of (2.3) are isomorphisms, so we have $T(\lambda) \cong T_\infty \cong T(\tau_0)$. $\square$

Twisted train tracks. Train tracks also provide a convenient description of twisted cycles on a geodesic lamination.

Let $\lambda \subset S$ be a geodesic lamination carried by a train track τ. Let

$$\pi : \widetilde{S} \to S$$

be a Galois covering space with abelian deck group G. We can then construct modules $T(\widetilde{\lambda})$ and $T(\widetilde{\tau})$ attached to the induced covering spaces of λ and τ. The deck group acts naturally on $\widetilde{\lambda}$ and $\widetilde{\tau}$, so we obtain modules over the group ring $\mathbb{Z}[G]$. The arguments of Theorem 2.2 can then be applied to the lift of a collapsing map $f : \lambda \to \tau$, to establish:

THEOREM 2.5. – *The $\mathbb{Z}[G]$-modules $T(\widetilde{\lambda})$ and $T(\widetilde{\tau})$ are naturally isomorphic. A choice of lifts for the edges and vertices (E, V) of τ to $\widetilde{\tau}$ determines a finite presentation*

$$\mathbb{Z}[G]^V \xrightarrow{D} \mathbb{Z}[G]^E \to T(\widetilde{\tau}) \to 0$$

for $T(\widetilde{\tau})$ as a $\mathbb{Z}[G]$-module.

Example. – Let S be a sphere with 4 disks removed. Let $\widetilde{S} \to S$ be the maximal abelian covering of S, with deck group

$$G = H_1(S, \mathbb{Z}) = \langle A, B, C \rangle \cong \mathbb{Z}^3$$

generated by counterclockwise loops around 3 boundary components of S.

Let $\tau \subset S$ be the train track shown in Fig. 4. Then for suitable lifts of the edges of τ, the module $T(\widetilde{\tau})$ is generated over $\mathbb{Z}[G]$ by $\langle a, b, c, d, e, f \rangle$, with the relations:

$$\begin{aligned} b &= a + d, \\ A^{-1}d &= a + e, \\ b &= c + f, \\ c &= B^{-1}e + Cf, \end{aligned}$$

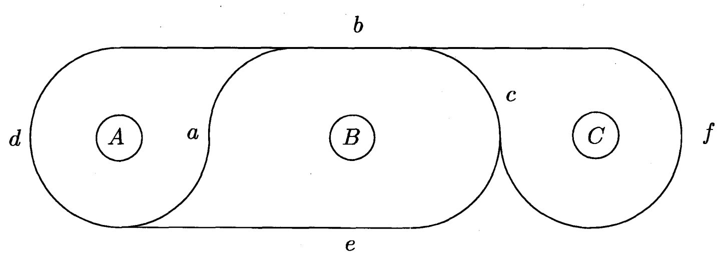

Fig. 4. Presenting a track track.

coming from the 4 vertices of τ. Simplifying, we find $T(\widetilde{\tau})$ is generated by $\langle a,b,c\rangle$ with the single relation

$$(1+A)a + AB(1+C)c = (1+ABC)b.$$

This relation shows, for example, that

$$\dim Z_1(\tau,\mathbb{C}_\rho) = \begin{cases} 3 & \text{if } \rho(A)=\rho(B)=\rho(C)=-1, \\ 2 & \text{otherwise,} \end{cases}$$

for any 1-dimensional representation $\rho: G \to \mathbb{C}^*$.

Notes.

(1) The usual (positive, countably-additive) transverse measures on a geodesic lamination λ generally span a *proper* subspace $M(\lambda)$ of the space of cycles $Z_1(\lambda,\mathbb{R})$. Indeed, a generic measured lamination λ on a closed surface cuts S into ideal triangles, so any train track τ carrying λ is the 1-skeleton of a triangulation of S. At the same time λ is typically uniquely ergodic, and therefore

$$\dim M(\lambda) = 1 < \dim Z_1(\lambda,\mathbb{R}) = \dim Z_1(\tau,\mathbb{R}) = 6g(S) - 6.$$

(2) Bonahon has shown that cycles $\mu \in Z_1(\lambda,\mathbb{R})$ correspond to transverse invariant *Hölder distributions*; that is, the pairing

$$\langle f,\mu\rangle = \int_T f(x)\,d\mu(x)$$

can be defined for any transversal T and Hölder continuous function $f: T \to \mathbb{R}$ [8, Theorem 17]. See also [8, Theorem 11] for a variant of Theorem 2.2, and [7] for additional results.

(3) One can also describe $Z_1(\lambda,\mathbb{R})$ as a space of closed *currents* carried by λ, since these cycles are distributional in nature and they need not be compactly supported (when λ is noncompact).

3. The Teichmüller polynomial

In this section we define the Teichmüller polynomial Θ_F of a fibered face F, and establish the *determinant formula*

$$\Theta_F(t,u) = \det\bigl(uI - P_E(t)\bigr) / \det\bigl(uI - P_V(t)\bigr).$$

We begin by introducing some notation that will be used throughout the sequel.

Let M^3 be a compact, connected, orientable, irreducible, atoroidal 3-manifold. Let $\pi: M \to S^1$ be a fibration with fiber $S \subset M$ and monodromy ψ. Then:

- S is a compact, orientable surface with $\chi(S) < 0$, and
- $\psi: S \to S$ is a pseudo-Anosov map, with an expanding invariant lamination
- $\lambda \subset S$, unique up to isotopy.

Adjusting ψ by isotopy, we can assume $\psi(\lambda) = \lambda$.

By the general theory of pseudo-Anosov mappings, there is a *positive* transverse measure $\mu \in Z_1(\lambda, \mathbb{R})$, unique up to scale, and $\psi_*(\mu) = k\mu$ for some $k > 1$. Then $[\Lambda] = [(\lambda, \mu)]$ is a fixed-point of ψ in the space of projective measured laminations $\mathbb{P}\mathcal{ML}(S)$. Moreover $[\psi^n(\gamma)] \to [\Lambda]$ for every simple closed curve $[\gamma] \in \mathbb{P}\mathcal{ML}(S)$.

Associated to (M, S) we also have:

- $\mathcal{L} \subset M$, the mapping torus of $\psi: \lambda \to \lambda$, and
- $F \subset H^1(M, \mathbb{R})$, the open face of unit ball in the Thurston norm with $[S] \in \mathbb{R}_+ \cdot F$.

We say F is a *fibered face* of the Thurston norm ball, since every point in $H^1(M, \mathbb{Z}) \cap \mathbb{R}_+ \cdot F$ is represented by a fibration of M over the circle [43, Theorem 5].

The flow lines of ψ. Using ψ we can present M in the form

$$M = (S \times \mathbb{R}) / \langle (s, t) \sim (\psi(s), t - 1) \rangle,$$

and the lines $\{s\} \times \mathbb{R}$ descend to the leaves of an oriented 1-dimensional foliation Ψ of M, the *flow lines* of ψ. The 2-dimensional lamination $\mathcal{L} \subset M$ is swept out by the leaves of Ψ passing through λ.

Invariance of $\mathcal{L}$. We now show $\mathcal{L}$ depends only on F.

THEOREM 3.1 (Fried). – *Let $[S'] \in \mathbb{R}_+ \cdot F$ be a fiber of M. Then after an isotopy,*

- *S' is transverse to the flow lines Ψ of ψ, and*
- *the first return map of the flow coincides with the pseudo-Anosov monodromy $\psi': S' \to S'$.*

For this result, see [17, Theorem 7 and Lemma] and [19].

COROLLARY 3.2. – *Any two fibers $[S], [S'] \in \mathbb{R}_+ \cdot F$ determine the same lamination $\mathcal{L} \subset M$ (up to isotopy).*

Proof. – Consider two fibers S and S' for the same face F. Let ψ, ψ' denote their respective monodromy transformations, λ, λ' their expanding laminations, and $\mathcal{L}, \mathcal{L}' \subset M$ the mapping tori of λ, λ'.

By the theorem above, we can assume S' is transverse to Ψ and hence transverse to $\mathcal{L}$.

Let $\mu' = \mathcal{L} \cap S'$. Then $\mu' \subset S'$ is a ψ'-invariant lamination with no isolated leaves. By invariance, μ' must contain the expanding or contracting lamination of ψ'. Since flowing along Ψ expands the leaves of $\mathcal{L}$, we find $\mu' \supset \lambda'$.

By irreducibility of ψ', the complementary regions $S' - \lambda'$ are n-gons or punctured n-gons. In such regions, the only geodesic laminations are isolated leaves running between cusps. Since μ' has no isolated leaves, we conclude that $\mu' = \lambda'$ and thus $\mathcal{L} = \mathcal{L}'$ (up to isotopy). □

Modules and the Teichmüller polynomial. By the preceding corollary, the lamination $\mathcal{L} \subset M$ depends only on F. Associated to the pair (M, F) we now have:

- $G = H_1(M, \mathbb{Z})/\text{torsion}$, a free abelian group;
- $\widetilde{M} \to M$, the Galois covering space corresponding to $\pi_1(M) \to G$;
- $\widetilde{\mathcal{L}} \subset \widetilde{M}$, the preimage of the lamination $\mathcal{L}$ determined by F; and
- $T(\widetilde{\mathcal{L}})$, the $\mathbb{Z}[G]$-module of transversals to $\widetilde{\mathcal{L}}$.

Since $\mathcal{L}$ is compact, $T(\mathcal{L})$ is finitely-generated and $T(\widetilde{\mathcal{L}})$ is finitely-presented over the ring $\mathbb{Z}[G]$.

Choose a presentation

$$\mathbb{Z}[G]^r \xrightarrow{D} \mathbb{Z}[G]^s \to T(\widetilde{\mathcal{L}}) \to 0,$$

and let $I \subset \mathbb{Z}[G]$ be the ideal generated by the $s \times s$ minors of D. The ideal I is the *Fitting ideal* of the module $T(\widetilde{\mathcal{L}})$, and it is independent of the choice of presentation; see [28, Ch. XIII, §10], [36].

Using the fact that $\mathbb{Z}[G]$ is a unique factorization domain, we define the *Teichmüller polynomial* of (M,F) by

$$\Theta_F = \gcd(f\colon\ f \in I) \in \mathbb{Z}[G]. \tag{3.1}$$

The polynomial Θ_F is well-defined up to multiplication by a unit $\pm g \in \mathbb{Z}[G]$, and it depends only on (M,F).

Note that $\mathbb{Z}[G]$ can be identified with a ring of complex algebraic functions on the character variety

$$\widehat{G} = \mathrm{Hom}(G,\mathbb{C}^*)$$

by setting $(\sum a_g \cdot g)(\rho) = \sum a_g \rho(g)$.

THEOREM 3.3. – *The locus $\Theta_F(\rho) = 0$ is the largest hypersurface $V \subset \widehat{G}$ such that* $\dim Z_2(\mathcal{L},\mathbb{C}_\rho) > 0$ *for all $\rho \in V$.*

Proof. – A character ρ belongs to the zero locus of the ideal $I \Leftrightarrow$ the presentation matrix $\rho(M)$ has rank $r < s \Leftrightarrow$ we have

$$\dim_{\mathbb{C}} Z_2(\mathcal{L},\mathbb{C}_\rho) = \dim \mathrm{Hom}\big(T\big(\widetilde{\mathcal{L}}\big),\mathbb{C}_\rho\big) = s - r > 0;$$

and the greatest common divisor of the elements of I defines the largest hypersurface contained in $V(I)$. □

Computing the Teichmüller polynomial. We now describe a procedure for computing Θ_F as an explicit Laurent polynomial.

Consider again a fiber $S \subset M$ with monodromy ψ and expanding lamination λ. Associated to this data we have:

- $H = \mathrm{Hom}(H^1(S,\mathbb{Z})^\psi,\mathbb{Z}) \cong \mathbb{Z}^b$, the dual of the ψ-invariant cohomology of S;
- $\widetilde{S} \to S$, the Galois covering space corresponding to the natural map

$$\pi_1(S) \to H_1(S,\mathbb{Z}) \to H;$$

- $\tau \subset S$, a ψ-invariant train track carrying λ; and
- $\widetilde{\lambda}, \widetilde{\tau} \subset \widetilde{S}$, the preimages of $\lambda, \tau \subset S$.

Note that pullback by $S \subset M$ determines a surjection $H^1(M,\mathbb{Z}) \to H^1(S,\mathbb{Z})^\psi$, and hence a natural inclusion

$$H \subset G = H_1(M,\mathbb{Z})/\text{torsion} = \mathrm{Hom}\big(H^1(M,\mathbb{Z}),\mathbb{Z}\big).$$

Alternatively, we can regard $\widetilde{S}$ as a component of the preimage of S in the covering $\widetilde{M} \to M$ with deck group G; then $H \subset G$ is the stabilizer of $\widetilde{S} \subset \widetilde{M}$.

Now *choose* a lift

$$\widetilde{\psi}\colon \widetilde{S} \to \widetilde{S}$$

of the pseudo-Anosov mapping ψ. Then we obtain a splitting

$$G = H \oplus \mathbb{Z}\widetilde{\Psi},$$

where $\widetilde{\Psi} \in G$ acts on $\widetilde{M} = \widetilde{S} \times \mathbb{R}$ by

$$\widetilde{\Psi}(s,t) = \big(\widetilde{\psi}(s), t-1\big). \tag{3.2}$$

If we further choose a basis $(t_1,\dots,t_b)$ for H, written multiplicatively, and set $u=[\widetilde{\Psi}]$, then we obtain an isomorphism

$$\mathbb{Z}[G]\cong\mathbb{Z}\big[t_1^{\pm 1},\dots t_b^{\pm 1},u^{\pm 1}\big]$$

between the group ring of G and the ring of integral Laurent polynomials in the variables t_i and u.

Remark. – Under the fibration $M\to S^1$, the element $u\in H_1(M,\mathbb{Z})/\text{torsion}$ maps to -1 in $H_1(S^1,\mathbb{Z})\cong\mathbb{Z}$, as can be seen from (3.2).

A presentation for $T(\widetilde{\mathcal{L}})$. The next step in the computation of Θ_F is to obtain a concrete description of the module $T(\widetilde{\mathcal{L}})$.

We begin by using the train track τ to give a presentation of $T(\widetilde{\lambda})$ over $\mathbb{Z}[H]$. Let E and V denote the sets of edges and vertices of the train track $\tau\subset S$. By choosing a lift of each edge and vertex to the covering space $\widetilde{S}\to S$ with deck group H, we can identify the edges and vertices of $\widetilde{\tau}$ with the products $H\times E$ and $H\times V$. These lifts yield a presentation

$$\mathbb{Z}[H]^V\xrightarrow{D}\mathbb{Z}[H]^E\to T(\widetilde{\tau})\to 0 \tag{3.3}$$

for $T(\widetilde{\tau})\cong T(\widetilde{\lambda})$ as a $\mathbb{Z}[H]$-module.

Since τ is ψ-invariant, there is an H-invariant collapsing map

$$\widetilde{\psi}(\widetilde{\tau})\to\widetilde{\tau}.$$

By expressing each edge in the target as a sum of the edges in the domain which collapse to it, we obtain a natural map of $\mathbb{Z}[H]$-modules

$$P_E:\mathbb{Z}[H]^E\to\mathbb{Z}[H]^E.$$

There is a similar map P_V on vertices.

We can regard P_E and P_V as matrices $P_E(t), P_V(t)$ whose entries are Laurent polynomials in $t=(t_1,\dots,t_b)$. In the terminology of Appendix A, such a matrix is *Perron–Frobenius* if it has a power such that every entry is a nonzero Laurent polynomial with positive coefficients.

THEOREM 3.4. – *$P_E(t)$ is a Perron–Frobenius matrix of Laurent polynomials.*

Proof. – For any $e,f\in E$, the matrix entry $(P_E)_{ef}$ is a sum of monomials t^α for all α such that $\widetilde{\psi}(\alpha\cdot e)$ collapses to f. Thus each nonzero entry is a positive, integral Laurent monomial, and since ψ is pseudo-Anosov there is some iterate $P_E^N(t)$ with every entry nonzero. □

The matrices $P_E(t)$ and $P_V(t)$ are compatible with the presentation (3.3) for $T(\widetilde{\tau})$, so we obtain a commutative diagram

$$\begin{array}{ccccccc}
\mathbb{Z}[H]^V & \longrightarrow & \mathbb{Z}[H]^E & \longrightarrow & T(\widetilde{\tau}) & \longrightarrow & 0\\
\downarrow{\scriptstyle P_V(t)} & & \downarrow{\scriptstyle P_E(t)} & & \downarrow{\scriptstyle P(t)} & & \\
\mathbb{Z}[H]^V & \longrightarrow & \mathbb{Z}[H]^E & \longrightarrow & T(\widetilde{\tau}) & \longrightarrow & 0.
\end{array} \tag{3.4}$$

Here $P(t)=\psi^*$ under the natural identification $T(\widetilde{\tau})=T(\widetilde{\lambda})$.

The next result makes precise the fact that twisted cycles on $\mathcal{L}$ correspond to ψ-invariant twisted cycles on λ (compare Proposition 2.1).

THEOREM 3.5. – *There is a natural isomorphism*

$$T(\widetilde{\mathcal{L}}) \cong \operatorname{Coker}(uI - P(t))$$

as modules over $\mathbb{Z}[G]$.

Here $uI - P(t)$ is regarded as an endomorphism of $T(\widetilde{\tau}) \otimes \mathbb{Z}[u]$ over $\mathbb{Z}[G] = \mathbb{Z}[H] \otimes \mathbb{Z}[u]$.

Proof. – The lamination $\mathcal{L}$ fibers over S^1 with fiber λ and monodromy $\psi : \lambda \to \lambda$, so we can regard $\widetilde{\mathcal{L}}$ as $\widetilde{\lambda} \times \mathbb{R}$, equipped with the action of $G = H \oplus \mathbb{Z}\widetilde{\Psi}$. The product structure on $\widetilde{\mathcal{L}}$ gives an isomorphism $T(\widetilde{\mathcal{L}}) \cong T(\widetilde{\lambda}) \cong T(\widetilde{\tau})$ as modules over $\mathbb{Z}[H]$, so to describe $T(\widetilde{\mathcal{L}})$ as a $\mathbb{Z}[G]$-module we need only determine the action of u under this isomorphism. But u acts on $\widetilde{\lambda} \times \mathbb{R}$ by $(x,t) \mapsto (\widetilde{\psi}(x), t-1)$, so for any transversal $T \in T(\widetilde{\lambda})$ we have $uT = \widetilde{\psi}^*(T) = P(t)T$, and the theorem follows. □

The determinant formula. The main result of this section is:

THEOREM 3.6. – *The Teichmüller polynomial of the fibered face* F *is given by*:

$$\Theta_F(t,u) = \frac{\det(uI - P_E(t))}{\det(uI - P_V(t))} \tag{3.5}$$

when $b_1(M) > 1$.

Remarks. –

(1) If $b_1(M) = 1$ then the numerator must be multiplied by $(u-1)$ if τ is orientable. Compare Corollary 2.4.

(2) To understand the determinant formula, recall that by Theorem 3.3, the locus $\Theta_F(t,u) = 0$ in $\widehat{G}$ consists of characters for which we have

$$\dim Z_2(\mathcal{L}, \mathbb{C}_\rho) > 0.$$

Now a cocycle for $\mathcal{L}$ is the same as a ψ-invariant cocycle for λ, so we expect to have $\Theta_F(t,u) = \det(uI - P(t))$. But the module $T(\widetilde{\lambda})$ is not quite free in general, so we need the formula above to make sense of the determinant.

Proof of Theorem 3.6. – To simplify notation, let $A = \mathbb{Z}[G]$, let T be the A-module $T(\widetilde{\lambda}) \otimes \mathbb{Z}[G]$, and let $P : T \to T$ be the automorphism $P = \widetilde{\psi}^*$.

Let K denote the field of fractions of A. For each $f \in A$, $f \neq 0$, we can invert f to obtain the ring $A_f = A[1/f] \subset K$, and there is a naturally determined A_f-module T_f with automorphism P_f coming from P (see e.g. [2, Ch. 3]). The presentation (3.3) for T determines a presentation

$$A_f^V \xrightarrow{D_f} A_f^E \to T_f \to 0 \tag{3.6}$$

for T_f.

Now let $\Theta = \Theta_F(t,u) \in A$ be the Teichmüller polynomial for (M,F) (defined by (3.1)), and define $\Delta \in K$ by

$$\Delta = \Delta(t,u) = \frac{\det(uI - P_E(t))}{\det(uI - P_V(t))}.$$

Our goal is to show $\Theta = \Delta$ up to a unit in A. The method is to show that $\Theta = \Delta$ up to a unit in A_f for many different f. We break the argument up into 5 main steps.

I. *The map $D_f : A_f^V \to A_f^E$ is injective whenever $f = (t_i^2 - 1)g$ for some i, $1 \leqslant i \leqslant b$, and some $g \neq 0$ in A.*

To see this assertion, we use the dynamics of pseudo-Anosov maps. It is enough to show that the transpose $D_f^* : A_f^E \to A_f^V$ is surjective — then D_f^* has a right inverse, so D_f has a left inverse. We prefer to work with D_f^* since it behaves like a geometric boundary map.

Given a basis element t_i for $H = \mathrm{Hom}(H^1(S,\mathbb{Z})^\psi, \mathbb{Z})$, choose an oriented simple closed curve $\gamma \subset S$ such that $[\gamma] = t_i$. (Such a γ exists because every t_i is represented by a primitive homology class on S, and every such class contains a simple closed curve.) Then $[\psi^n(\gamma)] = t_i$ as well, since ψ fixes all homology classes in H. On the other hand, for n sufficiently large, $\psi^n(\gamma)$ is close to the expanding lamination λ of ψ. Thus by replacing γ with $\psi^n(\gamma)$, $n \gg 0$, we can assume that γ is carried with full support by τ.

Now choose any vertex $v \in V$, and lift γ to an edge path $\tilde{\gamma} \subset \tilde{\tau}$, starting at the (previously fixed) lift $\tilde{v}$ of v. Since $[\gamma] = t_i$, the arc $\tilde{\gamma}$ connects v to $t_i v$. Letting $e \in A^E$ denote the weighted edges occurring in $\tilde{\gamma}$, we then have

$$D^*[e] = (\pm t_i - 1)v \in A^V,$$

where the sign depends on the orientation of the switch at v.

In any case, when $f = (t_i^2 - 1)g$, the factor $(\pm t_i - 1)$ is a unit in A_f, and thus D_f^* is surjective and D_f is injective.

II. *If T_f is a free A_f-module and D_f is injective, then $\Theta = \Delta$ up to a unit in A_f.*

Indeed, if T_f is free then

$$T_f \xrightarrow{uI-P} T_f \to T(\widetilde{\mathcal{L}})_f \to 0$$

presents $T(\widetilde{\mathcal{L}})_f$ as a quotient of free modules. It is not hard to check that the formation of the Fitting ideal commutes with the inversion of f, and thus $(\Theta) \subset A_f$ is the smallest principal ideal containing the Fitting ideal of $T(\widetilde{\mathcal{L}})_f$. From the presentation of $T(\widetilde{\mathcal{L}})_f$ above, we have $\Theta = \det(uI - P(t))$ up to a unit in A_f.

To bring Δ into play, note that by injectivity of D_f we have an exact sequence:

$$0 \to A_f^V \xrightarrow{D_f} A_f^E \to T_f \to 0.$$

Since T_f is free, this sequence splits, and thus P_E can be expressed as a block triangular matrix with P_V and P on the diagonal. Therefore

$$\det\bigl(uI - P_V(t)\bigr)\det\bigl(uI - P(t)\bigr) = \det\bigl(uI - P_E(t)\bigr),$$

which gives $\Theta = \Delta$ up to a unit in A_f.

III. *The set*

$$I' = \{f \in A\colon\ T_f \text{ is free and } D_f \text{ is injective}\}$$

generates an ideal $I \subset A$ containing $(t_i^2 - 1)$ for $i = 1, \ldots, b$.

Let $f = (t_i^2 - 1)$, so D_f is injective. Then the $|V| \times |V|$-minors of D generate the ideal (1) in A_f.

Consider a typical minor $(V \times E')$ of D with determinant $g \neq 0$, where $E = E' \sqcup E''$. Set $h = fg$. Then the composition

$$A_h^V \xrightarrow{D_h} A_h^E \to A_h^{E'}$$

is an isomorphism (since its determinant is now a unit). Therefore the projection $A_h^{E''} \to T_h$ is an isomorphism, so T_h is free.

Since the minor determinants g generate the ideal (1) in A_f, we conclude that $f = (t_i^2 - 1)$ belongs to the ideal I generated by all such $h = fg$.

IV. *There are $a, c \in A$ such that $(a) \supset I$, $(c) \supset I$ and*

$$a\Theta = c\Delta. \tag{3.7}$$

Write $\Delta/\Theta = a/c \in K$ as a ratio of $a, c \in A$ with no common factor. By definition, for any $f \in I'$ we have $\Theta = \Delta$ up to a unit in A_f; therefore $a/c = d/f^n$ for some unit $d \in A^*$ and $n \in \mathbb{Z}$. Since $\gcd(a, c) = 1$, a and c are divisors of f. As $f \in I'$ was arbitrary, the principal ideals generated by a and c both contain I', and hence I.

V. *We have $\Theta = \Delta$ up to a unit in A.*

Let (p) be the smallest principal ideal satisfying

$$(p) \supset I \supset \left(t_1^2 - 1, \ldots, t_b^2 - 1\right)$$

(the second inclusion by (III) above). If the rank b of $H^1(S, \mathbb{Z})^\psi$ is 2 or more, then $\gcd(t_1^2 - 1, \ldots, t_b^2 - 1) = 1$ and thus $(p) = 1$. Since a, c in (3.7) generate principal ideals containing I, they are both units and we are done.

To finish, we treat the case $b = 1$. In this case we have $(p) \supset (t_1^2 - 1)$, so we can only conclude that $\Theta = \Delta$ up to a factors of $(t_1 - 1)$ and $(t_1 + 1)$.

But Δ and Θ have no such factors. Indeed, Δ is a ratio of monic polynomials of positive degree in u, so it has no factor that depends only on t_1.

Similarly, if we specialize to $(t_1, u) = (1, n)$ (by a homomorphism $\phi : A \to \mathbb{Z}$), then $P : T \to T$ becomes an endomorphism of a finitely generated abelian group, and $T(\mathcal{L}) = \mathrm{Coker}(uI - P)$ specializes to the group $K = \mathrm{Coker}(nI - P)$. For $n \gg 0$, the image of $(uI - P)$ has finite index in T, so K is a finite group. Thus $(\phi(\Theta)) = (n)$, the annihilator of K; in particular, $\phi(\Theta) \neq 0$. This shows $(t_1 - 1)$ does not divide Θ. The same argument proves $\gcd(\Theta, t_1 + 1) = 1$, and thus $\Theta = \Delta$ up to a unit in A. □

Notes. The train track τ in Fig. 4 provides a typical example where the module $T(\widetilde{\tau})$ is not free over $\mathbb{Z}[H]$. Indeed, letting $H = H_1(S, \mathbb{Z}) \cong \mathbb{Z}^3$, we showed in Section 2 that the dimension of

$$Z_1(\tau, \mathbb{C}_\rho) = \mathrm{Hom}(T, \mathbb{C}_\rho)$$

jumps at $\rho = (-1, -1, -1)$, while its dimension would be constant if T were a free module. Thus $f \in \mathbb{Z}[H]$ must vanish at $\rho = (-1, -1, -1)$ for $T(\tau)_f$ to be free — showing the ideal I in the proof above contains $(t_1 + 1, t_2 + 1, t_3 + 1)$.

4. Symplectic symmetry

In this section we show the characteristic polynomial of a pseudo-Anosov map $\psi : S \to S$ is symmetric. This symmetry arises because ψ preserves a natural symplectic structure on $\mathcal{ML}(S)$.

We then show the Teichmüller polynomial Θ_F packages all the characteristic polynomials of fibers $[S] \in \mathbb{R}_+ \cdot F$, and thus Θ_F is also symmetric.

Symmetry. Let λ be the expanding lamination of a pseudo-Anosov mapping $\psi : S \to S$. The *characteristic polynomial* of ψ is given by $p(k) = \det(kI - P)$, where

$$P : Z_1(\lambda, \mathbb{R}) \to Z_1(\lambda, \mathbb{R})$$

is the induced map on cycles, $P = \psi_*$.

THEOREM 4.1. – *The characteristic polynomial $p(k)$ of a pseudo-Anosov mapping is symmetric; that is, $p(k) = k^d p(1/k)$ where $d = \deg(p)$.*

Proof. – Since ψ is pseudo-Anosov, each component of $S - \lambda$ is an ideal polygon, possibly with one puncture. Since these polygons and their ideal vertices are permuted by ψ, we can choose $n > 0$ such that ψ^n preserves each complementary component D of $S - \lambda$ and fixes its ideal vertices.

By Theorem 2.2, there is a natural isomorphism $Z_1(\lambda, \mathbb{R}) \cong Z_1(\tau, \mathbb{R})$, where τ is a ψ-invariant train track carrying λ. By [24, Theorem 1.3.6], there exists a *complete* train track τ' containing τ. The train track τ is completed to τ' by adding a maximal set of edges joining the cusps of the complementary regions $S - \tau$. Since ψ^n fixes these cusps, $\psi^n(\tau')$ is carried by τ'.

Now recall that the vector space $Z_1(\tau', \mathbb{R})$ can be interpreted as a tangent space to $\mathcal{ML}(S)$, and hence it carries a natural *symplectic form* ω. If τ' is orientable (which only happens on a punctured torus), then ω is just the pullback of the intersection form on S under the natural map

$$Z_1(\tau', \mathbb{R}) \to H_1(S, \mathbb{R}).$$

If τ' is nonorientable, then ω is defined using the intersection pairing on a covering of S branched over the complementary regions $S - \tau'$; see [24, §3.2].

For brevity of notation, let

$$(V \subset V') = \big(Z_1(\tau, \mathbb{R}) \subset Z_1(\tau', \mathbb{R})\big),$$

and let

$$P = \psi_* : V \to V, \qquad Q = \big(\psi^n\big)_* : V' \to V';$$

then $P^n = Q|V$.

Both P and Q respect the symplectic form ω on V'. If (V, ω) is symplectic — that is, if $\omega|V$ is non-degenerate — then P is a symplectic matrix and the symmetry of its characteristic polynomial $p(k)$ is immediate. Unfortunately, (V, ω) need not be symplectic — for example, V may be odd-dimensional.

To handle the general case, we first decompose V' into generalized eigenspaces for Q; that is, we write

$$V' \otimes \mathbb{C} = \bigoplus V_\alpha = \bigoplus_\alpha \bigcup_1^\infty \operatorname{Ker}(\alpha I - Q)^i.$$

Grouping together the eigenspaces with $|\alpha| = 1$, we get a Q-invariant decomposition $V' = U \oplus S$ with

$$U \otimes \mathbb{C} = \bigoplus_{|\alpha|=1} V_\alpha \quad \text{and} \quad S \otimes \mathbb{C} = \bigoplus_{|\alpha| \neq 1} V_\alpha.$$

For $x \in V_\alpha$ and $y \in V_\beta$, the fact that Q preserves ω implies

$$\omega(x, y) = \omega(Qx, Qy) = 0$$

unless $\alpha\beta = 1$. Thus U and S are ω-orthogonal, and therefore (U,ω) and (S,ω) are both symplectic.

Since ψ^n fixes all the edges in $\tau' - \tau$, Q acts by the identity on V'/V. Therefore S is a subspace of V, and

$$V = S \oplus (U \cap V) = S \oplus W.$$

Since $P^n = Q$, the splitting $V = S \oplus W$ is preserved by P; $P|S$ is symplectic; and the eigenvalues of $P|W$ are roots of unity. Therefore

$$p(k) = \det(kI - P|S) \cdot \det(kI - P|W).$$

The first term is symmetric because $P|S$ is a symplectic matrix, and the second term is symmetric because the eigenvalues of $P|W$ lie on S^1 and are symmetric about the real axis. Thus $p(k)$ is symmetric. □

Characteristic polynomials of fibers. We now return to the study of the Teichmüller polynomial $\Theta_F = \sum a_g \cdot g \in \mathbb{Z}[G]$. Given $\phi \in H^1(M,\mathbb{Z}) = \operatorname{Hom}(G,\mathbb{Z})$, we obtain a polynomial in a single variable k by setting

$$\Theta_F(k^\phi) = \sum a_g k^{\phi(g)}.$$

Recall that $\mathcal{L}$ denotes the mapping torus of the expanding lamination λ of any fiber $[S] \in \mathbb{R}_+ \cdot F$ (Corollary 3.2); and $\mathcal{L}$ is transversally orientable iff λ is.

THEOREM 4.2. – *The characteristic polynomial of the monodromy of a fiber* $[S] = \phi \in \mathbb{R}_+ \cdot F$ *is given by*

$$p(k) = \Theta_F\left(k^\phi\right) \cdot \begin{cases} (k-1) & \text{if } \mathcal{L} \text{ is transversally orientable,} \\ 1 & \text{otherwise,} \end{cases}$$

up to a unit $\pm k^n$.

Proof. – Let $t_i, u \in G$ be a basis adapted to the splitting $G = H \oplus \mathbb{Z}$ determined by the choice of a lift of the monodromy, $\widetilde{\psi} : \widetilde{S} \to \widetilde{S}$. Then $\phi(t_i) = 0$ and $\phi(u) = 1$, so $k^\phi : G \to \mathbb{C}^*$ has coordinates $(t,u) = (1,k) \in \widehat{G}$. Thus

$$\Theta_F\left(k^\phi\right) = \Theta_F(1,u)|_{u=k} = \det\bigl(kI - P_E(1)\bigr) / \det\bigl(kI - P_V(1)\bigr)$$

by the determinant formula (3.5).

Applying the functor $\operatorname{Hom}(\cdot,\mathbb{R})$ to the commutative diagram (3.4), with $t = 1$, we obtain the adjoint diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & Z_1(\tau,\mathbb{R}) & \longrightarrow & \mathbb{R}^E & \xrightarrow{D(1)^*} & \mathbb{R}^V & \longrightarrow & \mathbb{R}^m & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle P(1)^*} & & \downarrow{\scriptstyle P_E(1)^*} & & \downarrow{\scriptstyle P_V(1)^*} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & Z_1(\tau,\mathbb{R}) & \longrightarrow & \mathbb{R}^E & \xrightarrow{D(1)^*} & \mathbb{R}^V & \longrightarrow & \mathbb{R}^m & \longrightarrow & 0.
\end{array}$$

Here $m = 1$ if $\mathcal{L}$ (and hence τ) is orientable, and $m = 0$ otherwise (compare Corollary 2.4).

Since the rows of the diagram above are exact, the characteristic polynomial of $P = P(1)^*$ is given by the alternating product

$$p(k) = \frac{\det(kI - P_E(1))\,(k-1)^m}{\det(kI - P_V(1))} = \Theta_F\left(k^\phi\right)(k-1)^m. \qquad \square$$

COROLLARY 4.3. – *The Teichmüller polynomial is symmetric; that is,*

$$\Theta_F = \sum a_g \cdot g = \pm h \sum a_g \cdot g^{-1}$$

for some unit $\pm h \in \mathbb{Z}[G]$.

Proof. – Since $\mathbb{R}_+ \cdot F \subset H^1(M,\mathbb{R})$ is open, we can choose $[S] = \phi \in \mathbb{R}_+ \cdot F$ such that the values $\phi(g)$ over the finite set of g with $a_g \neq 0$ are all distinct. Then symmetry of Θ_F follows from symmetry of the characteristic polynomial $p(k) = \Theta_F(k^\phi) = \sum a_g k^{\phi(g)}$. □

Notes. Although the characteristic polynomial $f(u) = \det(uI - P)$ of a pseudo-Anosov mapping ψ is always symmetric, $f(u)$ may factor over $\mathbb{Z}$ into a product of non-symmetric polynomials. In particular, the minimal polynomial of a pseudo-Anosov expansion factor $K > 1$ need *not* by symmetric. For example, the largest root $K = 1.83929\ldots$ of the non-symmetric polynomial $x^3 - x^2 - x - 1$ is a pseudo-Anosov expansion factor; see [1], [20, §5].

5. Expansion factors

In this section we study the expansion factor $K(\phi)$ for a cohomology class $\phi \in \mathbb{R}_+ \cdot F$, and prove it is strictly convex and determined by Θ_F.

Definitions. Let $[S] = \phi \in \mathbb{R}_+ \cap F$ be a fiber with monodromy ψ and expanding measured lamination $\Lambda \in \mathcal{ML}(S)$. The *expansion factor* $K(\phi) > 1$ is the expanding eigenvalue of $\psi : \mathcal{ML}(S) \to \mathcal{ML}(S)$; that is, the constant such that

$$\psi \cdot \Lambda = K(\phi)\Lambda.$$

The function

$$L(\phi) = \log K(\phi)$$

gives the *Teichmüller length* of the unique geodesic loop in the moduli space of Riemann surfaces represented by

$$\psi \in \mathrm{Mod}(S) \cong \pi_1(\mathcal{M}_{g,n}).$$

(Compare [4].)

THEOREM 5.1. – *The expansion factor satisfies*

$$K(\phi) = \sup\{k > 1 \colon \Theta_F(k^\phi) = 0\} \tag{5.1}$$

for any fiber $[S] = \phi \in \mathbb{R}_+ \cdot F$.

Proof. – By Theorem 4.2, $p(k) = \Theta_F(k^\phi)$ is the characteristic polynomial of the map

$$P : Z_1(\lambda,\mathbb{R}) \to Z_1(\lambda,\mathbb{R})$$

determined by monodromy of S, and the largest eigenvalue of P is $K(\phi)$, with eigenvector the expanding measure associated to Λ. □

Since the right-hand side of (5.1) is defined for real cohomology classes, we will use it to extend the definition of $K(\phi)$ and $L(\phi)$ to the entire cone $\mathbb{R}_+ \cdot F$. Then we have the homogeneity properties:

$$K(a\phi) = K(\phi)^{1/a},$$
$$L(a\phi) = a^{-1}L(\phi).$$

Here is a useful fact established in [18, Theorem F].

THEOREM 5.2 Fried. – *The expansion factor $K(\phi)$ is continuous on F and tends to infinity as $\phi \to \partial F$.*

Next we derive some convexity properties of the expansion factor. These properties are illustrated in Fig. 7 of Section 11.

THEOREM 5.3. – *For any $k > 1$, the level set*

$$\Gamma = \{\phi \in \mathbb{R}_+ \cdot F\colon\ K(\phi) = k\}$$

is a convex hypersurface with $\mathbb{R}_+ \cdot \Gamma = \mathbb{R}_+ \cdot F$.

Proof. – By homogeneity, Γ meets every ray in $\mathbb{R}_+ \cdot F$, and thus $\mathbb{R}_+\Gamma = \mathbb{R}_+ \cdot F$. For convexity, it suffices to consider the level set Γ where $\log K(\phi) = 1$.

Choose a fiber $[S] \in \mathbb{R}_+ \cdot F$ and a lift $\widetilde{\psi}$ of its monodromy. Then we obtain a splitting $H^1(M,\mathbb{R}) = H^1(S,\mathbb{R})^{\psi} \oplus \mathbb{R}$ and associated coordinates (s,y) on $H^1(M,\mathbb{R})$ and $(t,u) = (e^s, e^y)$ on $\widehat{G} = \exp H^1(M,\mathbb{R})$.

By the determinant formula (3.5), $\Theta_F(t,u)$ is the ratio between the characteristic polynomials of $P_E(t)$ and $P_V(t)$. By Theorem 3.4, $P_E(t)$ is a Perron–Frobenius matrix of Laurent polynomials; let $E(t) > 1$ denote its leading eigenvalue for $t \in \mathbb{R}^b_+$. Since $P_V(t)$ is simply a permutation matrix, we have $\Theta_F(t, E(t)) = 0$ for all t. By Theorem A.1 of Appendix A, $y = \log E(e^s)$ is a convex function of s, so its graph Γ' is convex.

To complete the proof, we show $\Gamma' = \Gamma$. First note that $\Gamma' \subset \Gamma$. Indeed, if $\phi = (s,y) \in \Gamma'$, then $\Theta_F(e^s, e^y) = 0$ and so $K(\phi) \geqslant e$. But by Theorem A.1, the ray $\mathbb{R}_+ \cdot \phi$ meets Γ' at most once; since $u = E(t)$ is the *largest* zero of $\Theta_F(t,u)$, we have $K(\phi) = e$, and thus $(s,u) \in \Gamma$.

Since Γ' is a graph over $H^1(S,\mathbb{R})$, it is properly embedded in $H^1(M,\mathbb{R})$; but Γ is connected, so $\Gamma = \Gamma'$. □

COROLLARY 5.4. – *The function $y = 1/\log K(\phi)$ on the cone $\mathbb{R}_+ \cdot F$ is real-analytic, strictly concave, homogeneous of degree* 1, *and*

$$y(\phi) \to 0 \quad \textit{as } \phi \to \partial F.$$

Proof. – The homogeneity of $y(\phi)$ follows from that of $K(\phi)$.

Let Γ be the convex hypersurface on which $\log K(\phi) = 1$. Since Γ is a component of the analytic set $\Theta_F(e^{\phi}) = 0$, and $K(\phi)$ is homogeneous, $K(\phi)$ is real-analytic.

To prove concavity, let $\phi_3 = \alpha\phi_1 + (1-\alpha)\phi_2$ be a convex combination of $\phi_1, \phi_2 \in \mathbb{R}_+ \cdot F$, and let $y_i = 1/\log K(\phi_i)$, so $y_i^{-1}\phi_i \in \Gamma$. By convexity of Γ, the segment $[y_1^{-1}\phi_1, y_2^{-1}\phi_2]$ meets the ray through ϕ_3 at a point p which is farther from the origin than $y_3^{-1}\phi_3$. Since

$$p = \frac{\alpha y_1(y_1^{-1}\phi_1) + (1-\alpha)y_2(y_2^{-1}\phi_2)}{\alpha y_1 + (1-\alpha)y_2} = \frac{\phi_3}{\alpha y_1 + (1-\alpha)y_2},$$

we find

$$y_3^{-1} \leqslant \big(\alpha y_1 + (1-\alpha)y_2\big)^{-1}$$

and therefore $y(\phi)$ is concave.

Finally $y(\phi)$ converges to zero at ∂F by Theorem 5.2, so by real-analyticity it must be strictly concave. □

Notes.

(1) The concavity of $1/\log K(\phi)$ was established by Fried; see [18, Theorem E], [20, Proposition 8], as well as [31] and [32]. Our proof of concavity is rather different and uses only general properties of Perron–Frobenius matrices (presented in Appendix A).

(2) By Corollary 5.4, the expansion factor $K(\phi)$ assumes its minimum at a unique point $\phi \in F$, providing a *canonical center* for any fibered face of the Thurston norm ball. **Question.** Is the minimum always achieved at a *rational* cohomology class?

6. The Thurston norm

Let $F \subset H^1(M,\mathbb{R})$ be a fibered face of the Thurston norm ball. In this section we use the fact that $K(\phi)$ blows up at ∂F to show one can compute the cone $\mathbb{R}_+ \cdot F$ from the polynomial Θ_F. This observation is conveniently expressed in terms of a second norm on $H^1(M,\mathbb{R})$ attached to Θ_F.

Norms and Newton polygons. Write the Teichmüller polynomial $\Theta_F \in \mathbb{Z}[G]$ as

$$\Theta_F = \sum a_g \cdot g.$$

The *Newton polygon* $N(\Theta_F) \subset H_1(M,\mathbb{R})$ is the convex hull of the finite set of integral homology classes g with $a_g \neq 0$. We define the *Teichmüller norm* of $\phi \in H^1(M,\mathbb{R})$ (relative to F) by:

$$\|\phi\|_{\Theta_F} = \sup_{a_g \neq 0 \neq a_h} \phi(g-h).$$

The norm of ϕ measures the length of the projection of the Newton polygon, $\phi(N(\Theta_F)) \subset \mathbb{R}$. Multiplication of Θ_F by a unit just translates $N(\Theta_F)$, so the Teichmüller norm is well-defined.

THEOREM 6.1. – *For any fibered face F of the Thurston norm ball, there exists a face D of the Teichmüller norm ball,*

$$D \subset \{\phi\colon \|\phi\|_{\Theta_F} = 1\},$$

such that $\mathbb{R}_+ \cdot F = \mathbb{R}_+ \cdot D$.

Proof. – Pick a fiber $[S] \in \mathbb{R}_+ \cdot F$ with monodromy ψ. Choose coordinates $(t,u) = (e^s, e^y)$ on

$$H^1(M,\mathbb{R}_+) \cong \exp\bigl(H^1(S,\mathbb{R})^{\psi} \oplus \mathbb{R}\bigr),$$

and let $E(t)$ be the leading eigenvalue of the Perron–Frobenius matrix $P_E(t)$. As we saw in Section 5, we have $\mathbb{R}_+ \cdot \Gamma = \mathbb{R}_+ \cdot F$, where Γ is the graph of the function

$$y = f(s) = \log E(e^s).$$

Now the determinant formula (3.5) shows $\Theta_F(t,u)$ is a factor of $\det(uI - P_E(t))$ with $\Theta_F(t,E(t)) = 0$, so by Theorem A.1(C) of Appendix A, $\mathbb{R}_+ \cdot \Gamma$ coincides with the dual cone $C(u^d)$ of the leading term u^d of $\Theta_F(t,u)$. Equivalently, $\mathbb{R}_+ \cdot \phi$ meets the graph of $f(s)$ iff ϕ achieves its maximum on $N(\Theta_F)$ at the vertex $v \in N(\Theta_F)$ corresponding to u^d.

Since Θ_F is symmetric (Corollary 4.3), so is its Newton polygon, and thus the unit ball B of the Teichmüller norm is dual to the convex body $N(\Theta_F)$. Under this duality, the linear functionals ϕ achieving their maximum at v correspond to the cone over a face $D \subset B$; and therefore

$$\mathbb{R}_+ \cdot F = C(u^d) = \mathbb{R}_+ \cdot D. \qquad \square$$

Skew norms. Although in some examples the Thurston and Teichmüller norms actually agree (see Section 11), in general the norm faces F and D of Theorem 6.1 are skew to one another.

Here is a construction showing that F and D carry different information in general. Let $\lambda \subset S$ be the expanding lamination of a pseudo-Anosov mapping ψ, and let $\mathcal{L} \subset M$ be its mapping torus. Assume $b_1(M) \geqslant 2$.

Assume moreover that ψ has a fixed-point x in the center of an ideal n-gon of $S - \lambda$, with $n \geqslant 3$. (In the measured foliation picture, x is an n-prong singularity.) Then the mapping torus of x gives an oriented loop $X \subset M$ transverse to S. Construct a 3-dimensional submanifold

$$M' \overset{i}{\hookrightarrow} M$$

by removing a tubular neighborhood of $X \subset M$, small enough that we still have $\mathcal{L} \subset M'$. Let $S' = S \cap M'$; it is a fiber of M'.

Let F and F' be the faces of the Thurston norm balls whose cones contain $[S]$ and $[S']$. We wish to compare the norms of ϕ and $\phi' = i^*(\phi)$ for $\phi \in \mathbb{R}_+ \cdot F$.

First, the Teichmüller norms agree: that is,

$$\|\phi'\|_{\Theta'_F} = \|\phi\|_{\Theta_F}. \tag{6.1}$$

Indeed, the mapping torus of the expanding lamination is $\mathcal{L}' = \mathcal{L}$ for both M' and M, and therefore $i_*(\Theta_{F'}) = \Theta_F$, which gives (6.1).

On the other hand, the Thurston norms satisfy

$$\|\phi'\|_T = \|\phi\|_T + \phi(X). \tag{6.2}$$

Indeed, let $[R] = \phi$ be a fiber in M and let $[R'] = [R \cap M']$ be the corresponding fiber in M'. Then we have

$$\|\phi'\|_T = |\chi(R')| = |\chi(R - X)| = |\chi(R)| + |R \cap X| = \|\phi\|_T + \phi(X).$$

By (6.1) and (6.2), the Teichmüller and Thurston norms can agree on at most one of the cones $\mathbb{R}_+ \cdot F$ and $\mathbb{R}_+ \cdot F'$. With an appropriate choice of X, one can construct examples where the Thurston norm is not even a constant multiple of the Teichmüller norm on $\mathbb{R}_+ \cdot F$.

Notes.

(1) Theorem 6.1 provides an effective algorithm to determine a fibered face F of M from a single fiber S and its monodromy ψ.
The first step is to find a ψ-invariant train track τ. Bestvina and Handel have given an elegant algorithm to find such a train track, based on entropy reduction [5]. Versions of this algorithm have been implemented by T. White, B. Menasco — J. Ringland, T. Hall and P. Brinkman; see [9].

Once τ is found, it is straightforward to compute the matrices $P_E(t)$ and $P_V(t)$ giving the action of $\widetilde{\psi}$ on $\widetilde{\tau}$. The determinant formula

$$\Theta_F(t,u) = \det\big(uI - P_E(t)\big) / \det\big(uI - P_V(t)\big)$$

then gives the Teichmüller polynomial for F, and the Newton polygon of Θ_F determines the cone $\mathbb{R}_+ \cdot F$ as we have seen above. Finally F itself can be recovered as the intersection of $\mathbb{R}_+ \cdot F$ with the unit sphere $\|\phi\|_A = 1$ in the *Alexander norm* on $H^1(M,\mathbb{R})$ (see Section 7).

(2) For any fiber $[S] \in \mathbb{R}_+ \cdot F$ with expanding lamination λ, we have

$$\|[S]\|_{\Theta_F} = -\chi(\lambda),$$

where the Euler characteristic is computed with Čech cohomology. To verify this equation, use the determinant formula for Θ_F and observe that $\chi(\lambda) = \chi(\tau) = |V| - |E|$.

7. The Alexander norm

In this section we show that a fibered face F can be computed from the Alexander polynomial of M when λ is transversely orientable.

The Alexander polynomial and norm. Assume $b_1(M) > 1$, let $G = H_1(M,\mathbb{Z})/\text{torsion}$, and let $\widehat{G} = \operatorname{Hom}(G,\mathbb{C}^*)$.

Recall that the Teichmüller polynomial of a fibered face defines, via its zero set, the largest hypersurface $V \subset \widehat{G}$ such $\dim Z_2(\mathcal{L},\mathbb{C}_\rho) > 0$ for all $\rho \in V$ (Theorem 3.3). Similarly, the *Alexander polynomial* of M,

$$\Delta_M = \sum a_g \cdot g \in \mathbb{Z}[G],$$

defines the largest hypersurface on which $\dim H^1(M,\mathbb{C}_\rho) > 0$. (See [33, Corollary 3.2].) The *Alexander norm* on $H^1(M,\mathbb{R})$ is defined by

$$\|\phi\|_A = \sup_{a_g \neq 0 \neq a_h} \phi(g-h).$$

(By convention, $\|\phi\|_A = 0$ if $\Delta_M = 0$.)

THEOREM 7.1. – *Let F be a fibered face in $H^1(M,\mathbb{R})$ with $b_1(M) \geqslant 2$. Then we have:*

(1) *$F \subset A$ for a unique face A of the Alexander norm ball, and*

(2) *$F = A$ and Δ_M divides Θ_F if the lamination $\mathcal{L}$ associated to F is transversally orientable.*

Remark. – Transverse orientability of $\mathcal{L}$ is equivalent to transverse orientability of $\lambda \subset S$ for a fiber $S \in \mathbb{R}_+ \cdot F$, and to orientability of a train track τ carrying λ.

Proof of Theorem 7.1. – In [33] we show

$$\|\phi\|_A \leqslant \|\phi\|_T$$

for all $\phi \in H^1(M,\mathbb{R})$, with equality if ϕ comes from a fibration $M \to S^1$; this gives part (1) of the theorem.

For part (2), pick a fiber $[S] \in \mathbb{R}_+ \cdot F$ with monodromy ψ and invariant lamination λ. Let (t,u) be coordinates on the character variety $\widehat{G}$ adapted to the splitting $G = H \oplus \mathbb{Z}$ coming from the choice of a lift $\widetilde{\psi}$ of ψ.

If $\mathcal{L}$ is transversally orientable, then λ is carried by an orientable train track τ. Since τ fills the surface S, we obtain a surjective map:

$$\pi : Z_1(\tau, \mathbb{C}_t) \cong H_1(\tau, \mathbb{C}_t) \twoheadrightarrow H_1(S, \mathbb{C}_t) \tag{7.1}$$

for any character $t \in \widehat{H}$.

Let $P(t)$ and $Q(t)$ denote the action of $\widetilde{\psi}$ on $Z_1(\tau, \mathbb{C}_t)$ and $H_1(S, \mathbb{C}_t)$ respectively. Fixing a nontrivial character t, we have

$$\Delta_M(t,u) = \det\bigl(uI - Q(t)\bigr) \quad \text{and} \quad \Theta_F(t,u) = \det\bigl(uI - P(t)\bigr)$$

up to a unit in $\mathbb{Z}[G]$. By (7.1), $\Delta_M(t,u)$ is a divisor of $\Theta_F(t,u)$. It follows that Δ_M divides Θ_F (using an algebraic argument as in Section 3 to lift the divisibility to $\mathbb{Z}[G]$).

The action of $\widetilde{\psi}$ on $\operatorname{Ker}(\pi)$ corresponds to the action of ψ by permutations on the components of $S - \tau$, so it does not include the leading eigenvalue $E(t)$ of $P(t)$. Therefore $\Delta_M(t, E(t)) = 0$, so we can apply Theorem A.1(C) of the Appendix to conclude that there is a face A of the Alexander norm ball with $\mathbb{R}_+ \cdot A = \mathbb{R}_+ \cdot F$ (just as in Theorem 6.1). By (1) we have $F \subset A$, and therefore $F = A$. □

Note. Dunfield has given an example where the fibered face F is a *proper* subset of the Alexander face A; see [14].

8. Twisted measured laminations

In this section we add another interpretation to the Teichmüller polynomial, by showing Θ_F determines the eigenvalues of $\psi \in \operatorname{Mod}(S)$ on the space of twisted (or affine) measured laminations $\mathcal{ML}_s(S)$. We will establish:

THEOREM 8.1. – *A pseudo-Anosov mapping $\psi : S \to S$ has a unique pair of fixed-points*

$$\Lambda_+, \Lambda_- \in \mathbb{P}\mathcal{ML}_s(S)$$

for any $s \in H^1(S,\mathbb{R})^{\psi}$. The supporting geodesic laminations (λ_+, λ_-) of (Λ_+, Λ_-) coincide with the expanding and contracting laminations of ψ respectively, and we have

$$\psi \cdot \Lambda_+ = k\Lambda_+,$$

where $k > 0$ is the largest root of the equation $\Theta_F(e^s, k) = 0$.

$\mathcal{ML}_s(S)$. Fix a cohomology class $s \in H^1(S,\mathbb{R})$. We can interpret s as a homomorphism

$$s : H_1(S,\mathbb{Z}) \to \mathbb{R},$$

determining an element $t \in H^1(S,\mathbb{R}_+)$ by

$$t = e^s : H_1(S,\mathbb{Z}) \to \mathbb{R}_+ = SL_1(\mathbb{R}).$$

Thus s (or t) gives $\mathbb{R}$ the structure of a module $\mathbb{R}_s$ (or $\mathbb{R}_t$) over the ring $\mathbb{Z}[H_1(S,\mathbb{Z})]$.

The space of *twisted measured laminations*, $\mathcal{ML}_s(S)$, is the set of all $\Lambda = (\lambda, \mu)$ such that:

- $\lambda \subset S$ is a compact geodesic lamination,
- $\mu \in Z_1(\lambda, \mathbb{R}_s)$ is a cycle, and
- $\mu(T) > 0$ for every nonempty transversal T to λ.

Here μ can be thought of as a transverse measure taking values in a fixed flat $\mathbb{R}$-bundle $L_s \to S$. For $s = 0$, the bundle L_s is trivial, so $\mathcal{ML}_0(S)$ reduces to the space of ordinary measured laminations $\mathcal{ML}(S)$. Let $\mathbb{P}\mathcal{ML}_s(S) = \mathcal{ML}_s(S)/\mathbb{R}_+$ denote the projective space of rays in $\mathcal{ML}_s(S)$.

Using train tracks, one can give $\mathcal{ML}_s(S)$ local charts and a topology. A basic result from [25] is:

THEOREM 8.2 (Hatcher–Oertel). – *The spaces $\mathcal{ML}_s(S)$ form a fiber bundle over $H^1(M,\mathbb{R}_+)$. In particular, $\mathcal{ML}_s(S) \cong \mathbb{R}^n$ for all s.*

Perron–Frobenius eigenvectors. Let $\psi : S \to S$ be a pseudo-Anosov mapping with monodromy ψ and expanding lamination λ carried by an invariant train track τ. As in (3.4), we obtain a matrix

$$P_E(t) : \mathbb{Z}[H]^E \to \mathbb{Z}[H]^E$$

describing the action of $\widetilde{\psi}$ on the edges of $\widetilde{\tau}$, and $P_E(t)$ is a Perron–Frobenius matrix of Laurent polynomials by Theorem 3.4. We can think of $P_E(t)$ as a map

$$P_E : H^1(S,\mathbb{R}_+)^{\psi} \to \mathrm{End}\big(\mathbb{R}^E\big),$$

whose values are traditional Perron–Frobenius matrices over $\mathbb{R}$.

As in Section 4, we can apply the functor $\mathrm{Hom}(\cdot,\mathbb{R}_t)$ to (3.4) to obtain the adjoint diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & Z_1(\tau,\mathbb{R}_t) & \longrightarrow & \mathbb{R}^E & \xrightarrow{D(t)^*} & \mathbb{R}^V \\ & & \big\downarrow{\scriptstyle P(t)^*} & & \big\downarrow{\scriptstyle P_E(t)^*} & & \big\downarrow{\scriptstyle P_V(t)^*} \\ 0 & \longrightarrow & Z_1(\tau,\mathbb{R}_t) & \longrightarrow & \mathbb{R}^E & \xrightarrow{D(t)^*} & \mathbb{R}^V. \end{array} \tag{8.1}$$

For each t, the largest eigenvalue $E(t)$ of $P_E(t)^*$ is positive and simple, with a positive eigenvector $\mu(t) \in \mathbb{R}^E$.

THEOREM 8.3. – *For each $t \in H^1(S,\mathbb{R}_+)$, the leading eigenvalue $u = E(t)$ of $P_E(t)^*$ is the largest root of the polynomial equation*

$$\Theta_F(t,u) = 0,$$

and its positive eigenvector $\mu(t)$ belongs to $Z_1(\tau,\mathbb{R}_t)$.

Proof. – First suppose $t = 1$ is the trivial cohomology class. Then $P_E(1)$ is an integral Perron–Frobenius matrix, and hence $u = E(1) > 1$ is the largest root of the polynomial $\det(uI - P_E(1))$. On the other hand, $P_V(1)$ is a permutation matrix, with eigenvalues on the unit circle, so $\det(uI - P_V(1))$ has no root at $u = E(1)$. Since Theorem 3.6 expresses $\Theta_F(1,u)$ as the ratio of these two determinants, $E(1)$ is the largest root of the polynomial $\Theta_F(1,u) = 0$.

To see $\mu(1)$ is a cycle, just note that $D(1)^*\mu(1) = 0$ because (8.1) is commutative and $P_V(1)$ has no eigenvector with eigenvalue $E(1)$.

The same reasoning applies whenever $E(t)$ is not an eigenvalue of $P_V(t)$, and thus the Theorem holds for generic t. By continuity, it holds for all $t \in H^1(S, \mathbb{R}_+)$. □

Proof of Theorem 8.1. – Suppose $\psi \cdot \Lambda = E\Lambda$. As we saw in Corollary 3.2, the only possibilities for the support of Λ are the expanding and contracting geodesic laminations λ_+, λ_- of ψ. In the case $\Lambda = (\lambda_+, \mu)$, positivity of μ on transversals implies μ is a positive eigenvector of $P_E(t)^*$, $t = e^s$, under the isomorphism

$$Z_1(\lambda_+, \mathbb{R}_t) = Z_1(\tau, \mathbb{R}_t).$$

Since $P_E(t)^*$ is a Perron–Frobenius matrix, its positive eigenvector is unique up to scale, and thus $k = E(t)$. By Theorem 8.3, k is the largest root of $\Theta_F(t,k) = \Theta_F(e^s, k) = 0$. □

COROLLARY 8.4. – *Let $k(s)$ be the eigenvalue of*

$$\psi : \mathcal{ML}_s(S) \to \mathcal{ML}_s(S)$$

at Λ_+. Then $\log k(s)$ is a convex function on $H^1(S,\mathbb{R})^\psi$.

Proof. – Apply Theorem A.1 of Appendix A. □

Notes.

(1) It can happen that $\psi \cdot \Lambda_+ = k(s)\Lambda_+$ with $0 < k(s) < 1$, even though $\Lambda_+ \in \mathcal{ML}_s(S)$ is supported on the expanding lamination of ψ. Indeed, $k(s)$ depends on the choice of a lift $\widetilde{\psi}$ of ψ, and changing this lift by $h \in H$ changes $k(s)$ to $e^{\phi(h)}k(s)$.

(2) **Question.** Given a Riemann surface $X \in \operatorname{Teich}(S)$, is there a natural isomorphism $\mathcal{ML}_s(S) \cong Q_s(X)$ between the space of twisted measured laminations and the space of twisted quadratic differentials, defined as holomorphic sections of $K(X)^2 \otimes L_s$? Hubbard and Masur established this correspondence in the untwisted case [26].

(3) The existence of a fixed-point for ψ on $\mathcal{ML}_s(S)$ is also shown in [38, Proposition 2.3].

9. Teichmüller flows

We now turn to the study of measured foliations $\mathcal{F}$ of M.

Assume M is oriented and $\mathcal{F}$ is transversally oriented; then the leaves of $\mathcal{F}$ are also oriented. Measured foliations so oriented correspond bijectively to closed, nowhere-vanishing 1-forms ω on M, and we let $[\mathcal{F}] = [\omega] \in H^1(M,\mathbb{R})$. A flow $f : M \times \mathbb{R} \to M$ has *unit speed* (relative to $\mathcal{F}$) if it is generated by a vector field v with $\omega(v) = 1$. Such a flow preserves the foliation $\mathcal{F}$ and its transverse measure.

In this section we prove:

THEOREM 9.1. – *Let F be a fibered face of the Thurston norm ball for M. Then any $\phi \in \mathbb{R}_+ \cdot F$ determines:*

- *a measured foliation $\mathcal{F}$ of M with $[\mathcal{F}] = \phi$,*
- *a complex structure J on the leaves of $\mathcal{F}$, and*
- *a unit-speed Teichmüller flow*

$$f : (M, \mathcal{F}) \times \mathbb{R} \to (M, \mathcal{F})$$

with stretch factor $K(f_t) = K(\phi)^{|t|}$.

The data $(\mathcal{F}, J, f)$ is unique up to isotopy.

The idea of the proof is to use the results on twisted measured laminations in Section 8 to construct the analytic structure $(\mathcal{F}, J, f)$ from the purely combinatorial information provided by the cohomology class ϕ.

From measured laminations to quadratic differentials. As usual we choose a fiber $[S] \in \mathbb{R}_+ \cdot F$ with monodromy ψ and expanding and contracting laminations $\lambda_\pm$. Choose a lift $\widetilde{\psi}$ of ψ to the H-covering space $\widetilde{S}$ of S, and write

$$G = H_1(M, \mathbb{Z})/\text{torsion} = H \oplus \mathbb{Z}\widetilde{\psi}.$$

Let G act on $\widetilde{S}$ by

$$(h, i) \cdot s = \widetilde{\psi}^i\big(h(s)\big);$$

this action embeds G into the mapping-class group $\mathrm{Mod}(\widetilde{S})$.

THEOREM 9.2. – *There exist measured laminations $\widetilde{\Lambda}_\pm \in \mathcal{ML}(\widetilde{S})$, supported on $\widetilde{\lambda}_\pm$, such that for all $g \in G$ we have*

$$g \cdot \widetilde{\Lambda}_\pm = K^{\pm\phi(g)} \widetilde{\Lambda}_\pm, \tag{9.1}$$

where $K = K(\phi)$ is the expansion factor of ϕ.

Proof. – Writing $\phi = (s, y)$, the condition $K = K(\phi)$ means $y > 0$ is the largest solution to the equation $\Theta_F(K^s, K^y) = 0$. By Theorem 8.1 there exists a twisted measured lamination $\Lambda_+ \in \mathcal{ML}_{s \log K}(S)$, supported on λ_+, with $\psi \cdot \Lambda_+ = K^y \Lambda_+$. The lift of Λ_+ to $\widetilde{S}$ then gives a lamination $\widetilde{\Lambda}_+$ satisfying (9.1).

To construct Λ_-, note that $K(\phi) = K(-\phi)$ because the expansion and contraction factors of a pseudo-Anosov mapping are reciprocal. Thus the same construction applied to $-\phi$ yields $\widetilde{\Lambda}_-$ satisfying (9.1). □

Although $\mathrm{int}(\widetilde{S})$ has infinite topological complexity, it has a natural quasi-isometry type coming from the lift of a finite volume hyperbolic metric on $\mathrm{int}(S)$. Complex structures compatible with this quasi-isometry type are parameterized by the (infinite-dimensional) Teichmüller space $\mathrm{Teich}(\widetilde{S})$.

THEOREM 9.3. – *There is a Riemann surface $X \in \mathrm{Teich}(\widetilde{S})$ and a holomorphic quadratic differential $q(z)\,dz^2$ on X such that:*

1. *$G \subset \mathrm{Mod}(\widetilde{S})$ acts by commuting Teichmüller mappings $g(x)$ on X, preserving the foliations of q, and*
2. *The map $g(x)$ stretches the vertical and horizontal leaves of q by $(K^{-\phi(g)}, K^{+\phi(g)})$, where $K = K(\phi)$.*

Proof. – Integrating the transverse measures on $\widetilde{\Lambda}_\pm$, we will collapse their complementary regions and obtain a map $f : \widetilde{S} \to X$.

On any small open set $U_\alpha \subset \widetilde{S}$, we can introduce local coordinates (u, v) such that u and v are constant on the leaves of $\widetilde{\Lambda}_-$ and $\widetilde{\Lambda}_+$ respectively. Then there is a continuous map

$$f_\alpha : U_\alpha \to \mathbb{C}$$

given by $f_\alpha(u, v) = x(u) + iy(v)$, where $x(u)$ and $y(v)$ are monotone functions whose distributional derivatives $(x'(u), y'(v))$ are the transverse measures for $(\widetilde{\Lambda}_-, \widetilde{\Lambda}_+)$. The coordinate $z_\alpha = f_\alpha$ is unique up to

$$z_\alpha \mapsto \pm z_\alpha + b; \tag{9.2}$$

the sign ambiguity arises because the laminations are not oriented.

Since the coordinate change (9.2) is holomorphic, we can assemble the charts

$$V_\alpha = f_\alpha(U_\alpha)$$

to form a Riemann surface X. The forms dz_α^2 on U_α are invariant under (9.2), so they patch together to yield a holomorphic quadratic differential q on X. Finally the maps f_α piece together to give the collapsing map $f : \widetilde{S} \to X$.

The construction of $f : \widetilde{S} \to X$ is functorial in the measured laminations $(\widetilde{\Lambda}_-, \widetilde{\Lambda}_+)$. That is, if we apply the same construction to $(a^{-1}\widetilde{\Lambda}_-, a^{+1}\widetilde{\Lambda}_+)$, we obtain a new marked surface $f' : \widetilde{S} \to X'$ and a unique map $F : X \to X'$ such that $F \circ f = f'$. Moreover F is a Teichmüller mapping, stretching the vertical and horizontal leaves of q by a^{-1} and a^{+1} respectively.

Since $g \in G$ multiplies the laminations $(\widetilde{\Lambda}_-, \widetilde{\Lambda}_+)$ by $(K^{-\phi(g)}, K^{+\phi(g)})$, this functoriality provides the desired lifting of G to Teichmüller mappings on X. □

Isotopy. Finally we quote the following topological result of Blank and Laudenbach, recently treated by Cantwell and Conlon [29,35,11]:

THEOREM 9.4. – *Any two measured foliations $\mathcal{F}$, $\mathcal{F}'$ representing the same cohomology class on M are isotopic.*

Proof of Theorem 9.1. – We will construct $(\mathcal{F}, J, f)$ from the Riemann surface X, its quadratic differential q and the action of G given by Theorem 9.3.

Let $\widetilde{\mathcal{F}}$ be the measured foliation of $X \times \mathbb{R}$ with leaves $X_r = X \times \{r\}$ and with transverse measure dr. Let $\widetilde{f}_t : X \times \mathbb{R} \to X \times \mathbb{R}$ be the unit speed flow $\widetilde{f}_t(x,r) = (x, r+t)$. Let $\widetilde{J}$ be the unique complex structure on $T\widetilde{\mathcal{F}}$ such that $(X_0, \widetilde{J}_0) = X$ and such that $\widetilde{f}_t : X_0 \to X_t$ is a Teichmüller mapping stretching the vertical and horizontal leaves of q by (K^{-t}, K^{+t}). Finally, let G act on $X \times \mathbb{R}$ by

$$g \cdot (x,r) = \big(g(x), r + \phi(g)\big), \tag{9.3}$$

where $g(x)$ is the Teichmüller mapping of X to itself provided by Theorem 9.3.

With this action, G preserves the structure $(\widetilde{\mathcal{F}}, \widetilde{J}, \widetilde{f}_t)$, and therefore the quotient $N = (X \times \mathbb{R})/G$ carries a measured foliation $\mathcal{F}$, a complex structure J on $T\mathcal{F}$, and a unit speed Teichmüller flow $f_t : N \to N$.

To complete the construction, we will show N can be identified with M in such a way that $[\mathcal{F}] = \phi$. To construct a homeomorphism $N \cong M$, first note that ϕ pulls back to a trivial cohomology class on $X \cong \widetilde{S}$, so there exists a smooth function $\xi : X \to \mathbb{R}$ such that

$$\xi\big(h(x)\big) = \xi(x) + \phi(h)$$

for all $h \in H \subset G$. Set $a = \phi(\widetilde{\psi}) > 0$, so $\phi(h,i) = \phi(h) + ai$. Then the homeomorphism of $X \times \mathbb{R}$ given by

$$(x,r) \mapsto \big(x, ar + \xi(x)\big)$$

conjugates the action of $g = (h,i)$ by

$$g \cdot (x,r) = \big(g(x), r + i\big) \tag{9.4}$$

to the original action (9.3). Thus both actions have the same quotient space. On the other hand, the quotient of $X \times \mathbb{R}$ by the action of G given by (9.4) is:

$$N = (X \times \mathbb{R})/G = \big((X/H) \times \mathbb{R}\big)/\mathbb{Z} \cong M,$$

because $\mathbb{Z}$ acts on $X/H \cong S$ by a map isotopic to ψ.

Thus we have identified N with M. It is easy to see that $[\mathcal{F}] = \phi$ under this identification, so we have completed the construction of $(\mathcal{F}, J, f)$.

To prove uniqueness, the first step is to apply Theorem 9.4 to see that ϕ determines $\mathcal{F}$ up to isotopy. Then, given two Teichmüller flows f_1 and f_2 for the same foliation $\mathcal{F}$, we can pick a fiber S which is nearly parallel to the leaves of $\mathcal{F}$ and transverse to both flows. Each flow determines, via its distortion of complex structure, a pair of ψ-invariant twisted measured laminations $[\Lambda_\pm]$ for S. The uniqueness of $(\mathcal{F}, J, f)$ then follows from the uniqueness of these twisted laminations, guaranteed by Theorem 8.1. □

Note. Our original approach to Theorem 9.1 involved taking the geometric limit of the pseudo-Anosov flows known to exist for fibered classes in $H^1(M,\mathbb{Q})$ by ordinary Teichmüller theory. An examination of the expansion factor $K([\mathcal{F}])$ led to the more algebraic approach presented here.

10. Short geodesics on moduli space

Let S be a closed surface of genus $g \geqslant 2$, and let $\mathcal{M}_g = \text{Teich}(S)/\text{Mod}(S)$ be its moduli space, endowed with the Teichmüller metric. Then closed geodesics on $\mathcal{M}_g$ correspond bijectively to conjugacy classes of pseudo-Anosov elements $\psi \in \text{Mod}(S) \cong \pi_1(\mathcal{M}_g)$. The length $L(\psi)$ of the geodesic for ψ is given by

$$L(\psi) = \log K(\psi),$$

where $K(\psi) > 1$ is the pseudo-Anosov expansion factor for ψ. From [40] we have:

THEOREM 10.1 (Penner). – *The length of the shortest geodesic on the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g satisfies $L(\mathcal{M}_g) \asymp 1/g$.*

(Here $A \asymp B$ means we have $A/C \leqslant B \leqslant CA$ for a universal constant C.)

In this section we show any closed fibered hyperbolic 3-manifold with $b_1(M) \geqslant 2$ provides a source of short geodesics on moduli space as above.

Indeed, let $S \subset M$ be a fiber of genus $g \geqslant 2$ with monodromy ψ. The assumption $b_1(M) \geqslant 2$ is equivalent to the condition that ψ fixes a primitive cohomology class

$$\xi_0 \in H^1(S,\mathbb{Z}).$$

Let $\widetilde{S} \to S$ be the $\mathbb{Z}$-covering space corresponding to ξ_0, with deck group generated by $h : \widetilde{S} \to \widetilde{S}$, and let $\widetilde{\psi}$ be a lift of ψ to $\widetilde{S}$.

THEOREM 10.2. – *For all n sufficiently large,*

$$R_n = \widetilde{S}/\langle h^n \widetilde{\psi}\rangle$$

is a closed surface of genus $g_n \asymp n$, and $h : \widetilde{S} \to \widetilde{S}$ descends to a pseudo-Anosov mapping class $\psi_n \in \text{Mod}(R_n)$ with

$$L(\psi_n) = \frac{L(\psi)}{n} + O(n^{-2}) \asymp \frac{1}{g_n}. \tag{10.1}$$

Proof. – Corresponding to the commuting maps $\widetilde{\psi}$ and h on $\widetilde{S}$, we have a covering space

$$\widetilde{M} = \widetilde{S} \times \mathbb{R} \to M$$

with deck group $\mathbb{Z}H \oplus \mathbb{Z}\widetilde{\Psi}$, where

$$H(s,t) = (h(s),t) \quad \text{and} \quad \widetilde{\Psi}(s,t) = \big(\widetilde{\psi}(s), t-1\big).$$

Define a map

$$(\phi,\xi): H_1(M,\mathbb{Z}) \to \mathbb{Z}H \oplus \mathbb{Z}\widetilde{\Psi} \to \mathbb{Z}^2$$

by sending H to $(0,1)$ and $\widetilde{\Psi}$ to $(-1,0)$. Then the first factor $\phi: H_1(M,\mathbb{Z}) \to \mathbb{Z}$ is the same as the cohomology class corresponding to the fiber S.

Now ϕ belongs to the cone on a fibered face F, so $\phi_n = n\phi + \xi$ also comes from a fibration $\pi_n : M \to S^1$ for all $n \gg 0$. Since $\mathbb{Z}(H^n\widetilde{\Psi})$ corresponds to the kernel of ϕ_n, the $\mathbb{Z}$-covering space $M_n \to M$ corresponding to π_n is given by

$$M_n = \widetilde{M}/\langle H^n\widetilde{\Psi}\rangle \cong \widetilde{S}/\langle h^n\widetilde{\psi}\rangle \times \mathbb{R} = R_n \times \mathbb{R}.$$

Similarly, the monodromy of π_n is induced by the action of H^{-1} on $\widetilde{M}$, so it can be identified with $\psi_n^{-1}: R_n \to R_n$ (up to isotopy).

Now $\|\cdot\|_T$ is linear on $\mathbb{R}_+ \cdot F$, so we have

$$\|\phi_n\|_T = |\chi(R_n)| = 2g_n - 2 = n\phi(e) - \phi_0(e) \asymp n$$

for some $e \in H_1(M,\mathbb{Z})$ (the Euler class). Finally the expansion factor is differentiable and homogeneous of degree -1, so we have

$$K(\psi_n) = K(\phi_n) = K(\phi)^{1/n} + \mathrm{O}\big(n^{-2}\big),$$

giving (10.1). □

Notes.

(1) The exchange of deck transformations and dynamics in the statement of Theorem 10.2 is often called *renormalization*. Compare [46], where the same construction is used to analyze rotation maps.

(2) It is easy to see that $L(\mathcal{M}_1) = \log(3+\sqrt{5})/2$ is the log of the leading eigenvalue of $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. For genus 2 we have $L(\mathcal{M}_2) \leqslant 0.543533\ldots = \log k$, where $k^4 - k^3 - k^2 - k + 1 = 0$ [47], and in general $L(\mathcal{M}_g) \leqslant (\log 6)/g$ [3].

(3) It can be shown that the minimal expansion factor K_n for an $n \times n$ integral Perron–Frobenius matrix is the largest root of $x^n = x + 1$; it satisfies $K_n = 2^{1/n} + \mathrm{O}(1/n^2)$. The factor K_n is realized by the matrix

$$M_{ij} = \begin{cases} 1 & \text{if } j = i+1 \bmod n, \\ 1 & \text{if } (i,j) = (1,3), \\ 0 & \text{otherwise,} \end{cases}$$

which is the adjacency matrix of a cyclic graph with one shortcut; see Fig. 5 for the case $n = 8$. (For a detailed development of the Perron–Frobenius theory, see [30, §4].)

Since the expansion factor of ψ agrees with that of a Perron–Frobenius matrix attached to a train track with at most $6g - 6$ edges, we have $L(\mathcal{M}_g) \geqslant (\log 2)/(6g - 6)$.

(4) **Question.** Does $\lim_{g\to\infty} g \cdot L(\mathcal{M}_g)$ exist? What is its value?

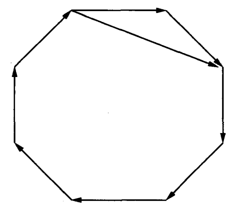

Fig. 5. An 8-vertex graph in which the number of paths of length n grows as slowly as possible.

11. Examples: Closed braids

Closed braids provide a natural source of fibered link complements $M^3 = S^3 - L(\beta)$. In this section we present the computation of Θ_F and the fibered face $F \subset H^1(M,\mathbb{R})$ for some simple braids.

Braids. Let $S = D^2 - \bigcup_1^n U_i$ be the complement of n disjoint round disks lying along a diameter of the closed unit disk D^2. Let $\mathrm{Diff}^+(S,\partial D)$ be the group of diffeomorphisms of S to itself, preserving orientation and fixing ∂D^2 *pointwise.*

The *braid group* B_n is the group of connected components of $\mathrm{Diff}^+(S,\partial D)$. It has standard generators σ_i, $i = 1,\dots,n-1$, which interchange ∂U_i and ∂U_{i+1} by performing a half Dehn twist to the left (see [6,10]).

There is a natural map $B_n \to \mathrm{Mod}(S)$ sending a braid $\beta \in B_n$ to a mapping class $\psi \in \mathrm{Mod}(S)$. Moreover β determines a *canonical* lift $\widetilde{\psi}$ of ψ to the H-covering space of S, by the requirement that $\widetilde{\psi}$ fixes the preimage of ∂D^2 pointwise.

There is a natural basis $t_i = [\partial U_i]$ for $H_1(S,\mathbb{Z})$, on which β acts by $\beta(t_i) = t_{\sigma i}$, and $b = \operatorname{rank} H$ is just the number of cycles of the permutation σ.

Links. Let M be the fibered 3-manifold with fiber S and monodromy ψ. There is a natural model for M as a link complement $M = S^3 - L(\beta)$ in the 3-sphere. To construct the link $L(\beta)$, simply close the braid β while passing it through an unknot α (see Fig. 1 of Section 1). The surface S embeds into M as a disk spanning α, punctured by the n strands of β.

The meridians of components of $L(\beta)$ give a natural basis for $H_1(M,\mathbb{Z})$; in particular the meridian of α corresponds to the natural lifting $\widetilde{\psi}$ of ψ.

Train tracks and braids on three strands. We will now compute $\Theta_F(t,u)$ and F in three examples, where F is the fibered face carrying S.

These examples all come from braids β in the semigroup of B_3 generated by σ_1 and σ_2^{-1}. This semigroup is easy to work with because it preserves a pair of train tracks τ_1,τ_2, where τ_1 is shown in Fig. 4 and τ_2 is the reflection of τ_1 through a vertical line.

As an additional simplification, each train track τ_i is a spine for S, and thus the Thurston and Teichmüller norms agree in these examples: we have

$$\|\phi\|_T = |\chi(S)| = |\chi(\lambda)| = |\chi(\tau)| = \|\phi\|_{\Theta_F}$$

for all fibers $[S] \in \mathbb{R}_+ \cdot F$ (see Note (2) of Section 6). In particular, the fibered face F coincides with a face of the Teichmüller norm ball, so it is easily computed from Θ_F.

I. The simplest pseudo-Anosov braid. For the first example, consider the simplest pseudo-Anosov braid, $\beta = \sigma_1\sigma_2^{-1}$. Its three strands are permuted cyclically, so $H = \mathrm{Hom}(H^1(S,\mathbb{Z})^{\psi},\mathbb{Z})$ is of rank one, generated by $t = t_1 + t_2 + t_3$.

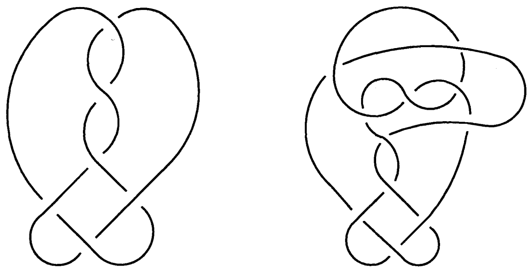

Fig. 6. The links $6_2^2 = L(\sigma_1\sigma_2^{-1})$ and $9_{51}^2 = L(\sigma_1\sigma_2^{-3})$.

The train tracks τ_1 and τ_2 differ only in their switching conditions, so their vertex and edge modules $\mathbb{Z}[t]^V$, $\mathbb{Z}[t]^E$ are naturally identified. Using this identification, we can express the action of σ_1, σ_2^{-1} on these modules as 4×4 and 6×6 matrices of Laurent polynomials.

Now the determinant formula gives Θ_F as the characteristic polynomial for the action of ψ on the 2-dimensional subspace

$$\operatorname{Ker} D(t)^* : \mathbb{Z}[t]^E \to \mathbb{Z}[t]^V.$$

By restricting σ_1 and σ_2^{-1} to this subspace, and projecting to the coordinates for the edge subset $E' = \{a, c\}$, we obtain the simpler 2×2 matrices:

$$\sigma_1(t) = \begin{pmatrix} t & t \\ 0 & 1 \end{pmatrix}, \qquad \sigma_2^{-1}(t) = \begin{pmatrix} 1 & 0 \\ t^{-1} & t^{-1} \end{pmatrix}.$$

Restricting to $\operatorname{Ker} D(t)^*$ removes the factor of $\det(uI - P_V(t))$ from $\det(uI - P_E(t))$, and therefore we have:

$$\Theta_F(t, u) = \det\bigl(uI - \beta(t)\bigr), \tag{11.1}$$

where $\beta(t)$ is the appropriate product of the matrices above.

Setting $\beta(t) = \sigma_1(t)\sigma_2^{-1}(t)$, we find the Teichmüller polynomial is given by

$$\Theta_F(t, u) = 1 - u\bigl(1 + t + t^{-1}\bigr) + u^2.$$

Its Newton polygon is a diamond, and its norm is:

$$\|(s, y)\|_{\Theta_F} = \max\bigl(|2s|, |2y|\bigr).$$

(Here (s, y) denotes the cohomology class evaluating to s and y on the meridian of α and β respectively.)

The fibered face $F \subset H^1(M, \mathbb{R})$ is the same as the face of the Teichmüller norm ball meeting $\mathbb{R}_+ \cdot [S] = \mathbb{R}_+ \cdot (0, 1)$, and therefore $F = \{1/2\} \times [-1/2, 1/2]$ in these (s, y)-coordinates.

The closed braid $L(\beta)$ can be simplified to a projection with 6 crossings (see Fig. 6), and it is denoted 6_2^2 in Rolfsen's tables [41]. In this projection, the two components of $L(\beta)$ are clearly interchangeable. In fact, the Thurston norm ball for $S^3 - L(\beta)$ has 4 faces, all fibered, and

$$\|(s, y)\|_T = 2|s| + 2|y|$$

for all $(s, y) \in H^1(M, \mathbb{R})$.

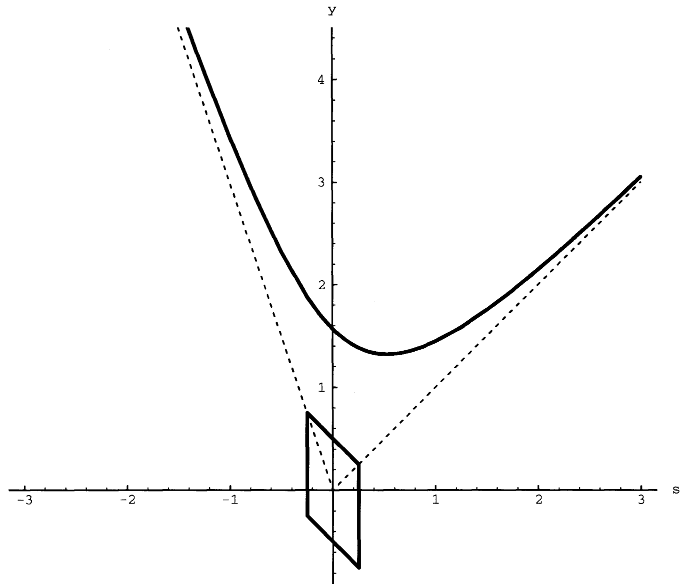

Fig. 7. Norm ball and expansion factor.

II. The Thurston and Alexander norms. The braid $\beta=\sigma_1\sigma_2^{-3}$ also permutes its strands cyclically. By (11.1) in this case we obtain

$$\Theta_F(t,u)=t^{-2}-u\big(t+1+t^{-1}+t^{-2}+t^{-3}\big)+u^2.$$

Fig. 7 shows the Teichmüller norm ball for this example in (s,y) coordinates, along with the graph $y=\log k(s)$, where $k(s)$ eigenvalue of ψ on $\mathcal{ML}_s(S)$ discussed in Section 8. The graph Γ is also the level set $\log K(\phi)=1$ of the expansion function on $\mathbb{R}_+\cdot F$. This picture illustrates the fact that Γ is convex, that the cones over F and Γ coincide, and that $K(\phi)$ tends to infinity at ∂F.

To compute the full Thurston norm ball for this example, we appeal to the inequality $\|\phi\|_A\leqslant\|\phi\|_T$ between the Alexander and Thurston norms (see Section 7). Because of this inequality, the two norms agree if they coincide on the extreme points of the Alexander norm ball. Now a straightforward computation gives

$$\Delta_M(t,u)=t^{-2}+u\big(t-1+t^{-1}-t^{-2}+t^{-3}\big)+u^2$$

in the present example. The polynomials Δ_M and Θ_F have the same Newton polygon, and thus the Alexander, Thurston and Teichmüller norms all coincide on F. But the endpoints of $\pm F$ are the extreme points of the Alexander norm ball, and therefore

$$\|(s,y)\|_T=\|(s,y)\|_A=\max\big(|2s+2y|,|4s|\big)$$

for all $(s,y)\in H^1(M,\mathbb{R})$.

For example, the simplest surface spanning both components of $L(\beta)$ has genus $g=2$, since $\|(\pm1,\pm1)\|_T=4$.

Finally we remark that the closed braid $L(\sigma_1\sigma_2^{-3})$ is actually the same as the link 9^2_{51} of Rolfsen's tables (see Fig. 6). We have thus established:

The Thurston and Alexander norms coincide for the link 9^2_{51}.

In [33] we found that the two norms coincide for all examples in Rolfsen's table of links with 10 or fewer crossings, except 9^3_{21}, and possibly 9^2_{41}, 9^2_{50}, 9^2_{51}, and 9^3_{15}. The link 9^2_{51} can now be removed from the list of possible exceptions.

III. Pure braids. We conclude by discussing *pure braids* β in the semigroup generated by the full twists $\sigma_1^2, \sigma_2^{-2}$. A pure braid acts trivially on $H_1(S,\mathbb{Z})$, and thus the Thurston norm ball is 4-dimensional. We take (t_1,t_2,t_3,u) as a basis for $H^1(M,\mathbb{Z})$, where t_i is the meridian of the ith strand of β and u is the meridian of α.

By cutting down to the kernel of $D(t)^*$ on $\mathbb{Z}[H]^E$ as before, we obtain an action of the full twists on a rank 2 module over $\mathbb{Z}[t_1,t_2,t_3]$. Setting $(t_1,t_2,t_3)=(a,b,c)$ to improve readability, we find that σ_1 and σ_2^{-2} act on this module by:

$$\sigma_1^2 = \begin{pmatrix} ab & ab+b \\ 0 & 1 \end{pmatrix}, \qquad \sigma_2^{-2} = \begin{pmatrix} 1 & 0 \\ b^{-1}+b^{-1}c^{-1} & b^{-1}c^{-1} \end{pmatrix}.$$

For a concrete example, we consider the pure braid $\beta = \sigma_1^2\sigma_2^{-6}$ whose link $L(\beta)$ appears in Fig. 1 of Section 1. Applying (11.1) with the matrices above, we find its Teichmüller polynomial is given by:

$$\begin{aligned} &\Theta_F(a,b,c,u) \\ &\quad = \frac{a}{b^2c^3} - \frac{u}{b^3c^3}\left(1 - b^4c^3(1+c+ac) + (a+1)b(1+c)(1+bc)\left(1+b^2c^2\right)\right) + u^2. \end{aligned}$$

The projection of the fibered face F for this example to $H^1(S,\mathbb{R})$ is shown in Fig. 2 of Section 1.

Since the coefficient of u^0 is $ab^{-2}c^{-3} = t^{(1,-2,-3)}$, we find the Thurston norm on $\mathbb{R}_+ \cdot F$ is given by

$$\|(s,y)\|_T = -s_1 + 2s_2 + 3s_3 + 2y.$$

For example, $\|(-1,1,-1,1)\|_T = 2$, showing that $L(\beta)$ is spanned by a Seifert surface of genus 0 running in alternating directions along the strands of β. It is interesting to locate this surface explicitly in Fig. 1.

Notes.

(1) For a general construction of pseudo-Anosov mappings, including the examples above as special cases, see [39,15].

(2) The Thurston norm of the 6^2_2 is also discussed in [17, p. 264] and [38, Ex. 2.2].

Appendix A. Positive polynomials and Perron–Frobenius matrices

This Appendix develops the theory of Perron–Frobenius matrices over a ring of Laurent polynomials. These results are used in Sections 5–8.

Laurent polynomials. Let $(s_1,\dots,s_b)$ be coordinates for $s \in \mathbb{R}^b$, and let

$$(t_1,\dots,t_b) = \left(e^{s_1},\dots,e^{s_b}\right)$$

be coordinates for $t = e^s$ in $\mathbb{R}^b_+$. An integral *Laurent polynomial* $p(t)$ is an element of the ring $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_b^{\pm 1}]$ generated by the coordinates t_i and their inverses. We can write such a polynomial as

$$p(t) = \sum_{\alpha \in A} a_\alpha t^\alpha, \tag{A.1}$$

where the exponents $\alpha = (\alpha_1, \ldots, \alpha_b)$ range over a finite set $A \subset \mathbb{Z}^b$, where $t^\alpha = t_1^{\alpha_1} \cdots t_b^{\alpha_b}$, and where the coefficients $a_\alpha \in \mathbb{Z}$ are nonzero.

Newton polygons. The *Newton polygon* $N(p) \subset \mathbb{R}^b$ of $p(t) = \sum_A a_\alpha t^\alpha$ is the convex hull of the set of exponents $A \subset \mathbb{Z}^b$.

If we think of (s_i) as a basis for an abstract real vector space V, then $N(p)$ also naturally resides in V. Each monomial t^α appearing in $p(t)$ determines an open *dual cone* $C(t^\alpha) \subset V^*$ consisting of the linear maps $\phi: V \to \mathbb{R}$ that achieve their maximum on $N(p)$ precisely at α. Equivalently,

$$C(t^\alpha) = \{\phi\colon\ \phi(\alpha) > \phi(\beta) \text{ for all } \beta \neq \alpha \text{ in } A\}.$$

Positivity and Perron–Frobenius. A Laurent polynomial $p(t) \neq 0$ is *positive* if it has coefficients $a_\alpha > 0$.

Let

$$P(t) = P_{ij}(t) \in M_n\left(\mathbb{Z}\left[t_1^{\pm 1}, \ldots, t_b^{\pm 1}\right]\right)$$

be an $n \times n$ matrix of Laurent polynomials, with each entry either zero or positive. If for some $k > 0$, *every* entry of $P^k_{ij}(t)$ is a positive Laurent polynomial, we say $P(t)$ is an (integral) *Perron–Frobenius* matrix. By convention, we exclude the case where $n = 1$ and $P(1) = [1]$.

The matrix $P(t)$ is a traditional Perron–Frobenius matrix for every fixed value $t \in \mathbb{R}^b_+$. In particular, the largest eigenvalue $E(t)$ of $P(t)$ is simple, real and positive [23]. Since $P(1)$ is an integral matrix ($\neq [1]$), we always have $E(1) > 1$.

The main result of this section is:

THEOREM A.1. – *Let $E(t)$ be the leading eigenvalue of a Perron–Frobenius matrix $P(t)$. Then*:

(A) *The function $f(s) = \log E(e^s)$ is a convex function of $s \in \mathbb{R}^b$.*
(B) *The graph of $y = f(s)$ meets each ray from the origin in $\mathbb{R}^b \times \mathbb{R}$ at most once.*
(C) *The rays passing through the graph of $y = f(s)$ coincide with the dual cone $C(u^d)$ of the polynomial*

$$\Theta_F(t,u) = u^d + b_1(t)u^{d-1} + \cdots + b_d(t),$$

for any factor $\Theta_F(t,u)$ of $\det(uI - P(t))$ *satisfying $\Theta_F(t, E(t)) = 0$.*

Positivity and convexity. In addition to Laurent polynomials, it is also useful to consider finite *power sums* $p(t) = \sum a_\alpha t^\alpha$ with *real exponents* $\alpha \in \mathbb{R}^b$, and real coefficients $a_\alpha \in \mathbb{R}$. As for a Laurent polynomial, we say a nonzero power sum is *positive* if its coefficients are positive.

PROPOSITION A.2. – *If $p(t) = \sum a_\alpha t^\alpha$ is a positive power sum, then*

$$f(s) = \log p(e^s)$$

is a convex function of $s \in \mathbb{R}^b$.

Proof. – By restricting $f(s)$ to a line and applying a translation, we are reduced to showing $f''(0) \geqslant 0$ when $p(t)$ is a power sum in one variable t. But then

$$f''(0) = \frac{(\sum a_\alpha)(\sum \alpha^2 a_\alpha) - (\sum \alpha a_\alpha)^2}{(\sum a_\alpha)^2} \geqslant 0,$$

by Cauchy–Schwarz. □

Proof of Theorem A.1(A). – Since $E(t)$ agrees with the spectral radius of $P(t)$, and $P_{ij}(t) \geqslant 0$, we have

$$E(t) = \lim_{n \to \infty} \left(\sum_{i,j} P^n_{ij}(t) \right)^{1/n}.$$

Therefore $\log E(e^s) = \lim n^{-1} \log E_n(e^s)$, where $E_n(t) = \sum_{i,j} P^n_{ij}(t)$. Since the nonzero entries of $P(t)$ are positive, $E_n(t)$ is a positive Laurent polynomial, and thus $\log E_n(e^s)$ is convex by the preceding result. Therefore the limit $f(s) = \log E(e^s)$ is also convex.

Proof of Theorem A.1(B). – Let (s, y) be coordinates on $\mathbb{R}^b \times \mathbb{R}$, and let R be a ray through the origin. (B) is immediate when R is contained in y-axis. Dispensing with that case, we can pass to functions of a single variable $t = e^s$ by restricting to the plane spanned R and the y-axis, and we can assume R is the graph of a linear function of the form $y = \gamma s$, for $s > 0$.

Now the function $f(s)$ is convex and real analytic. Thus $f(s)$ is either strictly convex or affine ($f(s) = as + b$).

To treat the affine case, note $b = f(0) = \log E(1) > 0$, since the leading eigenvalue of the integral Perron–Frobenius matrix $P(1)$ is greater than one. Thus the equation $y = \gamma s = f(s) = as + b$ has at most one solution, and we are done.

Now assume $f(t)$ is strictly convex. Recall that $f(t)$ is a limit of the convex functions $f_n(t) = n^{-1} \log E_n(t)$. If the ray R crosses the graph of $y = f(s)$ twice, then it also crosses the graph of $y = f_n(s)$ twice for some finite value of n.

Fixing such an n, let $\beta_n = \beta/n$ where $a_\beta t^\beta$ is the term with largest exponent appearing in the power sum $E_n(t)$. Then $f'_n(s) \to \beta_n$ as $s \to \infty$, so by strict convexity we have $f'_n(s) < \beta_n$ for all finite s. Since $f_n(s)$ has more than one term, and $a_\beta > 1$, we also have:

(A.2) $$f_n(s) = \frac{\log E_n(e^s)}{n} > \beta_n s + \frac{\log a_\beta}{n} \geqslant \beta_n s.$$

Now suppose $y = f_n(s)$ crosses the line $y = \gamma s$ twice. Then by convexity, the slopes satisfy $\beta_n > f'_n(s) > \gamma$ at the second intersection point. But (A.2) then implies $f_n(s) > \gamma s$ for all $s > 0$, so in fact the ray $y = \gamma s$ has *no* intersections with the graph of $y = f_n(s)$.

Proof of Theorem A.1(C). – Passing again to functions of a single variable $t = e^s$, we consider the condition that the ray $y = \gamma s$, $s > 0$, passes through the graph of $y = E(t)$.

By assumption, $u = E(t)$ is the largest root of the equation

$$\Theta_F(t, u) = \sum a_{\alpha i} t^\alpha u^i = u^d + b_1(t) u^{d-1} + \cdots + b_d(t) = 0.$$

Since the coefficients $b_i(t)$ are homogeneous of degree i in the roots of Θ, we have

$$E(t) \asymp \sup \left| b_i(t) \right|^{1/i}.$$

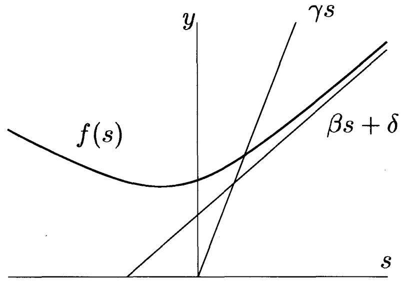

Fig. 8. A ray crossing the eigenvalue graph $y=f(s)=\log E(e^s)$.

In particular, as $t\to+\infty$, $E(t)$ grows like t^β with

$$\beta=\sup\alpha/(d-i), \tag{A.3}$$

the sup taken over all monomials $t^\alpha u^i$ appearing in Θ other than u^d. Thus as $s\to\infty$ the convex function $y=f(s)=\log E(e^s)$ is asymptotic to a linear function of the form $y=\beta s+\delta$.

Now consider the ray R through $(1,\gamma)$, with equation $y=\gamma s$, $s>0$. By (B), this ray meets $y=f(s)$ iff $\gamma>\beta$ (see Fig. 8). By (A.3), we have $\gamma>\beta$ iff

$$d\gamma>\alpha+i\gamma$$

for all monomials $t^\alpha u^i$ in Θ *other* than u^d. Thus R meets $y=f(s)$ iff the linear functional

$$\phi(\alpha,i)=1\cdot\alpha+\gamma\cdot i$$

achieves its maximum on the Newton polygon $N(\Theta)$ at the vertex $(\alpha,i)=(0,d)$ coming from u^d. This condition says exactly that R belongs to the dual cone $C(u^d)$. □

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