ANNALES SCIENTIFIQUES DE L'É.N.S.

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Annales scientifiques de l'É.N.S. 4^e série, tome 33, nº 4 (2000), p. 485-506

<http://www.numdam.org/item?id=ASENS_2000_4_33_4_485_0>

Ann. Scient. Éc. Norm. Sup.,
4[e] série, t. 33, 2000, p. 485 à 506.

NULL FORM ESTIMATES FOR $(1/2, 1/2)$ SYMBOLS AND LOCAL EXISTENCE FOR A QUASILINEAR DIRICHLET-WAVE EQUATION

BY HART F. SMITH AND CHRISTOPHER D. SOGGE

ABSTRACT. – We establish certain null form estimates of Klainerman–Machedon for parametrices of variable coefficient wave equations for the convex obstacle problem, and for wave equations with metrics of bounded curvature. These are then used to prove a local existence theorem for nonlinear Dirichlet-wave equations outside of convex obstacles. © 2000 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – Nous établissons certaines estimées de formes compatibles à la Klainerman–Machedon pour des parametrix d'équations d'ondes à coefficients variables dans le cas d'un obstacle convexe ou d'une métrique à courbure bornée. Ces estimées sont utilisées pour démontrer un théorème d'existence locale pour des équations d'ondes non linéaires avec conditions de Dirichlet en dehors d'obstacles convexes. © 2000 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

The purpose of this paper is to establish the following null form estimate

$$\|Q(du,dv)\|_{H^1(\mathbb{R}^{1+3}_{t,x})} \leqslant C\big(\|u_0\|_{H^2(\mathbb{R}^3)} + \|u_1\|_{H^1(\mathbb{R}^3)}\big)\big(\|v_0\|_{H^2(\mathbb{R}^3)} + \|v_1\|_{H^1(\mathbb{R}^3)}\big), \tag{1.1}$$

for solutions u and v to the Cauchy problem for certain wave equations

$$\begin{cases} \partial_t^2 u(t,x) = \Delta_{\mathbf{g}} u(t,x), \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x). \end{cases}$$

The null form Q may be of any one of the following forms

$$Q_0(du,dv) = \partial_t u(t,x)\partial_t v(t,x) - \sum_{i,j=1}^{3} \mathbf{g}^{ij}(x)\partial_{x_i}u(t,x)\partial_{x_j}v(t,x),$$
$$Q_{\alpha\beta}(du,dv) = \partial_{x_\alpha}u(t,x)\partial_{x_\beta}v(t,x) - \partial_{x_\beta}u(t,x)\partial_{x_\alpha}v(t,x),$$

where x_α and x_β may represent t or any x_i. Here $\sum_{1\leqslant i,j\leqslant 3} \mathbf{g}^{ij}(x)\,d\xi_i\,d\xi_j$ denotes the cometric associated with $\Delta_{\mathbf{g}}$.

For the Euclidean metric on $\mathbb{R}^3$, the estimate (1.1) was established globally by Klainerman and Machedon [2]. For smooth variable coefficient hyperbolic operators, local versions of (1.1) were established by the second author in [11].

This paper is concerned with two new cases. The first is the case that the wave equation is satisfied by u and v for x belonging to an open subset $\Omega \subset \mathbb{R}^3$ which has smooth boundary $\partial\Omega$, such that $\partial\Omega \subset \mathbb{R}^3$ is strictly geodesically concave with respect to $\mathbf{g}$. We then assume that u and v satisfy Dirichlet conditions on $\partial\Omega$,

$$u(t,x)|_{x\in\partial\Omega} = 0, \qquad v(t,x)|_{x\in\partial\Omega} = 0.$$

In this case we prove (1.1) for t in a small time interval and x in the intersection of a small ball with Ω. We point out that when Ω is the complement of a strictly convex obstacle in $\mathbb{R}^3$, with $\mathbf{g}$ the Euclidean metric, a partition of unity argument, together with the global Euclidean estimates of [2], implies (1.1) globally in x, for t in any bounded interval.

The second case that our results apply to is where $\mathbf{g}$ is a metric on a ball in $\mathbb{R}^3$, such that the components of the Riemann curvature tensor of $\mathbf{g}$ are bounded measurable functions, and such that the coordinate functions x_i are harmonic with respect to $\Delta_{\mathbf{g}}$. In such coordinates the metric coefficients $\mathbf{g}_{ij}$ have second derivatives belonging to $\mathrm{BMO}(\mathbb{R}^3)$, and the geodesic flow is uniquely determined and bilipschitz. The solution operator for the wave equation in this situation is studied in [8,9]. It can be written as the composition of an operator of Fourier integral type described below, with an operator which preserves the Sobolev spaces $H^j(\mathbb{R}^3)$, $j = 1, 2$. It then suffices to establish mapping properties for the Fourier integral part, which is the purpose of this paper. The results of this paper will then imply that (1.1) holds for such metrics provided that the norm is taken over a set of unit size.

In both of the above cases, the problem is reduced to establishing the following estimate

$$\big\|Q(dTf, dTg)\big\|_{L^2(\mathbb{R}^{1+3}_{t,x})} \leqslant C\|f\|_{H^1(\mathbb{R}^3)}\|g\|_{H^2(\mathbb{R}^3)}, \tag{1.2}$$

for an appropriate parametrix T of order 0.

For the obstacle problem, the main part of the parametrix takes the form

$$Tf(t,x) = \sum_{\pm} \int \mathrm{e}^{i\varphi^{\pm}(t,x,\xi)} a^{\pm}(t,x,\xi) \widehat{f}(\xi)\, d\xi,$$

where the phases $\varphi^{\pm}(t,x,\xi)$ satisfy the eikonal equation

$$\begin{aligned} \big|\partial_t\varphi^{\pm}(t,x,\xi)\big| &= \pm\big\|d_x\varphi^{\pm}(t,x,\xi)\big\|_{\mathbf{g}}, \\ \varphi^{\pm}(0,x,\xi) &= \langle x,\xi\rangle, \end{aligned} \tag{1.3}$$

and the symbols, which vanish for $|\xi| \leqslant 1$, satisfy the following modified $S^0_{2/3,1/3}$ estimates

$$\big|\langle\xi,\partial_\xi\rangle^N \partial^\beta_{t,x}\partial^\alpha_\xi a^{\pm}(t,x,\xi)\big| \leqslant C_{N,\alpha,\beta}\big(1+|\xi|\big)^{\frac{|\beta|}{3}-\frac{2|\alpha|}{3}}. \tag{1.4}$$

There is also a "diffractive" term, the estimation of which requires a modification of the argument for the main term, as will be discussed in Section 4.

In the case of the wave equation for metrics of bounded curvature tensor, the parametrix is more complicated. It takes the form

$$Tf(t,x) = \sum_{\pm}\sum_{k=1}^{\infty} \int \mathrm{e}^{i\varphi^{\pm}_k(t,x,\xi)} a^{\pm}_k(t,x,\xi) \widehat{f}_k(\xi)\, d\xi, \tag{1.5}$$

where $\widehat{f}(\xi)=\sum_{k=0}^{\infty}\widehat{f}_k(\xi)$, and for $k\geqslant 1$ the support of $\widehat{f}_k(\xi)$ lies in $2^{k-1}\leqslant|\xi|\leqslant 2^{k+1}$.

The phases $\varphi_k^{\pm}$, each of which is homogeneous of degree 1 in ξ, satisfy the eikonal equation (1.3) for a corresponding family of metrics $\mathbf{g}_k$, where $\mathbf{g}_k$ is a sequence of smooth metrics approximating the singular metric $\mathbf{g}$. This sequence of metrics satisfies the estimates

$$\left|\partial_x^{\alpha}\mathbf{g}_k(x)\right|\leqslant\begin{cases}C, & |\alpha|\leqslant 1,\\ Ck, & |\alpha|=2,\\ C_{\alpha}2^{k(|\alpha|-2)/2}, & |\alpha|\geqslant 3.\end{cases}\tag{1.6}$$

It also satisfies

$$\begin{aligned}\left|\mathbf{g}_k(x)-\mathbf{g}(x)\right|&\leqslant C2^{-k},\\ \left|\nabla_x\mathbf{g}_k(x)-\nabla_x\mathbf{g}(x)\right|&\leqslant C2^{-k/2}.\end{aligned}\tag{1.7}$$

The sequence of phases satisfies corresponding estimates

$$\sup_{|\xi|=1}\left|\partial_{t,x,\xi}^{\alpha}\varphi_k(t,x,\xi)\right|\leqslant\begin{cases}C, & |\alpha|\leqslant 2,\\ C,2^{k(|\alpha|-2)/2}, & |\alpha|\geqslant 2.\end{cases}\tag{1.8}$$

It also satisfies, for $k\geqslant j$,

$$\begin{aligned}\sup_{|\xi|=1}\left|\varphi_k^{\pm}(t,x,\xi)-\varphi_j^{\pm}(t,x,\xi)\right|&\leqslant C2^{-j},\\ \sup_{|\xi|=1}\left|\nabla_{t,x,\xi}\varphi_k^{\pm}(t,x,\xi)-\nabla_{t,x,\xi}\varphi_j^{\pm}(t,x,\xi)\right|&\leqslant C2^{-j/2}.\end{aligned}\tag{1.9}$$

Finally, the symbols satisfy the following modified $S^0_{1/2,1/2}$ estimates,

$$\left|\langle\xi,\partial_\xi\rangle^N\partial_{t,x}^{\beta}\partial_\xi^{\alpha}a_k^{\pm}(t,x,\xi)\right|\leqslant C_{N,\alpha,\beta}2^{k(\frac{|\beta|}{2}-\frac{|\alpha|}{2})}.\tag{1.10}$$

One of the main motivations for establishing the estimate (1.1) is that it gives local existence results for nonlinear wave equations with null form nonlinearities. Consider, for example, an N component system of the form

$$\begin{cases}\partial_t^2u-\Delta_{\mathbf{g}}u=F(u,du), & x\in\Omega,\\ u(0,\cdot)=u_0, \quad \partial_tu(0,\cdot)=u_1,\\ u(t,\cdot)|_{\partial\Omega}=0,\end{cases}\tag{1.11}$$

where Ω has geodesically concave boundary as discussed above. We assume that $F(u,du)=(F^1(u,du),\dots,F^N(u,du))$, and

$$F^i(u,du)=\sum_{j,k}a^i_{j,k}(t,x)\Gamma^i_{j,k}(u)B^i_{j,k}\big(du^j,du^k\big),$$

with $B^i_{j,k}$ being a null form associated with $\mathbf{g}$, $a^i_{j,k}\in C^\infty(\mathbb{R}\times\Omega)$, and $\Gamma^i_{j,k}\in C^\infty(\mathbb{C}^N)$.

If u is a solution of (1.11), then the vanishing of u and ∂_tu on $\partial\Omega$ imposes the following compatibility conditions on the data,

$$u_0(x)=u_1(x)=0, \quad \text{if } x\in\partial\Omega.\tag{1.12}$$

Conversely, under the hypotheses (1.12) on the data, we shall be able to obtain the following local existence result, generalizing results from [2] and [11].

THEOREM 1.1. – *Suppose that $u_j \in H^{2-j}(\Omega)$, $j = 0, 1$, have compact support and satisfy* (1.12). *Then there is a $T_* > 0$ and a unique solution $u \in H^2([0, T_*] \times \Omega)$ of* (1.11) *verifying*

$$\left\|Q\left(du^j, du^k\right)\right\|_{H^1([0,T_*]\times\Omega)} < \infty, \quad 1 \leqslant j, k \leqslant N.$$

We will return to this theorem in Section 4, in which we also discuss the reduction of the estimate (1.1) for the obstacle problem to that of (1.2), and handle the diffractive term. The main work of this paper, which occupies Sections 2 and 3, is to establish estimate (1.2) for parametrices of the above types. Since the main part of the parametrix associated with the convex obstacle problem is a special case of the type (1.5) that arises from bounded curvature metrics, we shall consider parametrices of the type (1.5) in Sections 2 and 3.

2. Further reductions

We begin by reducing the proof of estimate (1.2) to consideration of the case that $\widehat{f}(\xi)$ is supported in a dyadic annulus at scale 2^k, and $\widehat{g}(\xi)$ is supported in a ball of radius $c2^k$, where one may choose c arbitrarily small but fixed. To do this, we fix $\beta \in C_0^\infty((1/2, 2))$ so that $\sum_{-\infty}^{\infty} \beta(2^j s) = 1$, $s > 0$. We then set

$$\hat{f}_k(\xi) \doteq \beta\left(|\xi|/2^k\right)\widehat{f}(\xi)$$

so that $f = \sum f_k$ and supp $\widehat{f_k} \subset \{\xi \colon 2^{k-1} \leqslant |\xi| \leqslant 2^{k+1}\}$. We then write

$$\tilde{f}_j = \sum_{k<j+N} f_k, \qquad \tilde{g}_k = \sum_{j\leqslant k-N} g_j,$$

where N is a fixed number that is to be specified later. Recalling that the symbol of T vanishes for small $|\xi|$, we have the following identity,

$$Q(dTf, dTg) = \sum_{j=0}^{\infty} Q(dT\tilde{f}_j, dTg_j) + \sum_{k=0}^{\infty} Q(dTf_k, dT\tilde{g}_k) = I + II.$$

We consider I first. By the Strichartz estimates, which hold for the parametrix T by [10] and [7], [8], we may bound

$$\begin{aligned}\sum_{j=0}^{\infty}\left\|Q(dT\tilde{f}_j, dTg_j)\right\|_{L^2(\mathbb{R}^{1+3}_{t,x})} &\leqslant C\sum_{j=0}^{\infty}\|\tilde{f}_j\|_{H^{3/2}(\mathbb{R}^3)}\|g_j\|_{H^{3/2}(\mathbb{R}^3)}\\ &\leqslant C\left(\sum_{j=0}^{\infty}2^{-j}\|\tilde{f}_j\|^2_{H^{3/2}(\mathbb{R}^3)}\right)^{1/2}\|g\|_{H^2(\mathbb{R}^3)}\\ &\leqslant C\|f\|_{H^1(\mathbb{R}^3)}\|g\|_{H^2(\mathbb{R}^3)}.\end{aligned}$$

It thus remains to estimate II. To estimate its L^2 norm, we first observe that, for N large enough, the terms are essentially mutually orthogonal over k. This follows by a simple integration by parts argument, which yields

$$\left|\int Q(dTf_k, dT\tilde{g}_k)\overline{Q(dTf_{k'}, dT\tilde{g}_{k'})}\,dt\,dx\right| \leqslant C2^{-|k-k'|}\|f_k\|_2\|g_k\|_2\|f_{k'}\|_2\|g_{k'}\|_2,$$

provided $|k-k'| \geqslant 3$. Consequently,

$$\left\|\sum_k Q(dTf_k, dT\tilde{g}_k)\right\|^2_{L^2(\mathbb{R}^{1+3}_{t,x})} \leqslant C\sum_k \|Q(dTf_k, dT\tilde{g}_k)\|^2_{L^2(\mathbb{R}^{1+3}_{t,x})} + C\|f\|^2_{L^2(\mathbb{R}^3)}\|g\|^2_{L^2(\mathbb{R}^3)}.$$

Thus, to establish (1.2), it suffices to establish the following estimate, uniformly over k:

$$\|Q(dTf_k, dT\tilde{g}_k)\|_{L^2(\mathbb{R}^{1+3}_{t,x})} \leqslant C\|f_k\|_{H^1(\mathbb{R}^3)}\|\tilde{g}_k\|_{H^2(\mathbb{R}^3)}.$$

Finally, by the first estimate in (1.7), another application of the Strichartz estimates shows that we may replace the metric g in the form Q_0 by the metric g_k.

By writing $T = T^+ + T^-$, there are essentially two terms to consider: $Q(dT^+f, dT^+g)$, and $Q(dT^+f, dT^-g)$. In what follows we consider the term $Q(dT^+f, dT^+g)$; the arguments hold with minor modification for the latter term. To simplify notation we use $\varphi_k(t,x,\xi)$ to denote $\varphi_k^+(t,x,\xi)$.

In the formula for the operator dT^+, the terms where d hits the symbol $a(t,x,\xi)$ are easily handled by the Strichartz and energy estimates; thus it suffices to restrict attention to the term where d falls on the phase. Let

$$q_{kj}(t,x,\xi,\eta) = Q\big(d\varphi_k(t,x,\xi/|\xi|), d\varphi_j(t,x,\eta/|\eta|)\big).$$

The next reduction is to introduce polar coordinates for the η variable, $\eta = \rho\omega$, where $\rho \in \mathbb{R}^+$, and $\omega \in S^2$, the unit two-sphere. We now fix k and introduce the operator $T^\omega = T^\omega_k$ given by

$$T^\omega(f,g) = \sum_{j\leqslant k-N}\int \mathrm{e}^{i\varphi_k(t,x,\xi)+i\rho\varphi_j(t,x,\omega)}a_k(t,x,\xi)\,a_j(t,x,\rho\omega)q_{kj}(t,x,\xi,\omega)\widehat{f}_k(\xi)\widehat{g}_j(\rho)\,d\xi\,d\rho,$$

where $f \in L^2(\mathbb{R}^3)$ and $g \in L^2(\mathbb{R})$. A simple argument (see, e.g. [11]), now reduces the proof of (1.2) to showing that, for $g_\omega \in L^2(\mathbb{R}\times S^2)$, the following holds

$$\left\|\int T^\omega(f, g_\omega)\,d\omega\right\|_{L^2(dx\,dt)} \leqslant C\|f\|_{L^2(\mathbb{R}^3)}\|g_\omega\|_{L^2(\mathbb{R}\times S^2)}, \tag{2.1}$$

where the Fourier transforms of f and g_ω are restricted as above.

The next step, following [1], is to decompose phase space into regions on which the null form symbol q_{kj} is essentially constant. Since the phases φ_j depend on the scale j, this cannot be expressed simply in terms of the angle of ξ to η. To proceed, we set

$$\delta(l) = 2^{l-\frac{k}{4}},$$

and if β is as above, we write

$$q^l_{kj}(t,x,\xi,\eta) = \beta\big(\delta(l)^{-1}\times \mathrm{angle}\big[d_x\varphi_k(t,x,\xi), d_x\varphi_j(t,x,\eta)\big]\big)q_{kj}(t,x,\xi,\eta).$$

We then have

$$q_{kj}(t,x,\xi,\eta)=q^0_{kj}(t,x,\xi,\eta)+\sum_{l=1}^{\infty}q^l_{kj}(t,x,\xi,\eta),$$

where $q^0_{kj}(t,x,\xi,\eta)$ is supported in the region on which the angle is bounded by $2\cdot 2^{-k/4}$.

Using this decomposition, we write $T^\omega=\sum_l T^{l,\omega}$, where (recall that k is fixed)

$$\begin{aligned}&T^{l,\omega}(f,g)\\ (2.2)\quad &=\sum_{j\leqslant k-N}\int \mathrm{e}^{i\varphi_k(t,x,\xi)+i\rho\varphi_j(t,x,\omega)}a_k(t,x,\xi)a_j(t,x,\rho\omega)q^l_{kj}(t,x,\xi,\omega)\widehat{f}_k(\xi)\widehat{g}_j(\rho)\,d\xi\,d\rho.\end{aligned}$$

By (1.8), and (1.6) in the case of the null form Q_0 (recall that the metric $\mathbf{g}$ is replaced by $\mathbf{g}_k$), the following estimates are valid for $j\leqslant k$:

$$(2.3)\quad \begin{aligned}\left|\langle\xi,\partial_\xi\rangle^N\partial^\beta_{t,x}\partial^\alpha_\xi q^l_{kj}(t,x,\xi,\omega)\right| &\leqslant C_{N,\alpha,\beta}\delta(l)2^{\frac{k}{2}(|\beta|-|\alpha|)},\\ \left|\langle\xi,\partial_\xi\rangle^N\partial^\beta_{t,x}\partial^\alpha_\xi\left(q^l_{kj}(t,x,\xi,\omega)-q^l_{kk}(t,x,\xi,\omega)\right)\right| &\leqslant C_{N,\alpha,\beta}2^{-\frac{j}{2}}2^{\frac{k}{2}(|\beta|-|\alpha|)}.\end{aligned}$$

For the next step, if l is fixed, choose unit vectors $\xi^\mu\in S^2$ so that the balls $B(\xi^\mu,\delta(l))$ cover S^2 with bounded overlap (independent of $\delta(l)$). We then fix an associated partition of unity

$$1=\sum_\mu \Psi^\mu(\xi),\quad \xi\neq 0$$

consisting of $C^\infty(\mathbb{R}^3\backslash 0)$ functions that are homogeneous of degree zero, and which satisfy

$$\operatorname{supp}\Psi^\mu\cap S^2\subset B\big(\xi^\mu,2\delta(l)\big),\qquad D^\alpha\Psi^\mu(\xi)=\mathrm{O}\big(\delta(l)^{-|\alpha|}\big)\quad\text{if } |\xi|=1.$$

If we then write

$$\widehat{f}(\xi)=\sum_\mu \widehat{f}_\mu(\xi)$$

we have the following

LEMMA 2.1. – *For fixed N as above sufficiently large, the following holds for $l\geqslant 0$,*

$$\begin{aligned}&\left\|\sum_\mu\int T^{l,\omega}(f_\mu,g_\omega)\,d\omega\right\|^2_{L^2(dx\,dt)}\\ &\quad\leqslant C\sum_\mu\left\|\int T^{l,\omega}(f_\mu,g_\omega)\,d\omega\right\|^2_{L^2(dx\,dt)}+C\|f\|^2_{L^2(\mathbb{R}^3)}\|g_\omega\|^2_{L^2(\mathbb{R}\times S^2)}.\end{aligned}$$

Proof. – We shall show that if C is a large constant and $|\xi^\mu-\xi^{\mu'}|\geqslant C\delta(l)$, then for any $M>0$,

$$(2.4)\quad \int T^{l,\omega}(f_\mu,g)(t,x)\overline{T^{l,\omega'}(f_{\mu'},g')(t,x)}\,dt\,dx\leqslant C_M2^{-kM}\|f_\mu\|_2\|f_{\mu'}\|_2\|g\|_2\|g'\|_2.$$

This follows by considering the operator $(T^{l,\omega})^*T^{l,\omega'}$. If

$$q^l_{kj}(t,x,\xi,\rho\omega)q^l_{kj'}(t,x,\xi',\rho'\omega')\widehat{f}_\mu(\xi)\widehat{f}_{\mu'}(\xi')\neq 0,$$

then the angle of $\nabla_x\varphi_k(t,x,\xi) + \rho\nabla_x\varphi_j(t,x,\omega)$ to $\nabla_x\varphi_k(t,x,\xi') + \rho'\nabla_x\varphi_{j'}(t,x,\omega')$ is bounded below by $\delta(l)$, provided $|\xi^\mu - \xi^{\mu'}| \geqslant C\delta(l)$ for some large C, $|\xi|, |\xi'| \approx 2^k$, and $\rho, \rho' \leqslant 2^{k-N}$ with N sufficiently large. On account of this,

$$\big|\big(\nabla_x\varphi_k(t,x,\xi) + \rho\nabla_x\varphi_j(t,x,\omega)\big) - \big(\nabla_x\varphi_k(t,x,\xi') + \rho'\nabla_x\varphi_{j'}(t,x,\omega')\big)\big| \geqslant c2^k\delta(l) \geqslant c2^{3k/4}.$$

An easy integration by parts in x using (1.10) and the first part of (2.3) yields (2.4). □

LEMMA 2.2. –

$$\left\|\int T^{0,\omega}(f_\mu, g_\omega)\,d\omega\right\|_{L^2(dx\,dt)} \leqslant C\|f_\mu\|_{L^2(\mathbb{R}^3)}\|g_\omega\|_{L^2(\mathbb{R}\times S^2)}.$$

Proof. – For fixed j and fixed (t,x), the function $T^{0,\omega}(f_\mu, g_{j\omega})(t,x)$ vanishes unless ω is in a set of volume $\delta(0)^2 = 2^{-k/2}$. Thus,

$$\left\|\int T^{0,\omega}(f_\mu, g_\omega)\,d\omega\right\|_{L^2(dx\,dt)} \leqslant \sum_j 2^{-k/4}\big\|T^{0,\omega}(f_\mu, g_{j\omega})\big\|_{L^2(dx\,dt\,d\omega)}.$$

Because of (2.3), the operator

$$Af(x) = \int \mathrm{e}^{i\varphi_k(t,x,\xi)} a_k(t,x,\xi)\, q^0_{kj}(t,x,\xi,\omega)\widehat{f}(\xi)\,d\xi \tag{2.5}$$

has $L^2 \to L^2$ norm, for each fixed t, less than $C\,\delta(0) = C\,2^{-k/4}$, with C independent of t. For the obstacle problem, where, for all k, g_k equal a fixed smooth metric g, this just follows from standard L^2 estimates for Fourier integral operators. The general case where there is a k-dependence also follows from standard L^2 estimates along with (1.6). (See [8,9].)

The aforementioned bounds for Af immediately yield

$$\big\|T^{0,\omega}(f_\mu, g_{j\omega})\big\|_{L^2(dx\,dt)} \leqslant C2^{-k/4}\|f_\mu\|_{L^2(\mathbb{R}^3)}\|\widehat{g}_{j\omega}\|_{L^1(\mathbb{R})} \leqslant C2^{-k/4}2^{j/2}\|f_\mu\|_{L^2(\mathbb{R}^3)}\|g_{j\omega}\|_{L^2(\mathbb{R})}.$$

The lemma now follows since

$$\sum_{j\leqslant k} 2^{(j-k)/2}\|g_{j\omega}\|_{L^2(\mathbb{R}\times S^2)} \leqslant \|g_\omega\|_{L^2(\mathbb{R}\times S^2)}. \qquad \square$$

3. Null form estimates

In this section, we show that, for each fixed k, μ, and $l \geqslant 1$, the following holds:

$$\left\|\int T^{l,\omega}(f_\mu, g_\omega)\,d\omega\right\|_{L^2(dx\,dt)} \leqslant C\delta(l)^{1/4}\big|\log\delta(l)\big|^{1/2}\|f_\mu\|_{L^2(\mathbb{R}^3)}\|g_\omega\|_{L^2(\mathbb{R}\times S^2)}, \tag{3.1}$$

with constant C independent of k, l, and μ. Together with Lemmas 2.1 and 2.2, this implies estimate (2.1), after summing over l, which in turn implies the desired estimate (1.2).

We establish (3.1) by splitting the operator $T^{l,\omega}$ into two pieces. Let

(3.2)
$$\begin{aligned} T_1^{l,\omega}(f,g) &= \sum_{\{j:\ 2^j > 2^{k/2}\delta(l)^{-1}\}} T^{l,\omega}(f,g_j), \\ T_2^{l,\omega}(f,g) &= \sum_{\{j:\ 2^j \leqslant 2^{k/2}\delta(l)^{-1}\}} T^{l,\omega}(f,g_j). \end{aligned}$$

For the operator $T_1^{l,\omega}$, note that $2^{-j/2} \leqslant \delta(l)$ for the indices arising, since $2^{k/2} \geqslant \delta(l)^{-1}$. Hence, by the second part of (1.9) and the definition of q_{kj}^l, the symbol of $T_1^{l,\omega}$ vanishes unless

$$\operatorname{angle}\big(d_x\varphi_k(t,x,\xi), d_x\varphi_k(t,x,\omega)\big) \leqslant C\delta(l).$$

Since the map $\xi \to d_x\varphi_k(t,x,\xi)/|d_x\varphi_k(t,x,\xi)|$ is a C^1 diffeomorphism of the unit sphere, with uniform bounds over t, x, and k, for t small, it follows that the integrand vanishes unless $|\xi^\mu - \omega| \leqslant C\delta(l)$. Consequently, by the Schwarz inequality

$$\left\| \int T_1^{l,\omega}(f_\mu, g_\omega)\, d\omega \right\|_{L^2(dx\,dt)} \leqslant \delta(l) \big\| T_1^{l,\omega}(f_\mu, g_\omega)\big\|_{L^2(dx\,dt\,d\omega)}.$$

For the piece $T_1^{l,\omega}$, the estimate (3.1) is thus implied by the following

THEOREM 3.1. – *The following holds, with C independent of l, ω, k,*

$$\big\|T_1^{l,\omega}(f,g)\big\|_{L^2(dx\,dt)} \leqslant C\delta(l)^{-3/4}\big|\log\delta(l)\big|^{1/2}\|f\|_{L^2(\mathbb{R}^3)}\|g\|_{L^2(\mathbb{R})}.$$

We postpone the proof of Theorem 3.1, and first establish the somewhat easier

THEOREM 3.2. – *The following holds, with C independent of l, μ, k,*

$$\left\| \int T_2^{l,\omega}(f_\mu, g_\omega)\, d\omega \right\|_{L^2(dx\,dt)} \leqslant C\delta(l)\big|\log\delta(l)\big|\,\|f_\mu\|_{L^2(\mathbb{R}^3)}\|g_\omega\|_{L^2(\mathbb{R}\times S^2)}.$$

Proof. – We split the sum over j in (3.2) into three distinct cases: $2^j \leqslant \delta(l)^{-2}$, $\delta(l)^{-2} < 2^j \leqslant 2^{k/2}$, and $2^{k/2} < 2^j \leqslant \delta(l)^{-2}2^{k/2}$.

Case 1: $2^j \leqslant \delta(l)^{-2}$**.** In this case the index j runs over $O(|\log\delta(l)|)$ values. Also, for fixed j and fixed (t,x), the integrand vanishes unless ω lies in a set in S^2 of area $\delta(l)^2$. Hence, by the Schwarz inequality, it suffices to establish the following estimate, uniformly in j and ω:

(3.3)
$$\big\|T^{l,\omega}(f,g_j)\big\|_{L^2(dx\,dt)} \leqslant C\|f\|_{L^2(\mathbb{R}^3)}\|g_j\|_{L^2(\mathbb{R})}.$$

We now write $\widehat{f}(\xi) = \sum_\nu \widehat{f}_\nu(\xi)$ where $\widehat{f}_\nu$ is supported in a cone of angle $2^{-k/2}$ about a unit vector ξ^ν. The estimate (3.3) is a result of the following

(3.4)
$$\begin{aligned} &\left| \int T^{l,\omega}(f_\nu, g_j)\overline{T^{l,\omega}(f_{\nu'}, g_j)}\, dt\, dx \right| \\ &\qquad \leqslant C\big(1 + 2^{k/2}|\xi^\nu - \xi^{\nu'}|\big)^{-N}\|f_\nu\|_{L^2(\mathbb{R}^3)}\|f_{\nu'}\|_{L^2(\mathbb{R}^3)}\|g_j\|^2_{L^2(\mathbb{R})}. \end{aligned}$$

The (t,x) integrand in (3.4) is dominated by

$$\delta(l)^2 \left| \int \mathrm{e}^{i\varphi_k(t,x,\xi)-i\varphi_k(t,x,\xi')} a_{\nu,\nu'}(t,x,\xi,\xi') \widehat{f}_\nu(\xi) \widehat{f}_{\nu'}(\xi')\, d\xi\, d\xi' \right| \times \left| \int \mathrm{e}^{i\rho\varphi_j(t,x,\omega)} a_j(t,x,\rho\omega) \widehat{g}_j(\rho)\, d\rho \right|^2, \tag{3.5}$$

where, by (1.10) and (2.3),

$$\left|\partial^\beta_{t,x} \partial^\alpha_{\xi,\xi'} \langle \xi^\nu, \partial_\xi\rangle^m \langle \xi^{\nu'}, \partial_{\xi'}\rangle^{m'} a_{\nu,\nu'}(t,x,\xi,\xi')\right| \leqslant C 2^{\frac{k}{2}(|\beta|-|\alpha|)-k(m+m')}. \tag{3.6}$$

Since $\rho \leqslant \delta(l)^{-2} \leqslant 2^{k/2}$, the operator $2^{-k/2}\partial_x$ applied to the expression inside the absolute value sign in (3.5) leads to an expression of the same form. Furthermore, on the (ξ, ξ') support of the symbol in (3.5), the following holds,

$$2^{-k/2}\left|\partial_x \varphi_k(t,x,\xi) - \partial_x \varphi_k(t,x,\xi')\right| \geqslant c 2^{k/2} |\xi^\nu - \xi^{\nu'}|.$$

Integration by parts in x now bounds the left-hand side of (3.4) by

$$\frac{\delta(l)^2}{\left(1+2^{k/2}|\xi^\nu - \xi^{\nu'}|\right)^N} \int \left| \int \mathrm{e}^{i\varphi_k(t,x,\xi)-i\varphi_k(t,x,\xi')} a_{\nu,\nu'}(t,x,\xi,\xi') \widehat{f}_\nu(\xi) \widehat{f}_{\nu'}(\xi')\, d\xi\, d\xi' \right| \times \left| \int \mathrm{e}^{i\rho\varphi_j(t,x,\omega)} a_j(t,x,\rho\omega) \widehat{g}_j(\rho)\, d\rho \right|^2 dt\, dx,$$

where $a_{\nu,\nu'}(t,x,\xi,\xi')$ is a symbol satisfying the same estimates (3.6). Next, following [6], we replace the phase $\varphi_k(t,x,\xi)$ by $\langle \nabla_\xi \varphi_k(t,x,\xi^\nu), \xi\rangle$, modulo an error that is absorbed into the symbol, and similarly for $\varphi_k(t,x,\xi')$. The left-hand side of (3.4) is thus bounded by

$$\frac{\delta(l)^2}{\left(1+2^{k/2}|\xi^\nu - \xi^{\nu'}|\right)^N} \times \int f^*_\nu\left(\nabla_\xi \varphi_k(t,x,\xi^\nu)\right) f^*_{\nu'}\left(\nabla_\xi \varphi_k(t,x,\xi^{\nu'})\right) g^*_j\left(\varphi_j(t,x,\omega)\right)^2 dt\, dx, \tag{3.7}$$

where

$$f^*_\nu(y) = 2^{2k} \int \left(1 + 2^{k/2}|y-z| + 2^k |\langle \xi^\nu, y-z\rangle|\right)^{-4} |f_\nu(z)|\, dz,$$

and

$$g^*_j(s) = \int \left(1 + 2^j|s-r|\right)^{-2} |g_j(r)|\, ds,$$

hence

$$\|f^*_\nu\|_{L^2(dy)} \leqslant C\|f_\nu\|_{L^2(\mathbb{R}^3)}, \qquad \|g^*_j\|_{L^2(ds)} \leqslant C\|g_j\|_{L^2(\mathbb{R})}.$$

The change of variables $(t,x) \to (\varphi_j(t,x,\omega), \nabla_\xi \varphi_k(t,x,\xi^\nu))$ has Jacobian comparable to $\delta(l)^2$. An application of the Schwarz inequality to (3.7) thus yields (3.4).

Case 2: $\delta(l)^{-2} < 2^j \leqslant 2^{k/2}$. Consider the operator $\widetilde{T}^{l,\omega}$ obtained by replacing $q^l_{jk}(t,x,\xi,\omega)$ in Eq. (2.2) by $q^l_{jk}(t,x,\xi,\omega) - q^l_{kk}(t,x,\xi,\omega)$. The proof of the previous case, together with the second set of estimates in (2.3), shows that for $2^j \leqslant 2^{k/2}$ the following holds,

$$\left\| \int \widetilde{T}^{l,\omega}(f_\mu, g_{j\omega})\, d\omega \right\|_{L^2(dx\, dt)} \leqslant C 2^{-j/2} \|f_\mu\|_{L^2(\mathbb{R}^3)} \|g_{j\omega}\|_{L^2(\mathbb{R}\times S^2)}.$$

Applying the Schwarz inequality over j such that $2^j \geqslant \delta(l)^{-2}$ yields the estimate of Theorem 3.2 for this case if $T^{l,\omega}$ is replaced by $\widetilde{T}^{l,\omega}$. It thus remains to establish the same estimate for the term

$$S^{l,\omega}(f_\mu, g_\omega) = \left(\int e^{i\varphi_k(t,x,\xi)} a_k(t,x,\xi) q^l_{kk}(t,x,\xi,\omega) \widehat{f}_\mu(\xi)\, d\xi\right) \sum_j \int e^{i\rho\varphi_j(t,x,\omega)} a_j(t,x,\rho\omega) \widehat{g}_j(\rho)\, d\rho.$$

The ξ-integrand vanishes unless $|\omega - \xi^\mu| \leqslant \delta(l)$, hence

$$\left\| \int S^{l,\omega}(f_\mu, g_\omega)\, d\omega \right\|_{L^2(dx\,dt)} \leqslant C\delta(l) \left\| S^{l,\omega}(f_\mu, g_\omega)\right\|_{L^2(dx\,dt\,d\omega)}.$$

The proof of estimate (3.4) establishes the following bound,

$$\left| \int S^{l,\omega}(f_\nu, g) \overline{S^{l,\omega}(f_{\nu'}, g)}\, dt\, dx \right| \leqslant C\bigl(1 + 2^{k/2}|\xi^\nu - \xi^{\nu'}|\bigr)^{-N} \|f_\nu\|_{L^2(\mathbb{R}^3)} |f_{\nu'}\|_{L^2(\mathbb{R}^3)} \times \bigl(\sup_y \left\|Pg(\cdot, y)\right\|_{L^2(ds)}\bigr) \bigl(\sup_{y'} \left\|P'g(\cdot, y')\right\|_{L^2(ds')}\bigr),$$

where Pg is an operator of the form

$$Pg = \sum_j \int e^{i\rho\varphi_j(t,x,\omega)} a_j(t,x,\rho\omega) \widehat{g}_j(\rho)\, d\rho,$$

written in the new coordinates

$$(s, y) = \bigl(\varphi_k(t,x,\omega), \nabla_\xi \varphi_k(t,x,\xi^\nu)\bigr),$$

and $P'g$ is the same form with ν replaced by ν'. Using (1.9) we may write Pg in the form

$$Pg(s,y) = \sum_j \int e^{is\rho} a_j(s,y,\rho) \widehat{g}_j(\rho)\, d\rho,$$

where the new symbol satisfies

$$\left|\partial_s^m \partial_\rho^n a_j(s,y,\rho)\right| \leqslant C\bigl(2^{j/2}\delta(l)^{-2}\bigr)^m 2^{-jn}, \quad m \leqslant 1.$$

A simple integration by parts establishes the following bound,

$$\left| \int e^{is(\rho-\rho')} a_j(s,y,\rho) \widehat{g}_j(\rho) \overline{a_{j'}(s,y,\rho') \widehat{g}_{j'}(\rho')}\, d\rho\, d\rho'\, ds \right| \leqslant C\bigl(1 + \delta(l)^2\, 2^{\max(j,j')/2}\bigr)^{-1} \|g_j\|_{L^2(\mathbb{R})} \|g_{j'}\|_{L^2(\mathbb{R})}. \tag{3.8}$$

Summing over j, j' such that $2^j \geqslant \delta(l)^{-2}, 2^{j'} \geqslant \delta(l)^{-2}$, yields the following,

$$\sup_y \|Pg\|_{L^2(ds)} \leqslant C\bigl|\log \delta(l)\bigr| \|g\|_{L^2(\mathbb{R})},$$

which completes the proof for the second case.

Case 3: $2^{k/2} < 2^j \leqslant 2^{k/2}\delta(l)^{-1}$. There are $\mathrm{O}(|\log\delta(l)|)$ terms j, so as in the first case it suffices to establish the estimate (3.4) uniformly over j. Let

$$v(t,x) = \frac{1}{|\xi^\nu - \xi^{\nu'}|} \times \big(\text{projection of } \nabla_x\varphi_k(t,x,\xi^\nu) - \nabla_x\varphi_k(t,x,\xi^{\nu'}) \text{ onto } \nabla_x\varphi_k(t,x,\xi^\nu)^\perp\big).$$

It follows from (1.8) that

$$\big|\partial_{t,x}^\alpha v(t,x)\big| \leqslant C_\alpha 2^{\frac{k}{2}|\alpha|}.$$

Next note that, if $q_{kj}^l(t,x,\xi,\omega)f_\nu(\xi)$ is nonzero, then $|\xi^\nu - \omega| \leqslant C\,\delta(l)$. Also note that $2^{-j/2} \leqslant \delta(l)$. The following is thus seen to hold by (1.8),

$$\big|\partial_{t,x}^\alpha\langle v(t,x), \nabla_x\varphi_j(t,x,\omega)\rangle\big| = \big|\partial_{t,x}^\alpha\langle v(t,x), \nabla_x\varphi_j(t,x,\omega) - \nabla_x\varphi_k(t,x,\xi^\nu)\rangle\big| \leqslant C_\alpha\delta(l)2^{\frac{k}{2}|\alpha|}.$$

Since $\rho\delta(l) \leqslant 2^{k/2}$, it follows that for any N one may write

$$\big(2^{-k/2}\langle v(t,x), \nabla_x\rangle\big)^N\left(\int \mathrm{e}^{i\rho\varphi_j(t,x,\omega)}a_j(t,x,\rho\omega)\widehat{g}_j(\rho)\,d\rho\right)$$

as an expression of the same form as that in parentheses, but with a new symbol which satisfies the following estimates

$$\big|\partial_{t,x}^\alpha\partial_\rho^m\tilde{a}_j(t,x,\rho)\big| \leqslant C_{\alpha,m}2^{\frac{k}{2}|\alpha| - mj}.$$

These estimates imply the following bound,

$$\left|\int \mathrm{e}^{i\rho\varphi_j(t,x,\omega)}\tilde{a}_j(t,x,\rho\omega)\widehat{g}_j(\rho)\,d\rho\right| \leqslant Cg_j^*\big(\varphi_j(t,x,\omega)\big).$$

We next note that if $\widehat{f}_\nu(\xi)$ and $\widehat{f}_{\nu'}(\xi')$ are nonzero, and $|\xi^\nu - \xi^{\nu'}| \geqslant C\,2^{-k/2}$, then

$$\big\langle v(t,x), \nabla_x\varphi_k(t,x,\xi) - \nabla_x\varphi_k(t,x,\xi')\big\rangle \approx 2^k|\xi^\nu - \xi^{\nu'}|.$$

The proof of estimate (3.4) from the first case now carries over to the third case, where in establishing the estimate (3.7) for the third case, one integrates by parts using

$$\big(\big\langle v(t,x), \nabla_x\varphi_k(t,x,\xi) - \nabla_x\varphi_k(t,x,\xi')\big\rangle\big)^{-1}\big\langle v(t,x), \nabla_x\big\rangle.$$

Since we have handled all three cases, the proof of Theorem 3.2 is complete. □

The proof of Theorem 3.1 rests on the following two lemmas estimating the gradients of the phase function.

LEMMA 3.3. – *Let* $\Phi(t,x,t',x',\xi) = \varphi(t,x,\xi) - \varphi(t',x',\xi)$, *where* $\varphi = \varphi_k$ *for some* k. *Suppose that* ω *is a unit vector, and let* $\delta = \mathrm{angle}(\omega,\xi)$. *If* $|\xi| = 2^k$, *then for some* $c > 0$,

$$\begin{aligned}(3.9)\qquad 2^{k/2}\big|\nabla_\xi\Phi(t,x,t',x',\xi)\big| + \big|\Phi(t,x,t',x',\xi)\big| + 2^j\big|\Phi(t,x,t',x',\omega)\big| \\ \geqslant c\big(2^{k/2}\big|\nabla_\xi\Phi(t,x,t',x',\omega)\big| + 2^{k/2}\delta|t-t'|\big),\end{aligned}$$

for all j, k *and* δ *such that* $j \leqslant k$, *and* $2^j\delta \geqslant 2^{k/2}$.

Proof. – We have $2^k \geqslant 2^j \geqslant 2^{k/2}\delta^{-1}$, so introducing the new variables

$$y_j(t,x) = \partial_{\xi_j}\varphi(t,x,\xi), \qquad \mu = \xi/|\xi|,$$

the left-hand side of (3.9) is larger than

$$2^{k/2}|y-y'| + 2^{k/2}\delta^{-1}\big(\big|\langle \mu, y-y'\rangle\big| + \big|F(t,y)-F(t',y')\big|\big), \tag{3.10}$$

where $F(t,y)$ is $\varphi(t,x,\omega)$ written in the coordinates (t,y). We begin by showing that the quantity (3.10) is larger than

$$2^{k/2}\big(|y-y'| + \delta|t-t'|\big).$$

To see this, note that the C^1 distance of $F(t,y)$ to $\langle \mu, y\rangle$ is of size δ, so that

$$F(t',y) - F(t',y') = \langle \mu, y-y'\rangle + \mathrm{O}\big(\delta|y-y'|\big).$$

Thus (3.10) dominates

$$2^{k/2}\big(|y-y'| + \delta^{-1}\big|F(t,y)-F(t',y)\big|\big).$$

We will be done by establishing the following identity,

$$\partial_t F(t,y) = \big\|d_x\varphi(t,x,\omega)\big\|_{\mathbf{g}} - \mathbf{g}\big(d_x\varphi(t,x,\omega), d_x\varphi(t,x,\mu)\big)/\big\|d_x\varphi(t,x,\mu)\big\|_{\mathbf{g}} \approx \delta^2,$$

where $\mathbf{g} = \mathbf{g}_k$. To see this, let $x = x(t,y)$ denote x in the (t,y) coordinates. Then $(x, d_x\varphi(t,x,\mu))$ is the backwards hamiltonian curve through (y,μ). Hamilton's equations thus yield

$$\partial_t x_i = -\sum_{m=1}^{3} \mathbf{g}^{mi}(t,x)\partial_{x_m}\varphi(t,x,\mu)/\big\|d_x\varphi(t,x,\mu)\big\|_{\mathbf{g}}.$$

Thus,

$$\begin{aligned}
\partial_t F(t,y) &= \partial_t\varphi\big(t,x(t,y),\omega\big) \\
&= \partial_t\varphi(t,x,\omega) + \sum_{i=1}^{3} \partial_{x_i}\varphi(t,x,\omega)\partial_t x_i \\
&= \big\|d_x\varphi(t,x,\omega)\big\|_{\mathbf{g}} - \mathbf{g}\big(d_x\varphi(t,x,\omega), d_x\varphi(t,x,\mu)\big)/\big\|d_x\varphi(t,x,\mu)\big\|_{\mathbf{g}}.
\end{aligned}$$

To finish the proof of the lemma, let $f_j(t,y)$ denote $\partial_{\xi_j}\varphi(t,x,\omega)$ in the (t,y) coordinates. Then the C^1 distance of f_j to y_j is comparable to δ, so that $|\partial_t f_j(t,y)| \leqslant \delta$. Consequently,

$$\big|\nabla_\xi \Phi(t,x,t',x',\omega)\big| \leqslant \sum_{j=1}^{3} \big|f_j(t,y) - f_j(t',y')\big| \leqslant C\big(|y-y'| + \delta|t-t'|\big). \qquad \square$$

LEMMA 3.4. – *Let $\Phi(t,x,\xi,\xi') = \varphi(t,x,\xi) - \varphi(t,x,\xi')$, where $\varphi = \varphi_k$ for some k. Suppose that*

$$\left|\omega - \frac{\xi}{|\xi|}\right|, \quad \left|\omega - \frac{\xi'}{|\xi'|}\right| \in \big[C^{-1}\delta, C\delta\big],$$

that $|\xi|, |\xi'| \in [2^{k-1}, 2^{k+1}]$, *and that* $|\rho|, |\rho'| \in [0, 2^k]$. *Then for some* $c > 0$, *independent of* k, δ,

$$|\nabla_{t,x}\Phi(t,x,\xi,\xi') + \nabla_{t,x}\Phi(t,x,\rho\omega,\rho'\omega)| \geqslant c\big(2^k\delta \times \text{angle}(\xi,\xi') + \delta^2|\rho-\rho'|\big). \tag{3.11}$$

Proof. – Let $w = \nabla_x\varphi(t,x,\xi)$, $w' = \nabla_x\varphi(t,x,\xi')$, and $\mu = \nabla_x\varphi(t,x,\omega)$. Also let $\alpha = \rho - \rho'$. The conditions of the statement imply that the angle of w or w' to μ is comparable to δ.

By the eikonal equations, the left-hand side of (3.11) dominates

$$|w - w' + \alpha\mu| + \big|\|w\| - \|w'\| + \alpha\|\mu\|\big|, \tag{3.12}$$

where $\|\cdot\|$ denotes the norm in the metric $\mathbf{g}_k(t,x)$. We consider the case $\alpha \geqslant 0$; the case $\alpha \leqslant 0$ follows by symmetry upon exchanging ξ and ξ'. Also, by scaling α, we may assume that $\|\mu\| = 1$. The quantity (3.12) then dominates

$$\begin{aligned}\|w\| + \alpha - \|w + \alpha\mu\| &\geqslant c2^{-k}\big((\|w\| + \alpha)^2 - \|w + \alpha\mu\|^2\big)\\ &= c2^{-k}\|w\|\alpha\left\|\frac{w}{\|w\|} - \mu\right\|^2\\ &\geqslant c\delta^2\alpha.\end{aligned}$$

We next observe that (3.12) dominates the following quantity (recall that $\|\mu\| = 1$)

$$2^k\left\|\frac{w}{\|w\|} - \mu - r\left(\frac{w'}{\|w'\|} - \mu\right)\right\|, \tag{3.13}$$

where $r = \|w'\|/\|w\| \in [c, c^{-1}]$. By making a linear transformation, we may replace the g norm $\|\cdot\|$ by the Euclidean norm $|\cdot|$, and assume that

$$\mu = (1,0,0), \qquad \frac{w}{|w|} = (\cos\theta, \sin\theta, 0), \qquad \frac{w'}{|w'|} = \big(\sqrt{1-z^2}\cos\gamma, \sqrt{1-z^2}\sin\gamma, z\big),$$

where θ, γ, z are small. The quantity (3.13) is then comparable to

$$2^k\big(\big|1 - \cos\theta - r\big(1 - \sqrt{1-z^2}\cos\gamma\big)\big| + \big|\sin\theta - r\sqrt{1-z^2}\sin\gamma\big| + |z|\big),$$

which in turn is comparable to

$$2^k\big(\big|1 - \cos\theta - r(1 - \cos\gamma)\big| + \big|\sin\theta - r\sin\gamma\big| + |z|\big).$$

By the half angle formula, this equals

$$2^k\left(\left|\sin\theta\tan\frac{\theta}{2} - 2r\sin^2\frac{\gamma}{2}\right| + \left|\sin\theta - 2r\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}\right| + |z|\right).$$

Since $\cos(\gamma/2) \approx 1$, this in turn dominates

$$2^k\left(|\sin\theta|\left|\tan\frac{\theta}{2} - \tan\frac{\gamma}{2}\right| + |z|\right) \geqslant c2^k\big(\delta|\theta-\gamma| + |z|\big) \geqslant c2^k\delta \times \text{angle}(\xi,\xi'). \qquad \square$$

Proof of Theorem 3.1. – To complete the proof, we make a further decomposition

$$T_1^{l,\omega}=\sum_{\nu,j,s,z}T^{l,\omega}_{\nu,j,s,z},$$

where

$$T^{l,\omega}_{\nu,j,s,z}F(t,x)=\int \mathrm{e}^{i\varphi_k(t,x,\xi)+i\rho\varphi_k(t,x,\omega)}a^{l,\omega}_{\nu,j,s,z}(t,x,\xi,\rho)\widehat{F}(\xi,\rho)\,d\xi\,d\rho. \tag{3.14}$$

The index ν corresponds to a set of unit vectors ξ^ν evenly spaced by $c2^{-k/2}\delta(l)^{-1}$, for some small c to be determined independent of k and l. The index j runs over the integers such that

$$2^k\geqslant 2^j\geqslant 2^{k/2}\delta(l)^{-1}.$$

The indices z and s run over lattices such that

$$2^{k/2}z\in\mathbb{Z}^3,\qquad 2^{k/2}\delta(l)s\in\mathbb{Z}.$$

The symbol $a^{l,\omega}_{\nu,j,s,z}(t,x,\xi,\rho)$ is supported in the set where $|\xi|\in[2^{k-1},2^{k+1}]$, $\rho\in[2^{j-1},2^{j+1}]$, and where

$$\left|\frac{\xi}{|\xi|}-\xi^\nu\right|\leqslant c2^{-k/2}\delta(l)^{-1},\quad \left|\nabla_\xi\varphi_k(t,x,\omega)-z\right|\leqslant 2\cdot 2^{-k/2},\quad |t-s|\leqslant 2\cdot 2^{-k/2}\delta(l)^{-1}.$$

The symbol furthermore satisfies the estimates

$$\left|\partial^\beta_{t,x}\partial^\alpha_\xi\partial^m_\rho\langle\xi,\partial_\xi\rangle^i a^{l,\omega}_{\nu,j,s,z}(t,x,\xi,\rho)\right|\leqslant C\delta(l)2^{-jm+\frac{k}{2}(|\beta|-|\alpha|)}. \tag{3.15}$$

A few remarks are in order here. First, as a result of (1.8) and (1.9), the function

$$\mathrm{e}^{i\rho\varphi_j(t,x,\omega)-i\rho\varphi_k(t,x,\omega)}$$

satisfies the symbol estimates (3.15), which allowed us to replace the phase $\varphi_j(t,x,\omega)$ by $\varphi_k(t,x,\omega)$ in formula (3.14). Next, since $\delta(l)\geqslant 2^{-k/4}$,

$$2^{-j/2}\leqslant 2^{-k/8}\delta(l)\ll\delta(l).$$

It follows from (1.9) and the definition of q^l_{kj} that the angle of $\nabla_x\varphi_k(t,x,\omega)$ to $\nabla_x\varphi_k(t,x,\xi)$ is comparable to $\delta(l)$, hence that the angle of ω to ξ is comparable to $\delta(l)$. By making the number c above small, it follows that the angle of ω to ξ^ν is comparable to $\delta(l)$.

We begin by showing that

$$\sup_{s',z'}\sum_{s,z}\left\|T^{l,\omega}_{\nu,j,s,z}\left(T^{l,\omega}_{\nu,j,s',z'}\right)^*\right\|^{1/2}\leqslant C\delta(l)^{-1}. \tag{3.16}$$

This will follow from showing that

$$\left\|T^{l,\omega}_{\nu,j,s,z}\left(T^{l,\omega}_{\nu,j,s',z'}\right)^*\right\|\leqslant\frac{C_N\delta(l)^{-2}}{(1+2^{k/2}|z-z'|+2^{k/2}\delta(l)|s-s'|)^N}. \tag{3.17}$$

To establish (3.17), we express $T^{l,\omega}_{\nu,j,s,z}(T^{l,\omega}_{\nu,j,s',z'})^*$ as an integral kernel of the form

$$K(t,x;t',x') = \int \mathrm{e}^{i\varphi_k(t,x,\xi)-i\varphi_k(t',x',\xi)+i\rho(\varphi_k(t,x,\omega)-\varphi_k(t',x',\omega))} a^{l,\omega}_{\nu,j,s,z}(t,x,\xi,\rho)\overline{a}^{l,\omega}_{\nu,j,s',z'}(t',x',\xi,\rho)\,d\xi\,d\rho.$$

Integration by parts in ξ and ρ, together with the estimates (3.15) and the support conditions, shows that the kernel $|K(t,x;t',x')|$ is bounded by

$$\int_{R_{\nu,j}} \frac{C_N\delta(l)^2}{(1+2^{k/2}|\nabla_\xi \Phi(t,x,t',x',\xi)|+|\Phi(t,x,t',x',\xi)|+2^j|\Phi(t,x,t',x',\omega)|)^N}\,d\xi\,d\rho,$$

where Φ is as in Lemma 3.3. For each ξ in the domain of integration, the change of variables $(t,x)\to(\nabla_\xi\varphi_k(t,x,\xi),\varphi_k(t,x,\omega))$ has Jacobian comparable to $\delta(l)^2$; consequently, by Schur's Lemma and Lemma 3.3, for each fixed ξ and ρ the integrand is an operator on $L^2(dt\,dx)$ with norm bounded by

$$\frac{C_N 2^{-2k-j}}{(1+2^{k/2}|z-z'|+2^{k/2}\delta(l)|s-s'|)^N}.$$

The volume of $R_{\nu,j}$ is comparable to $2^{2k+j}\delta(l)^{-2}$, and the estimate (3.17) follows.

We next establish the following estimate

$$\sup_{j',\nu'}\sum_{j,\nu}\big\|\big(T^{l,\omega}_{\nu,j,s,z}\big)^* T^{l,\omega}_{\nu',j',s,z}\big\|^{1/2} \leqslant C\delta(l)^{-1/2}\big|\log\delta(l)\big|. \tag{3.18}$$

This will follow from showing that

$$\big\|\big(T^{l,\omega}_{\nu,j,s,z}\big)^* T^{l,\omega}_{\nu',j',s,z}\big\| \leqslant \frac{C_N\delta(l)^{-1}}{(1+2^{-k/2}\delta(l)^2|2^j-2^{j'}|+2^{k/2}\delta(l)|\xi^\nu-\xi^{\nu'}|)^N} \tag{3.19}$$

provided that $|j-j'|\geqslant 3$. For $|j-j'|\leqslant 2$, the estimate holds as if $j=j'$. That (3.18) is a result of (3.19) follows from that fact that

$$\sum_j \frac{1}{1+2^{-k/2}\delta(l)^2|2^j-2^{j'}|}\leqslant C|\log\delta(l)|,$$

where the sum is over j such that $2^j\geqslant 2^{k/2}\delta(l)^{-1}$.

To establish (3.19), we note that $(T^{l,\omega}_{\nu,j,s,z})^* T^{l,\omega}_{\nu',j',s,z}$ has an integral kernel of the form

$$K(\xi,\rho;\xi',\rho') = \int \mathrm{e}^{i\varphi_k(t,x,\xi)-i\varphi_k(t,x,\xi')+i\varphi_k(t,x,\rho\omega)-i\varphi_k(t,x,\rho'\omega)}\overline{a}^{l,\omega}_{\nu,j,s,z}(t,x,\xi,\rho)a^{l,\omega}_{\nu',j',s,z}(t,x,\xi',\rho')\,dt\,dx.$$

Integration by parts in (t,x) yields the following bound,

$$|K(\xi,\rho;\xi',\rho')|\leqslant \int_{R_{s,z}} \frac{C_N\delta(l)^2}{(1+2^{-k/2}|\nabla_{t,x}\Phi(t,x,\xi,\xi')+\nabla_{t,x}\Phi(t,x,\rho\omega,\rho'\omega)|)^N}\,dt\,dx,$$

where Φ is as in Lemma 3.4, and $R_{s,z}$ is a set of volume $2^{-2k}\delta(l)^{-1}$. The change of variables $(\xi,\rho)\to\nabla_{t,x}(\varphi_k(t,x,\xi)+\varphi_k(t,x,\rho\omega))$ has, for each fixed (t,x), Jacobian factor comparable to $\delta(l)^2$. The estimate (3.19) now follows from Schur's Lemma and Lemma 3.4.

To conclude the proof of Theorem 3.1, we split $T^{l,\omega}$ into a finite number of pieces so that we may assume that

$$\left(T^{l,\omega}_{\nu,j,s,z}\right)^* T^{l,\omega}_{\nu',j',s',z'} = 0$$

unless $z=z'$ and $s=s'$, and

$$T^{l,\omega}_{\nu',j',s',z'}\left(T^{l,\omega}_{\nu,j,s,z}\right)^* = 0,$$

unless $\nu=\nu'$ and $j=j'$. We now consider an arbitrary finite truncation of the following sum to M elements

$$T^{l,\omega} = \sum_{\nu,j,s,z} T^{l,\omega}_{\nu,j,s,z}.$$

The proof of the Cotlar–Stein Lemma yields the following,

$$\begin{aligned}\left\|T^{l,\omega}\right\|^{2N} \leqslant C\delta(l)^{-1}\sum \left\|\left(T^{l,\omega}_{\nu_1,j_1,s_1,z_1}\right)^* T^{l,\omega}_{\nu_2,j_2,s_1,z_1}\right\|^{1/2} \left\|T^{l,\omega}_{\nu_2,j_2,s_1 z_1}\left(T^{l,\omega}_{\nu_2,j_2,s_2 z_2}\right)^*\right\|^{1/2}\\ \times \left\|\left(T^{l,\omega}_{\nu_2,j_2,s_2 z_2}\right)^* T^{l,\omega}_{\nu_3,j_3,s_2 z_2}\right\|^{1/2}\cdots\left\|\left(T^{l,\omega}_{\nu_N,j_N,s_N,z_N}\right)^* T^{l,\omega}_{\nu_{N+1},j_{N+1},s_N,z_N}\right\|^{1/2},\end{aligned}$$

and by estimates (3.16) and (3.18) this implies

$$\left\|T^{l,\omega}\right\|^{2N} \leqslant MC^{2N}\delta(l)^{-3N/2}\left|\log\delta(l)\right|^N.$$

Letting $N\to\infty$ completes the proof of Theorem 3.1. □

4. Null form estimates for the wave equation on geodesically concave manifolds

In this section we work locally on a three-dimensional Riemannian manifold Ω with metric $\mathbf{g}$ and with smooth boundary $\partial\Omega$, such that Ω is strictly geodesically concave with respect to $\mathbf{g}$. The typical example is Ω the complement in $\mathbb{R}^3$ of a strictly convex open set, with the Euclidean metric understood. By the Cauchy problem on Ω with Dirichlet condition we understand the following system

$$\begin{cases} \partial_t^2 u(t,x) = \Delta_{\mathbf{g}} u(t,x) + F(t,x), \\ u(t,x) = 0 \quad \text{if } x\in\partial\Omega, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x). \end{cases}$$

We work in a local coordinate patch centered at the origin such that Ω is defined by $x_3 \geqslant 0$. For $k=1,2$ we set

$$H^k_D(\Omega) = \left\{f\in H^k(\Omega)\colon\ f|_{\partial\Omega}=0\right\},$$

where $H^k(\Omega)$ is the space of restrictions of elements of $H^k(\mathbb{R}^3)$.

THEOREM 4.1. – *Suppose that u and v satisfy the Cauchy problem on Ω with Dirichlet condition, with respective data*

$$u_0,v_0\in H^2_D(\Omega),\qquad u_1,v_1\in H^1_D(\Omega),\qquad F,G,DF,DG\in L^1_t\left([-\delta,\delta];L^2(\Omega)\right).$$

Suppose also that the data vanishes for $|x|\geqslant\delta$, where $\delta>0$ is a constant depending on Ω. Then the following hold, for any of the null forms Q,

$$\|DQ(du,dv)\|_{L^2_{t,x}([-\delta,\delta]\times\Omega)} \leqslant C\Big(\|u_0\|_{H^2_D(\Omega)} + \|u_1\|_{H^1_D(\Omega)} + \sum_{|\alpha|\leqslant 1}\|D^\alpha F\|_{L^1_tL^2_x([-\delta,\delta]\times\Omega)}\Big)$$
$$\times\Big(\|v_0\|_{H^2_D(\Omega)} + \|v_1\|_{H^1_D(\Omega)} + \sum_{|\alpha|\leqslant 1}\|D^\alpha G\|_{L^1_tL^2_x([-\delta,\delta]\times\Omega)}\Big),$$
$$\|Q(du,dv)\|_{L^2_{t,x}([-\delta,\delta]\times\Omega)} \leqslant C\Big(\|u_0\|_{H^1_D(\Omega)} + \|u_1\|_{L^2(\Omega)} + \|F\|_{L^1_tL^2_x([-\delta,\delta]\times\Omega)}\Big)$$
$$\times\Big(\|v_0\|_{H^2_D(\Omega)} + \|v_1\|_{H^1_D(\Omega)} + \sum_{|\alpha|\leqslant 1}\|D^\alpha G\|_{L^1_tL^2_x([-\delta,\delta]\times\Omega)}\Big).$$

Before proving this result, we should point out that it immediately yields Theorem 1.1. This just follows from the standard existence argument given in [2].

Proof of Theorem 4.1. – For convenience, in this proof we refer to the discussion in [10] regarding the parametrix for the Dirichlet problem; however, all of the results used are due to Melrose and Taylor [3–5], and Zworski [13]. Since we are working locally, we may assume that Ω is a compact manifold, hence that the Dirichlet Laplacian $-\Delta$ is strictly positive on L^2.

In the estimate for DQ, the terms where the D act on the coefficients of Q may be handled by energy estimates. Hence, by symmetry we may replace $\|DQ(du,dv)\|_{L^2([-\delta,\delta]\times\Omega)}$ by $\|Q(d\partial u,dv)\|_{L^2([-\delta,\delta]\times\Omega)}$, where ∂u is any space or time derivative of u. The next step is to reduce Theorem 4.1 to the following pair of estimates for the homogeneous problem,

$$\|Q(d\partial_x u,dv)\|_{L^2([-\delta,\delta]\times\Omega)} \leqslant C\big(\|u_0\|_{H^2_D(\Omega)} + \|u_1\|_{H^1_D(\Omega)}\big)\big(\|v_0\|_{H^2_D(\Omega)} + \|v_1\|_{H^1_D(\Omega)}\big),$$
$$\|Q(du,dv)\|_{L^2([-\delta,\delta]\times\Omega)} \leqslant C\big(\|u_0\|_{H^1_D(\Omega)} + \|u_1\|_{L^2(\Omega)}\big)\big(\|v_0\|_{H^2_D(\Omega)} + \|v_1\|_{H^1_D(\Omega)}\big). \tag{4.1}$$

To do this, we first reduce Theorem 4.1 to the case $G=0$. To this end, we integrate by parts to write the contribution to v from G as

$$\int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}G(s,x)\,ds \tag{4.2}$$
$$= \cos\big(t\sqrt{-\Delta}\big)\Delta^{-1}G(0,x) - \Delta^{-1}G(t,x) + \int_0^t \cos\big((t-s)\sqrt{-\Delta}\big)\Delta^{-1}\partial_s G(s,x)\,ds$$
$$= I + II + III,$$

where Δ^{-1} denotes the inverse Laplacian on Ω with Dirichlet conditions, which maps $H^k(\Omega)$ to $H^{k+2}(\Omega)$ by elliptic regularity.

To handle I, we note that

$$\|\Delta^{-1}G(0,\cdot)\|_{H^2_D(\Omega)} \leqslant C\|G(0,\cdot)\|_{L^2(\Omega)} \leqslant C\sum_{j\leqslant 1}\|\partial_t^j G\|_{L^1_tL^2_x([-\delta,\delta]\times\Omega)}.$$

This term can thus be absorbed into the initial data v_0.

Next, let $\tilde v(t,x,s) = \cos((t-s)\sqrt{-\Delta})\Delta^{-1}\partial_s G(s,x)$. Then $\tilde v(t,x,s)$ is a solution of the homogeneous wave equation in (t,x) for each s, with initial data satisfying

$$\|\tilde v(0,\cdot,s)\|_{H^2_D(\Omega)} + \|\partial_t\tilde v(0,\cdot,s)\|_{H^1_D(\Omega)} \leqslant C\|\partial_s G(s,\cdot)\|_{L^2(\Omega)}.$$

Note that the t-derivative of II cancels the term in the t-derivative of III coming from the upper limit of integration. Hence, we may write

$$d(II+III)=\int_0^t d\tilde{v}(t,x,s)\,ds+d_x(II).$$

Assuming that the second estimate of Theorem 4.1 holds in the case $G=0$, we may bound

$$\begin{aligned}\left\|Q\left(du,\int_0^t d\tilde{v}(\,\cdot\,,s)\,ds\right)\right\|_{L^2([-\delta,\delta]\times\Omega)} &\leqslant \int_{-\delta}^{\delta}\left\|Q(du,d\tilde{v}(\,\cdot\,,s))\right\|_{L^2([-\delta,\delta]\times\Omega)}\,ds\\ &\leqslant C\big(\|u_0\|_{H^1_D(\Omega)}+\|u_1\|_{L^2(\Omega)}+\|F\|_{L^1_tL^2_x}\big)\|\partial_tG\|_{L^1_tL^2_x}.\end{aligned}$$

The first estimate of the theorem is handled identically.

It remains to handle the term $d_x(II)$. We do this by showing that

$$\left\|d_x\Delta^{-1}G\right\|_{L^2_tL^\infty_x([-\delta,\delta]\times\Omega)}\leqslant C\sum_{|\alpha|\leqslant 1}\left\|D^\alpha G\right\|_{L^1_tL^2_x([-\delta,\delta]\times\Omega)}. \tag{4.3}$$

Energy estimates show that $\|d\partial_x u\|_{L^\infty_tL^2_x}$ and $\|du\|_{L^\infty_tL^2_x}$ are bounded by the appropriate norms of u_0,u_1, and F, yielding the desired estimate.

The proof of (4.3) is based on the following estimate, which holds globally on $\mathbb{R}^3$ for functions f such that $\widehat{f}(\xi)\in L^1_{\mathrm{loc}}$,

$$\|f\|^2_{L^\infty(\mathbb{R}^3)}\leqslant C\big\||D|f\big\|_{L^2(\mathbb{R}^3)}\|\Delta f\|_{L^2(\mathbb{R}^3)}.$$

This estimate is verified by noting that it is dilation invariant, so that one may reduce to the case $\|\Delta f\|_{L^2(\mathbb{R}^3)}=\||D|f\|_{L^2(\mathbb{R}^3)}=1$, for which it follows easily by separately considering the low and high frequencies of f. We then bound

$$\begin{aligned}\left\|d_x\Delta^{-1}G\right\|_{L^2_tL^\infty_x([-\delta,\delta]\times\Omega)} &\leqslant C\int_{-\delta}^{\delta}\left\|G(t,\cdot)\right\|_{L^2(\Omega)}\left\|G(t,\cdot)\right\|_{H^1(\Omega)}\,dt\\ &\leqslant C\|G\|_{L^\infty_tL^2_x([-\delta,\delta]\times\Omega)}\|G\|_{L^1_tH^1_x([-\delta,\delta]\times\Omega)}\\ &\leqslant C\bigg(\sum_{|\alpha|\leqslant 1}\left\|D^\alpha G\right\|_{L^1_tL^2_x([-\delta,\delta]\times\Omega)}\bigg)^2,\end{aligned}$$

which concludes the proof of (4.3), and the reduction of the theorem to the case $G=0$.

It remains to reduce Theorem 4.1 to the case $F=0$. Consider the second estimate of the theorem. We note that

$$d\int_0^t\frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}F(s,x)\,ds=\int_0^t d\left(\frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}F(s,x)\right)ds,$$

which reduces the second estimate to the case $F=0$; that is, the second estimate of (4.1).

As we have remarked previously, the first estimate of the theorem is reduced to considering $\|Q(d\partial u,dv)\|_{L^2([-\delta,\delta]\times\Omega)}$. To handle $Q(d\,\partial_t u,dv)$, we note that $\partial_t u$ solves the Cauchy problem with data in $H^1_D(\Omega)\times L^2(\Omega)$, with inhomogeneity in $L^1_tL^2_x$, thus controlling $\|Q(d\partial_t u, dv)\|_{L^2}$ is reduced to the second estimate of Theorem 4.1, which we have already reduced to (4.1).

Next consider $Q(d\partial_x u, dv)$. We apply the identity (4.2) with G replaced by F, and as before reduce to considering the term $Q(\partial_x^2(II), dv)$. To bound the $L^2_{t,x}$ norm of this term, we note that

$$\begin{aligned}
&\big\|Q\big(\partial_x^2\Delta^{-1}F, dv\big)\big\|^2_{L^2_{t,x}([-\delta,\delta]\times\Omega)}\\
&\leqslant \big\|\partial_x^2\Delta^{-1}F\big\|^2_{L^2_tL^3_x([-\delta,\delta]\times\Omega)}\|dv\|^2_{L^\infty_tL^6_x([-\delta,\delta]\times\Omega)}\\
&\leqslant C\|F\|_{L^\infty_tL^2_x([-\delta,\delta]\times\Omega)}\|F\|_{L^1_tL^6_x([-\delta,\delta]\times\Omega)}\|dv\|^2_{L^\infty_tL^6_x([-\delta,\delta]\times\Omega)}\\
&\leqslant C\Big(\sum_{|\alpha|\leqslant 1}\|D^\alpha F\|_{L^1_tL^2_x([-\delta,\delta]\times\Omega)}\Big)^2\big(\|v_0\|_{H^1_D(\Omega)}+\|v_1\|_{L^2(\Omega)}\big)^2.
\end{aligned}$$

This concludes the reduction of Theorem 4.1 to the pair of estimates (4.1).

To establish the estimates (4.1), we note that, as discussed in [10] immediately preceding formulas (2.12) and (2.24) of that paper, for some δ as in the statement of the theorem, the solution v may be written, modulo smoothing operators acting on the data, as a finite sum of terms of the form

$$Tg(t,x)=\int \mathrm{e}^{i\varphi^\pm(t,x,\xi)}a(t,x,\xi)\widehat{g}(\xi)\,d\xi,$$

where the phases are the solutions to the eikonal equation for some smooth extension of the metric g to an open neighborhood of the origin in $\mathbb{R}^3$, and the data $g\in H^2(\mathbb{R}^3)$ satisfies

$$\|g\|_{H^2(\mathbb{R}^3)}\leqslant C\big(\|v_0\|_{H^2_D(\Omega)}+\|v_1\|_{H^1_D(\Omega)}\big).$$

The solution u may be similarly written, with data f belonging respectively to $H^2(\mathbb{R}^3)$ or $H^1(\mathbb{R}^3)$, in the cases of the two estimates (4.1). The amplitude $a(t,x,\xi)$, which is smooth in all variables and vanishes for $|x|\geqslant C\,\delta$, is of one of two types. Either it satisfies the modified $S^0_{2/3,1/3}$ estimates (1.4) of this paper, or it satisfies the following estimates:

$$(4.4)\qquad \big|x_3^j\partial_{x_3}^k\langle\xi,\partial_\xi\rangle^N\partial^\beta_{t,x_1,x_2}\partial^\alpha_\xi a(t,x,\xi)\big|\leqslant C_{j,k,N,\alpha,\beta}\big(1+|\xi|\big)^{\frac{2}{3}(k-j-|\alpha|)+\frac{1}{3}|\beta|}.$$

(We remark that in [10] these estimates on the symbol were shown to hold for $N=0$; that the estimates hold for general N follows from the fact that these modified estimates are preserved under the equivalence of phase theorem of Hörmander as seen, for example, by the asymptotic formula for the transformed symbol, and the fact that the symbol in our case is obtained by a change of phase from the product of a standard symbol with cutoff functions that satisfy (4.4).)

In either case, the operator $\partial_x T$ is an operator of the same type, with a symbol of one higher order, hence the estimates (4.1), and consequently Theorem 4.1, are reduced to verifying the following estimate

$$(4.5)\qquad \big\|Q(dTf,dTg)\big\|_{L^2_{t,x}([-\delta,\delta]\times\Omega)}\leqslant C\|f\|_{H^1(\mathbb{R}^3)}\|g\|_{H^2(\mathbb{R}^3)},$$

for T an operator as above with a symbol satisfying either (1.4) or (4.4).

We remark that in [10], the Strichartz estimates were shown to hold for both symbol types:

$$\|Tf\|_{L^4_tL^4_x([-\delta,\delta]\times\Omega)}\leqslant C\|f\|_{H^{1/2}(\mathbb{R}^3)}.$$

We first verify that the reductions of the second section of this paper hold for symbols satisfying the estimates (4.4). There are two places where the arguments need to be modified. The first is to verify that the estimate (4.5) holds if, in the formula for dT, the d acts on the symbol $a(t,x,\xi)$. Consider the term dTf, where the d hits the symbol satisfying (4.4). In this case, one obtains an operator Sf of the same form but with symbol of order $2/3$. The resulting contribution to the left hand side of (4.5) is controlled by noting that

$$\big\|(Sf)(dTg)\big\|_{L^2_{t,x}} \leqslant \|Sf\|_{L^6_t L^3_x} \|dTg\|_{L^\infty_t L^6_x} \leqslant C\|f\|_{H^1}\|g\|_{H^2},$$

where the last estimate for Sf follows by interpolating the following estimates

$$\|Sf\|_{L^4_t L^4_x} \leqslant C\|f\|_{H^{7/6}(\mathbb{R}^3)},$$
$$\|Sf\|_{L^\infty_t L^2_x} \leqslant C\|f\|_{H^{2/3}(\mathbb{R}^3)}.$$

Similarly one may bound

$$\big\|(dTf)(Sg)\big\|_{L^2_{t,x}} \leqslant \|dTf\|_{L^\infty_t L^2_x} \big\||D_x|^{5/6} Sg\big\|_{L^4_t L^4_x} \leqslant C\|f\|_{H^1}\|g\|_{H^2}.$$

The other modification is to verify that the operator (2.5) has norm of order $2^{-k/4}$, if now the symbol $a_k(t,x,\xi)$ satisfies (4.4). This follows by expressing

$$Af(x) = \int_0^{x_n} 2^{2k/3}\big(1+2^{4k/3}r^2\big)^{-1} A_r f(x)\,dr,$$

where A_r is the operator obtained by replacing $a_k(t,x,\xi)$ by the symbol

$$a_{k,r}(t,\overline{x},\xi) = 2^{-2k/3}\big(1+2^{4k/3}r^2\big)\partial_{x_3} a_k(t,\overline{x},r,\xi), \quad \overline{x} = (x_1,x_2)$$

which satisfies, for each r, the estimates (1.4), with constants independent of r. One then has the bound

$$\|Af\|_{L^2(\mathbb{R}^3_x)} \leqslant \sup_r \|A_r f\|_{L^2(\mathbb{R}^3_x)} \leqslant C2^{-k/4}\|f\|_{L^2},$$

with, as before, the $2^{-k/4} = \delta(0)$ factor coming from (2.3). This procedure of "freezing the x_3 coefficient" will be used in subsequent steps.

We are thus reduced to establishing estimate (3.1). The above technique of freezing the x_3 coefficient reduces to the case that the symbol $a_k(t,x,\xi)$ in formula (2.2) satisfies the good estimates (1.4), and the symbol $a_j(t,x,\rho\omega)$ satisfies the estimates (4.4) above. (Note that one cannot freeze the x_3 coefficient of $a_j(t,x,\rho\omega)$, since $\widehat{g}(\rho)$ is not localised to a dyadic interval.)

We next note that the proofs of Theorems 3.1 and 3.2 go through if $\widehat{g}(\rho)$ is supported in the region where $\rho \leqslant 2^{3k/4}$. This follows since, in this case, we have $2^{2j/3} \leqslant 2^{k/2}$, hence ∂_x loses at most $2^{k/2}$ against the symbol $a_j(t,x,\rho\omega)$. The only step in the proof that needs to be modified is to replace the right hand side of (3.8) by

$$C\big(1+\delta(l)^2 2^{\max(j,j')/3}\big)^{-1}\|g_j\|_{L^2(\mathbb{R})}\|g_{j'}\|_{L^2(\mathbb{R})},$$

to reflect the $(\frac{2}{3},\frac{2}{3})$ estimates on $a_j(t,x,\rho\omega)$.

We thus assume that $\widehat{g}(\rho)$ is supported in the region where $\rho \geqslant 2^{3k/4}$. Notice that $\rho \geqslant 2^{k/2}\delta(l)^{-1}$, since $\delta(l) \geqslant 2^{-k/4}$. Consequently $T^{l,\omega}(f,g) = T_1^{l,\omega}(f,g)$. We will show that

$$\|T^{l,\omega}(f,g)\|_{L^2(dx\,dt)} \leqslant C\delta(l)^{-3/4}\left|\log\delta(l)\right|^{3/2}\|f\|_{L^2(\mathbb{R}^3)}\|g\|_{L^2(\mathbb{R})}. \tag{4.6}$$

We do this by setting

$$\widetilde{T}_0^{l,\omega}(f,g) = \sum_{\{j:\ 2^{3k/4}\leqslant 2^j\leqslant 2^{2k/3}\delta(l)^{-1}\}} T^{l,\omega}(f,g_j),$$
$$\widetilde{T}_1^{l,\omega}(f,g) = \sum_{\{j:\ 2^j > 2^{2k/3}\delta(l)^{-1}\}} T^{l,\omega}(f,g_j).$$

For the term $\widetilde{T}_0^{l,\omega}(f,g)$, the index j runs over at most $|\log\delta(l)|$ terms. Thus, the bound (4.6) for this term results from the following bound (uniform over j)

$$\|T^{l,\omega}(f,g_j)\|_{L^2(dx\,dt)} \leqslant C\delta(l)^{-3/4}\left|\log\delta(l)\right|^{1/2}\|f\|_{L^2(\mathbb{R}^3)}\|g_j\|_{L^2(\mathbb{R})}.$$

This estimate follows from the argument for $(\frac{1}{2},\frac{1}{2})$ symbols by freezing the x_3 coefficient in $a_j(t,x,\rho\omega)$, which is possible now that the index j is fixed.

To handle the term $\widetilde{T}_1^{l,\omega}(f,g)$, we modify the argument of Theorem 3.1 by taking the partition of unity such that the symbol $a^{l,\omega}_{\nu,j,s,z}(t,x,\xi,\rho)$ is supported in the set

$$\left|\frac{\xi}{|\xi|}-\xi^\nu\right| \leqslant c2^{-k/3}\delta(l)^{-1}, \quad \left|\nabla_\xi\varphi_k(t,x,\omega)-z\right| \leqslant 2\cdot 2^{-2k/3}, \quad |t-s| \leqslant 2\cdot 2^{-2k/3}\delta(l)^{-1},$$

and adjusting the spacing of the index points (ν,s,z) accordingly. With these changes, and using the modified $S_{2/3,2/3}$ estimates for the symbol, estimates (3.17) and (3.19) are respectively replaced by

$$\left\|T^{l,\omega}_{\nu,j,s,z}\left(T^{l,\omega}_{\nu,j,s',z'}\right)^*\right\| \leqslant \frac{C_N\delta(l)^{-2}}{(1+2^{2k/3}|z-z'|+2^{2k/3}\delta(l)|s-s'|)^N},$$
$$\left\|\left(T^{l,\omega}_{\nu,j,s,z}\right)^*T^{l,\omega}_{\nu',j',s,z}\right\| \leqslant \frac{C_N\delta(l)^{-1}}{(1+2^{-2k/3}\delta(l)^2|2^j-2^{j'}|+2^{k/3}\delta(l)|\xi^\nu-\xi^{\nu'}|)^N},$$

where we use the appropriate modification of Lemma 3.3. Since the indices now run over $2^j\,\delta(l) \geqslant 2^{2k/3}$, the rest of the proof of Theorem 3.1 goes through. □

In the case that Ω is the complement in $\mathbb{R}^3$ of a strictly convex obstacle, with the Euclidean metric understood, a partition of unity argument allows one to extend Theorem 4.1 to hold globally on Ω (but still over a finite time interval). Precisely, from the result of Klainerman–Machedon [2] that the conclusion of the theorem holds globally on Minkowski space, together with finite propagation velocity and energy estimates, we may conclude the following extension.

THEOREM 4.2. – *Let Ω be the complement in $\mathbb{R}^3$ of a strictly convex, smoothly bounded compact subset. Suppose that u and v satisfy the Cauchy problem for the Euclidean metric on Ω with Dirichlet condition, with respective data*

$$u_0,v_0\in H^2_D(\Omega), \qquad u_1,v_1\in H^1_D(\Omega), \qquad F,G,DF,DG\in L^1_t\big([-1,1];L^2(\Omega)\big).$$

Then the following hold, for any of the null forms Q,

$$\|DQ(du,dv)\|_{L^2_{t,x}([-1,1]\times\Omega)} \leqslant C\Bigg(\|u_0\|_{H^2_D(\Omega)} + \|u_1\|_{H^1_D(\Omega)} + \sum_{|\alpha|\leqslant 1} \|D^\alpha F\|_{L^1_t L^2_x([-1,1]\times\Omega)}\Bigg) \times \Bigg(\|v_0\|_{H^2_D(\Omega)} + \|v_1\|_{H^1_D(\Omega)} + \sum_{|\alpha|\leqslant 1} \|D^\alpha G\|_{L^1_t L^2_x([-1,1]\times\Omega)}\Bigg),$$

$$\|Q(du,dv)\|_{L^2_{t,x}([-1,1]\times\Omega)} \leqslant C\big(\|u_0\|_{H^1_D(\Omega)} + \|u_1\|_{L^2(\Omega)} + \|F\|_{L^1_t L^2_x([-1,1]\times\Omega)}\big) \times \Bigg(\|v_0\|_{H^2_D(\Omega)} + \|v_1\|_{H^1_D(\Omega)} + \sum_{|\alpha|\leqslant 1} \|D^\alpha G\|_{L^1_t L^2_x([-1,1]\times\Omega)}\Bigg).$$

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(Manuscript received January 19, 1999;
accepted, after revision, September 17, 1999.)

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