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GABRIEL P. PATERNAIN

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TOPOLOGICAL PRESSURE FOR GEODESIC FLOWS

BY GABRIEL P. PATERNAIN

Dedicated to Detlef Gromoll for his sixtieth birthday

ABSTRACT. – We give a Riemannian formula for the topological pressure of the geodesic flow of a closed Riemannian manifold. As a consequence, we derive an asymptotic formula for the stable norm in cohomology in terms of geodesic arcs for manifolds with topological entropy $h_{\text{top}} = 0$. We introduce a Poincaré series also in terms of geodesic arcs and we show that it defines a holomorphic function on the half plane given by those complex numbers with real part $> h_{\text{top}}$. © 2000 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – On donne une formule riemannienne pour la pression topologique du flot géodésique sur une variété riemannienne compacte. On en déduit, pour les variétés à entropie topologique nulle, une formule asymptotique pour la norme stable en cohomologie en termes des courbes géodésiques. On introduit une séries de Poincaré, dépendant également des courbes géodésiques, dont on montre qu'elle définit une fonction holomorphe sur le demi-plan constitué des nombres complexes de partie réelle supérieure à l'entropie topologique. © 2000 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

Let M^n be a closed connected C^∞ manifold and let g be a Riemannian metric of class C^r with $r \geqslant 3$. Let $\phi_t : SM \to SM$ be the geodesic flow of g acting on the unit sphere bundle SM. Given a continuous function $f : SM \to \mathbb{R}$, let $P(f)$ be the topological pressure of the function f with respect to the geodesic flow ϕ. We recall its definition. Given $T > 0$ and a point $(x, v) \in SM$, set

$$f_T(x,v) := \int_0^T f\big(\phi_t(x,v)\big)\,\mathrm{d}t.$$

We say that a set $E \subset SM$ is (T,ε)-separated if, given $(x_1,v_1) \neq (x_2,v_2) \in E$, there exists $t \in [0,T]$ for which the distance between $\phi_t(x_1,v_1)$ and $\phi_t(x_2,v_2)$ is at least ε. We set

$$r(T,\varepsilon,f) := \sup\left\{ \sum_{(x,v)\in E} e^{f_T(x,v)} \colon E \text{ is } (T,\varepsilon)\text{-separated} \right\},$$

$$r(\varepsilon,f) := \limsup_{T\to\infty} \frac{1}{T} \log r(T,\varepsilon,f).$$

The topological pressure is defined to be:

$$P(f) = \lim_{\varepsilon\to 0} r(\varepsilon,f).$$

The topological entropy h_{top} of ϕ is $P(0)$. The variational principle extends to the case of topological pressure as:

$$P(f)=\sup_{\mu\in\mathcal{M}(\phi)}\left(h_\mu+\int_{SM} f\,\mathrm{d}\mu\right), \tag{1}$$

where $\mathcal{M}(\phi)$ is the set of all ϕ-invariant Borel probability measures and h_μ is the entropy of the measure μ [8]. The study of the map $f\mapsto P(f)$ is important since it determines the members of $\mathcal{M}(\phi)$, and when the entropy map $\mu\mapsto h_\mu$ is upper semicontinuous on $\mathcal{M}(\phi)$, the knowledge of $P(f)$ for all f is equivalent to the knowledge of $\mathcal{M}(\phi)$ and h_μ for all $\mu\in\mathcal{M}(\phi)$ [8]. Also, the variational principle gives a natural way of selecting interesting members of $\mathcal{M}(\phi)$.

Our aim is to give a formula for $P(f)$ in terms of geodesic arcs between two points in M similar to Mañé's formula for h_{top} in [4]. For this we shall use the formula for the topological pressure of a C^∞ dynamical system recently obtained by O. Kozlovski in [3].

Given x and y in M, let $\gamma_{x,y}:[0,\ell(\gamma_{x,y})]\to M$ be a unit speed geodesic arc joining x to y with length $\ell(\gamma_{x,y})$. Given $T>0$ the set of all $\gamma_{x,y}$ with $\ell(\gamma_{x,y})\leqslant T$ is finite and its cardinality is locally constant for an open full measure subset of $M\times M$.

THEOREM A. – *If g is of class C^3 for any $\delta>0$ we have*

$$P(f)\leqslant\liminf_{T\to\infty}\frac{1}{T}\log\int_{M\times M}\left(\sum_{\{\gamma_{x,y}:\ T-\delta<\ell(\gamma_{x,y})\leqslant T\}} e^{\int_0^{\ell(\gamma_{x,y})} f(\gamma_{x,y}(t),\dot\gamma_{x,y}(t))\,\mathrm{d}t}\right)\mathrm{d}x\,\mathrm{d}y,$$

and if g is C^∞,

$$P(f)=\lim_{T\to\infty}\frac{1}{T}\log\int_{M\times M}\left(\sum_{\{\gamma_{x,y}:\ T-\delta<\ell(\gamma_{x,y})\leqslant T\}} e^{\int_0^{\ell(\gamma_{x,y})} f(\gamma_{x,y}(t),\dot\gamma_{x,y}(t))\,\mathrm{d}t}\right)\mathrm{d}x\,\mathrm{d}y.$$

If g is C^∞ and $P(f)\geqslant 0$ we also have

$$P(f)=\lim_{T\to\infty}\frac{1}{T}\log\int_{M\times M}\left(\sum_{\{\gamma_{x,y}:\ \ell(\gamma_{x,y})\leqslant T\}} e^{\int_0^{\ell(\gamma_{x,y})} f(\gamma_{x,y}(t),\dot\gamma_{x,y}(t))\,\mathrm{d}t}\right)\mathrm{d}x\,\mathrm{d}y.$$

When $f\equiv 0$, we recover Mañé's formula in [4]. Combining Theorem A with Lemma 4.3 in [4] (which is an application of the Borel–Cantelli Lemma) we obtain

COROLLARY 1. – *If g is C^∞, for almost every $(x,y)\in M\times M$ we have,*

$$\limsup_{T\to\infty}\frac{1}{T}\log\int_{M\times M}\left(\sum_{\{\gamma_{x,y}:\ T-\delta<\ell(\gamma_{x,y})\leqslant T\}} e^{\int_0^{\ell(\gamma_{x,y})} f(\gamma_{x,y}(t),\dot\gamma_{x,y}(t))\,\mathrm{d}t}\right)\leqslant P(f).$$

Theorem A is particularly appealing when f is the restriction to SM of a 1-form ω. We think of ω as a function $\omega:TM\to\mathbb{R}$. In this case we have for a C^∞ metric g,

$$P(\omega)=\lim_{T\to\infty}\frac{1}{T}\log\int_{M\times M}\left(\sum_{\{\gamma_{x,y}:\ T-\delta<\ell(\gamma_{x,y})\leqslant T\}} e^{\int_{\gamma_{x,y}}\omega}\right)\mathrm{d}x\,\mathrm{d}y.$$

The function $H^1(M,\mathbb{R}) \ni \omega \mapsto P(\omega) \in \mathbb{R}$ is closely related to Mather's function $\alpha : H^1(M,\mathbb{R}) \to \mathbb{R}$ [5]. Recall that α is given by

$$\alpha(\omega) = -\min\left\{ \int_{TM} \left(|v|^2/2 - \omega\right) \mathrm{d}\mu \colon \ \mu \in \mathcal{M}_{TM}(\phi) \right\}, \tag{2}$$

where $\mathcal{M}_{TM}(\phi)$ is the set of ϕ-invariant Borel probability measures of TM with compact support. Let SM_α be the set given by those points $(x,v) \in TM$ such that $|v| = a$. The support of a minimizing measure, that is, a measure for which the minimum in (2) is achieved, must be contained in the energy level $SM_{\sqrt{2\alpha(\omega)}}$ [1]. Hence if μ is minimizing,

$$2\alpha(\omega) = \int_{TM} \omega \,\mathrm{d}\mu.$$

Multiplication by the positive real a defines a natural bijection between the set of ϕ-Borel probability measures of TM which are supported on SM_a and $\mathcal{M}(\phi)$. Hence

$$\sqrt{2\alpha(\omega)} = \sup_{\mu \in \mathcal{M}(\phi)} \int_{SM} \omega \,\mathrm{d}\mu.$$

The function $\omega \mapsto \sqrt{2\alpha(\omega)}$ is a norm, dual to the stable norm [6]. From the variational principle (1) we obtain

$$\sqrt{2\alpha(\omega)} \leqslant P(\omega) \leqslant h_{\text{top}} + \sqrt{2\alpha(\omega)}.$$

Combining this inequality with Theorem A yields,

COROLLARY 2. – *If g is C^∞ and $h_{\text{top}} = 0$,*

$$\sqrt{2\alpha(\omega)} = \lim_{T\to\infty} \frac{1}{T} \log \int_{M\times M} \left(\sum_{\{\gamma_{x,y} \colon \ \ell(\gamma_{x,y}) \leqslant T\}} e^{\int_{\gamma_{x,y}} \omega} \right) \mathrm{d}x \,\mathrm{d}y.$$

The corollary applies, for example, to a C^∞ torus of revolution.

If (M,g) does not have conjugate points, we can prove the equality in Theorem A assuming that g is only of class C^3. More precisely we shall prove,

THEOREM B. – *Suppose that g is of class C^3 and (M,g) does not have conjugate points. Then for any $\delta > 0$ we have*

$$P(f) = \lim_{T\to\infty} \frac{1}{T} \log \int_{M\times M} \left(\sum_{\{\gamma_{x,y} \colon \ T-\delta < \ell(\gamma_{x,y}) \leqslant T\}} e^{\int_0^{\ell(\gamma_{x,y})} f(\gamma_{x,y}(t), \dot\gamma_{x,y}(t)) \,\mathrm{d}t} \right) \mathrm{d}x \,\mathrm{d}y.$$

If ω is a closed 1-form, then for any x and y in M we have:

$$P(\omega) = \lim_{T\to\infty} \frac{1}{T} \log \left(\sum_{\{\gamma_{x,y} \colon \ \ell(\gamma_{x,y}) \leqslant T\}} e^{\int_{\gamma_{x,y}} \omega} \right).$$

Finally, if $s \in \mathbb{R}$, then for any x and y in M we have:

$$h_{\text{top}} - s = \lim_{T\to\infty} \frac{1}{T} \log \left(\sum_{\{\gamma_{x,y}:\ T-\delta<\ell(\gamma_{x,y})\leqslant T\}} e^{-s\ell(\gamma_{x,y})} \right).$$

Consider those $s \in \mathbb{C}$ for which the following limit exists and is finite,

$$\eta_g(s) := \lim_{T\to\infty} \int_{M\times M} \left(\sum_{\{\gamma_{x,y}:\ \ell(\gamma_{x,y})\leqslant T\}} e^{-s\ell(\gamma_{x,y})} \right) \mathrm{d}x\, \mathrm{d}y.$$

We can regard $\eta_g(s)$ as the "Poincaré series" of the Riemannian metric g. Suppose that (x, y) is a pair of non-conjugate points and let

$$\ell^1_{x,y} \leqslant \ell^2_{x,y} \leqslant \cdots \leqslant \ell^n_{x,y} \leqslant \cdots,$$

be the lengths of all the geodesic arcs $\gamma_{x,y}$. Given $s \in \mathbb{C}$ we set

$$\eta_g(s, x, y) := \sum_{n=1}^{\infty} e^{-s\ell^n_{x,y}},$$

whenever the series converges. If $x = y$ and $\tilde{x}$ is any point in the universal covering $\widetilde{M}$ that projects to x, then $\eta_g(s, x, x)$ is exactly the Poincaré series of the action of $\pi_1(M)$ on $\widetilde{M}$ at the point $\tilde{x}$, provided that M has no conjugate points.

Using Theorems A and B we shall prove,

COROLLARY 3. – *Suppose that g is C^∞. Then $\eta_g(s)$ is a holomorphic function on the half plane* $\operatorname{Re}(s) > h_{\text{top}}$. *If g is of class C^3 and (M, g) does not have conjugate points, then for any x and y in M, $s \mapsto \eta_g(s, x, y)$ is a holomorphic function on the half plane* $\operatorname{Re}(s) > h_{\text{top}}$.

2. Kozlovski's formula

Let X be a closed Riemannian manifold and let $\phi_t : X \to X$ be a flow without singularities of class C^r with $r \geqslant 2$. Given a continuous function $f : X \to \mathbb{R}$, let $P(f)$ be the topological pressure of f with respect to the flow ϕ. Set

$$f_T(x) = \int_0^T f(\phi_t x)\, \mathrm{d}t.$$

Given a linear map $L : E \to F$ between finite-dimensional vector spaces with inner products, we define its *expansion* $\operatorname{ex}(L)$ by

$$\operatorname{ex}(L) = \max_S \left|\det(L|_S)\right|,$$

where the maximum is taken over all subspaces $S \subset E$.

THEOREM 2.1 ([3]). –

$$P(f) \leqslant \liminf_{T\to\infty} \frac{1}{T} \log \int_X e^{f_T(x)} \mathrm{ex}(d_x\phi_T)\,\mathrm{d}x,$$

and if ϕ is C^∞ *we have*

$$P(f) \leqslant \lim_{T\to\infty} \frac{1}{T} \log \int_X e^{f_T(x)} \mathrm{ex}(d_x\phi_T)\,\mathrm{d}x.$$

The first inequality, when $f \equiv 0$ is *Przytycki's inequality* for the topological entropy [7]. Kozlovski's proof in [3] for the equality case is based on Yomdin's work [9].

3. Area formula

Let $\psi: X \to Y$ be a smooth map between Riemannian manifolds of the same dimension. Suppose that X is compact and possibly with boundary. Let $u: X \to \mathbb{C}$ be a continuous function.

The following formula is a particular case of the area formula that holds for Lipschitz maps, manifolds with different dimensions and Hausdorff measures [2, Theorem 3.2.3].

THEOREM 3.1 (Area formula). –

$$\int_X u(x)|\det d_x\psi|\,\mathrm{d}x = \int_Y \Big(\sum_{x\in\psi^{-1}(y)} u(x)\Big)\,\mathrm{d}y.$$

Now, let us apply this formula to our context. Let $f: SM \to \mathbb{C}$ be a continuous function and set for brevity

$$F_T(x,y) := \sum_{\{\gamma_{x,y}:\ \ell(\gamma_{x,y})\leqslant T\}} e^{\int_0^{\ell(\gamma_{x,y})} f(\gamma_{x,y}(t),\dot\gamma_{x,y}(t))\,\mathrm{d}t},$$

$$F_{T,\delta}(x,y) := \sum_{\{\gamma_{x,y}:\ T-\delta<\ell(\gamma_{x,y})\leqslant T\}} e^{\int_0^{\ell(\gamma_{x,y})} f(\gamma_{x,y}(t),\dot\gamma_{x,y}(t))\,\mathrm{d}t}.$$

Let $\pi: SM \to M$ be the canonical projection and for $\theta = (x,y) \in SM$, let $V(\theta)$ be the vertical subspace at θ which is given by the kernel of $d_\theta\pi$. We endow SM with the Sasaki metric on SM and we let $\mathrm{d}\theta$ be its Riemannian measure. Let us consider the function $A: SM \times \mathbb{R} \to \mathbb{R}^+$ given by

$$A(\theta,t) = \big|\det d_\theta(\pi\circ\phi_t)|_{V(\theta)}\big|.$$

PROPOSITION 3.2. – *For all $T > 0$ we have*:

$$\int_0^T \int_{SM} e^{f_t(\theta)} A(\theta,t)\,\mathrm{d}\theta\,\mathrm{d}t = \int_{M\times M} F_T(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Similarly, for $T > \delta$,

$$\int_{T-\delta}^{T} \int_{SM} e^{f_t(\theta)} A(\theta, t)\, \mathrm{d}\theta\, \mathrm{d}t = \int_{M\times M} F_{T,\delta}(x, y)\, \mathrm{d}x\, \mathrm{d}y.$$

Proof. – Let us consider the function

$$\psi : SM \times [0, T] \to M \times M,$$

given by

$$\psi(\theta, t) = \bigl(x, \exp_x(tv)\bigr) = \bigl(\pi(\theta), \pi \circ \phi_t(\theta)\bigr). \qquad \square$$

LEMMA 3.3. –

$$|\det d_{(\theta,t)}\psi| = A(\theta, t).$$

Proof. – Fix $(\theta, t) \in SM \times [0, T]$ and identify $T_{(\theta,t)}(SM \times [0, T])$ with $T_\theta SM \times \mathbb{R}$. Let γ_θ be the unique geodesic defined by θ, i.e., $\gamma_\theta(t) = \pi \circ \phi_t(\theta)$. From the definition of ψ we have

$$d_{(\theta,t)}\psi(0, 1) = (0, \dot\gamma_\theta). \tag{3}$$

Let $\{e_1 = v_1, e_2, \ldots, e_n\}$ be an orthonormal basis of $T_x M$. For $1 \leqslant i \leqslant n$, let $e_i(t)$ be the parallel transport of e_i along γ_θ. Then $\{e_1(t) = \dot\gamma_\theta(t), e_2(t), \ldots, e_n(t)\}$ is an orthonormal basis of $T_{\gamma_\theta(t)}M$. Let $\{\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_{n-1}\}$ be an orthonormal basis of $T_\theta SM$ such that for all $1 \leqslant i \leqslant n$, the vector ξ_i is a horizontal vector with horizontal component e_i and for all $1 \leqslant i \leqslant n-1$, the vector η_i belongs to $V(\theta)$. From the definition of ψ we get

$$d_{(\theta,t)}\psi(\xi_i) = \bigl(e_i, d(\pi \circ \phi_t)(\xi_i)\bigr), \tag{4}$$

$$d_{(\theta,t)}\psi(\eta_i) = \bigl(0, d(\pi \circ \phi_t)(\eta_i)\bigr). \tag{5}$$

Also for all $1 \leqslant i \leqslant n-1$,

$$\bigl\langle \dot\gamma_\theta(t), d(\pi \circ \phi_t)(\eta_i)\bigr\rangle = 0. \tag{6}$$

The lemma follows easily by looking at the matrix of $d_{(\theta,t)}\psi$ with respect to the basis we selected and Eqs. (3), (4), (5) and (6). $\square$

Let $u : SM \times [0, T] \to \mathbb{C}$ be the continuous function given by

$$u(\theta, t) = e^{f_t(\theta)}.$$

The proposition follows right away from Lemma 3.3 and Theorem 3.1 applied to ψ and u as above.

Remark 3.4. – The same proof as above also shows that for all $x \in M$ we have

$$\int_0^T \int_{S_x} e^{f_t(\theta)} A(\theta, t)\, \mathrm{d}\theta\, \mathrm{d}t = \int_M F_T(x, y)\, \mathrm{d}y,$$

where S_x is the set of unit vectors in $T_x M$.

4. Time shifts

In this section we prove two lemmas which are consequences of the fact that the lift of the geodesic flow to the Grassmannian bundle of Lagrangian subspaces is transverse to the Maslov cycle defined by the vertical subspace. These two lemmas are clearly inspired by the arguments in [4].

Given $\theta \in SM$, let $S(\theta)$ be the orthogonal complement of the vector field of the geodesic flow at θ. Let $\Lambda(SM)$ be the Grassmannian bundle of Lagrangian subspaces, i.e., $(\theta, E) \in \Lambda(SM)$ if $\theta \in SM$ and $E \subset S(\theta)$ is a Lagrangian subspace. Given Lagrangian subspaces E_1, E_2 of $S(\theta)$ we define the *angle* between E_1 and E_2 as

$$\alpha(E_1, E_2) = \left|\det(P|_{E_1})\right|,$$

where $P: S(\theta) \to E_2^{\perp}$ is the orthogonal projection. Clearly α depends continuously on the subspaces and $\alpha(E_1, E_2) = 0$ if and only if $E_1 \cap E_2 \neq \{0\}$.

LEMMA 4.1. – *Given a positive real number q, there exist a constant $\gamma > 0$, an integer $m \geqslant 1$ and an upper semicontinuous function*

$$\tau: \Lambda(SM) \times \mathbb{R} \to \{0, q/m, 2q/m, \ldots, q\}$$

such that after abbreviating $\tau(\theta, E, T)$ to τ, we have that for all $(\theta, E) \in \Lambda(SM)$ and all T:

$$\alpha\big(d_\theta \phi_{T+\tau}(E), V(\phi_{T+\tau}\theta)\big) > \gamma.$$

Proof. – We shall show first that there exist a constant $\gamma > 0$ and an integer $m \geqslant 1$ such that for all $(\theta, E, T) \in \Lambda(SM) \times \mathbb{R}$ the set given by

$$Q(\theta, E, T) := \big\{i \in \mathbb{Z}:\ 0 \leqslant i \leqslant m,\ \alpha\big(d_\theta \phi_{T+iq/m}(E), V(\phi_{T+iq/m}\theta)\big) > \gamma\big\},$$

is not empty. Suppose that this is not the case. Then given any $m \geqslant 1$ there exists a sequence $(\theta_m, E_m, T_m) \in \Lambda(SM) \in \mathbb{R}$ such that

$$\alpha\big(d_{\theta_m} \phi_{T_m+s}(E_m), V(\phi_{T_m+s}\theta_m)\big) \leqslant 1/2^m, \tag{7}$$

for all $s \in A_m$ where

$$A_m := \big\{jq/2^m:\ j \in \mathbb{Z},\ 0 \leqslant j \leqslant 2^m\big\}.$$

Note that $A_m \subset A_{m+1}$. Since $\Lambda(SM)$ is compact the sequence

$$\big\{\big(\phi_{T_m}(\theta_m), d_{\theta_m}\phi_{T_m}(E_m)\big)\big\}$$

has a convergent subsequence

$$\big\{\big(\phi_{T_{m_k}}(\theta_{m_k}), d_{\theta_{m_k}}\phi_{T_{m_k}}(E_{m_k})\big)\big\}$$

that converges to a point $(\theta, E) \in \Lambda(SM)$. It follows from (7) and the continuity of α that given any k and any $s \in A_{m_k}$ we have

$$\alpha\big(d_\theta \phi_s(E), V(\phi_s\theta)\big) = 0.$$

Hence for all s in $[0,q]$ we have

$$d_\theta\phi_s(E)\cap V(\phi_s\theta)\neq\{0\}.$$

This contradicts the transversality of the lift of the geodesic flow to the Maslov cycle defined by the vertical.

Now define

$$\tau(\theta,E,T)=\min\{iq/m\colon\ i\in Q(\theta,E,T)\}.$$

Clearly τ is upper semicontinuous and given $(\theta,E,T)\in\Lambda(SM)\times\mathbb{R}$ we have

$$\alpha\big(d_\theta\phi_{T+\tau}(E),V(\phi_{T+\tau}\theta)\big)>\gamma. \qquad\square$$

Remark 4.2. – Observe that by the definition of angle saying that

$$\alpha\big(d_\theta\phi_{T+\tau}(E),V(\phi_{T+\tau}\theta)\big)>\gamma,$$

is equivalent to saying that

$$\big|\det(d_{\phi_{T+\tau}(\theta)}\pi|_S)\big|>\gamma,$$

where $S:=d_\theta\phi_{T+\tau}(E)$.

LEMMA 4.3. – *Given a positive real number q, there exist a constant $\kappa>0$, an integer $n\geqslant 1$ and an upper semicontinuous function*

$$\rho\colon\Lambda(SM)\to\{0,q/n,2q/n,\dots,q\}$$

such that after abbreviating $\rho(\theta,E)$ to ρ and setting $\theta_-=\phi_{-\rho}\theta$ we have that for all $(\theta,E)\in\Lambda(SM)$:

$$\alpha\big(E,d_{\theta_-}\phi_\rho\big(V(\theta_-)\big)\big)>\kappa.$$

Proof. – The proof is very similar to the last lemma. We shall show first that there exist a constant $\kappa>0$ and an integer $n\geqslant 1$ such that for all $(\theta,E)\in\Lambda(SM)$ the set given by

$$Q(\theta,E):=\big\{i\in\mathbb{Z}\colon\ 0\leqslant i\leqslant n,\ \alpha\big(E,d_{\phi_{-iq/n}\theta}\phi_{iq/n}\big(V(\phi_{-iq/n}\theta)\big)\big)>\kappa\big\},$$

is not empty. Suppose that this is not the case. Then given any $n\geqslant 1$ there exists $(\theta_n,E_n)\in\Lambda(SM)$ such that

$$\alpha\big(E_n,d_{\phi_{-s}\theta_n}\phi_s\big(V(\phi_{-s}\theta_n)\big)\big)\leqslant 1/2^n, \tag{8}$$

for all $s\in A_n$. Since $\Lambda(SM)$ is compact the sequence $\{(\theta_n,E_n)\}$ has a convergent subsequence that converges to a point $(\theta,E)\in\Lambda(SM)$. From (8) and the continuity of α we get that for all s in $[0,q]$ we have

$$\alpha\big(E,d_{\phi_{-s}\theta}\phi_s\big(V(\phi_{-s}\theta)\big)\big)=0.$$

Hence for all s in $[0,q]$ we have

$$d_\theta\phi_{-s}(E)\cap V(\phi_{-s}\theta)\neq\{0\}.$$

This contradicts the transversality of the lift of the geodesic flow to the Maslov cycle defined by the vertical.

Now define

$$\rho(\theta,E)=\min\{iq/n\colon\ i\in Q(\theta,E)\}.$$

Clearly ρ is an upper semicontinuous function with the desired properties. □

5. Proof of Theorem A

We shall follow closely Mañé's ideas in [4]. We need the following lemma [4, Lemma 2.2].

LEMMA 5.1. – *For each $\theta\in SM$ and $t\in\mathbb{R}$, there is a Lagrangian subspace $R_t(\theta)\subset S(\theta)$, which depends measurably on t and θ, and satisfies*:
(1) $|\det(d_\theta\phi_t|_{R_t(\theta)})|=\mathrm{ex}(d_\theta\phi_t)$;
(2) *if E is a Lagrangian subspace of $S(\theta)$, then*

$$\left|\det(d_\theta\phi_t|_E)\right|\geqslant\alpha\left(E,R_t^{\perp}(\theta)\right)\mathrm{ex}(d_\theta\phi_t).$$

We now show,

LEMMA 5.2. – *Given a positive real number q there exist $\beta>0$, an integer $m\geqslant 1$ and measurable functions $\tau_i\colon SM\times\mathbb{R}\to\{0,q/m,2q/m,\dots,q\}$, $i=1,2$, such that, after abbreviating $\tau_i(\theta,t)$ to τ_i for $i=1,2$, and setting*

$$\theta_1=\phi_{-\tau_1}(\theta),\quad \theta_2=\phi_{t+\tau_2}(\theta),\quad \textit{and}\quad V_i=V(\theta_i)\quad \textit{for } i=1,2,$$

we have for all θ and t:

$$(1)\quad \alpha\left(d_{\theta_1}\phi_{\tau_1}(V_1),R_t^{\perp}(\theta)\right)>\beta;\qquad (2)\quad \alpha\left(d_{\theta_1}\phi_{t+\tau_1+\tau_2}(V_1),V_2\right)>\beta.$$

Proof. – It suffices to prove that we can find $\beta_1,\beta_2>0$, integers $m_1,m_2\geqslant 1$ and measurable functions $\tau_i\colon SM\times\mathbb{R}\to\{0,q/m_i,2q/m_i,\dots,q\}$ such that properties (1) and (2) of Lemma 5.2 hold with β changed to β_1 in (1) and to β_2 in (2). Then we can easily obtain Lemma 5.2 with $m=m_1m_2$ and $\beta=\min(\beta_1,\beta_2)$.

Let κ, n and ρ be given by Lemma 4.3, and let us set

$$\beta_1:=\kappa,\qquad m_1:=n,\qquad \tau_1(\theta,t):=\rho\left(\theta,R_t^{\perp}(\theta)\right).$$

Since $R_t^{\perp}(\theta)$ is measurable and ρ is upper semicontinuous, the function τ_1 is measurable. Lemma 4.3 applied to $E=R_t^{\perp}(\theta)$ implies that

$$\alpha\left((d_{\theta_1}\phi_{\tau_1})V_1,R_t^{\perp}(\theta)\right)>\beta_1,$$

so property (1) is proved.

Now let γ, m and τ be given by Lemma 4.1, and let us set

$$\beta_2:=\gamma,\qquad m_2:=m,\qquad \tau_2(\theta,t):=\tau\left(\theta,d_{\theta_1}\phi_{\tau_1}(V_1),t\right).$$

Since τ_2 is a composition of measurable functions, it is measurable. Lemma 4.1 applied to $E=d_{\theta_1}\phi_{\tau_1}(V_1)$ implies that

$$\alpha\left(d_{\theta_1}\phi_{t+\tau_1+\tau_2}(V_1),V_2\right)>\beta_2,$$

so property (2) is proved. □

PROPOSITION 5.3. – *Given a positive real number q, there exist an integer $m \geqslant 1$, a constant $C > 0$ and two measurable functions*

$$\tau_1, \tau_2 : SM \times \mathbb{R} \to \{0, q/m, 2q/m, \dots, q\},$$

such that after abbreviating $\tau_i(\theta, t)$ to τ_i for $i = 1, 2$ we have

$$e^{f_{t+\tau_1+\tau_2}(\phi_{-\tau_1}\theta)} A\big(\phi_{-\tau_1}(\theta), t + \tau_1 + \tau_2\big) \geqslant C e^{f_t(\theta)} \mathrm{ex}(d_\theta \phi_t),$$

for all $\theta \in SM$ and all $t \in \mathbb{R}$.

Proof. – Let m, τ_1 and τ_2 be given by Lemma 5.2. Set $S = d_{\theta_1}\phi_{\tau_1}(V_1)$. By Lemma 5.1 and property (1) of Lemma 5.2 we have:

$$\big|\det(d_\theta \phi_t|_S)\big| \geqslant \alpha\big(S, R_t^{\perp}(\theta)\big)\mathrm{ex}(d_\theta \phi_t) \geqslant \beta\, \mathrm{ex}(d_\theta \phi_t). \tag{9}$$

Take $a > 0$ such that

$$\big|\det(d_\zeta \phi_s|_L)\big| \geqslant a, \tag{10}$$

for every $(\zeta, L) \in \Lambda(SM)$ and $s \in [0, q]$. Set $\hat{\theta} = \phi_t(\theta)$ and $\widehat{S} = d_\theta \phi_t S$. Then Eqs. (9) and (10) imply

$$\begin{aligned} \big|\det(d_{\theta_1}\phi_{t+\tau_1+\tau_2}|_{V_1})\big| &= \big|\det(\mathrm{d}_{\hat{\theta}}\phi_{\tau_2}|_{\widehat{S}})\big| \cdot \big|\det(d_\theta \phi_t|_S)\big| \cdot \big|\det(d_{\theta_1}\phi_{\tau_1}|_{V_1})\big| \\ &\geqslant a^2 \beta\, \mathrm{ex}(d_\theta \phi_t). \end{aligned} \tag{11}$$

Now set $S_2 = (d_{\theta_1}\phi_{t+\tau_1+\tau_2})V_1$. By property (2) of Lemma 5.2 and the definition of α we have respectively

$$\alpha(S_2, V_2) \geqslant \beta, \qquad |\det(d_{\theta_2}\pi|_{S_2})| \geqslant \beta,$$

which together with inequality (11) implies that

$$\begin{aligned} A\big(\phi_{-\tau_1}(\theta), t + \tau_1 + \tau_2\big) &= \big|\det(d_{\theta_1}(\pi \circ \phi_{t+\tau_1+\tau_2})|_{V_1})\big| \\ &= \big|\det(d_{\theta_2}\pi|_{S_2})\big| \cdot \big|\det(d_{\theta_1}\phi_{t+\tau_1+\tau_2}|_{V_1})\big| \\ &\geqslant \beta\big|\det(d_{\theta_1}\phi_{t+\tau_1+\tau_2}|_{V_1}\big| \geqslant \beta^2 a^2 \mathrm{ex}(d_\theta \phi_t). \end{aligned} \tag{12}$$

Now observe that,

$$\begin{aligned} f_{t+\tau_1+\tau_2}\big(\phi_{-\tau_1}(\theta)\big) &= \int_0^{t+\tau_1+\tau_2} f\big(\phi_{s-\tau_1}(\theta)\big)\,\mathrm{d}s = \int_{-\tau_1}^{t+\tau_2} f\big(\phi_s(\theta)\big)\,\mathrm{d}s \\ &= \int_0^t f\big(\phi_s(\theta)\big)\,\mathrm{d}s + \int_{-\tau_1}^0 f\big(\phi_s(\theta)\big)\,\mathrm{d}s + \int_t^{t+\tau_2} f\big(\phi_s(\theta)\big)\,\mathrm{d}s. \end{aligned}$$

Since f is bounded from below and $0 \leqslant \tau_1, \tau_2 \leqslant q$, there exists a constant k such that

$$\int_{-\tau_1}^0 f\big(\phi_s(\theta)\big)\,\mathrm{d}s + \int_t^{t+\tau_2} f\big(\phi_s(\theta)\big)\,\mathrm{d}s \geqslant k.$$

Hence

$$f_{t+\tau_1+\tau_2}\big(\phi_{-\tau_1}(\theta)\big) \geqslant f_t(\theta) + k. \tag{13}$$

The proposition follows from (12) and (13) by taking $C := e^k\beta^2 a^2$. □

Given $\delta > 0$ take q so that $\delta - 2q > 0$. For this q, let m, C, τ_1 and τ_2 be given by Proposition 5.3.

Given a closed interval $[a,b] \subset \mathbb{R}$ define

$$F : SM \times [a,b] \to SM \times [a, b+2q]$$

by

$$F(\theta,t) = \big(\phi_{-\tau_1(\theta,t)}(\theta), t + \tau_1(\theta,t) + \tau_2(\theta,t)\big).$$

Set $\mathrm{d}\mu = \mathrm{d}\theta\,\mathrm{d}t$. The next lemma is taken from [4]. We include its proof for the sake of completeness.

LEMMA 5.4. – *Let $\Phi : SM \times [a, b+2q] \to \mathbb{R}^+$ be an integrable function with respect to the measure μ. Then*

$$\int\limits_{SM\times[a,b]} (\Phi\circ F)\,\mathrm{d}\mu \leqslant (m+1)^2 \int\limits_{SM\times[a,b+2q]} \Phi\,\mathrm{d}\mu.$$

Proof. – Given integers $0 \leqslant i \leqslant m$, $0 \leqslant j \leqslant m$, define

$$A(i,j) = \big\{(\theta,t) \in SM \times [a,b]:\ \tau_1(\theta,t) = iq/m,\ \tau_2(\theta,t) = jq/m\big\}.$$

On each $A(i,j)$, F is injective and $\mu(F(S)) = \mu(S)$ for every Borel set $S \subset A(i,j)$. Hence for any integrable function $\Phi : SM \times [a, b+2q] \to \mathbb{R}$ we have

$$\int\limits_{F(A(i,j))} \Phi\,\mathrm{d}\mu = \int\limits_{A(i,j)} (\Phi\circ F)\,\mathrm{d}\mu.$$

Suppose now that $\Phi \geqslant 0$. Then

$$\begin{aligned}
\int\limits_{SM\times[a,b]} (\Phi\circ F)\,\mathrm{d}\mu &= \int\limits_{\bigcup_{i,j} A(i,j)} (\Phi\circ F)\,\mathrm{d}\mu = \sum_{i,j} \int\limits_{A(i,j)} (\Phi\circ F)\,\mathrm{d}\mu \\
&= \sum_{i,j} \int\limits_{F(A(i,j))} \Phi\,\mathrm{d}\mu \leqslant \sum_{i,j} \int\limits_{SM\times[a,b+2q]} \Phi\,\mathrm{d}\mu \\
&= (m+1)^2 \int\limits_{SM\times[a,b+2q]} \Phi\,\mathrm{d}\mu.
\end{aligned}$$
□

PROPOSITION 5.5. – *There exists a constant $D > 0$ such that for all T,*

$$\int\limits_{T+2q-\delta}^{T+2q} \int\limits_{SM} e^{f_t(\theta)} A(\theta,t)\,\mathrm{d}\theta\,\mathrm{d}t \geqslant D \int\limits_{T+2q-\delta}^{T} \int\limits_{SM} e^{f_t(\theta)} \mathrm{ex}(d_\theta\phi_t)\,\mathrm{d}\theta\,\mathrm{d}t.$$

Proof. – Let $\Phi : SM \times [T+2q-\delta, T+2q] \to \mathbb{R}$ be given by

$$\Phi(\theta,t) := e^{f_t(\theta)} A(\theta,t) \geqslant 0.$$

By Proposition 5.3 we have,

$$\Phi \circ F(\theta,t) = e^{f_{t+\tau_1+\tau_2}(\phi_{-\tau_1}(\theta))} A\big(\phi_{-\tau_1}(\theta), t+\tau_1+\tau_2\big) \geqslant C e^{f_t(\theta)} \mathrm{ex}(d_\theta \phi_t).$$

Lemma 5.4 and the last inequality give:

$$\int_{T+2q-\delta}^{T+2q} \int_{SM} e^{f_t(\theta)} A(\theta,t)\, \mathrm{d}\theta\, \mathrm{d}t \geqslant \frac{C}{(m+1)^2} \int_{T+2q-\delta}^{T} \int_{SM} e^{f_t(\theta)} \mathrm{ex}(d_\theta \phi_t)\, \mathrm{d}\theta\, \mathrm{d}t.$$

Now take $D := C/(m+1)^2$. □

Let us prove Theorem A. Note that if $f : [0,\infty) \to (0,\infty)$ is any continuous function, then

$$\liminf_{T\to\infty} \frac{1}{T} \log f(T) \leqslant \liminf_{T\to\infty} \frac{1}{T} \log \int_{T-\delta}^{T} f(t)\, \mathrm{d}t.$$

By Theorem 2.1 we have

$$\begin{aligned} P(f) &\leqslant \liminf_{T\to\infty} \frac{1}{T} \log \int_{SM} e^{f_T(\theta)} \mathrm{ex}(d_\theta \phi_T)\, \mathrm{d}\theta \\ &\leqslant \liminf_{T\to\infty} \frac{1}{T} \log \int_{T+2q-\delta}^{T} \int_{SM} e^{f_t(\theta)} \mathrm{ex}(d_\theta \phi_t)\, \mathrm{d}\theta\, \mathrm{d}t. \end{aligned}$$

Combining this with Proposition 5.5 yields

$$P(f) \leqslant \liminf_{T\to\infty} \frac{1}{T} \log \int_{T-\delta}^{T} \int_{SM} e^{f_t(\theta)} A(\theta,t)\, \mathrm{d}\theta\, \mathrm{d}t,$$

and from Proposition 3.2 we obtain

$$P(f) \leqslant \liminf_{T\to\infty} \frac{1}{T} \log \int_{M\times M} F_{T,\delta}(x,y)\, \mathrm{d}x\, \mathrm{d}y,$$

which proves the first inequality in Theorem A.

Now observe that by the definition of the expansion we have

$$A(\theta,t) = \big|\det d_\theta(\pi \circ \phi_t)|_{V(\theta)}\big| \leqslant \big|\det d_\theta \phi_t|_{V(\theta)}\big| \leqslant \mathrm{ex}(d_\theta \phi_t),$$

$$\int_{M\times M} F_{T,\delta}(x,y)\, \mathrm{d}x\, \mathrm{d}y = \int_0^T \int_{SM} e^{f_t(\theta)} A(\theta,t)\, \mathrm{d}\theta\, \mathrm{d}t \leqslant \int_0^T \int_{SM} e^{f_t(\theta)} \mathrm{ex}(d_\theta \phi_t)\, \mathrm{d}\theta\, \mathrm{d}t, \tag{14}$$

and

$$\int\limits_{M\times M} F_{T,\delta}(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int\limits_{T-\delta}^{T}\int\limits_{SM} e^{f_t(\theta)}A(\theta,t)\,\mathrm{d}\theta\,\mathrm{d}t \leqslant \int\limits_{T-\delta}^{T}\int\limits_{SM} e^{f_t(\theta)}\mathrm{ex}(d_\theta\phi_t)\,\mathrm{d}\theta\,\mathrm{d}t. \tag{15}$$

LEMMA 5.6. – *Given $\delta>0$, there exist constants $c_1,c_2>0$ such that for all $T\in\mathbb{R}$, $t\in[T-\delta,T]$ and $\theta\in SM$ we have*

$$c_1 e^{f_T(\theta)}\mathrm{ex}(d_\theta\phi_T) \leqslant e^{f_t(\theta)}\mathrm{ex}(d_\theta\phi_t) \leqslant c_2 e^{f_T(\theta)}\mathrm{ex}(d_\theta\phi_T)$$

Proof. – Observe that for any t and s in $\mathbb{R}$,

$$\mathrm{ex}(d_\theta\phi_t) \leqslant \mathrm{ex}(d_{\phi_{t+s}\theta}\phi_{-s})\mathrm{ex}(d_\theta\phi_{t+s}).$$

Given $t\in[T-\delta,T]$ write $T=t+s$ where $s\in[0,\delta]$. Since M is compact, there exists a constant $C>0$ such that for all $\theta\in SM$ and all $s\in[-\delta,0]$ we have

$$\mathrm{ex}(d_\theta\phi_s)\leqslant C,$$

hence,

$$\mathrm{ex}(d_\theta\phi_t)\leqslant C\,\mathrm{ex}(d_\theta\phi_T).$$

Also, since f is bounded there exists a constant D such that

$$f_t(\theta)\leqslant f_T(\theta)+D,$$

therefore

$$e^{f_t(\theta)}\mathrm{ex}(d_\theta\phi_t)\leqslant Ce^D e^{f_T(\theta)}\mathrm{ex}(d_\theta\phi_T).$$

Take $c_2 := Ce^D$. Similarly one finds a positive constant c_1. □

Using Lemma 5.6 and (15) we have

$$\int\limits_{M\times M} F_{T,\delta}(x,y)\,\mathrm{d}x\,\mathrm{d}y \leqslant c_2\delta \int\limits_{SM} e^{f_T(\theta)}\mathrm{ex}(d_\theta\phi_T)\,\mathrm{d}\theta,$$

and thus

$$\limsup_{T\to\infty}\frac{1}{T}\log\int\limits_{M\times M} F_{T,\delta}(x,y)\,\mathrm{d}x\,\mathrm{d}y \leqslant \limsup_{T\to\infty}\frac{1}{T}\log\int\limits_{SM} e^{f_T(\theta)}\mathrm{ex}(d_\theta\phi_T)\,\mathrm{d}\theta.$$

If g is C^∞, the equality in Theorem 2.1 ensures that

$$\limsup_{T\to\infty}\frac{1}{T}\log\int\limits_{M\times M} F_{T,\delta}(x,y)\,\mathrm{d}x\,\mathrm{d}y \leqslant P(f),$$

which combined with the inequality in Theorem A yields

$$\lim_{T\to\infty}\frac{1}{T}\log\int\limits_{M\times M} F_{T,\delta}(x,y)\,\mathrm{d}x\,\mathrm{d}y = P(f).$$

To obtain the second equality in Theorem A, observe that if $P(f) \geqslant 0$, then

$$\limsup_{T\to\infty} \frac{1}{T} \log \int_0^T \int_{SM} e^{f_t(\theta)} \mathrm{ex}(d_\theta \phi_t)\, \mathrm{d}\theta\, \mathrm{d}t \leqslant \limsup_{T\to\infty} \frac{1}{T} \log \int_{SM} e^{f_T(\theta)} \mathrm{ex}(d_\theta \phi_T)\, \mathrm{d}\theta,$$

which combined with (14) gives

$$\limsup_{T\to\infty} \frac{1}{T} \log \int_{M\times M} F_T(x,y)\, \mathrm{d}x\, \mathrm{d}y \leqslant P(f),$$

as desired. □

A useful consequence of the previous arguments is summarized in the next proposition.

PROPOSITION 5.7. – *For any $\delta > 0$ we have*

$$\liminf_{T\to\infty} \frac{1}{T} \log \int_{M\times M} F_{T,\delta}(x,y)\, \mathrm{d}x\, \mathrm{d}y = \liminf_{T\to\infty} \frac{1}{T} \log \int_{SM} e^{f_T(\theta)} \mathrm{ex}(d_\theta \phi_T)\, \mathrm{d}\theta,$$

and

$$\limsup_{T\to\infty} \frac{1}{T} \log \int_{M\times M} F_{T,\delta}(x,y)\, \mathrm{d}x\, \mathrm{d}y = \limsup_{T\to\infty} \frac{1}{T} \log \int_{SM} e^{f_T(\theta)} \mathrm{ex}(d_\theta \phi_T)\, \mathrm{d}\theta.$$

Proof. – It follows right away from Proposition 5.5, inequality (15) and Lemma 5.6. □

6. Proof of Theorem B

Let M be a closed Riemannian manifold without conjugate points. Let $\widetilde{M}$ be the universal covering of M with covering projection p. Given x and y in M, let $\tilde{x}$ and $\tilde{y}$ be points in $\widetilde{M}$ with $p(\tilde{x}) = x$ and $p(\tilde{y}) = y$. Since given two points in $\widetilde{M}$ there exists a unique geodesic connecting them, there is a one to one correspondence between the sets

$$\{\gamma_{x,y}\colon \ell(\gamma_{x,y}) \leqslant T\},$$
$$\mathcal{A}_T(\tilde{x},\tilde{y}) := \{\alpha \in \pi_1(M)\colon d(\tilde{x},\alpha\tilde{y}) \leqslant T\}.$$

Similarly for any $\delta > 0$ we have a one to one correspondence between the sets

$$\{\gamma_{x,y}\colon T-\delta < \ell(\gamma_{x,y}) \leqslant T\},$$
$$\mathcal{A}_{T,\delta}(\tilde{x},\tilde{y}) := \{\alpha \in \pi_1(M)\colon T-\delta < d(\tilde{x},\alpha\tilde{y}) \leqslant T\}.$$

Given a function $f: SM \to \mathbb{R}$ and a point $\tilde{x} \in \widetilde{M}$ we can define a new function $f_{\tilde{x}}: \widetilde{M} \to \mathbb{R}$ as follows. Given $\tilde{y} \in \widetilde{M}$, let $v \in T_{\tilde{x}}\widetilde{M}$ be the unique unit vector such that the geodesic defined by $(\tilde{x}, v)$ connects $\tilde{x}$ to $\tilde{y}$. Now set

$$f_{\tilde{x}}(\tilde{y}) := f_{d(\tilde{x},\tilde{y})}\big(x, d_{\tilde{x}}p(v)\big).$$

Clearly

$$F_T(x,y) = \sum_{\alpha\in\mathcal{A}_T(\tilde{x},\tilde{y})} e^{f_{\tilde{x}}(\alpha\tilde{y})}. \tag{16}$$

LEMMA 6.1. – *For all $\delta > 0$ we have,*

$$\limsup_{T\to\infty}\frac{1}{T}\log\int_{M\times M}F_{T,\delta}(x,y)\,\mathrm{d}x\,\mathrm{d}y\leqslant P(f).$$

Proof. – Choose a compact fundamental domain D and take $\tilde{x},\tilde{y}\in D$ with $p(\tilde{x})=x$ and $p(\tilde{y})=y$. Take a constant $c>0$ such that for all $\alpha\in\pi_1(M)$ different from the identity and all $\tilde{y}\in D$ we have $d(\tilde{y},\alpha\tilde{y})>c$. Fix $\delta>0$ such that $c-2\delta>0$. We shall prove the statement of the lemma for this fixed δ. By Proposition 5.7 this implies that the lemma holds for all $\delta>0$. Let $\widetilde{E}\subset S\widetilde{M}$ be the set given by those points of the form $(\tilde{x},v_\alpha)$, where $v_\alpha\in T_{\tilde{x}}\widetilde{M}$ is the unique unit vector such that the geodesic $\gamma_{(\tilde{x},v_\alpha)}$ defined by $(\tilde{x},v_\alpha)$ connects $\tilde{x}$ to $\alpha\in\mathcal{A}_{T,\delta}(\tilde{x},\tilde{y})$. Take $\alpha\neq\beta$ and set

$$A:=\gamma_{(\tilde{x},v_\alpha)}(T),\qquad B:=\gamma_{(\tilde{x},v_\beta)}(T).$$

By the triangle inequality we have

$$c<d(\alpha\tilde{y},\beta\tilde{y})\leqslant T-d(\tilde{x},\alpha\tilde{y})+d(A,B)+T-d(\tilde{x},\beta\tilde{y})<2\delta+d(A,B).$$

Hence the set $E=dp(\widetilde{E})$ is (T,ε)-separated for $\varepsilon:=\frac{1}{2}\min\{c-2\delta,r(M)\}$, where $r(M)$ is the injective radius of M. Since f is continuous, there exists a constant k such that $f_{\tilde{x}}(\alpha\tilde{y})\leqslant f_T(x,d_{\tilde{x}}p(v_\alpha))+k$ and hence

$$F_{T,\delta}(x,y)=F_T(x,y)-F_{T-\delta}(x,y)\leqslant e^k\sum_{(x,v)\in E}e^{f_T(x,v)}\leqslant e^k r(T,\varepsilon,f).$$

If we integrate with respect to x and y we obtain

$$\int_{M\times M}F_{T,\delta}(x,y)\,\mathrm{d}x\,\mathrm{d}y\leqslant \mathrm{Vol}(M\times M)e^k r(T,\varepsilon,f),$$

which yields

$$\limsup_{T\to\infty}\frac{1}{T}\log\int_{M\times M}F_{T,\delta}(x,y)\,\mathrm{d}x\,\mathrm{d}y\leqslant r(\varepsilon,f)\leqslant P(f).\qquad\square$$

LEMMA 6.2. – *Let $D\subset\widetilde{M}$ be a compact fundamental domain with diameter c. Given any $\tilde{x}_1,\tilde{y}_1,\tilde{x}_2,\tilde{y}_2\in D$ and $T>0$ we have*

$$\mathcal{A}_T(\tilde{x}_1,\tilde{y}_1)\subseteq\mathcal{A}_{T+4c}(\tilde{x}_2,\tilde{y}_2).$$

Similarly for any $\delta>0$.

$$\mathcal{A}_{T,\delta}(\tilde{x}_1,\tilde{y}_1)\subseteq\mathcal{A}_{T+4c,\delta+4c}(\tilde{x}_2,\tilde{y}_2).$$

Proof. – The triangle inequality implies right away that

$$\mathcal{A}_T(\tilde{x},\tilde{y})\subseteq\mathcal{A}_{T+c}(\tilde{x},\tilde{x}),$$
$$\mathcal{A}_T(\tilde{x},\tilde{x})\subseteq\mathcal{A}_{T+c}(\tilde{x},\tilde{y}),$$
$$\mathcal{A}_{T,\delta}(\tilde{x},\tilde{y})\subseteq\mathcal{A}_{T+c,\delta+c}(\tilde{x},\tilde{x}),$$
$$\mathcal{A}_{T,\delta}(\tilde{x},\tilde{x})\subseteq\mathcal{A}_{T+c,\delta+c}(\tilde{x},\tilde{y}).$$

Apply these inclusions twice to obtain the lemma. □

Suppose now that $f:SM\to\mathbb{R}$ is the restriction to SM of a closed smooth 1-form $\omega:TM\to\mathbb{R}$. Since M is simply connected and ω is closed there exists a smooth function $g:\widetilde{M}\to\mathbb{R}$ such that $p^*\omega=dg$. In this case $f_{\tilde{x}}$ is simply given by

$$f_{\tilde{x}}(\tilde{y})=g(\tilde{y})-g(\tilde{x}).$$

LEMMA 6.3. – *There exists a constant $C>0$ such that for any $x_1,y_1,x_2,y_2\in M$ and $T>0$ we have*

$$F_T(x_1,y_1)\leqslant CF_{T+4c}(x_2,y_2).$$

Proof. – Take points $\tilde{x}_1,\tilde{y}_1,\tilde{x}_2,\tilde{y}_2\in D$ such that they project under p to x_1,y_1,x_2,y_2. Using (16), since g is bounded on D, there exists a constant $C_1>0$ such that

$$F_T(x_1,y_1)=e^{-g(\tilde{x}_1)}\sum_{\alpha\in\mathcal{A}_T(\tilde{x}_1,\tilde{y}_1)}e^{g(\alpha\tilde{y}_1)}\leqslant C_1\sum_{\alpha\in\mathcal{A}_T(\tilde{x}_1,\tilde{y}_1)}e^{g(\alpha\tilde{y}_1)}.$$

Now observe that for any $\alpha\in\pi_1(M)$,

$$g(\alpha\tilde{y}_1)-g(\alpha\tilde{y}_2)=g(\tilde{y}_1)-g(\tilde{y}_2).$$

Since g is bounded on D there exists a constant C_2 such that $g(\alpha\tilde{y}_1)-g(\alpha\tilde{y}_2)\leqslant C_2$, hence

$$F_T(x_1,y_1)\leqslant C_1e^{C_2}\sum_{\alpha\in\mathcal{A}_T(\tilde{x}_1,\tilde{y}_1)}e^{g(\alpha\tilde{y}_2)}.$$

By Lemma 6.2

$$\sum_{\alpha\in\mathcal{A}_T(\tilde{x}_1,\tilde{y}_1)}e^{g(\alpha\tilde{y}_2)}\leqslant\sum_{\alpha\in\mathcal{A}_{T+4c}(\tilde{x}_2,\tilde{y}_2)}e^{g(\alpha\tilde{y}_2)},$$

which yields

$$F_T(x_1,y_1)\leqslant C_1e^{C_2}e^{g(\tilde{x}_2)}F_{T+4c}(x_2,y_2).$$

Once again, since g is bounded on D, there exists a constant $C>0$ such that

$$C_1e^{C_2}e^{g(\tilde{x}_2)}\leqslant C,$$

therefore

$$F_T(x_1,y_1)\leqslant CF_{T+4c}(x_2,y_2),$$

as desired. □

Suppose now that $f\equiv -s$ for $s\in\mathbb{R}$. In this case

$$f_{\tilde{x}}(\tilde{y})=-sd(\tilde{x},\tilde{y}).$$

LEMMA 6.4. – *There exists a constant $C>0$ such that for any $x_1,y_1,x_2,y_2\in M$ and $T>0$ we have*

$$F_{T,\delta}(x_1,y_1)\leqslant CF_{T+4c,\delta+4c}(x_2,y_2).$$

Proof. – Take points $\tilde{x}_1, \tilde{y}_1, \tilde{x}_2, \tilde{y}_2 \in D$ such that they project under p to x_1, y_1, x_2, y_2. By the triangle inequality,

$$d(\tilde{x}_1, \alpha\tilde{y}_1) \leqslant d(\tilde{x}_2, \alpha\tilde{y}_2) + 2c.$$

Suppose that $s \leqslant 0$; hence

$$F_{T,\delta}(x_1, y_1) = \sum_{\alpha \in \mathcal{A}_{T,\delta}(\tilde{x}_1, \tilde{y}_1)} e^{-sd(\tilde{x}_1, \alpha\tilde{y}_1)} \leqslant e^{-2cs} \sum_{\alpha \in \mathcal{A}_{T,\delta}(\tilde{x}_1, \tilde{y}_1)} e^{-sd(\tilde{x}_2, \alpha\tilde{y}_2)}.$$

By Lemma 6.2

$$\sum_{\alpha \in \mathcal{A}_{T,\delta}(\tilde{x}_1, \tilde{y}_1)} e^{-sd(\tilde{x}_2, \alpha\tilde{y}_2)} \leqslant \sum_{\alpha \in \mathcal{A}_{T+4c,\delta+4c}(\tilde{x}_2, \tilde{y}_2)} e^{-sd(\tilde{x}_2, \alpha\tilde{y}_2)} = F_{T+4c,\delta+4c}(x_2, y_2),$$

thus

$$F_{T,\delta}(x_1, y_1) \leqslant e^{-2cs} F_{T+4c,\delta+4c}(x_2, y_2),$$

as desired. If $s \geqslant 0$ we proceed in a similar manner. □

Theorem B is now a consequence of the first inequality in Theorem A and Lemmas 6.1, 6.3 and 6.4.

7. Proof of Corollary 3

Take $s \in \mathbb{C}$. Let us apply Proposition 3.2 to the case $f \equiv -s$. We obtain,

$$\int_{M \times M} \left(\sum_{\{\gamma_{x,y}\colon\, \ell(\gamma_{x,y}) \leqslant T\}} e^{-s\ell(\gamma_{x,y})} \right) \mathrm{d}x \, \mathrm{d}y = \int_0^T \int_{SM} e^{-st} A(\theta, t) \, \mathrm{d}\theta \, \mathrm{d}t.$$

Hence $\eta_g(s)$ is the Laplace transform of the continuous function

$$t \mapsto \int_{SM} A(\theta, t) \, \mathrm{d}\theta,$$

and therefore it defines a holomorphic function on the half plane $\mathrm{Re}(s) > \lambda$, where λ is the infimum of $\mathrm{Re}(s)$ where s runs over those complex numbers for which the integral

$$\int_0^\infty \int_{SM} e^{-st} A(\theta, t) \, \mathrm{d}\theta \, \mathrm{d}t$$

converges. Let us prove that $\lambda = h_{\mathrm{top}}$. Suppose that $s \in \mathbb{R}$. Observe that $P(f \equiv -s) = h_{\mathrm{top}} - s$. If $h_{\mathrm{top}} - s > 0$, Theorem A implies that

$$\int_{M \times M} \left(\sum_{\{\gamma_{x,y}\colon\, \ell(\gamma_{x,y}) \leqslant T\}} e^{-s\ell(\gamma_{x,y})} \right) \mathrm{d}x \, \mathrm{d}y,$$

has a positive exponential growth rate, and therefore $h_{\text{top}} \leqslant \lambda$. If $h_{\text{top}} - s < 0$, Theorem A implies that

$$\int\limits_{M\times M} \left(\sum_{\{\gamma_{x,y}:\ \ell(\gamma_{x,y})\leqslant T\}} e^{-s\ell(\gamma_{x,y})} \right) \mathrm{d}x\,\mathrm{d}y,$$

is bounded above and since it is increasing the limit when $T \to \infty$ exists and is finite. Hence $h_{\text{top}} \geqslant \lambda$. Similarly, using Theorem B one shows the claim for $\eta_g(s,x,y)$.

If (M,g) is a manifold all of whose geodesics are closed with common period τ, then

$$\eta_g(s) = \frac{\int_0^\tau \int_{SM} e^{-st} A(\theta,t)\,\mathrm{d}\theta\,\mathrm{d}t}{1 - e^{-s\tau}}.$$

If (M^n,g) is a flat manifold,

$$\eta_g(s) = \frac{c}{s^n},$$

where c is a constant. If (M,g) is a surface with constant curvature -1, then

$$\eta_g(s) = \frac{c}{s^2-1},$$

where c is a constant.

Note that in these three cases η_g admits a meromorphic extension to the complex plane.

REFERENCES

[1] M.J. DIAS CARNEIRO, On minimizing measures of the action of autonomous Lagrangians, *Nonlinearity* 8 (1995) 1077–1085.

[2] H. FEDERER, *Geometric Measure Theory*, Springer, New York, 1969.

[3] O.S. KOZLOVSKI, An integral formula for topological entropy of C^∞ maps, *Ergodic Theory Dynam. Systems* 18 (1998) 405–424.

[4] R. MAÑÉ, On the topological entropy of geodesic flows, *J. Differential Geom.* 45 (1997) 74–93.

[5] J. MATHER, Action minimizing measures for positive definite Lagrangian systems, *Math. Z.* 207 (1991) 169–207.

[6] D. MASSART, Normes stables pour les surfaces, Thèse, E.N.S. Lyon, 1996.

[7] F. PRZYTYCKI, An upper estimation for topological entropy of diffeomorphisms, *Invent. Math.* 59 (1980) 205–213.

[8] P. WALTERS, *An Introduction to Ergodic Theory*, Graduate Texts in Mathematics, Springer, Berlin, 1982.

[9] Y. YOMDIN, Volume growth and entropy, *Israel J. Math.* 57 (1987) 287–300.

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Gabriel P. PATERNAIN
Centro de Matemática,
Facultad de Ciencias, Iguá 4225,
Montevideo CP 11400, Uruguay
e-mail: gabriel@cmat.edu.uy